Optimal Control of Initial-Boundary Value Problems for Hyperbolic Balance Laws with Switching Controls and State Constraints

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Zusammenfassung

Die vorliegende Arbeit befasst sich mit der optimalen Steuerung hyperbolischer Bilanzgleichungen mit schaltenden Steuerungen und Zustandsschranken. Diese Thematik kann am Beispiel des Gastransports in Netzwerken motiviert werden. Die mathematische Beschreibung des Gasflusses erfolgt durch die kompressiblen Eulergleichungen, die ein gekoppeltes System hyperbolischer Bilanzgleichungen bilden. Netzwerkkomponenten wie Verzweigungen, Ventile, Schieber und Verdichter werden durch geeignete Knoten- bzw. Randbedingungen modelliert. Schaltende Steuerungen, welche abrupte Änderungen der Steuerungsparameter beschreiben, treten hier beispielsweise durch das Öffnen und Schließen von Ventilen auf. Ziel ist die Optimierung der Druck- und Geschwindigkeitsverteilung des transportierten Gases durch die zeitabhängige Steuerung der Netzwerkkomponenten, wobei als weitere Anforderung der resultierende Druck im gesamten Netzwerk innerhalb eines gewissen Toleranzbereichs liegen muss. Dieser Bereich wird mathematisch durch Zustandsschranken modelliert. Als weiteres Beispiel sei an dieser Stelle die Optimierung des Verkehrsflusses in Netzwerken genannt, der im LWR-Modell durch eine skalare hyperbolische Bilanzgleichung beschrieben wird. Schaltende Steuerungen treten hier in Form der Schaltzeitpunkte von Ampeln auf, durch deren Wahl der Verkehrsfluss optimiert werden soll. Zustandsschranken treten in Form der Forderung auf, dass in gewissen Bereichen die Verkehrdichte unterhalb eines festgelegten Wertes liegen soll.

Die hier aufgezählten Beispiele führen jeweils zu einem Optimalsteuerungsproblem mit hyperbolischen Bilanzgleichungen als Nebenbedingungen.

Die Frage nach der Existenz und Eindeutigkeit von Lösungen hyperbolischer Bilanzgleichungen auf unbeschränkten Gebieten mit Anfangsbedingungen wurde bereits in einer Vielzahl von Veröffentlichungen diskutiert. Die Schwierigkeit dieser Fragestellung beruht darauf, dass selbst bei glatten Daten die Lösungen nach endlicher Zeit Unstetigkeiten ausbilden können, sodass nur die Existenz schwacher Lösungen erwartet werden kann. Da diese jedoch aufgrund der auftretenden Unstetigkeiten nicht eindeutig sind, wird unter dieser Vielzahl schwacher Lösungen die physikalisch sinnvolle Lösung, welche durch den eindeutigen Grenzwert der parabolischen Regularisierung gegeben ist, mittels geeigneter Entropiebedingungen charakterisiert. Die Existenz und Eindeutigkeit solcher Entropielösungen wurde im Fall skalarer Bilanzgleichungen weitreichend beantwortet. Betrachtet man Systeme, so gibt es im Fall einer Ortsdimension entsprechende Resultate bezüglich der Existenz und Eindeutigkeit von Lösungen, wobei der Fall mehrerer Ortsdimensionen nach wie vor ein offenes Problem darstellt. Betrachtet man hyperbolische Bilanzgleichungen auf beschränkten Gebieten, so stellt sich zusätzlich die Frage nach einer

geeigneten Formulierung der Randbedingungen, die zu einem wohlgestellten Problem führt. Eine passende Formulierung wurde mittels Grenzwertbetrachtung der parabolischen Regularisierung gewonnen, in welcher die Spur der Lösung nur am Einströmrand mit den vorgegebenen Randdaten übereinstimmen muss.

Die einführenden Beispiele zeigen, dass neben der Frage der Existenz und Eindeutigkeit von Lösungen in vielen Anwendungen insbesondere die optimale Steuerung der hyperbolischen Bilanzgleichungen eine wichtige Rolle spielt. Das Auftreten von Unstetigkeiten in Entropielösungen führt dazu, dass die Steuerungs-Zustandsabbildung nur in sehr schwachen Räumen differenzierbar ist, sodass die Differenzierbarkeit des reduzierten Zielfunktionals nicht durch Verwendung von Standardargumenten hergeleitet werden kann. Es wurden bereits einige Konzepte zur Lösung dieser Problematik entwickelt. Die bisherigen Ansätze für den Systemfall setzen jedoch entweder starke Annahmen an die Lösungsstruktur voraus oder sind auf hyperbolische Erhaltungsgleichungen beschränkt und nicht auf Bilanzgleichungen erweiterbar. Daher beschränken wir uns in der vorliegenden Arbeit auf skalare Bilanzgleichungen in 1D.

Für diesen Fall wurde innerhalb der letzten Jahre in [69, 81] ein Sensitivitäts- und Adjungiertenkalkül entwickelt, wobei keine speziellen Forderungen an die Lösungsstruktur gestellt werden. Hauptwerkzeug ist dabei das in [81] eingeführte Konzept der *Shift-Differenzierbarkeit*. Es wurde gezeigt, dass die Komposition einer Reihe von Zielfunktionalen mit einer shift-differenzierbaren Funktion Fréchet-differenzierbar ist. Auf Grundlage des von Dafermos entwickelten Konzepts der verallgemeinerten Charakteristiken wurde in [81] die Lösungsstruktur von Entropielösungen mit beschränkter Variation analysiert. Unter Verwendung dieser Struktur wurde im Anschluss die Shift-Differenzierbarkeit der Steuerungs-Zustandsabbildung und somit die Fréchet-Differenzierbarkeit des reduzierten Zielfunktionals nachgewiesen. Diese Ergebnisse wurden in [69] auf Anfangs-Randwertprobleme erweitert.

Unter Verwendung des in [69, 81] entwickelten Sensitivitäts- und Adjungiertenkalküls sollen in der vorliegenden Arbeit notwendige Optimalitätsbedingungen für Probleme mit skalaren hyperbolischen Bilanzgleichungen und punktweisen Zustandsschranken hergeleitet werden. Hierbei stellen die punktweisen Zustandsschranken eine besondere Herausforderung dar. Der Grund hierfür liegt unter anderem darin, dass die benötigte Constraint-Qualification Stetigkeit der Steuerungs-Zustandsabbildung nach L^∞ voraussetzt. Aufgrund der Unstetigkeiten in der Entropielösung ist dies jedoch nicht gegeben. Mithilfe der besonderen Struktur von BV-Entropielösungen, welche bereits zur Herleitung der Shift-Differenzierbarkeit eine Schlüsselrolle spielte, werden in der vorliegenden Arbeit neue Zustandsvariablen eingeführt. Unter Verwendung dieser neuen Variablen werden in einem ersten Schritt notwendige Optimalitätsbedingungen hergeleitet. Da es sich bei den Lagrange-Multiplikatoren um Maße handelt, ist die Berechnung einer Lösung des Optimalitätssystems sehr kompliziert. Daher soll in einem zweiten Schritt der Moreau-Yosida-Regularisierungsansatz zur Behandlung der Zustandsschranken diskutiert werden. Aufbauend auf einer Reihe von Vorarbeiten zu dem elliptischen und parabolischen Fall soll für das in dieser Arbeit betrachtete Optimalsteuerungsproblem eine Konvergenzanalyse der Moreau-Yosida-Regularisierung durchgeführt werden.

Contents

CHAPTER 1

Introduction

The present thesis deals with the optimal control of hyperbolic balance laws with switching controls and state constraints. This is motivated by the example of gas transport in networks, where the gas flow is described by the solution of the compressible Euler equations which form a coupled system of hyperbolic balance laws. Network components like junctions, valves, compressor stations are modeled by appropriate node and boundary conditions, see, e.g., [6, 25]. Fast changes of parameters between different modes, e.g., the opening and closing of valves, are characterized by the term *switching controls*. The goal is to optimize the gas flow in the network through the time-dependent control of the network components such that in addition the pressure and velocity lies within lower and upper bounds. These bounds are mathematically described by state constraints.

Another motivation for the problem considered in this thesis is the optimal control of traffic flow, see, e.g., [36, 69]. The traffic flow is mathematically described by the so-called *LWR-model* which is named after Lighthill, Whitham and Richards who developed this model in [56] and [74]. The optimal control problem in [69] consists of choosing the switching times of a traffic light such that the resulting traffic density distribution is optimal with respect to the considered cost functional, see also [71]. The switching of the traffic light corresponds to the opening and closing of a valve in a gas network and can be considered as a switching control. If the traffic density is required to stay below a certain value on some parts of the road, we again have to deal with state constraints.

As we can see, both examples lead to an optimal control problem involving switching controls, state constraints and hyperbolic balance laws. In general, a *balance law* is a partial differential equations of the form

$$y_t + \sum_{i=1}^{d} ((f_i(y))_{x_i} = g(t, x, y) \quad \text{on } \Omega_T :=]0, T[\times\Omega, \tag{1.0.1}$$

where $\Omega \subset \mathbb{R}^d$, $f_1, \ldots, f_d : \mathbb{R}^p \to \mathbb{R}^p$ and $g : [0, \infty[\times\mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^p$. In the following, we say that (1.0.1) is *hyperbolic* if for all $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ with $\|v\|_2 = 1$ and all $u \in \mathbb{R}^p$ the matrix

$$A(v, u) := \sum_{i=1}^{d} v_i \mathrm{D} f_i(u)$$

has p real eigenvalues and is diagonalizable. If in addition the eigenvalues all have distinct values, then we call (1.0.1) *strictly hyperbolic*, cf. [51].

If $g \equiv 0$, then we say that (1.0.1) is a *conservation law*. For the case that $p = 1$, (1.0.1) is called *scalar balance law*, for $p > 1$, (1.0.1) is a *system of balance laws*, respectively. One can easily see that scalar balance laws are always strictly hyperbolic.

Solutions to partial differential equations of type (1.0.1) can develop discontinuities, so-called *shock-curves*, even if one considers smooth data, see, e.g., [47, 51]. Therefore, classical solutions fail to exist on the whole time interval such that one has to consider weak solutions, which are due to the shock-curves in general not unique, see for example [49]. In order to solve the problem of this nonuniqueness, Lax, Kružkov and Oleinik introduce in [49, 47, 67] admissible conditions that single out the physically relevant *entropy solution*, which is obtained by the so-called *vanishing viscosity method*, see [42]. The authors consider initial value problems, i.e., $\Omega = \mathbb{R}^d$ and (1.0.1) is endowed with initial conditions. In [50], Lax derives an explicit formula for the solution of scalar hyperbolic conservation laws. Considering scalar balance laws, Kružkov proves existence and uniqueness of entropy solutions.

In [7], Bardos, LeRoux and Nédélec examine scalar hyperbolic balance laws on bounded domains. Using the vanishing viscosity method, the authors derive a proper formulation of Dirichlet boundary conditions, which are firstly only well-defined for functions with bounded variation. The theory of Bardos, LeRoux and Nédélec is extended to the L^∞-setting in [68, 59]. Moreover, Otto proves in [68] that this formulation is equivalent to the one derived in [7] as long as the solution admits boundary traces. In [23], Coclite, Karlsen and Kwon show the existence of such boundary traces under additional assumptions, see also [84].

For the case of systems of one dimensional hyperbolic balance laws on unbounded domains, there are several publications on the existence and uniqueness of entropy solutions, see, e.g., [1, 12, 13, 14, 28, 29]. The case of systems with $d > 1$ in (1.0.1) is still an open problem. In order to compute entropy solutions, there are a variety of numerical schemes for hyperbolic balance laws, see, e.g., [44, 54, 66, 85].

As we have seen, there is a high number of contribution to the question of existence and uniqueness of solutions to (1.0.1). Nonetheless, the motivating example of gas networks shows that beyond the question of existence and uniqueness, the optimal control of hyperbolic balance laws is an important issue which has been discussed by many authors in the recent years, see e.g., [3, 8, 10, 15, 18, 21, 22, 26, 27, 36, 52, 70, 81, 78]. For problems without state constraints, the existence of optimal solutions is proved for example in [3, 4, 27].

In order to compute such an optimal solution with e.g., some descent method, one needs the gradient of the reduced cost functional. However, due to the shock-curves appearing in the solutions to hyperbolic balance laws, the corresponding control-to-state mapping is not differentiable to L^1, see for example [16, 15, 10], such that one cannot use standard methods to deduce the Fréchet-differentiability of the reduced cost functional.

There are several authors who developed different methods to cope with this problem. In the following, we list some of these methods and examine the underlying assumptions. Bressan and Marson introduce in [17] so-called *generalized tangent vectors* yielding an approximation in L^1 of the control-to-state mapping, see also [16, 18]. These generalized tangent vectors consists of the sensitivities of the smooth parts and of the shock-curves. The authors require the solution of the hyperbolic balance law to be piecewise smooth and only allow variations of the control which preserve the structure. In [55], it is proved that these requirements on the structure hold locally if one considers generalized Riemann problems. The variational calculus which is developed in [10] does not require piecewise smooth solutions, but is only valid for the case of conservation laws and seems not to be extendable to balance laws.

Since we want to examine hyperbolic balance laws with $g \not\equiv 0$, the analysis in the present thesis builds up on the notion of *shift-differentiability* which is developed by Ulbrich in [81] and implies the Fréchet-differentiability for the considered cost functional, see also [82]. These results are based on the structure of solutions with bounded variation. This structure is analyzed in [81] by using the concept of generalized characteristics of Dafermos in [31]. Moreover, Ulbrich derives an adjoint representation of the gradient of the reduced cost functional. In [69], Pfaff extends these results to scalar hyperbolic balance laws on bounded domains, see also [70, 71].

Apart from the difficult issue concerning the differentiability of the control-to-state mapping, also the treatment of the pointwise state constraints poses some challenges. The goal of the present thesis is to combine the concepts of [69, 81] with those that were developed for the analysis and approximation of optimal control problems with pointwise state constraints. The latter concepts are based on [19], where Casas considers a quadratic problem for elliptic equations and pointwise state constraints. He derives optimality conditions where the Lagrange multipliers turn out to be measures. In [20], the authors consider an elliptic problem where in addition to the pointwise state constraints also gradient constraints are included. In [72, 73], optimal control problems for parabolic equations with pointwise constraints are discussed and necessary optimality conditions are derived. One method to approximate a solution of this optimality system is the so-called *Moreau-Yosida regularization*, which is first introduced by Ito and Kunisch in [43], see also [9]. This approach is applied to elliptic problems, e.g., in [62, 63], to parabolic problems in [64] and to the Navier-Stokes equations in [32]. Moreover, primal-dual path following concepts in connection with the Moreau-Yosida regularization are discussed in [37, 38, 39]. This approach can in addition be used for the treatment of problems with gradient constraints, see, e.g., [38] or [86]. In the present thesis we apply this regularization method to optimal control problems with hyperbolic balance laws and pointwise state constraints. Besides the Moreau-Yosida regularization there are also alternative approaches like for example the *virtual control concept* which is proposed in [46] for the regularization of a linear elliptic problem with pointwise state constraints. Moreover, the authors in [61] propose a *Lavrentiev type regularization*. This concept is discussed in [40] for elliptic optimal control problems and in [65] for parabolic problems.

The first aim of this thesis is to derive necessary optimality conditions for optimal control problems with hyperbolic balance laws and state constraints. In a further step, we approximate the corresponding optimality system by using Moreau-Yosida type regularizations. This leads to the second aim of this thesis, namely finding the answer to the question in which sense the optimality systems of the regularized problems converge to the one of the original optimal control problem with pointwise state constraints. This is very challenging since the shock-curves appearing in solutions of hyperbolic balance laws prohibit the use of standard methods to derive the Fréchet-differentiability of the underlying cost functional. Furthermore, since we consider state constraints, the solution of the hyperbolic balance law must at least have L^∞-regularity in order to assure that a Robinson type constraint qualification is possible to be satisfied. However, due to the shock-curves, the control-to-state mapping to L^∞ is not even continuous. To cope with these problems, we use the concepts developed by Pfaff and Ulbrich in [69, 81].

This thesis is organized as follows: In Chapter 2, we introduce the notation as well as some basic definitions and results which are used in this thesis. In Chapter 3, we first recall some results on the existence, uniqueness and stability of entropy solutions on bounded domains. Moreover, we introduce the optimal control problem and use the concepts of [69, 81] to derive the Fréchet-differentiability of the reduced cost functional. Following the ideas of [71], the sensitivity and adjoint calculus developed in [69, 81] will be extended to the case that the shifting of rarefaction centers in the initial and boundary data is allowed. Building up on these results, we derive necessary optimality conditions for the state-constrained problem in Chapter 4. Although there are various contributions discussing optimal control problems with pointwise state constraints, to the best of our knowledge the present work is the first attempt to derive optimality conditions for problems governed by hyperbolic balance laws and pointwise state constraints.

In Chapter 5, we will apply the Moreau-Yosida regularization approach to the state-constrained optimal control problem and prove that the regularized problems converge to the optimal control problem with state constraints. As already mentioned, this approach has already been discussed by several authors. But to the best of our knowledge, this is the first work analyzing Moreau-Yosida regularization for problems governed by hyperbolic balance laws.

In the last chapter, we will give a summary of the main results in this thesis and give a brief outlook to possible extensions.

We note that a paper dealing with the results of this thesis is in preparation and will appear soon as a joint work with Stefan Ulbrich.

CHAPTER 2

Notation, definitions and basic results

The basic notation, definitions and results that are used in the present thesis are introduced in this chapter. Most of the definitions and notations are also used in [69] and [81]. The current chapter is organized as follows: In §2.1, we introduce the basic notation. The definitions of Banach spaces and Hilbert spaces as well as some basic results are provided in §2.2. In §2.3, we give an overview of some classical function spaces that are used in this thesis. In §2.4, the basic definition of distributions and the concept of distributional derivatives are introduced. The definitions of Borel and Radon measures are collected in §2.5 and the last section of this chapter is concerned with the concept of functions with bounded variation.

2.1 Notation

Let $d \in \mathbb{N}$. Then we define:

- $\|x\|_1 \coloneqq \sum_{k=1}^{d} |x_k| \quad$ for all $x \in \mathbb{R}^d$
- $\|x\|_2 \coloneqq \sqrt{\sum_{k=1}^{d} x_k^2} \quad$ for all $x \in \mathbb{R}^d$
- $x \cdot y \coloneqq \sum_{k=1}^{d} x_k y_k \quad$ for all $x, y \in \mathbb{R}^d$

Considering a set $\Omega \subset \mathbb{R}^d$, we introduce the following notation:

- $\operatorname{int}\Omega$: Interior of Ω
- Ω^{cl}: Closure of Ω
- The indicator function:

$$\mathbb{1}_\Omega : \mathbb{R}^d \to \{1,0\}, \quad x \mapsto \mathbb{1}_\Omega(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{else.} \end{cases}$$

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ denote a function and $A \subset \mathbb{R}^d$ some subset of $\mathbb{R}^d$.

- $\varphi|_A$: Restriction of the function φ to the set A
- $\operatorname{supp}\varphi := \left\{x \in \mathbb{R}^d : \varphi(x) \neq 0\right\}^{\text{cl}}$

We consider a function $\varphi : \mathbb{R} \to \mathbb{R}$. Provided that the following limits exist, which holds for example if $\varphi \in BV(I)$, where the space $BV(I)$ will be introduced later, we use the following notation:

- $\varphi(\bar{x}-) := \lim_{x \nearrow \bar{x}} \varphi(x), \quad \varphi(\bar{x}+) := \lim_{x \searrow \bar{x}} \varphi(x)$
- $[\varphi(x)] := \varphi(x-) - \varphi(x+)$
- $I(\alpha, \beta) := [\min(\alpha,\beta), \max(\alpha,\beta)]$

(cf. [69, §2])

2.2 Normed spaces, Banach-spaces and Hilbert spaces

The definitons and results that are stated in this section can be found for example in [41, Ch.1]. See also [88], [75] and [76].

Definition 2.2.1 (Banach spaces)**.** *Consider a real vector space X and a mapping $\|\cdot\| : X \to [0,\infty[$ such that the following is true:*

(i) $\|x\|_X = 0 \iff x = 0$,

(ii) $\|\lambda x\|_X = \lambda \|x\|_X \quad \forall x \in X, \lambda \in \mathbb{R}$,

(iii) $\|x+y\|_X \leq \|x\|_X + \|y\|_X$.

Then we call $(X, \|\cdot\|_X)$ a normed space.

If $(X, \|\cdot\|_X)$ is in addition complete, i.e., every Cauchy sequence $(x_k)_{k\in\mathbb{N}} \subset X$ has a limit $x \in X$, then we say that $(X, \|\cdot\|_X)$ is a Banach space.

For normed spaces X and Y, the canonical norm on the cartesian product $X \times Y$

is given by

$$\|(x,y)\|_{X\times Y} \coloneqq \|x\|_X + \|y\|_Y .$$

Definition 2.2.2 (Hilbert space). *Suppose that for a real vector space H and a mapping $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ it holds that*

(i) $(u,v)_H = (v,u)_H \quad \forall u,v \in H$,

(ii) for all $u \in H$ the mapping $H \ni v \mapsto (u,v)_H$ is linear,

(iii) $(u,u)_H = 0 \iff u = 0$ *and* $(u,u)_H \geq 0 \quad \forall u \in H$.

Then we call $(\cdot,\cdot)_H$ an inner product and the vector space $(H, \|\cdot\|_H)$, where

$$\|\cdot\|_H \coloneqq \sqrt{(\cdot,\cdot)_H},$$

a Pre-Hilbert space. If $(H, \|\cdot\|_H)$ is in addition complete, then we say that $(H, \|\cdot\|_H)$ is a Hilbert space.

Definition 2.2.3 (Dual space). *Consider a Banach space $(X, \|\cdot\|_X)$ and define*

$$X^* = \{f : X \to \mathbb{R} : f \text{ is linear and bounded}\} .$$

We call the elements of X^* the linear functionals on X and the Banach space $(X^*, \|\cdot\|_{X^*})$ the dual space of $(X, \|\cdot\|_X)$, where

$$\|x^*\|_{X^*} = \sup_{\|x\|_X = 1} |x^*(x)| .$$

For $x^* \in X^*$, we finally introduce the so-called dual pairing

$$\langle x^*, x\rangle_{X^*,X} = x^*(x) \quad \forall x \in X.$$

We say that a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ converges weakly to some $\bar{x} \in X$, i.e., $x_k \rightharpoonup \bar{x}$, if

$$\lim_{k\to\infty} \langle x^*, x_k\rangle_{X^*,X} = \langle x^*, \bar{x}\rangle_{X^*,X} \quad \forall x^* \in X^*.$$

Finally, we say that a sequence $(x_k^*)_{k\in\mathbb{N}} \subset X^*$ converges weakly-* to some $\bar{x}^* \in X^*$, i.e., $x_k^* \overset{*}{\rightharpoonup} \bar{x}^*$, if

$$\lim_{k\to\infty} \langle x_k^*, x\rangle_{X^*,X} = \langle \bar{x}^*, x\rangle_{X^*,X} \quad \forall x \in X.$$

Theorem 2.2.4 (Riesz representation theorem). *Let $(H, \|\cdot\|_H)$ be a Hilbert space.*

Then for all $f \in H^*$ there exists a unique $u \in H$ such that

$$f(v) = (u, v)_H$$

holds for all $v \in H$ and $\|f\|_{H^*} = \|u\|_H$. Moreover, for all $u \in H$ the mapping

$$H \ni v \mapsto (u, v)_H \in \mathbb{R}$$

is an element of H^*.

Definition 2.2.5. *We call a normed space $(X, \|\cdot\|_X)$ separable if it contains a countable dense subset.*

A proof of the following result can be found in [58, Theorem 5.10.1].

Theorem 2.2.6 (Sequential Banach-Alaoglu theorem)**.** *Consider a separable normed vector space $(X, \|\cdot\|_X)$ and denote by $(X^*, \|\cdot\|_{X^*})$ the corresponding dual space. Then every bounded sequence $(x_k^*)_{k\in\mathbb{N}} \subset X^*$ contains a subsequence that converges weakly-* to some $\bar{x}^* \in X^*$.*

2.3 Classical function spaces

In this section, we will introduce some classical function spaces that will be used in this thesis, cf. [41, §1]. See also [79], [75], [76] and [88].

2.3.1 Lebesgue spaces

Considering a set $\Omega \subset \mathbb{R}^d$ with $1 \leq p < \infty$, we introduce the function spaces

$$\mathcal{L}^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \|f\|_{p,\Omega} := \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\}.$$

Furthermore, we define the space

$$\mathcal{L}^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \|f\|_{\infty,\Omega} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty \right\}.$$

Remark 2.3.1. The spaces above are, strictly speaking, no normed spaces since there are measurable functions $u \in \mathcal{L}^p(\Omega)$ with $\|u\|_{p,\Omega} = 0$ but $u(x) \neq 0$ for some $x \in \Omega$. Therefore, the *Lebesgues spaces* are defined by the equivalence classes

$$L^p(\Omega) := \mathcal{L}^p(\Omega)/\sim,$$

where the equivalence relation is given by

$$u \sim v \quad \Longleftrightarrow \|u - v\|_{p,\Omega} = 0 \quad \Longleftrightarrow \quad u = v \text{ a.e. on } \Omega. \tag{2.3.1}$$

We note that the last equivalence in (2.3.1) is well-known, see, e.g., [41]. For $p \in [1, \infty]$, the norm of the Lebesgue spaces can alternatively be denoted by

$$\|\cdot\|_{L^p(\Omega)} := \|\cdot\|_{p,\Omega}.$$

Furthermore, we define the function spaces

$$L^p_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} \text{ measurable} : f \in L^p(K) \text{ for all compact } K \subset \Omega\}.$$

Next, we will collect some basic properties of Lebesgue spaces, see, e.g., [41]:

Theorem 2.3.2 (Fischer-Riesz)**.** *Let $\Omega \subset \mathbb{R}^d$, then for all $p \in [1, \infty]$ the spaces $L^p(\Omega)$ are Banach-spaces and $L^2(\Omega)$ is a Hilbert space with inner product*

$$(f, g)_{2,\Omega} = (f, g)_{L^2(\Omega)} := \int_\Omega f(x) g(x)\,\mathrm{d}x.$$

Theorem 2.3.3. *For $1 \le p < \infty$ the Banach-spaces $L^p(\Omega)$ are separable.*

Theorem 2.3.4. *Let $\Omega \subset \mathbb{R}^d$ and $p, q \in [1, \infty]$ be chosen such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{2.3.2}$$

is valid. Then for all $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ Hölder's Inequality holds

$$uv \in L^1(\Omega) \quad \textit{and} \quad \|uv\|_{1,\Omega} \le \|u\|_{p,\Omega} \|v\|_{q,\Omega}.$$

In addition, for $p \in [1, \infty[$ the dual space $\left(L^p(\Omega)^*, \|\cdot\|_{L^p(\Omega)^*}\right)$ can be identified with

$\left(L^q(\Omega), \|\cdot\|_{L^q(\Omega)}\right)$ *via the isometric isomorphism*

$$L^q(\Omega) \ni v \mapsto u^* \in L^p(\Omega)^*, \quad \textit{where } \langle u^*, u\rangle_{L^p(\Omega)^*, L^p(\Omega)} := \int_\Omega u(x)v(x)\,\mathrm{d}x$$

and $q \in]1, \infty]$ *is chosen such that* (2.3.2) *is satisfied.*

2.3.2 Spaces of continuous and continuously differentiable functions

For $\Omega \subset \mathbb{R}^d$, we introduce the function space

$$C(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ is continuous}\}.$$

We note that for bounded sets Ω the space $\left(C(\Omega^{\mathrm{cl}}), \|\cdot\|_{C(\Omega^{\mathrm{cl}})}\right)$ is a Banach space with

$$\|f\|_{C(\Omega^{\mathrm{cl}})} := \sup_{x \in \Omega^{\mathrm{cl}}} |f(x)|.$$

Furthermore, for an open set $\Omega \subset \mathbb{R}^d$ we introduce the function spaces

$$C^k(\Omega) := \left\{f : \Omega \to \mathbb{R} : D^\beta f \in C(\Omega) \text{ for } |\beta| \le k\right\},$$

where $\beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ denotes a *multi-index* with $|\beta| := \sum_{k=1}^d \beta_k$ and

$$D^\beta f(x) := \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \ldots \partial x_1^{\beta_d}}(x).$$

For an open and bounded set $\Omega \subset \mathbb{R}^d$ and some $k \in \mathbb{N}_0$ we introduce

$$C^k(\Omega^{\mathrm{cl}}) := \left\{f \in C^k(\Omega) : D^\beta f \text{ has a continuous extension to } \Omega^{\mathrm{cl}} \text{ for } |\beta| \le k\right\}.$$

We observe that the spaces $\left(C^k(\Omega^{\mathrm{cl}}), \|\cdot\|_{C^k(\Omega^{\mathrm{cl}})}\right)$ are Banach-spaces with

$$\|f\|_{C^k(\Omega^{\mathrm{cl}})} := \sum_{|\beta| \le k} \left\|D^\beta f\right\|_{C(\Omega^{\mathrm{cl}})}.$$

Considering the function spaces $C^k(\Omega)$, we further introduce the function spaces

$$C_c^k(\Omega) := \left\{ f \in C^k(\Omega) : f \text{ has a compact support in } \Omega \right\}.$$

Finally, we define the function spaces

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega)$$

and

$$C_c^\infty(\Omega) := \{ f \in C^\infty(\Omega) : f \text{ has a compact support in } \Omega \}.$$

Considering a closed interval $[a, b] \subset \mathbb{R}$, $k \in \mathbb{N}$ and some points $a = x_0 < x_1 < \ldots < x_{n_x} < x_{n_x+1} = b$, we introduce the set of piecewise k- times continuously differentiable functions:

$$PC^k(I; x_1, \ldots, x_{n_x}) := \left\{ f : I \to \mathbb{R} : f|_{[x_{i-1}, x_i]} \in C^k([x_{i-1}, x_i]) \ \forall i = 1, \ldots, n_x + 1 \right\}.$$

2.3.3 Hölder spaces

Let $\alpha \in]0, 1]$ and $k \in \mathbb{N}$. Then we introduce the so-called *Hölder spaces* by

$$C^{k,\alpha}(\Omega^{\mathrm{cl}}) := \left\{ f \in C^k(\Omega^{\mathrm{cl}}) : \|f\|_{C^{k,\alpha}(\Omega)} < \infty \right\},$$

where

$$\|f\|_{C^{k,\alpha}(\Omega^{\mathrm{cl}})} := \|f\|_{C^k(\Omega^{\mathrm{cl}})} + \sum_{|\beta|=k} \sup_{x,y \in \Omega, x \neq y} \frac{\left| D^\beta f(x) - D^\beta f(y) \right|}{\|x - y\|_2^\alpha}.$$

We note that for bounded $\Omega \subset \mathbb{R}^d$ the spaces $\left(C^{k,\alpha}(\Omega^{\mathrm{cl}}), \|\cdot\|_{C^{k,\alpha}(\Omega^{\mathrm{cl}})} \right)$ are Banach-spaces.

2.3.4 Sobolev spaces

Next, we introduce the so-called *Sobolev spaces*. To this end, we firstly introduce the concept of *weak derivatives*:

Definition 2.3.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set and consider some $f \in L^1_{\mathrm{loc}}(\Omega)$. If* *there exists a function* $g \in L^1_{\text{loc}}(\Omega)$ *such that*

$$\int_\Omega g\varphi \, \mathrm{d}x = (-1)^{|\beta|} \int_\Omega f D^\beta \varphi \, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(\Omega)$$

holds true, then $D^\beta f := g$ *is called the* β*-weak partial derivative of* f*.*

Using this definition, we can introduce the Sobolev spaces as follows:

Definition 2.3.6. *Consider an open set* $\Omega \subset \mathbb{R}^d$*,* $k \in \mathbb{N}_0$*,* $p \in [0, \infty]$ *and* $k \in \mathbb{N}$*. Then we define the space*

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) : f \text{ has weak derivatives } D^\beta f \in L^p(\Omega) \text{ for all } |\beta| \le k \right\}.$$

Introducing the norms

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\beta| \le k} \left\| D^\beta f \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \quad p \in [1, \infty[$$

$$\text{and } \|f\|_{W^{\infty,p}(\Omega)} := \sum_{|\beta| \le k} \left\| D^\beta f \right\|_{L^\infty(\Omega)},$$

we note that the spaces $\left(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)} \right)$ are Banach-spaces for $p \in [0, \infty]$. Moreover, $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space with inner product

$$(f, g)_{H^k(\Omega)} := \sum_{|\beta| \le k} \left(D^\beta f, D^\beta g \right)_{L^2(\Omega)}.$$

2.3.5 Fractional Sobolev spaces

Now we want to generalize the Sobolev spaces introduced in the last subsection and obtain the so-called *fractional Sobolev spaces*, see for example [79] or [33] :

Definition 2.3.7. *Consider an open set* $\Omega \subset \mathbb{R}^d$ *and fix some fractional exponent* $s \in]0, 1[$*. Then for any* $p \in [1, \infty[$ *we define the fractional Sobolev space*

$$W^{s,p}(\Omega) := \left\{ f \in L^p(\Omega) : \frac{|f(x) - f(y)|}{\|x - y\|_2^{\frac{d}{p} + s}} \in L^p(\Omega \times \Omega) \right\}$$

which is endowed with the norm

$$\|f\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |f(x)|^p \,\mathrm{d}x + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{\|x-y\|_2^{d+sp}} \,\mathrm{d}x \,\mathrm{d}y \right)^{\frac{1}{p}}.$$

For $m \in \mathbb{N}$, we finally define the space

$$W^{s+m,p}(\Omega) := \left\{ f \in W^{m,p} : D^\beta f \in W^{s,p}(\Omega) \text{ for any } \beta \text{ with } |\beta| = m \right\}$$

which is endowed with the norm

$$\|f\|_{W^{s+m,p}(\Omega)} := \left(\|f\|^p_{W^{m,p}(\Omega)} + \sum_{|\beta|=m} \left\|D^\beta f\right\|^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}}.$$

Remark 2.3.8. The function spaces that are introduced in this section are spaces of real-valued functions. These spaces can be generalized to Banach space-valued functions $f : \Omega \to X$, where $(X, \|\cdot\|_X)$ is a Banach space. Then we denote for example the space of continuous functions $f : \Omega \to X$ by $C(\Omega; X)$.

2.4 Distributions

The definitions that we introduce in this section can be found for example in [77, Ch.7].

Definition 2.4.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set. A linear functional T on $C_c^\infty(\Omega)$ is called a distribution on Ω, if for every sequence $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$satisfying*

(i) $\operatorname{supp} f_k \subset K \subset \Omega$*, where K is compact and does not depend on k,*

(ii) $\left\|D^\beta f_k\right\|_{\infty,\Omega} \to 0$ *for $k \to \infty$ is valid for all β with $|\beta| \geq 0$,*

it holds true that $T(f_k) \to 0$.

In addition, we define the space of distributions on Ω by

$$\mathcal{D}'(\Omega) := \{T \text{ is a distribution on } \Omega\}.$$

Finally, we introduce the concept of distributional derivatives:

Definition 2.4.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set. In addition, consider a distribution*

$T \in \mathcal{D}'(\Omega)$ and a multi-index $\beta \in \mathbb{N}_0^d$. Then the derivative $D^\beta T$ is defined by

$$D^\beta T \cdot \varphi = (-1)^{|\beta|} T \cdot D^\beta \varphi, \quad \varphi \in C_c^\infty(\Omega).$$

2.5 Borel and Radon measures

The following definitions and results can be found for example in [87, Ch. 4]. See also [48] and [34].

Definition 2.5.1. *Let $K \subset \mathbb{R}^d$ be a compact set and denote by $\mathcal{B}_K$ the Borel σ-algebra on K.*

(i) A measure $\mu : \mathcal{B}_K \to [0, \infty]$ on $(K, \mathcal{B}_K)$ is called a Borel measure on K. We call μ regular, if for all $A \in \mathcal{B}_K$ it holds true that

$$\mu(A) = \inf \{\mu(O) : A \subset O, O \text{ open}\}$$
$$\text{and} \quad \mu(A) = \sup \{\mu(C) : C \subset A, C \text{ compact}\}.$$

(ii) A regular Borel measure μ on K satisfying $\mu(K) < \infty$ is called a Radon measure.

(iii) A signed measure

$$\mu : \mathcal{B}_K \to]-\infty, \infty[$$

on $(K, \mathcal{B}_K)$ is called a signed Radon measure if its positive part μ^+ and its negative part μ^- are both Radon measures on $(K, \mathcal{B}_K)$. We finally define the space

$$\mathcal{M}(K) := \{\mu : \mathcal{B}_K \to \mathbb{R} : \mu \text{ is a signed Radon measure}\}$$

and the total variation of μ by

$$|\mu| : \mathcal{B}_K \to [0, \infty[, \quad A \mapsto |\mu|(A) := \mu^+(A) + \mu^-(A). \tag{2.5.1}$$

The following result is known as *Riesz representation theorem*:

Theorem 2.5.2 ([87], Ch.4, §19, Thm. 19.54. and Thm. 19.55.)**.** *Consider the Banach space $\left(C(K), \|\cdot\|_{C(K)}\right)$, where $K \subset \mathbb{R}^d$ is a compact set. Then for every linear functional $\lambda \in C(K)^*$ there exists a unique signed Radon measure $\mu \in \mathcal{M}(K)$*

such that

$$\langle \lambda, f \rangle_{C(K)^*, C(K)} = \int_K f(x)\,\mathrm{d}\mu(x) \quad \textit{for every } f \in C(K) \tag{2.5.2}$$

holds true and $\|\lambda\|_{C(K)^*} = |\mu|(K)$. *Finally, the dual space* $\left(C(K)^*, \|\cdot\|_{C(K)^*}\right)$ *can be identified with* $\left(\mathcal{M}(K), \|\cdot\|_{\mathcal{M}(K)}\right)$ *via* (2.5.2).

In the next theorem, we will provide a transformation formula which can be found in [34, Ch.V, §3, (3.1)].

Theorem 2.5.3. *Consider a measure space* $(X, \mathcal{B}_X, \mu)$ *and a measurable space* $(Y, \mathcal{B}_Y)$. *For a measurable mapping* $t : X \to Y$, *we define the pushforward measure*

$$t(\mu)(B) \coloneqq \mu(t^{-1}(B)) \quad \textit{for all } B \in \mathcal{B}_Y. \tag{2.5.3}$$

Then a $\mathcal{B}_Y$*-measurable function* $f : Y \to \mathbb{R}$ *is* $t(\mu)$*-integrable on* Y *if and only if* $f \circ t$ *is* μ*-integrable on* X. *In this case, it additionally holds true that*

$$\int_Y f\,\mathrm{d}t(\mu) = \int_X f \circ t\,\mathrm{d}\mu.$$

2.6 Functions of bounded variation

Definition 2.6.1. *Considering an open set* $\Omega \subset \mathbb{R}^d$ *and a function* $f : \Omega \to \mathbb{R}$, *we say that* f *has bounded variation if* $f \in L^1(\Omega)$ *and* $\|f\|_{TV,\Omega} < \infty$ *with*

$$\|f\|_{TV,\Omega} \coloneqq \sup\left\{ \int_\Omega f(x) \mathrm{div}\varphi(\mathrm{x})\,\mathrm{dx} : \varphi \in \mathrm{C}^1_\mathrm{c}(\Omega; \mathbb{R}^\mathrm{d}), \|\varphi\|_{\infty,\Omega} \leq 1 \right\}.$$

In addition, we define the norm $\|\cdot\|_{BV,\Omega} \coloneqq \|\cdot\|_{1,\Omega} + \|\cdot\|_{TV,\Omega}$ *and introduce the set*

$$BV(\Omega) \coloneqq \left\{ f : \Omega \to \mathbb{R} : \|f\|_{BV,\Omega} < \infty \right\}.$$

Theorem 2.6.2 ([5], Thm.10.1.1.). *The space* $BV(\Omega)$ *equipped with the norm* $\|\cdot\|_{BV,\Omega}$ *is a Banach space.*

Theorem 2.6.3 ([5], Thm.10.1.4.). *Let* $\Omega \subset \mathbb{R}^d$ *be an open and bounded set with*

Lipschitz boundary. For all p *satisfying* $1 \leq p < \frac{d}{d-1}$ *the embedding*

$$BV(\Omega) \hookrightarrow L^p(\Omega)$$

is compact.

Theorem 2.6.4 ([5], Thm. 10.2.1.)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be an open and bounded set with a Lipschitz boundary* Γ *and denote by* $\mathcal{H}^{d-1}$ *the* $(n-1)$*-dimensional Hausdorff measure. Then there exists a linear continuous mapping*

$$\gamma_0 : BV(\Omega) \to L^1_{\mathcal{H}^{d-1}}(\Gamma)$$

satisfying

$$\gamma_0(u) = u|_\Gamma \qquad \textit{for all } u \in C(\Omega^{cl}) \cap BV(\Omega).$$

Furthermore, Green's formula holds

$$\forall \varphi \in C^1(\Omega^{cl}; \mathbb{R}^n) : \qquad \int_\Omega \varphi \cdot \nabla u \, \mathrm{d}x = -\int_\Omega u \mathrm{div} \varphi \, \mathrm{dx} + \int_\Gamma \gamma_0(\mathrm{u}) \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{\mathrm{d}-1},$$

where $\nu(x)$ *denotes the outer unit normal at* $\mathcal{H}^{d-1}$*-almost all* $x \in \Gamma$*.*

The following result is a corollary of Theorem 2.6.4 and can be found in [69]:

Corollary 2.6.5 ([69], Cor. 2.2.10)**.** *Consider some functions* $f \in C^1(\mathbb{R})$*,* $y \in BV(Q)$ *and* $p \in C^{0,1}(Q^{cl})$ *for some open and bounded set* $Q \subset]0,T[\times \mathbb{R}$ *with a Lipschitz boundary* Γ*. Then it holds true that*

$$\begin{aligned} \int_Q p(t,x) \left(y_t + f(y)_x \right) \mathrm{d}x \, \mathrm{d}t = & \int_Q y p_t + f(y) p_x \, \mathrm{d}x \, \mathrm{d}t \\ & + \int_\Gamma p \left(y \cdot \nu_1 + f(y) \cdot \nu_2 \right) \mathrm{d}\mathcal{H}^1. \end{aligned}$$

CHAPTER 3

Optimal boundary control of hyperbolic balance laws

Chapter 3 is concerned with the analysis of optimal control problems governed by hyperbolic balance laws with initial and boundary conditions and state constraints. The main goal of this chapter is to prove continuous Fréchet-differentiability of the reduced cost functional and to derive an adjoint representation of the corresponding gradient.

In §3.1, we introduce a suitable notion of solutions to hyperbolic balance laws with initial and boundary conditions. Furthermore, we discuss the existence, uniqueness and L^1_{loc}-stability of solutions. We note that the definitions and results of §3.1 can be found in [69, §3.1.1], see also [70] and [81]. We first show existence, uniqueness and stability of solutions in the L^∞-setting. Then we consider BV-solutions whose existence can be ensured under stricter assumptions. The setting of functions with bounded variations allows us to apply Dafermos' theory of generalized characteristics in §3.2.

The goal of §3.2 is to analyze the structure of BV-solutions to hyperbolic balance laws with initial and boundary conditions. Using the concept of generalized characteristics also the behavior of the solution near the boundary of the space-time cylinder can be examined. The results of this section can be found in [69, §3.1.3] where the results of [81] are extended from initial value problems to initial-boundary value problems. For further reading, see also [70] and [71].

In §3.3, we introduce the underlying optimal control problem and state the basic assumptions. Due to the shock-curves appearing in solutions of hyperbolic balance laws, the derivation of the continuous Fréchet-differentiability of the reduced cost functional is nontrivial. Therefore, we introduce in §3.4 the concept of shift-differentiability of the control-to-state mapping which is developed in [81], see also [82]. A useful feature of this concept is the fact that it implies the Fréchet-differentiability of the reduced cost functional.

The main results of Chapter 3 are presented in §3.5. One of them is the continuous Fréchet-differentiability of the reduced cost functional, which can be derived by using the concept of shift-differentiability. The second important result is the derivation of an adjoint representation for the gradient of the reduced cost functional. To this end, we analyze in §3.6 the corresponding adjoint equation, introduce a suitable notion of a solution and show existence, uniqueness and stability. These results can be found in [81, §4.2], see also [11].

In §3.7, we prove the main results stated in §3.5. These proofs are based on the concepts that are introduced for initial value problems in [81] and extended to initial-boundary value problems in [69]. The results in this thesis are an extension of those in [69] to the case that the shifting of rarefaction centers is allowed.

3.1 Notion, existence and uniqueness of solutions of hyperbolic balance laws

Let $\Omega :=]\mathfrak{a}, \mathfrak{b}[\subset \mathbb{R}$ with $-\infty \leq \mathfrak{a} < \mathfrak{b} \leq \infty$ denote a spatial subset of $\mathbb{R}$ and $]0, T[\subset [0, \infty)$ with $0 < T < \infty$ a bounded time interval. Defining the space-time cylinder $\Omega_T :=]0, T[\times \Omega$, we consider *hyperbolic balance laws* of the form

$$y_t + f(y)_x = g(\cdot, y, u_1) \quad \text{on } \Omega_T, \tag{3.1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ denotes the so-called *flux function* and $g : [0, \infty[\times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ the *source term*. The aim is to find a unique solution of (3.1.1) which satisfies a given initial condition

$$y(0, \cdot) = u_0(\cdot) \quad \text{on } \Omega$$

and boundary conditions

$$''y(\cdot, \mathfrak{a}+) = u_{B,\mathfrak{a}}(\cdot)'' \quad \text{on }]0, T[\quad (\text{if } \mathfrak{a} > -\infty),$$
$$''y(\cdot, \mathfrak{b}-) = u_{B,\mathfrak{b}}(\cdot)'' \quad \text{on }]0, T[\quad (\text{if } \mathfrak{b} < \infty).$$

If $\mathfrak{a} = -\infty$ and $\mathfrak{b} = \infty$, then the boundary conditions are omitted and we only have to deal with the initial value problem (IVP)

$$\begin{aligned} y_t + f(y)_x &= g(\cdot, y, u_1), && \text{on } \Omega_T, \\ y(0, \cdot) &= u_0(\cdot) && \text{on } \Omega. \end{aligned}$$

This chapter is concerned with finding a notion of a solution such that the initial-boundary value problem (IBVP), which is given by

$$y_t + f(y)_x = g(\cdot, y, u_1) \qquad \text{on } \Omega_T, \tag{3.1.2a}$$
$$y(0, \cdot) = u_0(\cdot) \qquad \text{on } \Omega, \tag{3.1.2b}$$
$$"y(\cdot, \mathfrak{a}+) = u_{B,\mathfrak{a}}(\cdot)" \qquad \text{on }]0, T[\quad (\text{if } \mathfrak{a} > -\infty), \tag{3.1.2c}$$
$$"y(\cdot, \mathfrak{b}-) = u_{B,\mathfrak{b}}(\cdot)" \qquad \text{on }]0, T[\quad (\text{if } \mathfrak{b} < \infty), \tag{3.1.2d}$$

is well-defined under suitable assumptions. In particular, we have to make clear in which sense the initial condition (3.1.2b) and the boundary conditions (3.1.2c) and (3.1.2d) are supposed to hold. To this end, we first need some basic assumptions, which are similar to those in [69]:

(A1) Let the flux function satisfy $f \in C^2_{\text{loc}}(\mathbb{R})$ and be uniformly convex, i.e. there exists a positive constant $m_{f''} > 0$ such that $f'' \geq m_{f''}$. Concerning the source term, we assume that $g \in L^\infty(]0, T[\times\mathbb{R}; C^{0,1}_{\text{loc}}(\mathbb{R}\times\mathbb{R}^m))$ and that for all $M > 0$ there exist positive constants C_1 and C_2 such that

$$g(t, x, y, u_1)\text{sgn}(y) \leq C_1 + C_2|y|$$

is valid for all $(t, x, y, u_1) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times [-M, M]^m$.

The results and definitions that will be presented in this chapter are basically a brief collection of the results in [69, §3.1.1]. It is well-known that solutions of hyperbolic balance laws develop discontinuities after finite time even for smooth data, as we can see in the following example that can be found in [12, Example 1.4]:

Example 3.1.1. We consider a pure initial value problem, i.e., the case that $\mathfrak{a} = -\infty$ and $\mathfrak{b} = \infty$. Furthermore, let the flux function be given by $f(z) = \frac{z^2}{2}$ and the source term be equal to zero. Then the corresponding IVP reads

$$\begin{aligned} y_t + \left(\frac{y^2}{2}\right)_x &= 0 \quad \text{on } \Omega_T, \\ y(0, x) = u_0(x) &:= \frac{1}{1 + x^2} \quad \text{on } \Omega. \end{aligned} \tag{3.1.3}$$

Supposing that there exist a classical solution of (3.1.3), the author in [12] shows by using the method of characteristics that this solution is implicitly given by

$$y\left(t, z + \frac{t}{1+z^2}\right) = u_0(z) = \frac{1}{1+z^2}. \tag{3.1.4}$$

From (3.1.4) we can deduce that y is constant along the curves

$$(t, \xi(t,z)) := \left(t, z + \frac{t}{1+z^2}\right) \tag{3.1.5}$$

and hence $y(t,x)$ is given by

$$y(t,x) = u_0(z(t,x)) = \frac{1}{1+z(t,x)^2},$$

where $z(t,x)$ denotes the solution of the equation

$$x = z + \frac{t}{1+z^2} \tag{3.1.6}$$

in terms of z. This equation is uniquely solvable for all $(t,x) \in [0,\tilde{t}[\times\Omega$ with $\tilde{t} = 8/\sqrt{27}$, but when $t \geq \tilde{t}$, the curves defined in (3.1.5) start to intersect and hence (3.1.6) does not admit a unique solution anymore. Thus, a classical solution can only exist on $[0,\tilde{t}[\times\Omega$.

Example 3.1.1 shows that we have to consider weak solutions. This means that we are interested in solutions y satisfying (3.1.1) in the sense that

$$y_t + f(y)_x = g(\cdot, y, u_1) \quad \text{in } \mathcal{D}'(\Omega_T). \tag{3.1.7}$$

However, since weak solutions of hyperbolic balance laws are in general not unique due to the existence of shocks (see, e.g., [12]), we have to single out the physically meaningful solution. This solution can be characterized by the limit $y^\varepsilon \to y$ for $\varepsilon \to 0$, where y^ε is given by the solution of

$$\begin{aligned}
y_t + f(y)_x &= g(\cdot, y, u_1) + \varepsilon y_{xx}, && \text{on } \Omega_T, && (3.1.8a)\\
y(0,\cdot) &= u_0(\cdot) && \text{on } \Omega, && (3.1.8b)\\
y(\cdot, \mathfrak{a}+) &= u_{B,\mathfrak{a}}(\cdot) && \text{on }]0,T[\quad (\text{if } \mathfrak{a} > -\infty), && (3.1.8c)\\
y(\cdot, \mathfrak{b}-) &= u_{B,\mathfrak{b}}(\cdot) && \text{on }]0,T[\quad (\text{if } \mathfrak{b} < \infty), && (3.1.8d)
\end{aligned}$$

cf. [12, 7]. This method is called the *vanishing viscosity method* and (3.1.8) is the so-called *parabolic regularization* of (3.1.2). There are several ways to characterize this limit solution. In order to find a characterization being convenient for our setting, we will first have a look at the IVP, i.e., it holds that $\mathfrak{a} = -\infty$ and $\mathfrak{b} = \infty$. In this case, one can characterize the limit solution by using the so-called *(Kružkov-) entropy pair* (η_c, q_c), where $\eta_c(\lambda) := |\lambda - c|$ and the associated *entropy flux* q_c given by $q_c(\lambda) := \operatorname{sgn}(\lambda - c)(f(\lambda) - f(c))$ for $c \in \mathbb{R}$. Kružkov shows in [47, Theorem 4] that the limit solution of (3.1.8) for a pure IVP satisfies

$$\begin{aligned} (\eta_c(y))_t + (q_c(y))_x - \eta_c'(y)g(\cdot, y, u_1) &\leq 0 \quad \text{in } \mathcal{D}'(\Omega_T), \\ \operatorname*{ess\,lim}_{t \to 0+} \|y(t, \cdot) - u_0\|_{1,\Omega \cap (-R,R)} &= 0 \quad \text{for all } R > 0 \end{aligned} \tag{3.1.9}$$

for all $c \in \mathbb{R}$. In [47], Kružkov further proves under different assumptions than (A1) that (3.1.9) admits a unique solution, which is called *entropy solution*. Nevertheless, one can find a proof for an existence and uniqueness result of an entropy solution $y \in L^\infty(\Omega_T)$ for the setting in (A1) in [81]. Choosing $c = \|y\|_{\infty,\Omega_T}$ yields that a solution of (3.1.9) also satisfies (3.1.7).

Considering a genuine IBVP, i.e., $\mathfrak{a} > -\infty$ and/or $\mathfrak{b} < \infty$ hold true, one interesting issue arises in the question in which sense the limit solution of (3.1.8) satisfies the boundary conditions (3.1.2c) and (3.1.2d). Under stronger assumptions than (A1), Bardos, LeRoux and Nédélec have proved in [7] that this limit solution has BV-regularity and satisfies the so-called *BLN-conditions*

$$\min_{k \in I(y(\cdot,\mathfrak{a}+),u_{B,\mathfrak{a}})} \operatorname{sgn}(u_{B,\mathfrak{a}} - y(\cdot, \mathfrak{a}+))(f(y(\cdot, \mathfrak{a}+)) - f(k)) = 0 \qquad \text{a.e. on } [0, T], \tag{3.1.10a}$$

$$\min_{k \in I(y(\cdot,\mathfrak{b}-),u_{B,\mathfrak{b}})} \operatorname{sgn}(y(\cdot, \mathfrak{b}-) - u_{B,\mathfrak{b}})(f(y(\cdot, \mathfrak{b}-)) - f(k)) = 0 \qquad \text{a.e. on } [0, T]. \tag{3.1.10b}$$

The BV-regularity of y guarantees the existence of the boundary traces of y in (3.1.10), but since we will work in the L^∞-setting the BV-regularity of y cannot be guaranteed any more. In this thesis, we use the same notion of a solution to (3.1.2) as in [69], which was introduced in [60]. An entropy solution of (3.1.2) is characterized by using the so-called *semi-Kružkov entropy flux pairs* $(\eta_c^\pm, q_c^\pm)$ for some $c \in \mathbb{R}$, where

$$(\eta_c^\pm(\cdot), q_c^\pm(\cdot)) := \mathbb{1}_{\mathbb{R}^\pm}(\cdot - c)(\eta_c(\cdot), q_c(\cdot)) \quad \text{on } \mathbb{R}. \tag{3.1.11}$$

Definition 3.1.2. *We call $y \in L^\infty(\Omega_T)$ an entropy solution of the IBVP* (3.1.2) *if* *for all* $c \in \mathbb{R}$ *and all* $\phi \in \mathcal{D}(]-\infty, T[\times\mathbb{R})$ *with* $\phi \geq 0$ *it holds true that*

$$\int_{\Omega_T} \eta_c^{\pm}(y)\phi_t + q_c^{\pm}(y)\phi_x \pm \mathbb{1}_{\mathbb{R}_\pm}(y-c)g(\cdot, y, u_1)\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} \eta_c^{\pm}(u_0)\phi(0,\cdot)\,\mathrm{d}x$$
$$+ \mathcal{L}_{f,[-\|y\|_\infty, \|y\|_\infty]} \int_0^T \eta_c^{\pm}(u_{B,\mathfrak{b}})\phi(0,\mathfrak{b}) + \eta_c^{\pm}(u_{B,\mathfrak{a}})\phi(0,\mathfrak{a})\,\mathrm{d}t \geq 0. \tag{3.1.12}$$

Here, $\mathcal{L}_{f,[-\|y\|_\infty, \|y\|_\infty]}$ *denotes a Lipschitz constant of* $f|_{[-\|y\|_\infty, \|y\|_\infty]}$.

We observe that for the choice $c = \|y\|_{\infty,\Omega_T}$, a solution of (3.1.12) also satisfies (3.1.7).

The following existence and uniqueness result can be found in [69], see also [47], [66] and [85].

Proposition 3.1.3 (Prop. 3.1.2, [69]). *Let* (A1) *be satisfied and define the set*

$$\mathcal{U} := L^\infty(\Omega) \times L^\infty(]0,T[)^2 \times L^\infty(]0,T[\times\mathbb{R})^m.$$

Then for every $u = (u_0, u_{B,\mathfrak{a}}, u_{B,\mathfrak{b}}, u_1) \in \mathcal{U}$ *the IBVP* (3.1.2) *admits a unique entropy solution* $y(\cdot;u) \in L^\infty(\Omega_T)$ *according to Definition 3.1.2. Moreover, after a modification on a set of measure zero,* y *is an element of* $C([0,T]; L^1_{\text{loc}}(\Omega))$. *Furthermore, for all* $M_u > 0$ *there exist constants* $M_y > 0$ *and* $L_y > 0$ *such that for all* $u, \hat{u} \in \mathcal{U}$ *with* $\|u\|_{\mathcal{U}}, \|\hat{u}\|_{\mathcal{U}} \leq M_u$, *all* $\bar{t} \in]0,T[$ *and all* $a, b \in \Omega$ *with* $\mathfrak{a} < a < b < \mathfrak{b}$, *the following estimates are valid:*

$$\begin{aligned}
\|y(\bar{t},\cdot;u)\|_{\infty,\Omega} &\leq M_y, \\
\|y(\bar{t},\cdot;u) - y(\bar{t},\cdot;\hat{u})\|_{1,]a,b[} &\leq L_y\big(\|u_0 - \hat{u}_0\|_{1,\mathcal{K}^0_{\bar{t}}(a,b)} + \|u_{B,\mathfrak{a}} - \hat{u}_{B,\mathfrak{a}}\|_{1,\mathcal{K}^{\mathfrak{a}}_{\bar{t}}(a,b)} \\
&\quad + \|u_{B,\mathfrak{b}} - \hat{u}_{B,\mathfrak{b}}\|_{1,\mathcal{K}^{\mathfrak{b}}_{\bar{t}}(a,b)} + \|u_1 - \hat{u}_1\|_{1,\mathcal{K}^{\Omega}_{\bar{t}}(a,b)},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_{\bar{t}}(a,b) &:= \{(t,x) \in [0,\bar{t}] \times \mathbb{R} : a - M_{f'}(\bar{t}-t) \leq x \leq b + M_{f'}(\bar{t}-t)\}, \\
\mathcal{K}^{\Omega}_{\bar{t}}(a,b) &:= \mathcal{K}_{\bar{t}}(a,b) \cap \Omega_T, \\
\mathcal{K}^{0}_{\bar{t}}(a,b) &:= \{x \in \Omega : (0,x) \in \mathcal{K}_{\bar{t}}(a,b)\}, \\
\mathcal{K}^{\mathfrak{a}}_{\bar{t}}(a,b) &:= \{t \in]0,T[: (t,\mathfrak{a}) \in \mathcal{K}_{\bar{t}}(a,b)\}, \\
\mathcal{K}^{\mathfrak{b}}_{\bar{t}}(a,b) &:= \{t \in]0,T[: (t,\mathfrak{b}) \in \mathcal{K}_{\bar{t}}(a,b)\}.
\end{aligned}$$

As explained in [69], from the results in [68] and [60] one obtains that if a solution according to Definition 3.1.2 has bounded variation, then the BLN-conditions in (3.1.10) hold true. The next proposition can also be found in [69].

Proposition 3.1.4 (Prop. 3.1.4, [69])**.** *Let* (A1) *hold true and assume that* g *is locally Lipschitz continuous w.r.t.* x*. Then for all* $u \in \mathcal{U}$ *with additionally* $u_0 \in BV_{\mathrm{loc}}(\Omega), u_{B,\mathfrak{a}}, u_{B,\mathfrak{b}} \in BV(]0,T[)$ *and* $u_1 \in L^1(]0,T[; BV_{\mathrm{loc}}(\Omega))^m$*, the unique entropy solution of* (3.1.2) *satisfies* $y \in (BV_{\mathrm{loc}} \cap L^\infty)(\Omega_T)$ *and* $y(t,\cdot) \in BV_{\mathrm{loc}}(\Omega)$ *for all* $t \in [0,T]$*. Finally, the BLN-conditions in* (3.1.10) *hold true.*

Remark 3.1.5. In [84] and [24], the authors have proved that the entropy solution $y \in L^\infty(\Omega_T)$ admits also for the L^∞-setting a boundary trace that is reached by L^1-convergence.

As last result of this chapter, we recall a result which can be found in [69].

Lemma 3.1.6 (Lem. 3.1.11, [69])**.** *Let the assumptions of Proposition 3.1.4 hold and consider the case that* $\mathfrak{a} > -\infty$ *and/or* $\mathfrak{b} < \infty$*. For some* $u \in \mathcal{U}$ *satisfying* $u_0 \in BV_{\mathrm{loc}}(\Omega), u_{B,\mathfrak{a}}, u_{B,\mathfrak{b}} \in BV(]0,T[)$ *and* $u_1 \in L^1(]0,T[; BV_{\mathrm{loc}}(\Omega))^m$*, let* $y \in (BV_{\mathrm{loc}} \cap L^\infty)(\Omega_T)$ *denote the corresponding entropy solution of the IBVP* (3.1.2)*. Then the BLN-conditions in* (3.1.10) *can be rewritten as follows: The condition* (3.1.10a) *is satisfied if and only if for almost all* $t \in]0,T[$ *one of the following conditions is satisfied:*

$$
\begin{aligned}
& f'(u_{B,\mathfrak{a}}(t)) \le 0, \quad f'(y(t,\mathfrak{a}+)) \le 0 && \text{(3.1.13a)}\\
\text{or} \quad & f'(u_{B,\mathfrak{a}}(t)) \ge 0, \quad u_{B,\mathfrak{a}}(t) \ge y(t,\mathfrak{a}+), \quad f(u_{B,\mathfrak{a}}(t)) \le f(y(t,\mathfrak{a}+)) && \text{(3.1.13b)}
\end{aligned}
$$

Analogously, (3.1.10b) *holds if and only if for almost all* $t \in]0,T[$ *one of the following conditions is valid:*

$$
\begin{aligned}
& f'(u_{B,\mathfrak{b}}(t)) \ge 0, \quad f'(y(t,\mathfrak{b}-)) \ge 0 && \text{(3.1.14a)}\\
\text{or} \quad & f'(u_{B,\mathfrak{b}}(t)) \le 0, \quad u_{B,\mathfrak{b}}(t) \le y(t,\mathfrak{b}-), \quad f(u_{B,\mathfrak{b}}(t)) \le f(y(t,\mathfrak{b}-)) && \text{(3.1.14b)}
\end{aligned}
$$

The characterization of the BLN-conditions in Lemma 3.1.6, which is used in [53], stay the same if we replace $u_{B,\mathfrak{a}}$ by $\max\left\{u_{B,\mathfrak{a}}, f'^{-1}(0)\right\}$ and $u_{B,\mathfrak{b}}$ by $\min\left\{u_{B,\mathfrak{b}}, f'^{-1}(0)\right\}$, see [69]. Therefore, we can w.l.o.g. assume that

$$
f'(u_{B,\mathfrak{a}}) \ge 0 \quad \text{and} \quad f'(u_{B,\mathfrak{b}}) \le 0 \tag{3.1.15}
$$

are satisfied. In the subsequent analysis we will assume that (3.1.15) holds true. Due to (3.1.15), the cases (3.1.13a) and (3.1.14a) are impossible. Therefore, the BLN-conditions in (3.1.10a) and (3.1.10b) are equivalent to (3.1.13b) and (3.1.14b).

3.2 The structure of solutions to hyperbolic balance laws

In this chapter we will collect some results of [69] and [81] concerning the structure of entropy solutions to the IBVP (3.1.2). Those results are mainly based on Dafermos' theory of generalized characteristics that is introduced in [31] for pure initial value problems. The concept of generalized characteristics that is applied in [81] for the analysis of the pure IVP is adapted in [69] to treat initial-boundary value problems. We will recall some results of [69, §3.1.3.2] and work under assumptions which are equal to the assumptions $(A1)$ and $(A1'_{BV})$ in [69].

(A2) Let (A1) hold true and assume in addition that $g \in C([0,T]; C^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m))$. Moreover, assume that there exists a constant $\varepsilon_g > 0$ such that

$$\begin{aligned} g(t,\cdot,y,u_1) &\geq 0 \quad \text{on } [\mathfrak{a}-\varepsilon_g, \mathfrak{a}+\varepsilon_g], \\ g(t,\cdot,y,u_1) &\leq 0 \quad \text{on } [\mathfrak{b}-\varepsilon_g, \mathfrak{b}+\varepsilon_g] \end{aligned}$$

hold for all $(t,y,u_1) \in [0,T]\times\mathbb{R}\times\mathbb{R}^m$.

In order to apply Dafermos' theory of generalized characteristics, we need to assure that the entropy solution $y \in L^\infty(\Omega_T)$ of the IBVP (3.1.2) satisfies the following conditions for almost all $t \in [0,T]$, cf. [31]:

$$\begin{aligned} y(t,x-) \text{ and } y(t,x+) &\quad \text{exist for all } x \in \Omega, \\ y(t,x-) \geq y(t,x+) &\quad \text{for all } x \in \Omega \end{aligned} \tag{3.2.1}$$

From the following result, which can be found in [69], one can deduce that the conditions in (3.2.1) are satisfied.

Proposition 3.2.1 (Lem. 3.1.13, [69]). *Let* (A2) *hold true. Then there exist constants $C > 0$ and $M_u > 0$ such that for every*

$$u = (u_0, u_{B,\mathfrak{a}}, u_{B,\mathfrak{b}}, u_1) \in L^\infty(\Omega) \times BV(]0,T[)^2 \times L^\infty(]0,T[; C^1(\mathbb{R}))^m$$

with

$$\|u_0\|_{\infty,\Omega} \leq M_u, \quad \left\|u_{B,\mathfrak{a}/\mathfrak{b}}\right\|_{\infty,]0,T[} \leq M_u \quad \text{and} \quad \|u_1\|_{C(]0,T[;C^1(\Omega_T))} \leq M_u \tag{3.2.2}$$

the corresponding entropy solution $y(u)$ satisfies Oleinik's entropy condition

$$y_x(t,\cdot) \leq \frac{C}{1-e^{-m_{f''}C\min(t,C\varepsilon_1,C\varepsilon_2)}} \quad \text{on } [\mathfrak{a}+\varepsilon_1,\mathfrak{b}-\varepsilon_2] \tag{3.2.3}$$

for all $t \in]0,T]$ and all $\varepsilon_1,\varepsilon_2>0$ in the sense of distributions.

We note that Proposition 3.2.1 is an extension of Proposition 3.4.1 in [81] from initial value problems to initial-boundary value problems. Under assumption (A2), from the previous result we can deduce that the conditions in (3.2.1) are satisfied for all $t \in]0,T]$.

More precisely, with similar arguments as in the proof of [81, Proposition 3.4.1], one can show that (3.2.3) implies that $y(t,\cdot) \in BV_{\text{loc}}(\mathbb{R})$ holds for all $t \in]0,T]$ and hence the limits $y(t,x-)$ and $y(t,x+)$ exist for all $t \in]0,T]$ and all $x \in \Omega$. Furthermore, (3.2.3) yields that

$$y(t,x-) \geq y(t,x+) \quad \text{for all } x \in \Omega \tag{3.2.4}$$

is valid for all $t \in]0,T]$. Similarly to [81], we will consider the following pointwise defined representative of the entropy solution y:

Convention 3.2.2. We consider the representative of y satisfying $y \in C([0,T];L^1_{\text{loc}}(\Omega))$ and $y(t,x) = y(t,x-)$ for all $(t,x) \in [0,T]\times\Omega$.

Hence, we can apply the concept of generalized characteristics. A generalized characteristic is according to [31] defined as follows:

Definition 3.2.3. *A Lipschitz continuous curve $t \mapsto (t,\xi(t)) \to \Omega_T$ defined on an interval $[\alpha,\beta] \subset [0,T]$ and satisfying*

$$\dot{\xi}(t) \in [f'(y(t,\xi(t)+)), f'(y(t,\xi(t)-))] \quad \text{a.e. on } [\alpha,\beta] \tag{3.2.5}$$

is called a (generalized) characteristic on $[\alpha,\beta]$. In what follows, we also call ξ a generalized characteristic. If $f'(y(t,\xi(t)+)) = f'(y(t,\xi(t)-))$ holds true for almost all $t \in [\alpha,\beta]$, then ξ is called genuine. A characteristic that satisfies $\dot{\xi}_\pm(t) = f'(y(t,\xi(t)\pm))$ is called maximal/minimal characteristic.

A solution ξ to (3.2.5) is a so-called *contingent solution* of

$$\dot{\xi}(t) = f'(y(t,\xi(t)))$$

with discontinuous right-hand side in the sense of Filippov [35]. Therefore, according to the theory of contingent equations in [35], in every point $(\bar{t}, \bar{x}) \in \Omega_T$ there is at least one forward characteristic defined on a maximal interval $[\bar{t}, \bar{t} + \delta[$ for some $\delta > 0$ and at least one backward characteristic defined on a maximal interval $[s, \bar{t}[$ for some $s > 0$. According to [31, Theorem 3.4], the forward characteristic through any point $(\bar{t}, \bar{x}) \in \Omega_T$ with $\bar{t} > 0$ is unique, while the set of backward characteristics lies within the funnel between the minimal and the maximal characteristic through $(\bar{t}, \bar{x})$. Since the entropy solution y is uniformly bounded in L^∞ the maximal speed of the corresponding generalized characteristics is bounded as well. Moreover, since y is also a weak solution of (3.1.2), we obtain that a characteristic ξ defined on some interval $[\alpha, \beta]$ satisfies

$$\dot{\xi}(t) = \begin{cases} f'(y(t, \xi(t)\pm)) & \text{if } y(t, \xi(t)-) = y(t, \xi(t)+) \\ \frac{f(y(t,\xi(t)+)) - f(y(t,\xi(t)-))}{y(t,\xi(t)+) - y(t,\xi(t)-)} & \text{if } y(t, \xi(t)-) > y(t, \xi(t)+) \end{cases} \tag{3.2.6}$$

for almost all $t \in [\alpha, \beta]$. A function η $:]\alpha, \beta[\to \Omega_T$ satisfying the second case of (3.2.6), which describes the so-called *Rankine-Hugoniot condition*, is called a *shock-curve*. The next result is an extension of [31, Theorem 3.3] to initial-boundary value problems and can be found in [69].

