Generalized Derivatives for Solution Operators of Variational Inequalities of Obstacle Type

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To my family

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Zusammenfassung

Das Thema dieser Arbeit ist die Bestimmung und die Charakterisierung verallgemeinerter Ableitungen für Lösungsoperatoren von Hindernisproblemen.

Das klassische Hindernisproblem beschreibt die Gleichgewichtsposition einer elastischen Membran unter Krafteinwirkung, wobei die Membran am Rand eines Gebietes eingespannt ist und ober- oder unterhalb eines gegebenen undurchdringbaren Hindernisses bleiben muss. Verwandte Probleme lassen sich beispielsweise auch in der Physik, der Biologie oder dem Finanzwesen finden.

Mathematisch lassen sich Hindernisprobleme durch Variationsungleichungen beschreiben, bei denen die zulässige Menge durch ein oder mehrere Hindernisse begrenzt ist. Es ist bekannt, dass die betrachteten Variationsungleichungen für verschiedene Eingabewerte eindeutige Lösungen besitzen und der zugehörige Lösungsoperator Lipschitz-stetig ist. Die Hindernisbedingung führt jedoch dazu, dass die jeweiligen Lösungsoperatoren im Allgemeinen nicht differenzierbar sind. Eine Verallgemeinerung des Satzes von Rademacher auf genügend reguläre unendlichdimensionale Räume besagt nun, dass die betrachteten Lösungsoperatoren auf einer dichten Teilmenge Gâteaux-differenzierbar sind. Damit lassen sich sogenannte verallgemeinerte Differentiale in jedem Punkt definieren. Die verallgemeinerten Ableitungen in einem festen Punkt des Urbildraums sind definiert als Grenzwerte von Gâteaux-Ableitungen in approximierenden Punkten des Urbildraums. Hierbei können im Unendlichdimensionalen sowohl im Urbildraum als auch im Raum der stetigen linearen Operatoren unterschiedliche Topologien betrachtet werden.

Durch Kenntnis von verallgemeinerten Ableitungen können einerseits nichtglatte Optimierungsmethoden, zum Beispiel Bundle-Methoden, zur Lösung

von Optimalsteuerproblemen bezüglich der Hindernisprobleme angewandt werden und andererseits Kandidaten für Optimallösungen charakterisiert werden. Die Struktur der verallgemeinerten Differentiale ist aber auch aus theoretischer Sicht interessant.

Ausganspunkt für die Analyse in dieser Arbeit ist die Beschreibung der Richtungsableitung als Lösung einer Variationsungleichung, wie sie von Mignot [Mig76] gefunden wurde. Hieraus gewinnen wir eine Charakterisierung der Gâteaux-Ableitungen als Lösungsoperatoren von Variationsgleichungen auf quasi-offenen Mengen, die durch die Kontaktmenge zwischen Hindernis und Lösung festgelegt ist und so von dem betrachteten Punkt des Urbildraums abhängt.

Das Grenzverhalten dieser Operatoren wird nun für verschiedene Lösungsoperatoren von Hindernisproblemen untersucht. Unter Ausnutzung von Monotonieeigenschaften bestimmen wir für beliebige Punkte im Urbildraum zwei verallgemeinerte Ableitungen für ein allgemein formuliertes Hindernisproblem, bei dem die Eingabedaten mittels eines möglicherweise nichtlinearen Operators in die Variationsungleichung eingehen. Hierfür wird für geeignete konvergente Folgen im Urbildraum die Mosco-Konvergenz der zulässigen Mengen für die Gâteaux-Ableitungen, also die Mosco-Konvergenz von Sobolevräumen auf quasi-offenen Mengen, gezeigt und der Grenzwert charakterisiert. Betrachten wir das Hindernisproblem für den Laplace-Operator auf dem Gesamtraum, so können sogar alle Elemente der verallgemeinerten Differentiale bestimmt werden. Im Falle der schwachen Operatortopologie als Konvergenzbegriff auf dem Raum der Operatoren tauchen hier Lösungsoperatoren zu relaxierten Dirichletproblemen auf, die nicht mehr durch eine quasi-offene Menge bestimmt sind sondern durch ein kapazitäres Maß. Zudem betrachten wir auch ein allgemein formuliertes Hindernisproblem mit zwei Hindernissen und finden in diesem Fall verallgemeinerte Ableitungen. Schließlich untersuchen wir, wie numerisch ein Clarke-Subgradient im Finite-Elemente-Raum für ein Optimalsteuerproblem bezüglich des Hindernisproblems berechnet werden kann. Wir leiten einen Fehlerschätzer her, der beispielsweise für die Implementierung von inexakten Bundle-Verfahren verwendet werden kann.

Contents

Chapter 1

Introduction

Variational inequalities of obstacle type are the central objects that are considered within this thesis. The challenge inherent in these problems is the intrinsic nonsmoothness of the corresponding solution operators that is caused by the obstacle constraint.

As a prototype, we consider the variational inequality

$$\text{Find } y \in K_\psi : \quad \langle Ly - \zeta, z - y \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0 \quad \forall\, z \in K_\psi \tag{OP}$$

and the respective admissible set

$$K_\psi := \{z \in H_0^1(\Omega) : z \geq \psi \text{ a.e. in } \Omega\}.$$

Here, $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is a bounded, linear, coercive operator and $\psi \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is the obstacle function, which is chosen such that $K_\psi \neq \emptyset$ is guaranteed. We call the operator S mapping $\zeta \in H^{-1}(\Omega)$ to the unique solution of (OP) the solution operator of the above problem.

Without the obstacle constraint, e.g., when considering $\psi = -\infty$, and assuming that L is a differential operator, S is the bounded and linear solution operator to a partial differential equation. In particular, the corresponding operator is Gâteaux differentiable.

While being related to the unconstrained problem, the solution operator of (OP) with nontrivial obstacle does not share this property. One can show that it is a nonsmooth, but Lipschitz continuous operator between infinite dimensional spaces. A generalization of Rademacher's theorem to such spaces states that S is Gâteaux differentiable on a dense subset of $H^{-1}(\Omega)$. This motivates to consider generalized differentials consisting of limits of Gâteaux derivatives at converging points in the domain. Considering the weak operator topology for the convergence of the Gâteaux derivatives, one can a priori show that the obtained differential is nonempty. Now, on the one hand, the structure of the differentials is interesting from an analytic point of view. On the other hand, the availability of generalized derivatives allows to apply nonsmooth optimization methods, such as Bundle methods, to the optimal control of obstacle problems. The characterization of these generalized derivatives for solution operators of obstacle problems and the analysis of the obtained objects is the aim of this thesis.

In the literature, the optimal control of obstacle problems and of more general elliptic variational inequalities is considered by various authors, e.g., [Bar84, Ber97, BL04, Fri88, IK00, HW18, HK11, KKT03, KW12, MRW15, Mig76, MP84, SW13]. To overcome the difficulties caused by the nonsmooth operator, penalization, relaxation and regularization techniques are employed that approximate the problem by more regular formulations. Based on such approaches, optimality conditions are derived in [Bar84, Ber97, BL04, MP84, IK00, HK11] and numerical solution methods are developed in [IK00, HK11, KKT03, KW12, MRW15, SW13].

In addition, some authors also deal with nonsmooth operators in infinite dimensions. Different optimality systems for the optimal control of the obstacle problem are compared in [HW18]. In [CCMW18], generalized derivatives for the nonsmooth solution operator of a semilinear elliptic equation are characterized. For the usage of generalized derivatives, inexact bundle methods in Hilbert space can be employed, see, e.g., [HU19].

On the other hand, considering a finite dimensional version of the obstacle problem, a characterization of the entire Clarke subdifferential of the reduced objective function is obtained in [HR86].

In the classical reference [Mig76], Mignot establishes the directional differentiability of the solution operator and gives a characterization of the directional derivative as the solution of a another variational inequality. The analysis in the present thesis heavily relies on this representation. It allows to describe the Gâteaux derivatives at points of differentiability as solution operators of variational equations. Here, the admissible sets are Sobolev spaces on quasi-open domains, which are determined by the contact set of the solution $y = S(\zeta)$ and the obstacle ψ and by the properties of the mulitplier $Ly-\zeta$. Using the tool of Mosco convergence, we can perform the convergence analysis of the Gâteaux derivatives. Here, the monotonicity structures of the set-valued maps mapping ζ to the admissible Sobolev spaces are important. We also change our perspective and understand the Gâteaux derivatives as solution operators of superordinate relaxed Dirichlet problems. Using classical results and connecting them to the problem at hand, we gain new insights into the generalized differentials.

Within this thesis, we consider a variety of formulations related to (OP). On the one hand, we consider the composition of the solution operator of (OP) with a nonlinear monotone operator with domain space different from $H^{-1}(\Omega)$. In this case, we cannot rely on chain rules to characterize the generalized differentials since, in general, they do not apply. Moreover, we consider also a bilateral obstacle problem where, in addition to the lower obstacle ψ, the admissible set is constrained also by a second upper obstacle.

Our insights on generalized derivatives are used to derive error estimators for inexact Clarke subgradients of an objective functional that are computed on a discrete level.

Outline of the thesis

This work is structured as follows. In Chapter 2, we clarify the notation, recall basic concepts and provide auxiliary results that will be used later on. In particular, in Section 2.3 we introduce the sets of generalized derivatives that are the subject of our analysis in the context of solution operators of variational inequalities of obstacle type. Another important aspect that is addressed in Section 2.5 is the notion of capacity on Ω w.r.t. the space $H_0^1(\Omega)$ which is essential in studying the structure of the generalized differentials for the solution operator of the obstacle problem. This concept enables us to work with quasi-continuous representatives in $H^1(\Omega)$ that are defined more precisely than up to a set of Lebesgue measure zero. As a consequence, we can find adequate descriptions of subsets of Ω which are, for instance, defined as the preimage of a quasi-continuous representative, such as the active set.

We study the basic features of the obstacle problem in Chapter 3. For a general variational inequality we recall existence, uniqueness and continuity results in Section 3.1. In the subsequent analysis we also state conditions under which the solution operator of a variational inequality is monotone and directionally differentiable. These results are transferred to the variational inequality describing the obstacle problem in Section 3.2. The directional derivative is characterized as the solution operator of another variational inequality. For a suitable description of the admissible set, the active set and the strictly active set are introduced and illustrated.

Chapter 4 discusses the composition of the solution operator of the obstacle problem with a monotone, possibly nonlinear operator on a potentially smaller space than $H^{-1}(\Omega)$ and is based on the publication [RU19]. The assumptions on the differential operator in the variational inequality and on the obstacle are not very restrictive and the considered formulation covers a wide class of imaginable settings. Using the representation of the directional derivatives, we derive variational equations which describe the Gâteaux derivatives in points at which the corresponding solution operator is differentiable. The admissible sets depend on the active and strictly active sets. Exploiting the monotonicity structures, we can show the Mosco convergence of the admissible sets using increasing or decreasing sequences in the preimage space in Section 4.5. Additionally, by application of a generalization of Rademacher's theorem, we establish the existence of such increasing and decreasing sequences at which the solution operator is Gâteaux differentiable in Section 4.3. The chapter will be closed with the characterization of two generalized derivatives for the solution operator and with the representation of Clarke subgradients for corresponding reduced objective functions.

In Chapter 5, a selection of results from [RW20] is presented which originated from a collaboration with Gerd Wachsmuth. Here, the solution operator of an obstacle problem on the complete space $H^{-1}(\Omega)$ is considered. We demonstrate that the Gâteaux derivatives can be interpreted as solution operators of relaxed Dirichlet problems involving capacitary measures, which motivates the study of these objects that is performed and reviewed in Section 5.1. In Section 5.2, a characterization of the entire generalized differentials using the strong operator topology for the convergence of the Gâteaux derivatives is obtained. Under regularity assumptions we also characterize the generalized differential relative to the weak operator topology in the subsequent section. It consists of particular solution operators of relaxed Dirichlet problems involving capacitary measures.

Chapter 6 is dedicated to the analysis of the bilateral obstacle problem and presents the results from [RU20]. Unlike in the previous analysis of the unilateral obstacle problem, the corresponding multiplier in the variational inequality describing the bilateral obstacle problem is not nonnegative and, hence, cannot be identified with a nonnegative measure. Therefore, the analysis of the admissible set for the variational inequality describing the directional derivative is more involved and carried out in Section 6.2. Again, we show the convergence of the Gâteaux derivatives in points of suitable monotone sequences, although the respective admissible sets in the variational equations for the Gâteaux derivatives are nonmonotone. Two generalized derivatives are obtained in Section 6.5.

Finally, in Chapter 7, we deduce error estimates for Clarke subgradients that are computed on a discrete level. We address the inexactness that arises due to the lack of knowledge on the correct active and strictly active sets, which are, by the previous analysis, the sets determining the domain on which generalized derivatives can be computed. We focus on the generalized derivative that is obtained on the complement of the strictly active set. In Section 7.6.2, imposing a nondegeneracy condition that is well known in the literature concerning the analysis of free boundaries, we can show that the strictly active set and the weakly active set have a suitable structure. Now, we derive discrete approximations of the complement of the strictly active set from the interior to use it as the domain for the discrete subgradient and from the exterior to find an upper bound for the error. The construction of these approximations is performed in Section 7.6.3 and Section 7.6.4. Finally, we obtain an error estimate for an inexact Clarke subgradient and test our findings in a numerical example in Section 7.9.

CHAPTER 2

Mathematical framework

In this chapter, we recall basic results and concepts that are frequently used throughout this thesis. Moreover, the corresponding notation is introduced.

We start with notions of differentiability in Section 2.1 and continue with a generalization of Rademacher's theorem to infinite dimensional spaces in Section 2.2. The standard operator topologies on the space of linear bounded operators are recalled and sets of generalized derivatives are introduced in Section 2.3. Subsequently, in Section 2.4, we define the relevant Sobolev spaces and recall basic calculus rules. Section 2.5 serves as a brief introduction into capacity theory and we collect the concepts and results that are needed later on. Finally, in Section 2.6, we will see that nonnegative elements in the dual space of $H_0^1(\Omega)$ can be identified with specific measures.

2.1 Concepts of differentiability

Let us give the definition of directional differentiability, see also [BS00, Def. 2.44].

Definition 2.1 (Directional differentiability) Let X, Y be Banach spaces and consider a mapping $T\colon X \to Y$. We say that T is directionally differentiable at a point $x \in X$ in a direction $h \in X$ if the limit

$$T'(x;h) := \lim_{t \searrow 0} \frac{T(x+th) - T(x)}{t} \tag{2.1}$$

exists. If the limit in (2.1) exists for all directions $h \in X$, we say that T is directionally differentiable at x.

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Throughout this thesis, we use the notation

$$\mathcal{L}(X,Y) := \{T\colon X \to Y \mid T \text{ is linear and } \|T\|_{\mathcal{L}(X,Y)} < \infty\}$$

and for a linear operator $T\colon X \to Y$ we set

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X \leq 1} \|T(x)\|_Y.$$

Definition 2.2 (Gâteaux differentiability) Let X, Y be Banach spaces and consider a mapping $T\colon X \to Y$. We say that T is Gâteaux differentiable at a point $x \in X$ if T is directionally differentiable at x and if the directional derivative operator $T'(x;\cdot)$ is linear and bounded, i.e., $T'(x;\cdot) \in \mathcal{L}(X,Y)$. We also use the notation $T'(x)$ for the Gâteaux derivative of T.

The solution operators of variational inequalities that we consider in this thesis are directionally differentiable. We are interested in chain rules for the directional derivative of composite mappings. Since the chain rule does not hold for merely directionally differentiable mappings a stronger form of directional differentiability is needed. The following definition is taken from [BS00, Def. 2.45].

Definition 2.3 (Hadamard directional differentiability) Let X, Y be Banach spaces and consider a mapping $T\colon X \to Y$. Then T is directionally differentiable at $x \in X$ in the Hadamard sense (or Hadamard directionally differentiable) if the directional derivative $T'(x;h)$ exists for all $h \in X$ and fulfills

$$T'(x;h) = \lim_{n\to\infty} \frac{T(x + t_n h_n) - T(x)}{t_n}$$

for all sequences $(t_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ with $t_n \searrow 0$ and $h_n \to h$.

We also state the following continuity property of Hadamard directionally differentiable mappings. The result is taken from [BS00, Prop. 2.46].