Proposition 3.2.4 (Prop. 3.1.14, [69]). *Let* (A2) *and* (3.1.15) *hold true, consider a control $u \in \mathcal{U}$ additionally satisfying $u_0 \in L^\infty(\Omega) \cap BV_{\text{loc}}(\Omega)$, $u_{B,\mathfrak{a}/\mathfrak{b}} \in BV(]0, T[)$ and $u_1 \in C(]0, T[; C^1(\mathbb{R}))^m$. Furthermore, assume that* (3.2.2) *is satisfied and denote by $y(u)$ the corresponding entropy solution of the IBVP* (3.1.2). *Let ξ $:]\alpha, \beta[\subset]0, T[\to \Omega$ denote some generalized characteristic defined on a maximal interval $]\alpha, \beta[\subset]0, \bar{t}] \subset]0, T[$. Then the following assertion holds: If ξ is genuine, then $\xi(\cdot) = \zeta(\cdot)$ and $v(\cdot) = y(\cdot, \xi(\cdot))$ is valid on $]\alpha, \beta[$, where ζ $:]\alpha, \beta[\to \Omega$ and v $:]\alpha, \beta[\to \Omega$ solve the so-called characteristic equation*

$$\begin{aligned} \dot{\zeta}(t) &= f'(v(t)), \\ \dot{v}(t) &= g(t, \zeta(t), v(t), u_1(t, \zeta(t))). \end{aligned} \tag{3.2.7}$$

We further observe the following:

(i) Considering the case that ξ is genuine and $\alpha = 0$, set $z := \xi(0) := \lim_{t \searrow 0} \xi(t) \in \Omega$. Then it holds that

$$z = \zeta(0), \quad u_0(z-) \leq v(0) \leq u_0(z+).$$

(ii) Considering the case that ξ is genuine and $\xi(\alpha) = \mathfrak{a}$, define $\theta := \alpha$. Then

$$u_{B,\mathfrak{a}}(\theta+) \leq v(\theta) \leq u_{B,\mathfrak{a}}(\theta-)$$

is satisfied. In addition, if $\xi(\alpha) = \mathfrak{b}$*, then it holds that*

$$u_{B,\mathfrak{b}}(\theta+) \geq v(\theta) \geq u_{B,\mathfrak{b}}(\theta-).$$

(iii) If ξ *is genuine and* $\xi(\beta) := \lim_{t \nearrow \beta} \xi(t) \in \Omega$*, then*

$$\xi(\beta) = \zeta(\beta), \quad y(t, \xi(\beta)-) \geq v(\beta) \geq y(t, \xi(\beta)+)$$

is valid. Moreover, if $\xi(\beta) = \mathfrak{a}$*, then* $\theta := \beta$ *satisfies*

$$f'(v(\theta)) \leq 0, \quad f(v(\theta)) \geq f(u_{B,\mathfrak{a}}(\theta-)).$$

In addition, if $\xi(\alpha) = \mathfrak{b}$*, then it holds true that*

$$f'(v(\theta)) \geq 0, \quad f(v(\theta)) \geq f(u_{B,\mathfrak{b}}(\theta-)).$$

(iv) If $f'(y(\theta, \mathfrak{a}-)) < 0$ *holds true for some* $\theta \in]0, T[$*, then there exists a genuine characteristic* ξ *on* $]\alpha, \theta]$ *that satisfies*

$$\xi(\theta) := \lim_{t \nearrow \theta} \xi(t) = \mathfrak{a}, \quad \dot{\xi}(\theta) := \lim_{t \nearrow \theta} \dot{\xi}(t) = f'(y(\theta, \mathfrak{a}+)).$$

Furthermore, if $f'(y(\theta, \mathfrak{b}-)) > 0$ *holds true for some* $\theta \in]0, T[$*, then there exists a genuine characteristic* ξ *on* $]\alpha, \theta]$ *that satisfies*

$$\xi(\theta) = \mathfrak{b}, \quad \dot{\xi}(\theta) = f'(y(\theta, \mathfrak{b}-)).$$

(v) Finally, if $\xi = \xi_\pm$ *is a minimal/maximal backward characteristic through some point* $(\bar{t}, \bar{x})$*, then* $\xi_\pm$ *is genuine and it holds that* $\xi(\cdot) = \zeta(\cdot)$*, where* (ζ, v) *solves* (3.2.7) *for given end data* $(\zeta, v)(\bar{t}) = (\bar{x}, y(\bar{t}, \bar{x}\pm))$*.*

In the next result, which can be found in [69] and is based on [81, Lemma 3.4.6], we will collect some important properties of the solution (ξ, v) of the characteristic equation (3.2.7). Before stating the result, we first introduce the following notation. For given $(\theta, z, w, u_1) \in [0, T] \times \mathbb{R}^2 \times L^\infty(]0, T[; C^1(\mathbb{R})^m)$, let $(\zeta, v)(\cdot; \theta, z, w, u_1)$ denote the solution of (3.2.7) for initial data

$$(\zeta, v)(\theta; \theta, z, w, u_1) = (z, w).$$

Lemma 3.2.5 (Lem. 3.1.15, [69])**.** *Let* (A2) *hold true and define the sets*

$$\mathcal{B}_i := [0, T] \times \mathbb{R}^2 \times L^2(]0, T[; C^i(\mathbb{R})^m), \quad i = 0, 1,$$

$$\bar{\mathcal{B}} := \left\{(\theta, z, w, u_1) \in \mathcal{B}_1 : |w| \leq C_1, u_1 \in C([0,T]; C^1(\mathbb{R})^m), \|u_1\|_{C([0,T];C^1(\mathbb{R})^m)} \leq C_2\right\}$$

with constants $C_1, C_2 > 0$. Consider the mapping

$$(\theta, z, w, u_1) \in (\bar{\mathcal{B}}, \|\cdot\|_{\mathcal{B}_i}) \mapsto (\zeta, v)(\cdot; \theta, z, w, u_1) \in C([0,T])^2. \tag{3.2.8}$$

Then the right-hand side of (3.2.8) is on $\bar{\mathcal{B}}$ uniformly Lipschitz w.r.t. t. Furthermore, the mapping (3.2.8) is Lipschitz continuous for $i = 0$ and continuously differentiable for $i = 1$ with derivative

$$\mathrm{d}_{(\theta,z,w,u_1)}(\zeta, w) \cdot (\delta\theta, \delta z, \delta w, \delta u_1) = (\delta\zeta, \delta v)(\cdot; \theta, z, w, u_1; \delta\theta, \delta z, \delta w, \delta u_1),$$

where $(\delta\zeta, \delta v)(\cdot; \theta, z, w, u_1; \delta\theta, \delta z, \delta w, \delta u_1)$ denotes the solution of the linearized characteristic equation

$$\begin{aligned}
\dot{\delta\zeta}(t) &= f''(v(t))\delta v(t),\\
\dot{\delta v}(t) &= g_x(:)\delta\zeta(t) + g_w(:)\delta v(t) + g_{u_1}(:)u_{1x}(t, \zeta(t))\delta\zeta(t)\\
&\qquad + g_{u_1}(:)\delta u_1(t, \zeta(t)),\\
(\delta\zeta, \delta v)(\theta) &= (\delta z - f'(w)\delta\theta, \delta w - g(\theta, z, w, u_1(\theta, w))\delta\theta),
\end{aligned} \tag{3.2.9}$$

with $(:) = (t, \zeta(t), v(t), u_1(t, \zeta(t)))$. Consider a closed set $S \subset [0,T] \times \mathbb{R}$, a fixed $\bar{z} \in \mathbb{R}$ and an interval $\mathcal{T} \subset [0,T]$. Then the mapping

$$\begin{aligned}
(\theta, u_B, u_1) &\in C(S; \mathcal{T}) \times C^1(\mathcal{T}) \times C([0,T]; C^1(\mathbb{R})^m)\\
&\mapsto (\zeta, v)(\cdot_t, \theta, \bar{z}, u_B(\theta), u_1) \in C(S)^2
\end{aligned}$$

is continuously Fréchet-differentiable, where $\cdot_t$ denotes the t-part of a point $(t, x) \in S$.

Since we assume that (3.1.15) holds true, we have already observed that the BLN-conditions (3.1.10a) and (3.1.10b) are equivalent to (3.1.13b) and (3.1.14b). Moreover, one can deduce from (3.1.13b) and (3.1.14b) that if the characteristic speed of the boundary trace satisfies $f'(y(\hat{t}, \mathfrak{a}+)) \geq 0$ or $f'(y(\hat{t}, \mathfrak{b}-)) \leq 0$ for some $\hat{t} \in]0, T[$, then the boundary trace of the entropy solution y of the IBVP (3.1.2) coincides with the boundary data $u_{B,\mathfrak{a}}(\cdot)$ or $u_{B,\mathfrak{b}}(\cdot)$ at $t = \hat{t}$, respectively. In addition, if $f'(y(\tilde{t}, \mathfrak{a}+)) < 0$ or $f'(y(\tilde{t}, \mathfrak{b}-)) > 0$ holds for some $\tilde{t} \in]0, T[$, then the boundary trace of the entropy of the IBVP (3.1.2) differs from the boundary data $u_{B,\mathfrak{a}}(\cdot)$ or $u_{B,\mathfrak{b}}(\cdot)$ at $t = \tilde{t}$, due to (3.1.15). Hence, for all $t \in]0, T[$ it holds true that

$$\begin{aligned}
y(t, \mathfrak{a}+) = u_{B,\mathfrak{a}}(t) \quad &\Leftrightarrow \quad f'(y(t, \mathfrak{a}+)) \geq 0,\\
y(t, \mathfrak{b}+) = u_{B,\mathfrak{b}}(t) \quad &\Leftrightarrow \quad f'(y(t, \mathfrak{b}+)) \leq 0,
\end{aligned} \tag{3.2.10}$$

cf. [69]. The result in (3.2.10) yields that the sets of time points where the characteristic speeds at the boundaries change sign play an important role. In the following result, which can be found in [69], we will further analyze these points. Before we state this result, we first introduce the following definition:

Definition 3.2.6. *Let $y \in BV_{\text{loc}}(\Omega_T)$ and define the corresponding sets*

$$\begin{aligned}
\mathbb{T}^{\mathfrak{a}}_{\pm} &:= \{\theta \in [0,T] : f'(y(\theta+,\mathfrak{a}+) > 0 \text{ and } f'(y(\theta-,\mathfrak{a}+) \leq 0\}\,,\\
\mathbb{T}^{\mathfrak{a}}_{\mp} &:= \{\theta \in [0,T] : f'(y(\theta+,\mathfrak{a}+) \leq 0 \text{ and } f'(y(\theta-,\mathfrak{a}+) > 0\}\,,\\
\mathbb{T}^{\mathfrak{b}}_{\pm} &:= \{\theta \in [0,T] : f'(y(\theta+,\mathfrak{b}-) \geq 0 \text{ and } f'(y(\theta-,\mathfrak{b}-) < 0\}\,,\\
\mathbb{T}^{\mathfrak{b}}_{\mp} &:= \{\theta \in [0,T] : f'(y(\theta+,\mathfrak{b}-) < 0 \text{ and } f'(y(\theta-,\mathfrak{b}-) \geq 0\}\,.
\end{aligned} \tag{3.2.11}$$

Considering some $\theta \in \mathbb{T}^{\mathfrak{a}}_{\pm}$, let $\xi^{\theta}_{\mathfrak{a}}$ denote the maximal backward characteristic through $(\theta,\mathfrak{a})$ and $\vartheta^{\theta}_{\mathfrak{a}}$ the time where $\xi^{\theta}_{\mathfrak{a}}$ leaves the space-time cylinder Ω_T. Analogously, if $\theta \in \mathbb{T}^{\mathfrak{b}}_{\mp}$, then $\xi^{\theta}_{\mathfrak{b}}$ denotes the minimal backward characteristic through $(\theta,\mathfrak{b})$ and $\vartheta^{\theta}_{\mathfrak{b}}$ the time where $\xi^{\theta}_{\mathfrak{a}}$ leaves the space-time cylinder Ω_T. Moreover, define the sets

$$D^{\mathfrak{a}}_{-} := \bigcup_{\theta \in \mathbb{T}^{\mathfrak{a}}_{\pm}} \left\{(t,x) \in]\vartheta^{\theta}_{\mathfrak{a}},\theta[\times]\mathfrak{a},\xi^{\theta}_{\mathfrak{a}}(t)[\right\} \tag{3.2.12}$$

and

$$D^{\mathfrak{b}}_{-} := \bigcup_{\theta \in \mathbb{T}^{\mathfrak{b}}_{\mp}} \left\{(t,x) \in]\vartheta^{\theta}_{\mathfrak{b}},\theta[\times]\xi^{\theta}_{\mathfrak{b}}(t),\mathfrak{b}[\right\}. \tag{3.2.13}$$

If $\xi^{\theta}_{\mathfrak{a}}$ ends in a point $(\vartheta^{\theta}_{\mathfrak{a}},\mathfrak{b})$, we extend $D^{\mathfrak{a}}_{-}$ by adding $[0,\vartheta^{\theta}_{\mathfrak{a}}] \times \Omega$ to (3.2.12). We analogously add $[\mathfrak{a},\vartheta^{\theta}_{\mathfrak{b}}] \times \Omega$ to (3.2.13), if $\xi^{\theta}_{\mathfrak{b}}$ ends in a point $(\vartheta^{\theta}_{\mathfrak{b}},\mathfrak{a})$. We finally define

$$D_{-} := D^{\mathfrak{a}}_{-} \cup D^{\mathfrak{b}}_{-}.$$

The following result of [69] states that all points $(\theta,\mathfrak{a})$ with $\theta \in \mathbb{T}^{\mathfrak{a}}_{\pm}$ and $(\theta,\mathfrak{b})$ with $\theta \in \mathbb{T}^{\mathfrak{b}}_{\mp}$ are shock generating points.

Lemma 3.2.7 (Lem. 3.1.17, [69]). *Let* (A2) *hold true and consider a control $u \in \mathcal{U}$ with $\mathcal{U}$ defined in Proposition 3.1.3 that satisfies in addition $u_0 \in L^{\infty}(\Omega) \cap BV_{\text{loc}}(\Omega)$, $u_{B,\mathfrak{a}/\mathfrak{b}} \in BV(]0,T[)$ and $u_1 \in C(]0,T[;C^1(\mathbb{R}))^m$. Furthermore, assume that* (3.2.2) *as well as $f'(u_{B,\mathfrak{a}}) \geq \alpha > 0$ and $f'(u_{B,\mathfrak{b}}) \leq -\alpha < 0$ are valid for some constant $\alpha > 0$. Denote by $y(u)$ the corresponding entropy solution of the IBVP* (3.1.2)*. Then the sets defined in* (3.2.11) *are all finite. In addition all backward characteristics through any point $(\bar{t},\bar{x}) \in \Omega_T \setminus D_{-}$ do not intersect D_{-}. Finally, all points $(\theta,\mathfrak{a})$ with $\theta \in \mathbb{T}^{\mathfrak{a}}_{\pm}$ and $(\theta,\mathfrak{b})$ with $\theta \in \mathbb{T}^{\mathfrak{b}}_{\mp}$ are shock generating points, i.e., the unique* *forward characteristic is a shock curve.*

3.3 Statement of the optimal control problem and basic assumptions

We will now introduce the optimal control problem which will be analyzed in this thesis. Given a mapping of the form

$$J : \mathbb{R} \times \mathcal{U} \to \mathbb{R}, \tag{3.3.1}$$

our goal is to find a control $\bar{u} = (\bar{u}_0, \bar{u}_{B,\mathfrak{a}}, \bar{u}_{B,\mathfrak{b}}, \bar{u}_1) \in \hat{\mathcal{U}}$, where $\hat{\mathcal{U}}$ is a suitable subset of

$$\mathcal{U} := L^\infty(\Omega) \times L^\infty(]0,T[)^2 \times L^\infty(]0,T[\times\mathbb{R})^m,$$

such that

$$J(y(\bar{u})) \leq J(y(u)) \quad \text{for all } u \in \hat{\mathcal{U}} \tag{3.3.2}$$

holds true and the state constraints

$$y(\bar{t}, \cdot; \bar{u}) \leq \bar{y}(\cdot) \quad \text{on } [a,b] \tag{3.3.3}$$

are satisfied, where $\bar{y} \in C^1(\Omega^{\text{cl}})$. Here, $y(u)$ denotes the entropy solution of the IBVP (3.1.2). Moreover, a, b are constants with $-\infty < \mathfrak{a} < a < b < \mathfrak{b} < \infty$ and $\bar{t} \in]0,T[$ is a fixed time point. We will assume that

$$\begin{aligned}\hat{\mathcal{U}} \subset & PC^1(\Omega; x_1, \dots, x_{n_x}) \times PC^1([0,T]; t_1, \dots, t_{n_{t,\mathfrak{a}}}) \times PC^1([0,T]; t_1, \dots, t_{n_{t,\mathfrak{b}}}) \\ & \times C([0,T]; C^1(\mathbb{R})^m),\end{aligned}$$

where $n_x, n_{t,\mathfrak{a}}$ and $n_{t,\mathfrak{b}}$ describe the fixed numbers of switching points, where the initial data and the boundary data, respectively, switch between C^1-functions. In what follows, the control is given by the switching points and the smooth functions which are separated by them. More precisely, consider some

$$\mathbf{u} = (u^0, u^{B,\mathfrak{a}}, u^{B,\mathfrak{b}}, x^0, t^{\mathfrak{a}}, t^{\mathfrak{b}}, u_1) \in \mathbf{U}_{\text{ad}},$$

where

$$\mathbf{U}_{\text{ad}} \subset C^1(\Omega^{\text{cl}})^{n_x+1} \times C^1([0,T])^{n_{t,\mathfrak{a}}+1} \times C^1([0,T])^{n_{t,\mathfrak{b}}+1} \times \Omega^{n_x} \times [0,T]^{n_{t,\mathfrak{a}}} \times [0,T]^{n_{t,\mathfrak{b}}} \\ \times C([0,T];C^1(\mathbb{R})^m).$$

Then the initial and boundary data $u_0, u_{B,\mathfrak{a}}$ and $u_{B,\mathfrak{b}}$ associated with $\mathbf{u}$ are given by

$$u_0(x;\mathbf{u}) = \begin{cases} u_1^0(x) & \text{if } x \in I_0^1(\mathbf{u}) := (\mathfrak{a}, x_1^0], \\ u_j^0(x) & \text{if } x \in I_0^j(\mathbf{u}) := (x_{j-1}^0, x_j^0], \\ u_{n_x+1}^0(x) & \text{if } x \in I_0^{n_x+1}(\mathbf{u}) := (x_{n_x}^0, \mathfrak{b}), \end{cases} \tag{3.3.4}$$

where $2 \le j \le n_x$ and

$$u_{B,\mathfrak{a}/\mathfrak{b}}(t;\mathbf{u}) = \begin{cases} u_1^{B,\mathfrak{a}/\mathfrak{b}}(t) & \text{if } t \in I_{B,\mathfrak{a}/\mathfrak{b}}^1(\mathbf{u}) := [0, t_1^{\mathfrak{a}/\mathfrak{b}}], \\ u_j^{B,\mathfrak{a}/\mathfrak{b}}(t) & \text{if } t \in I_{B,\mathfrak{a}/\mathfrak{b}}^j(\mathbf{u}) := (t_{j-1}^{\mathfrak{a}/\mathfrak{b}}, t_j^{\mathfrak{a}/\mathfrak{b}}], \\ u_{n_{t,\mathfrak{a}/\mathfrak{b}}+1}^{B,\mathfrak{a}/\mathfrak{b}}(t) & \text{if } t \in I_{B,\mathfrak{a}/\mathfrak{b}}^{n_{t,\mathfrak{a}/\mathfrak{b}}+1}(\mathbf{u}) := (t_{n_{t,\mathfrak{a}/\mathfrak{b}}}, T], \end{cases} \tag{3.3.5}$$

where $2 \le j \le n_{t,\mathfrak{a}/\mathfrak{b}}$. From now on, we consider $\mathbf{u} \in \mathbf{U}$ as control and $\mathbf{U}_{\text{ad}}$ as the space of admissible controls. Furthermore, the entropy solution of the IBVP (3.1.2) associated with $\mathbf{u} \in \mathbf{U}$ will be denoted by $y(\mathbf{u})$. Since for all $\mathbf{u} \in \mathbf{U}$ it holds that $u_{B,\mathfrak{a}/\mathfrak{b}}(\mathbf{u}) \in BV(]0,T[)$, $u_0(\mathbf{u}) \in BV(\Omega)$ and $u_1 \in C([0,T];C^1(\mathbb{R})^m)$, we obtain by Proposition 3.1.4 that $y(\mathbf{u}) \in BV(\Omega_T)$ and that (3.1.10a) and (3.1.10b) are satisfied. Concerning the mapping in (3.3.2), we consider cost functionals of the form

$$J(y,\mathbf{u}) = \int_a^b \psi(y(\bar{t},x), y_d(x))\,\mathrm{d}x + R(\mathbf{u}),$$

where $\psi \in C_{\text{loc}}^{1,1}(\mathbb{R}^2)$, $y_d \in BV_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $R : \mathbf{U} \to [0,\infty)$ is a Fréchet-differentiable regularization term.

To summarize, we will consider the optimal control problem

$$\left.\begin{aligned} &\min_{\mathbf{u}\in\mathbf{U}_{\text{ad}}} J(y(\mathbf{u}),\mathbf{u}) = \int_a^b \psi(y(\bar{t},x;\mathbf{u}), y_d(x))\,\mathrm{d}x + R(\mathbf{u}), \\ &\qquad \text{where } y(\mathbf{u}) \text{ is the entropy solution of the IBVP (3.1.2)} \\ &\qquad \text{and } y(\bar{t},x;\mathbf{u}) \le \bar{y}\,(x) \quad \text{for all } x \in [a,b]. \end{aligned}\right\} \tag{P}$$

The main goal of this thesis is to derive necessary optimality conditions for (P). Since entropy solutions of hyperbolic balance laws develop shock-curves after finite time, it is not possible to prove Fréchet-differentiability of the reduced cost functional $\hat{J}(\mathbf{u}) := J(y(\mathbf{u}), \mathbf{u})$ with standard methods which can be seen in the following example from [81]:

Example 3.3.1. We consider a pure initial value problem, i.e., the case $\mathfrak{a} = -\infty$ and $\mathfrak{b} = \infty$. Furthermore, let the flux function be given by $f(z) = \frac{z^2}{2}$ and the source term be equal to zero. Then the corresponding IVP reads

$$\begin{aligned} y_t + \left(\frac{y^2}{2}\right)_x &= 0 \quad \text{on } \Omega_T, \\ y(0,\cdot) &= u_0(\cdot;\mathbf{u}) \quad \text{on } \Omega. \end{aligned}$$

Let $n_x = 1$ and consider some control $\bar{\mathbf{u}} = \left(\bar{u}_1^0, \bar{u}_2^0, \bar{x}_1^0\right) = (1, -1, 0)$ and some perturbation $\delta\mathbf{u} = (\varepsilon, 0, 0)$ for some $\varepsilon > 0$. Setting $\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u}$, the corresponding initial data is given by

$$u_0(x;\mathbf{u}) = \begin{cases} 1+\varepsilon & \text{if } x \leq 0, \\ -1 & \text{if } x > 0 \end{cases}$$

and the entropy solution of the IVP denoted by $y(\mathbf{u})$ is given by

$$y(t,x;\bar{\mathbf{u}}+\delta\mathbf{u}) = \begin{cases} 1+\varepsilon & \text{if } x \leq \frac{t\varepsilon}{2}, \\ -1 & \text{if } x > \frac{t\varepsilon}{2}. \end{cases}$$

One can see that $y(\bar{\mathbf{u}}+\delta\mathbf{u})$ has a shock-curve emanating from $(0,0)$ given by

$$\eta(t;\mathbf{u}) = \frac{\varepsilon}{2}t.$$

If we fix some $\bar{t} \in]0,T[$ and try to compute a directional derivative of the mapping $\mathbf{u} \mapsto y(\bar{t},\cdot;\mathbf{u}) \in L^1(\Omega)$ in $\mathbf{u} = \bar{\mathbf{u}}$ in the direction $(1,0,0)$, we observe that the limit

$$\lim_{\varepsilon \searrow 0} \frac{y(t,x;\bar{\mathbf{u}}+\delta\mathbf{u}) - y(t,x;\bar{\mathbf{u}})}{\varepsilon}$$

does not exist in L^1. Hence, the control-to-state mapping $\mathbf{u} \mapsto y(\bar{t},\cdot;\mathbf{u})$ is not differentiable to L^1_{loc}, but only to $\mathcal{M}_{\text{loc}}(\mathbb{R})$ w.r.t. the weak topology, where the derivative is given by

$$\mathrm{d}_{\mathbf{u}} y(\bar{t},\cdot)|_{\mathbf{u}=\bar{\mathbf{u}}} \cdot \delta\mathbf{u} = \mathbb{1}_{x<\eta(\bar{t};\bar{\mathbf{u}})}\varepsilon + [y(\bar{t},\eta(\bar{t};\bar{\mathbf{u}}),\bar{\mathbf{u}})]\,\delta\left(\cdot - \eta(\bar{t};\bar{\mathbf{u}})\right)\frac{t}{2}\varepsilon. \tag{3.3.6}$$

We note the weak topology of $\mathcal{M}_{\text{loc}}(\mathbb{R})$ is not strong enough to yield the Fréchet-differentiability of the reduced cost functional $\hat{J}(\cdot)$ w.r.t. $\mathbf{u}$. Nevertheless, in [81] the derivative in (3.3.6) serves as starting point to construct a first order approximation of $y(\bar{t},\cdot,\bar{\mathbf{u}}+\delta\mathbf{u})-y(\bar{t},\cdot,\bar{\mathbf{u}})$ in L^1_{loc}. Indeed, replacing the second term in (3.3.6) by $\operatorname{sgn}(\frac{\bar{t}}{2}\varepsilon)\,[y(\bar{t},\eta(\bar{t};\bar{\mathbf{u}}),\bar{\mathbf{u}})]\,\mathbb{1}_{I\left(\eta(\bar{t};\bar{\mathbf{u}}),\eta(\bar{t};\bar{\mathbf{u}})+\frac{\bar{t}}{2}\varepsilon\right)}$, we obtain

$$\begin{aligned}&S^{(\eta(t;\bar{\mathbf{u}}))}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathbb{1}_{x<\eta(\bar{t};\bar{\mathbf{u}})}\cdot\varepsilon,\frac{t}{2}\varepsilon\right)\\&:=\mathbb{1}_{x<\eta(\bar{t};\bar{\mathbf{u}})}\varepsilon+\operatorname{sgn}(\frac{\bar{t}}{2}\varepsilon)\,[y(\bar{t},\eta(\bar{t};\bar{\mathbf{u}}),\bar{\mathbf{u}})]\,\mathbb{1}_{I\left(\eta(\bar{t};\bar{\mathbf{u}}),\eta(\bar{t};\bar{\mathbf{u}})+\frac{\bar{t}}{2}\varepsilon\right)},\end{aligned}\tag{3.3.7}$$

which is a (nonlinear) first order approximation of $y(\bar{t},\cdot,\bar{\mathbf{u}}+\delta\mathbf{u})-y(\bar{t},\cdot,\bar{\mathbf{u}})$ in L^1_{loc}. This feature will be analyzed in a more general setting in §3.4.

Another problem arises in the fact that due to the state constraints (3.3.3), we have to require the state $y(\bar{t},\cdot)$ to have at least L^∞-regularity. However, the previous example shows that the control-to-state mapping $\mathbf{u}\mapsto y(\bar{t},\cdot,\mathbf{u})\in L^\infty(\Omega)$ is in general not even continuous. This is due to the appearance of discontinuities whose positions possibly change if the control varies. Before solving those problems, we first introduce the assumptions for the subsequent analysis.

(A3) Let (A2) hold true and assume in addition that $f\in C^3_{\mathrm{loc}}(\mathbb{R})$ and $f'^{-1}\in C^{2,\beta}_{\mathrm{loc}}(\mathbb{R})$ for some $\beta\in(0,1]$. We further assume that there exists a constant $\varepsilon_g>0$ such that

$$g(\cdot,y,u_1)=0\quad\text{on }[0,\varepsilon_g]\times\mathbb{R}\cup[0,T]\times\mathbb{R}\setminus(\mathfrak{a}+\varepsilon_g,\mathfrak{b}-\varepsilon_g)\tag{3.3.8}$$

is satisfied for all $(y,u_1)\in\mathbb{R}\times\mathbb{R}^m$. Finally we assume that g is Lipschitz w.r.t. x and affine linear w.r.t. y.

(A4) Let $\mathbf{U}_{\mathrm{ad}}$ denote the set of admissible controls. We assume that $\mathbf{U}_{\mathrm{ad}}$ is a nonempty and convex subset of the set $\mathbf{U}$, which is given by

$$\mathbf{U}:=C^1(\Omega^{\mathrm{cl}})^{n_x+1}\times U^\alpha_{B,\mathfrak{a}}\times U^\alpha_{B,\mathfrak{b}}\times\mathcal{X}\times\mathcal{T}^{\mathfrak{a}}\times\mathcal{T}^{\mathfrak{b}}\times C([0,T];C^1(\mathbb{R})^m),$$

where $\mathbf{U}$ is equipped with the norm

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{U}}:=&\|u^0\|_{C^1(\Omega^{\mathrm{cl}})^{n_x+1}}+\|u^{B,\mathfrak{a}}\|_{C^1([0,T])^{n_{t,\mathfrak{a}}+1}}+\|u^{B,\mathfrak{b}}\|_{C^1([0,T])^{n_{t,\mathfrak{b}}+1}}\\&+\|x^0\|_1+\|t^{\mathfrak{a}}\|_1+\|t^{\mathfrak{b}}\|_1+\|u_1\|_{C([0,t];C^1(\mathbb{R})^m)}.\end{aligned}$$

Here, we use the abbreviations

$$\begin{aligned}&\mathcal{X}:=\{\vec{x}\in\Omega^{n_x}:\mathfrak{a}<x_1<\ldots<x_{n_x}<\mathfrak{b}\},\\&\mathcal{T}^{\mathfrak{a}}:=\left\{\vec{t}\in[0,T]^{n_{t,\mathfrak{a}}}:0<t_1<\ldots<t_{n_{t,\mathfrak{a}}}<T\right\},\end{aligned}$$

$$\mathcal{T}^{\mathfrak{b}} := \left\{\vec{t} \in [0,T]^{n_{t,\mathfrak{b}}} : 0 < t_1 < \ldots < t_{n_{t,\mathfrak{b}}} < T\right\},$$
$$U^{\alpha}_{B,\mathfrak{a}} := \left\{u^{B,\mathfrak{a}} \in C^1([0,T])^{n_{t,\mathfrak{a}}+1} : f'(u^{B,\mathfrak{a}}_j) \geq \alpha, j = 1,\ldots,n_{t,\mathfrak{a}}+1\right\},$$
$$U^{\alpha}_{B,\mathfrak{b}} := \left\{u^{B,\mathfrak{b}} \in C^1([0,T])^{n_{t,\mathfrak{b}}+1} : f'(u^{B,\mathfrak{b}}_j) \leq -\alpha, j = 1,\ldots,n_{t,\mathfrak{b}}+1\right\}$$

for some $\alpha > 0$. Moreover, we assume that $\mathbf{U}_{\text{ad}}$ is sequentially compact in $\mathbf{U}$ and that there exists a constant $M_u > 0$ such that $\|u_1\|_{L^\infty(0,T;C^1(\Omega_T^{\text{cl}})^m)} \leq M_u$ holds true for all $\mathbf{u} \in \mathbf{U}_{\text{ad}}$. Finally, suppose that $\psi \in C^{1,1}_{\text{loc}}(\mathbb{R}^2)$, $y_d \in BV_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\bar{y} \in C^1(\Omega^{\text{cl}})$ and let $R : \mathbf{U} \to [0,\infty)$ be continuously Fréchet-differentiable.

Remark 3.3.2. In (A4) we assume inter alia that $\mathbf{U}_{\text{ad}}$ is sequentially compact in $\mathbf{U}$. According to [2, Theorem 5.1] and [33, Theorem 8.2] or alternatively [79, Theorem 4.6.1], this holds for example if we choose

$$\mathbf{U}_{\text{ad}} = \mathcal{V}_0 \times \mathcal{V}^{\alpha}_{B,\mathfrak{a}} \times \mathcal{V}^{\alpha}_{B,\mathfrak{b}} \times \mathcal{X}_\varepsilon \times \mathcal{T}^{\mathfrak{a}}_\varepsilon \times \mathcal{T}^{\mathfrak{b}}_\varepsilon \times \mathcal{V}_g,$$

with

$$\mathcal{V}_0 := \left\{u^0 \in W^{s+1,2}(\Omega)^{n_x+1} : \|u^0_j\|_{W^{s+1,2}(\Omega)} \leq C,\ j = 1,\ldots,n_x+1\right\},$$

$$\mathcal{V}^{\alpha}_{B,\mathfrak{a}} := \Big\{u^{B,\mathfrak{a}} \in W^{s+1,2}([0,T])^{n_{t,\mathfrak{a}}+1} : \|u^{B,\mathfrak{a}}_j\|_{W^{s+1,2}([0,T])} \leq C \text{ and } f'(u^{B,\mathfrak{a}}_j) \geq \alpha,$$
$$j = 1,\ldots,n_{t,\mathfrak{a}}+1\Big\},$$

$$\mathcal{V}^{\alpha}_{B,\mathfrak{b}} := \Big\{u^{B,\mathfrak{b}} \in W^{s+1,2}([0,T])^{n_{t,\mathfrak{b}}+1} : \|u^{B,\mathfrak{b}}_j\|_{W^{s+1,2}([0,T])} \leq C \text{ and } f'(u^{B,\mathfrak{b}}_j) \leq -\alpha,$$
$$j = 1,\ldots,n_{t,\mathfrak{b}}+1\Big\},$$

$$\mathcal{X}_\varepsilon := \{\vec{x} \in \mathcal{X} : x_j - x_{j-1} \geq \varepsilon,\ j = 1,\ldots,n_x+1\},$$

$$\mathcal{T}^{\mathfrak{a}}_\varepsilon := \left\{\vec{t} \in \mathcal{T}^{\mathfrak{a}} : t_j - t_{j-1} \geq \varepsilon,\ j = 1,\ldots,n_{t,\mathfrak{a}}+1\right\},$$

$$\mathcal{T}^{\mathfrak{b}}_\varepsilon := \left\{\vec{t} \in \mathcal{T}^{\mathfrak{b}} : t_j - t_{j-1} \geq \varepsilon,\ j = 1,\ldots,n_{t,\mathfrak{b}}+1\right\},$$

$$\mathcal{V}_g := \left\{u_1 \in W^{s,2}((0,T);W^{s+1,2}(\Omega)^m) : \|u_1\|_{W^{s,2}((0,T);W^{s+1,2}(\Omega)^m)} \leq C\right\}$$

for some $\varepsilon, C > 0$, $s > \frac{1}{2}$, where we set $x_0 := \mathfrak{a}$, $x_{n_x+1} = \mathfrak{b}$, $t_0 = 0$ and $t_{n_{t,\mathfrak{a}}+1} = t_{n_{t,\mathfrak{b}}+1} = T$.

We note that at the switching points, the initial and the boundary data have a discontinuity or a kink. We first consider a switching time $t^{\mathfrak{a}}_j$ at which the left boundary data $u_{B,\mathfrak{a}}$ is discontinuous. It is well known (see for example [50]) that if $[u_{B,\mathfrak{a}}(t_j^{\mathfrak{a}})] < 0$, i.e., the boundary data has an up-jump at $t_j^{\mathfrak{a}}$, then the corresponding entropy solution y has a shock-curve emanating from the point $(t_j^{\mathfrak{a}}, \mathfrak{a})$. On the other hand, if it holds that $[u_{B,\mathfrak{a}}(t_j^{\mathfrak{a}})] > 0$, i.e., the boundary data has a down-jump at $t_j^{\mathfrak{a}}$, then $(t_j^{\mathfrak{a}}, \mathfrak{a})$ is the center of a rarefaction wave. Considering the initial data and the right boundary data, we note that switching times produce shock-curves if they are down-jumps and rarefaction waves if they are up-jumps. Therefore, we introduce the following notation:

Notation 3.3.3. For a given $\mathbf{u} \in \mathbf{U}$, we associate the sets of indices:

$$
\begin{aligned}
I_{\mathrm{c},0}(\mathbf{u}) &:= \left\{ j \in \{1, \dots, n_x\} : [u_0(x_j^0)] = 0 \right\}, \\
I_{\mathrm{c},\mathfrak{a}}(\mathbf{u}) &:= \left\{ j \in \{1, \dots, n_{t,\mathfrak{a}}\} : [u_{B,\mathfrak{a}}(t_j^{\mathfrak{a}})] = 0 \right\}, \\
I_{\mathrm{c},\mathfrak{b}}(\mathbf{u}) &:= \left\{ j \in \{1, \dots, n_{t,\mathfrak{b}}\} : [u_{B,\mathfrak{b}}(t_j^{\mathfrak{b}})] = 0 \right\}, \\
I_{\mathrm{s},0}(\mathbf{u}) &:= \left\{ j \in \{1, \dots, n_x\} : [u_0(x_j^0)] > 0 \right\}, \\
I_{\mathrm{s},\mathfrak{a}}(\mathbf{u}) &:= \left\{ j \in \{1, \dots, n_{t,\mathfrak{a}}\} : [u_{B,\mathfrak{a}}(t_j^{\mathfrak{a}})] < 0 \right\}, \\
I_{\mathrm{s},\mathfrak{b}}(\mathbf{u}) &:= \left\{ j \in \{1, \dots, n_{t,\mathfrak{b}}\} : [u_{B,\mathfrak{b}}(t_j^{\mathfrak{b}})] > 0 \right\}, \\
I_{\mathrm{r},0}(\mathbf{u}) &:= \{1, \dots, n_x\} \setminus (I_{\mathrm{s},0}(\mathbf{u}) \cup I_{\mathrm{c},0}(\mathbf{u})), \\
I_{\mathrm{r},\mathfrak{a}}(\mathbf{u}) &:= 1, \dots, n_{\mathfrak{a}} \setminus (I_{\mathrm{s},\mathfrak{a}}(\mathbf{u}) \cup I_{\mathrm{c},\mathfrak{a}}(\mathbf{u})), \\
I_{\mathrm{r},\mathfrak{b}}(\mathbf{u}) &:= 1, \dots, n_{\mathfrak{b}} \setminus (I_{\mathrm{s},\mathfrak{b}}(\mathbf{u}) \cup I_{\mathrm{c},\mathfrak{b}}(\mathbf{u})).
\end{aligned}
$$

At the end of this chapter, we want to prove that the reduced cost functional $\hat{J}(\mathbf{u})$ is Lipschitz continuous w.r.t. the control $\mathbf{u}$. To this end, we will need the following two results:

Lemma 3.3.4. *Let* (A3) *and* (A4) *hold true and consider some controls* $\mathbf{u}, \bar{\mathbf{u}} \in \mathbf{U}_{ad}$. *Then the initial and boundary data* $u_0, u_{B,\mathfrak{a}}$ *and* $u_{B,\mathfrak{b}}$*, which are defined in* (3.3.4) *and* (3.3.5) *satisfy the following estimations*

$$
\begin{aligned}
\|u_0(\mathbf{u}) - u_0(\bar{\mathbf{u}})\|_{1,[\mathfrak{a},\mathfrak{b}]} &\le (|\mathfrak{b} - \mathfrak{a}| + \|\mathbf{u}\|_{\mathbf{U}} + \|\bar{\mathbf{u}}\|_{\mathbf{U}})\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}}, \\
\|u_{B,\mathfrak{a}}(\mathbf{u}) - u_{B,\mathfrak{a}}(\bar{\mathbf{u}})\|_{1,[0,T]} &\le (T + \|\mathbf{u}\|_{\mathbf{U}} + \|\bar{\mathbf{u}}\|_{\mathbf{U}})\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}}, \\
\|u_{B,\mathfrak{b}}(\mathbf{u}) - u_{B,\mathfrak{b}}(\bar{\mathbf{u}})\|_{1,[0,T]} &\le (T + \|\mathbf{u}\|_{\mathbf{U}} + \|\bar{\mathbf{u}}\|_{\mathbf{U}})\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}}.
\end{aligned}
$$

Proof. We will only prove the first inequality and notice that the remaining two inequalities can be proved analogously. Recalling the definition of u_0 in (3.3.4), we first observe that the locations of the ith switching point differ by $|x_i^0 - \bar{x}_i^0|$.

Furthermore, it holds that

$$\|u_0(x;\mathbf{u}) - u_0(x;\bar{\mathbf{u}})\|_{C(I(x_i^0,\bar{x}_i^0))} \leq \|u^0\|_{C(\Omega^{\mathrm{cl}})^{n_x+1}} + \|\bar{u}^0\|_{C(\Omega^{\mathrm{cl}})^{n_x+1}}.$$

We further note that outside the intervals $I\left(x_i^0, \bar{x}_i^0\right)$, $i = 1, \dots, n_x$, the difference $|u_0(x;\mathbf{u}) - u_0(x;\mathbf{u})|$ is bounded by $\|u^0 - \bar{u}^0\|_{C(\Omega^{\mathrm{cl}})^{n_x+1}}$. Therefore, we obtain

$$\begin{aligned}
&\|u_0(\mathbf{u}) - u_0(\bar{\mathbf{u}})\|_{1,[\mathfrak{a},\mathfrak{b}]} \\
&\leq |\mathfrak{b} - \mathfrak{a}|\|u^0 - \bar{u}^0\|_{C(\Omega^{\mathrm{cl}})^{n_x+1}} + (\|u^0\|_{C(\Omega^{\mathrm{cl}})^{n_x+1}} + \|\bar{u}^0\|_{C(\Omega^{\mathrm{cl}})^{n_x+1}})\|x^0 - \bar{x}^0\|_1 \\
&\leq (|\mathfrak{b} - \mathfrak{a}| + \|\mathbf{u}\|_{\mathbf{U}} + \|\bar{\mathbf{u}}\|_{\mathbf{U}})\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}}.
\end{aligned}$$

□

Using the estimations from Proposition 3.1.3 in combination with Lemma 3.3.4 and the fact that $\mathbf{U}_{\mathrm{ad}}$ is bounded in $\mathbf{U}$, we obtain the following two corollaries:

Corollary 3.3.5. *Let* (A3) *and* (A4) *hold true. Then for all* $\mathbf{u}, \bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *the following estimation holds true*

$$\|y(t,\cdot;\mathbf{u}) - y(t,\cdot;\bar{\mathbf{u}})\|_{1,[a,b]} \leq L_y\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}},$$

where $L_y > 0$ *is a sufficiently large constant.*

Corollary 3.3.6. *Let* (A3) *and* (A4) *hold true. Then there exists a constant* $L > 0$ *such that*

$$\left|J(y(\mathbf{u}),\mathbf{u}) - J(y(\bar{\mathbf{u}}),\bar{\mathbf{u}})\right| \leq L\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}} \quad \text{for all } \mathbf{u}, \bar{\mathbf{u}} \in \mathbf{U}_{ad}.$$

Proof. The proof can be found for example in [83] and is based on the estimation in Corollary 3.3.5, the regularity of the mapping R and the fact that $\psi \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ and $y_d \in BV([a,b])$. □

3.4 Shift-differentiability

Considering the optimal control problem (P), Example 3.3.1 shows that it is not possible to derive the Fréchet-differentiability of the reduced cost functional $\hat{J}(\cdot)$ by standard arguments. In [81], the concept of the *shift-differentiability* is introduced for the pure initial value problem to cope with this problem, see also [17]. This concept is extended to initial-boundary value problems in [69]. Using (3.3.7) in Example 3.3.1 as a motivation, the following definition of *shift-variations* is introduced in [81]:

Definition 3.4.1. *Consider an interval $[a,b]$ with $a<b$, a function $y \in BV([a,b])$ and points $a < x_1 < \ldots < x_N < b$. Then we associate with $(\delta y, \delta x) \in L^1([a,b]) \times \mathbb{R}^N$ the shift-variation*

$$S_y^{(x_i)}(\delta y, \delta x)(x) \coloneqq \delta y(x) + \sum_{i=1}^{N} [y(x_i)] \operatorname{sgn}(\delta x_i) \mathbb{1}_{I(x_i, x_i+\delta x_i)}(x) \tag{3.4.1}$$

and call $(\delta y, \delta x)$ a generalized variation of $\mathbf{u}$*. We recall that*

$$\begin{aligned} [y(x_i)] &\coloneqq y(x_i-) - y(x_i+), \\ I(x, \tilde{x}) &\coloneqq [\min(x,\tilde{x}), \max(x,\tilde{x})], \quad \text{for } x, \tilde{x} \in \mathbb{R}. \end{aligned}$$

In the original work [81], the definition of the shift-variation is slightly different:

$$S_y^{(x_i)}(\delta y, \delta x)(x) \coloneqq \delta y(x) + \sum_{i=1}^{N} [y(x_i)]_+ \operatorname{sgn}(\delta x_i) \mathbb{1}_{I(x_i, x_i+\delta x_i)}(x), \tag{3.4.2}$$

where $[\varphi(z)]_+ \coloneqq \max\{0, \varphi(z-) - \varphi(z+)\}$. In this definition, only down-jumps of the function $y(\cdot)$ are shifted, which on the one hand is motivated by the fact that considering a pure initial value problem, the entropy solution of (3.1.2) satisfies $y(t,x-) \geq y(t,x+)$ for all $x \in \mathbb{R}$ and almost all $t \in]0,T]$, due to (3.2.3). On the other hand, another reason behind the definition (3.4.2) is that in [81] the initial data is varied by shift-variations, where the centers of rarefaction waves, which are exactly the up-jumps in the initial data, are not allowed to be shifted. In [69], the shifting of centers of rarefaction waves is also prohibited, but nevertheless shift-variations are also defined as in (3.4.1), since the initial and boundary data is not varied by shift-variations. In the present thesis, we will extend the results of [81] and [69] to the case where the shifting of rarefaction waves is allowed.

We now recall a further definition of [81]:

Definition 3.4.2. *Consider a Banach space $\mathcal{U}$, an open subset $\mathcal{D} \subset \mathcal{U}$ and an interval $[a,b] \subset \mathbb{R}$ with $a<b$. A mapping $\mathcal{D} \ni u \to y(u) \in L^\infty(\mathbb{R})$ is called shift-differentiable in $\bar{u} \in \mathcal{D}$ with $y(\bar{u}) \in BV([a,b])$ if there exist $a < \bar{x}_1 < \ldots < \bar{x}_N < b$ and a linear bounded operator*

$$\mathrm{T}_s(y(\bar{u})) \in \mathcal{L}(\mathcal{U}; L^r([a,b]) \times \mathbb{R}^N)$$

for some $r \in]1,\infty]$ *such that*

$$\lim_{u\to\bar u} \frac{\left\| y(u) - y(\bar u) - S^{(\bar x_k)}_{y(\bar u)}\left(\mathrm{T}_s\left(y(\bar u)\right)(u-\bar u)\right)\right\|_{1,[a,b]}}{\|u-\bar u\|_{\mathcal{U}}} = 0 \tag{3.4.3}$$

holds true. Moreover, the mapping $u \mapsto y(u)$ *is called continuously shift-differentiable at* $\bar u$ *if it is shift-differentiable in all*

$$u \in B^{\mathcal{U}}_{\rho}(\bar u) := \{u \in \mathcal{U} : \|u-\bar u\|_{\mathcal{U}} \le \rho\}$$

with a sufficiently small constant $\rho > 0$ *and the corresponding mappings*

$$\mathrm{T}_s\left(y(u)\right), x_k(u) \text{ and } y(x_k(u)\pm; u) \quad k = 1,\dots,N,$$

are continuous in $\bar u$.

A useful feature is that the shift-differentiability of a mapping $\mathcal{D} \ni u \to y(u) \in L^\infty(\mathbb{R})$ in some $\bar u$ yields the Fréchet-differentiability of the mapping

$$u \mapsto J(y(u)) = \int_a^b \psi(y(x;u), y_d(x))\,\mathrm{d}x \tag{3.4.4}$$

if ψ and y_d satisfy certain assumptions as we can see in the next result, which can be found in [81]:

Lemma 3.4.3 (Lemma 3.2.3, [81])**.** *Consider an interval* $[a,b] \subset \mathbb{R}$ *with* $a < b$ *and let* $\mathcal{U}$ *be a Banach space with an open subset* $\mathcal{D} \subset \mathcal{U}$*. Consider in addition a mapping* $\mathcal{D} \ni u \to y(u) \in L^\infty(\mathbb{R})$ *satisfying* $y(\bar u) \in BV([a,b])$ *that is shift-differentiable in* $\bar u \in \mathcal{D}$ *according to Definition 3.4.2 with corresponding* $a < \bar x_1 < \ldots < \bar x_N < b$ *and linear bounded operator* $\mathrm{T}_s\left(y(\bar u)\right) \in \mathcal{L}(\mathcal{U}; L^r([a,b]) \times \mathbb{R}^N)$ *for some* $r \in]1,\infty]$*. Considering the mapping in* (3.4.4)*, assume that* $\psi \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ *and* $y_d \in L^\infty([a,b])$ *is approximately continuous at* $\bar x_1,\dots,\bar x_N$*. Then the mapping in* (3.4.4) *is Fréchet-differentiable in* $\bar u$ *with derivative*

$$\begin{aligned}\mathrm{d}_u J(y(\bar u))\cdot(u-\bar u) &= \left(\psi_y(y(\bar u), y_d), \delta y\right)_{2,[a,b]} \\ &\quad + \sum_{k=1}^{N} \int_0^1 \psi_y(y(\bar x_k+;\bar u) + \tau\left[y(\bar x_k;\bar u)\right], y_d(\bar x_k))\,\mathrm{d}\tau\left[y(\bar x_k;\bar u)\right]\delta x_k,\end{aligned}$$

where $(\delta y, \delta x_1, \dots, \delta x_N) = \mathrm{T}_s\left(y(\bar u)\right)\cdot \delta u$*. If* $u \mapsto y(u)$ *is continuously shift-differentiable in* $\bar u$ *and* y_d *is continuous in a neighborhood of* $\bar x_1,\dots,\bar x_N$*, then the mapping* $u \mapsto J(y(u))$ *is continuously Fréchet-differentiable at* $\bar u$*.*

If y_d possesses at least at one $\bar{x}_k$ an approximate discontinuity, then the mapping (3.4.4) *is still directionally differentiable, where the directional derivative in a direction $\delta u \in \mathcal{U}$ is equal to*

$$(\psi_y(y(\bar{u}), y_d), \delta y)_{2,[a,b]} + \sum_{k=1}^{N} \int_0^1 \psi_y(y(\bar{x}_k+;\bar{u}) + \tau\,[y(\bar{x}_k;\bar{u})]\,, y_d(\bar{x}_k + 0\cdot\operatorname{sgn}(\delta x_k)))\,\mathrm{d}\tau\,[y(\bar{x}_k;\bar{u})]\,\delta x_k,$$

where

$$y_d(\bar{x}_k + 0\cdot\operatorname{sgn}(\delta x_k)) := \begin{cases} y_d(\bar{x}_k-)) & \textit{if } \operatorname{sgn}(\delta x_k) < 0, \\ y_d(\bar{x}_k+)) & \textit{if } \operatorname{sgn}(\delta x_k) > 0. \end{cases}$$

Lemma 3.4.3 will play a key role in the proof of the Fréchet-differentiability of the reduced cost functional of (P), cf. [81, 69].

3.5 Main results of Chapter 3

Before we state the main results of Chapter 3, we need some further basic definitions and assumptions. We note that these results are only slightly different from those in [69, Ch. 5, §2] and are based on the corresponding results for the pure initial value problem in [81]. In order to prove Fréchet-differentiability of the reduced cost functional of (P) in some $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$, the control $\bar{\mathbf{u}}$ has to satisfy non-degeneracy conditions. In particular, we have to ensure that the points where the characteristic speed at the boundary changes its sign are controllable in some sense. As we will see in the following lemma, considering some $\mathbf{u} \in \mathbf{U}$, it is sufficient to claim that the conditions

$$\operatorname*{ess\,inf}_{t\,:\,u_{B,\mathfrak{a}}(t;\mathbf{u})\neq y(t,\mathfrak{a}+;\mathbf{u})} |f(u_{B,\mathfrak{a}}(t;\mathbf{u})) - f(y(t,\mathfrak{a}+;\mathbf{u}))| > 0, \tag{3.5.1a}$$

$$\operatorname*{ess\,inf}_{t\,:\,u_{B,\mathfrak{b}}(t;\mathbf{u})\neq y(t,\mathfrak{b}-;\mathbf{u})} |f(u_{B,\mathfrak{b}}(t;\mathbf{u})) - f(y(t,\mathfrak{b}-;\mathbf{u}))| > 0 \tag{3.5.1b}$$

are satisfied.

Lemma 3.5.1. *Suppose that* (A3) *and* (A4) *hold true and consider a control $\mathbf{u} \in \mathbf{U}$ such that the corresponding entropy solution of the IBVP* (3.1.2) *satisfies the conditions in* (3.5.1)*. Then, recalling the sets defined in* (3.2.11) *and in Notation 3.3.3,* *it holds*

$$\mathbb{T}^{\mathfrak{a}}_{\pm} \subset I_{s,\mathfrak{a}}(\mathbf{u}) \quad \textit{and} \quad \mathbb{T}^{\mathfrak{b}}_{\mp} \subset I_{s,\mathfrak{b}}(\mathbf{u}). \tag{3.5.2}$$

Proof. Throughout the proof, let $\rho > 0$ denote a constant which is always sufficiently small such that the corresponding results hold true. In addition, ρ may be reduced throughout the proof. We will only prove the first assertion and note that the second one can be proved analogously. Let $\theta \in \mathbb{T}^{\mathfrak{a}}_{\pm}$ and suppose that

$$\theta \notin I_{\mathrm{s},\mathfrak{a}}(\mathbf{u}). \tag{3.5.3}$$

We note that $(\theta, \mathfrak{a})$ is, by Lemma 3.2.7 a shock generating point, i.e., there exists a unique forward characteristic through $(\theta, \mathfrak{a})$ which is a shock-curve given by some function $\eta(t)$ defined on $]\theta, \theta + \tau]$ such that

$$\dot{\eta}(t) = \frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} > 0 \quad \text{and } y(t,\eta(t)-) > y(t,\eta(t)+) \tag{3.5.4}$$

is satisfied on $]\theta, \theta + \rho]$. This yields that $(\theta, \mathfrak{a})$ cannot be the center of a rarefaction wave, i.e., it holds that

$$\theta \notin I_{\mathrm{r},\mathfrak{a}}(\mathbf{u}). \tag{3.5.5}$$

From (3.5.3) and (3.5.5) we obtain that $u_{B,\mathfrak{a}}(\cdot)$ is continuous on the interval $I =]\theta - \rho, \theta + \rho[$. Further on, note that (A4) yields $f'(u_{B,\mathfrak{a}}) \geq \alpha > 0$ and in addition $f'(y(\cdot, \mathfrak{a}+)) \leq 0$ is satisfied on $[\theta - \rho, \theta[$ since $\theta \in \mathbb{T}^{\mathfrak{a}}_{\pm}$. Using this and the strict monotonicity of f', we obtain

$$y(t, \mathfrak{a}+) - u_{B,\mathfrak{a}}(t) < -\delta_1 \quad \forall t \in [\theta - \rho, \theta[, \tag{3.5.6a}$$

$$y(\theta-, \mathfrak{a}+) - y(\theta+, \mathfrak{a}+) = y(\theta-, \mathfrak{a}+) - u_{B,\mathfrak{a}}(\theta) < -\delta_2 \tag{3.5.6b}$$

for some constants $\delta_1, \delta_2 > 0$. Here, the equality in (3.5.6b) holds due to the BLN-conditions and the continuity of $u_{B,\mathfrak{a}}(\cdot)$ on the interval I.

Using (3.1.13b), (3.5.1a) and (3.5.6a), we obtain that

$$f(y(\theta-, \mathfrak{a}+)) - f(u_{B,\mathfrak{a}}(\theta)) > \varepsilon \tag{3.5.7}$$

holds for some constant $\varepsilon > 0$. We will now analyze all possible scenarios case by case and show that for all these cases (3.5.4) is violated so that we obtain a contradiction to (3.5.3).

We now consider the **first case** in which we assume that $y(\cdot; \mathbf{u})$ is continuous on the set

$$\mathcal{S}_\rho^{>\eta} := \{(t,x) \in \Omega_T : (t,x) \in B_2^\rho((\theta,\mathfrak{a})) \text{ and } x > \eta(t) \text{ for all } t \in]\theta,\theta+\tau]\}. \quad (3.5.8)$$

Note that, due to the smoothness of $u_{B,\mathfrak{a}}(\cdot)$ on I, $y(\cdot;\mathbf{u})$ is always continuous on the set

$$\mathcal{S}_\rho^{<\eta} := \{(t,x) \in \Omega_T : (t,x) \in B_2^\rho((\theta,\mathfrak{a})) \text{ and } x < \eta(t) \text{ for all } t \in]\theta,\theta+\tau]\}. \quad (3.5.9)$$

Next, we rewrite the first term of (3.5.4) on $]\theta,\theta+\rho]$ by

$$\begin{aligned} &\frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+)) - y(t,\eta(t)-)} \\ &= \frac{f(y(t,\eta(t)+)) - f(y(\theta-,\mathfrak{a}+))}{y(t,\eta(t)+)) - y(t,\eta(t)-)} + \frac{f(y(\theta-,\mathfrak{a}+)) - f(u_{B,\mathfrak{a}}(\theta))}{y(t,\eta(t)+)) - y(t,\eta(t)-)} \\ &\quad + \frac{f(u_{B,\mathfrak{a}}(\theta)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+)) - y(t,\eta(t)-)} =: I_1(t) + I_2(t) + I_3(t). \end{aligned} \quad (3.5.10)$$

Before analyzing the terms $I_1(t)$, $I_2(t)$ and $I_3(t)$ in (3.5.10), we note that (3.5.6b) and the continuity of y on the sets defined in (3.5.8) and (3.5.9) yield

$$y(t,\eta(t)+)) - y(t,\eta(t)-) \le -\delta_3 \quad \forall t \in [\theta,\theta+\rho] \quad (3.5.11)$$

for some constant $\delta_3 > 0$. This result and (3.5.7) together imply

$$I_2(t) \le -\frac{\varepsilon}{\delta_3} =: -\varepsilon_0 < 0 \quad \forall t \in [\theta,\theta+\rho]. \quad (3.5.12)$$

Using (3.5.11) and the continuity of y on the sets defined in (3.5.8) and (3.5.9), we further obtain

$$|I_1(t)| + |I_3(t)| \le \frac{\varepsilon_0}{2} \quad \forall t \in [\theta,\theta+\rho]. \quad (3.5.13)$$

Using (3.5.12) and (3.5.13), we deduce from (3.5.10) that

$$\frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+)) - y(t,\eta(t)-)} \le -\frac{\varepsilon_0}{2}$$

is valid on $]\theta,\theta+\rho]$, which is a contradiction to (3.5.4) and hence to (3.5.3).

We now consider the **second case** where $y(\cdot;\mathbf{u})$ possesses exactly one shock-curve denoted by $\eta_2 : [\theta-\rho_2,\theta] \to \Omega$ on the set (3.5.8) which ends in $(\theta,\mathfrak{a})$. We first observe

that

$$\dot{\eta}_2(t) = \frac{f(y(t,\eta_2(t)+)) - f(y(t,\eta_2(t)-))}{y(t,\eta_2(t)+) - y(t,\eta_2(t)-)} < 0 \quad \text{and} \quad y(t,\eta_2(t)+) < y(t,\eta_2(t)-) \tag{3.5.14}$$

holds on $]\theta-\rho,\theta[$, where the second inequality holds due to (3.2.4). We note that y is continuous on the sets

$$\begin{aligned} &\mathcal{S}_\rho^{<\eta}, \quad \mathcal{S}_\rho^{>\eta} \cap \{(t,x) \in]\theta-\rho,\theta[\times\Omega : x > \eta_2(t)\}, \\ &\mathcal{S}_\rho^{>\eta} \cap \{(t,x) \in]\theta-\rho,\theta[\times\Omega : x < \eta_2(t)\}. \end{aligned} \tag{3.5.15}$$

Next, we consider again the first term of (3.5.4) on $]\theta,\theta+\rho]$:

$$\begin{aligned} &\frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\ &= \frac{f(y(t,\eta(t)+)) - f(y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})+))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\ &\quad + \frac{f(y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})+)) - f(y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\ &\quad + \frac{f(y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})-)) - f(y(\theta-,\mathfrak{a}+))}{y(t,\eta(t)+)) - y(t,\eta(t)-)} \\ &\quad + \frac{f(y(\theta-,\mathfrak{a}+)) - f(u_{B,\mathfrak{a}}(\theta))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\ &\quad + \frac{f(u_{B,\mathfrak{a}}(\theta)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} =: I_4(t) + I_5(t) + I_6(t) + I_7(t) + I_8(t). \end{aligned} \tag{3.5.16}$$

Using (3.5.14) and the continuity of y on the sets in (3.5.15), analogously to the first case, one can show that for a possibly smaller $\delta_3 > 0$ the inequality (3.5.11) is valid. We observe again that (3.5.7) and (3.5.11) together yield

$$I_7(t) \leq -\frac{\varepsilon}{\delta_3} =: -\varepsilon_0 < 0 \quad \forall t \in [\theta,\theta+\rho]. \tag{3.5.17}$$

Moreover, (3.5.14) and (3.5.11) impy that

$$\begin{aligned} I_5(t) = &\frac{f(y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})+)) - f(y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})-))}{y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})+) - y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})-)} \\ &\cdot \frac{y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})+) - y(\theta-\frac{\rho}{2},\eta_2(\theta-\frac{\rho}{2})-)}{y(t,\eta(t)+) - y(t,\eta(t)-)} \leq 0 \end{aligned} \tag{3.5.18}$$

holds on $]\theta,\theta+\rho]$. Using the continuity of y on the sets defined in (3.5.15), we obtain

that

$$|I_4(t)| + |I_6(t)| + |I_8(t)| \leq \frac{\varepsilon_0}{2} \quad \forall t \in]\theta, \theta+\rho]. \tag{3.5.19}$$

Finally, (3.5.18),(3.5.17), (3.5.19) and (3.5.16) together yield that

$$\frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \leq -\frac{\varepsilon_0}{2} \tag{3.5.20}$$

holds true on $]\theta, \theta+\rho]$. However, this is again a contradiction to (3.5.4) and therefore to (3.5.3). This technique can be extended to the case that $y(\cdot;\mathbf{u})$ possesses an arbitrary number of shock-curves on the set in (3.5.8). Indeed, if there are N shock-curves, then in the estimation (3.5.16) for each shock-curve a term of the form I_5 will appear, which all can be estimated according to (3.5.18), such that one can show (3.5.20).

Finally, we consider the **third case** where $(\theta, \mathfrak{a})$ is the center of a centered compression wave, i.e., there are two genuine backward characteristics $\zeta_l(\cdot)$ and $\zeta_r(\cdot)$ through $(\theta, \mathfrak{a})$ such that all backward characteristics in the funnel confined by them are genuine. We can w.l.o.g. assume that y is continuous on the sets

$$\begin{aligned} &\mathcal{S}_\rho^{<\eta}, \quad , \mathcal{S}_\rho^{>\eta} \cap \{(t,x) \in]\theta-\rho, \theta[\times \Omega : x > \zeta_r(t)\}, \\ &\mathcal{S}_\rho^{>\eta} \cap \{(t,x) \in]\theta-\rho, \theta[\times \Omega : x < \zeta_l(t)\}. \end{aligned} \tag{3.5.21}$$

If there is, in addition, one or multiple shock-curves ending in $(\theta, \mathfrak{a})$, then one can use the same arguments as described in the second case. We further observe that

$$f'(y(\zeta_r(t))) < f'(y(\zeta_l(t))) \leq 0 \quad \forall t \in]\theta-\rho, \theta[\tag{3.5.22}$$

holds. From (3.5.22) and the strict monotonicity of f', we obtain that

$$y(\zeta_r(t)) < y(\zeta_l(t)) \quad \text{and} \quad f(y(\zeta_l(t))) < f(y(\zeta_r(t))) \tag{3.5.23}$$

hold for all $t \in]\theta-\rho, \theta[$. We rewrite again the right-hand side of the first term of (3.5.4) on $]\theta, \theta+\rho]$:

$$\begin{aligned} &\frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\ &= \frac{f(y(t,\eta(t)+)) - f(y(\theta-\frac{\rho}{2}, \zeta_r(\theta-\frac{\rho}{2})))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\ &\quad + \frac{f(y(\theta-\frac{\rho}{2}, \zeta_r(\theta-\frac{\rho}{2}))) - f(y(\theta-\frac{\rho}{2}, \zeta_l(\theta-\frac{\rho}{2})))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \end{aligned}$$

$$\begin{aligned}
&+ \frac{f(y(\theta - \frac{\rho}{2}, \zeta_l(\theta - \frac{\rho}{2}))) - f(y(\theta-, \mathfrak{a}+))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\
&+ \frac{f(y(\theta-, \mathfrak{a}+)) - f(u_{B,\mathfrak{a}}(\theta))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\
&+ \frac{f(u_{B,\mathfrak{a}}(\theta)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \\
&=: I_9(t) + I_{10}(t) + I_{11}(t) + I_{12}(t) + I_{13}(t).
\end{aligned}$$

As in the first the second case, one can show that for a possibly smaller $\delta_3 > 0$ the inequality (3.5.11) holds true:

$$y(t,\eta(t)+)) - y(t,\eta(t)-) \leq \delta_3 \quad \forall t \in [\theta, \theta + \rho]. \tag{3.5.24}$$

Then, (3.5.23) and (3.5.24) together yield

$$I_{10}(t) \leq 0 \quad \forall t \in]\theta, \theta + \rho[. \tag{3.5.25}$$

Using again (3.5.7) in connection with (3.5.24), we obtain

$$I_{12}(t) \leq -\frac{\varepsilon}{\delta_3} =: -\varepsilon_0 < 0 \quad \forall t \in]\theta, \theta + \rho[. \tag{3.5.26}$$

As in the second case, we use the continuity of y on the sets defined in (3.5.21) and get

$$|I_9(t)| + |I_{11}(t)| + |I_{13}(t)| \leq \frac{\varepsilon_3}{2} \quad \forall t \in [\theta, \theta + \rho]. \tag{3.5.27}$$

Finally, (3.5.25), (3.5.26) and (3.5.27) together impy

$$\frac{f(y(t,\eta(t)+)) - f(y(t,\eta(t)-))}{y(t,\eta(t)+) - y(t,\eta(t)-)} \leq -\frac{\varepsilon_0}{2} \quad \forall t \in [\theta, \theta + \rho],$$

which is a contradiction to (3.5.4).

We find, in summary, that under the assumptions of Lemma 3.5.1, (3.5.4) and hence (3.5.3) can never be valid such that (3.5.2) must be satisfied. □

In order to prove Fréchet-differentiability of the mapping $\mathbf{u} \mapsto J(y(\bar{t}, \cdot, \mathbf{u}))$ in some $\bar{\mathbf{u}} \in \mathbf{U}$, we have to require that $\bar{\mathbf{u}}$ satisfies the following non-degeneracy conditions, which are basically the conditions of Theorem 5.2.3 in [69]:

(ND) Consider a control $\mathbf{u} \in \mathbf{U}$, a fixed time point $\bar{t} \in]0, T[$ and a bounded interval $[a, b] \subset \Omega$. We say that $\mathbf{u}$ is non-degenerated if the following conditions hold for a sufficiently small constant $\rho > 0$: Considering the sets of indices defined in Notation 3.3.3, it holds that

$$I_{c,0}(\mathbf{u}) \cup I_{c,\mathfrak{a}}(\mathbf{u}) \cup I_{c,\mathfrak{b}}(\mathbf{u}) = \emptyset.$$

Moreover, the corresponding entropy solution $y(\bar{t}, \cdot; \mathbf{u})$ of (3.1.2) has no shock generation points on $[a, b]$ and a finite number of discontinuities $a < x_1 < \cdots < x_N < b$, which are no shock interaction points and non-degenerated according to Definition 3.7.3. Furthermore, the condition (3.5.1) is satisfied. For every $t_j^{\mathfrak{a}} \in \mathbb{T}_{\pm}^{\mathfrak{a}}$ let $\xi_{\mathfrak{a}}^j$ denote the maximal backward characteristic through the point $(t_j^{\mathfrak{a}}, \mathfrak{a})$. Then it is possible to construct a stripe $S \subset \Omega_T$ with $\xi_{\mathfrak{a}}^j \subset \operatorname{int} S$ and there exists a continuously differentiable local solution $Y : S \to \mathbb{R}$ such that

$$y(\cdot; \mathbf{u} + \delta\mathbf{u}) \equiv Y(\cdot; \mathbf{u} + \delta\mathbf{u}) \quad \text{on } S \cap \Omega \times [0, t_j^{\mathfrak{a}} + \delta t_j^{\mathfrak{a}}]$$

is satisfied for all $\delta\mathbf{u} \in B_\rho^{\mathbf{U}}(0_{\mathbf{U}})$ and $f(Y(\theta, \mathfrak{a}; \mathbf{u}) < f(u_{B,\mathfrak{a}}(t_j^{\mathfrak{a}}+))$ holds.
Considering some $t_j^{\mathfrak{b}} \in \mathbb{T}_{\mp}^{\mathfrak{b}}$ let $\xi_{\mathfrak{b}}^j$ denote the maximal backward characteristic through the point $(t_j^{\mathfrak{b}}, \mathfrak{b})$. Analogously to the left boundary, one can construct a stripe $S \subset \Omega_T$ with $\xi_{\mathfrak{b}}^j \subset \operatorname{int} S$ and there exists a continuously differentiable local solution $Y : S \to \mathbb{R}$ such that for all $\delta\mathbf{u} \in B_\rho^{\mathbf{U}}(0_{\mathbf{U}})$

$$y(\cdot; \mathbf{u} + \delta\mathbf{u}) \equiv Y(\cdot; \mathbf{u} + \delta\mathbf{u}) \quad \text{on } S \cap \Omega \times [0, t_j^{\mathfrak{b}} + \delta t_j^{\mathfrak{b}}]$$

holds true and Y satisfies $f(Y(\theta, \mathfrak{b}; \mathbf{u}) > f(u_{B,\mathfrak{b}}(t_j^{\mathfrak{b}}+))$. Finally, we require that $|u_0(\mathfrak{a}+, \mathbf{u}) - u_{B,\mathfrak{a}}(0+, \mathbf{u})| > 0$ and $|u_0(\mathfrak{b}-, \mathbf{u}) - u_{B,\mathfrak{b}}(0+, \mathbf{u})| > 0$.

Remark 3.5.2. The last condition mentioned in (ND) ensures a certain stability of the corresponding entropy solution $y(\mathbf{u})$. More precisely, supposing that e.g. $u_0(\mathfrak{a}+, \mathbf{u}) = u_{B,\mathfrak{a}}(0+, \mathbf{u})$, then a small perturbation of $\mathbf{u}$ can cause a shock or a rarefaction wave emanating from $(0, \mathfrak{a})$. In [69, Lemma 6.2.16], the author provides a method how to deal with this instability. However, as we will see later, due the just mentioned instability, the results of Lemma 3.7.1 do not hold anymore. Since Lemma 3.7.1 will be essential to derive necessary optimality conditions for (P), we have to ensure that the solution near the points $(0, \mathfrak{a})$ and $(0, \mathfrak{b})$ is stable, i.e., there is either a shock or a rarefaction wave emanating from these points, respectively, which is guaranteed by the last condition of (ND). Another possibility to cope with this problem is to choose some subspace $\tilde{\mathbf{U}}(\mathbf{u}) \subset \mathbf{U}$ such that for all $\delta\mathbf{u} \in \tilde{\mathbf{U}}$ it holds that $u_0(\mathfrak{a}+/\mathfrak{b}-, \mathbf{u}+\delta\mathbf{u}) = u_{B,\mathfrak{a}/\mathfrak{b}}(0+, \mathbf{u}+\delta\mathbf{u})$ if the case $u_0(\mathfrak{a}+/\mathfrak{b}-, \mathbf{u}) = u_{B,\mathfrak{a}/\mathfrak{b}}(0+, \mathbf{u})$ appears.

The next result is an extension of Theorem 3.3.6. in [81] to initial-boundary value problems:

Theorem 3.5.3 (Nondegeneracy of shocks holds for almost all $\bar{t} \in]0,T]$)**.** *In addition to* (A3) *and* (A4)*, assume that* $g \in L^\infty(]0,T[; C^2_{\text{loc}}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m))$ *holds true. Consider a control* $\mathbf{u} \in \mathbf{U}$ *with additionally* $u_0(\mathbf{u}) \in PC^2(\Omega; x_1, \ldots, x_{n_x})$, $u_{B,\mathfrak{a}}(\mathbf{u}) \in PC^2([0,T]; t_1, \ldots, t_{n_{t,\mathfrak{a}}})$ *and* $u_{B,\mathfrak{b}}(\mathbf{u}) \in PC^2([0,T]; t_1, \ldots, t_{n_{t,\mathfrak{b}}})$*. Then for almost all* $\bar{t} \in]0,T]$ *the corresponding entropy solution* $y(\bar{t}, \cdot; \mathbf{u})$ *of the IBVP* (3.1.2) *has no shock generation points on* $[a,b]$ *and a finite number of (according to Definition 3.7.3) non-degenerated discontinuities* $a < x_1 < \cdots < x_N < b$*, which are no shock interaction points.*

Proof. Using the results of §3.7.1 and Lemma 3.2.5, the proof of Theorem 3.5.3 is similar to the proof of Theorem 3.3.6. in [81]. □

In the next result, we will prove Fréchet-differentiability of the reduced cost functional $\hat{J}(\cdot)$ and derive an adjoint-based representation of the gradient. Considering an entropy solution y of the IBVP (3.1.2), the corresponding adjoint equation reads

$$
\begin{aligned}
p_t + f'(y)p_x &= -g_y(\cdot, y, u_1)p \quad \text{on } \Omega_{\bar{t}} \\
p(\bar{t}, \cdot) &= p^{\bar{t}}(\cdot) \quad \text{on } \Omega, \\
p(\cdot, \mathfrak{a}+) &= 0 \qquad \text{on } \{t \in]0,\bar{t}[: (t,\mathfrak{a}) \in D_-\} \\
p(\cdot, \mathfrak{b}-) &= 0 \qquad \text{on } \{t \in]0,\bar{t}[: (t,\mathfrak{b}) \in D_-\}
\end{aligned}
\tag{3.5.28}
$$

where $p^{\bar{t}}$ is the given end data. With the adjoint equation (3.5.28) we associate the *adjoint state* p, which is defined as in [69, Definition 5.2.2]:

Definition 3.5.4. *Consider a time point* $\bar{t} \in]0,T[$ *and let* $p^{\bar{t}}$ *be a bounded function that is pointwise everywhere the limit of a sequence* $(p^{\bar{t}}_n)_{n \in \mathbb{N}} \subset C^{0,1}(\Omega)$ *which is bounded in* $C(\Omega) \cap W^{1,1}_{loc}(\Omega)$*. Moreover, let* $y \in BV(\Omega_T) \cap C([0,T]; L^1(\Omega))$ *be an entropy solution of the IBVP* (3.1.2)*. Then we associate with* (3.5.28) *the adjoint state* p*, which is defined as follows:*

1. *For any generalized backward characteristic* ξ *through* $(\bar{t}, \bar{x}) \in \Omega_T^{cl}$*, the mapping* $t \mapsto p^\xi(t) := p(t, \xi(t))$ *solves the ordinary differential equation*

$$
\begin{aligned}
&\dot{p}^\xi(t) = -g_y(t, \xi(t), y(t, \xi(t)), u_1(t, \xi(t)))p^\xi(t), \quad t \in]0, \bar{t}] : \xi(t) \in \Omega, \\
&p^\xi(\bar{t}) = p^{\bar{t}}(\bar{x}).
\end{aligned}
$$

2. *The adjoint state* p *is equal to zero on* D_-*.*

The following result is an extension of Theorem 5.2.6 in [69] to the case that the shifting of rarefaction centers is allowed:

Theorem 3.5.5. *Suppose that* (A3) *and* (A4) *hold true. Considering a time point* $\bar{t} \in]0,T[$ *and an interval* $[a,b]$, *let* $\bar{\mathbf{u}} \in \mathbf{U}$ *satisfy the conditions in* (ND). *Denoting by* $y(\bar{\mathbf{u}})$ *the corresponding entropy solution of* (3.1.2), *let* y_d *be continuous in a neighborhood of the discontinuities* $\bar{x}_1, \dots, \bar{x}_N$ *of* $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ *on* $[a,b]$. *Then the reduced cost functional*

$$\mathbf{U} \ni \mathbf{u} \mapsto \hat{J}(\mathbf{u}) := J(y(\mathbf{u}), \mathbf{u}) \in \mathbb{R} \tag{3.5.29}$$

is continuously Fréchet-differentiable in a neighborhood $B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$, *where* $\rho > 0$ *is sufficiently small. The corresponding derivative in a direction* $\delta\mathbf{u} \in \mathbf{U}$ *is given by*

$$\begin{aligned}
\hat{J}'(\mathbf{u}) \cdot \delta\mathbf{u} = {} & R'(\mathbf{u})\delta\mathbf{u} + (p g_{u_1}(\cdot, y, u_1), \delta u_1)_{2,\Omega_{\bar{t}}} \\
& + \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0)_{2,I_0^i(\mathbf{u})} + \sum_{i \in I_{s,0}(\mathbf{u})} p(0, x_i^0)[u_0(x_i^0)]\delta x_i^0 \\
& + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, \mathfrak{a}+), f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i(\mathbf{u}) \cap]0,\bar{t}[} \\
& + \sum_{\substack{i \in I_{s,\mathfrak{a}}(\mathbf{u}): \\ t_i^{\mathfrak{a}} \le \bar{t}}} p(t_i^{\mathfrak{a}}, \mathfrak{a}+)[f(y(t_i^{\mathfrak{a}}, \mathfrak{a}+; \mathbf{u}))]\delta t_i^{\mathfrak{a}} \\
& - \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot, \mathfrak{b}-), f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2,I_{B,\mathfrak{b}}^i \cap]0,\bar{t}[} \\
& - \sum_{\substack{i \in I_{s,\mathfrak{b}}(\mathbf{u}): \\ t_i^{\mathfrak{b}} \le \bar{t}}} p(t_i^{\mathfrak{b}}, \mathfrak{b}-)[f(y(t_i^{\mathfrak{b}}, \mathfrak{b}-; \mathbf{u}))]\delta t_i^{\mathfrak{b}} \\
& - \sum_{i \in I_{r,0}(\mathbf{u})} p_i^{r,0}\delta x_i^0 + \sum_{\substack{i \in I_{r,\mathfrak{a}}(\mathbf{u}): \\ t_i^{\mathfrak{a}} \le \bar{t}}} p_i^{r,\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i \in I_{r,\mathfrak{b}}(\mathbf{u}): \\ t_i^{\mathfrak{b}} \le \bar{t}}} p_i^{r,\mathfrak{b}}\delta t_i^{\mathfrak{b}}
\end{aligned} \tag{3.5.30}$$

where p *denotes the adjoint state according to Definition 3.5.4 with end data*

$$p^{\bar{t}} = \begin{cases} \int_0^1 \psi_y\left(y(\bar{t}, x+; \bar{\mathbf{u}})) + \tau[y(\bar{t}, x; \bar{\mathbf{u}})], y_d(x+) + \tau[y_d(x)]\right) \mathrm{d}\tau & \text{if } x \in [a,b] \\ 0 & \text{else} \end{cases}$$

and

$$p_j^{r,0} := \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))} \lim_{t \searrow 0} p(t, zt + x_j^0) \frac{1}{f''(f'^{-1}(z))} \, \mathrm{d}z, \quad j \in I_{r,0}(\mathbf{u}),$$

$$p_j^{r,\mathfrak{a}} := \int_{f'(u_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))} \lim_{t \searrow t_j^{\mathfrak{a}}} p(t, z(t - t_j^{\mathfrak{a}})) \frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z, \quad j \in I_{r,\mathfrak{a}}(\mathbf{u}) : t_j^{\mathfrak{a}} \le \bar{t},$$
$$p_j^{r,\mathfrak{b}} := \int_{f'(u_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(u_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \lim_{t \searrow t_j^{\mathfrak{b}}} p(t, z(t - t_j^{\mathfrak{b}})) \frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z, \quad j \in I_{r,\mathfrak{b}}(\mathbf{u}) : t_j^{\mathfrak{b}} \le \bar{t}.$$

The proof of Theorem 3.5.5 can be found in the last section of Chapter 3.

3.6 The adjoint equation

In this section we will have a closer look at the adjoint equation that was introduced in (3.5.28), where we restrict ourselves to the case $\mathfrak{a} = -\infty$ and $\mathfrak{b} = \infty$. Given an entropy solution $y \in L^\infty(\Omega_T)$ and end data p^τ for some $\tau \in]0, T]$, the corresponding adjoint equation reads

$$p_t + f'(y)p_x = -g_y(\cdot, y, u_1)p \;\text{ on } \Omega_\tau :=]0, \tau[\times\Omega \quad\text{and}\quad p(\tau, \cdot) = p^\tau(\cdot) \;\text{ on } \Omega. \tag{3.6.1}$$

We note that (3.6.1) is a transport equation of the form

$$p_t + ap_x = -bp + c \;\text{ on } \Omega_\tau \quad\text{and}\quad p(\tau, \cdot) = p^\tau(\cdot) \;\text{ on } \Omega. \tag{3.6.2}$$

This section is concerned with the analysis of general transport equations of type (3.6.2). Here, we assume that $\Omega = \mathbb{R}$, $b, c \in L^\infty(]0, T[; C^{0,1}(\mathbb{R}))$ and that $a \in L^\infty(\Omega_T)$ satisfies the so-called *one-sided Lipschitz condition* (OSLC)

$$a_x(t, \cdot) \le \alpha(t), \quad \alpha \in L^1(]0, T[), \tag{3.6.3}$$

or at least the *weakened one-sided Lipschitz condition* (weakened OSLC)

$$a_x(t, \cdot) \le \alpha(t), \quad \alpha \in L^1(]\sigma, T[) \quad \text{for all } \sigma \in]0, T[. \tag{3.6.4}$$

We note that the inequalities in (3.6.3) and (3.6.4) have to be understood in the sense of distributions. Although in this thesis we deal with possibly bounded domains $\Omega \subset \mathbb{R}$, we study (3.6.2) for the case $\Omega = \mathbb{R}$ and adapt the results in a way such that they are applicable to the case of bounded domains, cf. [69], [70] and [71].

Setting $a = f'(y)$, we observe that due to Oleiniks entropy condition in (3.2.3) the OSLC (3.6.3) cannot hold true if there are rarefaction centers in the initial data. Nevertheless, considering the pure initial value problem, one can show that in the presence of rarefaction waves the weakened OSLC (3.6.4) is satisfied.

As we can see in the following example which was first discussed by Conway in [30] and also by Bouchut and James in [11, Example 4.1.1], transport equations with discontinuous coefficients of the type (3.6.2) do not admit unique solutions.

Example 3.6.1. Let $a(t,x) = \operatorname{sgn}(x)$, $b, c \equiv 0$ and consider end data $p^\tau \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$. Then for any function $h \in C^{0,1}([0,T])$ satisfying $h(0) = p^\tau(0)$, the function

$$p(t,x) = \begin{cases} p^\tau(x - (\tau - t)\operatorname{sgn}(x)) & \text{if } \tau - t \le |x|\,, \\ h(\tau - t - |x|) & \text{if } \tau - t \le |x|\,. \end{cases}$$

is a Lipschitz continuous solution to (3.6.2).

In order to achieve uniqueness, Bouchut and James introduce in [11] the concept of *reversible solutions* for the homogeneous version of (3.6.2) given by

$$p_t + a p_x = 0 \ \text{ on } \Omega_\tau \quad \text{and} \quad p(\tau,\cdot) = p^\tau(\cdot) \ \text{ on } \mathbb{R}. \tag{3.6.5}$$

In [81], Ulbrich extends the results of Bouchut and James to the inhomogeneous case with possibly discontinuous end data p^τ. In the rest of this section, we will collect some important results of [11] and [81]. We first present a characterization of reversible solutions to (3.6.5), which is introduced in [11]:

Definition 3.6.2. *Denote by $\mathcal{L}_{hom}$ the set of Lipschitz continuous solutions to* (3.6.5)*. Then $p \in \mathcal{L}_{hom}$ is a reversible solution of* (3.6.5)*, if there exist $p_1, p_2 \in \mathcal{L}_{hom}$ satisfying $(p_1)_x, (p_2)_x \ge 0$ such that $p = p_1 - p_2$.*

In [11], Bouchut and James prove existence and uniqueness of a reversible solution if the OSLC (3.6.3) is satisfied:

Theorem 3.6.3 (Theorem 4.1.5, [11])**.** *Assume that $a \in L^\infty(\Omega_T)$ satisfies the OSLC* (3.6.3) *for some $\alpha \in L^1(]0,T[)$. Then for all $p^\tau \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$* (3.6.5) *admits a unique reversible solution $p \in C^{0,1}_{\mathrm{loc}}(\Omega^{cl}_\tau)$ such that $p(\tau,\cdot) = p^\tau$. Finally, for arbitrary $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $t \in [0,\tau]$, it holds that*

$$\|p(t,\cdot)\|_{\infty,I} \le \|p^\tau\|_{\infty,J}\,, \quad \|p_x(t,\cdot)\|_{\infty,I} \le e^{\int_t^\tau \alpha}\,\|p^\tau_x\|_{\infty,J}\,,$$

where $I :=]x_1, x_2[$ and $J :=]x_1 - \|a\|_{\infty,}(\tau - t), x_2 + \|a\|_{\infty,}(\tau - t)[$.

As already mentioned in [81], the notion of reversible solutions according to Definition 3.6.2 is not extendable to the inhomogeneous case. In order to characterize reversible solutions to (3.6.2) for the case $b, c \not\equiv 0$, the so-called *generalized backward flow*, introduced in [11], plays an important role.

Definition 3.6.4. *Assume that $a \in L^\infty(\Omega_T)$ satisfies the OSLC* (3.6.3) *and define the set*

$$D_b := \left\{(s,t) \in \mathbb{R}^2 : 0 \le t \le s \le T\right\}.$$

Then the generalized backward flow $X \in C^{0,1}(D_b \times \mathbb{R})$ associated with a is defined as follows: For any $s \in]0,T]$, $X(s;\cdot,\cdot)$ is given by the unique reversible solution to

$$\begin{aligned}&X_t(s;\cdot,\cdot) + aX_x(s;\cdot,\cdot) = 0, \quad (t,x) \in]0,s[\times\mathbb{R},\\ &X(s;s,x) = x, \quad x \in \mathbb{R}.\end{aligned}$$

For $s = 0$, we set $X(0;0,x) = x$.

Remark 3.6.5. As remarked in [11], the generalized backward flow X satisfies the composition formula

$$X(s;t,X(t;\sigma,z)) = X(s;\sigma,z) \quad \text{for all} \quad 0 \le \sigma \le t \le s \le T,\ z \in \mathbb{R}$$

and for all $(t,x) \in \Omega_T$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}s}X(s;t,x) \in [a(s,X(s;t,x)+), a(s,X(s;t,x)-)] \quad \text{for a.a. } s \in]0,T[.$$

That means that $X(\cdot;t,x)$ solves the ODE

$$\frac{\mathrm{d}}{\mathrm{d}s}X(s;t,x) = a(s,X(s;t,x)), \quad s \in [t,T], \quad X(t;t,x) = x \tag{3.6.6}$$

in the sense of Filippov [35]. We note that (3.6.3) respectively (3.6.4) yield $a(t,\cdot) \in BV_{\mathrm{loc}}(\mathbb{R})$ for a.a. t. Hence the left- and right-sided limits in (3.6.6) are well-defined, cf. [81].

Furthermore, Bouchut and James show in [11] that the unique reversible solution p to (3.6.5) is given by

$$p(t,x) = p^\tau(X(\tau;t,x)) \tag{3.6.7}$$

and therefore solves (3.6.5) along the generalized characteristics in the sense of Dafermos. Thus, the reversible solution to the transport equation in Example 3.6.1 is given by (3.6.7), where the generalized backward flow associated with a in the considered example is equal to

$$X(s;t,x) = \begin{cases} x - \operatorname{sgn}(x)(s-t) & \text{if } |x| > s-t \\ 0 & \text{if } |x| \le s-t \end{cases}, \qquad s \in [t,\tau],$$

cf. [81].

Using this characterization of reversible solutions, Ulbrich developed the following definition of reversible solutions to the inhomogeneous case.

Definition 3.6.6. *Assume that $a \in L^\infty(\Omega_T)$ satisfies the OSLC* (3.6.3)*, i.e., $a_x(t,\cdot) \le \alpha(t)$ for some $\alpha \in L^1(]0,T[)$ and let $b,c \in L^\infty(]0,T[;C^{0,1}(\mathbb{R}))$. Then p is called a reversible solution to* (3.6.2)*, if for all $z \in \mathbb{R}$ it holds that*

$$\begin{aligned} &p(\tau, X(\tau;0,z)) = p^\tau(X(\tau;0,z)), \\ &\frac{\mathrm{d}}{\mathrm{d}t} p(t, X(t;0,z)) = (-bp + c)(t, X(t;0,z)), \quad \text{for a.a. } t \in]0,\tau[. \end{aligned} \tag{3.6.8}$$

For the case that only

$$\alpha \in L^1(]\sigma,T[) \quad \text{for all } \sigma > 0$$

holds, p can first be defined on the domains $]\sigma,\tau[\times\mathbb{R}$ and then on Ω_τ by exhaustion.

Additionally, Ulbrich extends in [81] Theorem 3.6.3 to the inhomogeneous case (3.6.2), see also [82]. Ulbrich first derives an existence and uniqueness result for reversible solutions to (3.6.2), assuming that a satisfies the OSLC (3.6.3) and p^τ is Lipschitz continuous. Building on this result, Ulbrich further proves existence and uniqueness of reversible solutions under the weakened OSLC (3.6.4) for possibly discontinuous end data lying in the set

$$B_{\mathrm{Lip}}(\mathbb{R}) := \left\{ v \in B(\mathbb{R}) : \begin{array}{c} v \text{ is the pointwise everywhere limit of a sequence} \\ (w_n)_{n\in\mathbb{N}} \subset C^{0,1}(\mathbb{R}), (w_n) \text{ bounded in } C(\mathbb{R}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}) \end{array} \right\},$$

where $B(\mathbb{R})$ denotes the Banach space of bounded functions equipped with the sup-norm. This is crucial for our analysis, since we have to deal with discontinuous end data p^τ due to the computation of gradients of the reduced cost functional in Theorem 3.5.5. The just mentioned existence and uniqueness results are given by the following theorems:

Theorem 3.6.7 (Thm. 4.2.10, [81]). *Assume that $a \in L^\infty(\Omega_T)$ satisfies the OSLC* (3.6.3) *and let $b,c \in L^\infty(]0,T[;C^{0,1}(\mathbb{R}))$. Given any $p^\tau \in C^{0,1}(\mathbb{R})$,* (3.6.2) *admits a unique reversible solution $p \in C^{0,1}(\Omega_\tau^{cl})$. Moreover, p solves* (3.6.2) *almost ev-*

erywhere on Ω_τ *and* $\|p\|_{C^{0,1}(\Omega_\tau^{cl})}$ *has a bound that depends on* $\|b\|_{L^\infty(]0,T[;C^{0,1}(\mathbb{R}))}$, $\|c\|_{L^\infty(]0,T[;C^{0,1}(\mathbb{R}))}$, $\|p^\tau\|_{C^{0,1}(\mathbb{R})}$, $\|a\|_{\infty,[0,T]}$ *and* $\|\alpha\|_{1,[0,T]}$, *but not on* τ.

In addition, for arbitrary $t \in [0,\tau]$, $x_1, x_2 \in \mathbb{R}$ *with* $x_1 < x_2$ *and* $0 \le t_1 < t_2 \le \tau$, *it holds that*

$$\|p(t,\cdot)\|_{B(I)} \le \left(\|p^\tau\|_{B(J)} + \|c\|_{L^1(]0,\tau[;B(J))}\right) e^{\|b\|_{L^1(]0,\tau[;B(J))}}, \tag{3.6.9a}$$

$$\|p_x(t,\cdot)\|_{1,I} \le \left(\|p_x^\tau\|_{1,J} + \|b_x\|_{1,[t,\tau]\times J}\,\|p\|_{\infty,[t,\tau]\times J} + \|c_x\|_{1,[t,\tau]\times J}\right) e^{\|b\|_{L^1(]0,\tau[;B(J))}}, \tag{3.6.9b}$$

$$\begin{aligned}\|p_t\|_{1,[t_1,t_2]\times I} \le (t_2 - t_1)\big(&\|bp - c\|_{L^\infty(]t_1,t_2[;L^1(I))} \\ &+ \|a\|_{\infty,[t_1,t_2]\times I}\,\|p_x\|_{L^\infty(]t_1,t_2[;L^1(I))}\big),\end{aligned} \tag{3.6.9c}$$

where $I \coloneqq [x_1, x_2]$ *and* $J \coloneqq [x_1 - \|a\|_{\infty,}(T-t), x_2 + \|a\|_{\infty,}(T-t)]$. *In particular,* (3.6.9) *yields*

$$\begin{aligned}\|p\|_{W^{1,1}(]0,\tau[\times I)} + \|p\|_{B(]0,\tau];W^{1,1}(I))} &\le C_1 \\ \|p(t_2) - p(t_1)\|_{1,I} &\le C_2(t_2 - t_1),\end{aligned} \tag{3.6.10}$$

where the constants C_1 *and* C_2 *depend on* $\|p^\tau\|_{W^{1,1}(J)}$, $\|a\|_{\infty,[0,\tau]\times J}$, $\|b\|_{L^\infty(]0,\tau[;W^{1,1}(J))}$ *and* $\|c\|_{L^\infty(]0,\tau[;W^{1,1}(J))}$, *but not on* α.

Theorem 3.6.8 (Cor. 4.2.11, [81]). *Let the assumptions of Theorem 3.6.7 hold with the relaxation that only the weakened OSLC* (3.6.4) *is satisfied for some* α *with* $\alpha \in L^1(]\sigma, T[)$ *for each fixed* $\sigma > 0$. *Then the following holds true:*

(i) Given any $p^\tau \in C^{0,1}(\mathbb{R})$, (3.6.2) *admits a unique reversible solution* p *that satisfies*

$$p \in B(\Omega_T) \cap C^{0,1}([\sigma,\tau]\times\mathbb{R}) \cap C^{0,1}([0,\tau]; L^1_{\mathrm{loc}}(\mathbb{R})) \cap B([0,\tau]; BV_{\mathrm{loc}}(\mathbb{R}))$$

for all $\sigma \in]0,\tau[$. *Furthermore,* (3.6.9) *and* (3.6.10) *hold for all* $t \in]0,\tau]$ *and* (3.6.8) *for all* $t \in]0,\tau[$. *Moreover, for any closed set* E *such that*

$$a_x(t,\cdot)|_{\mathbb{R}\setminus E} \le \hat\alpha(t)$$

with some $\hat\alpha \in L^1(]0,T[)$, *the reversible solution* p *satisfies in addition*

$$p \in C^{0,1}([0,\tau]\times(\mathbb{R}\setminus E_\varepsilon))$$

for any ε*-neighborhood* E_ε *of* E. *Moreover,* $\|p\|_{C^{0,1}([0,\tau]\times(\mathbb{R}\setminus E_\varepsilon))}$ *has a bound only depending on* ε, $\|b\|_{L^\infty(]0,T[;C^{0,1}(\mathbb{R}))}$, $\|c\|_{L^\infty(]0,T[;C^{0,1}(\mathbb{R}))}$, $\|p^\tau\|_{C^{0,1}(\mathbb{R})}$,

$\|a\|_{\infty,[0,T]}$ *and* $\|\hat{\alpha}\|_{1,[0,T]}$*, but not on* τ*. For the case that the OSLC* (3.6.3) *is satisfied, one can choose* $E = E_\varepsilon = \emptyset$*.*

(ii) For end data $p^\tau \in B_{Lip}(\mathbb{R})$*,* (3.6.2) *admits a unique reversible solution* $p \in B(\Omega_\tau)$ *satisfying* (3.6.9a) *and*

$$p \in B(\Omega_T) \cap C^{0,1}([0,\tau]; L^1_{\text{loc}}(\mathbb{R})) \cap B([0,\tau]; BV_{\text{loc}}(\mathbb{R})) \cap BV_{\text{loc}}(\Omega_\tau^{cl}).$$

Consider an arbitrary sequence $(p_n^\tau)_{n\in\mathbb{N}} \subset C^{0,1}(\mathbb{R})$ *that is bounded in* $C(\mathbb{R}) \cap W^{1,1}_{\text{loc}}(\mathbb{R})$ *and converges pointwise everywhere to* p^τ*. Then the corresponding reversible solutions* $p_n \in C^{0,1}_{\text{loc}}(\Omega_\tau) \cap C^{0,1}([0,\tau]; L^1_{\text{loc}}(\mathbb{R}))$ *of* (3.6.2) *satisfy*

$$p_n \to p \quad \textit{boundedly everywhere on }]0,\tau] \times \mathbb{R} \;\; \textit{and in} \;\; C([0,\tau]; L^1_{\text{loc}}(\mathbb{R})).$$

If even the OSLC (3.6.3) *holds for some* $\alpha \in L^1(]0,T[)$*, then the reversible solutions* $p_n \in C^{0,1}_{\text{loc}}(\Omega_\tau^{cl})$ *satisfy in addition*

$$p_n \to p \quad \textit{boundedly everywhere on } [0,\tau] \times \mathbb{R}.$$

We will close this section by a result of [81] studying the stability of reversible solutions under the OSLC (3.6.3). Additionally, one can also find a version assuming that only the weakened OSLC (3.6.4) is satisfied, see [81, Theorem 4.2.13].

Theorem 3.6.9 (Thm. 4.2.12, [81])**.** *Let the following assumptions be satisfied:*

1. *Consider a sequence* $(a_n)_{n\in\mathbb{N}}$ *which is bounded in* $L^\infty(\Omega_T)$ *and satisfies*

$$a_n \overset{*}{\rightharpoonup} a \qquad \textit{in } L^\infty(\Omega_T)$$

as well as the OSLCs

$$(a_n)_x(t,\cdot) \le \alpha_n(t), \quad a_x(t,\cdot) \le \alpha(t) \qquad \textit{for a.a. all } t \in]0,T[$$

for a bounded sequence $(\alpha_n)_{n\in\mathbb{N}} \subset L^1(]0,T[)$ *and* $\alpha \in L^1(]0,T[)$*.*

2. *Furthermore, let* $(b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \subset L^\infty(]0,T[; C^{0,1}(\mathbb{R}))$ *be sequences which are bounded in* $L^1(]0,T[; C(\mathbb{R}))$ *and satisfy*

$$b_n \to b, \quad c_n \to c \qquad \textit{in } L^1(]0,T[; C_{\text{loc}}(\mathbb{R})),$$

where $b, c \in L^\infty(]0,T[; C^{0,1}(\mathbb{R}))$*.*

3. *Finally, consider a sequence* $(p_n^\tau)_{n\in\mathbb{N}} \subset C^{0,1}(\mathbb{R})$ *which is bounded in* $C(\mathbb{R})$

and satisfies

$$p_n^\tau \to p^\tau \quad \textit{in } C_{\mathrm{loc}}(\mathbb{R}),$$

with $p^\tau \in C^{0,1}(\mathbb{R})$.

Then the sequence of reversible solutions $(p_n)_{n\in\mathbb{N}}$ *of*

$$p_t + a_n p_x = -b_n p + c_n \quad \textit{on } \Omega_\tau, \quad p(\tau,\cdot) = p_n^\tau(\cdot) \quad \textit{on } \mathbb{R}$$

satisfies

$$p_n \to p \quad \textit{in } C([0,\tau]\times[-R,R])$$

for all $R > 0$, *where* p *is the reversible solution of* (3.6.2).

3.7 Proof of the main results of Chapter 3

The goal of this last section of Chapter 3 is to prove Theorem 3.5.5. Since Theorem 3.5.5 is an extension of Theorem 5.2.6 in [69] to the case that the shifting of rarefaction centers is allowed, we will use the same techniques as in [69]. These techniques are based on the concepts that are developed in [81] for the case of pure initial value problems, see also [70]. Therefore, we will recall some results of [69] and show that they still hold true if rarefaction centers are shifted. One of the basic steps to prove Theorem 3.5.5 is to show that the control-to-state mapping

$$\mathbf{U} \ni \mathbf{u} \mapsto y(\bar{t},\cdot,\mathbf{u}) \in L^\infty(\Omega), \tag{3.7.1}$$

is continuously shift-differentiable according to Definition 3.4.2 and then use Lemma 3.4.3 to prove that the mapping defined in (3.5.29), i.e.,

$$\mathbf{U} \ni \mathbf{u} \mapsto \hat{J}(\mathbf{u}) := J(y(\mathbf{u}),\mathbf{u}) \in \mathbb{R},$$

is continuously Fréchet-differentiable, cf. [81], [82], [69] and [70]. In order to show that the mapping in (3.7.1) is shift-differentiable, we have to find a linear bounded operator

$$\mathrm{T_s}\left(y(\bar{t},\cdot,\bar{\mathbf{u}})\right) \in \mathcal{L}(\mathcal{U}; L^r([a,b])\times\mathbb{R}^N) \tag{3.7.2}$$

for some $r \in]1, \infty]$ such that

$$\lim_{\mathbf{u}\to\bar{\mathbf{u}}} \frac{\left\| y(\mathbf{u}) - y(\bar{\mathbf{u}}) - S^{(\bar{x}_k)}_{y(\bar{t},\cdot,\bar{\mathbf{u}})} \left(\mathrm{T_s}\left(y(\bar{t},\cdot,\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}})\right)\right\|_{1,[a,b]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathcal{U}}} = 0. \tag{3.7.3}$$

The following result will play a key-role in finding an operator (3.7.2) such that (3.7.3) is valid.

Lemma 3.7.1. *Suppose that* (A3) *and* (A4) *hold true. Considering a time point* $\bar{t} \in]0, T[$ *and an interval* $[a, b]$*, let* $\bar{\mathbf{u}} \in \mathbf{U}$ *satisfy the conditions in* (ND)*. Then for all* $\mathbf{u}$ *in a neighborhood* $B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}})$ *of* $\bar{\mathbf{u}}$ *the corresponding entropy solution* $y(\bar{\mathbf{u}})$ *of the IBVP* (3.1.2) *is at the time point* $\bar{t}$ *on the interval* $[a, b]$ *given by*

$$y\left(\bar{t}, x; \mathbf{u}\right)|_{[a,b]} = Y_1(\bar{t}, x; \mathbf{u}) \cdot \mathbb{1}_{[a,x_1(\mathbf{u})]}(x) + \sum_{k=2}^{K+1} Y_k(\bar{t}, x; \mathbf{u}) \cdot \mathbb{1}_{]x_{k-1}(\mathbf{u}),x_k(\mathbf{u})]}(x), \tag{3.7.4}$$

where

$$Y_k : (x, \mathbf{u}) \in I^{\varepsilon}_k \times B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}) \mapsto Y_k\left(\bar{t}, x; \mathbf{u}\right) \in \mathbb{R} \tag{3.7.5}$$

$$x_k : \mathbf{u} \in B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}) \mapsto x_k\left(\mathbf{u}\right) \in \left]x_k(\bar{\mathbf{u}}) - \frac{\varepsilon}{2}, x_k(\bar{\mathbf{u}}) + \frac{\varepsilon}{2}\right[, \tag{3.7.6}$$

are continuously differentiable mappings and $I^{\varepsilon}_k :=]x_{k-1}(\bar{\mathbf{u}}) - \varepsilon, x_k(\bar{\mathbf{u}}) + \varepsilon[$ *for* $k = 1, \dots, K+1,$ *and* $x_0 = a, x_{K+1} = b$*. Finally, the mappings*

$$\mathbf{u} \in B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}) \mapsto Y_k\left(\bar{t}, \cdot; \mathbf{u}\right) \in C\left(I^{\varepsilon}_k\right), \quad k = 1, \dots, K+1$$

are continuously differentiable.

Proof. The proof of this lemma will be given at the beginning of §3.7.5. □

We consider a control $\bar{\mathbf{u}} \in \mathbf{U}$ and observe that the unique entropy solution $y(\bar{\mathbf{u}})$ to the IBVP (3.1.2) has BV-regularity according to Proposition 3.1.4. Therefore, fixing some time point $\bar{t} \in]0, T[$, $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ has at most a countable number of discontinuities on Ω. In order to prove Lemma 3.7.1, it will be crucial to have a closer look at the continuity and shock points of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$. A classification and analysis of the continuity and shock points have been done for the case of a pure initial value problem in [81]. These results are extended to initial-boundary value problems in [69, Ch. 6], which we will recall in the following subsection. Since in this thesis the shifting of rarefaction centers is not prohibited, we will have to adapt some of the results using the ideas of [69, §8.2].

3.7.1 Classification of continuity points

Throughout this subsection suppose that the assumptions (A3) and (A4) are satisfied. Considering a continuity point $\bar{x} \in [a,b]$ of $y(\bar{t},\cdot,\bar{\mathbf{u}})$ for some $\bar{\mathbf{u}} \in \mathbf{U}$, we denote by $\bar{\xi}$ the unique genuine backward characteristic through $(\bar{t},\bar{x})$. Since $\|y(\bar{t},\cdot;u)\|_{\infty,\Omega} \leq M_y$ holds due to Proposition 3.1.3, the speed of the characteristic $\bar{\xi}$ is bounded as well. Therefore, $\bar{\xi}$ ends in some point $(\bar{\theta},\bar{z})$ with $\bar{z} \in \Omega^{\mathrm{cl}}$ and $\bar{\theta} \in [0,\bar{t}[$, where only three scenarios can occur: Either $\bar{\xi}$ ends in $t=0$ such that $\bar{\theta}=0$, $\bar{\xi}$ ends in the left boundary such that $\bar{z}=\mathfrak{a}$ or $\bar{\xi}$ ends in the right boundary such that $\bar{z}=\mathfrak{b}$. For each of these scenarios there are three further cases: The characteristic $\bar{\xi}$ can end in a point where the data is continuous, in the inner of a rarefaction wave or on the boundary of a rarefaction wave. Before having a look at these cases, we first note that since $\bar{u}_{B,\mathfrak{a}} \geq \alpha$ and $\bar{u}_{B,\mathfrak{b}} \leq -\alpha$ are satisfied due to (A4), it cannot happen that $\bar{\xi}$ touches the boundary and then returns into the inner of Ω_T. The fact that $\bar{\xi}$ is genuine implies $\bar{z} \notin I_{s,0}$ and $\bar{\theta} \notin I_{s,\mathfrak{a}} \cup I_{s,\mathfrak{b}}$. Moreover, from Proposition 3.2.4 we obtain that $(\bar{\xi}(\cdot), y(\cdot,\bar{\xi}(\cdot)))$ coincides on $]\bar{\theta},\bar{t}]$ with (ζ,v) denoting the solution of (3.2.7) with end data $(\bar{x}, y(\bar{t},\bar{x};\bar{\mathbf{u}}))$. In what follows, we set $v(\bar{\theta})=\bar{w}$.

We first consider the cases where $\bar{\xi}$ ends in $t=0$, cf. [81, §3.4.3]:

Case C: Let $\bar{\theta}=0$ and $\bar{z} \neq \bar{x}_i$ for $i=1,\ldots,n_x$. Since $\bar{\xi}$ is genuine and $\bar{u}_0(\cdot)$ is smooth at $\bar{z}$, Proposition 3.2.4 yields that $\bar{\xi}$ coincides on $[0,\bar{t}]$ with $\zeta(\cdot;0,\bar{z},\bar{u}_j^0(\bar{z}),\bar{u}_1)$ for some $j \in \{1,\ldots,n_x\}$, where $\bar{z} \in J$ for some open interval $J \subset [\bar{x}_{j-1}^0, \bar{x}_j^0]$ and (ζ,v) is given by the solution of (3.2.7) with initial data $(\bar{z},\bar{u}_j^0(\bar{z}))$. The fact that genuine characteristics may only intersect at their endpoints (cf. Proposition 3.2.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t;0,z,\bar{u}_j^0(z),\bar{u}_1)\bigg|_{z=\bar{z}} \geq 0 \quad \forall t \in [0,\bar{t}], \tag{3.7.7}$$

cf. [81]. If there exists a constant $\beta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t;0,z,\bar{u}_j^0(z),\bar{u}_1)\bigg|_{z=\bar{z}} \geq \beta \quad \forall t \in [0,\bar{t}], \tag{3.7.8}$$

then we say that $\bar{x}$ is of class C^c. We note that if $(\bar{t},\bar{x})$ is not a shock generation point, then $\bar{x}$ is of class C^c, see [81].

Case R: Let $\bar{\xi}$ end in the inner of a rarefaction wave emanating from $t=0$, i.e., $\bar{\theta}=0$, $\bar{z}=\bar{x}_j^0$ for some $j \in I_{r,0}(\bar{\mathbf{u}})$ and $\bar{w} \in]\bar{u}_{j-1}^0(\bar{x}_j^0), \bar{u}_j^0(\bar{x}_j^0)[$. Using the same arguments as in the case above, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;0,\bar{x}_j^0,w,\bar{u}_1)\bigg|_{w=\bar{w}} \geq 0 \quad \forall t\in[0,\bar{t}], \tag{3.7.9}$$

cf. [81]. If furthermore

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;0,\bar{x}_j^0,w,\bar{u}_1)\bigg|_{w=\bar{w}} \geq \beta t \quad \forall t\in]0,\bar{t}] \tag{3.7.10}$$

is satisfied for a constant $\beta>0$, then we say that $\bar{x}$ is of class R^c. Supposing that $(\bar{t},\bar{x})$ is not a shock generation point, one can show that $\bar{x}$ is of class R^c, see [81].

Case RB**:** Let $\bar{\xi}$ end on the boundary of a rarefaction wave emanating from $t=0$, i.e., $\bar{\theta}=0$, $\bar{z}=\bar{x}_j^0$ for some $j\in I_{r,0}(\bar{\mathbf{u}})$ and $\bar{w}\in\left\{\bar{u}_{j-1}^0(\bar{x}_j^0),\bar{u}_j^0(\bar{x}_j^0)\right\}$. Then the one-sided derivatives satisfy (3.7.7) and (3.7.9). If the one-sided derivatives even satisfy (3.7.8) and (3.7.10), then we say that $\bar{x}$ is of class RB^c. If $(\bar{t},\bar{x})$ is not a shock generation point, then $\bar{x}$ is of class RB^c, see [81].

Now we consider the case that $\bar{\xi}$ leaves Ω_T at $x=\mathfrak{a}$ or $x=\mathfrak{b}$ at a time point $\bar{\theta}\in]0,\bar{t}[$. Pfaff extended in [69, §6.1] the results of [81, §3.4.3] to initial-boundary value problems. We will first collect the results for the case that $\bar{\xi}$ leaves Ω_T at $x=\mathfrak{a}$ at a time point $\bar{\theta}\in]0,\bar{t}[$:

Case $C_{B,\mathfrak{a}}$**:** Let $\bar{z}=\mathfrak{a}$ and $\bar{\neq}\bar{t}_i^{\mathfrak{a}}$ for $i=1,\dots,n_{t,\mathfrak{a}}$. Analogously to the C-case above, one can show that there exists an open interval $J\subset[\bar{t}_{j-1}^{\mathfrak{a}},\bar{t}_j^{\mathfrak{a}}]$ for some $j\in\{1,\dots,n_{t,\mathfrak{a}}\}$ such that $\bar{\theta}\in J$ and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,\mathfrak{a},\bar{u}_j^{B,\mathfrak{a}}(\theta),\bar{u}_1)\bigg|_{\theta=\bar{\theta}} \leq 0 \quad \forall t\in[\bar{\theta},\bar{t}], \tag{3.7.11}$$

cf. [69]. If there exists a constant $\beta>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,\mathfrak{a},\bar{u}_j^{B,\mathfrak{a}}(\theta),\bar{u}_1)\bigg|_{\theta=\bar{\theta}} \leq -\beta \quad \forall t\in[\bar{\theta},\bar{t}], \tag{3.7.12}$$

is valid, then we $\bar{x}$ is of class $C_{B,\mathfrak{a}}^c$. One can show that if $(\bar{t},\bar{x})$ is not a shock generation point, then $\bar{x}$ is of class $C_{B,\mathfrak{a}}^c$. This can be found in [69].

Case $R_{B,\mathfrak{a}}$**:** Let $\bar{\xi}$ end in the inner of a rarefaction wave emanating from the left boundary, i.e., $\bar{z}=\mathfrak{a}$, $\bar{\theta}=\bar{t}_j^{\mathfrak{a}}$ for some $j\in I_{r,\mathfrak{a}}(\bar{\mathbf{u}})$ and $\bar{w}\in]\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}),\bar{u}_{j-1}^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})[$. Using the same arguments as in the R-case, one can prove that

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\bar{t}_j^{\mathfrak{a}},\mathfrak{a},w,\bar{u}_1)\bigg|_{w=\bar{w}} \geq 0 \quad \forall t\in[\bar{\theta},\bar{t}], \tag{3.7.13}$$

holds true, cf. [69]. Moreover, if

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\bar{t}_j^{\mathfrak{a}},\mathfrak{a},w,\bar{u}_1)\bigg|_{w=\bar{w}} \geq \beta(t-\bar{\theta}) \quad \forall t \in]\bar{\theta},\bar{t}], \tag{3.7.14}$$

is satisfied for some constant $\beta > 0$, then $\bar{x}$ is of class $R^c_{B,\mathfrak{a}}$. If $(\bar{t},\bar{x})$ is not a shock generation point, then one can show that $\bar{x}$ is of class $R^c_{B,\mathfrak{a}}$, see [69].

Case $RB_{B,\mathfrak{a}}$**:** Let $\bar{\xi}$ end on the boundary of a rarefaction wave emanating from the left boundary, i.e., $\bar{z} = \mathfrak{a}$, $\bar{\theta} = \bar{t}_j^{\mathfrak{a}}$ for some $j \in I_{r,\mathfrak{a}}(\bar{\mathbf{u}})$ and $\bar{w} \in \left\{\bar{u}_{j-1}^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}), \bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})\right\}$. Supposing that $(\bar{t},\bar{x})$ is not a shock generation point, the one-sided derivatives satisfy (3.7.11) and (3.7.13). If for the one-sided derivatives even (3.7.12) and (3.7.14) are valid, then $\bar{x}$ is of class $RB^c_{B,\mathfrak{a}}$. For the case that $(\bar{t},\bar{x})$ is not a shock generation point, one can show that $\bar{x}$ is of class $RB^c_{B,\mathfrak{a}}$, cf. [69].

For the case that $\bar{\xi}$ leaves Ω_T at $x = \mathfrak{b}$ at a time point $\bar{\theta} \in]0,\bar{t}[$, the procedure is similar:

Case $C_{B,\mathfrak{b}}$**:** Let $\bar{z} = \mathfrak{b}$ and $\bar{\theta} \neq \bar{t}_i^{\mathfrak{b}}$ for $i = 1,\dots,n_{t,\mathfrak{b}}$. Using the same arguments as in the $C_{B,\mathfrak{a}}$-case, one can show that there exists an open interval $J \subset [\bar{t}_{j-1}^{\mathfrak{b}}, \bar{t}_j^{\mathfrak{b}}]$ for some $j \in \{1,\dots,n_{t,\mathfrak{b}}\}$ such that $\bar{\theta} \in J$ and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,\mathfrak{b},\bar{u}_j^{B,\mathfrak{b}}(\theta),\bar{u}_1)\bigg|_{\theta=\bar{\theta}} \geq 0 \quad \forall t \in [\bar{\theta},\bar{t}] \tag{3.7.15}$$

holds, cf. [69]. If there is a constant $\beta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,\mathfrak{a},\bar{u}_j^{B,\mathfrak{b}}(\theta),\bar{u}_1)\bigg|_{\theta=\bar{\theta}} \geq \beta \quad \forall t \in [\bar{\theta},\bar{t}], \tag{3.7.16}$$

then we say that $\bar{x}$ is of class $C^c_{B,\mathfrak{b}}$. Similar to the $C_{B,\mathfrak{a}}$-case, (3.7.16) can be guaranteed if $(\bar{t},\bar{x})$ is not a shock generation point, see [69].

Case $R_{B,\mathfrak{b}}$**:** Let $\bar{\xi}$ end in the inner of a rarefaction wave emanating from the right boundary, i.e., $\bar{z} = \mathfrak{b}$, $\bar{\theta} = \bar{t}_j^{\mathfrak{b}}$ for some $j \in I_{r,\mathfrak{b}}(\bar{\mathbf{u}})$ and $\bar{w} \in]\bar{u}_{j-1}^{B,\mathfrak{b}}(\bar{t}_j^{\mathfrak{b}}), \bar{u}_j^{B,\mathfrak{b}}(\bar{t}_j^{\mathfrak{b}})[$. As in the $R_{B,\mathfrak{a}}$-case, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\bar{t}_j^{\mathfrak{b}},\mathfrak{b},w,\bar{u}_1)\bigg|_{w=\bar{w}} \geq 0 \quad \forall t \in [\bar{\theta},\bar{t}], \tag{3.7.17}$$

cf. [69]. If in addition

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\bar{t}_j^{\mathfrak{b}},\mathfrak{b},w,\bar{u}_1)\bigg|_{w=\bar{w}} \geq \beta(t-\bar{\theta}) \quad \forall t \in]\bar{\theta},\bar{t}], \tag{3.7.18}$$

is valid for some constant $\beta > 0$, then we say that $\bar{x}$ is of class $R^c_{B,\mathfrak{b}}$. If $(\bar{t},\bar{x})$ is not a shock generation point, then one can show that $\bar{x}$ is of class $R^c_{B,\mathfrak{b}}$, see [69].

Case $RB_{B,\mathfrak{b}}$**:** Let $\bar{\xi}$ end on the boundary of a rarefaction wave emanating from $t=0$, i.e., $\bar{z}=\mathfrak{b}$, $\bar{\theta}=\bar{t}^{\mathfrak{b}}_j$ for some $j \in I_{r,\mathfrak{b}}(\bar{\mathbf{u}})$ and $\bar{w} \in \left\{\bar{u}^{B,\mathfrak{b}}_{j-1}(\bar{t}^{\mathfrak{b}}_j), \bar{u}^{B,\mathfrak{b}}_j(\bar{t}^{\mathfrak{b}}_j)\right\}$. If $(\bar{t},\bar{x})$ is not a shock generation point, then the one-sided derivatives satisfy (3.7.15) and (3.7.17). If even (3.7.16) and (3.7.18) are valid, which can be guaranteed supposing that $(\bar{t},\bar{x})$ is not a shock generation point (cf. [69]), then we say that $\bar{x}$ is of class $RB^c_{B,\mathfrak{b}}$.

Finally, we consider the case that $\bar{\xi}$ ends in $(0,\mathfrak{a})$ or in $(0,\mathfrak{b})$:

Case $R_{0,\mathfrak{a}}$**:** Let $\bar{\xi}$ end in the inner of a rarefaction wave emanating from $(0,\mathfrak{a})$, i.e., $\bar{z}=\mathfrak{a}$, $\bar{\theta}=0$ and $\bar{w} \in]\bar{u}^{B,\mathfrak{a}}_1(0), \bar{u}^0_1(\mathfrak{a})[$. As in the $R_{B,\mathfrak{a}}$-case, setting $\bar{\theta}=0$, we obtain that (3.7.13) holds true and if even (3.7.14) is satisfied for some constant $\beta > 0$, then we say that $\bar{x}$ is of class $R^c_{0,\mathfrak{b}}$, see [69].

Case $RB_{0,\mathfrak{a}}$**:** Let $\bar{\xi}$ end on the boundary of a rarefaction wave emanating from the right boundary, i.e., $\bar{z}=\mathfrak{a}$, $\bar{\theta}=0$ and $\bar{w} \in \left\{\bar{u}^{B,\mathfrak{a}}_1(0), \bar{u}^0_1(\mathfrak{a})\right\}$. Then the one-sided derivatives satisfy (3.7.11) and (3.7.13) in $\bar{\theta}=0$ if $\bar{\xi}$ lies on the left boundary of the rarefaction wave or (3.7.7) in $\bar{z}=\mathfrak{a}$ and (3.7.13) if $\bar{\xi}$ lies on the right boundary. If the one-sided derivatives even satisfy (3.7.14) and (3.7.12) or (3.7.8), respectively, then we say that $\bar{x}$ is of class $RB^c_{0,\mathfrak{a}}$. If $(\bar{t},\bar{x})$ is not a shock generation point, then $\bar{x}$ is of class $RB^c_{0,\mathfrak{a}}$, see [69].

Case $R_{0,\mathfrak{b}}$**:** Let $\bar{\xi}$ end in the inner of a rarefaction wave emanating from $(0,\mathfrak{b})$, i.e., $\bar{z}=\mathfrak{b}$ and $\bar{\theta}=0$ and $\bar{w} \in]\bar{u}^0_1(\mathfrak{b}), \bar{u}^{B,\mathfrak{b}}_1(0)[$. As in the $R_{B,\mathfrak{a}}$-case, setting $\bar{\theta}=0$ (3.7.17) holds and if even (3.7.18) is satisfied for some constant $\beta > 0$, then we say that $\bar{x}$ is of class $R^c_{0,\mathfrak{b}}$, see [69].

Case $RB_{0,\mathfrak{b}}$**:** Let $\bar{\xi}$ end on the boundary of a rarefaction wave emanating from $(0,\mathfrak{b})$, i.e., $\bar{z}=\mathfrak{b}$, $\bar{\theta}=0$ and $\bar{w} \in \left\{\bar{u}^{B,\mathfrak{b}}_1(0), \bar{u}^0_1(\mathfrak{b})\right\}$. Supposing that $(\bar{t},\bar{x})$ is not a shock generation point, the one-sided derivatives satisfy (3.7.15) and (3.7.17) in $\bar{\theta}=0$ if $\bar{\xi}$ lies on the right boundary of the rarefaction wave or (3.7.7) in $\bar{z}=\mathfrak{b}$ and (3.7.17) if $\bar{\xi}$ lies on the left boundary. If even (3.7.18) and (3.7.16) or (3.7.8) are valid, respectively, then we say that $\bar{x}$ is of class $RB^c_{0,\mathfrak{b}}$, which is guaranteed if $(\bar{t},\bar{x})$ is not a shock generation point, cf. [69].

Remark 3.7.2. In the previous classifications the cases that $\bar{\xi}$ ends in a point $(0,\bar{x}^0_j)$ with $j \in I_{c,0}$ or in a point $(\bar{t}^{B,\mathfrak{a}/\mathfrak{b}}_j, \mathfrak{a}/\mathfrak{b})$ with $j \in I_{c,\mathfrak{a}/\mathfrak{b}}$, which can also occur, are not mentioned. In addition, the case that $\bar{\xi}$ ends in $(0,\mathfrak{a}/\mathfrak{b})$ where $u_0(\mathfrak{a}+/\mathfrak{b}-,\mathbf{u}) = u_{B,\mathfrak{a}/\mathfrak{b}}(0+,\mathbf{u})$ is not discussed. These cases are excluded in order to guarantee a certain stability of the solution which is needed to prove Lemma 3.7.1, see Remark 3.5.2. The analysis of these cases can be found in [69, 6.1].

3.7.2 Classification of shock points

In this subsection suppose that (A3) and (A4) are satisfied and denote by $y(\bar{\mathbf{u}})$ the entropy solution of the IBVP (3.1.2) for some $\bar{\mathbf{u}} \in \mathbf{U}$. In addition, let $(\bar{t}, \bar{x})$ be a shock point of $y(\bar{\mathbf{u}})$ lying on some shock-curve $\eta(t)$. Furthermore, denote by $\bar{\xi}_\pm$ the corresponding minimal and maximal backward characteristics through $(\bar{t}, \bar{x})$ and by $(\bar{\theta}_\pm, \bar{z}_\pm)$ the points where they leave the domain Ω_T. We recall that $\bar{\xi}_\pm$ are genuine due to Proposition 3.2.4. Further on, Proposition 3.2.4 yields

$$\bar{\xi}_\pm(\cdot) = \zeta_\pm(\cdot) \quad \text{and} \quad y(\cdot, \bar{\xi}_\pm(\cdot)) = v_\pm(\cdot) \quad \text{on }]\bar{\theta}_\pm, \bar{t}[,$$

where $(\zeta_\pm, v_\pm)$ are given by the solution of (3.2.7) with end data $(\bar{x}, y(\bar{t}, \bar{x}\pm; \bar{\mathbf{u}}))$. We set $v(\bar{\theta}_\pm) = \bar{w}_\pm$. Considering the genuine backward characteristics $\bar{\xi}_\pm$, one can classify the shock points analogously to the continuity points. Such a classification of shock points is carried out for initial value problems in [81] and extended to initial-boundary value problems in [69]. We will briefly collect these results. If $\bar{\xi}_-$ ends in a point where the initial data or the boundary data is smooth, then the corresponding function ζ_- satisfies (3.7.7), (3.7.11) or (3.7.15). Analogously, if $\bar{\xi}_-$ ends in the inner of a rarefaction wave, then (3.7.9), (3.7.13) or (3.7.17) is valid. The same holds for ζ_+. If ζ_- and ζ_+ satisfy (3.7.8), (3.7.12),(3.7.16), (3.7.10), (3.7.14) or (3.7.18), then we say that the shock point $(\bar{t}, \bar{x})$ is of class X_l/X_r with $X_l, X_r \in \{C^c, R^c, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}\}$. Based on these observations, Pfaff defines in [69, Definition 6.1.1] the non-degeneracy of shock points as follows:

Definition 3.7.3. *Suppose that* (A3) *and* (A4) *are satisfied and denote by* $y(\bar{\mathbf{u}}) \in BV(\Omega_T)$ *the entropy solution of the IBVP* (3.1.2) *for some* $\bar{\mathbf{u}} \in \mathbf{U}$. *We say that a shock point* $(\bar{t}, \bar{x})$ *is non-degenerated if it is not a shock interaction point and of class* X_l/X_r *with* $X_l, X_r \in \left\{C^c, R^c, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}, C^c_{B,\mathfrak{b}}, R^c_{B,\mathfrak{b}}\right\}$.

3.7.3 Differentiability in the neighborhood of continuity points

Throughout the whole subsection, let (A3) and (A4) hold true and denote by $y(\mathbf{u})$ the unique entropy solution to the IBVP (3.1.2) for some $\mathbf{u} \in \mathbf{U}$. Based on the results in [81] for the pure initial value problem, Pfaff considers continuity points of class

$$X \in \left\{C^c, R^c, RB^c, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}, RB^c_{B,\mathfrak{a}}, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}, RB^c_{B,\mathfrak{a}}, R^c_{0,\mathfrak{a}}, RB^c_{0,\mathfrak{a}}, R^c_{0,\mathfrak{b}}, RB^c_{0,\mathfrak{b}}\right\},$$

see §3.7.1. Then he constructs in [69, §6.2] case by case smooth local solutions Y around the corresponding genuine backward characteristics which depend continuously differentiably on $\mathbf{u}$ in a small neighborhood of $\bar{\mathbf{u}}$. In a second step, it is proved that the local solutions coincide for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ with the corresponding entropy solution $y(u)$ of the IBVP (3.1.2) if ρ is sufficiently small. We will collect those results, whereby we will pay attention to the fact that in this thesis the shifting of rarefaction centers is not prohibited in contrast to [69] and [81].

3.7.3.1 Local solutions around genuine characteristics ending in $t = 0$

Consider a control $\bar{\mathbf{u}} \in \mathbf{U}$ and a continuity point $(\bar{t}, \bar{x})$ of $y(\cdot, \bar{\mathbf{u}})$ of class C^c. Then (3.7.8) holds for some positive constant $\beta > 0$ and $j \in \{1, \ldots, n_x + 1\}$. Due to continuity of the mapping (3.2.8) in Lemma 3.2.5, there exist $z_l < \bar{z} < z_r$ and $\delta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t; 0, z, \bar{u}_j^0(z), \bar{u}_1) \geq \frac{\beta}{2} > 0 \quad \forall (t, z) \in [0, \bar{t}] \times J \tag{3.7.19}$$

is satisfied, where $J :=]z_l - \delta, z_r + \delta[\subset]\bar{x}_{j-1}^0, \bar{x}_j^0[$. In the next result we will construct a smooth function Y which is defined on a neighborhood around the genuine backward characteristic through $(\bar{t}, \bar{x})$. This result will play an important role as we will see later.

Lemma 3.7.4 (Local solution in a neighborhood of continuity points of class C^c, Lemma 3.5.1, [81])**.** *Let* (A3) *and* (A4) *hold true, consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *and suppose that* (3.7.19) *is satisfied for some constants* $\delta, \beta > 0$ *and* $j \in \{1, \ldots, n_x + 1\}$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t; 0, z, u_j^0(z), u_1) \geq \frac{\beta}{3} > 0 \quad \forall (t, z) \in [0, \bar{t} + \tau] \times J,\ \forall \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$$

is valid for some constants $\rho, \tau > 0$*. Defining the stripe*

$$S = S(\tau) := \{(t, x) \in [0, \bar{t} + \tau] \times \Omega : \xi_l(t) \leq x \leq \xi_r(t)\}$$

with $\xi_{l/r}(t) := \zeta(t; 0, z_{l/r}, \bar{u}_j^0(z_{l/r}), \bar{u}_1)$*, the equation*

$$\zeta(t; 0, z, u_j^0(z), u_1) = x$$

admits for all $(t, x) \in S(\tau)$ *and all* $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ *a unique solution* $z = Z(t, x, \mathbf{u}) \in J$*. Further defining the mapping*

$$Y(t, x, \mathbf{u}) = v(t; 0, Z(t, x, \mathbf{u}), u_j^0(Z(t, x, \mathbf{u})), u_1), \tag{3.7.20}$$

for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ *it holds that* $Z(\cdot,\mathbf{u}), Y(\cdot,\mathbf{u}) \in C^{0,1}(S)$. *Moreover, for all* $t \in [0, \bar{t}+\tau[$ *the mappings*

$$(x,\mathbf{u}) \in]\xi_l(t), \xi_r(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Z(t,x,\mathbf{u})$$
$$(x,\mathbf{u}) \in]\xi_l(t), \xi_r(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(t,x,\mathbf{u})$$

are continuously Fréchet-differentiable with corresponding derivatives

$$\begin{aligned} \mathrm{d}_{(x,\mathbf{u})} Z(t,x,\mathbf{u}) \cdot (\delta x, \delta \mathbf{u}) &= \frac{\delta x - \delta\zeta(t;0,z,u_j^0(z),u_1;0,0,\delta u_j^0(z),\delta u_1)}{\delta\zeta(t;0,z,u_j^0(z),u_1;0,1,u_j^{0\prime}(z),0)} \\ \mathrm{d}_{(x,\mathbf{u})} Y(t,x,\mathbf{u}) \cdot (\delta x, \delta \mathbf{u}) &= \delta v(t;0,z,u_j^0(z),u_1;0,0,\delta u_j^0(z),\delta u_1), \\ &\quad + \delta v(t;0,z,u_j^0(z),u_1;0,1,u_j^{0\prime}(z),0) \cdot \mathrm{d}_{(x,\mathbf{u})} Z(t,x,\mathbf{u}) \cdot (\delta x, \delta \mathbf{u}), \end{aligned}$$

where δv *and* $\delta\zeta$ *are determined by the solution of the linearized characteristic equation in* (3.2.9). *Further on, the mappings*

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Z(\cdot,\mathbf{u}) \in C(S)$$
$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot,\mathbf{u}) \in C(S)$$

are continuously Fréchet-differentiable and the derivatives are given by

$$\mathrm{d}_{\mathbf{u}}(Z,Y)(\cdot,\mathbf{u}) \cdot \delta\mathbf{u} = \mathrm{d}_{(x,\mathbf{u})}(Z,Y)(\cdot,\mathbf{u}) \cdot (0,\delta\mathbf{u}).$$

Setting $\delta y = \mathrm{d}_{\mathbf{u}} Y(\mathbf{u}) \cdot \delta u$, *we note that* δy *is the unique broad solution, i.e., the solution along characteristics of the linearized equation*

$$\begin{aligned} \delta y_t + (f'(Y(\mathbf{u}))\delta y)_x &= g_y(\cdot,y,u_1)\delta y + g_{u_1}(\cdot,y,u_1)\delta u_1 && \text{on } S, \\ \delta y(0,\cdot) &= \delta u_j^0 && \text{on } [z_l,z_r]. \end{aligned} \tag{3.7.21}$$

Remark 3.7.5. According to the original version of the above lemma in [81, Lemma 3.5.1], the mappings $Z, Y \in C^{0,1}(S)$ as well as their respective derivatives w.r.t. $\mathbf{u}$ and x only depend on u_j^0 and $u_1|_S$. Hence, despite the fact that the shifting of rarefaction waves is allowed in this thesis (in contrast to [81] and [69]), the results of [81, Lemma 3.5.1] are still valid.

Remark 3.7.6. The mapping $Y \in C^{0,1}(S)$ defined in (3.7.20) is a classical solution of (3.1.2) on the stripe S.

Remark 3.7.7. In [81, Rem. 3.5.4.], it is shown that δy is also a weak solution of (3.7.21) and that for any domain $D \subset S$ with Lipschitz boundary and any $p \in$

$C^{0,1}(D)$, it holds that

$$(p(n_1 + n_2 f'(Y)), \delta y)_{2,\partial D} = (p_t + f'(Y)p_x + g_y(t,x,Y,u_1)p, \delta y)_{2,D} + (p g_{u_1}(t,x,Y,u_1), \delta u_1)_{2,D}\,, \tag{3.7.22}$$

where $(n_1, n_2)^T$ is the outer normal of D.

In the following result, we will see that the mapping Y which is constructed in the previous result coincides with the entropy solution $y(u)$ on a neighborhood of the genuine backward characteristic through the continuity point $(\bar{t}, \bar{x})$.

Lemma 3.7.8 (Differentiability properties in continuity points of class C^c, Lemma 3.5.5, [81])**.** *Suppose that* (A3) *and* (A4) *are satisfied. Furthermore, consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *and a continuity point* $(\bar{t}, \bar{x}) \in \Omega_T$ *of* $y(\cdot, \bar{\mathbf{u}})$ *of class* C^c*. Then the following statements are valid:*

(i) There is a maximal nonempty open interval I *with* $\bar{x} \in I$ *such that* I *does not contain any discontinuity of* $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ *and for all* $x \in I$ *the unique genuine backward characteristic* ξ *through* $(\bar{t}, x)$ *does not intersect* $t = 0$ *in some* $z \in \{\bar{x}_1^0, \ldots, \bar{x}_{n_x}^0\}$*. Moreover, it holds that* $y(\bar{t}, \cdot, \bar{\mathbf{u}}) \in C^1(I)$*.*

(ii) Consider an arbitrary interval $]x_l, x_r[$ *with* $[x_l, x_r] \subset I$*. Denote by* $\xi_{l/r}$ *the genuine backward characteristics through* $(\bar{t}, x_{l/r})$ *and by* $z_{l/r}$ *the points where they intersect* $t = 0$*. Then there exist constants* $\delta, \beta > 0$ *and* $j \in \{1, \ldots, n_x + 1\}$ *such that* (3.7.19) *is satisfied. Consider the mapping*

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

from Lemma 3.7.4. Then after a possible reduction of ρ *and* τ *we obtain*

$$y(t, x; \mathbf{u}) = Y(t, x, \mathbf{u}) \quad \forall (t,x) \in S,\, \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}). \tag{3.7.23}$$

Next, we consider a control $\bar{\mathbf{u}} \in \mathbf{U}$ and a continuity point $(\bar{t}, \bar{x})$ of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ of class R^c. In contrast to [69], in this thesis the shifting of rarefaction centers is allowed. Similarly to the results in Lemma 3.7.4 and 3.7.8, our goal is to define a stripe $S \subset \Omega_T$ and a mapping $Y(t, x, \mathbf{u})$ such that (3.7.23) is satisfied. To this end, we will follow the concepts introduced in [71], where the shifting of rarefaction centers is allowed, and apply them to initial-boundary value problems, cf. [69, §8.2]. One of the main arguments is here the following: Since the source term on the right-hand side of (3.1.2) is equal to zero on $[0, \varepsilon] \times \mathbb{R}$, there exists a sufficiently small $s \in]0, \varepsilon[$ and $\rho > 0$ such that for any $j \in I_{r,0}(\bar{\mathbf{u}})$ it holds

$$y(s, \cdot; \mathbf{u})|_{I_j^s(\mathbf{u})} = \phi_j^0(x, x_j^0) \quad \forall \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \tag{3.7.24}$$

with

$$I_j^s(\mathbf{u}) :=]x_j^0 + f'(u_j^0(x_j^0))s, x_j^0 + f'(u_{j+1}^0(x_j^0))s[\quad \text{and} \quad \phi_j^0(x, x_j^0) := f'^{-1}\left(\frac{x - x_j^0}{s}\right).$$

Due to (A3), the mapping

$$q \in \mathbb{R} \mapsto \phi_j^0(\cdot, q) \in C^1_{\text{loc}}(\mathbb{R}) \tag{3.7.25}$$

is continuously Fréchet-differentiable with derivative

$$\mathrm{d}_q \phi_j^0(\cdot, q) \cdot \delta q = -\frac{\delta q}{f''\left(f'^{-1}\left(\frac{x-q}{s}\right)\right)}. \tag{3.7.26}$$

This result can be found in the proof of Lemma 7.2.10 in [69]. A Taylor expansion of the terms $f'(u_j^0(x_j^0))$ and $f'(u_{j+1}^0(x_j^0))$ in $\bar{u}_j^0(\bar{x}_j^0)$ and $\bar{u}_{j+1}^0(\bar{x}_j^0)$ yields a mapping

$$\rho \in]0, \infty[\mapsto \varepsilon(\rho) \in]0, \infty[\quad \text{with} \;\; \rho \to 0 \Rightarrow \varepsilon(\rho) \to 0 \tag{3.7.27}$$

such that

$$\begin{aligned} x_j^0 + f'(u_j^0(x_j^0))s &< \bar{x}_j^0 + f'(\bar{u}_j^0(\bar{x}_j^0))s + \varepsilon(\rho) \\ &< \bar{x}_j^0 + f'(\bar{u}_{j+1}^0(\bar{x}_j^0))s - \varepsilon(\rho) \\ &< x_j^0 + f'(u_{j+1}^0(x_j^0))s \quad \forall \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \end{aligned}$$

holds for sufficiently small $\rho > 0$. Therefore we obtain

$$I_{j,\rho}^s := [\bar{x}_j^0 + f'(\bar{u}_j^0(\bar{x}_j^0))s + 2\varepsilon(\rho), \bar{x}_j^0 + f'(\bar{u}_{j+1}^0(\bar{x}_j^0))s - 2\varepsilon(\rho)] \subset I_j^s(\mathbf{u})$$

which combined with (3.7.24) and (3.7.25) yields the continuous Fréchet-differentiability of the mapping

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto y(s, \cdot; \mathbf{u})|_{I_{j,\rho}^s} \in C^1(I_{j,\rho}^s). \tag{3.7.28}$$

The derivative is given by the right-hand side of (3.7.26) with $q = x_j^0$ and $\delta q = \delta x_j^0$, cf. [69, Lemma 7.2.10].

Since the genuine backward characteristic $\bar{\xi}$ through $(\bar{t}, \bar{x})$ ends in the inner of a rarefaction wave with center $(0, \bar{x}_j^0)$, we obtain $\bar{z} = \bar{\xi}(s) \in I_{j,\rho}^s$ for sufficiently small ρ. Furthermore, (3.7.10) yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t; s, z, \phi_j^0(z, \bar{x}_j^0), \bar{u}_1)\right|_{z=\bar{z}} \geq \beta > 0 \quad \forall t \in [s, \bar{t}]. \tag{3.7.29}$$

Analogously to the derivation of (3.7.19), we obtain from (3.7.29) the existence of constants $\beta, \delta > 0$ and $z_l < \bar{z} < z_r$ such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t; s, z, \phi_j^0(z, \bar{x}_j^0), \bar{u}_1) \geq \frac{\beta}{2} > 0 \quad \forall (t, z) \in [s, \bar{t}] \times J \tag{3.7.30}$$

holds, where $J :=]z_l - \delta, z_r + \delta[\subset I_{j,\rho}^s$. Therefore, considering the truncated initial-boundary value problem on the domain $]s, T[\times\Omega$, we can apply Lemma 3.7.4 and obtain:

Lemma 3.7.9 (Local solution in a neighborhood of continuity points of class R^c)**.** *let* (A3) *and* (A4) *be satisfied. Moreover, consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *and suppose that* (3.7.30) *is satisfied for some* $\delta, \beta > 0$ *and some* $j \in I_{r,0}(\bar{\mathbf{u}})$*. Then there exist constants* $\rho, \tau > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t; s, z, \phi_j^0(z, \bar{x}_j^0), u_1) \geq \frac{\beta}{3} > 0 \quad \forall (t, z) \in [s, \bar{t} + \tau] \times J,\ \forall \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \tag{3.7.31}$$

is valid. Defining the stripe

$$S = S(\tau) := \{(t, x) \in [s, \bar{t} + \tau] \times \Omega : \xi_l(t) \leq x \leq \xi_r(t)\},$$

where $\xi_{l/r}(t) := \zeta(t; s, z_{l/r}, \phi_j^0(z_{l/r}, \bar{x}_j^0), \bar{u}_1)$*, the equation*

$$\zeta(t; s, z, \phi_j^0(z, \bar{x}_j^0), u_1) = x$$

admits for all $(t, x) \in S(\tau)$ *and all* $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ *a unique solution* $z = Z(t, x, \mathbf{u}) \in J$*. Further on, we construct the mapping*

$$Y(t, x, \mathbf{u}) = v(t; s, Z(t, x, \mathbf{u}), \phi_j^0(Z(t, x, \mathbf{u}), x_j^0), u_1) \tag{3.7.32}$$

and observe that $Z(\cdot, \mathbf{u}), Y(\cdot, \mathbf{u}) \in C^{0,1}(S)$ *holds for all* $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$*. In addition, the mappings*

$$(x, \mathbf{u}) \in]\xi_l(t), \xi_r(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Z(t, x, \mathbf{u})$$
$$(x, \mathbf{u}) \in]\xi_l(t), \xi_r(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(t, x, \mathbf{u})$$

are continuously Fréchet-differentiable for all $t \in [s, \bar{t} + \tau[$ *with derivatives*

$$\mathrm{d}_{(x,\mathbf{u})}Z(t, x, \mathbf{u}) \cdot (\delta x, \delta \mathbf{u}) = \frac{\delta x - \delta\zeta(t; s, z, \phi_j^0(z, x_j^0), u_1; 0, 0, \mathrm{d}_q\phi_j^0(z, x_j^0)\delta x_j^0, \delta u_1)}{\delta\zeta(t; s, z, \phi_j^0(z, x_j^0), u_1; 0, 1, \mathrm{d}_x\phi_j^0(z, x_j^0), 0)},$$

$$\mathrm{d}_{(x,\mathbf{u})}Y(t, x, \mathbf{u}) \cdot (\delta x, \delta \mathbf{u}) = \delta v(t; s, z, \phi_j^0(z, x_j^0), u_1; 0, 0, \mathrm{d}_q\phi_j^0(z, x_j^0)\delta x_j^0, \delta u_1)$$

$$+ \delta v(t; s, z, \phi_j^0(z, x_j^0), u_1; 0, 1, \mathrm{d}_x\phi_j^0(z, x_j^0), 0) \cdot \mathrm{d}_{(x,\mathbf{u})}Z(t, x, \mathbf{u}) \cdot (\delta x, \delta \mathbf{u}),$$

where δv *and* $\delta\zeta$ *are given by the solution of the linearized characteristic equation in* (3.2.9). *Finally, the mappings*

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Z(\cdot, \mathbf{u}) \in C(S)$$
$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

are continuously Fréchet-differentiable and the derivatives can be computed by

$$\mathrm{d}_{\mathbf{u}}(Z, Y)(\cdot, \mathbf{u}) \cdot \delta\mathbf{u} = \mathrm{d}_{(x,\mathbf{u})}(Z, Y)(\cdot, \mathbf{u}) \cdot (0, \delta\mathbf{u})$$

and $\delta y = \mathrm{d}_{\mathbf{u}}Y(\mathbf{u}) \cdot \delta\mathbf{u}$ *is the unique broad solution of the linearized equation*

$$\begin{aligned} \delta y_t + (f'(Y(\mathbf{u}))\delta y)_x &= g_y(\cdot, y, u_1)\delta y + g_{u_1}(\cdot, y, u_1)\delta u_1 && \text{on } S, \\ \delta y(s, \cdot) &= \mathrm{d}_q\phi_j^0(\cdot, \bar{x}_j^0)\delta x_j^0 && \text{on } [z_l, z_r]. \end{aligned}$$

Finally, the assertions of the Remarks 3.7.6 and 3.7.7 hold true.

Following the ideas of the proof of Lemma 7.2.10 in [69], we observe that due to (3.7.24) and (3.7.30), a continuity point $(\bar{t}, \bar{x}) \in \Omega_T$ of $y(\cdot, \bar{\mathbf{u}})$ of class R^c can be treated as a continuity point of class C^c for the truncated IBVP on $[s, T] \times \Omega$. Therefore, we obtain the following counterpart of Lemma 3.7.8:

Lemma 3.7.10 (Differentiability properties in continuity points of class R^c, cf. Lemma 7.2.10 in [69])**.** *Suppose that* (A3) *and* (A4) *are satisfied. Furthermore, consider a control* $\mathbf{u} \in \mathbf{U}$ *and a continuity point* $(\bar{t}, \bar{x}) \in \Omega_T$ *of* $y(\cdot, \bar{\mathbf{u}})$ *of class* R^c *such that the genuine backward characteristic through* $(\bar{t}, \bar{x})$ *ends in the inner of a rarefaction wave with corresponding center in some point* $(0, \bar{x}_j^0)$. *Then the following statements are valid:*

(i) There exists a maximal nonempty open interval I *with* $\bar{x} \in I$ *such that* I *does not contain any discontinuity of* $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ *and for all* $x \in I$ *the unique genuine backward characteristic* ξ *through* $(\bar{t}, x)$ *intersects* $t = s$ *in*

$$z = \xi(s) \in]\bar{x}_j^0 + f'(\bar{u}_j^0(\bar{x}_j^0))s, \bar{x}_j^0 + f'(\bar{u}_{j+1}^0(\bar{x}_j^0))s[.$$

Moreover, we obtain that $y(\bar{t}, \cdot, \bar{\mathbf{u}}) \in C^1(I)$.

(ii) Let $]x_l, x_r[$ *be an arbitrary interval with* $[x_l, x_r] \subset I$ *and denote by* $\xi_{l/r}$ *the genuine backward characteristics through* $(\bar{t}, x_{l/r})$ *and by* $z_{l/r}$ *the points where they intersect* $t = s$, *respectively. Then there exist* $\delta, \beta > 0$ *such that* (3.7.30)

holds true. Consider the mapping

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

from Lemma 3.7.9. Then after a possible reduction of ρ and τ we get that

$$y(t, x; \mathbf{u}) = Y(t, x, \mathbf{u}) \quad \forall (t, x) \in S, \ \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}).$$

Proof. The proof is analogous to the proof of Lemma 7.2.10 in [69]. □

Next, we consider a continuity point $(\bar{t}, \bar{x})$ of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ that is of class RB^c and lies on the right boundary of a rarefaction wave, i.e., the genuine backward characteristic $\bar{\xi}$ through the point $(\bar{t}, \bar{x})$ intersects $t = 0$ in some point $\bar{z}_r = \bar{\xi}(0) = \bar{x}_j^0 \in \Omega$ for some $j \in I_{r,0}(\bar{\mathbf{u}})$ and coincides on $[0, \bar{t}]$ with the solution $\zeta(\cdot; 0, \bar{z}_r, \bar{u}_{j+1}^0(\bar{z}_r), \bar{u}_1)$ of (3.2.7). Let $z_l = \bar{\xi}(s)$ denote the point where $\bar{\xi}$ intersects $t = s$ for some $s \in]0, \varepsilon_g[$. Considering the local solution ϕ_j^0 near rarefaction centers defined in (3.7.24), $\bar{\xi}$ coincides on $[s, \bar{t}]$ with $\zeta(\cdot; s, \bar{z}_l, \phi_j^0(z_l, \bar{x}_j^0)\bar{u}_1)$. Since (3.7.10) is satisfied and due to the regularity of the solution to (3.2.7), there exist $z_{l,l/r}, z_{r,l/r}$ with $z_{l,l} < \bar{z}_l < z_{l,r}$, $z_{r,l} < \bar{z}_r < z_{r,r}$ and constants $\beta, \delta > 0$ such that setting $J_l :=]z_{l,l} - \delta, z_{l,r} + \delta[$ and $J_r :=]z_{r,l} - \delta, z_{r,r} + \delta[$, (3.7.29) and (3.7.19) hold true. Therefore, one can apply Lemma 3.7.4 and Lemma 3.7.9 yielding stripes

$$\begin{aligned} S_l = S_l(\tau) &:= \{(t, x) \in [s, \bar{t} + \tau] \times \Omega : \xi_{l,l}(t) \le x \le \xi_{l,r}(t)\}, \\ S_r = S_r(\tau) &:= \{(t, x) \in [0, \bar{t} + \tau] \times \Omega : \xi_{r,l}(t) \le x \le \xi_{r,r}(t)\}, \end{aligned}$$

where $\xi_{l,l/r}(t) := \zeta(t; s, z_{l,l/r}, \phi_j^0(z_{l,l/r}, \bar{x}_j^0), \bar{u}_1)$, $\xi_{r,l/r}(t) := \zeta(t; 0, z_{r,l/r}, \bar{u}_i^0(z_{r,l/r}), \bar{u}_1)$ and $\tau > 0$ is a positive constant. In addition, Lemma 3.7.4 and 3.7.9 yield the mappings

$$(x, \mathbf{u}) \in]\xi_{r,l}(t), \xi_{r,r}(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y_r(t, x, \mathbf{u}) \quad t \in [0, \bar{t}]$$

and

$$(x, \mathbf{u}) \in]\xi_{l,l}(t), \xi_{l,r}(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y_l(t, x, \mathbf{u}) \quad t \in [s, \bar{t}].$$

Analogously to [81, Lemma 3.5.12], we define the set

$$\hat{S} := \left(S_l \cap \left\{x \le \hat{\xi}(t; \mathbf{u})\right\}\right) \cup \left(S_r \cap \left\{x > \hat{\xi}(t; \mathbf{u})\right\}\right),$$

with $\hat{\xi}(t;\mathbf{u}) \coloneqq \zeta(t;0,x_j^0,u_{j+1}^0(x_j^0),u_1)$, and the mapping

$$\hat{Y}(t,x,\mathbf{u}) \coloneqq \begin{cases} Y_l(t,x,\mathbf{u}) & \text{if } x < \hat{\xi}(t;\mathbf{u}), \\ Y_r(t,x,\mathbf{u}) & \text{if } x \geq \hat{\xi}(t;\mathbf{u}), \end{cases} \qquad \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}),\ (t,x) \in \hat{S}. \tag{3.7.33}$$

A version of the following result, where the shifting of centers of rarefaction waves is prohibited can be found in [81, Lemma 3.5.12].