Lemma 2.4 *Let X, Y be Banach spaces. Suppose that $T\colon X \to Y$ is directionally differentiable at $x \in X$ in the Hadamard sense. Then the directional derivative $T'(x;\cdot)$ is continuous on X.*

The next proposition can be found in [BS00, Prop. 2.49]. It states that Lipschitz continuity and directional differentiability together ensure directional differentiability in the Hadamard sense.

Proposition 2.5 *Assume X, Y are Banach spaces. Suppose that $T\colon X \to Y$ is directionally differentiable at x and Lipschitz continuous in a neighborhood of x. Then T is directionally differentiable at x in the Hadamard sense.*

If the outer function in a composition is Hadamard directionally differentiable, the following chain rule holds, see [BS00, Prop. 2.47].

Lemma 2.6 *Let X, Y, Z be Banach spaces and assume that $T\colon X \to Y$ is directionally differentiable at x and that $R\colon Y \to Z$ is Hadamard directionally differentiable at $T(x)$. Then the composite mapping $R \circ T$ is directionally differentiable at x and the following chain rule holds*

$$(R \circ T)'(x;h) = R'\left(T(x); T'(x;h)\right).$$

We state the following auxiliary lemma.

Lemma 2.7 *Let X, Y and Z be Banach spaces and suppose that X is densely embedded into Y. Denote by $\iota\colon X \hookrightarrow Y$ the continuous and dense linear embedding. Assume the operator $T\colon Y \to Z$ is directionally differentiable at $\iota(x)$ and locally Lipschitz continuous in a neighborhood of $\iota(x)$. If $(T \circ \iota)$ is*

Gâteaux differentiable at $x \in X$, then T is Gâteaux differentiable at $\iota(x) \in Y$.

Proof. Observe that T is directionally differentiable at $\iota(x)$ in the sense of Hadamard, cf. Proposition 2.5. Thus, using the chain rule in Lemma 2.6, for every $h_X \in X$ we have

$$T'(\iota(x); \iota(h_X)) = (T \circ \iota)'(x; h_X).$$

Moreover, since $T \circ \iota$ is Gâteaux differentiable at x, the operator $T'(\iota(x); \iota(\cdot))\colon X \to Z$ is linear. Furthermore, the operator $T'(\iota(x); \cdot)$ is continuous on X since T is directionally differentiable in the sense of Hadamard, see Lemma 2.4. Now, density of $\iota(X)$ in Y implies that $T'(\iota(x); \cdot)$ is a bounded linear operator on Y. Hence, T is Gâteaux differentiable at $\iota(x)$. □

2.2 Generalization of Rademacher's theorem to infinite dimensions

Before we can present a meaningful definition of generalized differentials for locally Lipschitz continuous maps between infinite dimensional spaces, we have to make sure that a Gâteaux derivative exists at sufficiently many points. In finite dimension, a result of this form is well-known as Rademacher's theorem which states that for a locally Lipschitz continuous map $T\colon \mathbb{R}^d \to \mathbb{R}$, the set of points at which T is not differentiable is a set of Lebesgue measure zero in $\mathbb{R}^d$, see [Rad19].

Here, we use the following generalization of Rademacher's theorem to infinite dimensions. For a proof we refer to, e.g., [Aro76, Ch. II, Sect.2, Thm. 1], [BL00, Thm. 6.42]. If the space X in Theorem 2.8 is additionally a Hilbert space, a version can be found in [Mig76, Thm. 1.2].

Theorem 2.8 *Let X be a separable Banach space and let Y be a Hilbert space. Assume $T\colon X \to Y$ is locally Lipschitz continuous. Then the set $\mathcal{D}_T$ of points at which T is Gâteaux differentiable is a dense subset of X.*

In [Aro76], the map T is Lipschitz continuous and defined on an open subset of X. By considering neighborhoods of points separately, the formulation as in Theorem 2.8 can be obtained.

Note that there are different possibilities to generalize Rademacher's theorem to infinite dimensions. On the one hand, different concepts of differentiability can be considered and on the other hand, the size of the $X \setminus \mathcal{D}_T$ can be measured using various concepts. For a selection of results, we refer to [BL00].

2.3 Operator topologies and generalized differentials

In this thesis, we analyze generalized derivatives for solution operators of obstacle problems. Here, we define the sets of generalized derivatives which we consider in our studies. If $T\colon X \to Y$ is an operator, then the set of generalized derivatives in a point in X will be a subset of $\mathcal{L}(X,Y)$. In the definition, we will differentiate between different topologies on X and $\mathcal{L}(X,Y)$. We consider the following standard operator topologies on $\mathcal{L}(X,Y)$.

Definition 2.9 (Operator topologies) Let X and Y be Banach spaces and let $(\Xi_n)_{n\in\mathbb{N}}, \Xi \subseteq \mathcal{L}(X,Y)$.

1. We say that the sequence $(\Xi_n)_{n\in\mathbb{N}}$ converges to Ξ in the strong operator topology of $\mathcal{L}(X,Y)$ if and only if $(\Xi_n(x))_{n\in\mathbb{N}}$ converges to $\Xi(x)$ in Y for all $x \in X$.
2. We say that the sequence $(\Xi_n)_{n\in\mathbb{N}}$ converges to Ξ in the weak operator topology of $\mathcal{L}(X,Y)$ if and only if $(\Xi_n(x))_{n\in\mathbb{N}}$ converges weakly to $\Xi(x)$ in Y for all $x \in X$.

From the uniform boundedness principle, we obtain that a sequence of operators which converges in the weak operator topology has to be bounded.

Lemma 2.10 *Let X, Y be Banach spaces. Suppose $(\Xi_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ and assume that $\Xi_n \to \Xi$ in the weak operator topology of $\mathcal{L}(X,Y)$ for some*

$\Xi \in \mathcal{L}(X,Y)$. *Then there is a constant* $c > 0$ *such that* $\|\Xi_n\|_{\mathcal{L}(X,Y)} \leq c$ *for all* $n \in \mathbb{N}$.

Proof. The convergence of $(\Xi_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ to Ξ in the weak operator topology implies that

$$\sup_{n\in\mathbb{N}} |\langle y^*, \Xi_n(x)\rangle_{Y^*,Y}| < \infty$$

for fixed $x \in X$ and $y^* \in Y^*$. Let $x \in X$ and define $\varphi_n \in Y^{**}$ by

$$\langle \varphi_n, y^*\rangle_{Y^{**},Y^*} = \langle y^*, \Xi_n(x)\rangle_{Y^*,Y}$$

for $y^* \in Y^*$. By the uniform boundedness principle, it holds $\|\varphi_n\|_{Y^{**}} \leq c_x$ for a constant $c_x > 0$ and for all $n \in \mathbb{N}$. Since $\|\varphi_n\|_{Y^{**}} = \|\Xi_n(x)\|_Y$, it follows that $\|\Xi_n(x)\| \leq c_x$ for all $n \in \mathbb{N}$. Applying the uniform boundedness principle once more, we deduce the existence of a constant $c > 0$ with $\|\Xi_n\|_{\mathcal{L}(X,Y)} \leq c$. $\square$

The next lemma shows conditions under which a sequence $(\Xi_n(x_n))_{n\in\mathbb{N}}$ converges. The proof is taken from [RW20, Lem. 2.9].

Lemma 2.11 *Let* X, Y *be Banach spaces. Suppose* $(\Xi_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ *and* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *are sequences.*

1. *Assume that* $\Xi_n \to \Xi$ *in the strong operator topology of* $\mathcal{L}(X,Y)$ *and* $x_n \to x$ *in* X. *Then* $\Xi_n(x_n) \to \Xi(x)$ *holds in* Y.
2. *Assume that* $\Xi_n \to \Xi$ *in the weak operator topology of* $\mathcal{L}(X,Y)$ *and* $x_n \to x$ *in* X. *Then we conclude* $\Xi_n(x_n) \rightharpoonup \Xi(x)$ *in* Y.
3. *Assume that* $\Xi_n \to \Xi$ *in the weak operator topology of* $\mathcal{L}(X,Y)$, $\Xi_n^* \to \Xi^*$ *in the strong operator topology of* $\mathcal{L}(Y^*, X^*)$ *and* $x_n \rightharpoonup x$ *in* X. *Then this implies* $\Xi_n(x_n) \rightharpoonup \Xi(x)$ *in* Y. *Here,* Ξ_n^*, Ξ^* *denote the (Banachian) adjoint operators of* Ξ_n, Ξ, *respectively.*

Proof. In any case, the norm of the operators Ξ_n is uniformly bounded, see Lemma 2.10. Now, we use the identity

$$\Xi_n(x_n) - \Xi(x) = (\Xi_n(x) - \Xi(x)) + \Xi_n(x_n - x). \tag{2.2}$$

Then the claim of the first two statements follows immediately.

Let us also prove the third statement. The convergence $\Xi_n(x) - \Xi(x) \rightharpoonup 0$ is clear. To prove the weak convergence of the second term in (2.2), we take $y^* \in Y^*$ and oberve

$$\langle y^*, \Xi_n (x_n - x)\rangle_{Y^*,Y} = \langle \Xi_n^*(y^*), x_n - x\rangle_{X^*,X} \to 0$$

since $\Xi_n^* \to \Xi^*$ in the strong operator topology of $\mathcal{L}(Y^*, X^*)$ by assumption.

□

Now, let us define the *Bouligand generalized differentials* that we will deal with in this thesis. The definition is based on the concept of the Bouligand generalized differentials which is known in finite dimensions, see, e.g., [OKZ98, Def. 2.12], [FP03, Def. 4.6.2]. The below generalizations to infinite dimension are also used in [CCMW18]. We obtain four different differentials, since we consider combinations of strong and weak topologies. Note that in finite dimensions these concepts coincide.

Definition 2.12 (Generalized differentials) Let $T\colon X \to Y$ be a locally Lipschitz mapping from a separable Banach space X to a separable and reflexive Banach space Y. We denote the set of points in X at which T is Gâteaux differentiable by $\mathcal{D}_T$. For $x \in X$ we define the following generalized differentials

$$\begin{aligned}
\partial_{\mathrm{B}}^{\mathrm{ss}} T(x) := \{\Xi \in \mathcal{L}(X,Y) \,|\, \exists\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_T : {} & x_n \to x \text{ in } X,\\
& T'(x_n) \to \Xi \text{ in the strong op. top. of } \mathcal{L}(X,Y)\},\\
\partial_{\mathrm{B}}^{\mathrm{sw}} T(x) := \{\Xi \in \mathcal{L}(X,Y) \,|\, \exists\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_T : {} & x_n \to x \text{ in } X,\\
& T'(x_n) \to \Xi \text{ in the weak op. top. of } \mathcal{L}(X,Y)\},\\
\partial_{\mathrm{B}}^{\mathrm{ws}} T(x) := \{\Xi \in \mathcal{L}(X,Y) \,|\, \exists\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_T : {} & x_n \rightharpoonup x \text{ in } X,\\
& T(x_n) \rightharpoonup T(x) \text{ in } Y,\\
& T'(x_n) \to \Xi \text{ in the strong op. top. of } \mathcal{L}(X,Y)\},\\
\partial_{\mathrm{B}}^{\mathrm{ww}} T(x) := \{\Xi \in \mathcal{L}(X,Y) \,|\, \exists\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_T : {} & x_n \rightharpoonup x \text{ in } X,\\
& T(x_n) \rightharpoonup T(x) \text{ in } Y,\\
& T'(x_n) \to \Xi \text{ in the weak op. top. of } \mathcal{L}(X,Y)\}.
\end{aligned}$$

Note that the first superscript refers to the mode of convergence of the points $(x_n)_{n\in\mathbb{N}}$ in X, whereas the second superscript refers to the type of operator topology for the convergence of $(T'(x_n))_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$.

Lemma 2.13 *Let $T\colon X \to Y$ be a locally Lipschitz mapping from a separable Banach space X to a separable Hilbert space Y. Let $x \in X$ be arbitrary. Then the generalized differential $\partial_{\mathrm{B}}^{\mathrm{sw}}T(x)$ is nonempty. Moreover, it holds*

$$\partial_{\mathrm{B}}^{\mathrm{ss}}T(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{sw}}T(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{ww}}T(x) \qquad \text{and} \qquad \partial_{\mathrm{B}}^{\mathrm{ss}}T(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{ws}}T(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{ww}}T(x).$$

Moreover, if T is Gâteaux differentiable at x, then $T'(x)$ is an element of the generalized differentials $\partial_{\mathrm{B}}^{\mathrm{ss}}T(x), \partial_{\mathrm{B}}^{\mathrm{sw}}T(x), \partial_{\mathrm{B}}^{\mathrm{ws}}T(x), \partial_{\mathrm{B}}^{\mathrm{ww}}T(x)$.

Proof. Since $\mathcal{D}_T$ is dense in X, x is the limit of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_T \subseteq X$ in X, see Theorem 2.8. Now, since T is locally Lipschitz continuous, the Gâteaux derivatives $T'(x_n)$ are bounded by the Lipschitz constant of T in a neighborhood of x. The sequential compactness of the unit ball in $\mathcal{L}(X,Y)$ with respect to the weak operator topology, which follows from a generalization of the Banach-Alaoglu theorem to the weak operator topology, yields the existence of an accumulation point of the sequence $(T'(x_n))_{n\in\mathbb{N}}$. Thus, $\partial_{\mathrm{B}}^{\mathrm{sw}}T(x)$ is nonempty.

The inclusions of the differentials follows easily by the relation between the respective topologies.

It directly follows from the definition that if T is Gâteaux differentiable at x with Gâteaux derivative $T'(x)$, then $T'(x)$ belongs to the generalized differentials defined in Definition 2.12. □

In the following proposition, we address closedness properties of the generalized differentials. Related results can be found in [CCMW18, Prop. 3.4, Prop. 3.5]

Proposition 2.14 *Let $T\colon X \to Y$ be a globally Lipschitz continuous map from a separable Banach space X to a separable, reflexive Banach space Y.*

1. *Let $x \in X$. Suppose there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \to x$ in X and a sequence $(L_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ with $L_n \in \partial_{\mathrm{B}}^{\mathrm{ss}}T(x_n)$ for all $n \in \mathbb{N}$.*

Furthermore, assume that $L_n \to L$ *in the strong operator topology of* $\mathcal{L}(X,Y)$ *for some* $L \in \mathcal{L}(X,Y)$. *Then* L *is in* $\partial_{\mathrm{B}}^{\mathrm{ss}} T(x)$.

2. *Let* $x \in X$. *Suppose there is a sequence* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *with* $x_n \to x$ *in* X *and a sequence* $(L_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ *with* $L_n \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(x_n)$ *for all* $n \in \mathbb{N}$. *Furthermore, assume that* $L_n \to L$ *in the weak operator topology of* $\mathcal{L}(X,Y)$ *for some* $L \in \mathcal{L}(X,Y)$. *Then* L *is in* $\partial_{\mathrm{B}}^{\mathrm{sw}} T(x)$.

Proof. 1. This part can be found in [CCMW18, Prop. 3.4], one just hast to replace $L^2(\Omega)$ by an arbitrary separable Banach space X.