Lemma 3.7.11 (Differentiability in continuity points of class RB^c)**.** *Let* (A3) *and* (A4) *hold true. Furthermore, consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *and a continuity point* $(\bar{t},\bar{x})$ *of* $y(\cdot,\bar{\mathbf{u}})$ *of class* RB^c *lying on the right boundary of a rarefaction wave with corresponding center in some point* $(0,\bar{x}_j^0)$*. Then there exists a constant* $\rho > 0$ *and* $x_l, x_r \in]\mathfrak{a},\mathfrak{b}[$ *with* $x_l < \bar{x} < x_r$ *such that for all* $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$

$$y(\bar{t},x;\mathbf{u}) = \hat{Y}(\bar{t},x,\mathbf{u}) \quad \text{on }]x_l,x_r[$$

is satisfied with $\hat{Y}$ *given in* (3.7.33)*. Hence,* $y(\bar{t},\cdot;\mathbf{u}) \in C^{0,1}(]x_l,x_r[)$ *and* $y(\bar{t},\cdot;\mathbf{u})$ *is continuously differentiable on* $]x_l,x_r[\setminus \left\{\hat{\xi}(\bar{t},\mathbf{u})\right\}$*. Furthermore, the mapping*

$$\mathbf{u} \in \mathbf{U} \mapsto y(\bar{t},\cdot;\mathbf{u}) \in L^r([x_l,x_r]) \tag{3.7.34}$$

is continuously Fréchet-differentiable for all $r \in [1,\infty[$*, where the derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} y(\bar{t},\cdot;\mathbf{u}) = \mathbb{1}_{]x_l,\hat{\xi}(\bar{t};\mathbf{u})[} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_l(\bar{t},\cdot;\mathbf{u}) + \mathbb{1}_{]\hat{\xi}(\bar{t};\mathbf{u}),x_r[} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_r(\bar{t},\cdot;\mathbf{u}). \tag{3.7.35}$$

If $(\bar{t},\bar{x})$ *is a continuity point of class* RB^c *lying on the left boundary of a rarefaction wave, we obtain a similar result, whereby in the definition of* $\hat{Y}(t,x,\mathbf{u})$ *in* (3.7.33) $Y_l(t,x,\mathbf{u})$ *is computed according to Lemma 3.7.4 and* $Y_r(t,x,\mathbf{u})$ *as in Lemma 3.7.9.*

Proof. Using the definitions of the mappings $Y_{l/r}$, which can be obtained by Lemma 3.7.4 and Lemma 3.7.9, the proof of the result is similar to the proof of Lemma 3.5.12 in [81]. □

We will now have a look at the case, where the genuine backward characteristic through $(\bar{t},\bar{x})$ ends in the left boundary $x = \mathfrak{a}$. We collect some results of [69, §6.2.2] and extend them to the case that shiftings of rarefaction centers are allowed, where we follow the ideas of [70].

3.7.3.2 Local solutions around genuine characteristics ending in the left boundary

Consider a control $\bar{\mathbf{u}} \in \mathbf{U}$ and continuity point $(\bar{t}, \bar{x})$ of $y(\cdot, \bar{\mathbf{u}})$ that is of class $C^c_{B,\mathfrak{a}}$. Then (3.7.12) is valid for some positive constant $\beta > 0$ and $j \in \{1, \ldots, n_{t,\mathfrak{a}} + 1\}$. Due to the continuity of the mapping (3.2.8) in Lemma 3.2.5, there exist $\theta_r < \bar{\theta} < \theta_l$ and $\delta > 0$ such that $J :=]\theta_r - \delta, \theta_l + \delta[\subset]\bar{t}^{\mathfrak{a}}_{j-1}, \bar{t}^{\mathfrak{a}}_j[$ and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, \mathfrak{a}, \bar{u}^{B,\mathfrak{a}}_j(\theta), \bar{u}_1) \le -\frac{\beta}{2} < 0 \quad \forall (t, \theta) \in J_{\bar{t}}, \tag{3.7.36}$$

where $J_s := \{(t, \theta) \in [0, s] \times J : t \ge \theta\}$. The next result is the counterpart to Lemma 3.7.4.

Lemma 3.7.12 (Local solution in a neighborhood of continuity points of class $C^c_{B,\mathfrak{a}}$, Lemma 6.2.7, [69])**.** *Let* (A3) *and* (A4) *be satisfied and consider some control* $\mathbf{u} \in \mathbf{U}$. *In addition, assume that* (3.7.36) *is satisfied for some constants* $\delta, \beta > 0$ *and some* $j \in \{1, \ldots, n_{t,\mathfrak{a}} + 1\}$. *Then there exist constants* $\rho, \tau > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, \mathfrak{a}, u^{B,\mathfrak{a}}_j(\theta), u_1) \le -\frac{\beta}{3} < 0 \quad \forall (t, \theta) \in J_{\bar{t}+\tau},\ \mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}). \tag{3.7.37}$$

Further on, we define the stripe

$$S = S(\tau) := \{(t, x) \in [\theta_r, \bar{t} + \tau] \times \Omega : \xi_l(\max\{\theta_l, t\}) \le x \le \xi_r(t)\},$$

with $\xi_{l/r}(t) := \zeta(t; \theta_{l/r}, \mathfrak{a}, \bar{u}^{B,\mathfrak{a}}_j(\theta_{l/r}), \bar{u}_1)$. *Then the equation*

$$\zeta(t; \theta, \mathfrak{a}, u^{B,\mathfrak{a}}_j(\theta), u_1) = x$$

admits for all $(t, x) \in S(\tau)$ *and all* $\mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$ *a unique solution* $\theta = \Theta(t, x, \mathbf{u}) \in J$. *Moreover, defining the mapping*

$$Y(t, x, \mathbf{u}) = v(t; \Theta(t, x, \mathbf{u}), \mathfrak{a}, u^{B,\mathfrak{a}}_j(\Theta(t, x, \mathbf{u})), u_1), \tag{3.7.38}$$

it holds that $\Theta(\cdot, \mathbf{u}), Y(\cdot, \mathbf{u}) \in C^{0,1}(S)$ *for all* $\mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$. *Furthermore, for all* $t \in [\theta_r, \bar{t} + \tau[$ *the mappings*

$$(x, \mathbf{u}) \in]\xi_l(\max\{\theta_l, t\}), \xi_r(t)[\times B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}) \mapsto \Theta(t, x, \mathbf{u}),$$
$$(x, \mathbf{u}) \in]\xi_l(\max\{\theta_l, t\}), \xi_r(t)[\times B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}) \mapsto Y(t, x, \mathbf{u})$$

are continuously Fréchet-differentiable. The corresponding derivatives

$$\mathrm{d}_{(x,\mathbf{u})}\Theta(t,x,\mathbf{u})\cdot(\delta x,\delta\mathbf{u}) = \frac{\delta x - \delta\zeta(t;\theta,\mathfrak{a},u_j^{B,\mathfrak{a}}(\theta),u_1;0,0,\delta u_j^{B,\mathfrak{a}}(\theta),\delta u_1)}{\delta\zeta(t;\theta,\mathfrak{a},u_j^{B,\mathfrak{a}}(\theta),u_1;1,0,u_j^{B,\mathfrak{a}\prime}(\theta),0)},$$
$$\begin{aligned}\mathrm{d}_{(x,\mathbf{u})}Y(t,x,\mathbf{u})\cdot(\delta x,\delta\mathbf{u}) &= \delta v(t;\theta,\mathfrak{a},u_j^{B,\mathfrak{a}}(\theta),u_1;0,0,\delta u_j^{B,\mathfrak{a}}(\theta),\delta u_1)\\ &\quad+\delta v(t;\theta,\mathfrak{a},u_j^{B,\mathfrak{a}}(\theta),u_1;1,0,u_j^{B,\mathfrak{a}\prime}(\theta),0)\cdot\mathrm{d}_{(x,\mathbf{u})}\Theta(t,x,\mathbf{u})\cdot(\delta x,\delta\mathbf{u}),\end{aligned}$$

where δv and $\delta\zeta$ are determined by the solution of the linearized characteristic equation in (3.2.9). *Finally, the mappings*

$$\mathbf{u}\in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})\mapsto\Theta(\cdot,\mathbf{u})\in C(S)$$
$$\mathbf{u}\in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})\mapsto Y(\cdot,\mathbf{u})\in C(S)$$

are continuously Fréchet-differentiable and the derivatives are given by

$$\mathrm{d}_{\mathbf{u}}(\Theta,Y)(\cdot,\mathbf{u})\cdot\delta\mathbf{u} = \mathrm{d}_{(x,\mathbf{u})}(\Theta,Y)(\cdot,\mathbf{u})\cdot(0,\delta\mathbf{u}),$$

where $\delta y = \mathrm{d}_{\mathbf{u}}Y(\mathbf{u})\cdot\delta$ is the unique broad solution of

$$\begin{aligned}\delta y_t + (f'(Y(\mathbf{u}))\delta y)_x &= g_y(\cdot,y,u_1)\delta y + g_{u_1}(\cdot,y,u_1)\delta u_1 &&\text{on } S,\\ \delta y(\cdot,\mathfrak{a}+) &= \delta u_j^{B,\mathfrak{b}} &&\text{on } [\theta_r,\theta_l]\end{aligned}$$

and the assertions of the Remarks 3.7.6 and 3.7.7 hold true.

By the next result we obtain that the mapping $Y(\mathbf{u})$ constructed in the lemma above coincides with the entropy solution $y(\mathbf{u})$ of the IBVP (3.1.2) on a stripe around the genuine backward characteristic through the continuity point $(\bar{t},\bar{x})$ on a neighborhood of $\bar{\mathbf{u}}$.

Lemma 3.7.13 (Differentiability properties in continuity points of class $C^c_{B,\mathfrak{a}}$, Lemma 6.2.8, [69]). *Suppose that* (A3) *and* (A4) *are satisfied, consider a control $\bar{\mathbf{u}}\in\mathbf{U}$ and let $(\bar{t},\bar{x})\in\Omega_T$ be a continuity point of $y(\cdot,\bar{\mathbf{u}})$ of class $C^c_{B,\mathfrak{a}}$. Then the following statements are valid:*

(i) There is a maximal nonempty open interval I with $\bar{x}\in I$ such that I does not contain any discontinuity of $y(\bar{t},\cdot,\bar{\mathbf{u}})$ and for all $x\in I$ the unique genuine backward characteristic ξ through $(\bar{t},x)$ does not end in some $\theta\in\left\{\bar{t}_1^{\mathfrak{a}},\dots,\bar{t}_{n_{t,\mathfrak{a}}}^{B,\mathfrak{a}}\right\}$. Moreover, it holds that $y(\bar{t},\cdot,\bar{\mathbf{u}})\in C^1(I)$.

(ii) Consider an arbitrary interval $]x_l,x_r[$ with $[x_l,x_r]\subset I$. Let $\xi_{l/r}$ denote the genuine backward characteristics through the points $(\bar{t},x_{l/r})$ and $\theta_{l/r}$ the points

where they intersect $x = \mathfrak{a}$, respectively. Then there exist constants $\delta, \beta > 0$ and some $j \in \{1, \dots, n_{t,\mathfrak{a}} + 1\}$ such that (3.7.36) is satisfied. Consider the mapping

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

from Lemma 3.7.12. Then after a possible reduction of ρ and τ it holds that

$$y(t, x; \mathbf{u}) = Y(t, x, \mathbf{u}) \quad \forall (t, x) \in S,\ \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}).$$

Next, we consider a continuity point $(\bar{t}, \bar{x})$ of class $R^c_{B,\mathfrak{a}}$. Analogously to the case R^c, we follow the ideas of [71] to treat the $R^c_{B,\mathfrak{a}}$-case. Since the source term on the right-hand side of (3.1.2a) is by (A4) equal to zero on $[0, T] \times] - \infty, \varepsilon_g]$, for any $j \in I_{r,\mathfrak{a}}(\bar{\mathbf{u}})$ there exist sufficiently small $\hat{t}_j \in]\bar{t}_j^{\mathfrak{a}}, \bar{t}[$ and $\rho > 0$ such that

$$y(\hat{t}_j, \cdot; \mathbf{u})|_{I_{\mathfrak{a}}^{\hat{t}_j}(\mathbf{u})} = \phi_j^{B,\mathfrak{a}}(x, t_j^{\mathfrak{a}}) \quad \forall \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}), \tag{3.7.39}$$

where

$$I_{\mathfrak{a}}^{\hat{t}_j}(\mathbf{u}) :=]\mathfrak{a} + f'(u_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))(\hat{t}_j - t_j^{\mathfrak{a}}), \mathfrak{a} + f'(u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))(\hat{t}_j - t_j^{\mathfrak{a}})[$$
$$\text{and} \quad \phi_j^{B,\mathfrak{a}}(x, t_j^{\mathfrak{a}}) = f'^{-1}\left(\frac{x - \mathfrak{a}}{\hat{t}_j - t_j^{\mathfrak{a}}}\right).$$

Using (A3), we see that the mapping

$$t \in]0, \hat{t}_j[\mapsto \phi_j^{B,\mathfrak{a}}(\cdot, t) \in C^1_{\text{loc}}(\mathbb{R})$$

is continuously Fréchet-differentiable with derivative

$$\mathrm{d}_t \phi_j^{B,\mathfrak{a}}(\cdot, t) \cdot \delta t = \frac{\delta t}{\left(\hat{t}_j - t\right)^2 \cdot f''\left(f'^{-1}\left(\frac{x-\mathfrak{a}}{\hat{t}_j - t}\right)\right)},$$

cf. [69, proof of Lemma 7.2.10].

Analogously to the mapping in (3.7.28), we obtain that

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto y(\hat{t}_j, \cdot; \mathbf{u}) \in C^1(I_{\mathfrak{a},\rho}^{\hat{t}_j})$$

is continuously Fréchet-differentiable, where

$$I_{\mathfrak{a},\rho}^{\hat{t}_j} := [\mathfrak{a} + f'(\bar{u}_{j+1}^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\hat{t}_j - \bar{t}_j^{\mathfrak{a}}) + 2\varepsilon(\rho), \mathfrak{a} + f'(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\hat{t}_j - \bar{t}_j^{\mathfrak{a}}) - 2\varepsilon(\rho)].$$

with some mapping $\varepsilon(\rho)$ satisfying (3.7.27). Similar to the derivation of (3.7.30), one can prove that there exist constants $\beta, \delta > 0$ and $z_l < \bar{z} < z_l$ such that $J :=]z_l - \delta, z_r + \delta[\subset I^{\hat{t}_j}_{\mathfrak{a},\rho}$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t;\hat{t}_j, z, \phi_j^{B,\mathfrak{a}}(z,\bar{t}_j^{\mathfrak{a}}), \bar{u}_1) \geq \frac{\beta}{2} > 0 \quad \forall (t,z) \in [\hat{t}_j, \bar{t}] \times J \tag{3.7.40}$$

is satisfied. Using the same arguments as for the R^c-case, i.e., considering the truncated initial-boundary value problem on the domain $]\hat{t}_j, T[\times\Omega$, we treat $(\bar{t}, \bar{x})$ as a C^c-point. Therefore, applying Lemmas 3.7.4 and 3.7.8 yields similarly to the R^c-case the following results:

Lemma 3.7.14 (Local solution in a neighborhood of continuity points of class $R^c_{B,\mathfrak{a}}$). *Suppose that* (A3) *and* (A4) *are valid and consider a control* $\mathbf{u} \in \mathbf{U}$. *In addition, assume that* (3.7.40) *is satisfied for some constants* $\delta, \beta > 0$ *and some* $j \in I_{r,\mathfrak{a}}(\bar{\mathbf{u}})$. *If we replace* ϕ_j^0 *by the function* $\phi_j^{B,\mathfrak{a}}$ *and* s *by* $\hat{t}_j$, *the results of Lemma 3.7.9 are still valid.*

Lemma 3.7.15 (Differentiability properties in continuity points of class $R^c_{B,\mathfrak{a}}$). *Let* (A3) *and* (A4) *be satisfied. Furthermore, consider a control* $\mathbf{u} \in \mathbf{U}$ *and a continuity point* $(\bar{t}, \bar{x}) \in \Omega_T$ *of* $y(\cdot, \bar{\mathbf{u}})$ *of class* $R^c_{B,\mathfrak{a}}$ *such that the genuine backward characteristic through* $(\bar{t}, \bar{x})$ *ends in a point* $(\bar{t}_j^{\mathfrak{a}}, \mathfrak{a})$ *which is the center of a rarefaction wave. In addition, let* $\hat{t}_j \in]\bar{t}_j^{\mathfrak{a}}, \bar{t}[$ *be chosen small enough such that* (3.7.39) *is satisfied. Then the following statements hold true:*

(i) There is a maximal nonempty open interval I *with* $\bar{x} \in I$ *such that* I *does not contain any discontinuity of* $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ *and for all* $x \in I$ *the unique genuine backward characteristic* ξ *through* $(\bar{t}, x)$ *intersects* $t = \hat{t}_j$ *in*

$$z = \xi(\hat{t}_j) \in]\mathfrak{a} + f'(\bar{u}^{B,\mathfrak{a}}_{j+1}(\bar{t}_j^{\mathfrak{a}}))\left(\hat{t}_j - \bar{t}_j^{\mathfrak{a}}\right), \mathfrak{a} + f'(\bar{u}^{B,\mathfrak{a}}_{j}(\bar{t}_j^{\mathfrak{a}}))\left(\hat{t}_j - \bar{t}_j^{\mathfrak{a}}\right)[.$$

Moreover, it holds that $y(\bar{t}, \cdot, \bar{\mathbf{u}}) \in C^1(I)$.

(ii) Let $]x_l, x_r[$ *be an arbitrary interval with* $[x_l, x_r] \subset I$. *Moreover, denote by* $\xi_{l/r}$ *the genuine backward characteristics through the points* $(\bar{t}, x_{l/r})$ *and by* $z_{l/r}$ *the points where they intersect* $t = \hat{t}_j$. *Then there exist constants* $\delta, \beta > 0$ *such that* (3.7.40) *is satisfied. Considering the mapping*

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

given by Lemma 3.7.14, after a possible reduction of ρ and τ it holds true that

$$y(t,x;\mathbf{u}) = Y(t,x,\mathbf{u}) \quad \forall (t,x) \in S,\ \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}).$$

Let $(\bar{t},\bar{x})$ be a continuity point of $y(\bar{\mathbf{u}})$ of class $RB^c_{B,\mathfrak{a}}$ lying on the right boundary of a rarefaction wave, i.e., the genuine backward characteristic through the point $(\bar{t},\bar{x})$ denoted by $\bar{\xi}$ intersects $x=\mathfrak{a}$ in some point $\bar{\theta} = \bar{t}_j^{\mathfrak{a}} \in]0,T[$ for some $j \in I_{r,\mathfrak{a}}(\bar{\mathbf{u}})$ and coincides on $[\hat{t}_j,\bar{t}]$ with the solution $\zeta(\cdot;\bar{\theta},\mathfrak{a},\bar{u}_j^{B,\mathfrak{a}}(\bar{\theta}),\bar{u}_1)$ of (3.2.7). Moreover, we set $\bar{z} = \bar{\xi}(\hat{t}_j)$, where $\hat{t}_j \in]\bar{t}_j^{\mathfrak{a}},\bar{t}[$ is chosen sufficiently small such that (3.7.39) holds for small enough $\rho > 0$. Considering the local solution $\phi_j^{B,\mathfrak{a}}$ defined in (3.7.39), we obtain that $\bar{\xi}$ coincides on $[\hat{t}_j,\bar{t}]$ with $\zeta(\cdot;\hat{t}_j,\bar{z},\phi_j^{B,\mathfrak{a}}(\bar{z},\bar{t}_j^{\mathfrak{a}}),\bar{u}_1)$. Since $(\bar{t},\bar{x})$ is by assumption not a shock generation point and due to the regularity of the solution to (3.2.7), there exist $z_{l/r},\theta_{l/r}$ with $z_l < \bar{z} < z_r$ and $\theta_r < \bar{\theta} < \theta_l$ and constants $\beta,\delta > 0$ such that setting $J_l :=]z_l-\delta, z_r+\delta[$ and $J_r :=]\theta_r-\delta,\theta_l+\delta[$, we get that (3.7.12) and (3.7.14) are satisfied. Therefore, one can apply Lemma 3.7.12 and Lemma 3.7.14 yielding stripes

$$S_l = S_l(\tau) := \left\{(t,x) \in [\hat{t}_j,\bar{t}+\tau] \times \Omega : \xi_{l,l}(t) \le x \le \xi_{l,r}(t)\right\},$$
$$S_r = S_r(\tau) := \left\{(t,x) \in [\theta_r,\bar{t}+\tau] \times \Omega : \xi_{r,l}(\max(t,\bar{t}_j^{\mathfrak{a}})) \le x \le \xi_{r,r}(t)\right\},$$

where $\tau > 0$ is a constant and

$$\xi_{l,l/r}(t) := \zeta(t;s,z_{l/r},\phi_j^{B,\mathfrak{a}}(z_{l/r},\bar{t}_j^{\mathfrak{a}}),\bar{u}_1),$$
$$\xi_{r,l/r}(t) := \zeta(t;\theta_{l/r},\mathfrak{a},\bar{u}_j^{B,\mathfrak{a}}(\theta_{l/r}),\bar{u}_1).$$

Lemmas 3.7.12 and 3.7.14 further yield continuously Fréchet-differentiable mappings

$$(x,\mathbf{u}) \in]\xi_{r,l}(\max(t,\bar{t}_j^0)),\xi_{r,r}(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y_r(t,x,\mathbf{u}) \quad t \in [0,\bar{t}]$$

and

$$(x,\mathbf{u}) \in]\xi_{l,l}(t),\xi_{l,r}(t)[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y_r(t,x,\mathbf{u}) \quad t \in [\hat{t}_j,\bar{t}].$$

Analogously to [69, §6.22], we define the set

$$\hat{S} := \left(S_l \cap \left\{x \le \hat{\xi}(t;\mathbf{u})\right\}\right) \cup \left(S_r \cap \left\{x > \hat{\xi}(t;\mathbf{u})\right\}\right),$$

with $\hat{\xi}(t;\mathbf{u}) := \zeta(t; t_j^{\mathfrak{a}}, \mathfrak{a}, u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}), u_1)$ and the mapping

$$\hat{Y}(t,x,\mathbf{u}) := \begin{cases} Y_l(t,x,\mathbf{u}) & \text{if } x < \hat{\xi}(t;\mathbf{u}), \\ Y_r(t,x,\mathbf{u}) & \text{if } x \geq \hat{\xi}(t;\mathbf{u}), \end{cases} \qquad \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}),\ (t,x) \in \hat{S}. \tag{3.7.41}$$

Due to these considerations, we obtain similarly to the RB^c-case the following result:

Lemma 3.7.16 (Differentiability in continuity points of class $RB^c_{B,\mathfrak{a}}$)**.** *Let* (A3) *and* (A4) *be satisfied and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$*. In addition, let* $(\bar{t},\bar{x})$ *be a continuity point of* $y(\cdot,\bar{\mathbf{u}})$ *of class* $RB^c_{B,\mathfrak{a}}$ *lying on the right boundary of a rarefaction wave with center* $(\bar{t}_j^{\mathfrak{a}}, \mathfrak{a})$*. Then there exists a constant* $\rho > 0$ *and* $x_l, x_r \in]\mathfrak{a},\mathfrak{b}[$ *with* $x_l < \bar{x} < x_r$ *such that*

$$y(\bar{t},x;\mathbf{u}) = \hat{Y}(\bar{t},x,\mathbf{u}) \quad \text{on }]x_l,x_r[$$

is valid for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$*, where* $\hat{Y}(\bar{t},x,\mathbf{u})$ *is defined in* (3.7.41)*. Hence,* $y(\bar{t},\cdot;\mathbf{u}) \in C^{0,1}(]x_l,x_r[)$ *and* $y(\bar{t},\cdot;\mathbf{u})$ *is continuously differentiable on* $]x_l,x_r[\setminus \left\{\hat{\xi}(\bar{t},\mathbf{u})\right\}$*. Furthermore, the mapping*

$$\mathbf{u} \in \mathbf{U} \mapsto y(\bar{t},\cdot;\mathbf{u}) \in L^r([x_l,x_r]) \tag{3.7.42}$$

is continuously Fréchet-differentiable for all $r \in [1,\infty[$ *with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} y(\bar{t},\cdot;\mathbf{u}) = \mathbb{1}_{]x_l,\hat{\xi}(\bar{t};\mathbf{u})[} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_l(\bar{t},\cdot;\mathbf{u}) + \mathbb{1}_{]\hat{\xi}(\bar{t};\mathbf{u}),x_r[} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_r(\bar{t},\cdot;\mathbf{u}). \tag{3.7.43}$$

Considering the case that a continuity point $x = \bar{x} \in]\mathfrak{a},\mathfrak{b}[$ *of* $y(\bar{t},\cdot,\bar{\mathbf{u}})$ *lies on the left boundary of a rarefaction wave, we obtain a similar result, whereby in the definition of* $\hat{Y}(t,x,\mathbf{u})$ *in* (3.7.41) $Y_l(t,x,\mathbf{u})$ *is computed according to Lemma 3.7.12 and* $Y_r(t,x,\mathbf{u})$ *according to Lemma 3.7.14.*

Proof. Using the definitions of the mappings $Y_{l/r}$, which can be obtained by using Lemmas 3.7.12 and 3.7.14, the proof is similar to the proof of Lemma 6.2.12 in [81]. □

Remark 3.7.17. A version of Lemma 3.7.16 , where the shifting of centers of rarefaction waves is prohibited can be found in [69, Lemma 6.2.12].

3.7.3.3 Local solutions around genuine characteristics ending in the right boundary

The case that the genuine backward characteristic through a continuity point $(\bar{t}, \bar{x})$ of $y(\cdot, \bar{\mathbf{u}})$ ends in the right boundary can be treated analogously to the case that it ends in the left boundary at $x = \mathfrak{a}$. Therefore, we will only give an overview of the results, where further explanations can be found in the previous subsection. We start by examining continuity points of class $C^c_{B,\mathfrak{b}}$.

Lemma 3.7.18 (Local solution in a neighborhood of continuity points of class $C^c_{B,\mathfrak{b}}$)**.** *Suppose that* (A3) *and* (A4) *are satisfied and consider some* $\bar{\mathbf{u}} \in \mathbf{U}$. *Furthermore, assume that there exist* $\beta, \delta > 0$, $j \in \{1, \ldots, n_{t,\mathfrak{b}} + 1\}$ *and* $\theta_l < \bar{\theta} < \theta_r$ *with* $J :=]\theta_l - \delta, \theta_r + \delta[\subset]\bar{t}^{\mathfrak{b}}_{j-1}, \bar{t}^{\mathfrak{b}}_j[$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, \mathfrak{b}, \bar{u}^{B,\mathfrak{b}}_j(\theta), \bar{u}_1) \geq \frac{\beta}{2} > 0 \quad \forall (t, \theta) \in J_{\bar{t}} \tag{3.7.44}$$

is valid, where $J_s := \{(t, \theta) \in [0, s] \times J : t \geq \theta\}$.

Then there exist constants $\rho, \tau > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, \mathfrak{b}, u^{B,\mathfrak{b}}_j(\theta), u_1) \geq \frac{\beta}{3} > 0 \quad \forall (t, \theta) \in J_{\bar{t}+\tau},\ \mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}).$$

Defining the stripe

$$S = S(\tau) := \{(t, x) \in [\theta_r, \bar{t} + \tau] \times \Omega : \xi_l(\max\{\theta_l, t\}) \leq x \leq \xi_r(t)\},$$

where $\xi_{l/r}(t) := \zeta(t; \theta_{l/r}, \mathfrak{b}, \bar{u}^{B,\mathfrak{b}}_j(\theta_{l/r}), \bar{u}_1)$, *the equation*

$$\zeta(t; \theta, \mathfrak{b}, u^{B,\mathfrak{b}}_j(\theta), u_1) = x$$

admits for all $(t, x) \in S(\tau)$ *and all* $\mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$ *a unique solution* $\theta = \Theta(t, x, \mathbf{u}) \in J$. *In addition, for all* $\mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$ *it holds true that* $\Theta(\cdot, \mathbf{u}), Y(\cdot, \mathbf{u}) \in C^{0,1}(S)$ *with*

$$Y(t, x, \mathbf{u}) = v(t; \Theta(t, x, \mathbf{u}), \mathfrak{b}, u^{B,\mathfrak{b}}_j(\Theta(t, x, \mathbf{u})), u_1).$$

Further on, the mappings

$$(x, \mathbf{u}) \in]\xi_l(\max\{\theta_l, t\}), \xi_r(t)[\, \times B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}) \mapsto \Theta(t, x, \mathbf{u}),$$
$$(x, \mathbf{u}) \in]\xi_l(\max\{\theta_l, t\}), \xi_r(t)[\, \times B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}) \mapsto Y(t, x, \mathbf{u})$$

are continuously Fréchet-differentiable for all $t \in [\theta_r, \bar{t}+\tau[$ *with corresponding deriva-*

tives

$$
\begin{aligned}
\mathrm{d}_{(x,\mathbf{u})}\Theta(t,x,\mathbf{u})\cdot(\delta x,\delta\mathbf{u}) &= \frac{\delta x-\delta\zeta(t;\theta,\mathfrak{b},u_j^{B,\mathfrak{b}}(\theta),u_1;0,0,\delta u_j^{B,\mathfrak{b}}(\theta),\delta u_1)}{\delta\zeta(t;\theta,\mathfrak{b},u_j^{B,\mathfrak{b}}(\theta),u_1;1,0,u_j^{B,\mathfrak{b}\prime}(\theta),0)},\\
\mathrm{d}_{(x,\mathbf{u})}Y(t,x,\mathbf{u})\cdot(\delta x,\delta\mathbf{u}) &= \delta v(t;\theta,\mathfrak{b},u_j^{B,\mathfrak{b}}(\theta),u_1;0,0,\delta u_j^{B,\mathfrak{b}}(\theta),\delta u_1)\\
&\quad+\delta v(t;\theta,\mathfrak{b},u_j^{B,\mathfrak{b}}(\theta),u_1;1,0,u_j^{B,\mathfrak{b}\prime}(\theta),0)\cdot\mathrm{d}_{(x,\mathbf{u})}\Theta(t,x,\mathbf{u})\cdot(\delta x,\delta\mathbf{u}),
\end{aligned}
$$

where δv and $\delta\zeta$ are determined by the the solution of the linearized characteristic equation in (3.2.9)*. Moreover, the mappings*

$$
\begin{aligned}
\mathbf{u}\in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) &\mapsto \Theta(\cdot,\mathbf{u})\in C(S)\\
\mathbf{u}\in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) &\mapsto Y(\cdot,\mathbf{u})\in C(S)
\end{aligned}
$$

are continuously Fréchet-differentiable and the corresponding derivatives are given by

$$
\mathrm{d}_{\mathbf{u}}(\Theta,Y)(\cdot,\mathbf{u})\cdot\delta\mathbf{u}=\mathrm{d}_{(x,\mathbf{u})}(\Theta,Y)(\cdot,\mathbf{u})\cdot(0,\delta\mathbf{u}).
$$

Finally, $\delta y=\mathrm{d}_{\mathbf{u}}Y(\mathbf{u})\cdot\delta\mathbf{u}$ is obtained by the unique broad solution of the linearized equation

$$
\begin{aligned}
\delta y_t+(f'(Y(\mathbf{u}))\delta y)_x &= g_y(\cdot,y,u_1)\delta y+g_{u_1}(\cdot,y,u_1)\delta u_1 && \text{on } S,\\
\delta y(\cdot,\mathfrak{b}+) &= \delta u_j^{B,\mathfrak{b}} && \text{on } [\theta_r,\theta_l]
\end{aligned}
$$

and the assertions of the Remarks 3.7.6 and 3.7.7 are satisfied.

Lemma 3.7.19 (Differentiability properties in continuity points of class $C^c_{B,\mathfrak{b}}$)**.** *Suppose that* (A3) *and* (A4) *are satisfied and consider a control $\bar{\mathbf{u}}\in\mathbf{U}$. In addition, let $(\bar t,\bar x)$ be a continuity point of $y(\cdot,\bar{\mathbf{u}})$ of class $C^c_{B,\mathfrak{b}}$. Then the following statements are valid:*

(i) There is a maximal nonempty open interval I with $\bar x\in I$ such that I does not contain any discontinuity of $y(\bar t,\cdot,\bar{\mathbf{u}})$ and for all $x\in I$ the unique genuine backward characteristic ξ through $(\bar t,x)$ does not end in a point $\bar t_j^{\mathfrak{b}}\in\left\{\bar t_1^{\mathfrak{b}},\ldots,\bar t_{n_{t,\mathfrak{b}}}^{\mathfrak{b}}\right\}$. Moreover, it holds that $y(\bar t,\cdot,\bar{\mathbf{u}})\in C^1(I)$.

(ii) Consider an arbitrary interval $]x_l,x_r[$ with $[x_l,x_r]\subset I$. Let $\xi_{l/r}$ denote the genuine backward characteristics through the points $(\bar t,x_{l/r})$ and $\theta_{l/r}$ the points where they intersect $x=\mathfrak{b}$, respectively. Then there exist constants $\delta,\beta>0$

such that (3.7.44) *holds true for* $J =]\theta_r - \delta, \theta_l + \delta[$. *Consider the mapping*

$$\mathbf{u} \in B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

from Lemma 3.7.18. Then after a possible reduction of ρ *and* τ *it holds that*

$$y(t, x; \mathbf{u}) = Y(t, x, \mathbf{u}) \quad \forall (t, x) \in S,\, \mathbf{u} \in B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}).$$

Next, we consider a continuity point $(\bar{t}, \bar{x})$ of class $R^c_{B,\mathfrak{b}}$.

Analogously to the case $R^c_{B,\mathfrak{a}}$, we choose sufficiently small $\hat{t}_j \in]\bar{t}^{\mathfrak{b}}_j, \bar{t}[$ and $\rho > 0$ such that

$$y(\hat{t}_j, \cdot; \mathbf{u})|_{I^{\hat{t}_j}_{\mathfrak{b}}(\mathbf{u})} = \phi^{B,\mathfrak{b}}_j(x, t^{\mathfrak{b}}_j) \quad \forall \mathbf{u} \in B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}) \tag{3.7.45}$$

is satisfied for all $j \in I_{r,\mathfrak{b}}(\bar{\mathbf{u}})$, where

$$\begin{aligned} I^{\hat{t}_j}_{\mathfrak{b}}(\mathbf{u}) &:=]\mathfrak{b} + f'(u^{B,\mathfrak{b}}_j(t^{\mathfrak{b}}_j))s, \mathfrak{b} + f'(u^{B,\mathfrak{b}}_{j+1}(t^{\mathfrak{b}}_j))s[\\ \text{and} \quad \phi^{B,\mathfrak{b}}_j(x, t^{\mathfrak{b}}_j) &= f'^{-1}\left(\frac{x - \mathfrak{b}}{\hat{t}_j - t^{\mathfrak{b}}_j}\right). \end{aligned}$$

Recalling the assumptions in (A3), we conclude that

$$t \in]0, \hat{t}_j[\mapsto \phi^{B,\mathfrak{b}}_j(\cdot, t) \in C^1_{\mathrm{loc}}(\mathbb{R})$$

is continuously Fréchet-differentiable with derivative

$$\mathrm{d}_t \phi^{B,\mathfrak{b}}_j(\cdot, t) \cdot \delta t = \frac{\delta t}{\left(\hat{t}_j - t\right)^2 \cdot f''\left(f'^{-1}\left(\frac{x - \mathfrak{b}}{\hat{t}_j - t}\right)\right)}.$$

Further on, similar to (3.7.28), the mapping

$$\mathbf{u} \in B^{\mathbf{U}}_{\rho}(\bar{\mathbf{u}}) \mapsto y(\hat{t}_j, \cdot; \mathbf{u}) \in C^1(I^{\hat{t}_j}_{\mathfrak{b},\rho})$$

is continuously Fréchet-differentiable with

$$I^{\hat{t}_j}_{\mathfrak{b},\rho} := [\mathfrak{b} + f'(\bar{u}^{B,\mathfrak{b}}_j(\bar{t}^{\mathfrak{b}}_j))\left(\hat{t}_j - \bar{t}^{\mathfrak{b}}_j\right) + 2\varepsilon(\rho), \mathfrak{b} + f'(\bar{u}^{B,\mathfrak{b}}_{j+1}(\bar{t}^{\mathfrak{b}}_j))\left(\hat{t}_j - \bar{t}^{\mathfrak{b}}_j\right) - 2\varepsilon(\rho)],$$

where $\varepsilon(\rho)$ is some mapping satisfying (3.7.27). Using the same arguments as for the derivation of (3.7.19), we obtain constants $\beta, \delta > 0$ and $z_l < \bar{z} < z_r$ such that

$J :=]z_l - \delta, z_r + \delta[\subset I_{\mathfrak{b},\rho}^{\hat{t}_j}$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta(t; \hat{t}_j, z, \phi_j^{B,\mathfrak{b}}(z, \bar{t}_j^{\mathfrak{a}}), \bar{u}_1) \geq \frac{\beta}{2} > 0 \quad \forall (t,z) \in [\hat{t}_j, \bar{t}] \times J \tag{3.7.46}$$

is satisfied. Therefore, considering the truncated initial-boundary value problem on the domain $]\hat{t}_j, T[\times\Omega$, Lemmas 3.7.4 and 3.7.8 hold true. Similarly to the $R^c_{B,\mathfrak{a}}$-case we obtain the following two results:

Lemma 3.7.20 (Local solution in a neighborhood of continuity points of class $R^c_{B,\mathfrak{b}}$)**.** *Assume that* (A3) *and* (A4) *hold and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$*. In addition, suppose that* (3.7.46) *is satisfied for some constants* $\delta, \beta > 0$ *and* $j \in I_{r,\mathfrak{b}}(\bar{\mathbf{u}})$*. If we replace* ϕ_j^0 *by* $\phi_j^{B,\mathfrak{b}}$ *and* s *by* $\hat{t}_j$*, then the results of Lemma 3.7.9 are still valid.*

Lemma 3.7.21 (Differentiability properties in continuity points of class $R^c_{B,\mathfrak{b}}$)**.** *Suppose that* (A3) *and* (A4) *are satisfied. Furthermore, consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *and a continuity point* $(\bar{t}, \bar{x}) \in \Omega_T$ *of* $y(\cdot, \bar{\mathbf{u}})$ *of class* $R^c_{B,\mathfrak{b}}$ *such that the genuine backward characteristic through* $(\bar{t}, \bar{x})$ *ends in a point* $(\bar{t}_j^{\mathfrak{b}}, \mathfrak{b})$ *which is the center of a rarefaction wave. Moreover, choose* $\hat{t}_j \in]\bar{t}_j^{\mathfrak{b}}, \bar{t}[$ *such that* (3.7.45) *is satisfied. Then the following statements are valid:*

(i) There is a maximal nonempty open interval I *with* $\bar{x} \in I$ *such that* I *does not contain any discontinuity of* $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ *and for all* $x \in I$ *the unique genuine backward characteristic* ξ *through* $(\bar{t}, x)$ *intersects* $t = \hat{t}_j$ *in*

$$z = \xi(\hat{t}_j) \in]\mathfrak{b} + f'(\bar{u}_j^{B,\mathfrak{b}}(\bar{t}_j^{\mathfrak{b}}))\left(\hat{t}_j - \bar{t}_j^{\mathfrak{b}}\right), \mathfrak{b} + f'(\bar{u}_{j+1}^{B,\mathfrak{b}}(\bar{t}_j^{\mathfrak{a}}))\left(\hat{t}_j - \bar{t}_j^{\mathfrak{b}}\right)[.$$

Moreover, it holds that $y(\bar{t}, \cdot, \bar{\mathbf{u}}) \in C^1(I)$*.*

(ii) Let $]x_l, x_r[$ *be an arbitrary interval with* $[x_l, x_r] \subset I$*. Let* $\xi_{l/r}$ *denote the genuine backward characteristics through the points* $(\bar{t}, x_{l/r})$ *and* $z_{l/r}$ *the points where they intersect* $t = \hat{t}_j$*, respectively. Then there exist constants* $\delta, \beta > 0$ *such that* (3.7.46) *is satisfied. Consider the mapping*

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\cdot, \mathbf{u}) \in C(S)$$

given by (3.7.32) *with the adaptations described in Lemma 3.7.20. Then*

$$y(t, x; \mathbf{u}) = Y(t, x, \mathbf{u}) \quad \forall (t,x) \in S,\ \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$$

is valid after a possible reduction of ρ *and* τ*.*

To complete the case where the backward characteristic ends in the right boundary, we now consider a continuity point $(\bar{t}, \bar{x})$ of $y(\bar{\mathbf{u}})$ of class $RB^c_{B,\mathfrak{b}}$ that lies on the right boundary of a rarefaction wave, i.e., the genuine backward characteristic $\bar{\xi}$ through the point $(\bar{t}, \bar{x})$ denoted by $\bar{\xi}$ intersects $x = \mathfrak{b}$ in some point $\bar{\theta} = \bar{t}^{\mathfrak{b}}_j \in]0, T[$ for some $j \in I_{r,\mathfrak{b}}(\bar{\mathbf{u}})$.

Analogously to the $RB^c_{B,\mathfrak{a}}$- case, one can use the Lemmas 3.7.18 and 3.7.20 yielding stripes $S_{l/r}$ around $\bar{\xi}$ and mappings $Y_{l/r} : S_{l/r} \times B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}) \to \mathbb{R}$ such that we can define

$$\hat{Y}(t, x, \mathbf{u}) := \begin{cases} Y_l(t, x, \mathbf{u}) & \text{if } x < \hat{\xi}(t; \mathbf{u}), \\ Y_r(t, x, \mathbf{u}) & \text{if } x \geq \hat{\xi}(t; \mathbf{u}), \end{cases} \qquad \mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}),\ (t, x) \in \hat{S}, \tag{3.7.47}$$

where $\hat{\xi}(t; \mathbf{u}) := \zeta(t; t^{\mathfrak{b}}_j, \mathfrak{b}, u^{B,\mathfrak{b}}_j(t^{\mathfrak{b}}_j), u_1)$ and

$$\hat{S} := \left(S_l \cap \left\{x \leq \hat{\xi}(t; \mathbf{u})\right\}\right) \cup \left(S_r \cap \left\{x > \hat{\xi}(t; \mathbf{u})\right\}\right).$$

Lemma 3.7.22 (Differentiability in continuity points of class $RB^c_{B,\mathfrak{b}}$)**.** *Suppose that* (A3) *and* (A4) *hold true and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$. *Furthermore, let* $(\bar{t}, \bar{x})$ *be a continuity point of* $y(\cdot, \bar{\mathbf{u}})$ *of class* $RB^c_{B,\mathfrak{b}}$ *lying on the right boundary of a rarefaction wave with center* $(\bar{t}^{\mathfrak{b}}_j, \mathfrak{b})$. *Then there are constants* $\rho > 0$ *and* $x_l, x_r \in]\mathfrak{a}, \mathfrak{b}[$ *with* $x_l < \bar{x} < x_r$ *such that*

$$y(\bar{t}, x; \mathbf{u}) = \hat{Y}(\bar{t}, x, \mathbf{u}) \quad \text{on }]x_l, x_r[\quad \forall \mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$$

is satisfied with $\hat{Y}(\bar{t}, x, \mathbf{u})$ *as defined in* (3.7.47)*. Hence,* $y(\bar{t}, \cdot; \mathbf{u}) \in C^{0,1}(]x_l, x_r[)$ *and* $y(\bar{t}, \cdot; \mathbf{u})$ *is continuously differentiable on* $]x_l, x_r[\setminus \left\{\hat{\xi}(\bar{t}, \mathbf{u})\right\}$. *In addition, the mapping*

$$\mathbf{u} \in \mathbf{U} \mapsto y(\bar{t}, \cdot; \mathbf{u}) \in L^r([x_l, x_r]) \tag{3.7.48}$$

is continuously Fréchet-differentiable for all $r \in [1, \infty[$*, where the derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} y(\bar{t}, \cdot; \mathbf{u}) = \mathbb{1}_{]x_l, \hat{\xi}(\bar{t}; \mathbf{u})[} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_l(\bar{t}, \cdot; \mathbf{u}) + \mathbb{1}_{]\hat{\xi}(\bar{t}; \mathbf{u}), x_r[} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_r(\bar{t}, \cdot; \mathbf{u}). \tag{3.7.49}$$

If we consider the case that a continuity point $x = \bar{x} \in [\mathfrak{a}, \mathfrak{b}]$ *of* $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ *lies on the left boundary of a rarefaction wave, we obtain a similar result, whereby in the definition of* $\hat{Y}(t, x, \mathbf{u})$ *in* (3.7.47) $Y_l(t, x, \mathbf{u})$ *is computed according to Lemma 3.7.18 and* $Y_r(t, x, \mathbf{u})$ *according to Lemma 3.7.20.*

Proof. The proof is similar to the proof of Lemma 3.7.16. □

3.7.3.4 Local solutions around genuine characteristics ending in $(0, \mathfrak{a})$ or $(0, \mathfrak{b})$

Consider a control $\bar{\mathbf{u}} \in \mathbf{U}$ and denote by $y(\bar{\mathbf{u}})$ the corresponding entropy solution of (3.1.2). If $(\bar{t}, \bar{x})$ is a continuity point of class $R^c_{0,\mathfrak{a}}$, then one can apply Lemmas 3.7.14 and 3.7.15 with $\bar{t}^{\mathfrak{a}}_j = \bar{t}^{B,\mathfrak{a}}_0 = 0$. Now we consider the case that $(\bar{t}, \bar{x})$ is a continuity point of class $RB^c_{0,\mathfrak{a}}$. If $(\bar{t}, \bar{x})$ lies on the left boundary of the rarefaction wave, then one can use Lemma 3.7.16 with $\bar{t}^{\mathfrak{a}}_j = \bar{t}^{B,\mathfrak{a}}_0 = 0$. On the other hand, if $(\bar{t}, \bar{x})$ lies on the right boundary, one can apply Lemma 3.7.11 with $\bar{x}^0_j = \bar{x}^0_0 = \mathfrak{a}$. Analogously, for the case that $(\bar{t}, \bar{x})$ is a continuity point of class $R^c_{0,\mathfrak{b}}$ or $RB^c_{0,\mathfrak{b}}$, we can use Lemmas 3.7.20, 3.7.21 and 3.7.22 with $\bar{t}^{\mathfrak{b}}_j = \bar{t}^{B,\mathfrak{b}}_0 = 0$ or Lemma 3.7.11 with $\bar{x}^0_j = \bar{x}^0_0 = \mathfrak{b}$.

3.7.4 Differentiability of the shock position

In order to prove Theorem 3.5.5, which is the main result of §3.7, Lemma 3.7.1 will play a key-role. Further on, it will be essential to derive necessary optimality conditions for (P). Recall that for a given control $\mathbf{u} \in \mathbf{U}$ satisfying (ND), Lemma 3.7.1 guarantees the following representation for the entropy solution $y(\bar{t}, \cdot; \bar{\mathbf{u}})$ to (3.1.2) on the interval $[a, b]$:

$$y(\bar{t}, x; \mathbf{u})\,|_{[a,b]} = Y_1(\bar{t}, x; \mathbf{u}) \cdot \mathbb{1}_{[a, x_1(\mathbf{u})]}(x) + \sum_{k=2}^{K+1} Y_k(\bar{t}, x; \mathbf{u}) \cdot \mathbb{1}_{(x_{k-1}(\mathbf{u}), x_k(\mathbf{u})]}(x)$$

for some functions $Y_1, \ldots, Y_{K+1}$ and $x_1, \ldots, x_K$ given in (3.7.5) and (3.7.6) and $x_{K+1} = b$. One question that arises if we want to prove Lemma 3.7.1 is how to choose those functions. To give an answer to this question, we first note that the functions $Y_1, \ldots, Y_{K+1}$ can be obtained by using the results in §3.7.3. Moreover, we note that according to (ND), the interval $[a, b]$ contains finitely many non-degenerated discontinuities and points lying on boundaries of rarefaction waves, which we will choose for the functions $x_1, \ldots, x_K$ in Lemma 3.7.1. In order to prove Lemma 3.7.1, it remains to show that these points are functions depending continuously Fréchet-differentiably on the control. Concerning the points lying on the boundary of rarefaction waves this is not difficult to show since the continuously Fréchet-differentiable of continuity points lying on the boundary of a rarefaction wave with respect to the control is a direct consequence of Lemma 3.2.5.

The goal of this subsection is to obtain similar result for the case that x_k is a non-degenerated discontinuity of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ for some $k \in \{1, \ldots, K\}$. Such a result can be found in [81] for a pure initial value problem. Building on the results of [81], Pfaff provides an extension to initial-boundary value problems in [69, Lemma

6.3.1, 6.3.7], see also [70]. In contrast to this thesis, in [81] and [69] the shifting of rarefaction centers is prohibited. Therefore, we will adapt the results in [69, Lemma 6.3.1, 6.3.7] to the scenario considered in this thesis.

The following result is an extension of [69, Lemma 6.3.1] to the case that shiftings of rarefaction centers in the initial and boundary data are allowed, see also [81, Lemma 3.6.3].

Lemma 3.7.23 (Stability of the shock position)**.** *Let* (A3) *and* (A4) *be satisfied,* $\bar{\mathbf{u}} \in \mathbf{U}$ *be an arbitrary control and denote by* $y(\bar{\mathbf{u}})$ *the corresponding entropy solution of the IBVP* (3.1.2)*. Furthermore, consider a shock point* $(\bar{t}, \bar{x})$ *of* $y(\cdot; \bar{\mathbf{u}})$ *which is non-degenerated according to Definition 3.7.3 and lies on a shock-curve* $\eta(t)$*. Then there exist functions* $Y_{l/r}$ *which are constructed according to the Lemmas 3.7.4, 3.7.9, 3.7.12, 3.7.14, 3.7.18 or 3.7.20 along stripes around the minimal and maximal characteristics through* $(\bar{t}, \bar{x})$*. Moreover, we find an interval* $]x_l, x_r[$ *with* $\bar{x} \in]x_l, x_r[$ *and a mapping*

$$x_s : \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto x_s(\mathbf{u}) \in]x_l, x_r[\tag{3.7.50}$$

such that

$$y(\bar{t}, x; \mathbf{u}) = \begin{cases} Y_l(\bar{t}, x, \mathbf{u}) & \text{if } x \in]x_l, x_s(\mathbf{u})[, \\ Y_r(\bar{t}, x, \mathbf{u}) & \text{if } x \in]x_s(\mathbf{u}), x_r[, \end{cases} \qquad \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$$

holds if $\rho > 0$ *is chosen small enough.*

Proof. At first, we note that the case that $(\bar{t}, \bar{x})$ is a shock of class X_l/X_r with $X_l, X_r \in \left\{C^c, C^c_{B,\mathfrak{a}}, C^c_{B,\mathfrak{b}}\right\}$ can be found in the proof of Lemma 6.3.1 in [69], cf. [81, Lemma 3.6.3]. Therefore, only the case that either X_l or X_r is of class $\left\{R^c, R^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{b}}\right\}$ is of further interest. However, in this case one can compute the local solution $Y_{l/r}$ according to the Lemmas 3.7.9, 3.7.14 and 3.7.20 so that the remaining procedure is the same as in [69]. □

Theorem 3.7.24 (Differentiability of shock points of class $R^c_{B,\mathfrak{a}}/C^c$)**.** *Let* (A3) *and* (A4) *hold true and consider some* $\bar{\mathbf{u}} \in \mathbf{U}$ *satisfying* (ND) *and a non-degenerated shock point* $\bar{x} = x_s(\bar{\mathbf{u}})$ *of* $y(\bar{t}, \cdot; \bar{\mathbf{u}})$ *of class* $R^c_{B,\mathfrak{a}}/C^c$*. Then the mapping in* (3.7.50) *is continuously differentiable with derivative*

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{u}} x_s(\mathbf{u})\delta\mathbf{u} = {} & (p, g_{u_1}(\cdot, y, u_1)\delta u_1)_{2,\Omega_{\bar t}} \\
& + \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0)_{2,I_0^i(\mathbf{u})} + \sum_{i\in I_{s,0}(\mathbf{u})} p(0, x_i^0)[u_0(x_i^0)]\delta x_i^0 \\
& + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, \mathfrak{a}+), f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2, I^i_{B,\mathfrak{a}}(\mathbf{u})\cap]0,\bar t[} \\
& + \sum_{\substack{i\in I_{s,\mathfrak{a}}(\mathbf{u}):\\ t_i^{\mathfrak{a}}\le \bar t}} p(t_i^{\mathfrak{a}}, \mathfrak{a}+)[f(y(t_i^{\mathfrak{a}}, \mathfrak{a}+; \mathbf{u}))]\delta t_i^{\mathfrak{a}} \\
& - \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot, \mathfrak{b}-), f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2, I^i_{B,\mathfrak{b}}\cap]0,\bar t[} \\
& - \sum_{\substack{i\in I_{s,\mathfrak{b}}(\mathbf{u}):\\ t_i^{\mathfrak{b}}\le \bar t}} p(t_i^{\mathfrak{b}}, \mathfrak{b}-)[f(y(t_i^{\mathfrak{b}}, \mathfrak{b}-; \mathbf{u}))]\delta t_i^{\mathfrak{b}} \\
& - \sum_{i\in I_{r,0}(\mathbf{u})} p_i^{r,0}\delta x_i^0 + \sum_{\substack{i\in I_{r,\mathfrak{a}}(\mathbf{u}):\\ t_i^{\mathfrak{a}}\le \bar t}} p_i^{r,\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i\in I_{r,\mathfrak{b}}(\mathbf{u}):\\ t_i^{\mathfrak{b}}\le \bar t}} p_i^{r,\mathfrak{b}}\delta t_i^{\mathfrak{b}}
\end{aligned}
\tag{3.7.51}
$$

for $\delta\mathbf{u} \in \mathbf{U}$*, where* p *denotes the adjoint state with end data* $p^{\bar t} = \mathbb{1}_{\bar x}(\cdot)\frac{1}{[y(\bar t, \bar x; \bar{\mathbf{u}})]}$ *and*

$$
\begin{aligned}
p_j^{r,0} &:= \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))} \lim_{t\searrow 0} p(t, zt + x_j^0)\frac{1}{f''(f'^{-1}(z))}\,\mathrm{d}z, \quad j\in I_{r,0}(\mathbf{u}), \\
p_j^{r,\mathfrak{a}} &:= \int_{f'(u_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))} \lim_{t\searrow t_j^{\mathfrak{a}}} p(t, z(t-t_j^{\mathfrak{a}}))\frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z, \quad j\in I_{r,\mathfrak{a}}(\mathbf{u}): t_j^{\mathfrak{a}}\le \bar t, \\
p_j^{r,\mathfrak{b}} &:= \int_{f'(u_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(u_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \lim_{t\searrow t_j^{\mathfrak{b}}} p(t, z(t-t_j^{\mathfrak{b}}))\frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z, \quad j\in I_{r,\mathfrak{b}}(\mathbf{u}): t_j^{\mathfrak{b}}\le \bar t.
\end{aligned}
$$

Proof. In this proof we will first show Fréchet-differentiability of the mapping (3.7.50) in $\mathbf{u} = \bar{\mathbf{u}}$ and deduce the continuous Fréchet-differentiability from the stability of genuine characteristics and of the adjoint state (cf. [69, Proof of Lemma 6.2.7]). Let $\xi_{l/r}$ denote the minimal/maximal backward characteristic through $(\bar t, x_s(\bar{\mathbf{u}}))$. Since $(\bar t, x_s(\bar{\mathbf{u}}))$ is a shock point of class $R^c_{B,\mathfrak{a}}/C^c$, ξ_l ends in the interior of a rarefaction wave created by a discontinuity of the boundary data $u_{B,\mathfrak{a}}$ in $t = \bar t_m^{\mathfrak{a}}$ and ξ_r ends in a point $(0, z)$ where the initial data u_0 is smooth.

Let $\delta\mathbf{u} \in \mathbf{U}$ be arbitrarily chosen and denote by $\bar y := y(\bar{\mathbf{u}})$ and $y := y(\mathbf{u})$ the corresponding entropy solutions of the IBVP (3.1.2), where $\mathbf{u} := \bar{\mathbf{u}} + \delta\mathbf{u}$. Furthermore, set $\bar{u}_0 := u_0(\bar{\mathbf{u}})$, $\bar{u}_{B,\mathfrak{a}/\mathfrak{b}} := u_{B,\mathfrak{a}/\mathfrak{b}}(\bar{\mathbf{u}})$, $u_0 := u_0(\mathbf{u})$, $u_{B,\mathfrak{a}/\mathfrak{b}} := u_{B,\mathfrak{a}/\mathfrak{b}}(\mathbf{u})$ and define $\delta u_0 := u_0 - \bar{u}_0$, $\delta u_{B,\mathfrak{a}/\mathfrak{b}} := u_{B,\mathfrak{a}/\mathfrak{b}} - \bar{u}_{B,\mathfrak{a}/\mathfrak{b}}$, $\delta u_1 = u_1 - \bar{u}_1$ and $\Delta y := y - \bar{y}$.

To simplify the further steps we set $\mathfrak{a} = 0$ and denote by C a sufficiently large constant which may change its value throughout the proof. Analogously, let $\rho > 0$ denote a constant that also may change its value throughout the proof and which is always chosen small enough such that the corresponding results hold true. Similar to the proof of Lemma 6.3.7 in [69], one can use Lemma 3.7.23 to show that

$$\int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}} \Delta y(\bar{t}, x)\,\mathrm{d}x = (x_s(\mathbf{u}) - x_s(\bar{\mathbf{u}}))[y(\bar{t}, x_s(\bar{\mathbf{u}}))] + O((\bar{\varepsilon} + \|\delta\mathbf{u}\|_{\mathbf{U}})\,\|\delta\mathbf{u}\|_{\mathbf{U}}) \tag{3.7.52}$$

holds for all $\bar{\varepsilon} > 0$. The goal of the remaining part is to derive an adjoint representation for the term

$$\frac{1}{[\bar{y}(\bar{t}, x_s(\bar{\mathbf{u}}))]} \int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}} \Delta y(\bar{t}, x)\,\mathrm{d}x. \tag{3.7.53}$$

As in [69] and [70], we define

$$a(t,x) := f'(\bar{y}(t,x)), \quad b(t,x) := g_y(t,x,\bar{y}(t,x),\bar{u}_1(t,x)),$$
$$\tilde{a}(t,x) := \int_0^1 f'(\tau y(t,x) + (1-\tau)\bar{y}(t,x))\,\mathrm{d}\tau, \quad \tilde{b}(t,x) := g_y(t,x,y(t,x),u_1(t,x))$$

for all $(t,x) \in \Omega_{\bar{t}} :=]0,\bar{t}[\times]0,\mathfrak{b}[$. One can easily show that Δy is a weak solution of

$$\Delta y_t + (\tilde{a}\Delta y)_x = \tilde{b}\Delta y + g(\cdot,\bar{y},u_1) - g(\cdot,\bar{y},\bar{u}_1) \tag{3.7.54}$$

on $\Omega_{\bar{t}}$. In order to use the results of §3.6, we extend the the functions $a, \tilde{a}, b, \tilde{b}$ to the unbounded domain $[0,\bar{t}] \times \mathbb{R}$ by

$$a(t,x) = \tilde{a}(t,x) = M_{f'}, \text{ if } x < 0, \qquad (b,\tilde{b})(t,x) = (b,\tilde{b})(t,0+) \text{ if } x < 0,$$
$$a(t,x) = \tilde{a}(t,x) = -M_{f'}, \text{ if } x > \mathfrak{b}, \qquad (b,\tilde{b})(t,x) = (b,\tilde{b})(t,\mathfrak{b}-) \text{ if } x > \mathfrak{b},$$

cf. [70] and [69]. Since g_y does not depend on y by (A3), the regularity of g yields

$$\begin{aligned}&\tilde{b}, b \in L^\infty(]0,\bar{t}[; C^{0,1}(\mathbb{R})) \quad \text{and}\\ &\tilde{b} \to b \text{ in } L^\infty(]0,\bar{t}[; C(\mathbb{R})) \text{ as } \|\delta\mathbf{u}\|_{\mathbf{U}} \to 0,\end{aligned} \tag{3.7.55}$$

cf. [83]. Let $[t_1, t_2] \subset [0, T]$ be an interval such that $\bar{y}$ has no rarefaction wave creating discontinuity on $[t_1, t_2] \times [0, \mathfrak{b}]$. Then the same holds for $y(\mathbf{u})$ for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ and we obtain from Proposition 3.1.3 and Corollary 3.3.5 that

$$\begin{aligned} &\|\tilde{a}\|_{\infty,[t_1,t_2]\times\mathbb{R}}, \|a\|_{\infty,[t_1,t_2]\times\mathbb{R}} \leq M_y, \\ &\tilde{a} \to a \quad \text{in } L^1_{\text{loc}}([t_1,t_2]\times\mathbb{R}) \text{ as } \|\delta\mathbf{u}\|_{\mathbf{U}} \to 0, \\ &\tilde{a} \overset{*}{\rightharpoonup} a \quad \text{in } L^\infty([t_1,t_2]\times\mathbb{R}) \text{ as } \|\delta\mathbf{u}\|_{\mathbf{U}} \to 0. \end{aligned} \tag{3.7.56}$$

In addition, (3.2.3) together with [69, Lemma 6.3.3] assure that the coefficients $\tilde{a}$ and a satisfy on $[t_1, t_2] \times \mathbb{R}$ the OSLC (3.6.3) for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$.

Due to (ND), the points $x = x_s(\bar{\mathbf{u}}) - \bar{\varepsilon}$ and $x = x_s(\bar{\mathbf{u}}) + \bar{\varepsilon}$ are points of continuity of $\bar{y}(\bar{t}, \cdot)$ for sufficiently small $\bar{\varepsilon}$. For this reason we obtain unique genuine backward characteristics through the points $x_s(\bar{\mathbf{u}}) \mp \bar{\varepsilon}$ denoted by $\zeta_{l/r}$ and we can define the set

$$\begin{aligned} D^{\bar{\varepsilon}} := \Big(&\{(t,x) \in [0, \bar{t}_m^{\mathfrak{a}}] \times [0, \mathfrak{b}] : 0 \leq x \leq \zeta_r(t)\} \\ &\cup \{(t,x) \in [\bar{t}_m^{\mathfrak{a}}, \bar{t}] \times [0, \mathfrak{b}] : \zeta_l(t) \leq x \leq \zeta_r(t)\} \Big). \end{aligned}$$

Furthermore, the stability of backward characteristics yields

$$\begin{aligned} D := D^0 = \Big(&\{(t,x) \in [0, \bar{t}_m^{\mathfrak{a}}] \times [0, \mathfrak{b}] : 0 \leq x \leq \xi_r(t)\} \\ &\cup \{(t,x) \in [\bar{t}_m^{\mathfrak{a}}, \bar{t}] \times [0, \mathfrak{b}] : \xi_l(t) \leq x \leq \xi_r(t)\} \Big). \end{aligned} \tag{3.7.57}$$

We assume w.l.o.g. that $\bar{y}$ has outside the set $D^{\bar{\varepsilon}}$ no rarefaction center. Since for the subsequent analysis the points outside the set $D^{\bar{\varepsilon}}$ do not play any role, this is not a restriction. Therefore, let the rarefaction centers on the domain $[0, \bar{t}] \times \Omega$ be given by

$$\mathcal{R} := \left\{ \bar{t}_{j_1}^{\mathfrak{a}}, \ldots, \bar{t}_{j_{N_\mathfrak{a}}}^{\mathfrak{a}} \right\} = \{ \bar{t}_i^{\mathfrak{a}} : i \in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}) \},$$

where the constant $s \in]0, \varepsilon_g[$ is chosen sufficiently small. For each $t_{j_i}^{\mathfrak{a}} \in \mathcal{R}$, we introduce time points $\hat{t}_i < \bar{t}_{j_i}^{\mathfrak{a}} < \tilde{t}_i$ that are sufficiently close to $\bar{t}_{j_i}^{\mathfrak{a}}$, respectively, cf. Figure 3.1. In particular, we require that the points $\tilde{t}_k$ are sufficiently small such that

$$\tilde{x} := M_{f'}(\tilde{t}_k - \bar{t}_{j_k}^{\mathfrak{a}}) < \frac{\varepsilon_g}{2} \quad \text{and} \quad \bar{y}(\tilde{t}_k, \cdot) \in C^{0,1}([0, 2\tilde{x}]) \tag{3.7.58}$$

holds for all $k = 1, \dots, N_{\mathfrak{a}}$. Moreover, we assume w.l.o.g. that

$$D^{\bar{\varepsilon}} \cap D_- = \emptyset. \tag{3.7.59}$$

This assumption is for simplicity and the treatment of the more general case can be found in [70, Proof of Lem. 4.13.], see also [69].

Before giving a short overview of the main steps of the proof, we first define the following subdomains

$$\begin{aligned}
&D^1_{N_{\mathfrak{a}},\bar{\varepsilon}} := D^{\bar{\varepsilon}} \cap \left([\tilde{t}_{N_{\mathfrak{a}}}, \bar{t}] \times \mathbb{R}\right), \quad D^1_{N_{\mathfrak{a}}} := D \cap \left([\tilde{t}_{N_{\mathfrak{a}}}, \bar{t}] \times \mathbb{R}\right)\\
&D^1_k := D \cap \left([\tilde{t}_k, \hat{t}_{k+1}] \times \mathbb{R}\right), \quad k = 1, \dots, N_{\mathfrak{a}} - 1\\
&D^2_k := D \cap \left([\hat{t}_k, \tilde{t}_k] \times \mathbb{R}\right), \qquad k = 1, \dots, N_{\mathfrak{a}}\\
&D^1_0 := D \cap \left([s, \hat{t}_1] \times \mathbb{R}\right),\\
&D^2_0 := D \cap ([0, s] \times \mathbb{R}),
\end{aligned} \tag{3.7.60}$$

see also Figure 3.1.

We note that $\bar{y}$ has no rarefaction center on $D^1_{N_{\mathfrak{a}},\bar{\varepsilon}}$ and the sets D^1_k, $k = 0, \dots N_{\mathfrak{a}}$, such that the coefficients $\tilde{a}$ and a satisfy (3.7.55), (3.7.56) and the OSLC (3.6.3) on $[\tilde{t}_k, \hat{t}_{k+1}] \times \mathbb{R}$, $k = 0, \dots, N_{\mathfrak{a}} + 1$, where $\tilde{t}_0 = s$ and $\hat{t}_{N_{\mathfrak{a}}+1} = \bar{t}$. Hence, if we choose

$$\begin{aligned}
p^{\hat{t}_{N_{\mathfrak{a}}+1}} = p^{\bar{t}} &= \frac{1}{[\bar{y}(\bar{t}, x_s(\bar{\mathbf{u}}))]} \in C^{0,1}(\mathbb{R}),\\
p^{\hat{t}_{k+1}} &= p^r_{k+1}(\hat{t}_{k+1}, \cdot) \in C^{0,1}(\mathbb{R}), \quad \text{for } k = 0, \dots, N_{\mathfrak{a}} - 1
\end{aligned}$$

as end data, where the functions p^r_k for $k = 1, \dots, N_{\mathfrak{a}}$ are defined below, then Theorem 3.6.7 guarantees that for all $k = 0, \dots, N_{\mathfrak{a}}$ the equations

$$p_t + \tilde{a} p_x = -\tilde{b} p \quad \text{on }]\tilde{t}_k, \hat{t}_{k+1}[\times\mathbb{R}, \qquad p(\hat{t}_{k+1}, \cdot) = p^{\hat{t}_{k+1}}(\cdot) \quad \text{on } \mathbb{R} \tag{3.7.61}$$

$$\text{and} \quad p_t + a p_x = -b p \quad \text{on }]\tilde{t}_k, \hat{t}_{k+1}[\times\mathbb{R}, \qquad p(\hat{t}_{k+1}, \cdot) = p^{\hat{t}_{k+1}}(\cdot) \quad \text{on } \mathbb{R} \tag{3.7.62}$$

admit reversible solutions

$$\begin{aligned}
\tilde{p}^1_k &\in C^{0,1}([\tilde{t}_k, \hat{t}_{k+1}] \times \mathbb{R}),\\
p^1_k &\in C^{0,1}([\tilde{t}_k, \hat{t}_{k+1}] \times \mathbb{R}).
\end{aligned} \tag{3.7.63}$$

Furthermore, from Theorem 3.6.9 we obtain that

$$\tilde{p}^1_k \to p^1_k \ \text{ in } C([\tilde{t}_k, \hat{t}_{k+1}] \times [-R, R]) \quad \forall k = 0, \dots, N_{\mathfrak{a}} \tag{3.7.64}$$

holds true for all $R > 0$.

We note that the subdomains D_k^2 contain for all $k = 1, \dots, N_{\mathfrak{a}}$ exactly one rarefaction center such that the OSLC (3.6.3) is violated for the coefficients $\tilde{a}$ and a. Therefore, we cut out the rarefaction wave, respectively, and obtain for each $k = 1, \dots, N_{\mathfrak{a}}$ one subset of D_k^2 on the left-hand side of the rarefaction wave and one of its right-hand side. These subsets are given by

$$\begin{aligned} D_{\varepsilon,k}^l &:= \left\{(t,x) \in D_k^2 : x \le f'\left(\bar{u}_{j_k+1}^{B,\mathfrak{a}}(\bar{t}_{j_k}^{\mathfrak{a}} + \varepsilon)\right)(t - \bar{t}_{j_k}^{\mathfrak{a}} - \varepsilon)\right\}, \quad k = 1, \dots, N_{\mathfrak{a}} \\ D_{\varepsilon,k}^r &:= \left\{(t,x) \in D_k^2 : x \ge f'\left(\bar{u}_{j_k}^{B,\mathfrak{a}}(\bar{t}_{j_k}^{\mathfrak{a}} - \varepsilon)\right)(t - \bar{t}_{j_k}^{\mathfrak{a}} + \varepsilon)\right\}, \quad k = 1, \dots, N_{\mathfrak{a}}, \end{aligned} \tag{3.7.65}$$

where ε is a function depending on $\delta\mathbf{u}$ that satisfies

$$\varepsilon(\delta\mathbf{u}) \ge 2\,\|\delta\mathbf{u}\|_{\mathbf{U}} \quad \text{and} \quad \varepsilon(\delta\mathbf{u}) \to 0 \ \text{ if } \ \|\delta\mathbf{u}\|_{\mathbf{U}} \to 0. \tag{3.7.66}$$

Since the integration by parts will only be carried out on the sets defined in (3.7.65), we replace $\tilde{a}$ and a by coefficients which satisfy (3.7.56) as well as the OSLC (3.6.3) and coincide with $\tilde{a}$ and a on the sets in (3.7.65). Such coefficients are given by

$$\begin{aligned} \tilde{a}_k^l(t,x) &= -M_{f'}, \text{ if } (t,x) \in [\hat{t}_k, \tilde{t}_k] \times \Omega \setminus D_{\varepsilon,k}^l, \quad \tilde{a}_k^l(t,x) = \tilde{a}(t,x), \text{ else} \\ a_k^l(t,x) &= -M_{f'}, \text{ if } (t,x) \in [\hat{t}_k, \tilde{t}_k] \times \Omega \setminus D_{0,k}^l, \quad a_k^l(t,x) = a(t,x), \text{ else} \end{aligned}$$

and

$$\begin{aligned} \tilde{a}_k^r(t,x) &= M_{f'}, \text{ if } (t,x) \in D_k^2 \setminus D_{\varepsilon,k}^r, \quad \tilde{a}_k^r(t,x) = \tilde{a}(t,x), \text{ else} \\ a_k^r(t,x) &= M_{f'}, \text{ if } (t,x) \in D_k^2 \setminus D_{0,k}^r, \quad a_k^r(t,x) = a(t,x), \text{ else.} \end{aligned}$$

In what follows, we require that the function $\varepsilon(\delta\mathbf{u})$, which is defined in (3.7.66), is chosen such that

$$D_{\varepsilon,k}^{l/r} \subset D_{\frac{\varepsilon}{2},k}^{l/r} \subset D_{0,k}^{l/r} \quad \text{for all } \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}). \tag{3.7.67}$$

Due to their construction, for all $k = 1, \dots, N_{\mathfrak{a}}$ the new coefficients $\tilde{a}_k^{l/r}$ and $a_k^{l/r}$ satisfy (3.7.56) and the OSLC (3.6.3) on $[\hat{t}_k, \tilde{t}_k] \times \mathbb{R}$. Therefore, and due to (3.7.55), setting

$$p^{\tilde{t}_k} = p_k^1(\tilde{t}_k, \cdot) \in C^{0,1}(\mathbb{R}), \quad k = 1, \dots, N_{\mathfrak{a}}$$

as end data, Theorem 3.6.7 guarantees that

$$p_t + \tilde{a}_k^{l/r} p_x = -\tilde{b}p \qquad \text{on }]\hat{t}_k, \tilde{t}_k[\times\mathbb{R}, \qquad p(\tilde{t}_k, \cdot) = p^{\tilde{t}_k} \qquad \text{on } \mathbb{R} \tag{3.7.68}$$

$$\text{and } p_t + a_k^{l/r} p_x = -bp \qquad \text{on }]\hat{t}_k, \tilde{t}_k[\times\mathbb{R}, \qquad p(\tilde{t}_k, \cdot) = p^{\tilde{t}_k} \qquad \text{on } \mathbb{R} \tag{3.7.69}$$

admit for all $k = 1, \ldots, N_\mathfrak{a}$ reversible solutions

$$\begin{aligned} \tilde{p}_k^{l/r} &\in C^{0,1}([\hat{t}_k, \tilde{t}_k] \times \mathbb{R}), \\ p_k^{l/r} &\in C^{0,1}([\hat{t}_k, \tilde{t}_k] \times \mathbb{R}). \end{aligned} \tag{3.7.70}$$

Using Theorem 3.6.9 we further get that

$$\tilde{p}_k^{l/r} \to p_k^{l/r} \;\text{ in } C([\hat{t}_k, \tilde{t}_k] \times [-R, R]) \quad \forall k = 1, \ldots, N_\mathfrak{a} \tag{3.7.71}$$

holds for all $R > 0$. Moreover, (3.7.67) guarantees that

$$\tilde{a}_k^{l/r}|_{D_{\varepsilon,k}^{l/r}} \equiv \tilde{a} \quad \text{and} \quad a_k^{l/r}|_{D_{\varepsilon,k}^{l/r}} \equiv a \quad \text{for } k = 1, \ldots, N_\mathfrak{a}. \tag{3.7.72}$$

Moreover, according to [69, Lemma 6.3.5 (i)], it holds that

$$\begin{aligned} p_{N_\mathfrak{a}}^1|_{D_k^1} &= p \quad \forall k = 1, \ldots, N_\mathfrak{a}, \\ p_{N_\mathfrak{a}}^{l/r}|_{D_{0,k}^{l/r}} &= p \quad \forall k = 1, \ldots, N_\mathfrak{a}. \end{aligned} \tag{3.7.73}$$

We note that starting with end data $p^{\bar{t}} = \frac{1}{[\bar{y}(\bar{t}, x_s(\bar{\mathbf{u}}))]} \in C^{0,1}(\mathbb{R})$ ensures that $p^{\tilde{t}_{N_\mathfrak{a}}} \in C^{0,1}(\mathbb{R})$ and hence $p^{\hat{t}_{k+1}} \in C^{0,1}(\mathbb{R})$ for all $k = 0, \ldots, N_\mathfrak{a} - 1$ and $p^{\tilde{t}_k} \in C^{0,1}(\mathbb{R})$ for all $k = 0, \ldots, N_\mathfrak{a}$.

The proof consists of five main steps, cf. Figure 3.1. In **step 1** we consider (3.7.54) on $[\tilde{t}_{N_\mathfrak{a}}, \bar{t}] \times \Omega$, multiply it by $\tilde{p}_{N_\mathfrak{a}}^1$ and carry out integration by parts on $D_{N_\mathfrak{a},\bar{\varepsilon}}^1$. Then (3.7.64) yields that (3.7.53) can be rewritten in terms of the adjoint state, g and two integral terms $I_{N_\mathfrak{a},\varepsilon}^1$ and $I_{N_\mathfrak{a},\varepsilon}^2$ whose simplification will be done in step 2 and 3.

In **step 2**, we derive a representation of $I_{N_\mathfrak{a},\varepsilon}^1$ depending on the local solution near the rarefaction center and the adjoint state.

In **step 3**, we consider (3.7.54) on $[\tilde{t}_{N_\mathfrak{a}}, \bar{t}] \times \Omega$ and multiply it by $\tilde{p}_{N_\mathfrak{a}}^r$. Using integration by parts and (3.7.71), we obtain a representation of $I_{N_\mathfrak{a},\varepsilon}^2$ depending, inter alia, on an integral term $I_{\hat{t},N_\mathfrak{a}}$ which will be further simplified in the next step.

In **step 4**, we proceed as in step 1: Consider (3.7.54) on $[\tilde{t}_{N_\mathfrak{a}}, \bar{t}] \times \Omega$, multiply it by $\tilde{p}_{N_\mathfrak{a}-1}^1$ and use integration by parts on $D_{N_\mathfrak{a}-1}^1$. Then we use use (3.7.64) and obtain that $I_{\hat{t},N_\mathfrak{a}}$ can be rewritten in terms of the adjoint state, g and three integral terms $I_{N_\mathfrak{a}-1,\varepsilon}^1$, $I_{N_\mathfrak{a}-1,\varepsilon}^2$ and $I_{N_\mathfrak{a}-1,\varepsilon}^3$. These terms can again be reformulated by using the same techniques as in step 2 and 3. We continue this procedure until $k = 1$ yielding a presentation of the term $I_{\hat{t},N_\mathfrak{a}}$ which depends inter alia on an integral term $I_{\hat{t},1}$, which will be simplified in the last step.

In **step 5**, we consider (3.7.54) on $[s, \hat{t}_1] \times \Omega$, multiply it by $\tilde{p}_0^1$, apply integration by parts on D_0^1 and use (3.7.64). This yields a representation of $I_{\hat{t},1}$ depending inter alia on terms $I_{0,\varepsilon}^k$ for $k = 1, \ldots, N_0 + 1$ and $I_{r,\varepsilon}^j$ for $j = 1, \ldots, N_0$, where $N_0 = |I_{r,0}|$. These terms can be treated by using the methods of step 2 and 3. Finally, reinserting the terms and using (3.7.73) yields an adjoint representation for (3.7.53).

The idea of this proof is based on the proof of Lemma 7.3.4 in [69]. The procedure in step 1 and 4 can be found in [70, Proof of Lemma 4.10].

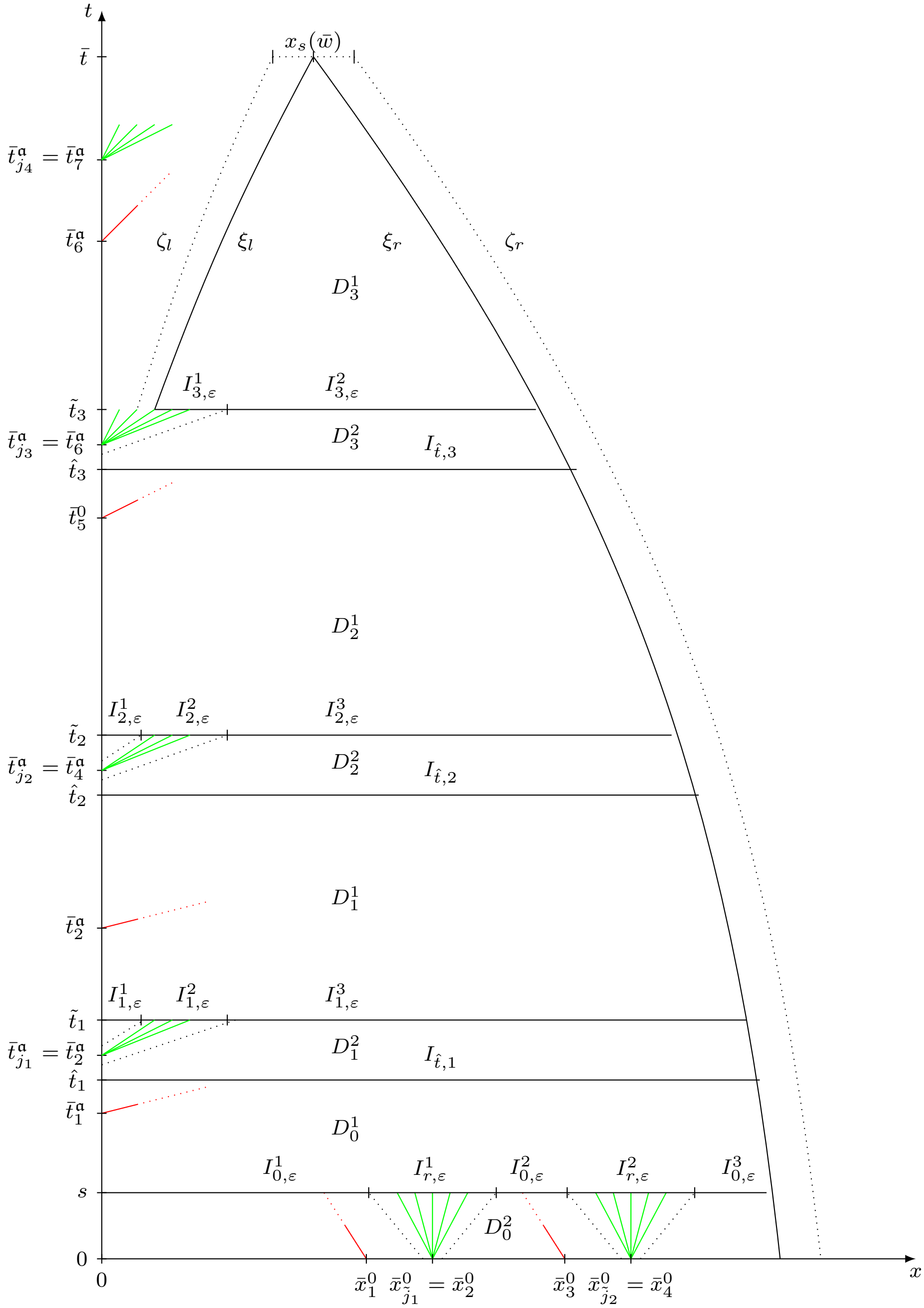

Figure 3.1. Proof of Theorem 3.7.24

Step 1: Considering (3.7.54) on $]\tilde{t}_{N_\mathfrak{a}}, \bar{t}[\times\Omega$, multiplying it with $\tilde{p}^1_{N_\mathfrak{a}}$, which denotes the reversible solution of (3.7.61), and applying integration by parts on the domain $D^1_{N_\mathfrak{a},\bar{\varepsilon}}$ gives

$$\begin{aligned}
&\frac{1}{[\bar{y}(\bar{t}, x_s(\bar{\mathbf{u}}))]}\int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}} \Delta y(\bar{t}, x)\,\mathrm{d}x \\
&= \int_{\zeta_l(\tilde{t}_{N_\mathfrak{a}})}^{\zeta_r(\tilde{t}_{N_\mathfrak{a}})} \tilde{p}^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}}, x)\Delta y(\tilde{t}_{N_\mathfrak{a}}, x)\,\mathrm{d}x \\
&\quad + \int_{D^1_{N_\mathfrak{a},\bar{\varepsilon}}} \tilde{p}^1_{N_\mathfrak{a}}(t,x)\big(g(t,x,\bar{y},u_1) - g(t,x,\bar{y},\bar{u}_1)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\tilde{t}_{N_\mathfrak{a}}}^{\bar{t}} \tilde{p}^1_{N_\mathfrak{a}}(t,\zeta_l(t))\Delta y(t,\zeta_l(t))(\tilde{a}(t,\zeta_l(t)) - a(t,\zeta_l(t)))\,\mathrm{d}t \\
&\quad + \int_{\tilde{t}_{N_\mathfrak{a}}}^{\bar{t}} \tilde{p}^1_{N_\mathfrak{a}}(t,\zeta_r(t))\Delta y(t,\zeta_r(t))(\tilde{a}(t,\zeta_r(t)) - a(t,\zeta_r(t)))\,\mathrm{d}t,
\end{aligned} \tag{3.7.74}$$

where we have used that $\tilde{p}^1_{N_\mathfrak{a}}$ solves (3.7.61) almost everywhere on $]\tilde{t}_{N_\mathfrak{a}}, \bar{t}[\times\Omega$. Using the regularity of g w.r.t. u_1 and (3.7.64), the second term on the right hand side of (3.7.74) can be reformulated

$$\begin{aligned}
&\int_{D^1_{N_\mathfrak{a},\bar{\varepsilon}}} \tilde{p}^1_{N_\mathfrak{a}}(t,x)\big(g(t,x,\bar{y},u_1) - g(t,x,\bar{y},\bar{u}_1)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{D^1_{N_\mathfrak{a},\bar{\varepsilon}}} p^1_{N_\mathfrak{a}}(t,x) g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t + o(\|\delta\mathbf{u}\|_{\mathbf{U}}) \\
&= \int_{D^1_{N_\mathfrak{a},\bar{\varepsilon}}\setminus D^1_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}}(t,x) g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{D^1_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}}(t,x) g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t + o(\|\delta\mathbf{u}\|_{\mathbf{U}}),
\end{aligned}$$

where $p^1_{N_\mathfrak{a}}$ denotes the reversible solution of (3.7.62) and the term $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$ is uniform w.r.t. $\bar{\varepsilon} > 0$. Using Lemma 3.2.5, we deduce

$$\|\xi_{l/r}(\cdot) - \zeta_{l/r}(\cdot)\|_{C([t_{N_\mathfrak{a}},\bar{t}])} \leq C\bar{\varepsilon}, \tag{3.7.75}$$

which yields together with the boundedness of $g_{u_1}(\cdot,\bar{y},\bar{u}_1)$ and $p^1_{N_\mathfrak{a}}$

$$\int_{D^1_{N_\mathfrak{a},\bar{\varepsilon}}\setminus D^1_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}}(t,x) g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t = \bar{\varepsilon}O(\|\delta\mathbf{u}\|_{\mathbf{U}})$$

and thus

$$\begin{aligned}
&\int_{D^1_{N_\mathfrak{a},\bar\varepsilon}} \tilde p^1_{N_\mathfrak{a}}(t,x)\big(g(t,x,\bar y,u_1)-g(t,x,\bar y,\bar u_1)\big)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{D^1_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}}(t,x)g_{u_1}(t,x,\bar y,\bar u_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\bar\varepsilon O(\|\delta\mathbf{u}\|_{\mathbf{U}})+o(\|\delta\mathbf{u}\|_{\mathbf{U}}).
\end{aligned}\tag{3.7.76}$$

Since $(\bar t,\bar x)$ is a shock point of class $R^c_{B,\mathfrak{a}}/C^c$, Lemma 3.7.15 and Lemma 3.7.10 guarantee smooth local solutions $\mathbf{u}\mapsto Y_{l/r}(\cdot,\mathbf{u})$ which are defined on some stripes $S_{l/r}$ containing $\xi_{l/r}$ and $\zeta_{l/r}$ for sufficiently small $\bar\varepsilon$. With the local solutions, the definitions of $\tilde a$ and a and the uniform boundedness of $\tilde p^1_{N_\mathfrak{a}}$, the last two integrals on the right hand side of (3.7.74) are equal to $O(\|\delta\mathbf{u}\|^2_{\mathbf{U}})$. In fact, using the definition of $\tilde a$ and a, we first note that the last two integrals of (3.7.74) can be rewritten by

$$\begin{aligned}
&-\int_{\tilde t_{N_\mathfrak{a}}}^{\bar t} \tilde p^1_{N_\mathfrak{a}}(t,\zeta_l(t))\Delta y(t,\zeta_l(t))\big(\tilde a(t,\zeta_l(t))-a(t,\zeta_l(t))\big)\,\mathrm{d}t\\
&+\int_{\tilde t_{N_\mathfrak{a}}}^{\bar t} \tilde p^1_{N_\mathfrak{a}}(t,\zeta_r(t))\Delta y(t,\zeta_r(t))\big(\tilde a(t,\zeta_r(t))-a(t,\zeta_r(t))\big)\,\mathrm{d}t\\
&\quad=-\int_{\tilde t_{N_\mathfrak{a}}}^{\bar t} \tilde p^1_{N_\mathfrak{a}}(t,\zeta_l(t))\big(f(y(t,\zeta_l(t)))-f(\bar y(t,\zeta_l(t)))-f'(\bar y(t,\zeta_l(t))\big)\Delta y(t,\zeta_r(t)))\,\mathrm{d}t\\
&\qquad+\int_{\tilde t_{N_\mathfrak{a}}}^{\bar t} \tilde p^1_{N_\mathfrak{a}}\big(t,\zeta_l(t))(f(y(t,\zeta_r(t)))-f(\bar y(t,\zeta_r(t)))-f'(\bar y(t,\zeta_r(t))\big)\Delta y(t,\zeta_r(t)))\,\mathrm{d}t
\end{aligned}\tag{3.7.77}$$

Using (3.7.64) and the regularity of $\tilde p^1_{N_\mathfrak{a}}$, we conclude

$$\big\|\tilde p^1_{N_\mathfrak{a}}\big\|_{\infty,[\tilde t_{N_\mathfrak{a}},\bar t]\times\mathbb{R}}<M_{\tilde p^1_{N_\mathfrak{a}}}\quad\forall\,\delta\mathbf{u}\in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}).$$

Moreover, the regularity of f and Proposition 3.1.3 yield $\|f''(y(\mathbf{u}))\|_{\Omega_{\bar t},\infty}\le M_{f''}$ for all $\delta\mathbf{u}\in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$. Therefore, the absolute value of the terms in (3.7.77) is bounded from above by

$$\begin{aligned}
&M_{f''}M_{\tilde p^1_{N_\mathfrak{a}}}\left(\int_{\tilde t_{N_\mathfrak{a}}}^{\bar t}|\Delta y(t,\zeta_l(t))|^2\,\mathrm{d}t+\int_{\tilde t_{N_\mathfrak{a}}}^{\bar t}|\Delta y(t,\zeta_r(t))|^2\,\mathrm{d}t\right)\\
&\le M_{f''}M_{\tilde p^1_{N_\mathfrak{a}}}\cdot(\bar t-\tilde t_{N_\mathfrak{a}})\Big(\|Y_l(\cdot,\mathbf{u})-Y_l(\cdot,\bar{\mathbf{u}})\|^2_{C(S_l)}+\|Y_r(\cdot,\mathbf{u})-Y_r(\cdot,\bar{\mathbf{u}})\|^2_{C(S_r)}\Big)\\
&\le C\,\|\delta\mathbf{u}\|^2_{\mathbf{U}}\,.
\end{aligned}$$

The last inequality holds due to Lemma 3.7.15 and Lemma 3.7.10. This estimation guarantees

$$\begin{aligned}&-\int_{\tilde{t}_{N_{\mathfrak{a}}}}^{\bar{t}} \tilde{p}^1_{N_{\mathfrak{a}}}(t,\zeta_l(t))\Delta y(t,\zeta_l(t))(\tilde{a}(t,\zeta_l(t)) - a(t,\zeta_l(t)))\,\mathrm{d}t\\ &+\int_{\tilde{t}_{N_{\mathfrak{a}}}}^{\bar{t}} \tilde{p}^1_{N_{\mathfrak{a}}}(t,\zeta_r(t))\Delta y(t,\zeta_r(t))(\tilde{a}(t,\zeta_r(t)) - a(t,\zeta_r(t)))\,\mathrm{d}t\\ &= O(\|\delta\mathbf{u}\|^2_{\mathbf{U}})\end{aligned} \tag{3.7.78}$$

Here, the term $O(\|\delta\mathbf{u}\|^2_{\mathbf{U}})$ is uniform w.r.t. $\bar{\varepsilon}$.

Next, we want to rewrite the first integral on the right hand side of (3.7.74). To this end, we first obtain from (3.7.64) and Corollary 3.3.5 that

$$\begin{aligned}&\left|\int_{\zeta_l(\tilde{t}_{N_{\mathfrak{a}}})}^{\zeta_r(\tilde{t}_{N_{\mathfrak{a}}})} (\tilde{p}^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x) - p^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x))\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x\right|\\ &\le \left\|\tilde{p}^1_{N_{\mathfrak{a}}} - p^1_{N_{\mathfrak{a}}}\right\|_{\infty,[\tilde{t}_{N_{\mathfrak{a}}},\bar{t}]\times\mathbb{R}} \cdot \|\Delta y\|_{1,[\tilde{t}_{N_{\mathfrak{a}}},\bar{t}]\times\Omega} = o(\|\delta\mathbf{u}\|_{\mathbf{U}})\end{aligned}$$

holds which leads to

$$\begin{aligned}&\int_{\zeta_l(\tilde{t}_{N_{\mathfrak{a}}})}^{\zeta_r(\tilde{t}_{N_{\mathfrak{a}}})} \tilde{p}^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x)\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x\\ &= \int_{\zeta_l(\tilde{t}_{N_{\mathfrak{a}}})}^{\zeta_r(\tilde{t}_{N_{\mathfrak{a}}})} p^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x)\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x + o(\|\delta\mathbf{u}\|_{\mathbf{U}}),\end{aligned} \tag{3.7.79}$$

where $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$ is uniform w.r.t. $\bar{\varepsilon}$.

Furthermore, we get

$$\begin{aligned}&\int_{\zeta_l(\tilde{t}_{N_{\mathfrak{a}}})}^{\zeta_r(\tilde{t}_{N_{\mathfrak{a}}})} p^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x)\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x\\ &= \int_{\zeta_l(\tilde{t}_{N_{\mathfrak{a}}})}^{\xi_l(\tilde{t}_{N_{\mathfrak{a}}})} p^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x)\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x\\ &\quad+ \int_{\xi_l(\tilde{t}_{N_{\mathfrak{a}}})}^{\xi_r(\tilde{t}_{N_{\mathfrak{a}}})} p^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x)\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x\\ &\quad+ \int_{\xi_r(\tilde{t}_{N_{\mathfrak{a}}})}^{\zeta_r(\tilde{t}_{N_{\mathfrak{a}}})} p^1_{N_{\mathfrak{a}}}(\tilde{t}_{N_{\mathfrak{a}}},x)\Delta y(\tilde{t}_{N_{\mathfrak{a}}},x)\,\mathrm{d}x.\end{aligned} \tag{3.7.80}$$

Recalling the local solutions $Y_{l/r}$ which are used to prove (3.7.78) and the boundedness of $p^1_{N_\mathfrak{a}}$, we obtain the following estimation

$$\begin{aligned}
&\left|\int_{\zeta_l(\tilde{t}_{N_\mathfrak{a}})}^{\xi_l(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x\right| \\
&\quad+\left|\int_{\xi_r(\tilde{t}_{N_\mathfrak{a}})}^{\zeta_r(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x\right| \\
&\le M_{p^1_{N_\mathfrak{a}}}\left\|Y_l(\cdot,\mathbf{u})-Y_l(\cdot,\bar{\mathbf{u}})\right\|_{C(S_l)}\cdot\left|\xi_l(\tilde{t}_{N_\mathfrak{a}})-\zeta_l(\tilde{t}_{N_\mathfrak{a}})\right| \\
&\quad+M_{p^1_{N_\mathfrak{a}}}\left\|Y_r(\cdot,\mathbf{u})-Y_r(\cdot,\bar{\mathbf{u}})\right\|_{C(S_r)}\cdot\left|\xi_r(\tilde{t}_{N_\mathfrak{a}})-\zeta_r(\tilde{t}_{N_\mathfrak{a}})\right| \\
&\le\left(|\xi_l(\tilde{t}_{N_\mathfrak{a}})-\zeta_l(\tilde{t}_{N_\mathfrak{a}})|+|\xi_r(\tilde{t}_{N_\mathfrak{a}})-\zeta_r(\tilde{t}_{N_\mathfrak{a}})|\right)\cdot C\left\|\delta\mathbf{u}\right\|_{\mathbf{U}},
\end{aligned}\tag{3.7.81}$$

where the last inequality is justified by Lemma 3.7.15 and Lemma 3.7.10. From (3.7.81) and (3.7.75) we deduce

$$\begin{aligned}
&\left|\int_{\zeta_l(\tilde{t}_{N_\mathfrak{a}})}^{\xi_l(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x\right|+\left|\int_{\xi_r(\tilde{t}_{N_\mathfrak{a}})}^{\zeta_r(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x\right| \\
&\le\bar{\varepsilon}O(\|\delta\mathbf{u}\|_{\mathbf{U}}).
\end{aligned}\tag{3.7.82}$$

Inserting (3.7.82) in (3.7.80) gives the following reformulation of (3.7.79):

$$\begin{aligned}
&\int_{\zeta_l(\tilde{t}_{N_\mathfrak{a}})}^{\zeta_r(\tilde{t}_{N_\mathfrak{a}})} \tilde{p}^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \\
&=\int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{\xi_r(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x+\bar{\varepsilon}O(\|\delta\mathbf{u}\|_{\mathbf{U}})+o(\|\delta\mathbf{u}\|_{\mathbf{U}})
\end{aligned}\tag{3.7.83}$$

Furthermore, inserting (3.7.76), (3.7.78) and (3.7.83) in (3.7.53) leads to

$$\begin{aligned}
&\frac{1}{[\bar{y}(\bar{t},x_s(\bar{\mathbf{u}}))]}\int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}} \Delta y(\bar{t},x)\,\mathrm{d}x \\
&=\int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{\xi_r(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \\
&\quad+\int_{D^1_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}}(t,x)g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t+o(\|\delta\mathbf{u}\|_{\mathbf{U}})+\bar{\varepsilon}O(\|\delta\mathbf{u}\|_{\mathbf{U}}).
\end{aligned}\tag{3.7.84}$$

For the first term of the right hand side of (3.7.84) we write

$$\begin{aligned}
&\int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{\xi_r(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \\
&= \int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}-\varepsilon)\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}+\varepsilon)} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \\
&\quad + \int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}-\varepsilon)\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}+\varepsilon)}^{\xi_r(\tilde{t}_{N_\mathfrak{a}})} p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \\
&= I^1_{N_\mathfrak{a},\varepsilon} + I^2_{N_\mathfrak{a},\varepsilon},
\end{aligned} \tag{3.7.85}$$

where ε is given according to (3.7.66). Using (3.7.85) and (3.7.73), we obtain that (3.7.84) reads

$$\begin{aligned}
&\frac{1}{[y(\bar{t},x_s(\bar{\mathbf{u}}))]}\int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}} \Delta y(\bar{t},x)\,\mathrm{d}x \\
&= I^1_{\bar{t},\varepsilon} + I^2_{N_\mathfrak{a},\varepsilon} + \int_{D^1_{N_\mathfrak{a}}} p(t,x) g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + o(\|\delta\mathbf{u}\|_{\mathbf{U}}) + \bar{\varepsilon} O(\|\delta\mathbf{u}\|_{\mathbf{U}}).
\end{aligned}$$

Step 2: First we have a closer look at $I^1_{\bar{t},\varepsilon}$. Since the minimal backward characteristic through $(\bar{t},\bar{x})$ ends in the inner of a rarefaction wave, we obtain from Lemma 3.7.16 and (3.7.58) that for all $\mathbf{u} \in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$ the corresponding entropy solution $y(\tilde{t}_{N_\mathfrak{a}},\cdot;\mathbf{u})$ is locally given by

$$y(\tilde{t}_{N_\mathfrak{a}},x;\mathbf{u}) = \begin{cases} f'^{-1}\left(\frac{x}{\tilde{t}_{N_\mathfrak{a}}-t_m^\mathfrak{a}}\right) & \text{if } x \in I_l, \\ Y(\tilde{t}_{N_\mathfrak{a}},x,\mathbf{u}) & \text{if } x \in I_r, \end{cases} \tag{3.7.86}$$

where

$$\begin{aligned}
I_l &:=]f'(u^{B,\mathfrak{a}}_{m+1}(t^\mathfrak{a}_m))\cdot(\tilde{t}_{N_\mathfrak{a}}-t^\mathfrak{a}_m), f'(u^{B,\mathfrak{a}}_m(t^\mathfrak{a}_m))\cdot(\tilde{t}_{N_\mathfrak{a}}-t^\mathfrak{a}_m)[\\
I_r &:= [f'(u^{B,\mathfrak{a}}_m(t^\mathfrak{a}_m))\cdot(\tilde{t}_{N_\mathfrak{a}}-t^\mathfrak{a}_m), f'(\bar{u}^{B,\mathfrak{a}}_m(\bar{t}^\mathfrak{a}_m))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}^\mathfrak{a}_m)+\delta[
\end{aligned}$$

and $\delta > 0$ is sufficiently small. Here, the continuously Fréchet-differentiable function

$$B^{\mathbf{U}}_\rho(\bar{\mathbf{u}}) \ni \mathbf{u} \mapsto Y(\tilde{t}_{N_\mathfrak{a}},\cdot;\mathbf{u}) \in C(I)$$

is obtained by Lemma 3.7.16 with

$$I =]f'(\bar{u}^{B,\mathfrak{a}}_m(\bar{t}^\mathfrak{a}_m))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}^\mathfrak{a}_m)-2\delta, f'(\bar{u}^{B,\mathfrak{a}}_m(\bar{t}^\mathfrak{a}_m))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}^\mathfrak{a}_m)+2\delta[.$$

In addition, we note that $y(\tilde{t}_{N_\mathfrak{a}},x)$ is continuous on I for all $\mathbf{u}\in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$.

A Taylor expansion of the term $f'(u_m^{B,\mathfrak{a}}(t_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-t_m^\mathfrak{a})$ in $u_m^{B,\mathfrak{a}}=\bar{u}_m^{B,\mathfrak{a}}$ and $t_m^\mathfrak{a}=\bar{t}_m^\mathfrak{a}$ yields the existence of a constant $C>0$ such that

$$\begin{aligned}&\xi_l(\tilde{t}_{N_\mathfrak{a}})<f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\left\|\delta\mathbf{u}\right\|_{\mathbf{U}}\quad\text{and}\\&[\xi_l(\tilde{t}_{N_\mathfrak{a}}),f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\left\|\delta\mathbf{u}\right\|_{\mathbf{U}}]\\&\quad\subset[\xi_l(\tilde{t}_{N_\mathfrak{a}}),f'(u_m^{B,\mathfrak{a}}(t_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-t_m^\mathfrak{a})]\end{aligned}\tag{3.7.87}$$

is valid for all $\mathbf{u}\in B^{\mathbf{U}}_\rho(\bar{\mathbf{u}})$. Using (3.7.86) and (3.7.87), we derive the following local approximation of the term $\Delta y(\tilde{t}_{N_\mathfrak{a}},\cdot)$ in $I^1_{\tilde{t},\varepsilon}$:

$$\begin{aligned}&\Delta y(\tilde{t}_{N_\mathfrak{a}},x)|_{[\xi_l(\tilde{t}_{N_\mathfrak{a}}),f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\|\delta\mathbf{u}\|_{\mathbf{U}}]}\\&=\frac{x\cdot\delta t_m^\mathfrak{a}}{f''\left(f'^{-1}\left(\frac{x}{\tilde{t}-\bar{t}_m^\mathfrak{a}}\right)\right)\cdot(\tilde{t}-\bar{t}_m^\mathfrak{a})^2}+o(\|\delta\mathbf{u}\|_{\mathbf{U}}).\end{aligned}$$

Using this local approximation, the term $I^1_{\tilde{t},\varepsilon}$ in (3.7.85) can be rewritten as follows:

$$\begin{aligned}I^1_{\tilde{t},\varepsilon}=&\int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\|\delta\mathbf{u}\|_{\mathbf{U}}}p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\frac{x\cdot\delta t_m^\mathfrak{a}}{f''\left(f'^{-1}\left(\frac{x}{\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}}\right)\right)(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})^2}\,\mathrm{d}x\\&+\int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\|\delta\mathbf{u}\|_{\mathbf{U}}}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}-\varepsilon))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}+\varepsilon)}p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x+o(\|\delta\mathbf{u}\|_{\mathbf{U}})\\=&\int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})}p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\frac{x\cdot\delta t_m^\mathfrak{a}}{f''\left(f'^{-1}\left(\frac{x}{\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}}\right)\right)(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})^2}\,\mathrm{d}x\\&-\int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\|\delta\mathbf{u}\|_{\mathbf{U}}}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})}p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\frac{x\cdot\delta t_m^\mathfrak{a}}{f''\left(f'^{-1}\left(\frac{x}{\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}}\right)\right)(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})^2}\,\mathrm{d}x\\&+\int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})-C\|\delta\mathbf{u}\|_{\mathbf{U}}}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}-\varepsilon))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}+\varepsilon)}p^1_{N_\mathfrak{a}}(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x+o(\|\delta\mathbf{u}\|_{\mathbf{U}})\end{aligned}\tag{3.7.88}$$

The regularity of $p^1_{N_\mathfrak{a}}$ and the uniform convexity of f imply that the second integral on the right hand side is equal to $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$. Using again the regularity of $p^1_{N_\mathfrak{a}}$, the choice of ε in (3.7.66) and the local mapping in (3.7.86), which is in particular Lipschitz continuous, the third integral is equal to $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$. More precisely, using the abbreviation

$$\xi(\mathbf{u}) \coloneqq f'(u_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t}_{N_\mathfrak{a}} - t_m^{\mathfrak{a}}),$$

the absolute value of the third integral can be estimated as follows

$$\begin{aligned}
&\left| \int_{\xi(\bar{\mathbf{u}})-C\|\delta\mathbf{u}\|_{\mathbf{U}}}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}-\varepsilon))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^{\mathfrak{a}}+\varepsilon)} p_{N_\mathfrak{a}}^1(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \right| \\
&= \left| \int_{\xi(\bar{\mathbf{u}})-C\|\delta\mathbf{u}\|_{\mathbf{U}}}^{\min(\xi(\bar{\mathbf{u}}),\xi(\mathbf{u}))} p_{N_\mathfrak{a}}^1(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \right| \\
&\quad + \left| \int_{\min(\xi(\bar{\mathbf{u}}),\xi(\mathbf{u}))}^{\max(\xi(\bar{\mathbf{u}}),\xi(\mathbf{u}))} p_{N_\mathfrak{a}}^1(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \right| \\
&\quad + \left| \int_{\max(\xi(\bar{\mathbf{u}}),\xi(\mathbf{u}))}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}-\varepsilon))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^{\mathfrak{a}}+\varepsilon)} p_{N_\mathfrak{a}}^1(\tilde{t}_{N_\mathfrak{a}},x)\Delta y(\tilde{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \right|.
\end{aligned} \tag{3.7.89}$$

We note that due to (3.7.86) and the boundedness of $p_{N_\mathfrak{a}}^1$, the first and the third integral on the right hand side of (3.7.89) are equal to $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$. Concerning the second integral, using the boundedness of $p_{N_\mathfrak{a}}^1$, the Lipschitz continuity of the mapping $\xi(\mathbf{u})$ w.r.t. $\mathbf{u}$ and the Lipschitz continuity of the mapping in (3.7.86) for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$, we obtain that the second integral on the right-hand side of (3.7.89) is equal to $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$. Hence, the third integral of the right hand side of (3.7.88) is equal to $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$.

Therefore, and since as already mentioned the second integral of the right-hand side of (3.7.88) is equal to $o(\|\delta\mathbf{u}\|_{\mathbf{U}})$, we obtain that

$$I_{N_\mathfrak{a},\varepsilon}^1 = \int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^{\mathfrak{a}})} p_{N_\mathfrak{a}}^1(\tilde{t}_{N_\mathfrak{a}},x) \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''\left(f'^{-1}\left(\frac{x}{\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^{\mathfrak{a}}}\right)\right)(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^{\mathfrak{a}})^2}\,\mathrm{d}x + o(\|\delta\mathbf{u}\|_{\mathbf{U}}).$$

From Lemma 6.3.5 in [69] we know that $p_{N_\mathfrak{a}}^1$ coincides on $D_{N_\mathfrak{a}}^1$ with the adjoint state

p. Therefore, we obtain

$$\begin{aligned}
I^1_{N_\mathfrak{a},\varepsilon} &= \int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))\cdot(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})} p(\tilde{t}_{N_\mathfrak{a}},x)\frac{x\cdot\delta t_m^\mathfrak{a}}{f''\left(f'^{-1}\left(\frac{x}{\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}}\right)\right)(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})^2}\,\mathrm{d}x \\
&\quad + o(\|\delta\mathbf{u}\|_\mathbf{U}) \\
&= \int_{\xi_l(\tilde{t}_{N_\mathfrak{a}})\setminus(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))} p(\tilde{t}_{N_\mathfrak{a}},z(\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}))\frac{\delta t_m^\mathfrak{a}\cdot z}{f''(f'^{-1}(z))}\,\mathrm{d}z + o(\|\delta\mathbf{u}\|_\mathbf{U}) \\
&= \int_{f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))} \lim_{t\searrow\bar{t}_m^\mathfrak{a}} p(t,z(t-\bar{t}_m^\mathfrak{a}))\frac{\delta t_m^\mathfrak{a}\cdot z}{f''(f'^{-1}(z))}\,\mathrm{d}z + o(\|\delta\mathbf{u}\|_\mathbf{U}).
\end{aligned} \tag{3.7.90}$$

We note that the last equality in (3.7.90) is valid since for sufficiently small $t > \bar{t}_m^\mathfrak{a}$, the term $p(\cdot, z(t-\bar{t}_m^\mathfrak{a}))$ is equal to zero if

$$z \in \left[f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a})), \frac{\xi_l(\tilde{t}_{N_\mathfrak{a}})}{\tilde{t}_{N_\mathfrak{a}}-\bar{t}_m^\mathfrak{a}}\right]$$

and constant on $]\bar{t}_m^\mathfrak{a}, \tilde{t}_{N_\mathfrak{a}}]$ for all $z \in [f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a})), f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}))]$.

Step 3: In order to rewrite $I^2_{N_\mathfrak{a},\varepsilon}$, let the equation (3.7.54) on $]\hat{t}_{N_\mathfrak{a}}, \tilde{t}_{N_\mathfrak{a}}[\times\Omega$ be multiplied with $\tilde{p}^r_{N_\mathfrak{a}}$, which denotes the reversible solution of (3.7.68). Then using (3.7.72) and integration by parts on $D^r_{\varepsilon,N_\mathfrak{a}}$ leads to

$$\begin{aligned}
I^2_{N_\mathfrak{a},\varepsilon} &= \int_{D^r_{N_\mathfrak{a},\varepsilon}} \tilde{p}^r_{N_\mathfrak{a}}(t,x)(g(t,x,\bar{y},u_1)-g(t,x,\bar{y},\bar{u}_1))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{\xi_r(\hat{t}_{N_\mathfrak{a}})} \tilde{p}^r_{N_\mathfrak{a}}(\hat{t}_{N_\mathfrak{a}},x)\Delta y(\hat{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x \\
&\quad + \int_{\bar{t}_m^\mathfrak{a}-\varepsilon}^{\tilde{t}_{N_\mathfrak{a}}} \tilde{p}^r_{N_\mathfrak{a}}(t,\gamma(t,\varepsilon))\Delta y(t,\gamma(t,\varepsilon))(f'(\bar{y}(t,\gamma(t,\varepsilon)))-\tilde{a}(t,\gamma(t,\varepsilon)))\,\mathrm{d}t \\
&\quad - \int_{\hat{t}_{N_\mathfrak{a}}}^{\tilde{t}_{N_\mathfrak{a}}} \tilde{p}^r_{N_\mathfrak{a}}(t,\xi_r(t))\Delta y(t,\xi_r(t))(f'(\bar{y}(t,\xi_r(t)))-\tilde{a}(t,\xi_r(t)))\,\mathrm{d}t \\
&\quad + \int_{\hat{t}_{N_\mathfrak{a}}}^{\bar{t}_m^\mathfrak{a}-\varepsilon} \tilde{p}^r_{N_\mathfrak{a}}(t,0)\tilde{a}(t,0+)\Delta y(t,0+)\,\mathrm{d}t \\
&=: I_{20} + I_{\hat{t},N_\mathfrak{a}} + I_{22} + I_{23} + I_{24},
\end{aligned} \tag{3.7.91}$$

where $\gamma(t,\varepsilon) := f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^\mathfrak{a}-\varepsilon))(t-\bar{t}_m^\mathfrak{a}+\varepsilon)$. Moreover, in the derivation of (3.7.91), we have used (3.7.72) and the fact that $\tilde{p}^r_{N_\mathfrak{a}}$ solves (3.7.68) almost everywhere on

$\tilde{p}^r_{N_\mathfrak{a}}$. We further note that

$$y(\cdot, 0+; \mathbf{u}) = u_m^{B,\mathfrak{a}} \quad \text{on } [\hat{t}_{N_\mathfrak{a}}, \bar{t}^\mathfrak{a}_m[\tag{3.7.92}$$

for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$.

First we have a closer look on I_{20} in (3.7.91). Since $g(\cdot, y, u_1) = 0$ on $D^2_{N_\mathfrak{a}} \setminus D^r_{N_\mathfrak{a},\varepsilon}$ for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ and due to the regularity of g and (3.7.71), it holds that

$$I_{20} = \left(p^r_{N_\mathfrak{a}}, g_{u_1}(\cdot, \bar{y}, \bar{u}_1)\delta u_1\right)_{2, D^2_{N_\mathfrak{a}}} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}), \tag{3.7.93}$$

where $p^r_{N_\mathfrak{a}}$ denotes the reversible solution of (3.7.69). For I_{22}, we will show that

$$|I_{22}| \leq C \, \|\delta \mathbf{u}\|^2_{\mathbf{U}} \tag{3.7.94}$$

holds true and note that one can analogously prove

$$|I_{23}| \leq C \, \|\delta \mathbf{u}\|^2_{\mathbf{U}} \, . \tag{3.7.95}$$

To this end, we rewrite I_{22} by

$$\begin{aligned} I_{22} = \int_{\bar{t}^\mathfrak{a}_m - \varepsilon}^{\tilde{t}_{N_\mathfrak{a}}} -\tilde{p}^r_{N_\mathfrak{a}}(t, \gamma(t,\varepsilon)) \Big(& f(y(t, \gamma(t,\varepsilon))) \\ & - f(\bar{y}(t, \gamma(t,\varepsilon))) - f'(\bar{y}(t, \gamma(t,\varepsilon)))\Delta y(t, \gamma(t,\varepsilon)) \Big) \, \mathrm{d}t. \end{aligned}$$

The uniform boundedness of $\tilde{p}^r_{N_\mathfrak{a}}$ and the regularity of f imply that

$$|I_{22}| \leq M_{\tilde{p}^r_{N_\mathfrak{a}}} M_{f''} \int_{\bar{t}^\mathfrak{a}_m - \varepsilon}^{\tilde{t}_{N_\mathfrak{a}}} |\Delta y(t, \gamma(t,\varepsilon))|^2 \, \mathrm{d}t. \tag{3.7.96}$$

In order to prove (3.7.94), we will estimate $|\Delta y(t, \gamma(t,\varepsilon))|$ from above. To this end, we first note that since the point

$$(\tilde{t}_{N_\mathfrak{a}}, f'(\bar{\mathbf{u}}^{B,\mathfrak{a}}_m(\bar{t}^\mathfrak{a}_m))(\tilde{t}_{N_\mathfrak{a}} - \bar{t}^\mathfrak{a}_m))$$

is not a shock generation point and lies on the right boundary of the rarefaction wave emanating from $(\bar{t}_m^{\mathfrak{a}}, 0)$, there exists a stripe S with

$$\begin{gathered}\big\{((t,x) \in [\bar{t}_m^{\mathfrak{a}}, \tilde{t}_{N_{\mathfrak{a}}}] \times \Omega : x = f'(\bar{\mathbf{u}}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))(t - \bar{t}_m^{\mathfrak{a}})\big\} \subset S, \\ \left\{(t,x) \in [\bar{t}_m^{\mathfrak{a}} - \delta, \tilde{t}_{N_{\mathfrak{a}}}] \times \Omega : x = \sup_{\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})} \{f'(\mathbf{u}_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}}))\}(t - \bar{t}_m^{\mathfrak{a}} + \delta))\right\} \subset S,\end{gathered} \tag{3.7.97}$$

for sufficiently small $\delta > 0$. Furthermore, we obtain a continuously Fréchet-differentiable mapping $B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \ni \mathbf{u} \mapsto Y_m(\cdot; \mathbf{u}) \in C(S)$ such that

$$y(\cdot; \mathbf{u})|_{S_r(\mathbf{u})} = Y_m(\cdot; \mathbf{u}) \text{ for all } \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \tag{3.7.98}$$

holds, where

$$S_r(\mathbf{u}) := \left\{(t,x) \in S : x > \max\big\{0, f'(\mathbf{u}_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (t - t_m^{\mathfrak{a}})\big\}\right\}.$$

The validity of this can be found in the considerations before Lemma 3.7.16. Due to (3.7.97), $S_r(\mathbf{u})$ is nonempty for all $u \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ if ρ is small enough.

Moreover, we can choose ε in (3.7.66) such that $(t, \gamma(t,\varepsilon)) \in S_r(\mathbf{u})$ holds for all $t \in [\bar{t}_m^{\mathfrak{a}} - \varepsilon, \tilde{t}_{N_{\mathfrak{a}}}]$ and all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ for sufficiently small ρ. Using (3.7.98), we deduce

$$\begin{aligned} &\|\Delta y(\cdot, \gamma(\cdot,\varepsilon))\|_{C([\bar{t}_m^{\mathfrak{a}} - \varepsilon, \tilde{t}_{N_{\mathfrak{a}}}])} \\ &= \|Y_m(\cdot, \gamma(\cdot,\varepsilon); \mathbf{u}) - Y_m(\cdot, \gamma(\cdot,\varepsilon); \bar{\mathbf{u}})\|_{C([\bar{t}_m^{\mathfrak{a}} - \varepsilon, \tilde{t}_{N_{\mathfrak{a}}}])} \\ &\le \|Y_m(\cdot; \mathbf{u}) - Y_m(\cdot; \bar{\mathbf{u}})\|_{C(S)} \\ &= \left\|\frac{\partial}{\partial \mathbf{u}} Y_m(\cdot; \bar{\mathbf{u}})\right\|_{C(S)} \cdot \|\delta \mathbf{u}\|_{\mathbf{U}} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}) \\ &\le C \, \|\delta \mathbf{u}\|_{\mathbf{U}} \quad \forall \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}). \end{aligned} \tag{3.7.99}$$

Finally, (3.7.96) and (3.7.99) yield (3.7.94). Next, we prove that

$$I_{24} = \big(p_{N_{\mathfrak{a}}}^r(\cdot, 0), f'(\bar{y}(\cdot, 0+))\delta u_m^{B,\mathfrak{a}}\big)_{2,[\hat{t}_{N_{\mathfrak{a}}}, \bar{t}_m^{\mathfrak{a}}]} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}). \tag{3.7.100}$$

To this end, we observe that I_{24} can be rewritten by using (3.7.92) as follows:

$$I_{24} = \int_{\hat{t}_{N_{\mathfrak{a}}}}^{\bar{t}_m^{\mathfrak{a}} - \varepsilon} \tilde{p}_{N_{\mathfrak{a}}}^r(t, 0) \tilde{a}(t, 0+) \delta u_m^{B,\mathfrak{a}}(t) \, \mathrm{d}t \tag{3.7.101}$$

With the uniform boundedness of $\tilde{p}^r_{N_\mathfrak{a}}$ and (3.7.56), we obtain from (3.7.101) that

$$I_{24} = \int_{\hat{t}_{N_\mathfrak{a}}}^{\bar{t}^\mathfrak{a}_m - \varepsilon} \tilde{p}^r_{N_\mathfrak{a}}(t,0) a(t,0+) \delta u^{B,\mathfrak{a}}_m(t)\,\mathrm{d}t + o(\|\delta \mathbf{u}\|_{\mathbf{U}}) \tag{3.7.102}$$

holds true. Finally, the boundedness of the integrands in (3.7.102) and (3.7.71) yields (3.7.100). Considering the term $I_{\hat{t},N_\mathfrak{a}}$ in (3.7.91), with (3.7.71) and Corollary 3.3.5 we can show that

$$I_{\hat{t},N_\mathfrak{a}} = \int_0^{\xi_r(\hat{t}_{N_\mathfrak{a}})} p^r_{N_\mathfrak{a}}(\hat{t}_{N_\mathfrak{a}},x) \Delta y(\hat{t}_{N_\mathfrak{a}},x)\,\mathrm{d}x + o(\|\delta \mathbf{u}\|_{\mathbf{U}}). \tag{3.7.103}$$

Inserting (3.7.93), (3.7.94), (3.7.95) and (3.7.100) in (3.7.91), $I^2_{N_\mathfrak{a},\varepsilon}$ reads

$$\begin{aligned} I^2_{N_\mathfrak{a},\varepsilon} &= \left(p^r_{N_\mathfrak{a}} g_{u_1}(\cdot,\bar{y},\bar{u}_1), \delta u_1\right)_{2,D^2_{N_\mathfrak{a}}} + I_{\hat{t},N_\mathfrak{a}} \\ &\quad + \left(p^r_{N_\mathfrak{a}}(\cdot,0), f'(\bar{y}(\cdot,0+))\delta u^{B,\mathfrak{a}}_m\right)_{2,[\hat{t}_{N_\mathfrak{a}},\bar{t}^\mathfrak{a}_m]} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}), \end{aligned} \tag{3.7.104}$$

where $I_{\hat{t},N_\mathfrak{a}}$ will be further simplified in the next step. Before starting with the next step, we observe that (3.7.73) yields

$$\begin{aligned} I^2_{N_\mathfrak{a},\varepsilon} &= (p g_{u_1}(\cdot,\bar{y},\bar{u}_1), \delta u_1)_{2,D^2_{N_\mathfrak{a}}} + I_{\hat{t},N_\mathfrak{a}} \\ &\quad + \left(p(\cdot,0), f'(\bar{y}(\cdot,0+))\delta u^{B,\mathfrak{a}}_m\right)_{2,[\hat{t}_{N_\mathfrak{a}},\bar{t}^\mathfrak{a}_m]} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}). \end{aligned} \tag{3.7.105}$$

Step 4: Consider (3.7.54) on $]\tilde{t}_{N_\mathfrak{a}-1}, \hat{t}_{N_\mathfrak{a}}[\times\Omega$, multiply it with $\tilde{p}^1_{N_\mathfrak{a}-1}$, which is the reversible solution of (3.7.61), and apply integration by parts on the set $D^1_{N_\mathfrak{a}}$ yielding

$$\begin{aligned} I_{\hat{t},N_\mathfrak{a}} &= \int_{D^1_{N_\mathfrak{a}}} \tilde{p}^1_{N_\mathfrak{a}-1}(t,x)(g(t,x,\bar{y},u_1) - g(t,x,\bar{y},\bar{u}_1))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\tilde{t}_{N_\mathfrak{a}-1}}^{\hat{t}_{N_\mathfrak{a}}} \tilde{p}^1_{N_\mathfrak{a}-1}(t,0)\tilde{a}(t,0+)\Delta y(t,0+))\,\mathrm{d}t \\ &\quad - \int_{\tilde{t}_{N_\mathfrak{a}-1}}^{\hat{t}_{N_\mathfrak{a}}} \tilde{p}^1_{N_\mathfrak{a}-1}(t,\xi_r(t))\Delta y(t,\xi_r(t))(a(t,\xi_r(t)) - \tilde{a}(t,\xi_r(t)))\,\mathrm{d}t \\ &\quad + \int_0^{\xi_r(\tilde{t}_{N_\mathfrak{a}-1})} \tilde{p}^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},x)\Delta y(\tilde{t}_{N_\mathfrak{a}-1},x)\,\mathrm{d}x + o(\|\delta \mathbf{u}\|_{\mathbf{U}}) \\ &=: I_{30} + I_{31} + I_{32} + I_{33} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}). \end{aligned} \tag{3.7.106}$$

Due to the regularity of g and (3.7.64), I_{30} can be rewritten by

$$I_{30} = \int_{D^1_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}-1}(t,x) g_{u_1}(t,x,\bar{y},\bar{u}_1)\delta u_1 \,\mathrm{d}t\,\mathrm{d}x + o(\|\delta\mathbf{u}\|_{\mathbf{U}}), \tag{3.7.107}$$

with $p^1_{N_\mathfrak{a}-1}$ denoting the reversible solution of (3.7.62). Since u_0 is continuously differentiable in $x = \xi_r(0)$, we adopt Lemma 3.7.4 and prove analogously to the estimation of I_{22} in (3.7.94) that

$$I_{32} = O(\|\delta\mathbf{u}\|^2_{\mathbf{U}}). \tag{3.7.108}$$

Regarding the term I_{33}, (3.7.64) and Corollary 3.3.5 together yield

$$I_{33} = \int_0^{\xi_r(\tilde{t}_{N_\mathfrak{a}-1})} p^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},x)\Delta y(\tilde{t}_{N_\mathfrak{a}-1},x)\,\mathrm{d}x + o(\|\delta\mathbf{u}\|_{\mathbf{U}}). \tag{3.7.109}$$

Next, we have a look at I_{31} and note that

$$\Delta y(t,0+) = \delta u_{B,\mathfrak{a}}(t) \quad \forall t \in [\tilde{t}_{N_\mathfrak{a}-1}, \hat{t}_{N_\mathfrak{a}}] \tag{3.7.110}$$

holds due to (3.7.59) and the BLN-conditions in (3.1.10). Therefore, Lemma 3.3.4 implies

$$\|\Delta y(\cdot,0+)\|_{1,]\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}[} = O(\|\delta\mathbf{u}\|_{\mathbf{U}}). \tag{3.7.111}$$

Using (3.7.64), (3.7.111) and the uniform boundedness of $\tilde{a}$, we deduce

$$I_{31} = \int_{\tilde{t}_{N_\mathfrak{a}-1}}^{\hat{t}_{N_\mathfrak{a}}} p^1_{N_\mathfrak{a}-1}(t,0)\tilde{a}(t,0+)\Delta y(t,0+)\,\mathrm{d}t + o(\|\delta\mathbf{u}\|_{\mathbf{U}}). \tag{3.7.112}$$

Further on,

$$\left\| \delta u_{B,\mathfrak{a}} - \sum_{i=1}^{n_{t,\mathfrak{a}}+1} \delta u_i^{B,\mathfrak{a}} \cdot \mathbb{1}_{I^i_{B,\mathfrak{a}}} - \sum_{i=1}^{n_{t,\mathfrak{a}}} \operatorname{sgn}(\delta t_i^\mathfrak{a})\mathbb{1}_{I(\bar{t}_i^\mathfrak{a},\bar{t}_i^\mathfrak{a}+\delta t_i^\mathfrak{a})}[\bar{u}_{B,\mathfrak{a}}(\bar{t}_i^\mathfrak{a})] \right\|_{1,]\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}[} = o(\|\delta\mathbf{u}\|_{\mathbf{U}}). \tag{3.7.113}$$

Now use (3.7.110), (3.7.113) and the uniform boundedness of $\tilde{a}$ and $p^1_{N_\mathfrak{a}-1}$ in order to rewrite (3.7.112) by

$$
\begin{aligned}
I_{31} = &\sum_{i=1}^{n_{t,\mathfrak{a}}+1} \left(p^1_{N_\mathfrak{a}-1}(\cdot,0), \tilde{a}(\cdot,0+)\delta u_i^{B,\mathfrak{a}}\right)_{2,I^i_{B,\mathfrak{a}}\cap[\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}]} \\
&+ \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}})\\ \tilde{t}_{N_\mathfrak{a}-1}<\bar{t}_i^\mathfrak{a}<\hat{t}_{N_\mathfrak{a}}}} \left(p^1_{N_\mathfrak{a}-1}(\cdot,0), \tilde{a}(\cdot,0+)\cdot \operatorname{sgn}(\delta t_i^\mathfrak{a})[\bar{u}_{B,\mathfrak{a}}(\bar{t}_i^\mathfrak{a})]\right)_{2,I(\bar{t}_i^\mathfrak{a},\bar{t}_i^\mathfrak{a}+\delta t_i^\mathfrak{a})} \\
&+ o(\|\delta \mathbf{u}\|_{\mathbf{U}}) \\
=:&\; I_{321} + I_{322} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}).
\end{aligned} \tag{3.7.114}
$$

Lemma 3.3.4, (3.7.110), the regularity of f and the definition of $\tilde{a}$ and a yield

$$
\tilde{a}(\cdot,0+) \to a(\cdot,0+) \quad \text{in } L^1([\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}]). \tag{3.7.115}
$$

Using (3.7.115) and the boundedness of $p^1_{N_\mathfrak{a}-1}$, we show that

$$
I_{321} = \sum_{i=1}^{n_{t,\mathfrak{a}}+1} \left(p^1_{N_\mathfrak{a}-1}(\cdot,0), a(\cdot,0+)\delta u_i^{B,\mathfrak{a}}\right)_{2,I^i_{B,\mathfrak{a}}\cap[\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}]} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}). \tag{3.7.116}
$$

Next, we regard I_{322} and pick out an arbitrary $i \in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}})$ with $\tilde{t}_{N_\mathfrak{a}-1} < \bar{t}_i^\mathfrak{a} < \hat{t}_{N_\mathfrak{a}}$ and assume w.l.o.g. that $\delta t_i^\mathfrak{a} > 0$. The case $\delta t_i^\mathfrak{a} < 0$ can be treated analogously and $\delta t_i^\mathfrak{a} = 0$ is trivial. Then the regularity of $\bar{u}_i^{B,\mathfrak{a}}$ and $\bar{u}_{i+1}^{B,\mathfrak{a}}$ leads to

$$
\|[\bar{u}_{B,\mathfrak{a}}(\bar{t}_i^\mathfrak{a})] - (\bar{u}_i^{B,\mathfrak{a}} - \bar{u}_{i+1}^{B,\mathfrak{a}})\|_{L^1(]\bar{t}_i^\mathfrak{a},\bar{t}_i^\mathfrak{a}+\delta t_i^\mathfrak{a}[)} = o(\|\delta \mathbf{u}\|_{\mathbf{U}}). \tag{3.7.117}
$$

Adopting the uniform boundedness of $\tilde{a}$ and $p^1_{N_\mathfrak{a}-1}$, we obtain from (3.7.117) that

$$
\begin{aligned}
I_{322} &= \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}})\\ \tilde{t}_{N_\mathfrak{a}-1}<\bar{t}_i^\mathfrak{a}<\hat{t}_{N_\mathfrak{a}}}} \left(p^1_{N_\mathfrak{a}-1}(\cdot,0), \tilde{a}(\cdot,0+)\cdot(\bar{u}_i^{B,\mathfrak{a}} - \bar{u}_{i+1}^{B,\mathfrak{a}})\right)_{2,]\bar{t}_i^\mathfrak{a},\bar{t}_i^\mathfrak{a}+\delta t_i^\mathfrak{a}[} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}) \\
&= \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}})\\ \tilde{t}_{N_\mathfrak{a}-1}<\bar{t}_i^\mathfrak{a}<\hat{t}_{N_\mathfrak{a}}}} \left(p^1_{N_\mathfrak{a}-1}(\cdot,0), f(\bar{u}_i^{B,\mathfrak{a}}) - f(\bar{u}_{i+1}^{B,\mathfrak{a}})\right)_{2,]\bar{t}_i^\mathfrak{a},\bar{t}_i^\mathfrak{a}+\delta t_i^\mathfrak{a}[} + o(\|\delta \mathbf{u}\|_{\mathbf{U}}).
\end{aligned}
$$

Further on, the regularity of $p^1_{N_\mathfrak{a}-1}$, f and $\bar{u}^{B,\mathfrak{a}}_j$ for $j=1,\dots,n_{t,\mathfrak{a}}$ gives

$$\begin{aligned} I_{322} &= \sum_{\substack{i\in I_{s,\mathfrak{a}}(\bar{\mathbf{u}})\\ \tilde{t}_{N_\mathfrak{a}-1}<\bar{t}^\mathfrak{a}_i<\hat{t}_{N_\mathfrak{a}}}} p^1_{N_\mathfrak{a}-1}(\bar{t}^\mathfrak{a}_i,0)(f(\bar{u}^{B,\mathfrak{a}}_i(\bar{t}^\mathfrak{a}_i)) - f(\bar{u}^{B,\mathfrak{a}}_{i+1}(\bar{t}^\mathfrak{a}_i)))\delta t^\mathfrak{a}_i + o(\|\delta\mathbf{u}\|_\mathbf{U}) \\ &= \sum_{\substack{i\in I_{s,\mathfrak{a}}(\bar{\mathbf{u}})\\ \tilde{t}_{N_\mathfrak{a}-1}<\bar{t}^\mathfrak{a}_i<\hat{t}_{N_\mathfrak{a}}}} p^1_{N_\mathfrak{a}-1}(\bar{t}^\mathfrak{a}_i,0)[f(\bar{y}(\bar{t}^\mathfrak{a}_i,0+))]\delta t^\mathfrak{a}_i + o(\|\delta\mathbf{u}\|_\mathbf{U}). \end{aligned} \tag{3.7.118}$$

Inserting (3.7.116) and (3.7.118) in (3.7.114) yields

$$\begin{aligned} I_{31} = &\sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p^1_{N_\mathfrak{a}-1}(\cdot,0), f'(\bar{u}^{B,\mathfrak{a}}_i)\delta u^{B,\mathfrak{a}}_i)_{2,I^i_{B,\mathfrak{a}}\cap[\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}]} \\ &+ \sum_{\substack{i\in I_{s,\mathfrak{a}}(\bar{\mathbf{u}})\\ \tilde{t}_{N_\mathfrak{a}-1}<\bar{t}^\mathfrak{a}_i<\hat{t}_{N_\mathfrak{a}}}} p^1_{N_\mathfrak{a}-1}(\bar{t}_i,0)[f(\bar{y}(\bar{t}^\mathfrak{a}_i,0+))]\delta t^\mathfrak{a}_i + o(\|\delta\mathbf{u}\|_\mathbf{U}). \end{aligned} \tag{3.7.119}$$

If we insert (3.7.107), (3.7.109),(3.7.108), (3.7.109) and (3.7.119) in (3.7.106), we finally obtain

$$\begin{aligned} I_{\hat{t},N_\mathfrak{a}} &= \big(p^1_{N_\mathfrak{a}-1} g_{u_1}(\cdot,\bar{y},\bar{u}_1), \delta u_1\big)_{2,D^1_{N_\mathfrak{a}}} \\ &+ (p^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},\cdot), \Delta y(\tilde{t}_{N_\mathfrak{a}-1},\cdot))_{2,]0,\xi_r(\tilde{t}_{N_\mathfrak{a}-1})[} \\ &+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p^1_{N_\mathfrak{a}-1}(\cdot,0), f'(\bar{u}^{B,\mathfrak{a}}_i)\delta u^{B,\mathfrak{a}}_i)_{2,I^i_{B,\mathfrak{a}}\cap[\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}]} \\ &+ \sum_{\substack{i\in I_{s,0}:\\ \bar{t}^\mathfrak{a}_i\in[\tilde{t}_{N_\mathfrak{a}-1},\hat{t}_{N_\mathfrak{a}}]}} p^1_{N_\mathfrak{a}-1}(\bar{t}^\mathfrak{a}_i,0)[f(y(\bar{t}^\mathfrak{a}_i,0+;\mathbf{u}))]\delta t^\mathfrak{a}_i + o(\|\delta\mathbf{u}\|_\mathbf{U}). \end{aligned} \tag{3.7.120}$$

The second term of the right side of (3.7.120) can be rewritten as follows:

$$\begin{aligned} &(p^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},\cdot), \Delta y(\tilde{t}_{N_\mathfrak{a}-1},\cdot))_{2,]0,\xi_r(\tilde{t}_{N_\mathfrak{a}-1})]} \\ &= (p^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},\cdot), \Delta y(\tilde{t}_{N_\mathfrak{a}-1},\cdot))_{2,]0,\gamma^1_{N_\mathfrak{a}-1,\varepsilon}(\tilde{t}_{N_\mathfrak{a}-1})]} \\ &\quad + (p^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},\cdot), \Delta y(\tilde{t}_{N_\mathfrak{a}-1},\cdot))_{2,]\gamma^1_{N_\mathfrak{a}-1,\varepsilon}(\tilde{t}_{N_\mathfrak{a}-1}),\gamma^2_{N_\mathfrak{a}-1,\varepsilon}(\tilde{t}_{N_\mathfrak{a}-1})]} \\ &\quad + (p^1_{N_\mathfrak{a}-1}(\tilde{t}_{N_\mathfrak{a}-1},\cdot), \Delta y(\tilde{t}_{N_\mathfrak{a}-1},\cdot))_{2,]\gamma^2_{N_\mathfrak{a}-1,\varepsilon}(\tilde{t}_{N_\mathfrak{a}-1}),\xi_r(\tilde{t}_{N_\mathfrak{a}-1})]} \\ &= I^1_{N_\mathfrak{a}-1,\varepsilon} + I^2_{N_\mathfrak{a}-1,\varepsilon} + I^3_{N_\mathfrak{a}-1,\varepsilon}, \end{aligned}$$

where

$$\gamma^1_{N_\mathfrak{a}-1,\varepsilon}(t) := f'(\bar{u}^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}+1}(\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}+\varepsilon)) \cdot (t-\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}-\varepsilon) \quad \text{and}$$
$$\gamma^2_{N_\mathfrak{a}-1,\varepsilon}(t) := f'(\bar{u}^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}}(\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}-\varepsilon)) \cdot (t-\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}+\varepsilon).$$

We will now have a closer look at the terms $I^i_{N_\mathfrak{a}-1,\varepsilon}$ for $i=1,2,3$. Consider (3.7.54) on $]\hat{t}_{N_\mathfrak{a}-1},\tilde{t}_{N_\mathfrak{a}-1}[\times\Omega$, multiply it with $\tilde{p}^l_{N_\mathfrak{a}-1}$ denoting the reversible solution of (3.7.68). Then we apply integration by parts and further (3.7.64). Analogously to the estimation of $I^2_{\bar{t},\varepsilon}$ in (3.7.104) one can show that

$$I^1_{N_\mathfrak{a}-1,\varepsilon} = \left(p^l_{N_\mathfrak{a}-1} g_{u_1}(\cdot,\bar{y},\bar{u}_1),\delta u_1\right)_{2,D^l_{N_\mathfrak{a}-1}} + \left(p^l_{N_\mathfrak{a}-1}(\cdot,0), f'(\bar{y}(\cdot,0+))\delta u^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}+1}\right)_{2,[\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}},\tilde{t}_{N_\mathfrak{a}-1}]} + o(\|\delta\mathbf{u}\|_\mathbf{U}), \tag{3.7.121}$$

where $p^l_{N_\mathfrak{a}-1}$ is the reversible solution of (3.7.69). Due to the choice of $\tilde{t}_{N_\mathfrak{a}-1}$ in (3.7.58) and using that the source term g is equal to zero for all $x\in[0,\varepsilon_g[$, the first term on the right-hand side of (3.7.121) is equal to zero leading to

$$I^1_{N_\mathfrak{a}-1,\varepsilon} = \left(p(\cdot,0), f'(\bar{y}(\cdot,0+))\delta u^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}+1}\right)_{2,[\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}},\tilde{t}_{N_\mathfrak{a}-1}]} + o(\|\delta\mathbf{u}\|_\mathbf{U}).$$

Next, we note that we can use the same arguments as for the simplification of the term $I^1_{\bar{t},\varepsilon}$ in (3.7.90) to justify that $I^2_{N_\mathfrak{a}-1,\varepsilon}$ is equal to

$$I^2_{N_\mathfrak{a}-1,\varepsilon} = \int_{f'(\bar{u}^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}+1}(\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}))}^{f'(\bar{u}^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}}(\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}))} \lim_{t\searrow\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}} p(t,z(t-\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}))\frac{\delta t^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}\cdot z}{f''(f'^{-1}(z))}\,\mathrm{d}z + o(\|\delta\mathbf{u}\|_\mathbf{U}).$$

Finally, we observe that $I^3_{N_\mathfrak{a}-1,\varepsilon}$ can be treated analogously as $I^2_{N_\mathfrak{a},\varepsilon}$ such that we obtain

$$I^3_{N_\mathfrak{a}-1,\varepsilon} = \left(p^r_{N_\mathfrak{a}-1} g_{u_1}(\cdot,\bar{y},\bar{u}_1),\delta u_1\right)_{2,D^2_{N_\mathfrak{a}-1}} + I_{\hat{t},N_\mathfrak{a}-1} + \left(p^r_{N_\mathfrak{a}-1}(\cdot,0), f'(\bar{y}(\cdot,0+))\delta u^{B,\mathfrak{a}}_{j_{N_\mathfrak{a}-1}}\right)_{2,[\hat{t}_{N_\mathfrak{a}-1},\bar{t}^\mathfrak{a}_{j_{N_\mathfrak{a}-1}}]} + o(\|\delta\mathbf{u}\|_\mathbf{U}),$$

where $p^r_{N_\mathfrak{a}-1}$ denotes the reversible solution of (3.7.69) and

$$I_{\hat{t},N_\mathfrak{a}-1} = \int_0^{\xi_r(\hat{t}_{N_\mathfrak{a}-1})} p^r_{N_\mathfrak{a}-1}(\hat{t}_{N_\mathfrak{a}-1},x)\Delta y(\hat{t}_{N_\mathfrak{a}-1},x)\,\mathrm{d}x + o(\|\delta\mathbf{u}\|_\mathbf{U}).$$

We note that the term $I_{\hat{t},N_{\mathfrak{a}}-1}$ can be treated similarly to $I_{\hat{t},N_{\mathfrak{a}}}$. Using (3.7.73) and continuing the previous steps until $k=1$ and reinserting successively the terms, we finally obtain

$$\begin{aligned}
&\frac{1}{[\bar{y}(\bar{t},x_s(\bar{\mathbf{u}}))]}\int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}}\Delta y(\bar{t},x)\,\mathrm{d}x\\
&=\sum_{k=2}^{N_{\mathfrak{a}}}(pg_{u_1}(\cdot,\bar{y},\bar{u}_1),\delta u_1)_{2,D_k^1\cup D_{k-1}^2\cup D_{N_{\mathfrak{a}}}^2\cup D_{N_{\mathfrak{a}}}^1}\\
&\quad+\sum_{i=1}^{n_{t,\mathfrak{a}}+1}(p(\cdot,0+),f'(\bar{u}_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i\cap[\hat{t}_1,\bar{t}_m^{\mathfrak{a}}]}\\
&\quad+\sum_{k=1}^{N_{\mathfrak{a}}}\int_{f'(\bar{u}_{j_k+1}^{B,\mathfrak{a}}(\bar{t}_{j_k}^{\mathfrak{a}}))}^{f'(\bar{u}_{j_k}^{B,\mathfrak{a}}(\bar{t}_{j_k}^{\mathfrak{a}}))}\lim_{t\searrow\bar{t}_{j_{N_k}}^{\mathfrak{a}}}p(t,z(t-\bar{t}_{j_k}^{\mathfrak{a}}))\frac{\delta t_{j_k}^{\mathfrak{a}}\cdot z}{f''(f'^{-1}(z))}\,\mathrm{d}z\\
&\quad+\sum_{\substack{i\in I_{s,0}:\\ \bar{t}_i^{\mathfrak{a}}\in[\hat{t}_1,\hat{t}_{N_{\mathfrak{a}}}]}}p(\bar{t}_i^{\mathfrak{a}},0+)[f(y(\bar{t}_i^{\mathfrak{a}},0+;\mathbf{u}))]\delta t_i^{\mathfrak{a}}\\
&\quad+I_{\hat{t},1}+o(\|\delta\mathbf{u}\|_{\mathbf{U}})+\bar{\varepsilon}O(\|\delta\mathbf{u}\|_{\mathbf{U}}),
\end{aligned}\tag{3.7.122}$$

where

$$I_{\hat{t},1}=\int_0^{\xi_r(\hat{t}_1)}p_1^r(\hat{t}_1,x)\Delta y(\hat{t}_1,x)\,\mathrm{d}x+o(\|\delta\mathbf{u}\|_{\mathbf{U}})\tag{3.7.123}$$

with p_1^r denoting the reversible solution of (3.7.69). The next step is concerned with finding a simplification of the term $I_{\hat{t},1}$ in (3.7.123).