2. We modify the proof of [CCMW18, Proposition 3.5] so it fits to our setting. Since $\Xi_n \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(x_n)$, there are sequences $\left(x_m^{(n)}\right) \subseteq \mathcal{D}_T$ with $x_m^{(n)} \to x_n$ as $m \to \infty$ and $T'\left(x_m^{(n)}\right) \to \Xi_n$ in the weak operator toplogy of $\mathcal{L}(X,Y)$ as $m \to \infty$. Since X is separable and since the properties of Y imply that Y^* is separable as well, we can find sequences $(h_n)_{n\in\mathbb{N}}$ and $(y_n^*)_{n\in\mathbb{N}}$ that are dense in X, respectively Y^*. For all $n \in \mathbb{N}$, fix $m(n) \in \mathbb{N}$ with

$$\begin{aligned} \left|\left\langle y_l^*, T'\left(x_{m(n)}^{(n)}; h_k\right) - \Xi_n(h_k)\right\rangle\right| &< 1/n \qquad \forall\, k,l = 1,\dots,n, \\ \left\| x_{m(n)}^{(n)} - x_n \right\| &\le 1/n. \end{aligned}$$

For fixed $h \in X$, $y^* \in Y^*$, and for all $n \in \mathbb{N}$ we define

$$\begin{aligned} \bar{h}_n &:= \arg\min\{\|h_k - h\|_X \mid 1 \le k \le n\}, \\ \bar{y}_n^* &:= \arg\min\{\|y_k^* - y^*\|_{Y^*} \mid 1 \le k \le n\}. \end{aligned}$$

These definitions imply that $\bar{h}_n \to h$ in X and $\bar{y}_n^* \to y^*$ in Y^*. We mention that all elements in $\partial_{\mathrm{B}}^{\mathrm{ww}} T(x)$ are bounded by the Lipschitz constant of T, see [CCMW18, Lem. 3.2(iii)]. In particular, $\left\| T'\left(x_{m(n)}^{(n)}\right) - \Xi_n \right\|_{\mathcal{L}(X,Y)}$ is bounded. This shows

$$\begin{aligned} &\left|\left\langle y^*, T'\left(x_{m(n)}^{(n)}; h\right) - \Xi(h)\right\rangle\right| \\ &\le \left|\left\langle \bar{y}_n^*, T'\left(x_{m(n)}^{(n)}; \bar{h}_n\right) - \Xi_n(\bar{h}_n)\right\rangle\right| + \left|\left\langle \bar{y}_n^* - y^*, T'\left(x_{m(n)}^{(n)}; \bar{h}_n\right) - \Xi_n(\bar{h}_n)\right\rangle\right| \\ &\quad + \left|\left\langle y^*, \left(T'\left(x_{m(n)}^{(n)}\right) - \Xi_n\right)(\bar{h}_n - h)\right\rangle\right| + |\langle y^*, \Xi_n(h) - \Xi(h)\rangle| \end{aligned}$$

$$\leq 1/n + \|\bar{y}_n^* - y^*\|_{Y^*} \left\| T\left(x_{m(n)}^{(n)}\right) - \Xi_n \right\|_{\mathcal{L}(X,Y)} \|\bar{h}_n\|_X$$
$$+ \|y^*\|_{Y^*} \left\| T'\left(x_{m(n)}^{(n)}\right) - \Xi_n \right\|_{\mathcal{L}(X,Y)} \|\bar{h}_n - h\|_X + |\langle y^*, \Xi_n(h) - \Xi(h)\rangle|$$
$$\to 0$$

as $n \to \infty$. Together with $x_{m(n)}^{(n)} \to x$, we obtain the desired $\Xi \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(x)$.

□

Lemma 2.15 *Let X and Y be separable Banach spaces and assume that X is densely embedded into Y. Denote by $\iota\colon X \hookrightarrow Y$ the continuous and dense linear embedding. Additionally, let Z be a separable Hilbert space and $T\colon Y \to Z$ be an operator that is locally Lipschitz continuous and directionally differentiable at some $x \in X$. Then*

1. $\partial_{\mathrm{B}}^{\mathrm{sw}}(T \circ \iota)(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{sw}} T(\iota(x)) \circ \iota := \{\Xi \circ \iota \mid \Xi \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(\iota(x))\}$ *and*
2. $\partial_{\mathrm{B}}^{\mathrm{ss}}(T \circ \iota)(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{ss}} T(\iota(x)) \circ \iota := \{\Xi \circ \iota \mid \Xi \in \partial_{\mathrm{B}}^{\mathrm{ss}} T(\iota(x))\}$.

Proof. 1. Let $\Xi_X \in \partial_{\mathrm{B}}^{\mathrm{sw}}(T \circ \iota)(x)$. Then there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \to x$ such that $T \circ \iota$ is Gâteaux differentiable at x_n for each $n \in \mathbb{N}$ and it holds $(T \circ \iota)'(x_n) \to \Xi_X$ in the weak operator topology of $\mathcal{L}(X, Z)$. By Lemma 2.7, for every $n \in \mathbb{N}$, T is Gâteaux differentiable at $\iota(x_n)$. Moreover, the Lipschitz continuity of T imples that

$$\|T'(\iota(x_n))\|_{\mathcal{L}(Y,Z)} \leq c \tag{2.3}$$

for all $n \in \mathbb{N}$, where c denotes the Lipschitz constant of T. Thus, for fixed $h \in X$ and all $n \in \mathbb{N}$, $(T \circ \iota)'(x_n; h) = T'(\iota(x_n); \iota(h))$ is an element of the closed and convex ball of radius $c\|\iota(h)\|_Y$. Since $\Xi_X(h)$ is the weak limit of these elements, Mazur's lemma yields

$$\|\Xi_X(h)\|_Z \leq c\|\iota(h)\|_Y. \tag{2.4}$$

Now, define $\Xi_Y : \iota(X) \to Z$ by

$$\Xi_Y(y) := \Xi_X(\iota^{-1}(y)).$$

Then Ξ_Y is well-defined on $\iota(X)$, linear, and (2.4) implies

$$\|\Xi_Y(y)\|_Z = \|\Xi_X(\iota^{-1}(y))\|_Z \le c\|y\|_Y$$

for y within the dense subset $\iota(X)$. Thus, by [BS00, Lem. 6.5], Ξ_Y has a unique linear and continuous extension over Y that will again be denoted by Ξ_Y. Also note that $\|\Xi_Y\|_{\mathcal{L}(Y,Z)} \le c$ and $\Xi_Y \circ \iota = \Xi_X$, i.e., it suffices to show convergence of $T'(\iota(x_n))$ to Ξ_Y.

To this end, let $\varepsilon > 0$, $y \in Y$ be arbitrary and fix $y_\varepsilon \in \iota(X)$ with $\|y - y_\varepsilon\|_Y < \frac{\varepsilon}{3c}$. In particular, we have

$$\begin{aligned}
&\|\Xi_Y(y) - \Xi_Y(y_\varepsilon)\|_Z + \|T'(\iota(x_n); y_\varepsilon) - T'(\iota(x_n); y)\|_Z \\
&\le c\|y - y_\varepsilon\|_Y + c\|y - y_\varepsilon\|_Y < \frac{2}{3}\varepsilon.
\end{aligned}$$

Then, for any $z^* \in Z^*$ with $\|z^*\|_{Z^*} = 1$, we may choose $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&|\langle z^*, \Xi_Y(y_\varepsilon) - T'(\iota(x_n); y_\varepsilon)\rangle_{Z^*,Z}| \\
&= |\langle z^*, \Xi_X(\iota^{-1}(y_\varepsilon)) - (T \circ \iota)'(x_n; \iota^{-1}(y_\varepsilon))\rangle_{Z^*,Z}| < \tfrac{\varepsilon}{3}
\end{aligned}$$

to find that

$$|\langle z^*, \Xi_Y - T'(\iota(x_n); y)\rangle_{Z^*,Z}| < \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, $T'(\iota(x_n))$ converges to Ξ_Y in the weak operator topology and $\Xi_Y \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(\iota(x))$.

2. If $\Xi_X \in \partial_{\mathrm{B}}^{\mathrm{ss}}(T \circ \iota)(x) \subseteq \partial_{\mathrm{B}}^{\mathrm{sw}}(T \circ \iota)(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converging to x such that $T \circ \iota$ is Gâteaux differentiable at each x_n and $(T \circ \iota)'(x_n) \to \Xi_X$ in the strong operator topology. For this sequence, the arguments above hold but in addition, within the last step we may find $n_0 \in \mathbb{N}$ such that $\|\Xi_X(\iota^{-1}(y_\varepsilon)) - T'(\iota(x_n); y_\varepsilon)\|_Z = \|\Xi_Y(y_\varepsilon) - (T \circ \iota)'(x_n; \iota^{-1}(y_\varepsilon))\|_Z < \frac{\varepsilon}{3}$ and conclude

$$\|\Xi_Y(y) - T'(\iota(x_n); y)\| < \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $T'(\iota(x_n)) \to \Xi_Y$ in the strong operator topology, i.e., $\Xi_Y \in \partial_{\mathrm{B}}^{\mathrm{ss}} T(\iota(x))$.

□

Finally, let us give a connection to Clarke's generalized gradient. It is defined as follows, see [Cla90].

Definition 2.16 (Clarke's generalized gradient) Let U be a Banach space and assume that $J\colon U \to \mathbb{R}$ is locally Lipschitz continuous. Then Clarke's generalized gradient $\partial_{\mathrm{C}} J(u)$ of J at $u \in U$ is defined as

$$\partial_{\mathrm{C}} J(u) := \{u^* \in U^* \mid J^\circ(u; h) \geq \langle u^*, h\rangle_{U^*,U} \text{ for all } h \in U\}.$$

Here, $J^\circ(u; h)$ denotes the generalized directional derivative in the sense of Clarke defined by

$$J^\circ(u; h) := \limsup_{\substack{v \to u \\ t \searrow 0}} \frac{J(v + th) - J(v)}{t}.$$

We do not need many results concerning Clarke's generalized gradient. The next lemma shows how a Clarke subgradient (or a Clarke generalized derivative) can be obtained for a reduced objective function when an element of $\partial_{\mathrm{B}}^{\mathrm{sw}} T(u)$ is available for the solution operator T of a state equation. The resulting subgradient for the reduced objective function can be used in nonsmooth optimization methods.

In a slightly different context, the following result can be found in [CCMW18, Prop. 4.6].

Lemma 2.17 *Let U be a separable Banach space and let Y be a separable Hilbert space. Assume $J\colon Y \times U \to \mathbb{R}$ is a continuously differentiable function and let $T\colon U \to Y$ be a locally Lipschitz continuous operator. Denote by $\hat{J}\colon U \to \mathbb{R}$ the map defined by*

$$\hat{J}(u) := J(T(u), u).$$

Let $u \in U$ be arbitrary. Then

$$\{\Xi^* J_y(T(u), u) + J_u(T(u), u) \mid \Xi \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(u)\} \subseteq \partial_{\mathrm{B}}^{\mathrm{ss}} \hat{J}(u) \subseteq \partial_{\mathrm{C}} \hat{J}(u) \tag{2.5}$$

holds.

Proof. Let $\Xi \in \partial_{\mathrm{B}}^{\mathrm{sw}} T(u)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \to u$ in U and $T'(u_n) \to \Xi$ in the weak operator topology of $\mathcal{L}(U, Y)$. Since J is continuously differentiable and thus Hadamard directionally differentiable, the chain rule holds, see Lemma 2.6. Since T is Gâteaux differentiable at u_n, $\hat{J}$ is also Gâteaux differentiable at u_n and it holds $\hat{J}'(u_n) = T'(u_n)^* J_y(T(u_n), u_n) + J_u(T(u_n), u_n)$. Taking the limit and using that $T'(u_n) \to \Xi$ in the weak operator topology of $\mathcal{L}(U, Y)$ and that J is continuously differentiable we obtain the first inclusion in (2.5).

Since $\hat{J}'(u_n) \in \partial_{\mathrm{C}} \hat{J}(u_n)$, see [Cla90, Prop. 2.2.2], the second inclusion is implied by weak*-closedness of Clarke's generalized gradient, compare [Cla90, Prop. 2.1.5b]. □

2.4 Sobolev functions

Let Ω be an open, bounded set. We denote by $C_{\mathrm{c}}(\Omega)$ the function space of continuous functions on Ω with compact support contained in Ω. The subspace of infinitely differentiable functions is denoted by $C_{\mathrm{c}}^\infty(\Omega)$. As usual, we define

$$H^1(\Omega) := \left\{ z \in L^2(\Omega) \mid \tfrac{\partial z}{\partial x_i} \in L^2(\Omega), i = 1, \dots, d \right\},$$

where $\frac{\partial z}{\partial x_i}$ is to be understood in the distributional sense. $H^1(\Omega)$ is equipped with the norm

$$\|z\|_{H^1(\Omega)} = \left(\int_\Omega z^2 + \sum_{i=1}^d \left(\frac{\partial z}{\partial x_i} \right)^2 d\lambda^d \right)^{1/2}.$$

Here, λ^d denotes the d-dimensional Lebesgue measure. The space $H_0^1(\Omega)$ is defined as the completion of $C_{\mathrm{c}}^\infty(\Omega)$ in $H^1(\Omega)$. Equipped with the scalar product

$$(v, w)_{H_0^1(\Omega)} := (\nabla v, \nabla w)_{L^2(\Omega)} \qquad (v, w \in H_0^1(\Omega)),$$

and the respective norm

$$\|z\|_{H_0^1(\Omega)} := \|\nabla z\|_{L^2(\Omega)},$$

$H_0^1(\Omega)$ is a Hilbert space.

We sometimes use the following product rules for the weak derivatives. The results follow from, e.g., [GT01, (7.18)].

Lemma 2.18 *Assume v, w are elements of $H_0^1(\Omega)$.*

1. *If $v, w \in L^\infty(\Omega)$, then $v\,w$ is in $H_0^1(\Omega)$ and it holds*

$$\nabla(v \cdot w) = v\,\nabla w + w\,\nabla v. \tag{2.6}$$

2. *Assume $v \in C^1(\overline{\Omega})$. Then $v\,w$ is in $H_0^1(\Omega)$ and the product rule in* (2.6) *holds.*

The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ and if $p \in H^{-1}(\Omega)$ and $z \in H_0^1(\Omega)$ we use the notation $\langle p, z\rangle$ for the dual pairing.

We do not identify $H_0^1(\Omega)$ with its dual space. Instead, we use the continuous and dense embeddings

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),$$

see, e.g., [BS00, Lem. 6.2 and 6.3]. Here, we regard $g \in L^2(\Omega)$ as an element of $H^{-1}(\Omega)$ via

$$\langle g, v\rangle = (g, v)_{L^2(\Omega)} \qquad (v \in H_0^1(\Omega)).$$

We denote by $H^1(\Omega)_+$, respectively by $H_0^1(\Omega)_+$, the respective subsets of nonnegative elements. Here, we define

$$v \geq 0 \qquad :\Leftrightarrow \qquad v \geq 0 \text{ a.e. on } \Omega.$$

It is well-known that for $v \in H_0^1(\Omega)$ also $v_+ := \max(0, v)$ and $v_- := -\min(0, v)$ are elements of $H_0^1(\Omega)$, see [ABM14, Sect. 5.8.1]. If $w \in H_0^1(\Omega)$, then also $\max(v, w), \min(v, w) \in H_0^1(\Omega)$. Let us define $|v| := v_+ + v_-$.

Proposition 2.19 *Let v be an element in $H_0^1(\Omega)$.*

1. *It holds $\||v|\|_{H_0^1(\Omega)} = \|v\|_{H_0^1(\Omega)}$.*
2. *Suppose that $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ is a sequence with $v_n \rightharpoonup v$ in $H_0^1(\Omega)$. Then it holds $|v_n| \rightharpoonup |v|$ in $H_0^1(\Omega)$.*
3. *Let $v_n, w, w_n \in H_0^1(\Omega)$, $n \in \mathbb{N}$, with $v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ in $H_0^1(\Omega)$. Then $\max(v_n, w_n) \rightharpoonup \max(v, w)$ in $H_0^1(\Omega)$.*

Proof. 1. This statement can be found in [ABM14, Cor. 5.8.1].
2. By the weak convergence of $(v_n)_{n\in\mathbb{N}}$ and since $\||v_n|\|_{H_0^1(\Omega)} = \|v_n\|_{H_0^1(\Omega)}$ for all $n \in \mathbb{N}$ by the first statement, $(|v_n|)_{n\in\mathbb{N}}$ is a bounded sequence and thus has a weakly convergent subsequence. By the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we conclude that the weak limit has to be $|v|$. This yields that the whole sequence $(|v_n|)_{n\in\mathbb{N}}$ converges weakly to $|v|$ in $H_0^1(\Omega)$.
3. This follows from (2.) by

$$\max(v_n, w_n) = \frac{1}{2}\left(v_n + w_n + |v_n - w_n|\right) \rightharpoonup \frac{1}{2}\left(v + w + |v - w|\right) = \max(v, w).$$

□

Let us shortly comment on the notation used in this thesis. In this thesis, we always write $\langle \cdot, \cdot \rangle$ for the dual pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. If we mean the dual pairing between another space X and its dual X^*, we write $\langle \cdot, \cdot \rangle_{X^*,X}$. Similarly, we write $(\cdot, \cdot)$ for the scalar product on $L^2(\Omega)$. If we mean the scalar product on another Hilbert space X, we use the notation $(\cdot, \cdot)_X$.

2.5 Capacity theory

We give a short introduction to capacity theory on Ω w.r.t. $H_0^1(\Omega)$.

For the definitions, see e.g. [ABM14, Sect. 5.8.2, 5.8.3], [DZ11, Def. 6.2] or [BS00, Def. 6.47].