Step 5: In the last step, we simplify the term $I_{\hat{t},1}$. To this end, we first note that $I_{\hat{t},1}$ in (3.7.123) can be simplified analogously to $I_{\hat{t},N_{\mathfrak{a}}}$ such that we obtain

$$\begin{aligned}
I_{\hat{t},1}&=\big(p_0^1 g_{u_1}(\cdot,\bar{y},\bar{u}_1),\delta u_1\big)_{2,D_0^1}+(p_0^1(s),\Delta y(s,\cdot))_{2,]0,\xi_r(s)[}\\
&\quad+\sum_{i=1}^{n_{t,\mathfrak{a}}+1}(p_0^1(\cdot,0),f'(\bar{u}_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i\cap[s,\hat{t}_1]}\\
&\quad+\sum_{i\in I_{s,0}:\,\bar{t}_i^{\mathfrak{a}}\in[s,\hat{t}_1]}p_0^1(\bar{t}_i^{\mathfrak{a}},0)[f(y(\bar{t}_i^{\mathfrak{a}},0+;\mathbf{u}))]\delta t_i^{\mathfrak{a}}+o(\|\delta\mathbf{u}\|_{\mathbf{U}}),
\end{aligned}\tag{3.7.124}$$

where p_0^1 denotes the reversible solution of (3.7.62). For the second term on the right-hand side of (3.7.124) we can write

$$\left(p^1_{N_\mathfrak{a}-1}(s), \Delta y(s,\cdot)\right)_{2,(0,\xi_r(s))} = \sum_{k=1}^{N_0+1} I^k_{0,\varepsilon} + \sum_{k=1}^{N_0} I^k_{r,\varepsilon} \tag{3.7.125}$$

with

$$\begin{aligned} I^1_{0,\varepsilon} &= (p^1_0(s), \Delta y(s,\cdot))_{2,]0,\gamma^1_{l,\varepsilon}(s)[}, \\ I^k_{0,\varepsilon} &= (p^1_0(s), \Delta y(s,\cdot))_{2,]\gamma^{k-1}_{r,\varepsilon}(s),\gamma^k_{l,\varepsilon}(s)[}, \quad k = 2,\dots,N_0 \\ I^{N_0+1}_{0,\varepsilon} &= (p^1_0(s), \Delta y(s,\cdot))_{2,]\gamma^{N_0}_{r,\varepsilon}(s),\xi_r(s)[}, \end{aligned} \tag{3.7.126}$$

and

$$I^k_{r,\varepsilon} = \left(p^1_{N_\mathfrak{a}-1}(s), \Delta y(s,\cdot)\right)_{2,]\gamma^k_{l,\varepsilon}(s),\gamma^k_{r,\varepsilon}(s)[}, \quad k = 1,\dots,N_0. \tag{3.7.127}$$

Here, the mappings $\gamma^k_{l,\varepsilon}(\cdot)$ and $\gamma^k_{r,\varepsilon}(\cdot)$ are given by

$$\begin{aligned} \gamma^k_{l,\varepsilon}(t) &= \bar{x}^0_{i_k} - \varepsilon + f'(u^0_{i_k}(\bar{x}^0_{i_k} - \varepsilon))t, \\ \gamma^k_{r,\varepsilon}(t) &= \bar{x}^0_{i_k} + \varepsilon + f'(u^0_{i_k+1}(\bar{x}^0_{i_k} + \varepsilon))t \end{aligned}$$

for $k = 1,\dots,N_0$ with $N_0 = |I_{r,0}|$ and

$$\left\{x^0_{i_1},\dots,x^0_{i_{N_0}}\right\} := \left\{x^0_j : j \in I_{r,0}\right\}.$$

Including (3.7.126) and (3.7.126) in (3.7.124) results in

$$\begin{aligned} I_{\hat{t},1} = &\left(p^1_0 g_{u_1}(\cdot,\bar{y},\bar{u}_1), \delta u_1\right)_{2,D^1_0} \\ &+ \sum_{k=1}^{N_0+1} I^k_{0,\varepsilon} + \sum_{k=1}^{N_0} I^k_{r,\varepsilon} \\ &+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p^1_0(\cdot,0), f'(\bar{u}^{B,\mathfrak{a}}_i)\delta u^{B,\mathfrak{a}}_i)_{2,I^i_{B,\mathfrak{a}}\cap[s,\hat{t}_1]\setminus[0,\bar{t}^\mathfrak{a}_{i_1}]} \\ &+ \sum_{\substack{i\in I_{s,0}:\\ \bar{t}^\mathfrak{a}_i\in[s,\hat{t}_1]\setminus[0,\bar{t}^\mathfrak{a}_{i_1}]}} p^1_0(\bar{t}^\mathfrak{a}_i,0)[f(y(\bar{t}^\mathfrak{a}_i,0+;\mathbf{u}))]\delta t^\mathfrak{a}_i + o(\|\delta\mathbf{u}\|_\mathbf{U}). \end{aligned} \tag{3.7.128}$$

Using (3.7.73), we see that (3.7.128) can be rewritten by

$$\begin{aligned} I_{\hat{t},1} &= (pg_{u_1}(\cdot,\bar{y},\bar{u}_1),\delta u_1)_{2,D_0^1} \\ &\quad + \sum_{k=1}^{N_0+1} I_{0,\varepsilon}^k + \sum_{k=1}^{N_0} I_{r,\varepsilon}^k \\ &\quad + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot,0), f'(\bar{u}_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i\cap[s,\hat{t}_1]} \\ &\quad + \sum_{i\in I_{s,0}:\bar{t}_i^{\mathfrak{a}}\in[0,\hat{t}_1]} p(\bar{t}_i^{\mathfrak{a}},0)[f(y(\bar{t}_i^{\mathfrak{a}},0+;\mathbf{u}))]\delta t_i^{\mathfrak{a}} + o(\|\delta\mathbf{u}\|_{\mathbf{U}}). \end{aligned} \tag{3.7.129}$$

We note that similar as for the estimation of $I_{N_{\mathfrak{a}},\varepsilon}^1$ in step 2, the terms $I_{r,\varepsilon}^k$ for $k=1,\ldots,N_0$ can be replaced by

$$I_{r,\varepsilon}^k = \int_{f'(\bar{u}_{i_k}^0(\bar{x}_{i_k}^0))}^{f'(\bar{u}_{i_k+1}^0(\bar{x}_{i_k}^0))} \lim_{t\searrow 0} p(t,zt+\bar{x}_{i_k}^0))\frac{\delta x_{i_k}^0}{f''(f'^{-1}(z))}\,\mathrm{d}z + o(\|\delta\mathbf{u}\|_{\mathbf{U}}). \tag{3.7.130}$$

Next, we consider the terms in (3.7.126) and observe that analogously to the estimation of the term $I_{N_{\mathfrak{a}},\varepsilon}^2$ in (3.7.105), one can show that

$$\begin{aligned} I_{0,\varepsilon}^k &= \sum_{i=1}^{n_x+1} (p(0,\cdot),\delta u_i^0)_{2,I_0^i\cap[\gamma_{r,\varepsilon}^{k-1}(0),\gamma_{l,\varepsilon}^k(0)]} \\ &\quad + \sum_{i=1}^{n_x} \mathbb{1}_{[\gamma_{r,\varepsilon}^{k-1}(0),\gamma_{l,\varepsilon}^k(0)]}(\bar{x}_i^0)p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 \\ &\quad + o(\|\delta\mathbf{u}\|_{\mathbf{U}}) \quad k=1,\ldots N_0-1 \\ I_{0,\varepsilon}^1 &= \sum_{i=1}^{n_x+1} (p(0,\cdot),\delta u_i^0)_{2,I_0^i\cap[0,\gamma_{l,\varepsilon}^1(0)]} + \sum_{i=1}^{n_x} \mathbb{1}_{[0,\gamma_{l,\varepsilon}^1(0)]}(\bar{x}_i^0)p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 \\ &\quad + o(\|\delta\mathbf{u}\|_{\mathbf{U}}), \\ I_{0,\varepsilon}^{N_0+1} &= \sum_{i=1}^{n_x+1} (p(0,\cdot),\delta u_i^0)_{2,I_0^i\cap[\gamma_{r,\varepsilon}^{N_0}(0),\mathfrak{b}]} + \sum_{i=1}^{n_x} \mathbb{1}_{[\gamma_{r,\varepsilon}^{N_0}(0),\mathfrak{b}]}(\bar{x}_i^0)p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 \\ &\quad + o(\|\delta\mathbf{u}\|_{\mathbf{U}}), \end{aligned}$$

where we have used that $g|_{[0,s]\times\mathbb{R}} \equiv 0$. Inserting (3.7.130) and the terms above in

(3.7.129) yields

$$
\begin{aligned}
I_{\hat{t},1} &= (pg_{u_1}(\cdot,\bar{y},\bar{u}_1),\delta u_1)_{2,D_0^1} \\
&\quad + \sum_{i=1}^{n_x+1} (p(0,\cdot),\delta u_i^0)_{2,I_0^i} + \sum_{i=1}^{n_x} p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 \\
&\quad + \sum_{k=1}^{N_0} \int_{f'(\bar{u}_{i_k}^0(\bar{x}_{i_k}^0))}^{f'(\bar{u}_{i_k+1}^0(\bar{x}_{i_k}^0))} \lim_{t\searrow 0} p(t,zt+\bar{x}_{i_k}^0))\frac{\delta x_{i_k}^0}{f''(f'^{-1}(z))}\,\mathrm{d}z \\
&\quad + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot,0+),f'(\bar{u}_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i\cap[0,\hat{t}_1]} \\
&\quad + \sum_{i\in I_{\mathrm{s},0}:\bar{t}_i^{\mathfrak{a}}\in[0,\hat{t}_1]} p(\bar{t}_i^{\mathfrak{a}},0+)[f(y(\bar{t}_i^{\mathfrak{a}},0+;\mathbf{u}))]\delta t_i^{\mathfrak{a}} + o(\|\delta\mathbf{u}\|_{\mathbf{U}}).
\end{aligned}
\tag{3.7.131}
$$

Next, we insert (3.7.131) in (3.7.122), set $\bar{\varepsilon} := \bar{\varepsilon}(\delta\mathbf{u})$ such that

$$\bar{\varepsilon}(\delta\mathbf{u}) \to 0 \quad \text{if} \quad \|\delta\mathbf{u}\|_{\mathbf{U}} \to 0 \tag{3.7.132}$$

and observe that p is equal to zero on $\Omega_{\bar{t}} \setminus (D_k^1 \cup D_{k-1}^2 \cup D_{N_\mathfrak{a}}^2 \cup D_{N_\mathfrak{a}}^1 \cup D_0^1)$. This yields

$$
\begin{aligned}
&\frac{1}{[y(\bar{t}, x_s(\bar{\mathbf{u}}))]} \int_{x_s(\bar{\mathbf{u}})-\bar{\varepsilon}}^{x_s(\bar{\mathbf{u}})+\bar{\varepsilon}} \Delta y(\bar{t}, x) \,\mathrm{d}x \\
&= (p g_{u_1}(\cdot, y, u_1), \delta u_1)_{2,\Omega_{\bar{t}}} \\
&\quad + \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0)_{2,I_0^i(\mathbf{u})} + \sum_{i\in I_{\mathrm{s},0}(\mathbf{u})} p(0, x_i^0)[u_0(x_i^0)]\delta x_i^0 \\
&\quad + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, \mathfrak{a}+), f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i(\mathbf{u})\cap]0,\bar{t}[} \\
&\quad + \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{a}}(\mathbf{u}):\\ t_i^{\mathfrak{a}}\le\bar{t}}} p(t_i^{\mathfrak{a}}, \mathfrak{a}+)[f(y(t_i^{\mathfrak{a}}, \mathfrak{a}+;\mathbf{u}))]\delta t_i^{\mathfrak{a}} \\
&\quad - \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot, \mathfrak{b}-), f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2,I_{B,\mathfrak{b}}^i\cap]0,\bar{t}[} \\
&\quad - \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{b}}(\mathbf{u}):\\ t_i^{\mathfrak{b}}\le\bar{t}}} p(t_i^{\mathfrak{b}}, \mathfrak{b}-)[f(y(t_i^{\mathfrak{b}}, \mathfrak{b}-;\mathbf{u}))]\delta t_i^{\mathfrak{b}} \\
&\quad - \sum_{i\in I_{\mathrm{r},0}(\mathbf{u})} p_i^{\mathrm{r},0}\delta x_i^0 + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{a}}(\mathbf{u}):\\ t_i^{\mathfrak{a}}\le\bar{t}}} p_i^{\mathrm{r},\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{b}}(\mathbf{u}):\\ t_i^{\mathfrak{b}}\le\bar{t}}} p_i^{\mathrm{r},\mathfrak{b}}\delta t_i^{\mathfrak{b}}
\end{aligned}
\tag{3.7.133}
$$

Finally, (3.7.52), (3.7.133) and the choice of $\bar{\varepsilon}$ in (3.7.132) yield the Fréchet-differentiability of the mapping in (3.7.50) in $\mathbf{u} = \bar{\mathbf{u}}$, where the derivative is given by (3.7.51). □

Analogously, one can treat the more general case that $(\bar{t}, \bar{x})$ is a non-degenerated shock point of class X_l/X_r with $X_l, X_r \in \{C^c, R^c, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}\}$:

Corollary 3.7.25 (Differentiability of shock points). *Suppose that* (A1) *and* (A4) *hold true and consider some* $\bar{\mathbf{u}} \in \mathbf{U}$ *satisfying* (ND). *Moreover, let* $\bar{x}$ *be a non-degenerated shock point of* $y(\bar{t}, \cdot; \bar{\mathbf{u}})$. *Then the mapping in* (3.7.50) *is continuously differentiable with derivative given by* (3.7.51).

3.7.5 Proof of Theorem 3.5.5

Considering a control $\bar{\mathbf{u}} \in \mathbf{U}$ satisfying (ND), our goal is to prove Theorem 3.5.5, i.e., the continuous Fréchet-differentiability of the cost functional $\mathbf{u} \mapsto J(y)$ in $\bar{\mathbf{u}}$. To this end, we follow the ideas of [81]: At first, we prove that the mapping $\mathbf{u} \mapsto y(\bar{t}, \cdot, \mathbf{u})$ is continuously shift-differentiable. Then we use Lemma 3.4.3 to show that the mapping $\mathbf{u} \mapsto J(y)$ is continuously Fréchet-differentiable in $\bar{\mathbf{u}}$, cf. [82, 70, 69]. As already mentioned, in order to derive the shift-differentiability of the control-to-state mapping, we will use Lemma 3.7.1 which we hence have to prove at first:

Proof of Lemma 3.7.1. Since $\bar{\mathbf{u}} \in \mathbf{U}$ satisfies by assumption the conditions in (ND), the requirements of Lemma 3.7.23, Corollary 3.7.25 and the Lemmas in §3.7.3 are satisfied.

The results of §3.7.3 show that the mappings in (3.7.6) consist of non-degenerated shock points $x_{s,1}, \ldots, x_{s,N}$, which depend continuously Fréchet-differentiably on the control due to Corollary 3.7.25, and of points $x_{r,1}, \ldots, x_{r,K-N}$ lying on the boundaries of rarefaction waves emanating from $t = 0$, $x = \mathfrak{a}$ or $x = \mathfrak{b}$. The continuous Fréchet-differentiability of the points $x_{r,1}, \ldots, x_{r,K-N}$ w.r.t. the control is a direct consequence of Lemma 3.2.5. The mappings in (3.7.5) and its derivatives can be computed according to the results in §3.7.3, these are the Lemmas 3.7.4, 3.7.9, 3.7.12, 3.7.14, 3.7.18 or 3.7.20. Finally, (3.7.4) holds due to Lemma 3.7.23, Lemmas 3.7.22, 3.7.16, 3.7.11 and the Lemmas 3.7.8, 3.7.13, 3.7.19, 3.7.10, 3.7.15 and 3.7.21. □

Using Lemma 3.7.1, one can show the following result:

Theorem 3.7.26 (Shift-differentiability of entropy solutions)**.** *Suppose that* (A3) *and* (A4) *hold true and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *satisfying* (ND) *for some* $\bar{t}$ *and an interval* $[a, b] \subset \Omega$*. Then the mapping*

$$\mathbf{U} \ni \mathbf{u} \mapsto y(\bar{t}, \cdot, \mathbf{u}) \in L^{\infty}([a, b]) \tag{3.7.134}$$

is continuously shift-differentiable on a neighborhood $B_{\rho}^{\mathbf{U}}(\bar{\mathbf{u}})$*, where* $\rho > 0$ *is chosen sufficiently small.*

Proof. The proof is basically similar to the proof of the Theorems 5.2.3 and 5.2.4 in [69, §6.4], see also [81]. We need to find points $a < \bar{x}_1 < \ldots < \bar{x}_N < b$ and a linear bounded operator

$$\mathrm{T_s}\left(y(\bar{t}, \cdot; \bar{\mathbf{u}})\right) \in \mathcal{L}(\mathbf{U}; L^r([a, b]) \times \mathbb{R}^N) \tag{3.7.135}$$

such that

$$\lim_{\mathbf{u} \to \bar{\mathbf{u}}} \frac{\left\| y(\bar{t}, \cdot; \mathbf{u}) - y(\bar{t}, \cdot; \bar{\mathbf{u}}) - S^{(\bar{x}_k)}_{y(\bar{t}, \cdot; \bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t}, \cdot; \bar{\mathbf{u}})\right)(\mathbf{u} - \bar{\mathbf{u}})\right) \right\|_{1,[a,b]}}{\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}}} = 0 \tag{3.7.136}$$

holds true. To this end, we choose as points the non-degenerated discontinuities of $y(\bar{t},\cdot,\bar{\mathbf{u}})$ on the interval $[a,b]$, which are the only discontinuities on $[a,b]$ due to (ND). Let $x_{s,1}(\mathbf{u}),\dots,x_{s,N}(\mathbf{u})$ denote the corresponding mappings which exist due to Lemma 3.7.23 and are continuously Fréchet-differentiable due to Corollary 3.7.25.

In order to find a mapping (3.7.135) such that (3.7.136) is satisfied, we recall the mappings defined in (3.7.5) and (3.7.6) in Lemma 3.7.1 and set

$$\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right) = \Bigg(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_1(\bar{t},x;\bar{\mathbf{u}})\mathbb{1}_{[a,\bar{x}_1(\bar{\mathbf{u}})]}(x) + \sum_{k=2}^{K+1}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_k(\bar{t},x;\bar{\mathbf{u}})\mathbb{1}_{(x_{k-1}(\bar{\mathbf{u}}),x_k(\bar{\mathbf{u}})]}(x), \\ \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{s,1}(\bar{\mathbf{u}}),\dots,\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{s,N}(\bar{\mathbf{u}})\Bigg). \tag{3.7.137}$$

According to Lemma 3.7.1, $\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)$ in (3.7.137) is a linear bounded operator. We note that $\{x_{s,1},\dots,x_{s,N}\}\subset\{x_1,\dots,x_K\}$ and that $\{x_1,\dots,x_K\}\setminus\{x_{s,1},\dots,x_{s,N}\}$ is equal to the set of points lying on a boundary of a rarefaction wave. The set of these points is denoted by

$$\{x_{r,1},\dots,x_{r,K-N}\} \coloneqq \{x_1,\dots,x_K\}\setminus\{x_{s,1},\dots,x_{s,N}\}.$$

Now, it remains to show that the limit in (3.7.136) holds if we choose the linear bounded operator in (3.7.135) according to (3.7.137). To this end, we choose a small constant $\delta>0$ and rewrite the left-hand side of (3.7.136) as follows:

$$\begin{aligned}
&\lim_{\mathbf{u}\to\bar{\mathbf{u}}}\frac{\left\|y(\bar{t},\cdot;\mathbf{u})-y(\bar{t},\cdot;\bar{\mathbf{u}})-S^{(\bar{x}_k)}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}}),\right)\right\|_{1,[a,b]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}}\\
&=\lim_{\mathbf{u}\to\bar{\mathbf{u}}}\Bigg[\frac{\left\|y(\bar{t},\cdot;\mathbf{u})-y(\bar{t},\cdot;\bar{\mathbf{u}})-S^{(\bar{x}_k)}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}}),\right)\right\|_{1,[a,x_1(\bar{\mathbf{u}})-\delta]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}}\\
&+\sum_{i=1}^{N}\frac{\left\|y(\bar{t},\cdot;\mathbf{u})-y(\bar{t},\cdot;\bar{\mathbf{u}})-S^{(\bar{x}_k)}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}}),\right)\right\|_{1,[x_{s,i}(\bar{\mathbf{u}})-\delta,x_{s,i}(\bar{\mathbf{u}})+\delta]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}}\\
&+\sum_{i=1}^{K-N}\frac{\left\|y(\bar{t},\cdot;\mathbf{u})-y(\bar{t},\cdot;\bar{\mathbf{u}})-S^{(\bar{x}_k)}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}}),\right)\right\|_{1,[x_{r,i}(\bar{\mathbf{u}})-\delta,x_{r,i}(\bar{\mathbf{u}})+\delta]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}}\\
&+\sum_{i=2}^{K}\frac{\left\|y(\bar{t},\cdot;\mathbf{u})-y(\bar{t},\cdot;\bar{\mathbf{u}})-S^{(\bar{x}_k)}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}}),\right)\right\|_{1,[x_{i-1}(\bar{\mathbf{u}})+\delta,x_i(\bar{\mathbf{u}})-\delta]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}}
\end{aligned}$$

$$+ \frac{\left\| y(\bar{t},\cdot;\mathbf{u}) - y(\bar{t},\cdot;\bar{\mathbf{u}}) - S^{(\bar{x}_k)}_{y(\bar{t},\cdot;\bar{\mathbf{u}})}\left(\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)(\mathbf{u}-\bar{\mathbf{u}}),\right)\right\|_{1,[x_K(\bar{\mathbf{u}})+\delta,b]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}}\Bigg]$$

$$=: \lim_{\mathbf{u}\to\bar{\mathbf{u}}}\left[I_1 + \sum_{i=1}^{N} I_{i,s} + \sum_{i=1}^{K-N} I_{i,r} + \sum_{i=2}^{K} I_i + I_{K+1}\right].$$

We first consider I_1 and note that for all $u \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ the term I_1 is equal to

$$I_1 = \frac{\left\|Y_1(\bar{t},\cdot;\mathbf{u}) - Y_1(\bar{t},\cdot;\bar{\mathbf{u}}) - \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_1(\bar{t},\cdot;\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})\right\|_{1,[a,x_1(\bar{\mathbf{u}})-\delta]}}{\|\mathbf{u}-\bar{\mathbf{u}}\|_{\mathbf{U}}},$$

where ρ is chosen sufficiently small. Due to the continuous Fréchet-differentiability of $\mathbf{u} \mapsto Y_1(\bar{t},\cdot;\mathbf{u}) \in C([a, x_1(\bar{\mathbf{u}}) - \delta])$, we obtain that

$$\lim_{\mathbf{u}\to\bar{\mathbf{u}}} I_1 = 0.$$

Analogously, one can show that

$$\lim_{\mathbf{u}\to\bar{\mathbf{u}}} \sum_{i=2}^{K} I_i = 0 \quad \text{and} \quad \lim_{\mathbf{u}\to\bar{\mathbf{u}}} I_{K+1} = 0.$$

Using the continuous Fréchet-differentiability of the mappings (3.7.34), (3.7.42) and (3.7.48) with derivatives (3.7.35), (3.7.43) and (3.7.49) in Lemmas 3.7.11, 3.7.16 and 3.7.22, we obtain that

$$\lim_{\mathbf{u}\to\bar{\mathbf{u}}} \sum_{i=1}^{K-N} I_{i,r} = 0. \tag{3.7.138}$$

Finally, using the stability and the continuous Fréchet-differentiability of shock points in Lemma 3.7.23 and Corollary 3.7.25, one can analogously to the proof of Theorem 5.2.3 in [69] show that

$$\lim_{\mathbf{u}\to\bar{\mathbf{u}}} \sum_{i=1}^{N} I_{i,s} = 0$$

holds true. Hence, we can conclude that the mapping in (3.7.134) is shift-differentiable in $\mathbf{u} = \bar{\mathbf{u}}$. The continuous shift-differentiability holds due the continuity of the mapping (3.7.137) w.r.t. $\mathbf{u}$ and the fact that the identity (3.7.4) in Lemma 3.7.1 holds in a neighborhood $B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ of $\bar{\mathbf{u}}$.

□

Remark 3.7.27. In the previous theorem, we have proved the shift-differentiability of the entropy solution, where the shift-derivative is given in (3.7.137) by

$$\begin{aligned}\mathrm{T_s}\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right) = \Bigg(&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_1(\bar{t},x;\bar{\mathbf{u}})\mathbb{1}_{[a,\bar{x}_1(\bar{\mathbf{u}})]}(x) + \sum_{k=2}^{K+1}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_k(\bar{t},x;\bar{\mathbf{u}})\mathbb{1}_{(x_{k-1}(\bar{\mathbf{u}}),x_k(\bar{\mathbf{u}})]}(x),\\ &\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{s,1}(\bar{\mathbf{u}}),\ldots,\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{s,N}(\bar{\mathbf{u}})\Bigg).\end{aligned} \tag{3.7.139}$$

The derivatives $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_1(\bar{t},x;\bar{\mathbf{u}}),\ldots,\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_{K+1}(\bar{t},x;\bar{\mathbf{u}})$ can be computed according to the Lemmas 3.7.4, 3.7.9, 3.7.12, 3.7.14, 3.7.18 and 3.7.20. The derivative of the shock position $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_k(\mathbf{u})\cdot\delta\mathbf{u}$ for any $k\in\{1,\ldots,N\}$ is given by Corollary 3.7.25.

Using Theorem 3.7.26 in connection with Lemma 3.4.3, we obtain the following result which is an extension of Corollary 5.2.5 in [69] to the case that the shifting of rarefaction centers is allowed:

Theorem 3.7.28. *Suppose that* (A3) *and* (A4) *hold true, consider a time point* $\bar{t}\in]0,T[$*, an interval* $[a,b]$ *and let* $\bar{\mathbf{u}}\in\mathbf{U}$ *satisfy the conditions in* (ND) *for this choice. Denote by* $y(\bar{\mathbf{u}})$ *the corresponding entropy solution of the IBVP* (3.1.2) *and let* y_d *be continuous in a neighborhood of the discontinuities* $\bar{x}_1,\ldots,\bar{x}_N$ *of* $y(\bar{t},\cdot,\bar{\mathbf{u}})$ *on* $[a,b]$*. Then the reduced cost functional*

$$\mathbf{U}\ni\mathbf{u}\mapsto\hat{J}(\mathbf{u}):=J(y(\mathbf{u}),\mathbf{u})\in\mathbb{R}$$

is continuously Fréchet-differentiable in a neighborhood $B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$*, where* $\rho>0$ *is sufficiently small. The corresponding derivative in a direction* $\delta\mathbf{u}\in\mathbf{U}$ *is given by*

$$\begin{aligned}\hat{J}'(\mathbf{u})\cdot\delta\mathbf{u} = R'(\bar{\mathbf{u}})\delta\mathbf{u} + &\left(\psi_y(y(\bar{t},\cdot),y_d),\delta y^{\bar{t}}\right)_{2,[a,b]}\\ &+\sum_{j=k}^{N}\int_0^1\psi_y(y(\bar{t},x_k(\mathbf{u}))+\tau[y(\bar{t},x_k(\mathbf{u}))],y_d(x_k(\mathbf{u})))\,\mathrm{d}\tau[y(\bar{t},x_k(\mathbf{u}))]\bar{s}_k,\end{aligned} \tag{3.7.140}$$

where

$$(\delta y^{\bar{t}},\bar{s}_1,\ldots,\bar{s}_N)=\mathrm{T}_s\left(y(\bar{t},\cdot;\bar{\mathbf{u}})\right)\cdot\delta\mathbf{u} \tag{3.7.141}$$

is given according to (3.7.139)*.*

Similarly to Corollary 3.3.8 in [81], inserting the formula of the shock sensitivities

$\bar{s}_k$ in (3.7.140), one can show that the derivative of the cost functional in (3.7.140) has the representation

$$\begin{aligned}\hat{J}'(\mathbf{u})\cdot\delta\mathbf{u} =& R'(\bar{\mathbf{u}})\delta\mathbf{u} \\ &+ \left(\psi_y(y(\bar{t},\cdot),y_d),\delta y^{\bar{t}}\right)_{2,[a,b]} \\ &+ (pg_{u_1}(\cdot,y,u_1),\delta u_1)_{2,\Omega_{\bar{t}}} \\ &+ \sum_{i=1}^{n_x+1}(p(0,\cdot),\delta u_i^0)_{2,I_0^i(\mathbf{u})} \\ &+ \sum_{i\in I_{\mathrm{s},0}(\mathbf{u})} p(0,x_i^0)[u_0(x_i^0)]\delta x_i^0 \\ &+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1}(p(\cdot,\mathfrak{a}+),f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I^i_{B,\mathfrak{a}}(\mathbf{u})\cap]0,\bar{t}[} \\ &+ \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{a}}(\mathbf{u}):\\ t_i^{\mathfrak{a}}\le\bar{t}}} p(t_i^{\mathfrak{a}},\mathfrak{a}+)[f(y(t_i^{\mathfrak{a}},\mathfrak{a}+;\mathbf{u}))]\delta t_i^{\mathfrak{a}} \\ &- \sum_{i=1}^{n_{t,\mathfrak{b}}+1}(p(\cdot,\mathfrak{b}-),f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2,I^i_{B,\mathfrak{b}}\cap]0,\bar{t}[} \\ &- \sum_{\substack{i\in I_{\mathrm{s},\mathfrak{b}}(\mathbf{u}):\\ t_i^{\mathfrak{b}}\le\bar{t}}} p(t_i^{\mathfrak{b}},\mathfrak{b}-)[f(y(t_i^{\mathfrak{b}},\mathfrak{b}-;\mathbf{u}))]\delta t_i^{\mathfrak{b}} \\ &- \sum_{i\in I_{\mathrm{r},0}(\mathbf{u})} p_i^{\mathrm{r},0}\delta x_i^0 + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{a}}(\mathbf{u}):\\ t_i^{\mathfrak{a}}\le\bar{t}}} p_i^{\mathrm{r},\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{b}}(\mathbf{u}):\\ t_i^{\mathfrak{b}}\le\bar{t}}} p_i^{\mathrm{r},\mathfrak{b}}\delta t_i^{\mathfrak{b}},\end{aligned} \tag{3.7.142}$$

where p denotes the adjoint state with end data

$$p^{\bar{t}}(x)=\begin{cases}\int_0^1\psi_y(y(\bar{t},x_k(\mathbf{u}))+\tau[y(\bar{t},x_k(\mathbf{u}))],y_d(x_k(\mathbf{u})))\,\mathrm{d}\tau & \text{if } x\in\{x_1(\bar{\mathbf{u}}),\dots,x_N(\bar{\mathbf{u}})\},\\ 0 & \text{else}\end{cases} \tag{3.7.143}$$

and

$$p_j^{\mathrm{r},0} := \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))}\lim_{t\searrow 0}p(t,zt+x_j^0)\frac{1}{f''(f'^{-1}(z))}\,\mathrm{d}z,\quad j\in I_{\mathrm{r},0}(\mathbf{u}),$$

$$p_j^{\mathrm{r},\mathfrak{a}} := \int_{f'(u_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}\lim_{t\searrow t_j^{\mathfrak{a}}}p(t,z(t-t_j^{\mathfrak{a}}))\frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z,\quad j\in I_{\mathrm{r},\mathfrak{a}}(\mathbf{u}):t_j^{\mathfrak{a}}\le\bar{t},$$

$$p_j^{\mathrm{r},\mathfrak{b}} := \int_{f'(u_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(u_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \lim_{t \searrow t_j^{\mathfrak{b}}} p(t, z(t - t_j^{\mathfrak{b}})) \frac{z}{f''(f'^{-1}(z))} \, \mathrm{d}z, \quad j \in I_{\mathrm{r},\mathfrak{b}}(\mathbf{u}) : t_j^{\mathfrak{b}} \leq \bar{t}.$$

Using (3.7.142), we can finally prove Theorem 3.5.5.

Proof of Theorem 3.5.5. The proof is similar to the proof of Theorem 3.3.9 in [81]. We will therefore only sum up the main arguments and assume for simplicity that y_d is continuous on the interval $[a, b]$, where the treatment of the case that y_d possesses discontinuities on $[a, b]$ can be found in [81]. First of all, we note that the continuous differentiability of the mapping

$$B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \ni \mathbf{u} \mapsto \hat{J}(\mathbf{u}) \coloneqq J(y(\mathbf{u}), \mathbf{u})$$

holds due to Theorem 3.7.28. It remains to prove that the corresponding derivative is given by (3.5.30). We will show that for $\mathbf{u} = \bar{\mathbf{u}}$ and deduce from the stability of genuine characteristics and of the adjoint state that the representation in (3.5.30) is valid for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ if ρ is sufficiently small. Since $\bar{\mathbf{u}} \in \mathbf{U}$ satisfies (ND), the corresponding entropy solution $y(\bar{t}, \cdot; \bar{\mathbf{u}})$ of (3.1.2) has no shock generation points on $[a, b]$ and a finite number of non-degenerated discontinuities $a < x_1(\bar{\mathbf{u}}) < \cdots < x_N(\bar{\mathbf{u}}) < b$, which are no shock interaction points. We introduce the following notation: For each $x_k(\bar{\mathbf{u}})$ with $k \in \{1, \ldots, N\}$, denote by ξ_k^- the minimal backward characteristic through the point $(\bar{t}, x_k(\bar{\mathbf{u}}))$ and by ξ_k^+ the corresponding maximal backward characteristic. Finally, let ξ^a and ξ^b denote the genuine backward characteristic through $(\bar{t}, a)$ and $(\bar{t}, b)$, respectively. If any of these characteristics ends in points $(\tilde{t}, \mathfrak{a})$ or $(\tilde{t}, \mathfrak{b})$ with $\tilde{t} > 0$, then we extend it until $t = 0$ by setting $\xi^i|_{[0,\tilde{t}]} = \mathfrak{a}$ or $\xi^i|_{[0,\tilde{t}]} = \mathfrak{b}$, where $i \in \{a, b, +, -\}$. Then, we define the following domains:

$$D_k \coloneqq \left\{(t, x) \in \Omega_{\bar{t}} : \xi_k^-(t) < x < \xi_k^+(t)\right\}, \quad k = 1, \ldots, N, \quad D \coloneqq \bigcup_{k=1}^{N} D_k$$

and

$$\begin{aligned} S_k &\coloneqq \left\{(t, x) \in \Omega_{\bar{t}} : \xi_{k-1}^+(t) \leq x \leq \xi_k^-(t)\right\}, \quad k = 2, \ldots, N, \\ S_1 &\coloneqq \left\{(t, x) \in \Omega_{\bar{t}} : \xi^a \leq x \leq \xi_1^-(t)\right\}, \\ S_{N+1} &\coloneqq \left\{(t, x) \in \Omega_{\bar{t}} : \xi_{N+1}^+(t) \leq x \leq \xi^b\right\}. \end{aligned} \tag{3.7.144}$$

As in [81], we denote by $\tilde{p}$ the adjoint state according to Definition 3.5.4 with end data (3.7.143) and by p the corresponding adjoint state with end data

$$p^{\bar{t}} = \begin{cases} \int_0^1 \psi_y \left(y(\bar{t}, x+; \bar{\mathbf{u}})\right) + \tau[y(\bar{t}, x; \bar{\mathbf{u}})], y_d(x+) + \tau[y_d(x)]) \, \mathrm{d}\tau & \text{if } x \in [a, b], \\ 0 & \text{else.} \end{cases}$$

Therefore, (3.7.142) can be rewritten as

$$\begin{aligned} \hat{J}'(\bar{\mathbf{u}}) \cdot \delta\mathbf{u} =& R'(\bar{\mathbf{u}})\delta\mathbf{u} \\ &+ \left(p^{\bar{t}}, \delta y^{\bar{t}}\right)_{2,[a,b]} \\ &+ (\tilde{p} g_{u_1}(\cdot, y, u_1), \delta u_1)_{2,\Omega_{\bar{t}}} \\ &+ \sum_{i=1}^{n_x+1} (\tilde{p}(0, \cdot), \delta u_i^0)_{2, I_0^i(\bar{\mathbf{u}})} \\ &+ \sum_{i \in I_{\mathrm{s},0}(\bar{\mathbf{u}})} \tilde{p}(0, x_i^0)[u_0(x_i^0)]\delta x_i^0 \\ &+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (\tilde{p}(\cdot, \mathfrak{a}+), f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^i(\bar{\mathbf{u}}) \cap]0,\bar{t}[} \\ &+ \sum_{\substack{i \in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{a}} \leq \bar{t}}} \tilde{p}(t_i^{\mathfrak{a}}, \mathfrak{a}+)[f(y(t_i^{\mathfrak{a}}, \mathfrak{a}+; \bar{\mathbf{u}}))]\delta t_i^{\mathfrak{a}} \\ &- \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (\tilde{p}(\cdot, \mathfrak{b}), f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2, I_{B,\mathfrak{b}}^i \cap]0,\bar{t}[} \\ &- \sum_{\substack{i \in I_{\mathrm{s},\mathfrak{b}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{b}} \leq \bar{t}}} \tilde{p}(t_i^{\mathfrak{b}}, \mathfrak{b})[f(y(t_i^{\mathfrak{b}}, \mathfrak{b}-; \bar{\mathbf{u}}))]\delta t_i^{\mathfrak{b}} \\ &- \sum_{i \in I_{\mathrm{r},0}(\bar{\mathbf{u}})} \tilde{p}_i^{\mathrm{r},0}\delta x_i^0 + \sum_{\substack{i \in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{a}} \leq \bar{t}}} \tilde{p}_i^{\mathrm{r},\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i \in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{a}} \leq \bar{t}}} \tilde{p}_i^{\mathrm{r},\mathfrak{b}}\delta t_i^{\mathfrak{b}}, \end{aligned} \tag{3.7.145}$$

where $\delta y^{\bar{t}}$ is given in (3.7.141) and

$$\begin{aligned}
\tilde{p}_j^{\mathrm{r},0} &:= \int_{f'(\bar{u}_j^0(x_j^0))}^{f'(\bar{u}_{j+1}^0(x_j^0))} \lim_{t\searrow 0} \tilde{p}(t, zt + x_j^0) \frac{1}{f''(f'^{-1}(z))} \,\mathrm{d}z, \\
\tilde{p}_j^{\mathrm{r},\mathfrak{a}} &:= \int_{f'(\bar{u}_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(\bar{u}_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))} \lim_{t\searrow t_j^{\mathfrak{a}}} \tilde{p}(t, z(t - t_j^{\mathfrak{a}})) \frac{z}{f''(f'^{-1}(z))} \,\mathrm{d}z, \\
\tilde{p}_j^{\mathrm{r},\mathfrak{b}} &:= \int_{f'(\bar{u}_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(\bar{u}_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \lim_{t\searrow t_j^{\mathfrak{b}}} \tilde{p}(t, z(t - t_j^{\mathfrak{b}})) \frac{z}{f''(f'^{-1}(z))} \,\mathrm{d}z.
\end{aligned} \tag{3.7.146}$$

We note that

$$p|_{D^{\mathrm{cl}}} = \tilde{p} \quad \text{and} \quad \operatorname{supp} \tilde{p} \subset D^{\mathrm{cl}} \tag{3.7.147}$$

holds due to the definition of the adjoint state. Furthermore, the terms for which we take the limits in (3.7.146) are constant if t is sufficiently close to $t_j^{\mathfrak{b}}$. Therefore, choosing a sufficiently small $s > 0$, the terms in (3.7.146) can be rewritten by

$$\begin{aligned}
\tilde{p}_j^{\mathrm{r},0} &:= \int_{f'(\bar{u}_j^0(x_j^0))}^{f'(\bar{u}_{j+1}^0(x_j^0))} \mathbb{1}_D(s, zs + x_j^0) p(s, zs + x_j^0) \frac{1}{f''(f'^{-1}(z))} \,\mathrm{d}z, \\
\tilde{p}_j^{\mathrm{r},\mathfrak{a}} &:= \int_{f'(\bar{u}_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(\bar{u}_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))} \mathbb{1}_D(t_j^{\mathfrak{a}} + s, z \cdot s) p(t_j^{\mathfrak{a}} + s, z \cdot s) \frac{z}{f''(f'^{-1}(z))} \,\mathrm{d}z, \\
\tilde{p}_j^{\mathrm{r},\mathfrak{b}} &:= \int_{f'(\bar{u}_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(\bar{u}_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \mathbb{1}_D(t_j^{\mathfrak{b}} + s, z \cdot s) p(t_j^{\mathfrak{b}} + s, z \cdot s) \frac{z}{f''(f'^{-1}(z))} \,\mathrm{d}z.
\end{aligned} \tag{3.7.148}$$

Introducing the sets $A_{0,k} := D_k^{\mathrm{cl}} \cap \{t = 0\}$ and $A_{\mathfrak{a}/\mathfrak{b},k} := D_k^{\mathrm{cl}} \cap \{x = \mathfrak{a}/\mathfrak{b}\}$, we observe that

$$\begin{aligned}
&\left\{\bar{x}_i^0 : i \in I_{\mathrm{s},0}(\bar{\mathbf{u}})\right\} \subset \bigcup_{k=1}^N \operatorname{int} A_{0,k}, \\
&\left\{\bar{t}_i^{\mathfrak{a}/\mathfrak{b}} : i \in I_{\mathrm{s},\mathfrak{a}/\mathfrak{b}}(\bar{\mathbf{u}}) \text{ and } t_i^{\mathfrak{a}/\mathfrak{b}} \le \bar{t}\right\} \subset \bigcup_{k=1}^N \operatorname{int} A_{\mathfrak{a}/\mathfrak{b},k}.
\end{aligned} \tag{3.7.149}$$

We note that (3.7.149) holds since the sets S_k in (3.7.144) do not contain shock-curves due to their construction and the fact that $S_k \setminus \cap D_- = \emptyset$ due to Lemma 3.2.7.

From (3.7.145), (3.7.147) and (3.7.149) we obtain

$$\begin{aligned}
\hat{J}'(\bar{\mathbf{u}}) \cdot \delta\mathbf{u} =& R'(\bar{\mathbf{u}})\delta\mathbf{u} \\
&+ \left(p^{\bar{t}}, \delta y^{\bar{t}}\right)_{2,[a,b]} \\
&+ \sum_{k=1}^{N} \Bigg[(p g_{u_1}(\cdot, y, u_1), \delta u_1)_{2,\Omega_{\bar{t}}} \\
&+ \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0)_{2, I_0^i(\bar{\mathbf{u}}) \cap A_{0,k}} \\
&+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, \mathfrak{a}+), f'(u_i^{B,\mathfrak{a}}) \delta u_i^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^i(\bar{\mathbf{u}}) \cap A_{\mathfrak{a},k}} \\
&- \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot, \mathfrak{b}-), f'(u_i^{B,\mathfrak{b}}) \delta u_i^{B,\mathfrak{b}})_{2, I_{B,\mathfrak{b}}^i \cap A_{\mathfrak{b},k}} \Bigg] \\
&+ \sum_{i \in I_{\mathrm{s},0}(\bar{\mathbf{u}})} p(0, x_i^0)[u_0(x_i^0)] \delta x_i^0 \\
&+ \sum_{\substack{i \in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{a}} \le \bar{t}}} p(t_i^{\mathfrak{a}}, \mathfrak{a}+)[f(y(t_i^{\mathfrak{a}}, \mathfrak{a}+; \bar{\mathbf{u}}))] \delta t_i^{\mathfrak{a}} \\
&- \sum_{\substack{i \in I_{\mathrm{s},\mathfrak{b}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{b}} \le \bar{t}}} p(t_i^{\mathfrak{b}}, \mathfrak{b}-)[f(y(t_i^{\mathfrak{b}}, \mathfrak{b}-; \bar{\mathbf{u}}))] \delta t_i^{\mathfrak{b}} \\
&- \sum_{i \in I_{\mathrm{r},0}(\bar{\mathbf{u}})} \tilde{p}_i^{\mathrm{r},0} \delta x_i^0 + \sum_{\substack{i \in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{a}} \le \bar{t}}} \tilde{p}_i^{\mathrm{r},\mathfrak{a}} \delta t_i^{\mathfrak{a}} + \sum_{\substack{i \in I_{\mathrm{r},\mathfrak{b}}(\bar{\mathbf{u}}): \\ t_i^{\mathfrak{b}} \le \bar{t}}} \tilde{p}_i^{\mathrm{r},\mathfrak{b}} \delta t_i^{\mathfrak{b}}.
\end{aligned} \tag{3.7.150}$$

In order to derive the representation of $\hat{J}'(\bar{\mathbf{u}}) \cdot \delta\mathbf{u}$ in (3.5.30), we next observe that

$$\delta y^{\bar{t}}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_1(\bar{t}, x; \bar{\mathbf{u}}) \cdot \mathbb{1}_{[a, \bar{x}_1(\bar{\mathbf{u}})]}(x) + \sum_{k=2}^{K+1} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_k(\bar{t}, x; \bar{\mathbf{u}}) \cdot \mathbb{1}_{(x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})]}(x) \right) \tag{3.7.151}$$

and therefore

$$\delta y^{\bar{t}}|_{]x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})[} = \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_k(\bar{t}, x; \bar{\mathbf{u}}) \delta\mathbf{u} \quad \forall k \in \{1, \ldots, N+1\}.$$

We set $\bar{x}_0(\bar{\mathbf{u}}) = a$ and $\bar{x}_{N+1}(\bar{\mathbf{u}}) = b$. Using the formula (3.7.22) in Remark 3.7.7, one can show for all $k \in \{1, \ldots, N+1\}$ that

$$\begin{aligned}
&\left(p^{\bar t}, \delta y^{\bar t}\right)_{2,]x_{k-1}(\bar{\mathbf{u}}),x_k(\bar{\mathbf{u}})[} = \left(p^{\bar t}, \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_k(\bar t, x; \bar{\mathbf{u}})\delta \mathbf{u}\right)_{2,]x_{k-1}(\bar{\mathbf{u}}),x_k(\bar{\mathbf{u}})[}\\
&= \left(p(\cdot), g_{u_1}^T(\cdot, y, u_1)\delta u_1\right)_{2,S_k} + \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2,I_0^i(\bar{\mathbf{u}})\cap B_{0,k}}\\
&+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot,\mathfrak{a}), f'(u_i^{B,\mathfrak{a}}(\cdot))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2,I_{B,\mathfrak{a}}^i(\bar{\mathbf{u}})\cap B_{\mathfrak{a},k}}\\
&- \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot,\mathfrak{b}), f'(u_i^{B,\mathfrak{b}}(\cdot))\delta u_i^{B,\mathfrak{b}}(\cdot))_{2,I_{B,\mathfrak{b}}^i\cap B_{\mathfrak{b},k}}\\
&- \sum_{i\in I_{\mathrm{r},0}(\bar{\mathbf{u}})} \bar p_{k,i}^{\mathrm{r},0}\delta x_i^0 + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}):\\ t_i^{\mathfrak{a}}\le \bar t}} \bar p_{k,i}^{\mathrm{r},\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{b}}(\bar{\mathbf{u}}):\\ t_i^{\mathfrak{b}}\le \bar t}} \bar p_{k,i}^{\mathrm{r},\mathfrak{b}}\delta t_i^{\mathfrak{b}}.
\end{aligned} \tag{3.7.152}$$

where $B_{0,k} := S_k^{\mathrm{cl}} \cap \{t = 0\}$, $B_{\mathfrak{a}/\mathfrak{b},k} := S_k^{\mathrm{cl}} \cap \{x = \mathfrak{a}/\mathfrak{b}\}$ and

$$\begin{aligned}
\bar p_{k,j}^{\mathrm{r},0} &:= \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))} \mathbb{1}_{S_k}(s, zs + x_j^0)p(s, zs + x_j^0)\frac{1}{f''(f'^{-1}(z))}\,\mathrm{d}z,\\
\bar p_{k,j}^{\mathrm{r},\mathfrak{a}} &:= \int_{f'(u_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))} \mathbb{1}_{S_k}(t_j^{\mathfrak{a}} + s, z\cdot s)p(t_j^{\mathfrak{a}} + s, z\cdot s)\frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z,\\
\bar p_{k,j}^{\mathrm{r},\mathfrak{b}} &:= \int_{f'(u_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(u_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \mathbb{1}_{S_k}(t_j^{\mathfrak{b}} + s, z\cdot s)p(t_j^{\mathfrak{b}} + s, z\cdot s).\frac{z}{f''(f'^{-1}(z))}\,\mathrm{d}z.
\end{aligned}$$

Here, s is chosen as in (3.7.148). From (3.7.151) and (3.7.152), we obtain that

$$\begin{aligned}
&\left(p^{\bar t}, \delta y^{\bar t}\right)_{2,[a,b]}\\
&= \left(p(\cdot), g_{u_1}^T(\cdot, y, u_1)\delta u_1\right)_{2,S} + \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0)_{2,I_0^i(\bar{\mathbf{u}})\cap(S\cap\{t=0\})}\\
&+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot,\mathfrak{a}), f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2,I_{B,\mathfrak{a}}^i(\bar{\mathbf{u}})\cap(S\cap\{x=\mathfrak{a}\})}\\
&- \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot,\mathfrak{b}), f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2,I_{B,\mathfrak{b}}^i\cap(S\cap\{x=\mathfrak{b}\})}\\
&- \sum_{i\in I_{\mathrm{r},0}(\bar{\mathbf{u}})} \bar p_i^{\mathrm{r},0}\delta x_i^0 + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}):\\ t_i^{\mathfrak{a}}\le \bar t}} \bar p_i^{\mathrm{r},\mathfrak{a}}\delta t_i^{\mathfrak{a}} + \sum_{\substack{i\in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}):\\ t_i^{\mathfrak{a}}\le \bar t}} \bar p_i^{\mathrm{r},\mathfrak{b}}\delta t_i^{\mathfrak{b}},
\end{aligned} \tag{3.7.153}$$

where

$$\bar{p}_j^{\mathrm{r},0} := \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))} \mathbb{1}_S(s, zs + x_j^0) p(s, zs + x_j^0) \frac{1}{f''(f'^{-1}(z))} \,\mathrm{d}z,$$

$$\bar{p}_j^{\mathrm{r},\mathfrak{a}} := \int_{f'(u_{j+1}^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))}^{f'(u_j^{B,\mathfrak{a}}(t_j^{\mathfrak{a}}))} \mathbb{1}_S(t_j^{\mathfrak{a}} + s, z \cdot s) p(t_j^{\mathfrak{a}} + s, z \cdot s) \frac{z}{f''(f'^{-1}(z))} \,\mathrm{d}z,$$

$$\bar{p}_j^{\mathrm{r},\mathfrak{b}} := \int_{f'(u_j^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))}^{f'(u_{j+1}^{B,\mathfrak{b}}(t_j^{\mathfrak{b}}))} \mathbb{1}_S(t_j^{\mathfrak{b}} + s, z \cdot s) p(t_j^{\mathfrak{b}} + s, z \cdot s) \frac{z}{f''(f'^{-1}(z))} \,\mathrm{d}z$$

and $S = \bigcup_{k=1}^{N+1} S_k$. We note that

$$\Omega_{\bar{t}} \setminus D = S = \bigcup_{k=1}^{N+1} S_k. \tag{3.7.154}$$

Due to (3.7.154), inserting (3.7.153) in (3.7.150) finally yields the representation of $\hat{J}'(\bar{\mathbf{u}})\delta\mathbf{u}$ in Theorem 3.5.5.

□

CHAPTER 4

Optimality theory

We recall the optimal control problem that is introduced in §3.3:

$$\left.\begin{aligned} &\min_{\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}} J(y(\mathbf{u}),\mathbf{u}) = \int_a^b \psi(y(\bar{t},x;\mathbf{u}),y_d(x))\,\mathrm{d}x + R(\mathbf{u}),\\ &\qquad \text{where } y(\mathbf{u}) \text{ is the entropy solution of the IBVP (3.1.2)}\\ &\qquad \text{and } y(\bar{t},x) \le \bar{y}\,(x) \quad \text{for all } x\in[a,b]. \end{aligned}\right\} \tag{P}$$

The goal of this chapter is to analyze (P). Therefore, we will first prove the existence of an optimal solution in §4.1 and then derive first-order necessary optimality conditions for (P) in §4.2. The proof of the existence of optimal controls follows by standard arguments, see, e.g., [57]. However, the standard techniques which are used to derive first-order necessary optimality conditions for state-constrained optimal control problems, for example in [19], are not applicable to (P) for the following reason: Due to the pointwise state constraints

$$y(\bar{t},x) \le \bar{y}\,(x) \quad \text{for all } x\in[a,b], \tag{4.0.1}$$

we have to consider the state y at least in L^∞ to assure that Robinson's CQ is possible to be satified, cf. [19]. However, since entropy solutions of hyperbolic balance laws may develop moving discontinuities, the control-to-state mapping is in general not continuous to L^∞ and therefore not differentiable. This poses a difficulty since first order optimality conditions for problems with state constraints as well as the Robinson's CQ require the first derivative of the control-to-state mapping. To cope with this problem, we will use Lemma 3.7.1 in order to introduce auxiliary state variables in §4.2.2. In terms of these new states, we will formulate first-order necessary optimality conditions in §4.2.3, which can be reformulated in terms of the original state y.

Remark 4.0.1. The state constraints in (4.0.1) are only considered for a fixed time point $t = \bar{t}$ and not for all $t \in [0, \bar{t}]$. This restriction is due to the fact that the results in Lemma 3.7.1, which are crucial in this chapter, hold for fixed time points $\bar{t} \in]0, T]$, respectively. Nevertheless, since by Theorem 3.5.3 the results of Lemma 3.7.1 hold for almost all time points $t \in]0, T]$, it seems be possible to extend the results of the remaining part of this thesis to the case that the state constraints are considered on the whole interval $[0, \bar{t}]$.

In the remaining chapters of this thesis we will use the following convention, which is slightly different from Convention 3.2.2.

Convention 4.0.2. We consider the representative of y satisfying
$y \in C([0, T]; L^1_{\text{loc}}(\Omega))$, $y(t, x) = y(t, x-)$ for all $(t, x) \in [0, T] \times \Omega \setminus \{a\}$ and
$y(t, a) = y(t, a+)$ for all $t \in [0, T]$.

We note that one can easily check that the results of the previous sections are still valid under Convention 4.0.2.

4.1 Existence of optimal controls

In the following result, we will prove the existence of an optimal control for (P).

Theorem 4.1.1. *Let* (A3) *and* (A4) *hold true and assume that there exists* $\tilde{\mathbf{u}} \in \mathbf{U}_{ad}$ *such that* $y(\bar{t}, x; \tilde{\mathbf{u}}) \leq \bar{y}\,(x)$ *is satisfied for all* $x \in [a, b]$. *Then* (P) *admits a globally optimal solution.*

Proof. The proof uses standard techniques, cf. [57]. We firstly prove compactness of the set $\tilde{\mathbf{U}}_{\text{ad}} := \{\mathbf{u} \in \mathbf{U}_{\text{ad}} : y(\bar{t}, x; \mathbf{u}) \leq \bar{y}\,(x)\ \forall x \in [a, b]\}$. Since $\tilde{\mathbf{U}}_{\text{ad}}$ is by assumption nonempty, we can consider a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}} \subset \tilde{\mathbf{U}}_{\text{ad}} \subset \mathbf{U}$. Due to the compactness of $\mathbf{U}_{\text{ad}}$ in $\mathbf{U}$, there exists a subsequence, again denoted by $(\mathbf{u}_n)_{n\in\mathbb{N}}$, converging to some $\bar{\mathbf{u}} \in \mathbf{U}_{\text{ad}}$ w.r.t. $\|\cdot\|_{\mathbf{U}}$. Corollary 3.3.5 implies that the sequence $(y(\bar{t}, \cdot; \mathbf{u}_n))_{n\in\mathbb{N}}$ converges in $L^1([a, b])$ to $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ and hence there exists a convergent subsequence, again denoted by $(y(\bar{t}, \cdot; \mathbf{u}_n))_{n\in\mathbb{N}}$, converging pointwise almost everywhere to $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ on $[a, b]$. Therefore and since $y(\bar{t}, x; \mathbf{u}_n) \leq \bar{y}\,(x)$ holds for all $x \in [a, b]$ and all $n \in \mathbb{N}$, we obtain that

$$y(\bar{t}, x; \bar{\mathbf{u}}) \leq \bar{y}\,(x) \quad \text{for a.a. } x \in [a, b]. \tag{4.1.1}$$

In order to show that (4.1.1) holds true for all $x \in [a,b]$, let $\hat{x} \in]a,b]$ be arbitrarily chosen. Due to (4.1.1), we can choose a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \nearrow \hat{x}$ for $n \to \infty$ such that for all x_n (4.1.1) is satisfied. By Convention 4.0.2, we obtain

$$y(\bar{t},\hat{x};\bar{\mathbf{u}}) = \lim_{n\to\infty} y(\bar{t},x_n;\bar{\mathbf{u}}) \leq \lim_{n\to\infty} \bar{y}(x_n) = \bar{y}(\hat{x}),$$

where the last equality holds due to the continuity of $\bar{y}(\cdot)$. One can analogously prove that (4.1.1) holds for $\hat{x} = a$ by taking a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \searrow a$ for $n \to \infty$ and using Convention 4.0.2. Thus, (4.1.1) holds for all $x \in [a,b]$ yielding that $\bar{\mathbf{u}} \in \tilde{\mathbf{U}}_{\text{ad}}$. Therefore, $\tilde{\mathbf{U}}_{\text{ad}}$ is compact. We now consider a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}} \subset \tilde{\mathbf{U}}_{\text{ad}}$ satisfying

$$\hat{J}(\mathbf{u}_n) \to \inf_{\mathbf{u}\in\tilde{\mathbf{U}}_{\text{ad}}} \hat{J}(\mathbf{u}) \quad \text{for } k \to \infty.$$

Since $\tilde{\mathbf{U}}_{\text{ad}}$ is compact, there exists a convergent subsequence, again denoted by $(\mathbf{u}_n)_{n\in\mathbb{N}}$, with $\mathbf{u}_n \to \bar{\mathbf{u}} \in \tilde{\mathbf{U}}_{\text{ad}}$. Since $\hat{J}$ is Lipschitz continuous w.r.t. $\mathbf{u}$ by Corollary 3.3.6, we obtain $\hat{J}(\bar{\mathbf{u}}) = \inf_{\mathbf{u}\in\tilde{\mathbf{U}}_{\text{ad}}} \hat{J}(\mathbf{u})$ and hence, $\bar{\mathbf{u}}$ is a global minimum for (P). $\square$

4.2 Optimality conditions

In this section, we want to derive optimality conditions for (P). To this end, we firstly consider a general optimal control problem ($\mathcal{P}$) in §4.2.1. Then we introduce auxiliary state variables in §4.2.2. Using these new states and the results for the general problem ($\mathcal{P}$), we can derive first-order necessary optimality conditions for (P) in §4.2.

4.2.1 Optimality conditions for a general optimal control problem

Our aim is to derive necessary optimality conditions for (P). To this end, we firstly consider a more general optimization problem (cf. [41, Subsection 1.7.3]):

$$\min_{z\in Z} f(z) \quad \text{subject to} \quad G(z) \in \mathcal{K}, \quad z \in \mathcal{C}. \tag{$\mathcal{P}$}$$

Theorem 4.2.1. *[Karush-Kuhn-Tucker conditions, [89]] Let $\bar{z} \in Z$ be a local optimum of $(\mathcal{P})$. Assume that Z and V are Banach-spaces and that the mappings $f : Z \to \mathbb{R}$ and $G : Z \to V$ are continuously differentiable in $\bar{z}$. In addition, let $\mathcal{C} \subset Z$ be closed, convex and nonempty and $\mathcal{K} \subset V$ be a closed convex cone. Finally, assume that Robinson's CQ is satisfied at $\bar{z} \in Z$,i.e.,*

$$0 \in \operatorname{int}\left(G\left(\bar{z}\right) + G'\left(\bar{z}\right)\left(\mathcal{C} - \bar{z}\right) - \mathcal{K}\right). \tag{4.2.1}$$

Then there exists a Lagrange multiplier $\bar{q} \in V^*$ such that the Karush-Kuhn-Tucker conditions

$$\begin{aligned}&G\left(\bar{z}\right) \in \mathcal{K}, \quad \bar{q} \in \mathcal{K}^{\circ} := \{q \in V^* : \langle q, v\rangle_{V^*,V} \leq 0 \quad \forall v \in \mathcal{K}\},\\ &\langle \bar{q}, G\left(\bar{z}\right)\rangle_{V^*,V} = 0, \quad \bar{z} \in \mathcal{C}, \quad \left\langle f'\left(\bar{z}\right) + G'\left(\bar{z}\right)^* \bar{q}, z - \bar{z}\right\rangle_{Z^*,Z} \geq 0 \quad \forall z \in \mathcal{C}\end{aligned}$$

hold true.

Proof. See [89]. □

In order to derive optimality conditions for (P), our goal is to bring (P) in the form of $(\mathcal{P})$ and then use Theorem 4.2.1. To this end, we choose $Z = \mathbf{U}$, $\mathcal{C} = \mathbf{U}_{\text{ad}}$ and $f = \hat{J}$. A naive choice for the mapping G could be

$$G(\mathbf{u}) = y(\bar{t}, \cdot; \mathbf{u}) - \bar{y}(\cdot) \in V, \tag{4.2.2}$$

where y denotes the entropy solution of the IBVP (3.1.2).

In order to assure that the set on the right-hand side of (4.2.1) has an inner point, we have to choose V such that $V \subset C(\Omega)$ or at least $V \subset L^\infty(\Omega)$ holds true. However, since y may develop shocks after finite time, the mapping in (4.2.2) is not even continuous. In order to cope with this problem, we have to choose G differently from (4.2.2). To this end, we will reformulate the state variable y.

4.2.2 Reformulation of the state variable

We consider a control $\bar{\mathbf{u}} \in \mathbf{U}$ such that the conditions (ND) are satisfied and we can apply Lemma 3.7.1. More precisely, (3.7.4) yields that for all $\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ the corresponding entropy solution $y(\bar{t}, \cdot, u)|_{[a,b]}$ can be rewritten by

$$y\left(\bar{t}, x; \mathbf{u}\right)|_{[a,b]} = Y_1(\bar{t}, x; \mathbf{u}) \cdot \mathbb{1}_{[a,x_1(\mathbf{u})]}(x) + \sum_{k=2}^{K+1} Y_k(\bar{t}, x; \mathbf{u}) \cdot \mathbb{1}_{(x_{k-1}(\mathbf{u}),x_k(\mathbf{u})]}(x).$$

Lemma 3.7.1 further yields that the mappings

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y_k\left(\bar{t}, \cdot; \mathbf{u}\right) \in C\left(I_k^\varepsilon\right), \qquad k = 1, \ldots, K+1, \tag{4.2.3a}$$

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto x_k\left(\mathbf{u}\right) \in \mathbb{R}, \qquad k = 1, \ldots, K \tag{4.2.3b}$$

are continuously Fréchet-differentiable. We recall the notation $x_0 = a$, $x_{K+1} = b$ and $I_k^\varepsilon := (x_{k-1}(\bar{\mathbf{u}}) - \varepsilon, x_k(\bar{\mathbf{u}}) + \varepsilon)$ for $k = 1, \ldots, K+1$, where $\varepsilon > 0$ is a sufficiently small constant. The main idea is to introduce the mappings in (4.2.3) as new state variables, which are transformed to the unit interval $[0,1]$ by using the variable transformations

$$\begin{aligned} &\varphi_{k;\bar{\mathbf{u}}} : [0,1] \to \big[x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})\big],\ \lambda \mapsto x_{k-1}(\bar{\mathbf{u}}) + \lambda\big(x_k(\bar{\mathbf{u}}) - x_{k-1}(\bar{\mathbf{u}})\big), \\ &\varphi_{k;\bar{\mathbf{u}}}^{-1} : \big[x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})\big] \to [0,1],\ x \mapsto \frac{x - x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}}) - x_{k-1}(\bar{\mathbf{u}})}. \end{aligned} \tag{4.2.4}$$

Hence, we introduce the mappings

$$\begin{gathered} \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto y_k(\mathbf{u}) \in C([0,1]), \quad k = 1, \ldots, K+1, \\ \mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto x_k(\mathbf{u}) \in \mathbb{R}, \quad k = 1, \ldots, K \end{gathered} \tag{4.2.5}$$

as new state variables, where

$$y_k(\lambda; \mathbf{u}) := Y_k\big(\bar{t}, \varphi_{k;\bar{\mathbf{u}}}(\lambda); \mathbf{u}\big), \quad \lambda \in [0,1]. \tag{4.2.6}$$

Using the variable transformations in (4.2.4), we also transform the state constraints in (4.0.1) on the unit interval:

$$y_k(\lambda; \mathbf{u}) \leq \bar{y}\big(\bar{t}, \varphi_{k;\bar{\mathbf{u}}}(\lambda)\big) =: \bar{y}_k(\lambda; \mathbf{u}), \quad \lambda \in [0,1]. \tag{4.2.7}$$

In the next Lemma we will show that the state constraints in (4.0.1) and in (4.2.7) are equivalent:

Lemma 4.2.2. *We assume that* (A3) *and* (A4) *hold true and consider some control* $\bar{\mathbf{u}} \in \mathbf{U}$ *that satisfies the conditions in* (ND). *Then the state constraints in* (4.0.1) *and in* (4.2.7) *are equivalent, i.e.,*

$$y(\bar{t}, \cdot; \bar{\mathbf{u}}) \leq \bar{y}(\cdot) \text{ on } [a,b]$$

holds if and only if

$$y_k(\cdot; \bar{\mathbf{u}}) \leq \bar{y}_k(\cdot; \bar{\mathbf{u}}) \text{ on } [0,1]$$

is satisfied for all $k = 1, \ldots, K+1$.

Proof. We firstly suppose that

$$y_k(\lambda; \bar{\mathbf{u}}) \leq \bar{y}_k(\lambda; \bar{\mathbf{u}}) \text{ on } [0,1] \quad \text{for all } k = 1, \ldots, K+1. \tag{4.2.8}$$

Using the definitions of $y_k(\lambda; \mathbf{u})$ in (4.2.6) and $\bar{y}_k(\lambda; \bar{\mathbf{u}})$ in (4.2.7), we obtain from (4.2.8) that

$$Y_k\big(\bar{t}, \varphi_{k;\bar{\mathbf{u}}}(\lambda); \mathbf{u}\big) \leq \bar{y}(\varphi_{k;\bar{\mathbf{u}}}(\lambda)) \text{ on } [0,1] \quad \text{for all } k = 1, \ldots, K+1,$$

which is equivalent to

$$Y_k\big(\bar{t}, x; \mathbf{u}\big) \leq \bar{y}(x) \text{ on } [x_{k-1}, x_k] \quad \text{for all } k = 1, \ldots, K+1. \tag{4.2.9}$$

Using the representation of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ in (3.7.4) and Convention 3.2.2, we see that (4.2.9) yields

$$y(\bar{t}, x; \bar{\mathbf{u}}) \leq \bar{y}(x) \text{ on } [a,b].$$

Now we suppose that $y(\bar{t}, x; \bar{\mathbf{u}}) \leq \bar{y}(x)$ holds on $[a,b]$. From (3.7.4) we obtain that

$$\begin{aligned} &Y_k\big(\bar{t}, x; \mathbf{u}\big) \leq \bar{y}(x) \text{ on }]x_{k-1}, x_k] \quad \text{for all } k = 2, \ldots, K+1. \\ &Y_1\big(\bar{t}, x; \mathbf{u}\big) \leq \bar{y}(x) \text{ on } [a, x_1] \end{aligned}$$

Next, we consider some $x_{\tilde{k}}$ with $\tilde{k} \in \{1, \ldots, K+1\}$ where $y(\bar{t}, x; \bar{\mathbf{u}})$ has a discontinuity. Due to (3.2.3), all discontinuities of $y(\bar{t}, \cdot; \bar{\mathbf{u}})$ are down jumps such that we get

$$Y_k\big(\bar{t}, x; \mathbf{u}\big) \leq \bar{y}(x) \text{ on } [x_{k-1}, x_k] \quad \text{for all } k = 1, \ldots, K+1. \tag{4.2.10}$$

Using again the definitions of $y_k(\lambda; \mathbf{u})$ in (4.2.6) and $\bar{y}_k(\lambda; \bar{\mathbf{u}})$ in (4.2.7), (4.2.10) finally yields

$$y_k(\cdot; \bar{\mathbf{u}}) \leq \bar{y}_k(\cdot; \bar{\mathbf{u}}) \text{ on } [0,1] \quad \text{for all } k = 1, \ldots, K+1.$$

□

Using the definition of the new states in (4.2.6) and Lemma 3.7.1, we obtain the following result:

Theorem 4.2.3 (Continuous differentiability of the state)**.** *We assume that* (A3) *and* (A4) *hold true. In addition, consider a control* $\bar{\mathbf{u}} \in \mathbf{U}$ *such that the assumptions* *in* (ND) *are satisfied. Then the mapping*

$$\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto \big(y_1(\mathbf{u}), \ldots, y_{K+1}(\mathbf{u}), x_1(\mathbf{u}), \ldots, x_K(\mathbf{u})\big) \in C\left([0,1]\right)^{K+1} \times \mathbb{R}^K$$

is well-defined and continuously Fréchet-differentiable if $\rho > 0$ is sufficiently small. The derivatives of $y_1, \ldots, y_{K+1}$ w.r.t. $\mathbf{u}$ are given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} y_k(\lambda;\mathbf{u})\delta\mathbf{u} = {} & \frac{\mathrm{d}}{\mathrm{d}x} Y_k\big(\bar{t}, x_{k-1}(\mathbf{u}) + \lambda(x_k(\mathbf{u}) - x_{k-1}(\mathbf{u})); \mathbf{u}\big) \cdot \Big[\lambda \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_k(\mathbf{u})\delta\mathbf{u} \\ & + (1-\lambda)\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{k-1}(\mathbf{u})\delta\mathbf{u}\Big] + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_k\big(\bar{t}, x_{k-1}(\mathbf{u}) + \lambda(x_k(\mathbf{u}) - x_{k-1}(\mathbf{u})); \mathbf{u}\big)\delta\mathbf{u}.\end{aligned} \tag{4.2.11}$$

The derivatives of $x_k(\mathbf{u})$, $k = 1, \ldots, K$ w.r.t. $\mathbf{u}$ can be computed according to Theorem 3.7.24 if $(\bar{t}, x_k)$ is a discontinuity of $y(\bar{t}, \cdot; \mathbf{u})$ or with the help of (3.2.9) *if $(\bar{t}, x_k)$ lies on the boundary of a rarefaction wave.*

Moreover, the derivatives of Y_k w.r.t. $\mathbf{u}$ can be computed according to the Lemmas 3.7.4, 3.7.9, 3.7.12, 3.7.14, 3.7.18 or 3.7.20.

4.2.3 Optimality conditions for the optimal control problem considered in this thesis

Let $\bar{\mathbf{u}} \in \mathbf{U}$ be a locally optimal solution of (P) such that (ND) is satisfied. Recalling the general optimization problem in ($\mathcal{P}$), we want to bring (P) in the same form such that we can derive necessary optimality conditions by using Theorem 4.2.1. Since in $\bar{\mathbf{u}} \in \mathbf{U}$ the conditions of (ND) are satisfied, we can introduce the mappings in (4.2.5) as new state variables and set:

$$Z = \mathbf{U},\ V = C\left([0,1]\right)^{K+1} \times \mathbb{R}^K, \tag{4.2.12a}$$

$$G_k(\mathbf{u}) = y_k(\lambda;\mathbf{u}) - \bar{y}_k(\lambda;\mathbf{u}), \quad k = 1, \ldots, K+1, \tag{4.2.12b}$$

$$G_{k+K+1}(\mathbf{u}) = x_k(\mathbf{u}), \quad k = 1, \ldots, K, \tag{4.2.12c}$$

$$\mathcal{K} = C([0,1];]-\infty,0])^{K+1} \times \mathbb{R}^K, \ \text{ and } \ \mathcal{C} = \mathbf{U}_{\mathrm{ad}}. \tag{4.2.12d}$$

We firstly observe that due to the choice in (4.2.12) Robinson's CQ in (4.2.1) reads

$$\mathbf{0} \in \operatorname{int}\left(\begin{pmatrix} y_1(\cdot;\mathbf{u}) - \bar{y}_1(\cdot;\mathbf{u}) \\ \vdots \\ y_{K+1}(\cdot;\mathbf{u}) - \bar{y}_{K+1}(\cdot;\mathbf{u}) \\ x_1(\mathbf{u}) \\ \vdots \\ x_K(\mathbf{u}) \end{pmatrix} + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} \begin{pmatrix} y_1(\cdot;\mathbf{u}) - \bar{y}_1(\cdot;\mathbf{u}) \\ \vdots \\ y_{K+1}(\cdot;\mathbf{u}) - \bar{y}_{K+1}(\cdot;\mathbf{u}) \\ x_1(\mathbf{u}) \\ \vdots \\ x_K(\mathbf{u}) \end{pmatrix} (\mathbf{U} - \bar{\mathbf{u}}) - \begin{pmatrix} C([0,1];]-\infty,0]) \\ \vdots \\ C([0,1];]-\infty,0]) \\ \mathbb{R} \\ \vdots \\ \mathbb{R} \end{pmatrix}\right)$$

which is obviously equivalent to

$$\mathbf{0} \in \operatorname{int}\left(\begin{pmatrix} y_1(\cdot;\mathbf{u}) - \bar{y}_1(\cdot;\mathbf{u}) \\ \vdots \\ y_{K+1}(\cdot;\mathbf{u}) - \bar{y}_{K+1}(\cdot;\mathbf{u}) \end{pmatrix} + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} \begin{pmatrix} y_1(\cdot;\mathbf{u}) - \bar{y}_1(\cdot;\mathbf{u}) \\ \vdots \\ y_{K+1}(\cdot;\mathbf{u}) - \bar{y}_{K+1}(\cdot;\mathbf{u}) \end{pmatrix} (\mathbf{U} - \bar{\mathbf{u}}) - \begin{pmatrix} C([0,1];]-\infty,0]) \\ \vdots \\ C([0,1];]-\infty,0]) \end{pmatrix}\right) \tag{4.2.13}$$

Remark 4.2.4. Technically speaking, the mappings $\mathbf{u} \mapsto G_k(\mathbf{u})$ are only well-defined on a neighborhood $B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ of $\bar{\mathbf{u}}$. Therefore, in order to guarantee that the mappings in (4.2.12b)-(4.2.12c) are well-defined on the whole Banach space $Z = \mathbf{U}$, we choose continuously differentiable extensions such that

$$\begin{aligned} G_k(\mathbf{u})\Big|_{B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})} &= (y_k(\lambda;\mathbf{u}) - \bar{y}_k(\lambda;\mathbf{u})), && k = 1,\dots,K+1 \\ G_{k+K+1}(\mathbf{u})\Big|_{B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})} &= x_k(\mathbf{u}), && k = 1,\dots,K. \end{aligned}$$

is satisfied.