Definition 2.20 (Capacity theory)

1. For every set $E \subseteq \Omega$ the capacity (in the sense of $H_0^1(\Omega)$) is defined as

$$\operatorname{cap}(E) := \inf\{\|z\|^2_{H_0^1(\Omega)} \,|z \in H_0^1(\Omega),\\ z \geq 1 \text{ a.e. on a neighborhood of } E\}.$$

2. A subset $O \subseteq \Omega$ is called quasi-open if for all $\varepsilon > 0$ there is an open set $O_\varepsilon \subseteq \Omega$ with $\operatorname{cap}(O_\varepsilon) < \varepsilon$ such that $O \cup O_\varepsilon$ is open. The relative complement of a quasi-open set in Ω is called quasi-closed.
3. A function $v\colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is called quasi-continuous (quasi lower-semicontinuous, quasi upper-semicontinuous, respectively) if for all $\varepsilon > 0$ there is an open set $O_\varepsilon \subseteq \Omega$ with $\operatorname{cap}(O_\varepsilon) < \varepsilon$ such that v is continuous (lower-semicontinuous, upper-semicontinuous, respectively) on $\Omega \setminus O_\varepsilon$.

If a property holds on Ω except on a set of zero capacity, we say that this property holds *quasi-everywhere* (q.e.) in Ω.

Let us note that the capacity as defined above is an outer measure but not a measure, since it is not σ-additive. The family of quasi-open sets does *not* define a topology on Ω, since arbitrary unions of quasi-open sets are not necessarily quasi-open.

It is straightforward to observe that a set of capacity zero has Lebesgue measure zero. The converse is not true. Nevertheless, let us make the following observation. We refer to [Wac14, Lem. 2.3] for a proof.

Lemma 2.21 *Let $O \subseteq \Omega$ be quasi-open and assume $v\colon \Omega \to \mathbb{R}$ is quasi-continuous. Then $v \geq 0$ a.e. on O is equivalent to $v \geq 0$ q.e. on O.*

The following statements can be found in [Fug71, Lem. 3.3]. We also refer to [BB05, Thm. 4.16] and [ABM14, Thm. 5.8.6].

Lemma 2.22 *Let $v\colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Then the following assertions are equivalent.*

(i) *The function v is quasi lower-semicontinuous.*

(ii) *The sets $\{v > c\}$ are quasi-open for all $c \in \mathbb{R}$.*

(iii) *The function $-v$ is quasi upper-semicontinuous.*

Recall that if $1 - d/2 \geq \beta$ for some $0 < \beta < 1$, and if $\partial\Omega$ is sufficiently regular, then we have the embedding $H^1(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega})$. Here, $C^{0,\beta}(\overline{\Omega})$ denotes a usual Hölder space. Hence, if $d = 1$, each element in $H^1(\Omega)$ has a continuous representative.

We will see that, independent from the space dimension d, there is a quasi-continuous representative. Note that in dimension $d = 1$, the concept of quasi-continuity coincides with the classical concept of continuity. The following result can be found in, e.g., [DZ11, Chap. 8, Thm. 6.1].

Lemma 2.23 *Each $v \in H^1(\Omega)$ possesses a quasi-continuous representative, which is uniquely determined up to values on a set of zero capacity.*

The above lemma reveals why concepts from capacity theory are important for our analysis. We can always identify $v \in H_0^1(\Omega)$ with its quasi-continuous representative which is essentially unique (up to a set of capacity zero). In particular, this allows to talk about the pointwise behavior of elements in $H_0^1(\Omega)$ even on subsets of Ω of Lebesgue measure zero and positive capacity. Throughout this thesis, when dealing with elements of $H^1(\Omega)$ we always consider the quasi-continuous representative.

Since we are often dealing with such representatives, definitions and relations between subsets of Ω such as equalities and inclusions are often meant to hold only up to a set of capacity zero. To indicate this, we use a subscript q throughout this thesis. For example, for $v \in H^1(\Omega)$ we could define the set $E_c :=_{\mathrm{q}} \{v > c\}$ for some $c \in \mathbb{R}$. Then, E_c is defined up to a set of capacity zero and quasi-open, cf. Lemma 2.22.

By the following lemma, we can also consider Borel measurable quasi-continuous representatives of $H_0^1(\Omega)$ and that will be our convention for this thesis. Moreover, each quasi lower- and upper-semicontinuous function is Borel measurable after a modification on a set of zero capacity. The proof of the second statement can be found in [RW20, Sec. 2.1].

Lemma 2.24 1. *Each $v \in H_0^1(\Omega)$ has a quasi-continuous representative that is Borel measurable.*

2. *Let $v\colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ be quasi lower- or quasi-upper semicontinuous. Then, there is a Borel measurable quasi lower-, respectively quasi upper-semicontinuous function w such that $v(\omega) = w(\omega)$ for quasi-all $\omega \in \Omega$.*

Proof. 1. This is implied by the proof of [BS00, Lem. 6.50].

2. We show the statement for a quasi upper-semicontinuous function v. By Lemma 2.22, the sets $\{v < c\}$ are quasi-open for all $c \in \mathbb{Q}$. Hence, there are open sets O_n^c with $\mathrm{cap}(O_n^c) < \frac{1}{n}$ such that the sets $\{v < c\} \cup O_n^c$ are open. The set $O_c := \bigcap_{n\in\mathbb{N}} O_n^c$ is a Borel set and satisfies $\mathrm{cap}(O_c) = 0$, by monotonicity of the capacity. In addition,

$$\bigcap_{n\in\mathbb{N}} (\{v < c\} \cup O_n) = \{v < c\} \cup \bigcap_{n\in\mathbb{N}} O_n = \{v < c\} \cup O_c,$$

i.e., $\{v < c\} \cup O_c$ is a Borel set aswell. Define

$$w(\omega) := \begin{cases} -\infty & \text{if } \omega \in \bigcup_{c\in\mathbb{Q}} O_c, \\ v(\omega) & \text{else.} \end{cases}$$

Then w is still quasi upper-semicontinuous and, by construction, Borel measurable. Here, we have used countable sub-additivity of the capacity, see [BS00, Lem. 6.48].

□

The following lemma states that quasi-open sets have Borel measurable representatives. A proof is given in [Wac14, Lem. 2.2].

Lemma 2.25 *Let $O \subseteq \Omega$ be a quasi-open set. Then there exists a set $M \subseteq \Omega$ with $\mathrm{cap}(M) = 0$ such that $O \cup M$ is a Borel set.*

Now, we argue that quasi lower- and upper-semicontinuous functions can be approximated pointwise quasi-everywhere by functions in $H_0^1(\Omega)$.

Lemma 2.26 *Let $v\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ be quasi lower-semicontinuous. and* *assume that v is nonnegative. Then there exists an increasing sequence $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)_+$ with $v_n \to v$ pointwise quasi-everywhere.*

Proof. We want to use the result in [Dal83, Lem. 1.5]. Since the statement is posed for quasi lower-semicontinuous functions on $\mathbb{R}^d$, and not on Ω, we need to work with a capacity on all of $\mathbb{R}^d$. This can be defined as in [Dal83, Sect. 1]. The function $y - \psi$ is nonnegative and quasi lower-semicontinuous. Moreover, if we extend this function by 0, it is quasi lower-semicontinuous on all of $\mathbb{R}^d$. Now, [Dal83, Lem. 1.5] implies the existence of an increasing sequence $(z_m)_{m\in\mathbb{N}} \subseteq H^1(\mathbb{R}^d)$ with $0 \leq z_m$ and $z_m \nearrow y - \psi$ pointwise q.e. on $\mathbb{R}^d$. From $y - \psi = 0$ on $\mathbb{R}^d \setminus \Omega$, we have $z_m = 0$ q.e. on $\mathbb{R}^d \setminus \Omega$. Thus, $z_m \in H_0^1(\Omega)$, see [HKM93, Thm 4.5]. Moreover, the capacity on $\mathbb{R}^d$ is such that also $z_m \nearrow y - \psi$ pointwise q.e. on Ω w.r.t. the capacity we use on Ω. $\square$

The following lemma is taken from [HW18, Lem. 3.4].

Lemma 2.27 *For every set $E \subseteq \Omega$ it holds*

$$\operatorname{cap}(E) = \inf\{\|v\|^2_{H_0^1(\Omega)} \mid v \in H_0^1(\Omega) \text{ and } v \geq 1 \text{ q.e. on } E\}. \tag{2.7}$$

If $\operatorname{cap}(E)$ is finite, the infimum in (2.7) is attained by a nonnegative function v with $v = 1$ q.e. on E.

For the following result, we refer to [BS00, Lem. 6.52]. It states that convergent sequences in $H_0^1(\Omega)$ have subsequences which converge pointwise quasi-everywhere on Ω.

Lemma 2.28 *Let $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$, $v \in H_0^1(\Omega)$ and assume that $v_n \to v$ in $H_0^1(\Omega)$. Then there is a subsequence of $(v_n)_{n\in\mathbb{N}}$, such that the sequence (of quasi-continuous representatives of) $(v_n)_{n\in\mathbb{N}}$ converges pointwise quasi-everywhere to (the quasi-continuous representative of) v.*

Lemma 2.29 *Let $O \subseteq \Omega$ be quasi-open and assume there is a sequence of quasi-open sets $(O_n)_{n\in\mathbb{N}}$ such that $(O_n)_{n\in\mathbb{N}}$ is increasing in n and such that $O =_{\mathrm{q}} \bigcup_{n\in\mathbb{N}} O_n$. Let $v \in H_0^1(O)$. Then there is a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in H_0^1(O_n)$ for each $n \in \mathbb{N}$ such that $v_n \to v$ in $H_0^1(\Omega)$. Furthermore, it holds $\sup|v_n| \leq \sup|v|$.*

Proof. The sequence $(O_n)_{n\in\mathbb{N}}$ represents a quasi-covering of O, see Remark 2.30 therefore, combining [KM92, Thm. 2.10 and Lem. 2.4], we find a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \to v$ in $H_0^1(\Omega)$ and such that each v_n is a finite sum of elements in $\bigcup_{m\in\mathbb{N}} H_0^1(O_m)$. Furthermore, $\sup|v_n| \leq \sup|v|$. Since the sets O_n are increasing, for each $n \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that $v_n \in \bigcap_{m=j}^{\infty} H_0^1(O_m)$. We extend the sequence by adding copies of elements in $(v_n)_{n\in\mathbb{N}}$ to the original sequence. This yields a sequence with the desired properties. □

Remark 2.30 Assume J is an index set and $(O_j)_{j\in J}$ a family of quasi-open subsets of Ω. Let $E \subseteq \Omega$ be a set. If there is a countable subfamily $(O_{j_n})_{n\in\mathbb{N}}$ with the property that

$$E \cap \bigcup_{n\in\mathbb{N}} O_{j_n} =_{\mathrm{q}} E,$$

then $(O_j)_{j\in J}$ is also called a quasi-covering in the literature, see, e.g., [KM92].

The following lemma can be found in [HW18, Lem. 3.6]. We also refer to [Vel15, Prop. 2.3.14].

Lemma 2.31 *Let $O \subseteq \Omega$ be quasi-open. Then there exists a function $v \in H_0^1(\Omega)_+$ such that $\{v > 0\} =_{\mathrm{q}} O$.*

Definition 2.32 (Sobolev spaces on quasi-open domains) Let $O \subseteq \Omega$ be a quasi-open set. Then we define

$$H_0^1(O) := \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. outside } O\}.$$

By Lemma 2.28, for a quasi-open set $O \subseteq \Omega$, the set $H_0^1(O)$ is a closed subspace of $H_0^1(\Omega)$. Moreover, for an open set $O \subseteq \Omega$, the definition of $H_0^1(O)$ as in Definition 2.32 coincides (up to extension by 0 onto Ω) with the classical definition of $H_0^1(O)$ as the closure of $C_{\mathrm{c}}^{\infty}(O)$ in $H^1(O)$, see, e.g., [HKM93, Thm. 4.5].

Note that if $O_1, O_2 \subseteq \Omega$ are quasi-open sets which coincide up to a set of capacity zero, then $H_0^1(O_1) = H_0^1(O_2)$ follows.

2.6 Identification of elements in $H^{-1}(\Omega)_+$ with measures

We use the following partial ordering on $H^{-1}(\Omega)$. For $\xi \in H^{-1}(\Omega)$ we say that

$$\xi \geq 0 \qquad :\Leftrightarrow \qquad \langle \xi, v\rangle \geq 0 \quad \forall\, v \in H_0^1(\Omega)_+.$$

We denote the subset of nonnegative elements in $H^{-1}(\Omega)$ by $H^{-1}(\Omega)_+$.

Note that the composition of elements $H^{-1}(\Omega)$ into positive and negative parts is in general not possible, see also Example 6.8.

In this section we will see that elements in $H^{-1}(\Omega)_+$ can be identified with nonnegative measures.

Let μ be a nonnegative Borel measure on Ω. Then μ is called regular, if

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B \text{ is compact}\} = \inf\{\mu(U) \mid B \subseteq U \subseteq \Omega \text{ is open}\}$$

holds for every Borel set $B \subseteq \Omega$. We say that μ is a nonnegative Radon measure on Ω if μ is a nonnegative regular Borel measure which is finite on compact subsets of Ω. We denote by $\mathcal{M}_+(\Omega)$ the set

$$\mathcal{M}_+(\Omega) := \{\mu \mid \mu \text{ is the completion of a nonnegative Radon measure on } \Omega\}. \tag{2.8}$$

The following lemma states that we can identify functionals in $H^{-1}(\Omega)_+$ with measures in $\mathcal{M}_+(\Omega)$. For the proof we refer to [BS00, Thm. 6.54, Lem. 6.55, Lem. 6.56], see also [Wac14, Lem. 2.4].

Lemma 2.33 *Let $\xi \in H^{-1}(\Omega)_+$. Then ξ can be identified with a unique element $\tilde{\xi}$ of $\mathcal{M}_+(\Omega)$. For every Borel set $E \subseteq \Omega$ with $\operatorname{cap}(E) = 0$ it holds $\tilde{\xi}(E) = 0$. In addition, the quasi-continuous representatives of $v \in H_0^1(\Omega)$ are $\tilde{\xi}$-integrable and it holds*

$$\langle \xi, v\rangle = \int_\Omega v \, d\tilde{\xi}.$$

For the proof of the following result we refer to [HW18, Lem. 3.7].

Lemma 2.34 *Let* $\xi \in H^{-1}(\Omega)_+$. *Then there exists a quasi-closed set* $\text{f-supp}(\tilde{\xi}) \subseteq \Omega$ *such that for all* $v \in H_0^1(\Omega)$ *it holds* $\tilde{\xi}(\{v \neq 0\}) = 0$ *if and only* $v = 0$ *q.e. on* $\text{f-supp}(\tilde{\xi})$. *The set* $\text{f-supp}(\tilde{\xi})$ *is unique up to a set of capacity zero.*

Remark 2.35 The quasi-closed set in Lemma 2.34 is called $\text{f-supp}(\tilde{\xi})$ since it can be defined as the *fine support* of the measure $\tilde{\xi}$, i.e., the support of $\tilde{\xi}$ w.r.t. the fine topology on Ω. We refer to [Wac14, App. A] for the details.

CHAPTER 3

The obstacle problem

In this chapter, we introduce the variational inequality describing the obstacle problem. We formulate and collect basic results for general variational inequalities in Hilbert spaces and transfer these properties to the variational inequality describing the obstacle problem.

Let $\zeta \in H^{-1}(\Omega)$. We consider the following formulation of the obstacle problem

$$\text{Find } y \in K_\psi : \quad \langle Ly - \zeta, z - y \rangle \geq 0 \quad \forall\, z \in K_\psi. \tag{OP$_{\mathrm{id}}$}$$

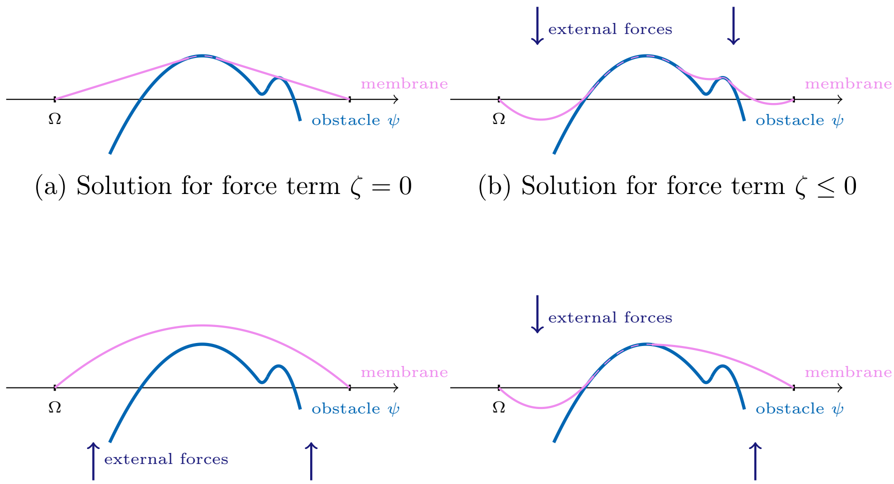
Here, the closed convex admissible set K_ψ is defined as

$$K_\psi := \{z \in H_0^1(\Omega) \mid z \geq \psi \text{ q.e. in } \Omega\} \tag{3.1}$$

and ψ is quasi upper-semicontinuous and chosen such that $K_\psi \neq \emptyset$. Note that if $\psi \in H^1(\Omega)$, then ψ has a quasi-continuous representative, see Lemma 2.23. Since quasi-continuous functions are quasi upper-semicontinuous this setting is also covered by our formulation. The operator $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is coercive, i.e., there is a constant $\alpha > 0$ such that

$$\langle Ly, y \rangle \geq \alpha \|y\|^2_{H_0^1(\Omega)}$$

holds for all $y \in H_0^1(\Omega)$.



(a) Solution for force term $\zeta = 0$ (b) Solution for force term $\zeta \leq 0$

(c) Solution for force term $\zeta \geq 0$ (d) Solution for force term ζ

Figure 3.1. Obstacle problem for different force terms

If $L = -\Delta$, i.e.,

$$\langle Lv, w \rangle = \int_\Omega \nabla v^T \nabla w \, \mathrm{d}\lambda^d,$$

the solution of ($\mathrm{OP}_{\mathrm{id}}$) coincides with the solution of the problem

$$\min_{y \in K_\psi} \frac{1}{2} \int_\Omega |\nabla y|^2 \, \mathrm{d}\lambda^d - \langle \zeta, y \rangle$$

Now, from a physical point of view, the solution of the obstacle problem satisfies the principle of energy minimization. Figure 3.1 illustrates the solutions of an obstacle problem for four different exemplary force terms $\zeta \in H^{-1}(\Omega)$.

The outline of the present chapter is as follows. We consider a general variational inequality in Hilbert space and show uniqueness and existence in Section 3.1. Global Lipschitz continuity of the corresponding solution operator is also established. In Section 3.1.1, assuming that the Hilbert space is equipped with a suitable partial ordering and that the differential operator behaves well with this ordering, we derive that the solution operator is increasing. Under assumptions on the admissible set, the directional differentiability is derived in Section 3.1.2. The directional derivative itself is given as the solution of a variational inequality on the critical cone. This famous result is due to Mignot (see [Mig76]). In Section 3.2, we convince ourself that the uniqueness, existence, Lipschitz continuity and monotonicity results also apply to the variational inequality of interest, the obstacle problem. Finally, in Section 3.2.1, the directional differentiability of the solution operator of the obstacle problem is obtained. We introduce the active and strictly active sets which are useful in the description of the critical cone.

3.1 Properties of solution operators of variational inequalities

We consider the following formulation of an abstract variational inequality in Hilbert space which includes the formulation of the obstacle problem ($\mathrm{OP}_{\mathrm{id}}$).

We fix the following assumptions.

Assumption 3.1 *We assume that H is a real Hilbert space and that $K \subseteq H$ is a nonempty closed convex subset. Furthermore, let $L \in \mathcal{L}(H, H^*)$ be coercive, i.e., let $\alpha > 0$ be a constant such that*

$$\langle Lh, h\rangle_{H^*,H} \geq \alpha \|h\|_H^2 \quad \forall\, h \in H.$$

For $\zeta \in H^*$, we consider the variational inequality

$$\text{Find } y \in K: \quad \langle Ly - \zeta, z - y\rangle_{H^*,H} \geq 0 \quad \forall\, z \in K. \tag{VI}$$

The following result states that the variational inequality (VI) has a unique solution and that the solution depends Lipschitz continuously on $f \in H^*$. This classical result is originally shown in [Sta64, LS67].

Theorem 3.1 *Suppose the conditions of Assumption 3.1 are satisfied. Then* *for each $\zeta \in H^*$ the variational inequality* (VI) *has a unique solution. Moreover, the solution operator $S\colon H^* \to H$ of* (VI) *is globally Lipschitz continuous with Lipschitz constant α^{-1}.*

Proof. The idea of this proof is based on the proof given in [Rod87, Thm. 4:3.1]. The existence and uniqueness part of the proof relies on an application of Banach's fixed point theorem. We denote by $\iota\colon H \to H^*$ the canonical isomorphism operator defined by

$$\langle \iota(h), g \rangle_{H^*,H} = (h, g)_H.$$

Let $\zeta \in H^*$ be arbitrary. Moreover, for $\rho > 0$, we define the operator $T_\rho\colon H \to H^*$ as

$$T_\rho(v) = \iota(v) - \rho(Lv - \zeta).$$

Then we can rewrite the variational inequality (VI) as

$$\text{Find } y \in K: \quad (y - \iota^{-1} T_\rho(y), z - y)_H \geq 0 \quad \forall\, z \in K. \tag{3.2}$$

Now, $y \in K$ is a solution of (3.2) and thus of (VI) if and only if $y \in K$ is the projection of $\iota^{-1}(T_\rho(y))$ onto the closed convex set K which we denote by

$$y = \mathcal{P}_K(\iota^{-1}(T_\rho(y))).$$

Therefore, we show that for appropriate $\rho > 0$ the operator $R_\rho := \mathcal{P}_K \circ \iota^{-1} \circ T_\rho$ has a unique fixed point in K. To apply Banach's fixed point theorem, we argue that R_ρ is a strict contraction for appropriately chosen $\rho > 0$.

Let $v, w \in K$. Since $\mathcal{P}_K$ is a contraction,

$$\|R_\rho(v) - R_\rho(w)\|_H \leq \|\iota^{-1}(T_\rho(v)) - \iota^{-1}(T_\rho(w))\|_H$$

holds. Moreover, using coercivity and continuity of L, we estimate

$$\begin{aligned}
&\|\iota^{-1}(T_\rho(v)) - \iota^{-1}(T_\rho(w))\|_H^2 \\
&= (\iota^{-1}(T_\rho(v)) - \iota^{-1}(T_\rho(w)), \iota^{-1}(T_\rho(v)) - \iota^{-1}(T_\rho(w)))_H \\
&= (v - w - \rho\iota^{-1}(Lv - Lw), v - w - \rho\iota^{-1}(Lv - Lw))_H
\end{aligned}$$

$$
\begin{aligned}
&= \|v - w\|_H^2 - 2\rho\langle Lv - Lw, v - w\rangle_{H^*,H} + \rho^2\|Lv - Lw\|_{H^*}^2 \\
&\leq (1 - 2\alpha\rho + \|L\|_{\mathcal{L}(H,H^*)}^2\rho^2)\|v - w\|_H^2.
\end{aligned}
$$

This shows that for $0 < \rho < \frac{2\alpha}{\|L\|_{\mathcal{L}(H,H^*)}^2}$ the operator R_ρ is a strict contraction. Now, Banach's fixed point theorem yields the existence of a unique fixed point of R_ρ in K and thus the existence of a solution of (VI) for arbitrary $\zeta \in H^*$.

Denote by y_i solutions of (VI) for $\zeta_i \in H^*$, $i = 1, 2$. Testing the variational inequalities describing y_i with y_j, $i \neq j$, gives

$$\langle Ly_i - \zeta_i, y_j - y_i\rangle_{H^*,H} \geq 0.$$

Summing up and using coercivity yields

$$
\begin{aligned}
\alpha\|y_1 - y_2\|_H^2 &\leq \langle Ly_1 - Ly_2, y_1 - y_2\rangle_{H^*,H} \\
&\leq \langle \zeta_1 - \zeta_2, y_1 - y_2\rangle_{H^*,H} \\
&\leq \|\zeta_1 - \zeta_2\|_{H^*}\|y_1 - y_2\|_H.
\end{aligned}
$$

This shows

$$\|y_1 - y_2\|_H \leq \alpha^{-1}\|\zeta_1 - \zeta_2\|_{H^*}.$$

We obtain the global Lipschitz continuity of the solution operator $S_{\mathrm{id}} \colon H^* \to H$. □

3.1.1 Monotonicity

The next proposition states that the solution operator of the variational inequality (VI) is increasing. Of course, we have to make sure that (VI) is formulated in a suitable setting allowing for monotonicity statements.

Therefore, we assume that $H = (H, \geq_H)$ is a partially ordered Hilbert space. We assume that for each $h, g \in H$ with $h \geq_H g$ it holds $h + j \geq g + j$ for each $j \in H$ and $t\,g \geq t\,h$ for each $t \geq 0$. Moreover, suppose that for any $h, g \in H$ the elements $\sup(h, g) \in H$ and $\inf(h, g) \in H$ exist. Here, it holds

$z = \sup(h, g)$ if and only if

$$z \geq_H h, \quad z \geq_H g \quad \text{and} \quad j \geq_H h, j \geq_H g \quad \Rightarrow \quad j \geq_H z.$$

Analogously, we have $z = \inf(h, g)$ if and only if

$$z \leq_H h, \quad z \leq_H g \quad \text{and} \quad j \leq_H h, j \leq_H g \quad \Rightarrow \quad j \leq_H z,$$

i.e., $\inf(h, g) = -\sup(-h, -g)$. Note that under these assumptions, H is often called a vector lattice in the literature, see e.g. [Bou04, Ch. 2, Def. 1], [Rod87, Ch. 4:5]. For arbitrary $h \in H$ we introduce the notation $h_+ := \sup(h, 0)$. Then one can easily show that $\sup(h, g) = h + (g - h)_+$ and $\inf(h, g) = g - (g - h)_+$, see [Rod87, Ch. 4:5].

The partial ordering on H induces a partial ordering on the dual space H^*. For $p \in H^*$ we say that $p \geq_{H^*} 0$ if and only if $\langle p, h \rangle_{H^*,H} \geq 0$ holds for all $h \in H$ satisfying $h \geq_H 0$.

Let $K \subseteq H$ be a closed convex and nonempty set. We assume that for all $k, l \in K$ we have $\sup(k, l), \inf(k, l) \in K$.

Moreover, we assume that the operator $L \in \mathcal{L}(H, H^*)$ is strictly T-monotone, i.e., for any $h, g \in H$ with $(h - g)_+ \neq 0$ it holds

$$\langle Lh - Lg, (h - g)_+ \rangle_{H^*,H} > 0. \tag{3.3}$$

Of course, the condition in (3.3) can be simplified, but since the notion of strict T-monotonicity exists also for nonlinear operators, we use the formulation as commonly used in the literature.

The following result is taken from [Rod87, Thm. 5.1] and we also state the proof given in this reference.

Proposition 3.2 *Assume the conditions of Assumption 3.1 and, in addition, let H be a vector lattice in the above sense and let L be strictly T-monotone. We suppose that for all $k, l \in K$ we have* $\sup(k, l), \inf(k, l) \in K$. *Then the solution operator $S\colon H^* \to H$ of the variational inequality* (VI) *is increasing, i.e., if $\zeta_1, \zeta_2 \in H^*$ satisfy $\zeta_1 \geq_{H^*} \zeta_2$, then it holds $S(\zeta_1) \geq_H S(\zeta_2)$.*

Proof. For $i = 1, 2$, let $\zeta_i \in H^*$ such that $\zeta_1 - \zeta_2 \geq_{H^*} 0$ and set $y_i := S(\zeta_i)$.

We test the variational inequality characterizing y_1 with $z_1 = \sup(y_1, y_2) = y_1 + (y_2 - y_1)_+ \in K$ and the variational inequality characterizing y_1 with $z_2 = \inf(y_1, y_2) = y_2 - (y_2 - y_1)_+ \in K$, respectively, and obtain

$$0 \leq \langle Ly_1 - \zeta_1, z_1 - y_1 \rangle = \langle Ly_1 - \zeta_1, (y_2 - y_1)_+ \rangle$$

and

$$0 \leq \langle Ly_2 - \zeta_2, z_2 - y_2 \rangle = \langle Ly_2 - \zeta_2, -(y_2 - y_1)_+ \rangle.$$

Adding up both inequalities we obtain

$$\langle Ly_1 - Ly_2, (y_2 - y_1)_+ \rangle \geq \langle \zeta_1 - \zeta_2, (y_2 - y_1)_+ \rangle \geq 0.$$

By strict T-monotonicity, we have $(y_2 - y_1)_+ = 0$, i.e., $y_1 \geq y_2$. □

3.1.2 Directional differentiability

In the following, the directional differentiability of the solution operator S of (VI) is derived and the directional derivative is characterized as the solution of another variational inequality under the assumption that the closed convex set $K \neq \emptyset$ in (VI) is polyhedric.

To this end, we define the radial and tangent cone of a convex set as well as the annihilator of a functional. For a convex set $K \subseteq H$ we define the *radial cone* to K at $y \in K$ and the *tangent cone* to K at $y \in K$ via

$$\mathcal{R}_K(y) := \{h \in H \mid \exists\, t > 0, y + th \in K\} \qquad \text{and} \qquad \mathcal{T}_K(y) := \overline{\mathcal{R}_K(y)}. \tag{3.4}$$

For $\xi \in H^*$, the *annihilator* $\xi^\perp$ is defined as

$$\xi^\perp := \{h \in H \mid \langle \xi, h \rangle_{H^*,H} = 0\}.$$

The subsequent result is originally due to [Mig76]. We also mention the related reference [Har77]. Both references examine the differentiability of metric projections onto closed convex sets and also the extension to variational inequalities. In particular, if L is symmetric (and induces a scalar product $(\cdot,\cdot)_L$ on H), the variational inequality (VI) describes the projection of the Riesz representative of $\zeta \in H^{-1}(\Omega)$ in $(H,(\cdot,\cdot)_L)$ onto K.

Theorem 3.3 *Suppose the conditions of Assumption 3.1 are satisfied and let $\zeta \in H^*$. We denote by S the solution operator of* (VI) *and set $y := S(\zeta)$, $\xi := Ly - \zeta$. Suppose that K is polyhedric at $(y,-\xi)$, i.e., we assume that*

$$\mathcal{T}_K(y) \cap \xi^\perp = \overline{\mathcal{R}_K(y) \cap \xi^\perp}.$$

Then S is directionally differentiable at $\zeta \in H^*$. The directional derivative $S'(\zeta;p)$ in direction $p \in H^*$ is the solution δ of the variational inequality

$$\textit{Find } \delta \in \mathcal{K}_K(y,\xi): \quad \langle L\delta - p, z - \delta\rangle_{H^*,H} \geq 0 \quad \forall\, z \in \mathcal{K}_K(y,\xi), \tag{3.5}$$

here $\mathcal{K}_K(y,\xi) := \mathcal{T}_K(y) \cap \xi^\perp$ denotes the critical cone.

Proof. We give a similar proof to the one that can be found in [Wac19, Thm. 5.2].

Let $\zeta, h \in H^*$ be arbitrary. For $t \geq 0$ we set $y(t) := S(\zeta + tp)$, i.e., $y(t)$ is the solution of the variational inequality

$$\text{Find } y(t) \in K: \quad \langle Ly(t) - \zeta - tp, z - y(t)\rangle_{H^*,H} \geq 0 \quad \forall\, z \in K. \tag{VI$_t$}$$

We observe

$$\frac{y(t) - y(0)}{t} = \frac{y(t) - y}{t} \in \mathcal{R}_K(y),$$

cf. (3.4). By Lipschitz continuity of S, see Theorem 3.1, we conclude

$$\|y(t) - y(s)\|_H = \|S(\zeta + tp) - S(\zeta + sp)\|_H \leq \alpha^{-1}\|p\|_H |t - s|,$$

i.e., $t \mapsto y(t)$ is Lipschitz continuous and it holds

$$\left\|\frac{y(t) - y}{t}\right\|_H \leq \alpha^{-1}\|p\|_H.$$

Thus, since $\left(\frac{y(t)-y}{t}\right)_{t>0}$ is bounded, there is a subsequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \searrow 0$ such that

$$\frac{y(t_k)-y}{t_k} \rightharpoonup \delta$$

for some $\delta \in H$.