Remark 4.2.5. Using Theorem 2.5.2, we obtain that

$$V^* = \left(C([0,1])^{K+1} \times \mathbb{R}^K\right)^* = \left(C([0,1])^*\right)^{K+1} \times \mathbb{R}^K = \mathcal{M}([0,1])^{K+1} \times \mathbb{R}^K. \tag{4.2.14}$$

Using (4.2.14), we obtain the following representation of the polar cone:

Lemma 4.2.6. *The polar cone of $\mathcal{K}$ can be characterized as follows:*

$$q \in \mathcal{K}^\circ \iff q = (\mu_1, \ldots, \mu_{K+1}, 0, \ldots, 0),$$

where $\mu_1, \ldots, \mu_{K+1} \in \mathcal{M}([0,1])$ are nonnegative.

Using (4.2.4), we can apply Theorem 4.2.1 in order to derive first order necessary optimality conditions for (P):

Theorem 4.2.7. *Suppose that* (A3) *and* (A4) *are valid and let $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ be a locally optimal solution for* (P). *We additionally require that $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ satisfies the conditions in* (ND) *and that Robinson's CQ in* (4.2.13) *holds. Then there exist nonnegative $\mu_1, \ldots, \mu_{K+1} \in \mathcal{M}([0,1])$ such that*

$$y_k(\cdot, \bar{\mathbf{u}}) \le \bar{y}_k(\cdot, \bar{\mathbf{u}}) \quad \text{on } [0,1] \quad \textit{for all } k = 1, \ldots, K+1, \tag{4.2.15a}$$

$$\sum_{k=1}^{K+1} \int_{[0,1]} \left(y_k(\lambda, \bar{\mathbf{u}}) - \bar{y}_k(\lambda, \bar{\mathbf{u}})\right) \mathrm{d}\mu_k(\lambda) = 0, \tag{4.2.15b}$$

$$\hat{J}'(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) + \sum_{k=1}^{K+1} \int_{[0,1]} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}(y_k(\lambda, \bar{\mathbf{u}}) - \bar{y}_k(\lambda, \bar{\mathbf{u}}))(\mathbf{u} - \bar{\mathbf{u}}) \, \mathrm{d}\mu_k(\lambda) \ge 0 \;\; \forall \mathbf{u} \in \mathbf{U}_{ad}. \tag{4.2.15c}$$

Proof. We firstly observe that $\bar{\mathbf{u}}$ is also a locally optimal solution for $(\mathcal{P})$ with the setting in (4.2.12) due to Lemma 4.2.2. Then we use Theorem 4.2.1 and Lemma 4.2.6 to show that (4.2.15) is satisfied. We finally note that Theorem 4.2.1 is applicable since its requirements hold due to Theorem 4.2.3 and (A4). □

The following result will provides a further characterization of Robinson's CQ in (4.2.13).

Lemma 4.2.8. *Let* (A3) *and* (A4) *hold true and consider some $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ satisfying* (ND). *Then* (4.2.13) *is satisfied if and only if there exists $\tilde{\mathbf{u}} \in \mathbf{U}_{ad}$ such that for a* *constant* $\varepsilon > 0$ *the following assertion is valid for all* $k = 1, \dots, K+1$*:*

$$y_k(\lambda, \bar{\mathbf{u}}) - \bar{y}_k(\lambda, \bar{\mathbf{u}}) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\big(y_k(\lambda, \bar{\mathbf{u}}) - \bar{y}_k(\lambda, \bar{\mathbf{u}})\big)(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \leq -\varepsilon \quad \forall \lambda \in [0,1]. \qquad (4.2.16)$$

Proof. We firstly suppose that (4.2.16) is satisfied and prove that (4.2.13) holds true. In order to show (4.2.13), it is sufficient to prove that there exists a constant $\rho > 0$ such that for all $f := (f_1, \dots, f_{K+1}) \in B_\rho^{C([0,1])^{K+1}}(\mathbf{0})$ there exists some $\tilde{\mathbf{u}} \in \mathbf{U}$ and $(h_1, \dots, h_{K+1}) \in C([0,1];]-\infty, 0])^{K+1}$ such that for all $k \in \{1, \dots, K+1\}$ it holds that

$$f_k(\cdot) = y_k(\cdot; \mathbf{u}) - \bar{y}_k(\cdot; \mathbf{u}) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\left(y_k(\cdot; \mathbf{u}) - \bar{y}_k(\cdot; \mathbf{u})\right)(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) - h_k(\cdot) \quad \text{on } [0,1].$$

To this end, we choose

$$\rho = \frac{\varepsilon}{2} \qquad (4.2.17)$$

and $\tilde{u}$ as in (4.2.16). Next, we consider an arbitrary $\tilde{k} \in \{1, \dots, K+1\}$ and set

$$h_{\tilde{k}}(\cdot) = y_{\tilde{k}}(\cdot; \mathbf{u}) - \bar{y}_{\tilde{k}}(\cdot; \mathbf{u}) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\big(y_{\tilde{k}}(\cdot; \mathbf{u}) - \bar{y}_{\tilde{k}}(\cdot; \mathbf{u})\big)(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) - f_{\tilde{k}}(\cdot). \qquad (4.2.18)$$

Due to (4.2.16), the fact that

$$(f_1, \dots, f_{K+1}) \in B_\rho^{C([0,1])^{K+1}}(\mathbf{0})$$

and the choice of ρ in (4.2.17), we obtain $h_{\tilde{k}}(\lambda) \leq -\frac{\varepsilon}{2}$ and hence $h_{\tilde{k}} \in C([0,1];]-\infty, 0])$. Since $\tilde{k}$ is arbitrarily chosen, (4.2.18) is satisfied for all $k \in \{1, \dots, K+1\}$ with functions $h_{\tilde{k}} \in C([0,1];]-\infty, 0])$. Since f was arbitrarily chosen, (4.2.13) holds true.

Now we suppose that (4.2.16) does not hold true and prove that (4.2.13) cannot be satisfied. Since (4.2.16) is supposed not to hold we obtain that for all $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$ there exists some $k \in \{1, \dots, K+1\}$ and $\hat{\lambda} \in [0,1]$ such that

$$y_k(\hat{\lambda}, \bar{\mathbf{u}}) - \bar{y}_k(\hat{\lambda}, \bar{\mathbf{u}}) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\big(y_k(\hat{\lambda}, \bar{\mathbf{u}}) - \bar{y}_k(\hat{\lambda}, \bar{\mathbf{u}})\big)(\mathbf{u} - \bar{\mathbf{u}}) \geq 0. \qquad (4.2.19)$$

Next, we choose an arbitrary small constant $\rho > 0$ and observe that choosing

$$f = (f_1, \dots, f_{K+1}) = \Big(-\frac{\rho}{2K+2}, \dots, -\frac{\rho}{2K+2}\Big) \qquad (4.2.20)$$

yields

$$f \in B_\rho^{C([0,1])^{K+1}}(\mathbf{0}).$$

If (4.2.13) is satisfied, then there exists $\tilde{u} \in \mathbf{U}_{\text{ad}}$ and

$$h = (h_1, \dots, h_{K+1}) \in C([0,1];]-\infty, 0])$$

such that

$$f_k(\cdot) = y_k(\cdot; \mathbf{u}) - \bar{y}_k(\cdot; \mathbf{u}) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\left(y_{\tilde{k}}(\cdot; \mathbf{u}) - \bar{y}_k(\cdot; \mathbf{u})\right)(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) - h_k(\cdot) \quad \text{on } [0,1]$$

is satified for all $k = 1, \dots, K+1$ which is equivalent to

$$h_k(\cdot) = y_k(\cdot; \mathbf{u}) - \bar{y}_k(\cdot; \mathbf{u}) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\left(y_{\tilde{k}}(\cdot; \mathbf{u}) - \bar{y}_k(\cdot; \mathbf{u})\right)(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) - f_k(\cdot) \quad \text{on } [0,1]$$

for all $k = 1, \dots, K+1$. Since (4.2.16) does not hold true, there exists some $\hat{k} \in \{1, \dots, K+1\}$ and $\hat{\lambda} \in [0,1]$ such that (4.2.19) is satisfied for $k = \hat{k}$. From this and the choice of f in (4.2.20), we obtain that $h_{\hat{k}}(\hat{\lambda}) \geq \frac{\rho}{2K+2}$ and hence $h \notin C([0,1];]-\infty, 0])^{K+1}$. Hence, (4.2.13) is not satisfied. □

The optimality conditions in Theorem 4.2.15 are formulated in terms of the new state variables which are defined in (4.2.5). Since it is convenient to have a formulation of the optimality conditions in Theorem 4.2.15 in terms of the original state $y(\bar{t}, \cdot; \bar{\mathbf{u}})$, in the remaining part of this chapter we will reformulate the optimality conditions in (4.2.15). As a first step, we rewrite the optimality conditions in terms of the mappings $Y_1, \dots, Y_{K+1}$ which are introduced in Lemma 3.7.1. We recall that the new state variables in (4.2.5) are obtained by using the variable transformations in (4.2.4). Using (4.2.4) in connection with (4.2.11), the optimality conditions in (4.2.15a)-(4.2.15c) can be written as follows

$$Y_k(\bar{t}, \varphi_{k;\bar{\mathbf{u}}}(\cdot), \bar{\mathbf{u}}) \leq \bar{y}(\varphi_{k;\bar{\mathbf{u}}}(\cdot)) \quad \text{on } [0,1] \quad \forall k = 1, \dots, K+1, \tag{4.2.21a}$$

$$\sum_{k=1}^{K+1} \int_{[0,1]} \left(Y_k(\bar{t}, \varphi_{k;\bar{\mathbf{u}}}(\lambda), \bar{\mathbf{u}}) - \bar{y}(\varphi_{k;\bar{\mathbf{u}}}(\lambda))\right) \mathrm{d}\mu_k(\lambda) = 0, \tag{4.2.21b}$$

$$\hat{J}'(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) + \sum_{k=1}^{K+1} \int_{[0,1]} \Bigg[\frac{\mathrm{d}}{\mathrm{d}x}\left(Y_k(\bar{t}, \varphi_{k;\bar{\mathbf{u}}}(\lambda); \bar{\mathbf{u}}) - \bar{y}(\varphi_{k;\bar{\mathbf{u}}}(\lambda))\right)$$

$$\cdot \left(\varphi_{k,\bar{\mathbf{u}}}^{-1}(\varphi_{k,\bar{\mathbf{u}}}(\lambda))\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_k(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) + \left(1 - \varphi_{k,\bar{\mathbf{u}}}^{-1}(\varphi_{k,\bar{\mathbf{u}}}(\lambda))\right)\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{k-1}(\mathbf{u})(\mathbf{u} - \bar{\mathbf{u}})\right)$$

$$+\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\big(Y_k(\bar{t},\varphi_{k;\bar{\mathbf{u}}}(\lambda);\bar{\mathbf{u}})-\bar{y}(\varphi_{k;\bar{\mathbf{u}}}(\lambda))\big)\cdot(\mathbf{u}-\bar{\mathbf{u}})\Big]\,\mathrm{d}\mu_k(\lambda)\geq 0\quad\forall\,\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}. \tag{4.2.21c}$$

Theorem 2.5.3 yields the existence of nonnegative measures $\bar{\mu}_k\in\mathcal{M}(I_k)$ with

$$\begin{aligned}&I_k:=[x_{k-1}(\bar{\mathbf{u}}),x_k(\bar{\mathbf{u}})]\qquad\text{for }k=1,\dots,K+1\\&x_0(\bar{\mathbf{u}}):=a,\quad x_{K+1}(\bar{\mathbf{u}}):=b,\end{aligned}\tag{4.2.22}$$

which are defined according to (2.5.3) by

$$\bar{\mu}_k(A):=\mu_k(\varphi_{k;\bar{\mathbf{u}}}^{-1}(A))\quad\forall A\subset I_k\text{ measurable, }k=1,\dots,K+1$$

such that

$$\int_{I_k}f\,\mathrm{d}\bar{\mu}_k=\int_{[0,1]}f\circ\varphi_{k;\bar{\mathbf{u}}}\,\mathrm{d}\mu_k\quad\forall k=1,\dots,K+1 \tag{4.2.23}$$

holds for all measurable functions f. Applying (4.2.23) on (4.2.21) yields

$$Y_k(\bar{t},\cdot,\bar{\mathbf{u}})\leq\bar{y}\,(\cdot)\quad\text{on }I_k\quad\forall k=1,\dots,K+1, \tag{4.2.24a}$$

$$\sum_{k=1}^{K+1}\int_{I_k}(Y_k(\bar{t},x,\bar{\mathbf{u}}))-\bar{y}(x)\,\mathrm{d}\bar{\mu}_k(x)=0, \tag{4.2.24b}$$

$$\begin{aligned}&\hat{J}'(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})+\sum_{k=1}^{K+1}\Bigg[\int_{I_k}\frac{\mathrm{d}}{\mathrm{d}x}[Y_k(\bar{t},x,\bar{\mathbf{u}})-\bar{y}(x)]\frac{x-x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\,\mathrm{d}\bar{\mu}_k(x)\\&\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_k(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})+\int_{I_k}\frac{\mathrm{d}}{\mathrm{d}x}[Y_k(\bar{t},x,\bar{\mathbf{u}})-\bar{y}(x)]\frac{x_k(\bar{\mathbf{u}})-x}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\,\mathrm{d}\bar{\mu}_k(x)\\&\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{k-1}(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})+\int_{I_k}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_k(\bar{t},x,\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})\,\mathrm{d}\bar{\mu}_k(x)\Bigg]\geq 0\quad\forall\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}.\end{aligned}\tag{4.2.24c}$$

We finally rewrite Robinson's CQ in (4.2.13) in terms of $Y_1,\dots,Y_{K+1}$:

$$\mathbf{0}\in\operatorname{int}\left(\begin{pmatrix}Y_1\,(\bar{t},\cdot;\mathbf{u})-\bar{y}\,(\cdot)\\\vdots\\Y_{K+1}\,(\bar{t},\cdot;\mathbf{u})-\bar{y}\,(\cdot)\end{pmatrix}+\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\begin{pmatrix}Y_1\,(\bar{t},\cdot;\mathbf{u})\,(\cdot)\\\vdots\\Y_{K+1}\,(\bar{t},\cdot;\mathbf{u})\,(\cdot)\end{pmatrix}(\mathbf{U}-\bar{\mathbf{u}})\right.$$

$$
+\begin{pmatrix}
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y_1\left(\bar{t},\cdot;\mathbf{u}\right)-\bar{y}\left(\cdot\right)\right)\left(\frac{\cdot-x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_k(\bar{\mathbf{u}})+\frac{x_k(\bar{\mathbf{u}})-\cdot}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{k-1}(\bar{\mathbf{u}})\right)\\
\vdots\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y_1\left(\bar{t},\cdot;\mathbf{u}\right)-\bar{y}\left(\cdot\right)\right)\left(\frac{\cdot-x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_k(\bar{\mathbf{u}})+\frac{x_k(\bar{\mathbf{u}})-\cdot}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{k-1}(\bar{\mathbf{u}})\right)
\end{pmatrix}
$$

$$
(\mathbf{U}-\bar{\mathbf{u}})-\begin{pmatrix} C(I_1;]-\infty,0]) \\ \vdots \\ C(I_{K+1};]-\infty,0]) \end{pmatrix}\Bigg) \tag{4.2.25}
$$

Hence, we can rewrite Theorem 4.2.7 as follows:

Theorem 4.2.9. *Suppose that* (A3) *and* (A4) *are satisfied and let* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *be a locally optimal solution of* (P) *that satisfies* (ND) *and Robinson's CQ* (4.2.25)*. Then there exist nonnegative measures* $\bar{\mu}_k \in \mathcal{M}(I_k)$*,* $k = 1, \dots, K+1$*, such that the conditions in* (4.2.24) *are satisfied, where the intervals* I_k *are defined in* (4.2.22)*.*

It is very important to show that Robinson's CQ in (4.2.25) is always satisfied under suitable assumptions. To this end, we firstly derive an equivalent characterization of (4.2.25) in the following lemma which is a direct consequence of the definition of the new states in (4.2.5) and Lemma 4.2.8:

Lemma 4.2.10. *Assume that* (A3) *and* (A4) *hold true and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *that satisfies* (ND)*. Then Robinson's CQ* (4.2.25) *is satisfied if and only if there exists* $\delta\mathbf{u} \in \mathbf{U}_{ad} - \bar{\mathbf{u}}$ *such that*

$$
Y_k(\bar{t},x,\bar{\mathbf{u}})-\bar{y}(x)+\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_k(\bar{t},x,\bar{\mathbf{u}})\delta\mathbf{u}+\frac{\mathrm{d}}{\mathrm{d}x}[Y_k(\bar{t},x,\bar{\mathbf{u}})-\bar{y}(x)]\cdot \tag{4.2.26}
$$
$$
\left(\frac{x-x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_k(\bar{\mathbf{u}})\delta\mathbf{u}+\frac{x_k(\bar{\mathbf{u}})-x}{x_k(\bar{\mathbf{u}})-x_{k-1}(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{k-1}(\bar{\mathbf{u}})\delta\mathbf{u}\right)\leq-\varepsilon
$$

holds true for all $x \in I_k$ *and* $k = 1, \dots, K+1$*, where* $\varepsilon > 0$ *is a sufficiently small constant and the intervals* I_k *are defined in* (4.2.22)*.*

Using Lemma 4.2.10, we will prove that Robinson's CQ in (4.2.25) is always satisfied under suitable assumptions:

Theorem 4.2.11. *Assume that* (A3) *and* (A4) *are satisfied and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *such that* (ND) *and the state constraints* (4.0.1) *are satisfied. Furthermore, suppose that* $\bar{\mathbf{u}} + \delta\mathbf{u} \in \mathbf{U}_{ad}$ *holds for all*

$$
\delta\mathbf{u} \in B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \cap \{\mathbf{u} \in \mathbf{U} : \mathbf{u} \leq 0 \wedge u_1 = 0\}, \tag{4.2.27}
$$

if $\rho > 0$ *is small enough. The inequality* $\mathbf{u} \leq 0$ *in* (4.2.27) *has to be understood componentwise. Finally, let*

$$g|_{\Omega_T} \equiv 0 \quad \textit{and} \quad \bar{y}(\cdot) \equiv \textit{constant.}$$

Then (4.2.25) *is satisfied in* $\bar{\mathbf{u}}$.

Proof. We use Lemma 4.2.10 to prove the theorem. To this end, we show the existence of a constant $\varepsilon > 0$ and $\delta\mathbf{u} \in \mathbf{U}_{\mathrm{ad}} - \bar{\mathbf{u}}$ such that (4.2.26) is satisfied for all $x \in [x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})]$ and $k = 1, \ldots, K+1$. We note that $x_k(\bar{\mathbf{u}}) \in [a, b]$ with $k \in \{1, \ldots, K+1\}$ is by assumption either a discontinuity of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ or lies on the boundary of a rarefaction wave. Here, the discontinuities of $y(\bar{t}, \cdot, \bar{\mathbf{u}})$ on the interval $[a, b]$ are nondegenerated according to Definition 3.7.3. Therefore, the discontinuities are all of type $X_l X_r$ with $X_l, X_r \in \left\{C^c, R^c, C^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{a}}, C^c_{B,\mathfrak{b}}, R^c_{B,\mathfrak{b}}\right\}$, i.e., the minimal respectively maximal backward characteristic through a discontinuity ends in a point where the initial or boundary data is smooth, or in the inner of a rarefaction wave. Recalling the Lemmas 3.7.10 and 3.7.15, we observe that the cases $X_{l/r} \in \left\{R^c, R^c_{B,\mathfrak{a}}, R^c_{B,\mathfrak{b}}\right\}$ can be treated analogously to the case $X_{l/r} \in \{C^c\}$. Moreover, the case $C^c_{B,\mathfrak{a}}$ can be treated similarly to the case $C^c_{B,\mathfrak{b}}$. Therefore, in order to discuss all relevant cases, we can firstly restrict ourselves to the following case:

Let $K = 2$ and assume that $(\bar{t}, x_1(\bar{\mathbf{u}}))$ lies on the right boundary of a rarefaction wave being created in a discontinuity $\bar{t}^{\mathfrak{a}}_j$ of the left boundary data $u_{B,\mathfrak{a}}(\cdot; \bar{\mathbf{u}})$. Moreover, let $(\bar{t}, x_2(\bar{\mathbf{u}}))$ be a nondegenerated shock, where the minimal backward characteristic through $(\bar{t}, x_2(\bar{\mathbf{u}}))$ ends in a continuity point $\tilde{t}^{\mathfrak{a}} \in (\bar{t}^{\mathfrak{a}}_{j-1}, \bar{t}^{\mathfrak{a}}_j)$ of $u_{B,\mathfrak{a}}(\cdot; \bar{\mathbf{u}})$ and the maximal backward characteristic ends in a continuity point $\tilde{x} \in (\bar{x}^0_{l-1}, \bar{x}^0_l)$ of the initial data $u_0(\cdot; \bar{\mathbf{u}})$. Then the genuine backward characteristic through $(\bar{t}, a)$ also ends in $(\bar{t}^{\mathfrak{a}}_j, \mathfrak{a})$ and the one through $(\bar{t}, b)$ ends within the interval $(\bar{x}^0_{l-1}, \bar{x}^0_l)$. If that was not the case, more discontinuities or points lying on the boundary of rarefaction wave would occur on $]a, b[$ which would be a contradicion to $K = 2$.

The further part of the proof is divided in two steps: **Step 1** will be concerned with deriving representations for the terms $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_1(\bar{t}, \cdot; \bar{\mathbf{u}})$, $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_2(\bar{t}, \cdot; \bar{\mathbf{u}})$, $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_3(\bar{t}, \cdot; \bar{\mathbf{u}})$, $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_1(\bar{\mathbf{u}})$ and $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_2(\bar{\mathbf{u}})$. In **Step 2**, we will construct some $\delta\mathbf{u} \in \mathbf{U}_{\mathrm{ad}} - \bar{\mathbf{u}}$ and prove that (4.2.26) is satisfied for all $k = 1, 2, 3$.

Step 1: With $g|_{\Omega_T} \equiv 0$, we obtain that $Y_1(\bar{t}, x, \bar{\mathbf{u}}) = f'^{-1}\left(\frac{x}{\bar{t} - \bar{t}^{\mathfrak{a}}_j}\right)$ and in addition

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_1(\bar{t}, x, \bar{\mathbf{u}}) \delta x = \frac{\delta x}{(\bar{t} - \bar{t}^{\mathfrak{a}}_j) \cdot f''(f'^{-1}(\frac{x}{\bar{t} - \bar{t}^{\mathfrak{a}}_j}))}, \tag{4.2.28}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_1(\bar{t}, x, \bar{\mathbf{u}})\delta\mathbf{u} = \frac{x \cdot \delta t_j^{\mathfrak{a}}}{(\bar{t} - \bar{t}_j^{\mathfrak{a}})^2 f''(f'^{-1}(\frac{x}{\bar{t}-\bar{t}_j^{\mathfrak{a}}}))}. \tag{4.2.29}$$

Since the minimal backward characteristic through $(\bar{t}, x_2(\bar{\mathbf{u}}))$ ends in a point $\tilde{t}^{\mathfrak{a}} \in (\bar{t}_i^{\mathfrak{a}}, \bar{t}_{i+1}^{\mathfrak{a}})$ where $u_{B,\mathfrak{a}}(\cdot; \bar{\mathbf{u}})$ is smooth, we obtain from Lemma 3.7.15 that the derivatives of Y_2 are given by

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_2(\bar{t}, x, \bar{\mathbf{u}})\delta x = \frac{(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:)) \cdot \delta x}{f''(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:))(\bar{t} - \Phi(:)) - f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))}, \tag{4.2.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_2(\bar{t}, x, \bar{\mathbf{u}})\delta\mathbf{u} = -\frac{f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:))) \cdot \delta u_j^{B,\mathfrak{a}}(\Phi(:))}{f''(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:))(\bar{t} - \Phi(:)) - f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))}, \tag{4.2.31}$$

where $(:) = (\bar{t}, x, \bar{\mathbf{u}})$, $(\bar{u}_j^{B,\mathfrak{a}})'(\cdot)$ denotes the derivative of $\bar{u}_j^{B,\mathfrak{a}}(\cdot)$ and $\Phi(\bar{t}, x, \bar{\mathbf{u}})$ is the unique solution of the equation $x = f'(\bar{u}_j^{B,\mathfrak{a}}(\phi))(\bar{t} - \phi) + \mathfrak{a}$ w.r.t. ϕ. We note that

$$\Phi(\bar{t}, x, \bar{\mathbf{u}}) \in [\tilde{t}^{\mathfrak{a}}, \bar{t}_j^{\mathfrak{a}}] \quad \text{for all } x \in [x_1(\bar{\mathbf{u}}), x_2(\bar{\mathbf{u}})].$$

is satisfied. The estimation (3.7.37) in Lemma 3.7.13 implies that

$$q_1(:) := f''\left(\bar{u}_j^{B,\mathfrak{a}}\right)(\Phi(:)))(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:))(\bar{t} - \Phi(\cdot)) - f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:))) < -\beta \tag{4.2.32}$$

holds for all $x \in (x_1(\bar{\mathbf{u}}) - \delta_0, x_2(\bar{\mathbf{u}}) + \delta_0)$ for some constants $\delta_0, \beta > 0$.

It remains to examine the term Y_3. Since the maximal backward characteristic through $(\bar{t}, x_2(\bar{\mathbf{u}}))$ ends in a point $\tilde{x} \in (\bar{x}_{l-1}^0, \bar{x}_l^0)$ where the initial data $u_0(\cdot; \bar{\mathbf{u}})$ is smooth, Lemma 3.7.8 yields

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_3(\bar{t}, x, \bar{\mathbf{u}})\delta x = \frac{(\bar{u}_l^0)'(Z(:)) \cdot \delta x}{f''(\bar{u}_l^0(Z(:)))(\bar{u}_l^0)'(Z(:))\bar{t} + 1}, \tag{4.2.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_3(\bar{t}, x, \bar{\mathbf{u}})\delta\mathbf{u} = \frac{\delta u_l^0(Z(:))}{f''(\bar{u}_l^0(Z(:)))(\bar{u}_l^0)'(Z(:))\bar{t} + 1}, \tag{4.2.34}$$

where $Z(:) = Z(\bar{t}, x, \bar{\mathbf{u}})$ denotes the unique solution of the equation

$$x = f'\left(\bar{u}_l^0(z)\right)\bar{t} + z$$

w.r.t. z. The estimation (3.7.31) in Lemma 3.7.13 yields the existence of a constant $\delta_0 > 0$ such that

$$q_2(:) = f''(\bar{u}_l^0(Z(:)))u_l^{0'}(Z(:))\bar{t} + 1 > \beta \tag{4.2.35}$$

is satified for $x \in (x_2(\bar{\mathbf{u}}) - \delta_0, b)$ and a possibly smaller constant $\beta > 0$.

Now we consider the term $x_1(\bar{\mathbf{u}})$. Since $(\bar{t}, x_1(\bar{\mathbf{u}}))$ lies on the right boundary of a rarefaction wave and the source term $g \equiv 0$, we see that $x_1(\bar{\mathbf{u}}) = f'(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})) \cdot (\bar{t} - \bar{t}_j^{\mathfrak{a}}) + \mathfrak{a}$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_1(\bar{\mathbf{u}})\delta\mathbf{u} = f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t} - \bar{t}_j^{\mathfrak{a}})\big[\delta u_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}) + (\bar{u}_j^{B,\mathfrak{a}})'(\bar{t}_j^{\mathfrak{a}})\delta t_j^{\mathfrak{a}}\big] - f'(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))\delta t_j^{\mathfrak{a}}. \tag{4.2.36}$$

Concerning the shock position $x_2(\bar{\mathbf{u}})$, we obtain from Corollary 3.7.25 that its derivative w.r.t. $\mathbf{u}$ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_2(\bar{\mathbf{u}})\delta\mathbf{u} = &\sum_{k=1}^{l} \big(p(0,\cdot), \delta u_k^0\big)_{2, I_0^k \cap]0,\tilde{x}[} + \sum_{k=1}^{j} \Big(p(\cdot,\mathfrak{a}), f'(\bar{u}_k^{B,\mathfrak{a}})\delta u_k^{B,\mathfrak{a}}\Big)_{2, I_{B,\mathfrak{a}}^k \cap]0,\tilde{t}^{\mathfrak{a}}[} \\ &+ \sum_{k \in I_{\mathrm{s},0} : \bar{x}_k^0 \in]0,\tilde{x}[} p(0,\bar{x}_k^0)[\bar{u}_0(\bar{x}_k^0)]\delta x_k + \sum_{k \in I_{\mathrm{s},\mathfrak{a}}(\bar{\mathbf{u}}) : \bar{t}_k^{\mathfrak{a}} \in]0,\tilde{t}^{\mathfrak{a}}[} p(\bar{t}_k^{\mathfrak{a}},\mathfrak{a})[f(y(\bar{t}_k^{\mathfrak{a}},\mathfrak{a}+;\bar{\mathbf{u}}))]\delta t_k^{\mathfrak{a}} \\ &- \sum_{k \in I_{\mathrm{r},0} : \bar{x}_k^0 \in]0,\tilde{x}[} p_k^{\mathrm{r},0}\delta x_k^0 + \sum_{k \in I_{\mathrm{r},\mathfrak{a}}(\bar{\mathbf{u}}) : \bar{t}_k^{\mathfrak{a}} \in]0,\tilde{t}^{\mathfrak{a}}[} p_k^{\mathrm{r},\mathfrak{a}}\delta t_k^{\mathfrak{a}},\end{aligned} \tag{4.2.37}$$

where p denotes the adjoint state with end data $p(\bar{t},\cdot) = \frac{1}{[y(\bar{t},x_2(\bar{\mathbf{u}});\bar{\mathbf{u}}]}$. According to Definition 3.5.4, p is given by $p(t,x)|_{\Omega_T \setminus D_-} \equiv \frac{1}{[y(\bar{t},x_2(\bar{\mathbf{u}});\bar{\mathbf{u}}]} > 0$ and equal to zero on D_-.

Step 2: In this step we contruct some $\delta\mathbf{u}$ such that (4.2.26) holds for all $k \in \{1,2,3\}$. First of all, all components of $\delta\mathbf{u}$ except $\delta u_j^{B,\mathfrak{a}}$ and δu_l^0 are set equal to zero and $\delta u_j^{B,\mathfrak{a}}$ and δu_l^0 are chosen as follows

$$\begin{aligned}\delta u_j^{B,\mathfrak{a}}(t) &= \begin{cases} 0 & \text{if } \bar{t}_{j-1}^{\mathfrak{a}} \le t < \tilde{t}^{\mathfrak{a}} - \rho_1, \\ \phi_1(t) & \text{if } \tilde{t}^{\mathfrak{a}} - \rho_1 \le t < \tilde{t}^{\mathfrak{a}}, \\ -\varepsilon_0 & \text{if } \tilde{t}^{\mathfrak{a}} \le t < \bar{t}_j^{\mathfrak{a}} - \rho_2, \\ \phi_2(t) & \text{if } \bar{t}_j^{\mathfrak{a}} - \rho_2 \le t < \bar{t}_j^{\mathfrak{a}}, \\ \frac{-\varepsilon_0}{f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t}-\bar{t}_j^{\mathfrak{a}})N} & \text{if } t = \bar{t}_j^{\mathfrak{a}}, \end{cases} \\ \delta u_l^0(x) &= \begin{cases} 0 & \text{if } x < \tilde{x} - \rho_3, \\ \phi_3(t) & \text{if } \tilde{x} - \rho_3 \le x < \tilde{x}, \\ -\varepsilon_0 & \text{if } \tilde{x} \le x, \end{cases}\end{aligned} \tag{4.2.38}$$

where $0 < \varepsilon_0 < \rho$ and $\rho_1, \rho_2, \rho_3 > 0$ are constants. In addition, we choose a natural number $N \in \mathbb{N}$ such that $f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t} - \bar{t}_j^{\mathfrak{a}}) \cdot N > 1$ holds. We note that ρ_1, ρ_2 and ρ_3 are chosen independently from ε_0. We will identify these constants later. Further on, choose ϕ_1, ϕ_2 and ϕ_3 such that $-\varepsilon_0 \leq \phi_i \leq 0$ and $\bar{\mathbf{u}} + \delta\mathbf{u} \in \mathbf{U}_{\text{ad}}$ holds for sufficiently small ε_0.

Now, we prove that (4.2.26) holds for $k = 1$. Since $f'' \geq m_{f''}$ holds by assumption, we see that the function $Y_1(\bar{t}, \cdot; \bar{\mathbf{u}})$ is monotonously increasing. Thus, the only point on the interval $[a, x_1(\bar{\mathbf{u}})]$ where Y_1 may touch the upper bound is at $x_1(\bar{\mathbf{u}})$. Moreover, we obtain from (4.2.36) and (4.2.38) that $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_1(\bar{\mathbf{u}}) = \frac{-\varepsilon_0}{N}$. Using the state constraints (4.0.1), the derivatives of Y_1 in (4.2.28)-(4.2.29), the fact that $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_1(\bar{\mathbf{u}}) = \frac{-\varepsilon_0}{N}$ and the construction of $\delta\mathbf{u}$ in (4.2.38), we conclude that the term on left-hand side of (4.2.26) is at $x = x_1(\bar{\mathbf{u}})$ bounded from above by

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_1(\bar{t}, x_1(\bar{\mathbf{u}}), \bar{\mathbf{u}}) \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_1(\bar{\mathbf{u}}) \delta\mathbf{u} = \frac{-\varepsilon_0}{N(\bar{t} - \bar{t}_j^{\mathfrak{a}}) f''(f'^{-1}(\frac{x_1(\bar{\mathbf{u}})}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}))} \leq \frac{-\varepsilon_0}{NTM_{f''}} =: -\varepsilon_{11}. \tag{4.2.39}$$

The inequality in (4.2.39) is valid since $0 < m_{f''} \leq f''(\cdot) \leq M_{f''}$.

Due to (4.2.39) and the continuity of the left term of (4.2.26) w.r.t. x, we obtain that the left term of (4.2.26) is on the interval $]x_1 - \delta_1, x_1]$ bounded from above by $-\frac{\varepsilon_{11}}{2}$, where δ_1 is some positive constant. Using that $Y_1(\bar{t}, \cdot; \bar{\mathbf{u}})$ is strictly monotonous increasing, we conclude that

$$Y_1(\bar{t}, x; \bar{\mathbf{u}}) - \bar{y} \leq -\varepsilon_{12} \quad \text{for all } x \in [a, x_1(\bar{\mathbf{u}}) - \delta_1] \tag{4.2.40}$$

holds for some constant $\varepsilon_{12} > 0$. Using the representation derivative of Y_1 w.r.t. x in (4.2.28), the choice of $\delta\mathbf{u}$ in (4.2.38) and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_1(\bar{\mathbf{u}}) = \frac{-\varepsilon_0}{N}, \tag{4.2.41}$$

we obtain that the term of the left-hand side of (4.2.26) is bounded from above by

$$-\varepsilon_{12} - \frac{\varepsilon_0}{N \cdot (\bar{t} - \bar{t}_j^{\mathfrak{a}}) \cdot f''(f'^{-1}(\frac{x}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}))} \frac{x - a}{x_1(\bar{\mathbf{u}}) - a} \leq -\varepsilon_{12} \quad \forall x \in [a, x_1(\bar{\mathbf{u}}) - \delta_1]. \tag{4.2.42}$$

Therefore, choosing $\varepsilon = \varepsilon_1 := \min\{\frac{\varepsilon_{11}}{2}, \varepsilon_{12}\}$, one can see that (4.2.26) is satisfied for $k = 1$.

Now we consider the case $k = 2$. At first, we observe that the state constraints (4.0.1) imply that the the term on the left-hand side of (4.2.26) is bounded from above by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_2(\bar{t},x,\bar{\mathbf{u}})\delta\mathbf{u}+\frac{\mathrm{d}}{\mathrm{d}x}Y_2(\bar{t},x,\bar{\mathbf{u}})\Bigg[\frac{x_2(\bar{\mathbf{u}})-x}{x_2(\bar{\mathbf{u}})-x_1(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_1(\bar{\mathbf{u}})\delta\mathbf{u}\\&+\frac{x-x_1(\bar{\mathbf{u}})}{x_2(\bar{\mathbf{u}})-x_1(\bar{\mathbf{u}})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_2(\bar{\mathbf{u}})\delta\mathbf{u}\Bigg]\end{aligned}\tag{4.2.43}$$

for all $x\in[x_1(\bar{\mathbf{u}}),x_2(\bar{\mathbf{u}})]$. Then, using the derivatives of Y_2 in (4.2.30)-(4.2.31) and the derivative of $x_1(\mathbf{u})$ in (4.2.36), inserting $x=x_1(\bar{\mathbf{u}})$ in (4.2.43) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_2(\bar{t},x_1(\bar{\mathbf{u}}),\bar{\mathbf{u}})\delta\mathbf{u}+\frac{\mathrm{d}}{\mathrm{d}x}Y_2(\bar{t},x_1(\bar{\mathbf{u}}),\bar{\mathbf{u}})\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_1(\bar{\mathbf{u}})\delta\mathbf{u}=\delta u_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})\\&=\frac{-\varepsilon_0}{f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t}-\bar{t}_j^{\mathfrak{a}})N}.\end{aligned}\tag{4.2.44}$$

From (4.2.44) and the continuity of (4.2.43) w.r.t. x we obtain that the term in (4.2.43) is bounded on $[x_1(\bar{\mathbf{u}}),x_1(\bar{\mathbf{u}})+\delta]$ from above by $\frac{\delta u_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})}{2}:=-\varepsilon_{21}<0$. Here, $\delta>0$ is chosen sufficiently small and does not depend on ε_0. Now, we choose ρ_2 such that $\Phi(x_1(\bar{\mathbf{u}})+\delta)=\bar{t}_j^{\mathfrak{a}}-\rho_2$,

$$\rho_1=\frac{[y(\bar{t},x_2(\bar{\mathbf{u}});\bar{\mathbf{u}})]}{2N\|f'(\bar{u}_j^{B,\mathfrak{a}}(\cdot))\|_{\infty,[0,\tilde{t}^{\mathfrak{a}}]}}>0\quad\text{and}\quad\rho_3=\frac{[y(\bar{t},x_2(\bar{\mathbf{u}});\bar{\mathbf{u}})]}{2N}>0.$$

Then the formula for derivative of $x_2(\bar{\mathbf{u}})$ in (4.2.37), the choice of $\delta\mathbf{u}$ in (4.2.38) and the fact $\|\phi_i(\cdot)\|_\infty\le\varepsilon_0$ yield

$$\left|\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_2(\bar{\mathbf{u}})\delta\mathbf{u}\right|\le\frac{\varepsilon_0}{N}.\tag{4.2.45}$$

We note that using (4.2.31), (4.2.32) and (4.2.38) yield

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_2(\bar{t},\cdot;\bar{\mathbf{u}})\le\frac{-\alpha\varepsilon_0}{\|q_1(\cdot)\|_{\infty,[x_1(\bar{\mathbf{u}})+\delta,x_2(\bar{\mathbf{u}})]}}\qquad\text{on }[x_1(\bar{\mathbf{u}})+\delta,x_2(\bar{\mathbf{u}})],$$

where α is given by the constant in (A4). Due to the boundedness of $\frac{\mathrm{d}}{\mathrm{d}x}Y_2(\bar{t},x,\bar{\mathbf{u}})$ and the derivatives of $x_1(\bar{\mathbf{u}})$ and $x_2(\bar{\mathbf{u}})$ in (4.2.41) and (4.2.45), we choose N large enough such that (4.2.43) is bounded on the interval $[x_1(\bar{\mathbf{u}})+\delta,x_2(\bar{\mathbf{u}})]$ from above by

$$-\varepsilon_{22}:=\frac{-\alpha\varepsilon_0}{2\|q_1(\bar{t},\cdot,\bar{\mathbf{u}})\|_{\infty,[x_1(\bar{\mathbf{u}})+\delta,x_2(\bar{\mathbf{u}})]}}.$$

Now we can see that (4.2.26) is satisfied for $k = 2$, if we choose $\varepsilon = \min\{\varepsilon_{21}, \varepsilon_{22}\}$.

Now we consider the last case $k = 3$. At first, we observe that (4.2.34), (4.2.35) and (4.2.38) yield the estimation

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_3(\bar{t}, \cdot; \bar{\mathbf{u}}) \leq \frac{-\varepsilon_0}{\|q_2(\bar{t}, \cdot, \bar{\mathbf{u}})\|_{\infty,[x_2(\bar{\mathbf{u}}),b]}} \quad \text{on } [x_2(\bar{\mathbf{u}}), b]. \tag{4.2.46}$$

Using the boundedness of $\frac{\mathrm{d}}{\mathrm{d}x} Y_2(\bar{t}, x, \bar{\mathbf{u}})$ and the estimation of $|\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_2(\bar{\mathbf{u}})\delta\mathbf{u}|$ in (4.2.45), (4.0.1) and (4.2.46), we obtain that the left term of (4.2.26) is on the interval $[x_2(\bar{\mathbf{u}}), b]$ bounded from above by the term

$$-\frac{\varepsilon_0}{2\|q_2(\bar{t}, \cdot, \bar{\mathbf{u}})\|_{\infty,[x_2(\bar{\mathbf{u}}),b]}} =: -\varepsilon_3$$

if N is sufficiently large. Note that ε_3 is finite due to the continuity of $q_2((\bar{t}, \cdot, \bar{\mathbf{u}})$. Thus, choosing $\varepsilon = \varepsilon_3$, we see that (4.2.39) is satisfied for $k = 3$.

We finally observe that (4.2.39) is satisfied for $k = 1, 2, 3$ if we choose $\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $\delta\mathbf{u}$ according to (4.2.38).

The case that $K > 2$ can be treated by continuing the procedure and using similar arguments. □

Remark 4.2.12. The main concept of the proof above is to construct $\delta u \in \mathbf{U}_{\mathrm{ad}} - \bar{\mathbf{u}}$ such that the mappings in (4.2.29), (4.2.31) and (4.2.34) are negative and the absolute values of the terms in (4.2.36) and (4.2.37) sufficiently small.

One can see that the proof is extendable to source terms $g \not\equiv 0$ guaranteeing that if one chooses componentwise negative u^0, $\delta u^{B,\mathfrak{a}}$ and $\delta u^{B,\mathfrak{b}}$, then the terms in (4.2.29), (4.2.31) and (4.2.34) are also negative. Studying the results in §3.7.3 and Corollary 3.7.25, one can see that this holds e.g., for source terms only depending on the state y and satisfying (A3). Furthermore, the proof is still valid for non-constant upper bounds $\bar{y}$.

Finally, we further simplify the optimality conditions in (4.2.24) by using the following result.

Lemma 4.2.13. *Suppose that* (A3) *and* (A4) *are satisfied and consider a control* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *and nonnegative measures* $\mu_k \in \mathcal{M}(I_k)$, $k = 1, \ldots, K+1$, *such that* (4.2.24a) *and* (4.2.24b) *are satisfied, where the intervals* I_k *are defined in* (4.2.22). *Then for all measurable sets* $A \subset]x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})[$ *the following holds true:*

$$\int_A \frac{\mathrm{d}}{\mathrm{d}x} \left[Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x)\right] \left(\frac{x - x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}}) - x_{k-1}(\bar{\mathbf{u}})}\right) \mathrm{d}\mu_k(x) = 0 \tag{4.2.47}$$

$$\int_A \frac{\mathrm{d}}{\mathrm{d}x}\left[Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x)\right]\left(\frac{x_k(\bar{\mathbf{u}}) - x}{x_k(\bar{\mathbf{u}}) - x_{k-1}(\bar{\mathbf{u}})}\right)\mathrm{d}\mu_k(x) = 0 \tag{4.2.48}$$

Proof. Considering an arbitrary $k \in \{1, \ldots, K+1\}$ and an arbitrary measurable set $A \subset \,]x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})[$, we define the following subsets of A:

$$A_1 := \{x \in A : Y_k(\bar{t}, x, \bar{\mathbf{u}}) < \bar{y}\,(x)\}\,,$$
$$A_2 := \{x \in A : Y_k(\bar{t}, x, \bar{\mathbf{u}}) = \bar{y}\,(x)\}\,.$$

Due to the regularity of the functions $Y_1, \ldots, Y_{K+1}$ and $\bar{y}(\cdot)$, the sets A_1 and A_2 are both measurable. We further note that (4.2.24a) yields that $A = A_1 \cup A_2$. In order to prove the lemma, we show

$$\mu_k(A_1) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x)\right]\bigg|_{A_2} \equiv 0. \tag{4.2.49}$$

For the proof of the first assertion of (4.2.49), we suppose that

$$\mu_k(A_1) \neq 0.$$

Due to the nonnegativity of μ_k, we obtain that $\mu_k(A_1) > 0$. Then (4.2.24a) yields

$$\sum_{j=1}^{K+1}\int_{I_j}(Y_j(\bar{t}, x, \bar{\mathbf{u}})) - \bar{y}(x)\,\mathrm{d}\mu_j(x) \leq \int_{I_k}(Y_k(\bar{t}, x, \bar{\mathbf{u}})) - \bar{y}(x)\,\mathrm{d}\mu_k(x)$$
$$\leq \int_{I_k \cap A_1}(Y_k(\bar{t}, x, \bar{\mathbf{u}})) - \bar{y}(x)\,\mathrm{d}\mu_k(x) < 0,$$

which is a contradiction to (4.2.24b) and hence the first assertion of (4.2.49) is proved.

Next, we prove the second assertion. To this end, we suppose that

$$\exists \tilde{x} \in A_2 : \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[Y_k(\bar{t}, \tilde{x}, \bar{\mathbf{u}}) - \bar{y}(\tilde{x})\right] \neq 0.$$

Here, we assume w.l.o.g. that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Y_k(\bar{t}, \tilde{x}, \bar{\mathbf{u}}) - \bar{y}(\tilde{x})\right] > 0 \tag{4.2.50}$$

and note that the other case can be treated analogously. Since $\tilde{x} \in A_2$, we obtain

that

$$[Y_k(\bar{t}, \tilde{x}, \bar{\mathbf{u}}) - \bar{y}(\tilde{x})] = 0. \tag{4.2.51}$$

Next, we deduce that (4.2.50) and (4.2.51) yield the existence of a constant $\varepsilon > 0$ such that $[\tilde{x}, \tilde{x} + \varepsilon] \subset]x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})[$ and

$$Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x) > 0 \quad \text{for all } x \in]\tilde{x}, \tilde{x} + \varepsilon]$$

which is a contradiction to (4.2.24a) and hence to the assumptions of the lemma. For the case that instead of (4.2.50) it holds that

$$\frac{\mathrm{d}}{\mathrm{d}x} [Y_k(\bar{t}, \tilde{x}, \bar{\mathbf{u}}) - \bar{y}(\tilde{x})] < 0, \tag{4.2.52}$$

one can use similar arguments. More precisely, as already mentioned, since $\tilde{x} \in A_2$, (4.2.51) is valid which together with (4.2.52) implies the existence of a constant $\varepsilon > 0$ such that $[\tilde{x} - \varepsilon, \tilde{x}] \subset]x_{k-1}(\bar{\mathbf{u}}), x_k(\bar{\mathbf{u}})[$ and

$$Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x) > 0 \quad \text{for all } x \in [\tilde{x} - \varepsilon, \tilde{x}[,$$

which again is a contradiction to (4.2.24a). Therefore, we obtain that the second assertion of (4.2.49) holds. Using $A = A_1 \cup A_2$ and (4.2.49), we can deduce that

$$\begin{aligned}
&\int_A \frac{\mathrm{d}}{\mathrm{d}x} [Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x)] \left(\frac{x - x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}}) - x_{k-1}(\bar{\mathbf{u}})} \right) \mathrm{d}\mu_k(x) \\
&= \sum_{i=1}^{2} \int_{A_i} \frac{\mathrm{d}}{\mathrm{d}x} [Y_k(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x)] \left(\frac{x - x_{k-1}(\bar{\mathbf{u}})}{x_k(\bar{\mathbf{u}}) - x_{k-1}(\bar{\mathbf{u}})} \right) \mathrm{d}\mu_k\,(x) = 0.
\end{aligned}$$

Using similar arguments, we can prove (4.2.48). □

With the help of the previous Lemma, we can further simplify the optimality conditions in Theorem 4.2.9:

Corollary 4.2.14. *Suppose that* (A3) *and* (A4) *are satisfied. In addition, consider some control* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *and nonnegative measures* $\mu_k \in \mathcal{M}\,(I_k)$, $k = 1, \ldots, K+1$ *such that* (4.2.25) *is satisfied, where the intervals* I_k *are defined in* (4.2.22). *Then* (4.2.24c) *can be rewritten by*

$$\hat{J}'(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) + \sum_{k=1}^{K+1} \left[\frac{\mathrm{d}}{\mathrm{d}x} [Y_k(\bar{t}, x_k(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - \bar{y}(x_k(\bar{\mathbf{u}}))] \cdot \bar{\mu}_k(\{x_k(\bar{\mathbf{u}})\}) \right.$$

$$\cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_k(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) + \frac{\mathrm{d}}{\mathrm{d}x}[Y_k(\bar{t}, x_{k-1}(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - \bar{y}(x_{k-1}(\bar{\mathbf{u}}))] \cdot \bar{\mu}_k(\{x_{k-1}(\bar{\mathbf{u}})\})$$
$$\left. \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{k-1}(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) + \int_{I_k} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_k(\bar{t}, x, \bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})\, \mathrm{d}\mu_k(x) \right] \geq 0, \quad \forall \mathbf{u} \in \mathbf{U}_{ad}.$$

As a final result of this chapter, we rewrite the optimality conditions in Theorem 4.2.9 respectively in Corollary 4.2.14 in terms of the original state $y(\bar{t}, \cdot; \bar{\mathbf{u}})$. Using the representation of y in (3.7.4) in Lemma 3.7.1 and Convention 3.2.2, the optimality conditions in (4.2.24) can be rewritten by

$$y(\bar{t}, x-, \bar{\mathbf{u}}) \leq \bar{y}\,(x) \quad \forall x \in [a, b], \tag{4.2.53a}$$

$$\sum_{k=1}^{K+1} \int_{I_k} (y(\bar{t}, x-, \bar{\mathbf{u}}) - \bar{y}(x))\, \mathrm{d}\bar{\mu}_k(x) = 0, \tag{4.2.53b}$$

$$\hat{J}'(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) + \sum_{k=1}^{K+1} \left[\frac{\mathrm{d}}{\mathrm{d}x}[y(\bar{t}, x_k(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - \bar{y}(x_k(\bar{\mathbf{u}}))] \cdot \bar{\mu}_k(\{x_k(\bar{\mathbf{u}})\}) \right.$$
$$\cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_k(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) + \frac{\mathrm{d}}{\mathrm{d}x}[y(\bar{t}, x_{k-1}(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - \bar{y}(x_{k-1}(\bar{\mathbf{u}}))] \cdot \bar{\mu}_k(\{x_{k-1}(\bar{\mathbf{u}})\})$$
$$\left. \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{k-1}(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) + \int_{I_k} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} y(\bar{t}, x, \bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})\, \mathrm{d}\mu_k(x) \right] \geq 0, \quad \forall \mathbf{u} \in \mathbf{U}_{\mathrm{ad}}. \tag{4.2.53c}$$

Hence, Theorem 4.2.24 can be reformulated as follows:

Theorem 4.2.15. *Suppose that* (A3) *and* (A4) *are satisfied and let* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *be a locally optimal solution of* (P) *that satisfies* (ND) *and* (4.2.25)*. Then there exist nonnegative measures* $\bar{\mu}_k \in \mathcal{M}(I_k)$ *such that the conditions in* (4.2.53) *are satisfied, where the intervals* I_k *are defined in* (4.2.22)*.*

CHAPTER 5

Moreau-Yosida regularization

Since the Lagrange multipliers in the optimality system in Theorem 4.2.9 are measures, the direct computation of a solution to this system is quite involved. To cope with this problem we will apply a well-known approach, the so-called *Moreau-Yosida regularization approach*. This approach was developed by Ito and Kunisch in [43] at the example of an elliptic problem with state constraints, cf. [62, 63]. Later on, the approach was used for parabolic problems in [64] and for the Navier-Stokes equations in [32]. In this chapter we want to apply this approach to the state-constrained optimal control problem considered in this thesis. Although the techniques in the analysis of hyperbolic balance laws are different from those which are used in the elliptic or parabolic case, we can adopt many of the concepts developed in the just mentioned contributions.

The main idea of this approach is to replace the state constraints by a suitable penalty function $P(y(\mathbf{u}))$ which is weighted by a penalty parameter γ, where $\gamma > 0$ and added to the cost functional such that we obtain

$$\left.\begin{aligned} &\min_{\mathbf{u}\in\mathbf{U}} J_\gamma(y(\mathbf{u}),\mathbf{u}) := J(y(\mathbf{u}),\mathbf{u}) + \frac{\gamma}{2}\int_a^b (y(\bar{t},x;\mathbf{u}) - \bar{y}(x))_+^2 \,\mathrm{d}x \\ &\text{s.t.} \quad \mathbf{u}\in\mathbf{U}_{\mathrm{ad}} \text{ and } y(\mathbf{u}) \text{ solves the (IBVP)} \end{aligned}\right\} \qquad (P_\gamma)$$

with $(\cdot)_+ := \max\{\cdot, 0\}$. Having a look at the regularized problem (P_γ), we see that if the state constraints are violated, then the penalty function will add a positive value to the cost functional. Note that punishment of a violation of the state constraints is increased, the larger the penalty parameter $\gamma > 0$ is chosen. The idea is to consider a sequence of penalty parameter $(\gamma_k)_{k\in\mathbb{N}}$ approaching infinity and compute for each $k \in \mathbb{N}$ a corresponding optimal solution $\mathbf{u}_{\gamma_k}$ for the regularized problem (P_γ). One of the main goals of this chapter is to give an answer to the question if a sequence

$(\mathbf{u}_k)_{k\in\mathbb{N}}$ of optimal solutions of (P_γ) converges to an optimal solution for the original problem (P). Considering a sequence of optimal solutions to (P_γ) converging to an optimal solution of (P) in which the assumptions of Theorem 4.2.9 satisfied, another question that arises is in which sense the optimality conditions of the regularized problems converge to the optimality system in (4.2.24).

This chapter is organized as follows: In §5.1 we will first of all prove that for all $\gamma > 0$ the problem (P_γ) is well-defined i.e., it admits an optimal solution. We will further show that the cost functional in (P_γ) is continuously Fréchet-differentiable w.r.t. the control $\mathbf{u}$ under suitable assumptions. In §5.2, we will analyze a sequence $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ of optimal solutions to (P_γ) for $\gamma_k \to \infty$. In §5.3 we will derive first order necessary optimality conditions for the regularized problems (P_γ) and examine in which sense these optimality conditions approach the optimality system in (4.2.24).

5.1 Basic results

We first note that the cost functional $J_\gamma(\cdot,\cdot)$ of the regularized problem (P_γ) is of same form as $J(\cdot,\cdot)$. Therefore, we are able to apply Theorem 3.5.5 to the regularized problem (P_γ) and obtain the following result.

Corollary 5.1.1. *Let the assumptions of Theorem 3.5.5 be satisfied for some control* $\bar{\mathbf{u}} \in \mathbf{U}$. *Then the mapping*

$$B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \ni \mathbf{u} \mapsto \hat{J}_\gamma(\mathbf{u}) := J_\gamma(y(\mathbf{u}), \mathbf{u}) \in \mathbb{R}$$

is continuously differentiable for a sufficiently small neighborhood $B_\rho^{\mathbf{U}}(\bar{\mathbf{u}})$ *of* $\bar{\mathbf{u}}$. *In addition, the derivative in a direction* $\delta\mathbf{u} \in \mathbf{U}$ *can be computed according to Theorem 3.5.5.*

Given the differentiability of the regularized cost functional, we show that (P_γ) admits for each $\gamma > 0$ a global solution:

Corollary 5.1.2. *Assume that* (A3) *and* (A4) *hold true. Then for each penalty parameter* $\gamma > 0$ *the regularized problem* (P_γ) *admits a globally optimal solution* $\mathbf{u}_\gamma \in \mathbf{U}_{ad}$.

Proof. Using the compactness of $\mathbf{U}_{\mathrm{ad}}$ in $\mathbf{U}$ and the regularity of the cost functional in (P_γ), the proof is similar to the proof of Theorem 4.1.1. □

5.2 Convergence of solutions

In the next result, we will examine the behavior of a sequence of global solutions of (P_γ) for penalty parameters $(\gamma_k)_{k\in\mathbb{N}}$ converging to infinity. The concepts of the proofs in this section are based on the ideas developed in [43].

Theorem 5.2.1. *Assume that* (A3) *and* (A4) *hold true and let* $(\gamma_k)_{k\in\mathbb{N}}$ *be a sequence of penalty parameters satisfying* $\lim_{k\to\infty}\gamma_k=\infty$. *Then denote by* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}\subset\mathbf{U}_{ad}$ *the corresponding sequence of global solutions to* (P_γ). *Assume in addition that* $x=a$ *is a point of continuity of* $y(\bar{t},\cdot;\mathbf{u})$ *for all* $\mathbf{u}\in\mathbf{U}_{ad}$. *Then there exists a subsequence again denoted by* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$, *and a control* $\mathbf{u}^*\in\mathbf{U}_{ad}$ *such that*

$$\mathbf{u}_{\gamma_k}\to\mathbf{u}^*\quad \textit{w.r.t.}\ \|\cdot\|_{\mathbf{U}}\,, \tag{5.2.1}$$

where $\mathbf{u}^*$ *is a global solution for* (P).

The idea of the proof is standard, see, e.g., the proof of Theorem 3.1 in [43].

Proof. Let $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ denote a sequence of globally optimal solutions to (P_γ). We will prove the existence of a subsequence converging to a globally optimal solution for (P). Due to compactness of the set $\mathbf{U}_{\mathrm{ad}}\subset\mathbf{U}$, there exists a convergent subsequence , which is again denoted by $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$, such that

$$\mathbf{u}_{\gamma_k}\to\mathbf{u}^*\quad \text{w.r.t.}\ \|\cdot\|_{\mathbf{U}} \tag{5.2.2}$$

holds. In order to show that $\mathbf{u}^*$ is a global solution for (P), we need to prove that $y(\mathbf{u}^*)$ fulfills the state constraints i.e.,

$$y(\bar{t},\cdot,\mathbf{u}^*)\le\bar{y}(\cdot)\quad \text{on } [a,b]. \tag{5.2.3}$$

To this end, we obverve that (P) admits a globally optimal solution according to Theorem 4.1.1 which we denote by $\bar{\mathbf{u}}\in\mathbf{U}_{\mathrm{ad}}$. Since $\mathbf{u}_{\gamma_k}\in\mathbf{U}_{\mathrm{ad}}$ is a globally optimal solution of (P_{γ_k}) and $\bar{\mathbf{u}}$ satisfies the state constraints, it holds true that

$$J_{\gamma_k}(y(\mathbf{u}_{\gamma_k}),\mathbf{u}_{\gamma_k})\le J(y(\bar{\mathbf{u}}),\bar{\mathbf{u}})\ \text{ for all } k\in\mathbb{N}. \tag{5.2.4}$$

By the definition of J_{γ_k} in (P_{γ_k}), we obtain that (5.2.4) is equivalent to

$$0\le\frac{\gamma_k}{2}\int_a^b\left(y(\bar{t},x;\mathbf{u}_{\gamma_k})-\bar{y}(x)\right)_+^2\,\mathrm{d}x\le J(y(\bar{\mathbf{u}}),\bar{\mathbf{u}})-J(y(\mathbf{u}_{\gamma_k}),\mathbf{u}_{\gamma_k}). \tag{5.2.5}$$

By the continuity of $J(y(\cdot),\cdot)$ w.r.t. $\mathbf{u}$ and (5.2.1), we can deduce that the right-hand side of (5.2.5) is bounded from above by a constant $C > 0$ which is uniform in $k \in \mathbb{N}$ such that the following holds true:

$$0 \leq \int_a^b \left(y(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x)\right)_+^2 \, \mathrm{d}x \leq 2 \cdot \frac{1}{\gamma_k} \cdot C. \tag{5.2.6}$$

Using (5.2.6), we will prove (5.2.3):

First, we observe that (5.2.6) and $\gamma_k \to \infty$ together yield

$$\left(y(\bar{t}, \cdot; \mathbf{u}_{\gamma_k}) - \bar{y}(\cdot)\right)_+ \to 0 \quad \text{ in } L^2([a,b])$$

which implies that

$$\left(y(\bar{t}, \cdot; \mathbf{u}_{\gamma_k}) - \bar{y}(\cdot)\right)_+ \to 0 \quad \text{pointwise a.e. on } [a,b] \tag{5.2.7}$$

holds for a subsequence which is again denoted by $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$. Moreover, we obtain by Corollary 3.3.5 and (5.2.2) that

$$y(\bar{t}, \cdot; \mathbf{u}_{\gamma_k}) \to y(\bar{t}, \cdot; \mathbf{u}^*) \quad \text{ in } L^1([a,b])$$

and therefore

$$y(\bar{t}, \cdot; \mathbf{u}_{\gamma_k}) \to y(\bar{t}, \cdot; \mathbf{u}^*) \quad \text{pointwise a.e. on } [a,b]. \tag{5.2.8}$$

for another subsequence. With this subsequence, (5.2.7) and (5.2.8) we get for almost all $x \in [a,b]$

$$\begin{aligned}
&\big(y(\bar{t}, x; \mathbf{u}^*) - \bar{y}(x)\big)_+ = \big(y(\bar{t}, x; \mathbf{u}^*) - y(\bar{t}, x; \mathbf{u}_{\gamma_k}) + y(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x)\big)_+ \\
&\leq \big(|y(\bar{t}, x; \mathbf{u}^*) - y(\bar{t}, x; \mathbf{u}_{\gamma_k})| + y(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x)\big)_+ \\
&\leq |y(\bar{t}, x; \mathbf{u}^*) - y(\bar{t}, x; \mathbf{u}_{\gamma_k})| + \big(y(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x)\big)_+ \to 0.
\end{aligned}$$

Hence, we obtain that

$$\big(y(\bar{t}, x; \mathbf{u}^*) - \bar{y}(x)\big)_+ = 0 \quad \text{ for a.a. } x \in [a,b].$$

which is equivalent to

$$y(\bar{t}, x; \mathbf{u}^*) \leq \bar{y}(x) \quad \text{ for a.a. } x \in [a,b]. \tag{5.2.9}$$

Using the same arguments as in the proof of Theorem 4.1.1, one can show that (5.2.9) holds true for all $x \in [a,b]$ and hence (5.2.3) is satisfied. It remains to show that $\mathbf{u}^* \in \mathbf{U}_{\text{ad}}$ is a globally optimal solution to (P). To this end, we observe that due to (5.2.3), (5.2.4) and the construction of J_γ in (P_γ), it holds true that

$$J(y(\mathbf{u}^*), \mathbf{u}^*) \leq J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}).$$

Using that $\bar{\mathbf{u}}$ is a globally optimal solution for (P) and for $\mathbf{u}^*$ the state constraints in (5.2.3) are satisfied, we obtain that $J(y(\mathbf{u}^*), \mathbf{u}^*) = J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}})$. Hence, $\mathbf{u}^*$ is a globally optimal solution for (P). □

Remark 5.2.2. The penalty term in the cost functional of the regularized problem (P_γ) is chosen as

$$\frac{\gamma}{2} \int_a^b (y(\bar{t}, x; \mathbf{u}) - \bar{y}(x))_+^2 \, \mathrm{d}x. \tag{5.2.10}$$

In [80] more general penalization terms of the form

$$\frac{1}{\gamma} \int_a^b \phi(\gamma(y(\bar{t}, x; \mathbf{u}) - \bar{y}(x))) \, \mathrm{d}x \tag{5.2.11}$$

are considered, where $\phi : \mathbb{R} \to \mathbb{R}$ is a convex, continuously differentiable mapping for which

$$\begin{aligned} &\phi(0) = 0, \ \phi'(0) = \sigma \geq 0, \ \phi'(s) \geq 0 \quad \forall s \in \mathbb{R} \\ &\lim_{s\to\infty} \phi'(s) = +\infty, \ \lim_{s\to-\infty} \phi'(s) = 0 \end{aligned}$$

holds. Choosing $\phi(s) = \frac{1}{2}\max^2\{0, s\}$, we obtain the penalty term considered in this thesis, see [80]. Now, let $(\mathbf{u}_k)_{k\in\mathbb{N}}$ be a sequence of controls converging to some $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$ such that

$$\lim_{k\to\infty} \int_a^b (y(\bar{t}, x; \mathbf{u}_k) - \bar{y}(x))_+^p \, \mathrm{d}x = 0 \tag{5.2.12}$$

for some $p \in [2, \infty[$. As we have seen in the proof of Theorem 5.2.1, (5.2.12) holds for example if $(\mathbf{u}_k)_{k\in\mathbb{N}}$ is a sequence of global solutions to (P_γ), where $(\gamma_k)_{k\in\mathbb{N}}$ is a sequence of penalty parameters satisfying $\lim\limits_{k\to\infty} \gamma_k = \infty$.

We assume that $(\bar{t}, a)$ is a point of continuity and not a shock generation point of $y(\cdot; \bar{\mathbf{u}})$. Using this assumption, Lemma 3.7.8 (respectively Lemma 3.7.13 or Lemma 3.7.19) yields the existence of a C^1 mapping

$$Y : (x, \mathbf{u}) \in [a, a+\rho[\times B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \mapsto Y(\bar{t}, x, \mathbf{u}) \in \mathbb{R}$$

with some constants $\rho, \tilde{\rho} > 0$ such that

$$y(\bar{t}, x; \mathbf{u}) = Y(\bar{t}, x, \mathbf{u}) \quad \forall x \in [a, a+\rho[, \mathbf{u} \in B_{\tilde{\rho}}^{\mathbf{U}}(\bar{\mathbf{u}}), \tag{5.2.13}$$

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} Y(\bar{t}, x, \mathbf{u}) \right| \leq L \quad \forall x \in [a, a+\rho[, \mathbf{u} \in B_{\tilde{\rho}}^{\mathbf{U}}(\bar{\mathbf{u}}). \tag{5.2.14}$$

Throughout this remark, $L > 0$ denotes a constant that is sufficiently large such that the respective results hold. For any $k \in \mathbb{N}$, we use the abbreviation

$$\varepsilon_k = \left\| (y(\bar{t}, \cdot; \mathbf{u}_k) - \bar{y}(\cdot))_+ \right\|_{L^\infty([a,b])}.$$

Moreover, since $y(\bar{t}, \cdot; \mathbf{u})$ satisfies (3.2.1) for all $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$, $\bar{y} \in C^1([\mathfrak{a}, \mathfrak{b}])$ and due to Convention 3.2.2, for all $k \in \mathbb{N}$ there exists $\tilde{x}_k \in [a, b]$ such that

$$\left| (y(\bar{t}, \tilde{x}_k; \mathbf{u}_k) - \bar{y}(\tilde{x}_k))_+ \right| = \varepsilon_k.$$

Hereby, we first consider the case that $\tilde{x}_k \in [a+\rho, b]$, where the case $\tilde{x}_k \in [a, a+\rho[$ will be discussed later. Proposition 3.2.1 and the regularity of $\bar{y}$ yield that

$$\frac{\mathrm{d}}{\mathrm{d}x} (y(\bar{t}, \cdot; \mathbf{u}) - \bar{y}(\cdot)) \leq L \quad \text{on } [\mathfrak{a} + \delta, \mathfrak{b} - \delta], \ \forall \mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$$

holds in the sense of distributions, where $\delta > 0$ is some small constant. Due to (A4), there is no rarefaction center in some neighborhoods of the points $(\bar{t}, \mathfrak{a})$ and $(\bar{t}, \mathfrak{b})$ for all $u \in \mathbf{U}_{\mathrm{ad}}$. Hence, even

$$\frac{\mathrm{d}}{\mathrm{d}x} (y(\bar{t}, \cdot; \mathbf{u}) - \bar{y}(\cdot)) \leq L \quad \text{on } [\mathfrak{a}, \mathfrak{b}], \ \forall \mathbf{u} \in \mathbf{U}_{\mathrm{ad}}, \tag{5.2.15}$$

holds in the sense of distributions, see [69, Lemma 6.3.3]. Using (5.2.15) and assuming w.l.o.g. that $\tilde{x}_k - \frac{\varepsilon_k}{2L} \geq a$, we can show for any $p \in [2, \infty[$ that

$$\begin{aligned} &\left\| (y(\bar{t}, \cdot; \mathbf{u}_k) - \bar{y}(\cdot))_+ \right\|_{L^p([a,b])} \geq \left\| (y(\bar{t}, \cdot; \mathbf{u}_k) - \bar{y}(\cdot))_+ \right\|_{L^p([\tilde{x}_k - \frac{\varepsilon_k}{2L}, \tilde{x}_k])} \geq \left(\left(\frac{\varepsilon_k}{2} \right)^p \cdot \frac{\varepsilon_k}{2L} \right)^{\frac{1}{p}} \\ &= \frac{1}{2^{\frac{p+1}{p}} L^{\frac{1}{p}}} \left\| (y(\bar{t}, \cdot; \mathbf{u}_k) - \bar{y}(\cdot))_+ \right\|_{L^\infty([a,b])}^{\frac{p+1}{p}}. \end{aligned}$$

see also [80, Proof of Theorem 8.21]. From this, we get

$$\left\| (y(\bar{t}, \cdot; \mathbf{u}_k) - \bar{y}(\cdot))_+ \right\|_{L^\infty([a,b])} \leq 2L^{\frac{1}{p+1}} \left\| (y(\bar{t}, \cdot; \mathbf{u}_k) - \bar{y}(\cdot))_+ \right\|_{L^p([a,b])}^{\frac{p}{p+1}}. \tag{5.2.16}$$

For the case that $\tilde{x}_k - \frac{\varepsilon_k}{2L} < a$, using similar arguments and in particular $\tilde{x}_k \in$

$[a+\rho, b]$, we obtain that

$$\varepsilon_k = \left\|(y(\bar{t},\cdot;\mathbf{u}_k) - \bar{y}(\cdot))_+\right\|_{L^\infty([a,b])} \leq \frac{2}{\rho^{\frac{1}{p}}} \left\|(y(\bar{t},\cdot;\mathbf{u}_k) - \bar{y}(\cdot))_+\right\|_{L^p([a,b])}.$$

Due to (5.2.12) and $\tilde{x}_k \in [a+\rho, b]$, this shows that for $k \in \mathbb{N}$ large enough $\tilde{x}_k - \frac{\varepsilon_k}{2L} \geq a$ is always satisfied. Therefore, considering a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ satisfying (5.2.12), we conclude that (5.2.16) yields

$$\left\|(y(\bar{t},\cdot;\mathbf{u}_k) - \bar{y}(\cdot))_+\right\|_{L^\infty([a,b])} \to 0 \quad \text{for } k \to \infty. \tag{5.2.17}$$

For the case that $\tilde{x}_k \in [a, a+\rho[$, we can use (5.2.13) and (5.2.14) and the regularity of $\bar{y}$ to prove that (5.2.16) is valid for sufficiently large $k \in \mathbb{N}$. Therefore, we finally obtain that (5.2.16) is always satisfied, which in particular yields

$$\lim_{k\to\infty} \int_a^b (y(\bar{t},x;\mathbf{u}_k) - \bar{y}(x))_+^p \,\mathrm{d}x = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \left\|(y(\bar{t},\cdot;\mathbf{u}_k) - \bar{y}(\cdot))_+\right\|_{L^\infty(a,b)} = 0$$

for all $p \in [2,\infty[$. Therefore, in this thesis, we can restrict ourselves to the penalty term in (5.2.10). Nevertheless, the results in this chapter are also valid for more general penalization terms as in (5.2.11).

In the previous theorem we considered a sequence of globally optimal solutions of the regularized problems in (P_γ). Since it is quite complex to compute global optima of (P_γ), we will follow the ideas of [62] to examine the case of a sequence of local solutions. More precisely, the authors in [62] introduce the following auxiliary problems which will play a key-role in the analysis of this thesis:

$$\min_{\mathbf{u}\in\mathbf{U}} J(y(\mathbf{u}),\mathbf{u}) \text{ s.t. } \begin{array}{l} y(\mathbf{u}) \text{ solves the (IBVP), } \mathbf{u} \in \mathbf{U}^r := \mathbf{U}_{\mathrm{ad}} \cap B_r^{\mathbf{U}}(\bar{\mathbf{u}}) \\ y(\bar{t},x;\mathbf{u}) \leq \bar{y}(x) \quad \forall x \in [a,b] \end{array} \tag{P^r}$$

and

$$\min_{\mathbf{u}\in\mathbf{U}} J_\gamma(y(\mathbf{u}),\mathbf{u}) \text{ s.t. } y(\mathbf{u}) \text{ solves the (IBVP), } \mathbf{u} \in \mathbf{U}^r, \tag{P_γ^r}$$

where the problems are additionally restricted on a ball around $\bar{\mathbf{u}}$ with a sufficiently small radius $r > 0$ is a sufficiently small constant.

Using the same arguments as in the proof of Theorem 4.1.1, one can show that (P^r) and (P_γ^r) admit global solutions which we denote by $\bar{\mathbf{u}}^r$ and $\mathbf{u}_\gamma^r$, respectively.

In the following theorem, which is based on Theorem 5.1 in [62], we will show that for each local optimum $\bar{\mathbf{u}}$ for (P) satisfying a quadratic growth condition, there exists a sequence of local solutions to (P_γ) converging to $\bar{\mathbf{u}}$.