Testing (VI) with $z = y(t_k)$ and dividing by $t_k > 0$ we obtain

$$\left\langle Ly - \zeta, \frac{y(t_k)-y}{t_k} \right\rangle_{H^*,H} \geq 0 \tag{3.6}$$

and taking the limit yields

$$\langle Ly - \zeta, \delta \rangle_{H^*,H} \geq 0.$$

In particular, this implies

$$\lim_{k\to\infty} \left\langle Ly(t_k) - \zeta - t_k p, \frac{y(t_k)-y}{t_k} \right\rangle_{H^*,H} = \langle Ly - \zeta, \delta \rangle_{H^*,H} \geq 0.$$

On the other hand, using (VI_t), we have

$$0 \leq \left\langle Ly(t_k) - \zeta - t_k p, \frac{y - y(t_k)}{t_k} \right\rangle_{H^*,H} \to \langle Ly - \zeta, -\delta \rangle_{H^*,H}. \tag{3.7}$$

We conclude $\langle \xi, \delta \rangle_{H^*,H} = 0$ and, using Mazur's lemma, $\delta \in \mathcal{T}_K(y) \cap \xi^\perp$.

We want to show that the weak limit δ is the solution of (3.5). Using the inequalities in (3.6) and (3.7), we obtain

$$\left\langle L\frac{y(t_k)-y}{t_k} - p, \frac{y - y(t_k)}{t_k} \right\rangle_{H^*,H} \geq 0. \tag{3.8}$$

Note that $\|\cdot\|_L := \sqrt{\langle L\,\cdot, \cdot\rangle_{H^*,H}}$ defines an equivalent norm on H. Thus, using weak lower semicontinuity of norms, we derive

$$\langle L\delta - p, -\delta \rangle_{H^*,H} \geq 0. \tag{3.9}$$

Now, let $z \in \mathcal{R}_K(y) \cap \xi^\perp$. Note that we find $s > 0$ such that $z = s(v - y)$ for some $v \in K$. Using this, as well as (VI_t) and $\langle Ly - \zeta, z\rangle_{H^*,H} = 0$, we estimate

$$\begin{aligned}
&\left\langle L\frac{y(t_k) - y}{t_k} - p, z\right\rangle_{H^*,H} \\
&= \left\langle Ly(t_k) - \zeta - t_k p, \frac{s(v - y(t_k) + y(t_k) - y)}{t_k}\right\rangle_{H^*,H} \\
&\geq s\left\langle Ly(t_k) - \zeta - t_k p, \frac{y(t_k) - y}{t_k}\right\rangle_{H^*,H} \overset{k\to\infty}{\to} s\langle Ly - \zeta, \delta\rangle_{H^*,H} = 0.
\end{aligned}$$

This shows

$$\left\langle L\frac{y(t_k) - y}{t_k} - p, z\right\rangle_{H^*,H} \overset{k\to\infty}{\to} \langle L\delta - p, z\rangle_{H^*,H} \geq 0.$$

In particular, we have $\langle L\delta - p, z\rangle_{H^*,H} \geq 0$ for all $z \in \overline{\mathcal{R}_K(y) \cap \xi^\perp} = \mathcal{K}_K(y, \xi)$ by polyhedricity. Putting this together with (3.9), we have shown that δ is the unique solution of the variational inequality (3.5).

This also shows that δ is the unique weak limit of the complete sequence $\left(\frac{y(t)-y}{t}\right)_{t>0}$. We show that it is also the strong limit of this sequence. Set $d_k := \frac{y(t_k)-y}{t_k}$ for a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \searrow 0$. From the variational inequality (3.5) we obtain $\langle L\delta - p, \delta\rangle_{H^*,H} = 0$. Using this and weak lower semicontinuity of norms we derive

$$\begin{aligned}
\liminf_{k\to\infty} \|d_k\|_L^2 &= \liminf_{k\to\infty} \langle Ld_k, d_k\rangle_{H^*,H} \\
&\geq \langle L\delta, \delta\rangle_{H^*,H} \\
&= \langle p, \delta\rangle_{H^*,H} \\
&= \lim_{k\to\infty} \langle p, d_k\rangle_{H^*,H} \\
&\geq \limsup_{k\to\infty} \|d_k\|_L^2.
\end{aligned}$$

Here, in the last step the inequality in (3.8) was used.

This consideration shows $\|d_k\|_L \to \|\delta\|_L$. Since $\|\cdot\|_L$ is equivalent to $\|\cdot\|_H$ and since $(d_k)_{k\in\mathbb{N}}$ converges weakly to δ, we obtain that $\lim\limits_{k\to\infty} d_k = \delta$ strongly in H. □

3.2 Properties of the solution operator of the obstacle problem

We consider again the variational inequality ($\mathrm{OP}_{\mathrm{id}}$) describing the obstacle problem. Let us first collect our assumptions on the data.

Assumption 3.2 *Let $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be quasi upper-semicontinuous and assume that the admissible set K_ψ in* (3.1) *is nonempty. Suppose that $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is a coercive operator and let $\alpha > 0$ be a constant such that*

$$\langle Lz, z\rangle \geq \alpha \|z\|^2_{H_0^1(\Omega)}$$

holds for all $z \in H_0^1(\Omega)$.

The following theorem is a corollary of Theorem 3.1 and states that for each $\zeta \in H^{-1}(\Omega)$ the obstacle problem in ($\mathrm{OP}_{\mathrm{id}}$) has a unique solution and that the corresponding solution operator $S_{\mathrm{id}}\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ is globally Lipschitz continuous. This shows that we are in a setting where generalized derivatives in the sense of Definition 2.12 for the operator S_{id} can be defined.

Theorem 3.4 *Suppose the conditions of Assumption 3.2 are satisfied. Then, for each $\zeta \in H^{-1}(\Omega)$, the variational inequality* ($\mathrm{OP}_{\mathrm{id}}$) *has a unique solution. Moreover, the solution operator*

$$S_{\mathrm{id}}\colon H^{-1}(\Omega) \to H_0^1(\Omega)$$

is globally Lipschitz continuous with Lipschitz constant α^{-1}.

We use the notation S_{id} for the solution operator of ($\mathrm{OP}_{\mathrm{id}}$) on $H^{-1}(\Omega)$ to distinguish it from solution operator of the obstacle problem on subspaces, see Chapter 4.

Next, we apply Proposition 3.2 to the obstacle problem. First recall that $H_0^1(\Omega)$ is a partially ordered space together with the partial ordering

$$v \geq w \quad \Leftrightarrow \quad v \geq w \text{ a.e. on } \Omega \quad \Leftrightarrow \quad v \geq w \text{ q.e. on } \Omega \quad (v, w \in H_0^1(\Omega)),$$

compare Section 2.4 and Lemma 2.21. Moreover, $\max(v, w)$ and $\min(v, w)$ exist for all $v, w \in H_0^1(\Omega)$. In particular, K_ψ is closed under taking suprema and infima of its elements, i.e., from $k, l \in K_\psi$ it follows $\max(k, l), \min(k, l) \geq \psi$ q.e. on Ω.

As usual, we use the ordering

$$p \geq 0 \quad \Leftrightarrow \quad \langle p, v \rangle \geq 0 \quad \forall\, v \in H_0^1(\Omega)_+ \qquad (p \in H^{-1}(\Omega))$$

in the dual space $H^{-1}(\Omega)$, see Section 2.6. Assuming strict T-monotonicity of the operator $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ we obtain the following monotonicity result for the solution operator S_{id} of (OP_{id}) as a corollary of Proposition 3.2.

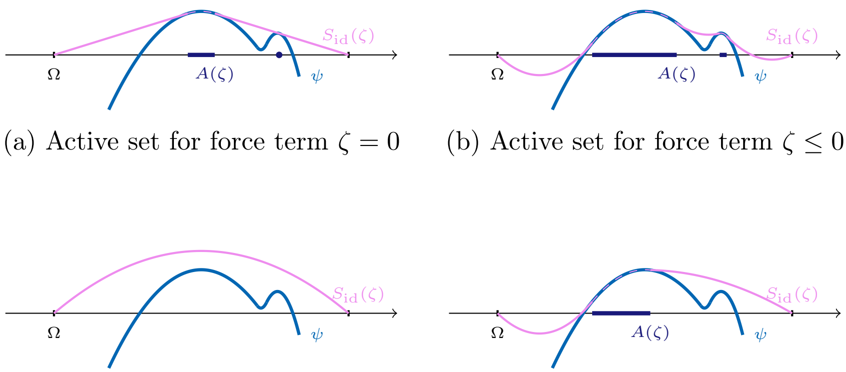
Proposition 3.5 *Suppose the conditions of Assumption 3.2 are satisfied and, in addition, assume that L is strictly T-monotone. Then the solution operator* S_{id} *of* (OP_{id}) *is increasing, i.e., if* ζ_1, ζ_2 *are elements of* $H^{-1}(\Omega)$ *satisfying* $\zeta_1 \geq \zeta_2$*, then it holds* $S_{\text{id}}(\zeta_1) \geq S_{\text{id}}(\zeta_2)$ *in* $H_0^1(\Omega)$*, i.e., a.e. and q.e. in* Ω.

3.2.1 Directional differentiability

We define the following subsets of Ω that are useful to describe the critical cone and essential for the analysis of generalized derivatives in the upcoming chapters. For $\zeta \in H^{-1}(\Omega)$, we define the active set $A(\zeta)$ by

$$A(\zeta) :=_{\text{q}} \{\omega \in \Omega \mid S_{\text{id}}(\zeta)(\omega) = \psi(\omega)\}.$$

As described in Lemma 2.23, by considering quasi-continuous representatives of $S_{\text{id}}(\zeta) \in H_0^1(\Omega)$, the set $A(\zeta)$ is quasi-closed and defined up to a set of capacity zero, see also Lemma 2.22. Since we can assume that ψ is Borel measurable, cf. Lemma 2.24, $A(\zeta)$ is Borel measurable. We also define the inactive set $I(\zeta) :=_{\text{q}} \Omega \setminus A(\zeta)$, which is a quasi-open set.



(a) Active set for force term $\zeta = 0$ (b) Active set for force term $\zeta \leq 0$

(c) Active set for force term $\zeta \geq 0$ (d) Active set for force term ζ

Figure 3.2. Active sets for different force terms, compare Fig. 3.1

Figure 3.2 shows an example where the corresponding active sets for different values of $\zeta \in H^{-1}(\Omega)$ are depicted.

Note that we have the following consequences of Proposition 3.5 on the inclusion of active and inactive sets.

Lemma 3.6 *Suppose the conditions of Assumption 3.2 are satisfied and, in addition, assume that L is strictly T-monotone. Suppose $\zeta_1, \zeta_2 \in H^{-1}(\Omega)$ satisfy $\zeta_1 \geq \zeta_2$. Then it holds*

1. $A(\zeta_1) \subseteq_{\mathrm{q}} A(\zeta_2)$,
2. $I(\zeta_1) \supseteq_{\mathrm{q}} I(\zeta_2)$.

Let $\zeta \in H^{-1}(\Omega)$ and $y := S_{\mathrm{id}}(\zeta)$. In the analysis of the obstacle problem, the multiplier $Ly - \zeta$ appearing in the variational inequality ($\mathrm{OP}_{\mathrm{id}}$) is relevant. In fact, the admissible set in the variational inequality for the directional derivative, the critical cone, see Theorem 3.3, is the intersection of the tangent cone $\mathcal{T}_{K_\psi}(y)$ and the annihilator of the functional $Ly - \zeta$. Before we state the corresponding directional differentiability result for the solution operator S_{id} of the obstacle problem ($\mathrm{OP}_{\mathrm{id}}$), let us therefore collect some features of the corresponding functional $Ly - \zeta \in H^{-1}(\Omega)$. The properties established in the following lemma will be the basis to find a characterization of the critical cone $\mathcal{K}_{K_\psi}(y)$. The result can be found in [Wac14, Prop. 2.5] in case the obstacle ψ is in $H^1(\Omega)$. Since we assume ψ to be merely quasi upper-semicontinuous, we use another argument to show the result in the second statement of the lemma.

Lemma 3.7 *Suppose the conditions of Assumption 3.2 are satisfied. Let $\zeta \in H^{-1}(\Omega)$ be arbitrary and denote $y := S_{\mathrm{id}}(\zeta)$ and $\xi := Ly - \zeta \in H^{-1}(\Omega)$. Then the following statements hold.*

1. *The functional ξ is nonnegative. It can be identified with measure $\tilde{\xi} \in \mathcal{M}_+(\Omega)$ fulfilling the properties described in Lemma 2.33, i.e., for every Borel set $E \subseteq \Omega$ with $\operatorname{cap}(E) = 0$ it holds $\tilde{\xi}(E) = 0$ and the quasi-continuous representative of $v \in H_0^1(\Omega)$ is $\tilde{\xi}$-integrable and it holds*

$$\langle \xi, v \rangle = \int_\Omega v \, d\tilde{\xi}.$$

2. *The measure $\tilde{\xi}$ satisfies $\tilde{\xi}(I(\zeta)) = 0$.*

Proof. 1. Let $v \in H_0^1(\Omega)_+$. Then it holds $y + v \geq \psi$, i.e. $y + v \in K_\psi$. Now, since y is the solution of $(\mathrm{OP}_{\mathrm{id}})$, it holds

$$\langle \xi, v \rangle = \langle \xi, (y + v) - y \rangle \geq 0.$$

Since $v \in H_0^1(\Omega)_+$ was arbitrary, this shows that ξ is nonnegative. Applying Lemma 2.33 we obtain the statement.
2. The function $y - \psi$ is quasi lower-semicontinuous. By Lemma 2.26, since $y - \psi$ is nonnegative, there exists an increasing sequence $(v_n)_{n \in \mathbb{N}} \subseteq H_0^1(\Omega)_+$ with $v_n \to y - \psi$ pointwise quasi-everywhere. We have $v_n \leq y - \psi$ q.e. in Ω and thus $-v_n + y \in K_\psi$. This implies for all $n \in \mathbb{N}$

$$0 \leq \langle \xi, -v_n + y - y \rangle = \int_\Omega -v_n \, \mathrm{d}\tilde{\xi}.$$

Since $-v_n \leq 0$ q.e. on Ω and thus $\tilde{\xi}$-a.e., see (1.), we conclude $v_n = 0$ $\tilde{\xi}$-a.e. on Ω. By pointwise q.e. convergence of $(v_n)_{n \in \mathbb{N}}$ to $y - \psi$ we have

$$\bigcup_{n \in \mathbb{N}} \{v_n > 0\} =_{\mathrm{q}} I(\zeta).$$

Thus, since $\tilde{\xi}(\{v_n > 0\}) = 0$ for all $n \in \mathbb{N}$ and since $\tilde{\xi}$ vanishes on sets of capacity zero we conclude

$$\tilde{\xi}(I(\zeta)) = \tilde{\xi}\left(\bigcup_{n\in\mathbb{N}} \{v_n > 0\}\right) = 0$$

by σ-subadditivity of $\tilde{\xi}$.

$\square$

In fact, the properties of the lemma hold for all elements in $-\mathcal{T}_{K_\psi}(y)^\circ$, compare [Wac14, Prop. 2.5]. Note that ξ as in Lemma 3.7 is an element of $-\mathcal{T}_{K_\psi}(y)$.

The polyhedricity of the admissible set K_ψ in (3.1) is established in [Mig76, Thm. 3.2] and a version of this result is stated in the following lemma. We also refer to [Wac19, Thm. 4.18] and [Har77]. The formulation in Lemma 3.8 will also include the polyhedricity of the admissible set for the bilateral obstacle problem, see Chapter 6.

Lemma 3.8 *Suppose $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is a quasi upper-semicontinuous function and $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ is quasi lower-semicontinuous. Assume that the admissible set*

$$K_\psi^\varphi := \{z \in H_0^1(\Omega) \mid \psi \le z \le \varphi \text{ q.e. in } \Omega\}$$

is nonempty. Let $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ be coercive. For an arbitrary $\zeta \in H^{-1}(\Omega)$ let y be the unique solution of

$$\text{Find } y \in K_\psi^\varphi: \quad \langle Ly - \zeta, z - y\rangle \ge 0 \quad \forall\, z \in K_\psi^\varphi \tag{3.10}$$

and $\xi := Ly - \zeta$. Then K_ψ^φ is polyhedric at $(y, -\xi)$.

Note that by choosing $\varphi := +\infty$ on Ω the variational inequality (3.10) is equivalent to the unilateral obstacle problem as in ($\mathrm{OP}_{\mathrm{id}}$). We state the result in Lemma 3.8 in this more general fashion so that it can also be applied in the analysis of the bilateral obstacle problem in Chapter 6.

Using the polyhedricity of K_ψ as in Lemma 3.8, the directional differentiability of the solution operator S_{id} of (OP_{id}) can be obtained and the variational inequalities describing the directional derivatives can be characterized. We refer again to [Mig76] for this result. The structure of the critical cone was analyzed in [Wac14, Lem. 3.1]. We state the proof from [Wac14, Lem. 3.1] and modify it to the case where the obstacle is a general quasi lower-semicontinuous function.

Theorem 3.9 *Suppose the conditions of Assumption 3.2 are satisfied. Then the solution operator* $S_{\text{id}}\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ *of* (OP_{id}) *is directionally differentiable. Let* $\zeta \in H^{-1}(\Omega)$ *be arbitrary and set* $y := S_{\text{id}}(\zeta)$, $\xi := Ly - \zeta$. *Then the directional derivative* $S'_{\text{id}}(\zeta; p)$ *for arbitrary* $p \in H^{-1}(\Omega)$ *is the solution* δ *of the variational inequality*

$$\text{Find } \delta \in \mathcal{K}_{K_\psi}(y,\xi): \quad \langle L\delta - p, z - \delta\rangle \geq 0 \quad \forall\, z \in \mathcal{K}_{K_\psi}(y,\xi). \tag{3.11}$$

Here, the critical cone $\mathcal{K}_{K_\psi}(y,\xi) = \mathcal{T}_{K_\psi}(y) \cap \xi^\perp$ *has the following structure. There exists a quasi-closed set* $A_{\text{s}}(\zeta) \subseteq_{\text{q}} A(\zeta)$ *which is unique up to a set of capacity zero such that*

$$\begin{aligned}\mathcal{K}_{K_\psi}(y,\xi) &= \{z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A(\zeta) \text{ and } \langle \xi, z\rangle = 0\}\\ &= \{z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A(\zeta) \text{ and } z = 0 \text{ q.e. in } A_{\text{s}}(\zeta)\}.\end{aligned} \tag{3.12}$$

Proof. Using Lemma 3.8 and Theorem 3.3, we immediately obtain the directional differentiability of the solution operator S_{id} on H^*. Moreover, for arbitrary $\zeta, h \in H^{-1}(\Omega)$, the directional derivative $S'_{\text{id}}(\zeta; h)$ is the unique solution of the variational inequality in (3.11).

Now, we verify the characterization of the critical cone stated in (3.12). Therefore, we proceed as in the proof of [Wac14, Lem. 3.1].

By [Mig76, Lem. 3.2], the first equation in (3.12) holds. Since $\xi = Ly - \zeta$ is a nonnegative functional in $H^{-1}(\Omega)$ it can be identified with a regular Borel measure $\tilde{\xi}$, see Lemma 3.7. Let $z \in H_0^1(\Omega)$ with $z \geq 0$ q.e. in $A(\zeta)$. Lemma 3.7(1.) implies $z \geq 0$ $\tilde{\xi}$-a.e. in $A(\zeta)$. Using the integral representation of ξ in Lemma 3.7(1.) and $\tilde{\xi}(I(\zeta)) = 0$, see Lemma 3.7(2.), we conclude

$$\langle \xi, z \rangle = \int_\Omega z \, \mathrm{d}\tilde{\xi} = \int_{A(\zeta)} z \, \mathrm{d}\tilde{\xi}.$$

Since $z \geq 0$ $\tilde{\xi}$-a.e. in $A(\zeta)$, it holds

$$\langle \xi, z \rangle = \int_{A(\zeta)} z \, \mathrm{d}\tilde{\xi} = 0$$

if and only if $z = 0$ $\tilde{\xi}$-a.e. on $A(\zeta)$ and, using again $\tilde{\xi}(I(\zeta)) = 0$, see Lemma 3.7, if and only if $z = 0$ $\tilde{\xi}$-a.e. on Ω.

This shows

$$\mathcal{K}_{K_\psi}(y, \xi) = \{z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A(\zeta) \text{ and } z = 0 \ \tilde{\xi}\text{-a.e.}\}.$$

Now, Lemma 2.34 yields the existence of a set $A_\mathrm{s}(\zeta) :=_\mathrm{q} \text{f-supp}(\tilde{\xi})$ which is unique up to a set of capacity zero such that the critical cone has the structure as in (3.12).

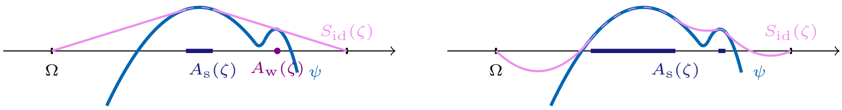
Finally, we show the inclusion $A_\mathrm{s}(\zeta) \subseteq_\mathrm{q} A(\zeta)$. Since $y - \psi$ is quasi lower-semicontinuous and nonnegative, we find an increasing sequence $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)_+$ with $v_n \to y - \psi$ pointwise quasi-everywhere, see Lemma 2.26. With the arguments as in the proof of Lemma 3.7, we have

$$\bigcup_{n\in\mathbb{N}} \{v_n > 0\} =_\mathrm{q} I(\zeta).$$

Now, $\tilde{\xi}(I(\zeta)) = 0$, see Lemma 3.7, implies $v_n = 0$ $\tilde{\xi}$-a.e. on Ω for all $n \in \mathbb{N}$ and using Lemma 2.34 we conclude $v_n = 0$ q.e. in $A_\mathrm{s}(\zeta)$. By pointwise q.e. convergence of $(v_n)_{n\in\mathbb{N}}$ to $y - \psi$ we conclude $y = \psi$ q.e. on $A_\mathrm{s}(\zeta)$. This shows $A_\mathrm{s}(\zeta) \subseteq_\mathrm{q} A(\zeta)$. □

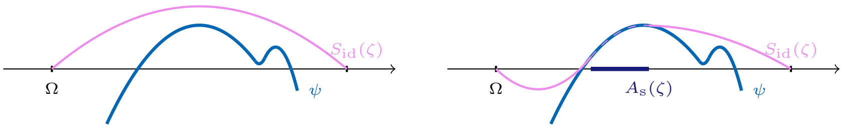
The second equality in (3.12) gives an implicit characterization of the strictly active set $A_\mathrm{s}(\zeta)$ and the second characterization of the critical cone describes the pointwise behaviour of its elements.

We often need the following properties of the strictly active set $A_\mathrm{s}(\zeta)$ which are also contained in the proof of Theorem 3.9.



(a) Strictly and weakly active set for force term $\zeta = 0$

(b) Strictly and weakly active set for force term $\zeta \leq 0$



(c) Strictly and weakly active set for force term $\zeta \geq 0$

(d) Strictly and weakly active set for force term ζ

Figure 3.3. Strictly and weakly active sets for different force terms, compare Fig. 3.1

Corollary 3.10 *Suppose the conditions of Assumption 3.2 are satisfied. Let $\zeta \in H^{-1}(\Omega)$ be arbitrary and denote by $y := S_{\mathrm{id}}(\zeta)$ and by $\xi := Ly - \zeta$. Let $z \in H_0^1(\Omega)$. Then the following statements are equivalent*

(i) *It holds $z = 0$ q.e. in $A_{\mathrm{s}}(\zeta) =_{\mathrm{q}} \operatorname{f-supp}(\tilde{\xi})$.*

(ii) *It holds $z = 0$ $\tilde{\xi}$-a.e. in Ω.*

and imply $\langle \xi, z \rangle = 0$. Here, $\tilde{\xi} \in \mathcal{M}_+(\Omega)$ is as in Lemma 3.7.

Proof. Since $A_{\mathrm{s}}(\zeta) = \operatorname{f-supp}(\tilde{\xi})$, the equivalence of the two statements is described in Lemma 2.34. The implication follows from the identification of ξ with the measure $\tilde{\xi}$, see Lemma 3.7(1.). □

For $\zeta \in H^{-1}(\Omega)$, we also define the weakly active set $A_{\mathrm{w}}(\zeta)$ as the complement of the strictly active set in the active set, i.e., we set

$$A_{\mathrm{w}}(\zeta) :=_{\mathrm{q}} A(\zeta) \setminus A_{\mathrm{s}}(\zeta).$$

In Fig. 3.3, an example is shown where the strictly active and the weakly active sets are illustrated for four different elements $\zeta \in H^{-1}(\Omega)$. We can interpret the strictly active set as the part of the active set where there is a pressure between the membrane and the obstacle.

CHAPTER 4

Generalized derivatives for the composition with an operator

In this chapter, we construct generalized derivatives in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(\cdot)$ (see Definition 2.12) for the composition of the solution operator of the obstacle problem with a general monotone and continuously differentiable operator $f \colon U \to H^{-1}(\Omega)$ on a control Banach space U with suitable properties. This composition is the solution operator $S_f := S_{\mathrm{id}} \circ f \colon U \to H_0^1(\Omega)$ of the generalized obstacle problem

$$\text{Find } y \in K_\psi : \quad \langle Ly - f(u), z - y \rangle \geq 0 \quad \forall\, z \in K_\psi \qquad (\mathrm{OP}_f)$$

with forces $f(U) \subseteq H^{-1}(\Omega)$. In particular, only a subset of $H^{-1}(\Omega)$, more precisely the image $f(U)$ in $H^{-1}(\Omega)$, can be realized as input data of the variational inequality (OP_f). In optimization problems, it is often the case that the set of admissible right-hand sides is a restricted subset of $H^{-1}(\Omega)$ and this situation can be covered by the formulation in (OP_f). Of course, the formulation (OP_f) includes also the basic obstacle problem with distributed controls by considering $f = \mathrm{id} \colon H^{-1}(\Omega) \to H^{-1}(\Omega)$.

The admissible set K_ψ in (OP_f) is defined as

$$K_\psi := \{z \in H_0^1(\Omega) \mid z \geq \psi \text{ q.e. in } \Omega\}.$$

Throughout this chapter, we assume that $\Omega \subseteq \mathbb{R}^d$ is open and bounded, and $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is coercive. The obstacle $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is a quasi upper-semicontinuous function which is chosen such that the admissible set K_ψ is nonempty.

In addition, we sometimes use that L is strictly T-monotone, which is the statement of the following assumption. Recall the definition of strict T-monotonicity from (3.3).

Assumption 4.1 *We assume that $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is strictly T-monotone operator.*

The assumptions on the obstacle ψ are minimal and allow for a wide class of possible functions including thin obstacles, where the inequality constraints $y, z \geq \psi$ are prescribed only on a small subset of Ω, possibly a subset of capacity zero. Recall that coercivity of the operator L is needed to show existence and uniqueness of a solution of (OP_f) and strict T-monotonicity implies that the solution operator S_{id} of (OP_f) with $f = \mathrm{id}\colon H^{-1}(\Omega) \to H^{-1}(\Omega)$ is increasing, see Section 3.2. Of course, if $f\colon U \to H^{-1}(\Omega)$ is defined on a partially ordered Banach space and increasing then also the composition $S_f = S_{\mathrm{id}} \circ f$, the solution operator of (OP_f), is increasing. This monotonicity of S_f is a central property in our arguments in this chapter, since the respective active sets and strictly active sets inherit such a monotonicity property in the sense of inclusions.

In this chapter, U is a always a Banach space and $f\colon U \to H^{-1}(\Omega)$ is an operator that is included in the variational inequality (OP_f). For the main results, we impose the following additional assumptions on U and f which are collected in the subsequent Assumption 4.2.

Assumption 4.2

1. *We assume that the operator $f\colon U \to H^{-1}(\Omega)$ is defined on a partially ordered Banach space $(U, \geq_U)$. In addition, let f be increasing, i.e., $u_1 \geq_U u_2$ implies $f(u_1) \geq f(u_2)$ in $H^{-1}(\Omega)$.*
2. *The operator f is continuously differentiable.*
3. *U is separable and there is a partially ordered Banach space $(V, \geq_V)$ such that the positive cone $\mathcal{P} = \{v \in V : v \geq_V 0\}$ has nonempty interior*

and V is embedded into U. The order relation $\geq_V$ has the property that for all $v, w \in V$ with $v \geq_V w$ it holds $v + z \geq_V w + z$ for all $z \in V$ and $t\,v \geq_V t\,w$ for all $t \geq 0$. We assume that the linear embedding $\iota\colon V \to U$ is continuous, dense and increasing, i.e., compatible with the order structures in V and U. This means that $v \in V$ with $v \geq_V 0$ implies $\iota(v) \geq_U 0$ in U.

We give some examples of operators f and (control) spaces U, V that can enter the optimal control problem (OP_f). For example, the operator f can realize controls given as $L^2(\Omega)$ functions with support in an open subset $\tilde{\Omega}$ of Ω by choosing $U = L^2(\tilde{\Omega})$ and f the embedding of $L^2(\tilde{\Omega})$ into $H^{-1}(\Omega)$. In this case, one may choose $V = C_{\mathrm{c}}(\tilde{\Omega})$. Next, the range of f can consist of weighted sums $\sum_{i=1}^{n} u_i \zeta_i$ with fixed nonnegative functionals $\zeta_i \in H^{-1}(\Omega)$, $i = 1, \ldots, n$ by setting $U = V = \mathbb{R}^n$ and $f(u_1, \ldots, u_n) := \sum_{i=1}^{n} u_i \zeta_i$. Moreover, U can be a closed linear subspace of $H^{-1}(\Omega)$ satisfying Assumption 4.2, where f is the embedding of U into $H^{-1}(\Omega)$. Hence, in particular also different types of sparse controls are possible. In particular, the assumptions on f include also the choices $U = H^{-1}(\Omega)$, $f(u) = \mathrm{id}(u) = u$ and $U = L^2(\Omega)$, $f\colon L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, which are interesting from a theoretical point of view.

At the beginning of our analysis, we will see that the Gâteaux derivatives of S_f in points $u \in U$ at which S_f is Gâteaux differentiable are characterized as solution operators of variational equations. The admissible sets in these variational equations are sets $H_0^1(D)$ where the domain D is any quasi-open set satisfying $I(f(u)) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$. Here, we use the notation $I(f(u))$ for the inactive set in $f(u) \in H^{-1}(\Omega)$ as introduced in Section 3.2.1 and $A_{\mathrm{s}}(f(u))$ for the strictly active set in $f(u)$ introduced in Theorem 3.9.

Now, for an arbitrary element $u \in U$, the goal is to construct an element in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$, which is defined as in Definition 2.12. This means, we are looking for limits of the derivatives $(S_f'(u_n))_{n\in\mathbb{N}}$ in the strong operator topology for sequences $(u_n)_{n\in\mathbb{N}} \subseteq U$ of points at which S_f is Gâteaux differentiable and which converge to u. The tool to analyze the convergence of $(S_f'(u_n))_{n\in\mathbb{N}}$ and to find a characterization of the limit is Mosco convergence. This notion of convergence for sequences of nonempty closed convex (admissible) sets is related to the convergence of the solutions of the respective variational inequality, see [Mos69]. In fact, the convergence of $(H_0^1(D_n))_{n\in\mathbb{N}}$ in the sense of Mosco implies the convergence of the Gâteaux derivatives $(S'_f(u_n))_{n\in\mathbb{N}}$ which are shown to be solution operators of variational equations.

Thus, one of the main tasks in this chapter is to analyze the Mosco convergence of the admissible sets. We will see that a particular monotonicity of the sequence $(u_n)_{n\in\mathbb{N}}$ ensures the Mosco convergence of the sets $H_0^1(I(f(u_n)))$ to $H_0^1(I(f(u)))$ and the opposite monotonicity ensures the Mosco convergence of the sets $H_0^1(\Omega \setminus A_s(f(u_n)))$ to $H_0^1(\Omega \setminus A_s(f(u)))$. Since the assumptions on the positive cone in U allow us to apply Rademacher's theorem and to construct suitable increasing and decreasing sequences $(u_n)_{n\in\mathbb{N}}$ of points at which the locally Lipschitz continuous operator S_f is Gâteaux differentiable and that converge to u, these considerations lead us to generalized derivatives in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$.