Theorem 5.2.3. *Assume that* (A3) *and* (A4) *hold true and consider a locally optimal solution* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *to* (P). *In addition, assume that the quadratic growth condition*

$$J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) + \frac{\delta}{2}\|\mathbf{u} - \bar{\mathbf{u}}\|_H^2 \leq J(y(\mathbf{u}), \mathbf{u}) \quad \forall \mathbf{u} \in \tilde{\mathbf{U}}_{ad} \text{ with } \|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{U}} < \varepsilon \tag{5.2.18}$$

is satisfied for some constants $\varepsilon, \delta > 0$, *where*

$$\tilde{\mathbf{U}}_{ad} \coloneqq \{\mathbf{u} \in \mathbf{U}_{ad} : y(\bar{t}, x; \mathbf{u}) \leq \bar{y}\,(x) \ \ \forall x \in [a, b]\}$$

and H *is a Hilbert space such that* $\mathbf{U}$ *is continuously embedded in* H. *Then there exists a sequence of local solutions for* (P_γ) *that converges to* $\bar{\mathbf{u}}$ *w.r.t.* $\|\cdot\|_{\mathbf{U}}$.

The concept of the proof can be found in the proof of Theorem 5.1 in [62].

Proof. Set $r = \frac{\varepsilon}{2}$ and let $(\mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}} \subset \mathbf{U}^r$ denote a sequence of globally optimal solutions for (P_γ^r) with $\gamma_k \to \infty$.

Carefully reading the proof of Theorem 4.1.1, we observe that $\mathbf{U}^r$ is compact in $\mathbf{U}$. Therefore $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ possesses a convergent subsequence without change of notation

$$\mathbf{u}_{\gamma_k} \to \mathbf{u}^* \in \mathbf{U}^r \quad \text{w.r.t. } \|\cdot\|_{\mathbf{U}}$$

for $k \to \infty$. Similar to the proof of Theorem 5.2.1, one can prove that $\mathbf{u}^*$ is a globally optimal solution for (P^r). Furthermore, the condition (5.2.18) yields that that $\mathbf{u}^* = \bar{\mathbf{u}}$. Therefore, $\mathbf{u}_{\gamma_k}^r \in \operatorname{int} B_r^W(\bar{\mathbf{u}})$ holds and hence $\mathbf{u}_{\gamma_k}^r$ is a locally optimal solution for (P_γ) for k large enough, cf. [62, Proof of Lemma 5.2]. □

Before stating the next result, we will have a look at some simple example in which, as we will check, the quadratic growth condition in (5.2.18) holds.

Example 5.2.4. We set $\Omega = \mathbb{R}$ and consider the following optimal control problem

$$\min_{\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}} J(y(\mathbf{u}), \mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}} \left(y(\bar{t}, x; \mathbf{u}) - y_d(x)\right)^2 \,\mathrm{d}x + \frac{\kappa}{2}\left\|u_1\right\|^2_{L^2(]-\infty, x_1])} + \frac{\kappa}{2}\left\|u_2\right\|^2_{L^2([x_1, x_2])},$$

where $\kappa > 0$ is a positive constant and y is the entropy solution of

$$
\begin{aligned}
y_t + \left(\frac{1}{2}y^2\right)_x &= 0 && \text{on } (0,T)\times\mathbb{R},\\
y(0,\cdot) &= u_1(\cdot) && \text{on }]-\infty, x_1],\\
y(0,\cdot) &= u_2(\cdot) && \text{on }]x_1, x_2],\\
y(0,\cdot) &= u_3(\cdot) && \text{on }]x_2, \infty[.
\end{aligned}
$$

The control is given by $\mathbf{u} = (u_1, u_2)$ with $u_1 \in C^1(]-\infty, x_1])$ and $u_2 \in C^1([x_1, x_2])$, where $-\infty < x_1 < x_2 < \infty$ and $u_3 \in C^1([x_2, \infty[)$ are fixed such that

$$\mathbf{U}_{\mathrm{ad}} = \mathbf{U} = C^1(]-\infty, x_1]) \times C^1([x_1, x_2]).$$

One can show that for suitable y_d, u_3, x_1 and x_2 this optimal control problem admits an optimal solution $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2) \in \mathbf{U}_{\mathrm{ad}}$ with $J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) < \infty$ and such that the following holds: The state $y(\bar{\mathbf{u}})$ has a rarefaction center in $(0, x_1)$ and a shock-curve ξ emanating from $(0, x_2)$ such that the minimal backward characteristic ξ_- through $(\bar{t}, x_s(\bar{\mathbf{u}}))$ with $x_s(\bar{\mathbf{u}}) := \xi(\bar{t}, \bar{\mathbf{u}})$ ends in the inner of the rarefaction wave emanating from $(0, x_1)$. We set $y := y(\mathbf{u})$ for some $\mathbf{u} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$. In what follows, let $\delta > 0$ be chosen sufficiently small such that the respective results hold. We assume that $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$ satisfies (ND) and therefore also all $\mathbf{u} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$. For a suitable choice of y_d this assumption can be guaranteed to hold. Due to Lemma 3.7.1, for all $\mathbf{u} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$ $y(\bar{t}, x; \mathbf{u})$ has the structure

$$
y(\bar{t}, x; \mathbf{u}) = \begin{cases} Y_1(\bar{t}, x, \mathbf{u}), & \text{if } x \in]-\infty, x_1 + u_1(x_1)\bar{t}] \\ Y_2(\bar{t}, x) = \frac{x - x_1}{\bar{t}}, & \text{if } x \in]x_1 + u_1(x_1)\bar{t}, x_s(\mathbf{u})] \\ Y_3(\bar{t}, x), & \text{if } x \in]x_s(\mathbf{u}), \infty] \end{cases}, \tag{5.2.19}
$$

where Y_1 depends continuously differentiable on x and $\mathbf{u}$, Y_2 and Y_3 are independent of $\mathbf{u}$. The reduced cost functional is Fréchet-differentiable according to Theorem 3.5.5 for any $\mathbf{u} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$ with derivative

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}), \mathbf{u})\delta\mathbf{u} &= \int_{-\infty}^{x_1} p(0,x)\delta u_1(x)\,\mathrm{d}x \\
+ \int_{x_1}^{x_2} p(0,x)\delta u_2(x)\,\mathrm{d}x &+ \kappa \int_{-\infty}^{x_1} u_1(x)\delta u_1(x)\,\mathrm{d}x + \kappa \int_{x_1}^{x_2} u_2(x)\delta u_2(x)\,\mathrm{d}x
\end{aligned} \tag{5.2.20}
$$

where p denotes the adjoint state according to Definition 3.5.4 with end data

$$p^{\bar{t}}(x) = \int_0^1 \psi_y \left(y(\bar{t}, x+; \mathbf{u})) + \tau[y(\bar{t}, x; \mathbf{u})], y_d(x+) + \tau[y_d(x)] \right) \mathrm{d}\tau.$$

We first rewrite the end data to

$$p^{\bar{t}}(x) = \begin{cases} y(\bar{t}, x; \mathbf{u}) - y_d(x) & \text{if } x \in \mathbb{R} \setminus \{x_s(\mathbf{u})\} \\ \frac{[(y(\bar{t}, x_s(\mathbf{u}); \mathbf{u}) - y_d(x_s(\mathbf{u})))^2]}{2[y(\bar{t}, x_s(\mathbf{u}); \mathbf{u})]} & \text{if } x = x_s(\mathbf{u}) \end{cases}$$

and then further obtain

$$p^{\bar{t}}(x) = \begin{cases} Y_1(\bar{t}, x, \mathbf{u}) - y_d(x) & \text{if } x \in]-\infty, x_s(\mathbf{u})[\\ Y_3(\bar{t}, x, \mathbf{u}) - y_d(x) & \text{if } x \in]x_s(\mathbf{u}), \infty[\\ \frac{1}{2} \left(y(\bar{t}, x_s(\mathbf{u})-) + y(\bar{t}, x_s(\mathbf{u})+) - 2y_d(x_s(\mathbf{u})) \right) & \text{if } x = x_s(\mathbf{u}) \end{cases} . \tag{5.2.21}$$

Using the definition of p in Definition 3.5.4, the construction of Y_1 in Lemma 3.7.4 and the presentation of the end data in (5.2.21), we get

$$p(0, x) = \begin{cases} u_1(x) - y_d(x + \bar{t}u_1(x)) & \text{if } x \in]-\infty, x_1[\\ \frac{1}{2} \left(y(\bar{t}, x_s(\mathbf{u})-) + y(\bar{t}, x_s(\mathbf{u})+) - 2y_d(x_s(\mathbf{u})) \right) & \text{if } x \in [x_1, x_2] \end{cases} ,$$

which inserting in (5.2.20) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}), \mathbf{u}) \delta\mathbf{u} &= \int_{-\infty}^{x_1} \left((1+\kappa) u_1(x) - y_d(x + \bar{t}u_1(x)) \right) \delta u_1(x) \, \mathrm{d}x \\ &+ \int_{x_1}^{x_2} \left[\frac{1}{2} \left(y(\bar{t}, x_s(\mathbf{u})-) + y(\bar{t}, x_s(\mathbf{u})+) - 2y_d(x_s(\mathbf{u})) \right) + \kappa u_2(x) \right] \delta u_2(x) \, \mathrm{d}x. \end{aligned} \tag{5.2.22}$$

Using the representation of the gradient in (5.2.22), we will see below that the reduced cost functional is for all $\mathbf{u} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$ twice continuously differentiable with Hessian $\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}), \mathbf{u}) \delta\mathbf{u} \right) \delta\mathbf{u}$ given in (5.2.32). The main idea to prove the quadratic growth condition (5.2.18) is the following: We show that

$$J(y(\hat{\mathbf{u}}), \hat{\mathbf{u}}) - J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) \geq \frac{\tilde{\kappa}}{2} \left(\|\hat{u}_1 - \bar{u}_1\|^2_{2,]-\infty,x_1]} + \|\hat{u}_2 - \bar{u}_2\|^2_{2,]x_1,x_2]} \right) \tag{5.2.23}$$

holds for all $\hat{\mathbf{u}} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$ with some constant $\tilde{\kappa} > 0$ which does not depend on $\hat{\mathbf{u}}$. To this end, let $\hat{\mathbf{u}} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$ be arbitrarily chosen. Since $J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) < \infty$, we obtain

$$\lim_{x \to -\infty} |\bar{u}_1(x)| = 0. \tag{5.2.24}$$

Therefore, in our setting y_d must also satisfy

$$\lim_{x \to -\infty} |y_d(x)| = 0. \tag{5.2.25}$$

We observe that if

$$\lim_{x \to -\infty} |\hat{u}_1(x)| > 0, \tag{5.2.26}$$

then $J(y(\hat{\mathbf{u}}), \hat{\mathbf{u}}) = \infty$ and (5.2.23) holds trivially. Hence, we can w.l.o.g. assume that

$$\lim_{x \to -\infty} |\hat{u}_1(x)| = 0. \tag{5.2.27}$$

Next, we observe that there exists $s \in [0, 1]$ such that

$$\begin{aligned} J(y(\hat{\mathbf{u}}), \hat{\mathbf{u}}) - J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) =& \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}})(\hat{\mathbf{u}} - \bar{\mathbf{u}}) \\ &+ \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}), \mathbf{u})(\hat{\mathbf{u}} - \bar{\mathbf{u}}) \right) (\hat{\mathbf{u}} - \bar{\mathbf{u}}) \end{aligned} \tag{5.2.28}$$

with $\mathbf{u} = \bar{\mathbf{u}} + s(\hat{\mathbf{u}} - \bar{\mathbf{u}})$. Using (5.2.24) and (5.2.27), we can deduce that

$$\lim_{x \to -\infty} |u_1(x)| = 0. \tag{5.2.29}$$

Due to the optimality of $\bar{\mathbf{u}}$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) = 0. \tag{5.2.30}$$

Therefore, in order to prove the quadratic growth condition (5.2.23) it is sufficient to show that

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}), \mathbf{u})(\hat{\mathbf{u}} - \bar{\mathbf{u}}) \right) (\hat{\mathbf{u}} - \bar{\mathbf{u}}) \\ &\geq \frac{\tilde{\kappa}}{2} \left(\|\hat{u}_1 - \bar{u}_1\|_{2,]-\infty,x_1]}^2 + \|\hat{u}_2 - \bar{u}_2\|_{2,]x_1,x_2]}^2 \right). \end{aligned} \tag{5.2.31}$$

To this end, we first observe that due to the regularity of y_d, Y_1, Y_2 and Y_3, the gradient of the reduced cost functional in (5.2.22) is again continuously differentiable in a neighborhood of $\bar{\mathbf{u}}$ such that the Hessian of the reduced cost functional is given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}J(y(\mathbf{u}),\mathbf{u})\delta\mathbf{u}\right)\delta\mathbf{u} \\
&= \int_{-\infty}^{x_1} \left((1+\kappa) - y_d'(x+\bar{t}u_1(x))\bar{t}\right)\delta u_1(x)^2\,\mathrm{d}x \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}x}\left(y(\bar{t},x_s(\mathbf{u})-;\mathbf{u}) + y(\bar{t},x_s(\mathbf{u})+;\mathbf{u}) - 2y_d(x_s(\mathbf{u}))\right)\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_s(\mathbf{u})\delta\mathbf{u} \\
&\qquad \cdot\int_{x_1}^{x_2}\delta u_2(x)\,\mathrm{d}x + \kappa\left\|\delta u_2\right\|_{2,[x_1,x_2]}^2 \\
&= \int_{-\infty}^{x_1} \left((1+\kappa) - y_d'(x+\bar{t}u_1(x))\bar{t}\right)\delta u_1(x)^2\,\mathrm{d}x \\
&\quad + [y(\bar{t},x_s(\mathbf{u}))]\frac{\mathrm{d}}{\mathrm{d}x}\left(y(\bar{t},x_s(\mathbf{u})-;\mathbf{u}) + y(\bar{t},x_s(\mathbf{u})+;\mathbf{u}) - 2y_d(x_s(\mathbf{u}))\right)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_s(\mathbf{u})\delta\mathbf{u}\right)^2 \\
&\quad + \kappa\left\|\delta u_2\right\|_{2,[x_1,x_2]}^2,
\end{aligned} \tag{5.2.32}$$

where we have used that

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_s(\mathbf{u})\delta\mathbf{u} = \frac{1}{[y(\bar{t},x_s(\mathbf{u}))]}\int_{x_1}^{x_2}\delta u_2(x)\,\mathrm{d}x,$$

which holds due to Theorem 3.7.24. Assuming that

$$y_d'(x+\bar{t}\bar{u}_1(x))\bar{t} < 1 + \frac{\kappa}{2} \quad \forall x \in]-\infty, x_1] \tag{5.2.33}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y(\bar{t},x_s(\bar{\mathbf{u}})-;\bar{\mathbf{u}}) + y(\bar{t},x_s(\bar{\mathbf{u}})+,\bar{\mathbf{u}}) - 2y_d(x_s(\bar{\mathbf{u}}))\right) > 0, \tag{5.2.34}$$

we can deduce from (5.2.32) that (5.2.31) holds with a constant $\tilde{\kappa} = \frac{\kappa}{2}$ that does not depend on $\hat{\mathbf{u}}$. Hereby, we have used that $[y(\bar{t},x_s(\mathbf{u}))] > 0$. Therefore, and since $\hat{\mathbf{u}} \in B_\delta^{\mathbf{U}}(\bar{\mathbf{u}})$ was arbitrarily chosen, the quadratic growth condition (5.2.18) is satisfied with $H = L^2$.

In the following result we examine the convergence of a sequence of local solutions for (P_γ):

Theorem 5.2.5. *Let* (A3) *and* (A4) *hold true and let* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ *be a sequence of local solutions for* (P_{γ_k}) *with* $\gamma_k \to \infty$ *for* $k \to \infty$. *Assume in addition that the* *condition*

$$J_{\gamma_k}(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k}) + \frac{\delta}{2}\|\mathbf{u} - \mathbf{u}_{\gamma_k}\|_H^2 \leq J_{\gamma_k}(y(\mathbf{u}), \mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{U}_{ad} \text{ with } \|\mathbf{u} - \mathbf{u}_{\gamma_k}\|_{\mathbf{U}} < \varepsilon \tag{5.2.35}$$

is satisfied for all $k \in \mathbb{N}$ *large enough, where* $\|\cdot\|_H$ *is defined as in Theorem 5.2.3 and constants* $\varepsilon, \delta > 0$.

Finally, assume that $x = a$ *is a point of continuity of* $y(\bar{t}, \cdot; \mathbf{u})$ *for all* $\mathbf{u} \in \mathbf{U}_{ad}$. *Then there exists a subsequence again denoted by* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ *such that*

$$\mathbf{u}_{\gamma_k} \to \bar{\mathbf{u}} \in \mathbf{U}_{ad} \quad \text{w.r.t. } \|\cdot\|_{\mathbf{U}},$$

where $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *is a local solution for* (P).

Proof. Due to the compactness of the space $\mathbf{U}_{\text{ad}}$, we obtain that $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ has a identically denoted convergent subsequence with limit $\bar{\mathbf{u}} \in \mathbf{U}_{\text{ad}}$. For $\bar{\mathbf{u}}$ we define the corresponding problems (P^r) and (P^r_γ), where we set $r = \frac{\varepsilon}{2}$.

From the condition (5.2.35), we obtain that $\mathbf{u}_{\gamma_k}$ is the unique globally optimal solution for $(P^{\frac{\varepsilon}{2}}_{\gamma_k})$, if k is sufficiently large. Then we can use the same arguments as in the proof of Theorem 5.2.1 to show that $\bar{\mathbf{u}} \in \mathbf{U}_{\text{ad}}$ is a globally optimal solution for $(P^{\frac{\varepsilon}{2}})$ and hence a local solution for (P). □

5.3 Analysis of the optimality system for the regularized problem

In the following we analyze the optimality system of the regularized problems (P_γ) which, as we will show, converges under suitable assumptions to the optimality system of the state-constrained optimal control problem in Theorem 4.2.9. To this end, we will need the following technical lemma.

Lemma 5.3.1. *Assume that* (A3) *and* (A4) *are satisfied and consider a sequence of* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}} \subset \mathbf{U}$ *converging to some control* $\mathbf{u}^* \in \mathbf{U}$ *satisfying the requirements in* (ND). *Then for all sufficiently large* k, *the controls* $\mathbf{u}_{\gamma_k}$ *also satisfy the requirements in* (ND), *respectively. In addition, there exists* $\varepsilon > 0$ *such that for all* $j = 1, \dots, K+1$

it holds true that

$$\lim_{k\to\infty} Y_j(\bar{t},\cdot;\mathbf{u}_{\gamma_k}) = Y_j(\bar{t},\cdot;\mathbf{u}^*) \qquad \text{in } C^1\left((x_{j-1}(\mathbf{u}^*)-\varepsilon, x_j(\mathbf{u}^*)+\varepsilon)\right), \tag{5.3.1a}$$

$$\lim_{k\to\infty} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t},\cdot;\mathbf{u}_{\gamma_k}) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t},\cdot;\mathbf{u}^*) \quad \text{in } C\left((x_{j-1}(\mathbf{u}^*)-\varepsilon, x_j(\mathbf{u}^*)+\varepsilon)\right), \tag{5.3.1b}$$

where $Y_1,\dots,Y_{K+1}$ denote the functions given by Lemma 3.7.1.

Proof. At first, the fact that $\mathbf{u}_{\gamma_k}$ satisfies (ND) for sufficiently large k is a direct consequence of (3.7.4) in Lemma 3.7.1. Now we consider the two limits in (5.3.1). The limit in (5.3.1b) follows from Lemma 3.7.1. Recalling the construction of the functions $Y_1,\dots,Y_{K+1}$ and the formulas for their derivatives w.r.t. x given in the Lemmas 3.7.4, 3.7.9, 3.7.12, 3.7.14, 3.7.18 and 3.7.20, we use Lemma 3.2.5 and obtain that also (5.3.1a) holds true. □

Since $\mathbf{U}_{\mathrm{ad}} \subset \mathbf{U}$ is nonempty and convex by (A4) and due to Corollary 5.1.1, we obtain the following necessary optimality conditions. A proof can be found in [41, Theorem 1.46].

Theorem 5.3.2. *Assume that* (A3) *and* (A4) *are satisfied and let $\mathbf{u}_\gamma \in \mathbf{U}_{ad}$ be a local solution for (P_γ) satisfying* (ND)*. Then the variational inequality*

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J_\gamma(y(\mathbf{u}_\gamma),\mathbf{u}_\gamma)\cdot(\mathbf{u}-\mathbf{u}_\gamma) \geq 0 \quad \forall \mathbf{u}\in\mathbf{U}_{ad}$$

holds true.

Consider a control $\mathbf{u}\in\mathbf{U}$ satisfying the requirements of (ND). Considering the representation of the corresponding solution $y(\bar{t},\cdot,\mathbf{u})$ to the IBVP in (3.7.4) in Lemma 3.7.1, we can rewrite $J_\gamma(y(\mathbf{u}),\mathbf{u})$ in terms of the functions $Y_1,\dots,Y_{K+1}$:

$$J_\gamma(y(\mathbf{u}),\mathbf{u}) = J(y(\mathbf{u}),\mathbf{u}) + \sum_{j=1}^{K+1} z_j(\mathbf{u})\cdot\frac{\gamma}{2z_j(\mathbf{u})}\int_{x_{j-1}(\mathbf{u})}^{x_j(\mathbf{u})} \left(Y_j(\bar{t},x;\mathbf{u})-\bar{y}(x)\right)_+^2 \,\mathrm{d}x, \tag{5.3.2}$$

where $z_j(\mathbf{u}) \coloneqq (x_j(\mathbf{u})-x_{j-1}(\mathbf{u}))$. The derivative of $J_\gamma(y(\mathbf{u}),\mathbf{u})$ in a direction

$\delta\mathbf{u} \in \mathbf{U}$ can be written as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J_\gamma(y(\mathbf{u}),\mathbf{u})\delta\mathbf{u} = {} & \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}),\mathbf{u})\delta\mathbf{u} \\
& + \sum_{j=1}^{K+1} z_j(\mathbf{u}) \Bigg[\frac{\gamma}{z_j(\mathbf{u})} \int_{x_{j-1}(\mathbf{u})}^{x_j(\mathbf{u})} \big(\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t},x,\mathbf{u})\delta\mathbf{u} (Y_j(\bar{t},x,\mathbf{u}) - \bar{y}(x))_+ \big) \,\mathrm{d}x \\
& \quad + \frac{-\gamma}{2z_j(\mathbf{u})^2} \int_{x_{j-1}(\mathbf{u})}^{x_j(\mathbf{u})} (Y_j(\bar{t},x,\mathbf{u}) - \bar{y}(x))_+^2 \,\mathrm{d}x \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} z_j(\mathbf{u})\delta\mathbf{u} \\
& \quad + \frac{\gamma}{2z_j(\mathbf{u})} \cdot \Bigg((Y_j(\bar{t},x_j(\mathbf{u});\mathbf{u}) - \bar{y}(x_j(\mathbf{u})))_+^2 \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\mathbf{u})\delta\mathbf{u} \\
& \quad - (Y_j(\bar{t},x_{j-1}(\mathbf{u});\mathbf{u}) - \bar{y}(x_{j-1}(\mathbf{u})))_+^2 \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\mathbf{u})\delta\mathbf{u} \Bigg) \Bigg] \\
& + \frac{\gamma}{2z_j(\mathbf{u})} \int_{x_{j-1}(\mathbf{u})}^{x_j(\mathbf{u})} (Y_j(\bar{t},x,\mathbf{u}) - \bar{y}(x))_+^2 \,\mathrm{d}x \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} z_j(\mathbf{u})\delta\mathbf{u}
\end{aligned}
\tag{5.3.3}
$$

Adopting the concepts developed in [43], we introduce for $j = 1, \ldots, K+1$ the *Lagrange multiplier estimates* by

$$
\lambda_j(x;\mathbf{u}) := \begin{cases} \gamma \left(Y_j(\bar{t},x,\mathbf{u}) - \bar{y}(x)\right)_+, & \text{for } x_{j-1}(\mathbf{u}) \le x \le x_j(\mathbf{u}) \\ 0, & \text{for } x \in [a,b] \setminus [x_j(\mathbf{u}), x_{j+1}(\mathbf{u})], \end{cases}
\tag{5.3.4}
$$

and in addition the abbreviation

$$
r_j(\mathbf{u},\gamma) := \frac{\gamma}{2(x_j(\mathbf{u}) - x_{j-1}(\mathbf{u}))} \int_{x_{j-1}(\mathbf{u})}^{x_j(\mathbf{u})} (Y_j(\bar{t},x,\mathbf{u}) - \bar{y}(x))_+^2 \,\mathrm{d}x.
\tag{5.3.5}
$$

Inserting (5.3.4) and (5.3.5) in (5.3.3), we can rewrite the optimality conditions of Theorem 5.3.2 as follows:

Theorem 5.3.3. *Let* (A3) *and* (A4) *hold true. Consider a local solution* $\mathbf{u}_\gamma \in \mathbf{U}_{ad}$

of (P_γ) *which satisfies the requirements of* (ND). *Then for all* $\mathbf{u} \in \mathbf{U}_{ad}$ *it holds that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}_\gamma), \mathbf{u}_\gamma)(\mathbf{u} - \mathbf{u}_\gamma) + \sum_{j=1}^{K+1} \Bigg[\int_{x_{j-1}(\mathbf{u}_\gamma)}^{x_j(\mathbf{u}_\gamma)} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t}, x; \mathbf{u}_\gamma)(\mathbf{u} - \mathbf{u}_\gamma) \lambda_j(x; \mathbf{u}_\gamma)\,\mathrm{d}x \\
&+ \int_{x_{j-1}(\mathbf{u}_\gamma)}^{x_j(\mathbf{u}_\gamma)} \frac{\partial}{\partial x}(Y_j(\bar{t}, x; \mathbf{u}_\gamma) - \bar{y}(x)) \frac{x - x_{j-1}(\mathbf{u}_\gamma)}{x_j(\mathbf{u}_\gamma) - x_{j-1}(\mathbf{u}_\gamma)} \lambda_j(x; \mathbf{u}_\gamma)\,\mathrm{d}x \\
&\quad \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\mathbf{u}_\gamma)(\mathbf{u} - \mathbf{u}_\gamma) \\
&+ \int_{x_{j-1}(\mathbf{u}_\gamma)}^{x_j(\mathbf{u}_\gamma)} \frac{\partial}{\partial x}(Y_j(\bar{t}, x; \mathbf{u}_\gamma) - \bar{y}(x)) \frac{x_j(\mathbf{u}_\gamma) - x}{x_j(\mathbf{u}_\gamma) - x_{j-1}(\mathbf{u}_\gamma)} \lambda_j(x; \mathbf{u}_\gamma)\,\mathrm{d}x \\
&\quad \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\mathbf{u}_\gamma)(\mathbf{u} - \mathbf{u}_\gamma) \\
&+ r_j(\mathbf{u}_\gamma, \gamma) \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} (x_j(\mathbf{u}_\gamma) - x_{j-1}(\mathbf{u}_\gamma))(\mathbf{u} - \mathbf{u}_\gamma) \Bigg] \geq 0.
\end{aligned} \tag{5.3.6}$$

The terms $r_j(\mathbf{u}_\gamma, \gamma)$ for $j = 1, \ldots, K+1$ go to zero for $\gamma \to \infty$ which we will show in the following lemma.

Lemma 5.3.4. *Assume that* (A3) *and* (A4) *hold true. Moreover, consider a sequence* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ *of local solutions* $\mathbf{u}_{\gamma_k}$ *of* (P_{γ_k}) *with* $\gamma_k \to \infty$ *for* $k \to \infty$ *that converges to a local solution* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *of* (P) *satisfying* (ND). *In addition, assume that there exists a constant* $\varepsilon > 0$ *such that the condition*

$$J_{\gamma_k}(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k}) \leq J_{\gamma_k}(y(\mathbf{u}), \mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{U}_{ad} \text{ with } \|\mathbf{u} - \mathbf{u}_{\gamma_k}\|_{\mathbf{U}} < \varepsilon \tag{5.3.7}$$

is satisfied for sufficiently large $k \in \mathbb{N}$. *Then it holds true that*

$$\lim_{k\to\infty} r_j(\mathbf{u}_{\gamma_k}, \gamma_k) = 0 \quad \text{for all } j = 1, \ldots, K+1.$$

Proof. Using (5.3.7) and the fact that $\mathbf{u}_{\gamma_k} \to \bar{\mathbf{u}}$ for $k \to \infty$, we obtain for sufficiently large $k \in \mathbb{N}$ that

$$J_{\gamma_k}(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k}) \leq J_{\gamma_k}(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) = J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}),$$

where the last equality is valid since in $y(\bar{\mathbf{u}})$ the state constraints are satisfied by assumption. This yields

$$\frac{\gamma_k}{2} \int_a^b (y(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x))_+^2 \,\mathrm{d}x \leq J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - J(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k}). \tag{5.3.8}$$

Moreover, since $\bar{\mathbf{u}}$ satisfies (ND), we obtain from Lemma 5.3.1 that $\mathbf{u}_{\gamma_k}$ satisfies (ND) for sufficiently large $k \in \mathbb{N}$. Using this and (5.3.8), we obtain for sufficiently large $k \in \mathbb{N}$ that

$$\begin{aligned} 0 \leq & \lim_{k\to\infty} \sum_{j=1}^{K+1} \frac{\gamma_k}{2} \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} (Y_j(\bar{t}, x, \mathbf{u}_{\gamma_k}) - \bar{y}(x))_+^2 \, \mathrm{d}x \\ = & \lim_{k\to\infty} \frac{\gamma_k}{2} \int_a^b (y(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x))_+^2 \, \mathrm{d}x \\ \leq & J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - \lim_{k\to\infty} J(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k}) = 0, \end{aligned} \tag{5.3.9}$$

where the last equality holds due to the continuity of $J(y(\mathbf{u}), \mathbf{u})$ with respect to the control $\mathbf{u}$ and the fact that $\mathbf{u}_{\gamma_k}$ converges to $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$. Since the integrands in (5.3.9) are nonnegative, (5.3.9) yields

$$\lim_{k\to\infty} \frac{\gamma_k}{2} \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} (Y_j(\bar{t}, x, \mathbf{u}_{\gamma_k}) - \bar{y}(x))_+^2 = 0 \quad \text{for all } j = 1, \ldots, K+1. \tag{5.3.10}$$

Furthermore, we note that since $x_1(\bar{\mathbf{u}}) < \ldots < x_{K+1}(\bar{\mathbf{u}})$ holds by the assumptions in (ND) and

$$x_j(\mathbf{u}_{\gamma_k}) \to x_j(\bar{\mathbf{u}}) \quad \text{for } k \to \infty \quad \text{for all } j = 1, \ldots, K+1,$$

we obtain that the terms $(x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k}))$, $j = 1, \ldots, K+1$, are positive and uniformly bounded away from zero for sufficiently large k, which together with (5.3.10) implies

$$\begin{aligned} \lim_{k\to\infty} r_j(\mathbf{u}_{\gamma_k}, \gamma_k) = & \lim_{k\to\infty} \frac{\gamma_k}{2(x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k}))} \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} (Y_j(\bar{t}, x, \mathbf{u}_{\gamma_k}) - \bar{y}(x))_+^2 \, \mathrm{d}x \\ = & \, 0 \end{aligned}$$

for all $j = 1, \ldots, K+1$. □

Remark 5.3.5. The quadratic growth condition (5.2.35) implies the condition (5.3.7) in Lemma 5.3.4.

Considering the sequence from Lemma 5.3.4, we assume in addition that the Robinson's CQ (4.2.13) is satisfied in $\bar{\mathbf{u}}$. Then the corresponding sequences of Lagrange multiplier estimates are bounded in $L^1([a, b])$ as we will see in the following lemma, cf. [45, Lemma 9].

Lemma 5.3.6. *Suppose that* (A3) *and* (A4) *hold and let* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ *denote a sequence of local solutions of* (P_{γ_k}) *with* $\gamma_k \to \infty$ *for* $k \to \infty$ *satisfying* (5.3.7) *for sufficiently large* $k \in \mathbb{N}$ *and converging to a local solution* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *to* (P) *which satisfies* (ND) *and the Robinson's CQ* (4.2.13). *Then the corresponding sequences of Lagrange multiplier estimates* $(\lambda_j(\cdot;\mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}}$ *are uniformly bounded in* $L^1([a,b])$.

Proof. At first, we observe that $\mathbf{u}_{\gamma_k}$ satisfies (ND) for sufficiently large k by Lemma 5.3.1. For the rest of the proof, we will always assume that k is sufficiently large such that the corresponding result holds true. Thus, we obtain from Theorem 5.3.3 that (5.3.6) is satisfied in $\mathbf{u}_{\gamma_k}$ and we obtain that the following inequality is valid for all $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$:

$$\begin{aligned}\sum_{j=1}^{K+1} \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} -\Big[&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t},x;\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k}) + \frac{\partial}{\partial x}(Y_j(\bar{t},x;\mathbf{u}_{\gamma_k}) - \bar{y}(x)) \\ &\cdot \frac{x - x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k}) \\ &+ \frac{\partial}{\partial x}(Y_j(\bar{t},x;\mathbf{u}_{\gamma_k}) - \bar{y}(x)) \cdot \frac{x_j(\mathbf{u}_{\gamma_k}) - x}{x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})} \\ &\cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\mathbf{u}_{\gamma_k}) \cdot (\mathbf{u}-\mathbf{u}_{\gamma_k})\Big] \lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\ \le \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\mathbf{u}_{\gamma_k}),\mathbf{u}_{\gamma_k}) \cdot (\mathbf{u}-\mathbf{u}_{\gamma_k}) + \sum_{j=1}^{K+1} r_j(\mathbf{u}_{\gamma_k},\gamma_k) \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}&\left(x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})\right) \cdot (\mathbf{u}-\mathbf{u}_{\gamma_k}).\end{aligned}$$

Due to Theorem 3.5.5, Lemma 5.3.4 and the compactness of $\mathbf{U}_{\mathrm{ad}}$, we obtain that the right-hand side of the previous inequality is uniformly bounded w.r.t. k and $\mathbf{u}$

by a constant $C > 0$:

$$\begin{aligned}\sum_{j=1}^{K+1} \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} -\Big[&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t}, x; \mathbf{u}_{\gamma_k})(\mathbf{u} - \mathbf{u}_{\gamma_k}) \\ &+ \frac{\partial}{\partial x}(Y_j(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x)) \\ &\cdot \frac{x - x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\mathbf{u}_{\gamma_k})(\mathbf{u} - \mathbf{u}_{\gamma_k}) \\ &+ \frac{\partial}{\partial x}(Y_j(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x)) \\ &\cdot \frac{x_j(\mathbf{u}_{\gamma_k}) - x}{x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\mathbf{u}_{\gamma_k}) \cdot (\mathbf{u} - \mathbf{u}_{\gamma_k})\Big] \\ \cdot \lambda_j(x; \mathbf{u}_{\gamma_k}) \, \mathrm{d}x \leq C.\end{aligned} \tag{5.3.11}$$

We define the sets of points where the functions $Y_j(\bar{t}, \cdot, \bar{\mathbf{u}})$ for $j = 1, \ldots, K+1$ exceed the upper bound $\bar{y}$ by

$$\hat{I}_j := \{x \in [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0] : Y_j(\bar{t}, x, \bar{\mathbf{u}}) \geq \bar{y}(x)\}. \tag{5.3.12}$$

We observe that $\hat{I}_j \subset [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0]$ holds for $j = 1, \ldots, K+1$. The constant $\varepsilon_0 > 0$ is here chosen sufficiently small such that the functions $Y_j(\bar{t}, \cdot, \bar{\mathbf{u}})$ from Lemma 3.7.1 are well-defined on the intervals

$$[x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0] \quad , j = 1, \ldots, K+1.$$

In addition, considering a constant $\varepsilon_1 > 0$, we introduce

$$\hat{I}_{j,\varepsilon_1} = \bigcup_{x \in \hat{I}_j} (x - \varepsilon_1, x + \varepsilon_1) \cap [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0], \quad j = 1, \ldots, K+1$$

and note that the sets given by

$$[x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0] \setminus \hat{I}_{j,\varepsilon_1} = [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0] \setminus \bigcup_{x \in \hat{I}_j} (x - \varepsilon_1, x + \varepsilon_1)$$

are closed and bounded and thus compact for all $j = 1, \ldots, K+1$. This compactness, the construction of the sets $\hat{I}_{j,\varepsilon_1}$ and the continuity of $(Y_j(\bar{t},\cdot;\bar{\mathbf{u}}) - \bar{y}(\cdot))$ yields

$$Y_j(\bar{t},x,\bar{\mathbf{u}}) - \bar{y}(x) \leq -\delta_0 < 0 \quad \text{for all } x \in [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_0, x_j(\bar{\mathbf{u}}) + \varepsilon_0] \setminus \hat{I}_{j,\varepsilon_1}$$

with a constant $\delta_0 > 0$. Since the functions $Y_j(\bar{t},x;\mathbf{u})$ and $x_j(\mathbf{u})$ are continuous w.r.t. $\mathbf{u}$ and the sequence $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ converges to $\bar{\mathbf{u}}$ by assumption, we can further conclude

$$Y_j(\bar{t},x;\mathbf{u}_{\gamma_k}) - \bar{y}(x) \leq -\frac{\delta_0}{2} < 0 \;\text{ for all }\; x \in [x_{j-1}(\mathbf{u}_{\gamma_k}), x_j(\mathbf{u}_{\gamma_k})] \setminus \hat{I}_{j,\varepsilon_1}. \tag{5.3.13}$$

Thus, using (5.3.13) and the definition of the Lagrange multiplier estimates in (5.3.4), it turns out that

$$\lambda_j(x,\mathbf{u}_{\gamma_k}) = 0 \quad \text{for all } x \in \mathbb{R} \setminus \hat{I}_{j,\varepsilon_1},\; j = 1,\dots,K+1 \tag{5.3.14}$$

if k is large enough.

We observe that since the Robinson's CQ (4.2.13) is satisfied in $\bar{\mathbf{u}}$, Lemma 4.2.10 yields the existence of some control $\tilde{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$ and a constant $\varepsilon_2 > 0$ such that the estimation

$$\begin{aligned} &Y_j(\bar{t},x,\bar{\mathbf{u}}) - \bar{y}(x) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t},x,\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \\ &+ \frac{\mathrm{d}}{\mathrm{d}x}\left[Y_j(\bar{t},x,\bar{\mathbf{u}}) - \bar{y}(x)\right]\left(\frac{x - x_{j-1}(\bar{\mathbf{u}})}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \right. \\ &\qquad\qquad \left. + \frac{x_j(\bar{\mathbf{u}}) - x}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}})\right) \\ &\leq -\varepsilon_2 \end{aligned} \tag{5.3.15}$$

is satisfied on the interval $[x_{j-1}(\bar{\mathbf{u}}), x_j(\bar{\mathbf{u}})]$ for all $j = 1,\dots,K+1$

Due to the continuity of the left-hand side of (5.3.15) w.r.t. x, there exists a constant ε_3 with $0 < \varepsilon_3 < \varepsilon_0$ such that on the extended interval $x \in [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_3, x_j(\bar{\mathbf{u}}) + \varepsilon_3]$ it holds that

$$\begin{aligned} &Y_j(\bar{t},x,\bar{\mathbf{u}}) - \bar{y}(x) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t},x,\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \\ &+ \frac{\mathrm{d}}{\mathrm{d}x}\left[Y_j(\bar{t},x,\bar{\mathbf{u}}) - \bar{y}(x)\right]\left(\frac{x - x_{j-1}(\bar{\mathbf{u}})}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \right. \\ &\qquad\qquad \left. + \frac{x_j(\bar{\mathbf{u}}) - x}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}})\right) \\ &\leq -\frac{\varepsilon_2}{2}. \end{aligned} \tag{5.3.16}$$

Further on, we note that

$$(Y_j(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}(x)) \mid_{\hat{I}_j \cap [x_{j-1}(\bar{\mathbf{u}})-\varepsilon_3, x_j(\bar{\mathbf{u}})+\varepsilon_3]} \geq 0 \tag{5.3.17}$$

holds due to the construction of the set $\hat{I}_j$ in (5.3.12). Using (5.3.17) and (5.3.16), we can conclude that

$$R_j(x, \bar{\mathbf{u}}) \leq -\frac{\varepsilon_2}{2} \quad \forall x \in \hat{I}_j \cap [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_3, x_j(\bar{\mathbf{u}}) + \varepsilon_3]$$

holds true, where

$$\begin{aligned} R_j(x, \mathbf{u}) :=& \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t}, x; \mathbf{u})(\tilde{\mathbf{u}} - \mathbf{u}) \\ &+ \frac{\mathrm{d}}{\mathrm{d}x} [Y_j(\bar{t}, x; \mathbf{u}) - \bar{y}(x)] \Bigg(\frac{x - x_{j-1}(\mathbf{u})}{x_j(\mathbf{u}) - x_{j-1}(\mathbf{u})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\mathbf{u})(\tilde{\mathbf{u}} - \mathbf{u}) \\ &\qquad + \frac{x_j(\mathbf{u}) - x}{x_j(\mathbf{u}) - x_{j-1}(\mathbf{u})} \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\mathbf{u})(\tilde{\mathbf{u}} - \mathbf{u}) \Bigg). \end{aligned}$$

Since the terms $R_j(\cdot, \bar{\mathbf{u}})$ are continuous w.r.t. x, we obtain

$$R_j(x, \bar{\mathbf{u}}) \leq -\frac{\varepsilon_2}{4} \quad \forall x \in \hat{I}_{j,\varepsilon_1} \cap [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon_3, x_j(\bar{\mathbf{u}}) + \varepsilon_3]$$

for for sufficiently small ε_1.

Moreover, the continuity of the functions $R_j(x, \mathbf{u})$ and $x_j(\mathbf{u})$ w.r.t. the control $\mathbf{u}$ and the fact that the sequence $(\mathbf{u}_{\gamma_k})_{k \in \mathbb{N}}$ converges to $\bar{\mathbf{u}}$ for $k \to \infty$ together yield

$$R_j(x, \mathbf{u}_{\gamma_k}) \leq -\frac{\varepsilon_2}{8} \quad \forall x \in \hat{I}_{j,\varepsilon_1} \cap [x_{j-1}(\mathbf{u}_{\gamma_k}) - \frac{\varepsilon_3}{2}, x_j(\mathbf{u}_{\gamma_k}) + \frac{\varepsilon_3}{2}]. \tag{5.3.18}$$

Using (5.3.14), we obtain for all $j = 1, \ldots, K+1$

$$\begin{aligned} &\int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} -R_j(x; \mathbf{u}_{\gamma_k}) \lambda_j(x; \mathbf{u}_{\gamma_k}) \, \mathrm{d}x \\ &= \int_{[x_{j-1}(\mathbf{u}_{\gamma_k}), x_j(\mathbf{u}_{\gamma_k})] \cap \hat{I}_{j,\varepsilon_1}} -R_j(x; \mathbf{u}_{\gamma_k}) \lambda_j(x; \mathbf{u}_{\gamma_k}) \, \mathrm{d}x. \end{aligned} \tag{5.3.19}$$

holds true. Since the terms $\lambda_j(x; \mathbf{u}_{\gamma_k})$ are nonnegative, we obtain from (5.3.18) and

(5.3.19)

$$\begin{aligned}
&\int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} -R_j(x;\mathbf{u}_{\gamma_k})\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&= \int_{[x_{j-1}(\mathbf{u}_{\gamma_k}),x_j(\mathbf{u}_{\gamma_k})]\cap\hat{I}_{j,\varepsilon_1}} -R_j(x;\mathbf{u}_{\gamma_k})\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&\geq \int_{[x_{j-1}(\mathbf{u}_{\gamma_k}),x_j(\mathbf{u}_{\gamma_k})]\cap\hat{I}_{j,\varepsilon_1}} \frac{\varepsilon_2}{8}\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x.
\end{aligned} \tag{5.3.20}$$

Using (5.3.14) again, we can conclude that

$$\begin{aligned}
&\int_{[x_{j-1}(\mathbf{u}_{\gamma_k}),x_j(\mathbf{u}_{\gamma_k})]\cap\hat{I}_{j,\varepsilon_1}} \frac{\varepsilon_2}{8}\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&= \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} \frac{\varepsilon_2}{8}\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&= \int_{[a,b]} \frac{\varepsilon_2}{8}\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x.
\end{aligned} \tag{5.3.21}$$

From (5.3.20) and (5.3.21) we obtain that there exists a constant $\bar{k} \in \mathbb{N}$ such that

$$\begin{aligned}
&\int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} -R_j(x;\mathbf{u}_{\gamma_k})\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&\geq \int_{[a,b]} \frac{\varepsilon_2}{8}\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&\geq 0 \quad \forall k \geq \bar{k}
\end{aligned} \tag{5.3.22}$$

is valid for all $j = 1, \ldots, K+1$. Note that the second inequality in (5.3.22) holds due to $\lambda_j(x;\mathbf{u}_{\gamma_k}) \geq 0$.

Finally, we note that the left-hand side of (5.3.22) is equal to the jth summand of the left side of (5.3.11). Therefore, using (5.3.11) and (5.3.22) results in

$$\sum_{j=1}^{K+1} \int_{[a,b]} |\lambda_j(x;\mathbf{u}_{\gamma_k})|\,\mathrm{d}x \leq \frac{8C}{\varepsilon_2} := \tilde{C} \quad \forall k \geq \bar{k}$$

and the sequences $(\lambda_j(x;\mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}}$ are uniformly bounded in $L^1([a,b])$. □

In the last theorem of this chapter, we show that if a sequence of local solutions to the regularized problems (P_γ) converges to a local solution to (P) such that (ND) and Robinson's CQ hold, then the corresponding Lagrange multiplier estimates converge to measures for which the optimality system in (P) is satisfied. A similar result for the elliptic case can be found in [43, Thm. 3.1].

Theorem 5.3.7. *Suppose that* (A3) *and* (A4) *are satisfied and let* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ *denote a sequence of local solutions to* (P_{γ_k}) *with* $\gamma_k \to \infty$ *for* $k \to \infty$ *satisfying* (5.3.7) *for sufficiently large* $k \in \mathbb{N}$ *and converging to a local solution* $\bar{\mathbf{u}} \in \mathbf{U}_{ad}$ *of* (P). *In addition, assume that* $\bar{\mathbf{u}}$ *satisfies* (ND) *and Robinson's CQ* (4.2.13). *Then there exists a subsequence* $(\mathbf{u}_{\gamma_k})_{k\in\mathbb{N}}$ *without change of notation such that the corresponding sequence of Lagrange multiplier estimates satisfies*

$$\lambda_j(\cdot;\mathbf{u}_{\gamma_k}) \overset{*}{\rightharpoonup} \mu_j \quad \text{in } \mathcal{M}([a,b]) \quad \forall j=1,\dots,K+1.$$

The measures $\mu_j \in \mathcal{M}([a,b])$ *are nonnegative and the optimality conditions in Theorem 4.2.9 are satisfied in* $\bar{\mathbf{u}}$ *for* $\bar{\mu}_j = \mu_j|_{I_j}$ *for all* $j = 1,\dots,K+1$.

The main idea of this proof is standard and can be found for example in the proof of Theorem 3.1 in [43]. We will use the boundedness of the Lagrange multiplier estimates in L^1 and then use Theorem 2.2.6 yielding a weakly-* convergent subsequences with measures as limits. In a second step, we will prove that for these measures the conditions in (4.2.13) hold true.

Proof. At first, we obtain from Lemma 5.3.6 that the sequences of Lagrange multiplier estimates $(\lambda_j(\cdot;\mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}}$ are uniformly bounded in $L^1([a,b])$. Therefore, there exist subsequences $(\lambda_j(\cdot;\mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}}$ that satisfy

$$\lambda_j(\cdot;\mathbf{u}_{\gamma_k}) \overset{*}{\rightharpoonup} \mu_j \quad \text{in } \mathcal{M}([a,b]) \quad \forall j=1,\dots,K+1.$$

Since $(\lambda_j(\cdot;\mathbf{u}_{\gamma_k})) \geq 0$ holds for all $k \in \mathbb{N}$, we obtain that the measures $\mu_j \in \mathcal{M}([a,b])$ are nonnegative.

It remains to prove that the optimality conditions in Theorem 4.2.9, i.e. (4.2.24a), (4.2.24b) and (4.2.24c) are satisfied for the choice $\bar{\mu}_j = \mu_j|_{I_j}$ for $j = 1,\dots,K+1$.

First, we note that (4.2.24a) is trivially satisfied since $\bar{\mathbf{u}}$ is a local solution to (P). Next, we show that (4.2.24b) is also satisfied. To this end, we note that

$$0 \leq J_\gamma(y(\mathbf{u}_{\gamma_k}),\mathbf{u}_{\gamma_k}) \leq J_\gamma(y(\bar{\mathbf{u}}),\bar{\mathbf{u}}) = J(y(\bar{\mathbf{u}}),\bar{\mathbf{u}}) \tag{5.3.23}$$

holds due to (5.3.7) for sufficiently large $k \in \mathbb{N}$. Using the definition of $J_{\gamma_k}(y(\mathbf{u}),\mathbf{u})$ in (P_γ) and $\lambda_j(\cdot;\mathbf{u})$ in (5.3.4), (5.3.23) is equivalent to

$$0 \leq \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (Y_j(\bar{t},x;\mathbf{u}_{\gamma_k}) - \bar{y}(x))\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x$$

$$\leq 2 \cdot (J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - J(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k})).$$

Since $\mathbf{u}_{\gamma_k}$ converges to $\bar{\mathbf{u}}$, the continuity of the terms above w.r.t. $\mathbf{u}$ yields

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (Y_j(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}(x))\lambda_j(x; \mathbf{u}_{\gamma_k})\,\mathrm{d}x \\ &\leq 2 \cdot \lim_{k\to\infty} (J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - J(y(\mathbf{u}_{\gamma_k}), \mathbf{u}_{\gamma_k})) = 0. \end{aligned}$$

From this result we further obtain

$$\begin{aligned} 0 &= \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (Y_j\,(\bar{t}, x; \mathbf{u}_{\gamma_k}) - \bar{y}\,(x)) \cdot \lambda_j(x; \mathbf{u}_{\gamma_k})\,\mathrm{d}x \\ &= \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (Y_j\,(\bar{t}, x, \bar{\mathbf{u}}) - \bar{y}\,(x)) \cdot \lambda_j(x; \mathbf{u}_{\gamma_k})\,\mathrm{d}x \\ &\quad + \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (Y_j\,(\bar{t}, x; \mathbf{u}_{\gamma_k}) - Y_j\,(\bar{t}, x, \bar{\mathbf{u}})) \cdot \lambda_j(x; \mathbf{u}_{\gamma_k})\,\mathrm{d}x. \end{aligned} \tag{5.3.24}$$

Since the sequences $(\lambda_j(\cdot; \mathbf{u}_{\gamma_k})))_{k\in\mathbb{N}}$ are uniformly bounded in $L^1([a, b])$ and the mappings

$$B_\rho^{\mathbf{U}}(\bar{\mathbf{u}}) \ni w \mapsto Y_j\,(\bar{t}, \cdot; \mathbf{u}) \in C([x_{j-1}(\bar{\mathbf{u}}), x_j(\bar{\mathbf{u}})])$$

are continuous for sufficiently small $\rho > 0$ and $\mathbf{u}_{\gamma_k} \to \bar{\mathbf{u}}$, we get

$$\lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (Y_j\,(\bar{t}, x; \mathbf{u}_{\gamma_k}) - Y_j\,(\bar{t}, x, \bar{\mathbf{u}})) \cdot \lambda_j(x; \mathbf{u}_{\gamma_k})\,\mathrm{d}x = 0. \tag{5.3.25}$$

If we insert (5.3.25) in (5.3.24) and use that

$$\lambda_j(\cdot; \mathbf{u}_{\gamma_k}) \to \mu_j \quad \text{in } \mathcal{M}([a, b])\ \forall j = 1, \ldots, K+1,$$

we can conclude

$$\begin{aligned} 0 &= \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (\bar{y}\,(x) - Y_j\,(\bar{t}, x, \bar{\mathbf{u}})) \cdot \lambda_j(x; \mathbf{u}_{\gamma_k})\,\mathrm{d}x \\ &= \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} (\bar{y}\,(x) - Y_j\,(\bar{t}, x, \bar{\mathbf{u}}))\,\mathrm{d}\mu_j(x) \end{aligned} \tag{5.3.26}$$

and thus (4.2.24b) is proven.

It remains to show that (4.2.24c) also holds true. To this end, we first introduce the following abbreviations:

$$\bar{I}_1^j(x;\mathbf{u}) := \frac{\partial}{\partial x}\left(\bar{y}(x) - Y_j(\bar{t},x;\mathbf{u})\right)\cdot\frac{x - x_{j-1}(\mathbf{u})}{x_j(\mathbf{u}) - x_{j-1}(\mathbf{u})}$$
$$\bar{I}_2^j(x;\mathbf{u}) := \frac{\partial}{\partial x}\left(\bar{y}(x) - Y_j(\bar{t},x;\mathbf{u})\right)\cdot\frac{x_j(\mathbf{u}) - x}{x_j(\mathbf{u}) - x_{j-1}(\mathbf{u})}$$

We note that since the optimal solutions $\mathbf{u}_{\gamma_k}$ to (P_{γ_k}) satisfy (ND) if k is sufficiently large, Lemmas 5.3.1, 5.3.3 and 5.3.4 together yield

$$\begin{aligned}\lim_{k\to\infty}\Bigg[&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}J(y(\mathbf{u}_{\gamma_k}),\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\\
&+\sum_{j=1}^{K+1}\Bigg(\int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})}\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_j(\bar{t},x;\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x\\
&+\int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})}\bar{I}_1^j(\mathbf{u}_{\gamma_k},x)\cdot\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_j(\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\\
&+\int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})}\bar{I}_2^j(\mathbf{u}_{\gamma_k},x)\cdot\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{j-1}(\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\Bigg)\Bigg]\geq 0\end{aligned}$$

for all $\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}$. As a next step, we want to replace the integration limits $x_j(\mathbf{u}_{\gamma_k})$ by $x_j(\bar{\mathbf{u}})$ for all $j = 1,\ldots,K+1$. To this end, we use the variable transformation

$$x = x_{j-1}(\mathbf{u}_{\gamma_k}) + \frac{\tilde{x} - x_{j-1}(\bar{\mathbf{u}})}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})}\big(x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})\big) \tag{5.3.27}$$

and deduce that

$$
\begin{aligned}
\lim_{k\to\infty}\Bigg[&\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}J(y(\mathbf{u}_{\gamma_k}),\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})+\\
&\sum_{j=1}^{K+1}\Bigg(\int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})}\tilde{Y}_j(\bar{t},\tilde{x};\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\frac{x_j(\mathbf{u}_{\gamma_k})-x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\bar{\mathbf{u}})-x_{j-1}(\bar{\mathbf{u}})}\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x}\\
&\quad+\int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})}\tilde{I}_1^j(\tilde{x};\mathbf{u}_{\gamma_k})\cdot\frac{x_j(\mathbf{u}_{\gamma_k})-x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\bar{\mathbf{u}})-x_{j-1}(\bar{\mathbf{u}})}\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x}\\
&\qquad\qquad\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_j(\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\\
&\quad+\int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})}\tilde{I}_2^j(\tilde{x};\mathbf{u}_{\gamma_k})\cdot\frac{x_j(\mathbf{u}_{\gamma_k})-x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\bar{\mathbf{u}})-x_{j-1}(\bar{\mathbf{u}})}\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x}\\
&\qquad\qquad\cdot\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}x_{j-1}(\mathbf{u}_{\gamma_k})(\mathbf{u}-\mathbf{u}_{\gamma_k})\Bigg)\Bigg]\geq 0.
\end{aligned}
\tag{5.3.28}
$$

Hereby, we use the abbreviations

$$
\begin{aligned}
&\vdots = x_{j-1}(\mathbf{u}_{\gamma_k})+\frac{\tilde{x}-x_{j-1}(\bar{\mathbf{u}})}{x_j(\bar{\mathbf{u}})-x_{j-1}(\bar{\mathbf{u}})}\big(x_j(\mathbf{u}_{\gamma_k})-x_{j-1}(\mathbf{u}_{\gamma_k})\big),\\
&\tilde{Y}_j(\bar{t},\tilde{x};\mathbf{u}_{\gamma_k}) := \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_j(\bar{t},\vdots;\mathbf{u}_{\gamma_k}),\ \tilde{I}_i^j(\tilde{x};\mathbf{u}_{\gamma_k}) := \bar{I}_i^j(\vdots;\mathbf{u}_{\gamma_k}),\\
&\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k}) := \lambda_j(\vdots;\mathbf{u}_{\gamma_k})\quad j=1,\dots,K+1,\quad i=1,2.
\end{aligned}
\tag{5.3.29}
$$

Due to the boundedness of the sequences $(\lambda_j(x;\mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}}$ in $L^1([a,b])$ by Lemma 5.3.6, the sequences $\left(\tilde{\lambda}_j(\cdot;\mathbf{u}_{\gamma_k})\right)_{k\in\mathbb{N}}$ are also bounded in $L^1([a,b])$.

Therefore, using Theorem 2.2.6, we obtain a further subsequence $\left(\tilde{\lambda}_j(\cdot;\mathbf{u}_{\gamma_k})\right)_{k\in\mathbb{N}}$ satisfying

$$
\tilde{\lambda}_j(\cdot;\mathbf{u}_{\gamma_k}) \overset{*}{\rightharpoonup} \tilde{\mu}_j \text{ in } \mathcal{M}([a,b]) \quad \text{for all } j=1,\dots,K+1, \tag{5.3.30}
$$

where $\tilde{\mu}_j\in\mathcal{M}([a,b])$ are nonnegative measures. Lemma 5.3.1 further yields

$$
\tilde{Y}_j(\bar{t},\cdot;\mathbf{u}_{\gamma_k})\to\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}}Y_j(\bar{t},\cdot;\bar{\mathbf{u}}),\ \tilde{I}_i^j(\cdot;\mathbf{u}_{\gamma_k})\to\bar{I}_i^j(\cdot;\bar{\mathbf{u}}) \text{ in } C([x_{j-1}(\bar{\mathbf{u}}),x_j(\bar{\mathbf{u}})]) \tag{5.3.31}
$$

for $k\to\infty$. Using (5.3.30), (5.3.31) and $\frac{x_j(\mathbf{u}_{\gamma_k})-x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\bar{\mathbf{u}})-x_{j-1}(\bar{\mathbf{u}})}\to 1$, we can rewrite

(5.3.28) by

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} J(y(\bar{\mathbf{u}}), \bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) + \sum_{j=1}^{K+1} \Bigg[\int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} Y_j(\bar{t}, x, \bar{\mathbf{u}})(w-\bar{\mathbf{u}}) \,\mathrm{d}\tilde{\mu}_j(x) \\ + \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} \bar{I}_1^j(\bar{\mathbf{u}}, x) \,\mathrm{d}\tilde{\mu}_j(x) \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_j(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) \\ + \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} \bar{I}_2^j(\bar{\mathbf{u}}, x) \,\mathrm{d}\tilde{\mu}_j(x) \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{u}} x_{j-1}(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}}) \Bigg] \geq 0.$$

Thus, (4.2.24c) is satisfied for the choice

$$\bar{\mu}_j = \tilde{\mu}_j|_{I_j} \quad \text{for all } j = 1, \ldots, K+1.$$

It remains to show that (4.2.24c) also holds for the choice $\bar{\mu}_j = \mu_j|_{I_j}$. To this end, we prove

$$\tilde{\mu}_j = \mu_j \quad \text{in } \mathcal{M}(I_j) \quad \text{for all } j = 1, \ldots, K+1. \tag{5.3.32}$$

Due to the continuity of the functions $x_j(\mathbf{u})$ with $j = 1, \ldots, K+1$ and the fact that $\mathbf{u}_{\gamma_k} \to \bar{\mathbf{u}}$, choosing a small constant $\varepsilon > 0$ yields for sufficiently large k

$$x_{j-1}(\bar{\mathbf{u}}) - \varepsilon \leq x_{j-1}(\mathbf{u}_{\gamma_k}) < x_j(\mathbf{u}_{\gamma_k}) \leq x_j(\bar{\mathbf{u}}) + \varepsilon \quad \text{for all } j = 1, \ldots, K+1$$

which is equivalent to

$$(x_{j-1}(\mathbf{u}_{\gamma_k}), x_j(\mathbf{u}_{\gamma_k})) \subset J_{j,\varepsilon} \quad \text{for all } j = 1, \ldots, K+1 \tag{5.3.33}$$

where $J_{j,\varepsilon} = [x_{j-1}(\bar{\mathbf{u}}) - \varepsilon, x_j(\bar{\mathbf{u}}) + \varepsilon]$.

We note that the weak-* convergence of $(\lambda_j(\cdot; \mathbf{u}_{\gamma_k}))_{k\in\mathbb{N}}$ to $\mu_j \in \mathcal{M}([a,b])$ implies that for all $j = 1, \ldots, K+1$ and for arbitrary $\varphi_j \in C(J_{j,\varepsilon})$ it holds that

$$\int_{J_{j,\varepsilon}} \varphi_j(x) \,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \varphi_j(x) \lambda_j(x; \mathbf{u}_{\gamma_k}) \,\mathrm{d}x.$$

Next we observe that (5.3.33) and (5.3.4) yield

$$\begin{aligned}
&\int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) \\
&= \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \varphi_j(x)\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x \\
&= \lim_{k\to\infty} \int_{x_{j-1}(\mathbf{u}_{\gamma_k})}^{x_j(\mathbf{u}_{\gamma_k})} \varphi_j(x)\lambda_j(x;\mathbf{u}_{\gamma_k})\,\mathrm{d}x.
\end{aligned} \tag{5.3.34}$$

Next, we apply the variable transformation in (5.3.27) to the right-hand side of (5.3.34) and obtain

$$\int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x},$$

where

$$\begin{aligned}
\tilde{\varphi}_j(\tilde{x}) :=& \varphi_j\left(x_{j-1}(\mathbf{u}_{\gamma_k}) + \frac{\tilde{x} - x_{j-1}(\bar{\mathbf{u}})}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})} \big(x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k}) \big) \right) \\
&\cdot \frac{x_j(\mathbf{u}_{\gamma_k}) - x_{j-1}(\mathbf{u}_{\gamma_k})}{x_j(\bar{\mathbf{u}}) - x_{j-1}(\bar{\mathbf{u}})}
\end{aligned}$$

and $\tilde{\lambda}$ is defined according to (5.3.29). Using this definition, we further obtain

$$\begin{aligned}
&\int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{x_{j-1}(\bar{\mathbf{u}})}^{x_j(\bar{\mathbf{u}})} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x} \\
&= \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x}.
\end{aligned} \tag{5.3.35}$$

Since $\varphi_j \in C(J_{j,\varepsilon})$ and $x_j(\mathbf{u}_{\gamma_k}) \to x_j(\bar{\mathbf{u}})$ for $k \to \infty$, we can conclude that

$$\tilde{\varphi}_j(\cdot) \to \varphi_j(\cdot) \quad \text{in } C(J_{j,\varepsilon}) \quad \forall k = 1, \ldots, K+1. \tag{5.3.36}$$

Using the uniform boundedness of $\left(\tilde{\lambda}_j(\cdot;\mathbf{u}_{\gamma_k})\right)_{k\in\mathbb{N}}$ in $L^1([a,b])$, (5.3.30) and (5.3.36), we obtain from (5.3.35) for all $j = 1, \ldots, K+1$ that

$$\begin{aligned}
&\int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x};\mathbf{u}_{\gamma_k})\,\mathrm{d}\tilde{x} \\
&= \int_{J_{j,\varepsilon}} \varphi_j(\tilde{x})\tilde{\mu}_j(\tilde{x})\,\mathrm{d}\tilde{x}
\end{aligned}$$

holds true. Since the test functions $\varphi_j \in C(J_{j,\varepsilon})$ are arbitrarily chosen, we can conclude that (5.3.32) holds and therefore (4.2.24c) is also satisfied if we choose $\bar{\mu}_j = \mu_j|_{I_j}$ for $j = 1, \ldots, K+1$. □

CHAPTER 6

Conclusion and outlook

We have analyzed optimal control problems governed by scalar balance laws and pointwise state constraints, where the partial differential equation is considered on bounded domains and subject to initial as well as boundary conditions. For the optimal control problem we have shown not only the existence of a global optimal control, but moreover, we have derived necessary optimality conditions supposing that the optimal control is non-degenerated and satisfies Robinson's CQ. We could further show that Robinson's CQ always holds under suitable assumptions on the source term and the set of admissible controls. More precisely, we have applied the results on the structure of solutions to hyperbolic balance laws in [69, 81] yielding the continuous Fréchet-differentiability of the reduced cost functional via the concept of shift-differentiability. Following the ideas of [69, 71], we could reduce the restrictions of the setting in [69] to the case that rarefaction centers can be shifted. On the other hand, these results on the structure played a key-role in deriving the optimality conditions. Particularly, evaluated at almost all time points, the entropy solution $y(\bar{t}, \cdot, \mathbf{u})$ is piecewise smooth, where the smooth parts $Y_1, \ldots, Y_{K+1}$ are separated by points $x_1, \ldots, x_K$ consisting of discontinuities and points lying on boundaries of rarefaction waves. The analysis of the structure of entropy solutions in [69, 81] shows that $Y_1, \ldots, Y_{K+1}$ and $x_1, \ldots, x_K$ are functions depending continuously Fréchet-differentiably on the control. We introduced these functions as auxiliary state variables which we have used to derive optimality conditions. In the end, we succeeded to reformulate the optimality system in terms of the original state.

In order to approximate the state-constrained problem, we have discussed the Moreau-Yosida regularization approach and carried out a convergence analysis. For a sequence of optimal solutions to the regularized problems where the penalty parameter goes to zero, we have proven strong convergence to an optimal solution of the original problem with pointwise state constraints. Finally, we could prove convergence of the optimality conditions for the regularized problems to the optimality system of the state-constrained optimal control problem. More precisely, supposing a sequence of solutions to the regularized problems converging to a local solution $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$ to the state-constrained problem, we have constructed Lagrange multipliers estimates which we have proven to be bounded in L^1. Adopting the Banach-Alaoglu Theorem, we have proven that a subsequence converges weakly-* to nonnegative measures $\mu_1, \ldots, \mu_{K+1}$. Choosing these measures as Lagrange multipliers, we have finally shown that $(\bar{\mathbf{u}}, \mu_1, \ldots, \mu_{K+1})$ is a solution to the optimality system for the problem with state constraints. The basic structure of the proofs bases on the auxiliary state variables and on standard techniques which are used in the convergence analysis of the Moreau-Yosida type regularization for state-constrained optimal control problems with elliptic or parabolic equations as constraints, see, e.g., [43, 62, 63, 64].

One possible extension of the results in this thesis is to apply the developed method to networks with node conditions, for example traffic light problems, where the traffic flow is modeled by the LWR-model as proposed in [56, 74]. Building up on the sensitivity and adjoint calculus introduced in [71], one can derive optimality conditions for state-constrained optimal control problems on networks with node conditions.

In this thesis we have restricted ourselves to scalar hyperbolic balance laws. But since several physical models involve systems of hyperbolic balance laws, e.g., the compressible Euler equations which describe the gas flow in a pipe, it is desirable to extend the developed methods to consider systems of balance laws. The analysis of the scalar case strongly relies on the structural properties of entropy solutions yielding the auxiliary state variables $Y_1, \ldots, Y_{K+1}$ and $x_1, \ldots, x_K$. One can show that the concept of shift-differentiability developed in [81] can be generalized to systems, but unfortunately Theorem 3.3 in [31] which was crucial for the derivation of the structural properties for the scalar case in [81] does not hold in the case of systems. Another possibility is to adopt the results developed in [17] where the authors consider piecewise Lipschitz continuous solutions to systems of hyperbolic balance laws and only allow variations of the control preserving this structure. Although this seems very restrictive, the concepts developed [17] may be applied to generalized Riemann problems, since Li and Yu have proved in [55] that solutions to this class of problems are for a sufficiently small time interval piecewise smooth and the structure is stable under small variations of the initial data. Developing a sensitivity and adjoint calculus for generalized Riemann problems would be the first step to treat more complex problems with systems of hyperbolic balance laws like the optimal control of the gas flow in a network.

List of symbols

$\|x\|_1$	$\|x\|_1 \coloneqq \sum_{k=1}^{d} \|x_k\|$ for all $x \in \mathbb{R}^d$.
$\|x\|_2$	$\|x\|_2 \coloneqq \sqrt{\sum_{k=1}^{d} x_k^2}$ for all $x \in \mathbb{R}^d$.
$B_\varepsilon^X(f)$	$B_\varepsilon^X(f) = \{g \in X : \|g - f\|_X \leq \varepsilon\}$, where X is a Banach space and $\varepsilon > 0$.
$x \cdot y$	$x \cdot y \coloneqq \sum_{k=1}^{d} x_k y_k$ for all $x, y \in \mathbb{R}^d$.
$\operatorname{int} \Omega$	interior of Ω.
Ω^{cl}	closure of Ω.
$\mathbb{1}_\Omega$	indicator function for the set Ω.
$\varphi\|_A$	restriction of the function φ to the set A.
$\operatorname{supp} \varphi$	support of a function φ.
$[\varphi(x)]$	$[\varphi(x)] = \varphi(x-) - \varphi(x+)$.
$I(\alpha, \beta)$	$I(\alpha, \beta) \coloneqq [\min(\alpha, \beta), \max(\alpha, \beta)]$.
$(\cdot)_+$	$(\cdot)_+ \coloneqq \max(\cdot, 0)$.
$L^p(\Omega)$	Lebesgue space with $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^d$ measurable.
$\|\cdot\|_{L^p(\Omega)}$	L^p-Norm with $\|f\|_{p,\Omega} \coloneqq \left(\int_\Omega \|f(x)\|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$ for $p \in [1, \infty[$ and $\|f\|_{\infty,\Omega} \coloneqq \operatorname{ess\,sup}_{x \in \Omega} \|f(x)\|$ for $p = \infty$.

$L^p_{\text{loc}}(\Omega)$	set of all measurable $f : \Omega \to \mathbb{R}$ with $f\vert_K \in L^p(\Omega)$ for all compact $K \subset \Omega$, where $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^d$ measurable.
$L^p(\Omega; X)$	Lebesgue-Bochner space, where X is a Banach space, $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^d$ measurable.
$C(\Omega)$	space of continuous functions, where $\Omega \subset \mathbb{R}^d$.
$\Vert f\Vert_{C(\Omega)}$	sup norm, $\Vert f\Vert_{C(\Omega^{\text{cl}})} \coloneqq \sup_{x \in \Omega^{\text{cl}}} \vert f(x)\vert$.
$C^k(\Omega)$	space of k-times continously differentiable functions, where $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0$.
$\Vert f\Vert_{C^k(\Omega)}$	$\Vert f\Vert_{C^k(\Omega)} \coloneqq \sum_{\vert\beta\vert \le k} \left\Vert D^\beta f \right\Vert_{C(\Omega)}$
$C^k(\Omega^{\text{cl}})$	space of k-times continously differentiable functions such that all derivatives admit continous extensions to Ω^{cl}, where $\Omega \subset \mathbb{R}^d$ is open and bounded.
$C^{k,\alpha}(\Omega^{\text{cl}})$	denotes the usual Hölder space with $k \in \mathbb{N}_0$, $\alpha \in]0, 1]$ and $\Omega \subset \mathbb{R}^d$ open.
$C^k_c(\Omega)$	$C^k_c(\Omega) \coloneqq \left\{ f \in C^k(\Omega) : f \text{ has compact support in } \Omega \right\}$
$C^\infty(\Omega)$	$C^\infty(\Omega) \coloneqq \bigcap_{k \in \mathbb{N}} C^k(\Omega)$
$C^\infty_c(\Omega)$	$C^\infty_c(\Omega) \coloneqq \{ f \in C^\infty(\Omega) : f \text{ has compact support in } \Omega \}$
$C^k(\Omega; X)$	analogously defined as $C^k(\Omega)$, where X is a Banach space.
$PC^k(I; x_1, \dots, x_N)$	space of piecewise k-times continuously differentiable functions f with possible discontinuities at points $a < x_1 < \cdots < x_N < b$ for a closed interval $I \supset [a, b]$: $f\vert_{I_k} \in C^k(I_k)$ for $k = 1, \dots, N + 1$ with $I_k = [x_{i-1}, x_i]$ if $k \in \{2, \dots N\}$ and $I_1 = I \cap \{x \le x_1\}$, $I_{N+1} = I \cap \{x \ge x_N\}$.
$W^{k,p}(\Omega)$	denotes for $\Omega \subset \mathbb{R}^d$ open, $p \in]0, \infty]$ and $k \ge 0$ the usual (fractional) Sobolev space.
$\mathcal{D}(\Omega)$	denotes the space of test functions for distributions, where $\Omega \subset \mathbb{R}^d$ is open.

$\mathcal{D}'(\Omega)$	denotes the space of distributions, where $\Omega \subset \mathbb{R}^d$ is open.
$\mathcal{M}(\Omega)$	denotes the space of signed Radon measures on a compact set $\Omega \subset \mathbb{R}^d$.
$BV(\Omega)$	denotes the space of functions with bounded variation on an open set $\Omega \subset \mathbb{R}^d$.
$\|\cdot\|_{BV,\Omega}$	$\|\cdot\|_{BV,\Omega} = \|\cdot\|_{1,\Omega} + \|\cdot\|_{TV,\Omega}$
$(\cdot,\cdot)_H$	denotes the inner product of a Hilbert space H where we set $(\cdot,\cdot)_{2,\Omega}$ if $H = L^2(\Omega)$.

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