The outline of the chapter is as follows. We derive characterizations of Gâteaux derivatives $S'_f(u)$ as solution operators of variational equations in points u at which S_f is Gâteaux differentiable. The admissible sets in the variational equations are Sobolev spaces $H_0^1(D)$ for quasi-open sets D satisfying $I(f(u)) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$. Moreover, the relation between Gâteaux differentiability of S_f in u and the validity of the strict complementarity condition $A(f(u)) =_{\mathrm{q}} A_{\mathrm{s}}(f(u))$ is discussed in Section 4.1.1. In Section 4.2, the tool of Mosco convergence is introduced and its relevance in the analysis of convergence of solutions to variational inequalities is recapitulated. Using Rademacher's theorem in infinite dimensions, we show in Section 4.3 that increasing and decreasing convergent sequences in U where the solution operator S_f is Gâteaux differentiable exist for arbitrary limits. The set-valued maps $u \mapsto H_0^1(I(f(u)))$ and $u \mapsto H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ are analyzed in Section 4.4. In Section 4.4.1, it is shown that these maps are also monotone which implies that $(H_0^1(I(f(u_n))))_{n\in\mathbb{N}}$ and $(H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))))_{n\in\mathbb{N}}$ are increasing or decreasing sequences if $(u_n)_{n\in\mathbb{N}}$ is chosen increasing, respectively decreasing. The role of suitable monotonicity properties in the study of Mosco convergence is motivated and demonstrated by means of an example in Section 4.4.2. The Mosco convergence of the admissible sets $(H_0^1(D_n))_{n\in\mathbb{N}}$ for $D_n :=_{\mathrm{q}} I(f(u_n))$ or $D_n :=_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u_n))$ using increasing, respectively, decreasing sequences $(u_n)_{n\in\mathbb{N}}$ is established in Section 4.5. We summarize our results and obtain generalized derivatives in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$ in Section 4.6. From the characterization of these generalized derivatives, we easily obtain subgradients in Clarke's generalized differential when considering the reduced objective function w.r.t. a continuously differentiable objective function $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ and the state equation $y = S_f(u)$. A representation of such Clarke subgradients is obtained in Section 4.7.

A huge part of the results in this chapter is based on the paper [RU19]. The newer results analyzing the monotonicity and the Mosco convergence properties of the sets $H_0^1(\Omega \setminus A_\mathrm{s}(f(u)))$ appear in [RU20].

4.1 Characterization of Gâteaux derivatives

In Theorem 3.9, we have seen that the directional derivatives $S_\mathrm{id}'(\zeta; p)$ for the solution operator S_id of $(\mathrm{OP}_\mathrm{id})$ on $H^{-1}(\Omega)$ and for $\zeta, p \in H^{-1}(\Omega)$ are given by the solution of the variational inequality

$$\text{Find } \delta \in \mathcal{K}_{K_\psi}(y,\xi): \quad \langle L\delta - p, z - \delta\rangle \geq 0 \quad \forall\, z \in \mathcal{K}_{K_\psi}(y,\xi). \tag{4.1}$$

Here $y := S_\mathrm{id}(\zeta)$ and $\xi := Ly - \zeta$. Moreover, we have seen that the critical cone $\mathcal{K}_{K_\psi}(y,\xi)$ has the following structure

$$\mathcal{K}_{K_\psi}(y,\xi) = \left\{ z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. on } A(\zeta), z = 0 \text{ q.e. on } A_\mathrm{s}(\zeta) \right\}. \tag{4.2}$$

In order to obtain the directional derivative for the solution operator of (OP_f) on U, i.e., for the composite mapping $S_f = S_\mathrm{id} \circ f$ for an operator $f\colon U \to H^{-1}(\Omega)$ as specified before, we will apply a chain rule for the directional derivatives.

For arbitrary directionally differentiable mappings the chain rule does not hold. Thus, we use the stronger form of Hadamard directional differentiability, as introduced in Definition 2.3.

We obtain the following corollary on directional differentiability in the Hadamard sense for the solution operator S_id of (OP_f) for $f = \mathrm{id}$.

Proposition 4.1 *The solution operator* $S_\mathrm{id}\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ *of* $(\mathrm{OP}_\mathrm{id})$ *is* *directionally differentiable in the Hadamard sense.*

Proof. Since S_{id} is Lipschitz continuous, see Theorem 3.4, the statement follows from Proposition 2.5. □

Applying the chain rule from Lemma 2.6, we obtain the following result.

Proposition 4.2 *Assume that the operator $f\colon U \to H^{-1}(\Omega)$ is directionally differentiable. Then the solution operator $S_f\colon U \to H_0^1(\Omega)$ of (OP_f) is directionally differentiable. For given $u, h \in U$, the directional derivative $S_f'(u; h)$ is given by the solution of the variational inequality*

$$\textit{Find } \delta \in \mathcal{K}_{K_\psi}(y,\xi): \quad \langle L\delta - f'(u;h), z - \delta\rangle \geq 0 \quad \forall\, z \in \mathcal{K}_{K_\psi}(y,\xi). \tag{4.3}$$

Here, $y := S_f(u)$, $\xi := LS_f(u) - f(u)$ and $f'(u;h)$ denotes the directional derivative of f in $u \in U$ and in direction $h \in U$. Moreover, we have

$$\mathcal{K}_{K_\psi}(y,\xi) = \left\{z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. on } A(f(u)), z = 0 \text{ q.e. on } A_{\mathrm{s}}(f(u))\right\}. \tag{4.4}$$

Proof. This follows from the Hadamard directional derivative of S_{id}, see (4.1), (4.2), and the chain rule in Lemma 2.6. □

We now specify the behavior of S_f' in points where S_f is Gâteaux differentiable.

Theorem 4.3 *Assume that $f\colon U \to H^{-1}(\Omega)$ is directionally differentiable. Suppose that the solution operator S_f of (OP_f) is Gâteaux differentiable at $u \in U$ and let $h \in U$ be arbitrary. Then the directional derivative $S_f'(u;h)$ is determined by the solution of the variational equation*

$$\textit{Find } \delta \in H_0^1(D): \quad \langle L\delta - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(D). \tag{4.5}$$

Here, any quasi-open set D with $I(f(u)) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$ is admissible and provides the same solution δ.

Proof. The assumption that u is a point where S_f is Gâteaux differentiable implies that $S_f'(u;\cdot)$ is linear and the image is a linear subspace of $H_0^1(\Omega)$.

By the characterization (4.3) in Proposition 4.2, the image of $S_f(u;\cdot)$ lies in a linear subspace of the critical cone $\mathcal{K}_{K_\psi}(y,\xi)$ for $y = S_f(u)$, $\xi = Ly - f(u)$. The structure of the critical cone, cf. (4.4), implies that $S'_f(u;h) \in H^1_0(I(f(u)))$ for all $h \in U$, since $H^1_0(I(f(u)))$ is the largest linear subset contained in the critical cone. Thus, for all $h \in U$ it holds

$$\langle LS'_f(u;h) - f'(u;h), z - S'_f(u;h)\rangle \geq 0 \quad \forall\, z \in H^1_0(I(f(u))) \subseteq \mathcal{K}_{K_\psi}(y,\xi).$$

Since $H^1_0(I(f(u)))$ is a linear subspace, the variational inequality becomes a variational equation and thus $S'_f(u;h)$ is determined by the unique solution of the variational equation (4.5) for $D =_{\mathrm{q}} I(f(u))$.

On the other hand, the image of $S_f(u;\cdot)$ is also contained in the linear hull of the critical cone $\mathcal{K}_{K_\psi}(y,\xi)$, the set $H^1_0(\Omega \setminus A_{\mathrm{s}}(f(u)))$. We argue that the inequality

$$\langle LS'_f(u;h) - f'(u;h), z - S'_f(u;h)\rangle \geq 0$$

is fulfilled for all test functions z from $H^1_0(\Omega \setminus A_{\mathrm{s}}(f(u)))$, and not only from the subset $\mathcal{K}_{K_\psi}(y,\xi)$.

Fix $z \in \mathcal{K}_{K_\psi}(y,\xi)$ and take an arbitrary $h \in U$. Then the test function z fulfills the variational inequality for the direction $-h$, namely

$$\langle LS'_f(u;-h) - f'(u;-h), z - S'_f(u;-h)\rangle \geq 0$$

or, equivalently,

$$\langle LS'_f(u;h) - f'(u;h), -z - S'_f(u;h)\rangle \geq 0.$$

This shows that $-z$ is also an admissible test function for the direction h. Now, consider an arbitrary $z \in H^1_0(\Omega \setminus A_{\mathrm{s}}(f(u)))$. Then we can write $z = z_+ - z_-$, where $z_+ = \max(0,z)$ denotes the positive part and $z_- = -\min(0,z)$ the negative part of z. We conclude $z_+, z_- \in \mathcal{K}_{K_\psi}(y,\xi)$. Since $\mathcal{K}_{K_\psi}(y,\xi)$ is a cone, $2z_+$, respectively $2z_-$, are in $\mathcal{K}_{K_\psi}(y,\xi)$ and it holds

$$\langle LS'_f(u;h) - f'(u;h), 2z_+ - S'_f(u;h)\rangle \geq 0$$

and

$$\langle LS'_f(u;h) - f'(u;h), -2z_- - S'_f(u;h)\rangle \geq 0$$

for all $h \in U$. Adding up both inequalities and dividing by 2 yields

$$\langle LS'_f(u;h) - f'(u;h), z - S'_f(u;h)\rangle \geq 0.$$

Therefore, each $z \in H_0^1(\Omega \setminus A_\mathrm{s}(f(u)))$ is a valid test function and $S'_f(u;h)$ is the unique solution of the variational equation (4.5) for $D =_\mathrm{q} \Omega \setminus A_\mathrm{s}(f(u))$.

Consider now an arbitrary quasi-open set D with $I(u) \subseteq_\mathrm{q} D \subseteq_\mathrm{q} \Omega \setminus A_\mathrm{s}(f(u))$. Then we have $H_0^1(I(f(u))) \subseteq H_0^1(D) \subseteq H_0^1(\Omega \setminus A_\mathrm{s}(f(u)))$ and together with what we have shown already, this implies that for arbitrary $h \in U$, $S'_f(u;h)$ is the solution of (4.5). □

4.1.1 Relation to strict complementarity

In this subsection, we discuss the connection between Gâteaux differentiability of the solution operator S_f of (OP$_f$) and the validity of the so-called strict complementarity condition. We say that the *strict complementarity condition* holds in $f(u) \in H^{-1}(\Omega)$ if the equality $A(f(u)) =_\mathrm{q} A_\mathrm{s}(f(u))$ holds (up to a set of capacity zero).

From Proposition 4.2 and in particular, from the characterization of the critical cone in (4.4), it follows that strict complementarity in $f(u)$ implies Gâteaux differentiability of S_f in $u \in U$, since the critical cone turns into a linear subset. Beyond that, we obtain the following result, which states that for certain operators f the strict complementarity condition is also necessary for Gâteaux differentiability. For parts of this result, we refer to [RW20, Lem. 2.6].

Lemma 4.4 *Assume $f\colon U \to H^{-1}(\Omega)$ is Gâteaux differentiable at $u \in U$. If the strict complementarity condition holds in $f(u) \in H^{-1}(\Omega)$, then the solution operator $S_f\colon U \to H_0^1(\Omega)$ of* (OP$_f$) *is Gâteaux differentiable at $u \in U$.*

Moreover, if ι is a continuous and dense linear embedding $\iota\colon U \hookrightarrow H^{-1}(\Omega)$, then the solution operator $S_\iota\colon U \to H_0^1(\Omega)$ is Gâteaux differentiable at $u \in U$ if and only if the strict complementarity conditions holds in $\iota(u)$.

Proof. Let us assume $A(f(u)) =_{\mathrm{q}} A_{\mathrm{s}}(f(u))$. Then the critical cone $\mathcal{K}_{K_\psi}(y,\xi) = \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } A(f(u))\} = H_0^1(I(f(u)))$ is a linear subspace and the variational inequality (4.3) for the directional derivative $S_f'(u;h)$ reduces to

$$\text{Find } \delta \in \mathcal{K}_{K_\psi}(y,\xi): \quad \langle L\delta - f'(u;h), z\rangle = 0 \quad \forall z \in \mathcal{K}_{K_\psi}(y,\xi).$$

Now, linearity of $S_f'(u;\cdot)$ follows from linearity of $f'(u;\cdot)$ and from the uniqueness of solutions of variational inequalities. Moreover, boundedness of $S_f'(u;\cdot)$ follows from boundedness of $f'(u;\cdot)$, since

$$\begin{aligned}\alpha^2 \|S_f'(u;h)\|^2_{H_0^1(\Omega)} &\le \langle L S_f'(u;h), S_f'(u;h)\rangle \\ &= \langle f'(u;h), S_f'(u;h)\rangle \\ &\le \|f'(u;h)\|_{H^{-1}(\Omega)} \|S_f'(u;h)\|_{H_0^1(\Omega)}.\end{aligned}$$

Thus, if the strict complementarity condition holds in $f(u)$, then S_f is Gâteaux differentiable at $u \in U$.

For the second statement of the lemma, we first show that if S_{id} is Gâteaux differentiable at $\zeta \in H^{-1}(\Omega)$, then the strict complementarity condition $A(\zeta) =_{\mathrm{q}} A_{\mathrm{s}}(\zeta)$ holds. Thus, let us assume that S_{id} is Gâteaux differentiable at $\zeta \in H^{-1}(\Omega)$. Denote $y := S_{\mathrm{id}}(\zeta)$ and $\xi := Ly - \zeta$. Arguing as in the proof of Theorem 4.3, we see that the image of $S_{\mathrm{id}}'(\zeta;\cdot)$ is contained in $\mathcal{K}_{K_\psi}(y,\xi)$. Conversely, let $v \in \mathcal{K}_{K_\psi}(y,\xi)$ be arbitrary. We know that $S_{\mathrm{id}}'(\zeta; Lv)$ is the solution of

$$\text{Find } \delta \in \mathcal{K}_{K_\psi}(y,\xi): \quad \langle L\delta - Lv, z - \delta\rangle \ge 0 \quad \forall\, z \in \mathcal{K}_{K_\psi}(y,\xi). \tag{4.6}$$

Thus, we can derive $S_{\mathrm{id}}'(\zeta; Lv) = v$, since v solves (4.6). This implies that $\mathcal{K}_{K_\psi}(y,\xi)$ coincides with the image of $S_{\mathrm{id}}'(\zeta;\cdot)$. Thus, $\mathcal{K}_{K_\psi}(y,\xi)$ is a linear subspace of $H_0^1(\Omega)$ and we have shown $\mathcal{K}_{K_\psi}(y,\xi) = H_0^1(I(\zeta))$, since $H_0^1(I(\zeta))$

is the largest linear subset of $\mathcal{K}_{K_\psi}(y,\xi)$. In summary, we have

$$H_0^1(I(\zeta)) = \mathcal{K}_{K_\psi}(y,\xi) \subseteq H_0^1(\Omega \setminus A_\mathrm{s}(\zeta)).$$

Now, for $v \in H_0^1(\Omega \setminus A_\mathrm{s}(\zeta))_+$, we have $v \geq 0$ q.e. in Ω. This implies $v \in \mathcal{K}_{K_\psi}(y,\xi)$ by the structure of the critical cone, see (4.2), and, since $\mathcal{K}_{K_\psi}(y,\xi)$ is a linear subspace, we also have $-v \in \mathcal{K}_{K_\psi}(y,\xi)$. This leads to $v \geq 0$ and $v \leq 0$ q.e. on $A(\zeta)$. Therefore, $v \in H_0^1(I(\zeta))$. This shows

$$H_0^1(I(\zeta)) = \mathcal{K}_{K_\psi}(y,\xi) = H_0^1(\Omega \setminus A_\mathrm{s}(\zeta)). \tag{4.7}$$

By Lemma 2.31, we find $v \in H_0^1(\Omega)_+$ satisfying $\{v > 0\} =_\mathrm{q} \Omega \setminus A_\mathrm{s}(u)$. In particular, v is an element of $H_0^1(\Omega \setminus A_\mathrm{s}(\zeta))$ and by (4.7) also of $H_0^1(I(\zeta))$, i.e., $v = 0$ q.e. on $A(\zeta) \supseteq_\mathrm{q} A(\zeta) \setminus A_\mathrm{s}(\zeta)$. By choice of v this implies $A(\zeta) =_\mathrm{q} A_\mathrm{s}(\zeta)$.

We have shown that the strict complementarity condition holds in $\zeta \in H^{-1}(\Omega)$ if and only if S_id is Gâteaux differentiable at ζ.

Assume now S_ι is Gâteaux differentiable at $u \in U$. We apply Lemma 2.7 and deduce that S_id is Gâteaux differentiable at $\iota(u)$.

By what we have shown, the strict complementarity condition holds in $\iota(u)$, i.e., $A(\iota(u)) =_\mathrm{q} A_\mathrm{s}(\iota(u))$. □

Remark 4.5 In particular, if $\Omega \subseteq \mathbb{R}^d$ is bounded, any space $L^q(\Omega)$ with $q \geq 2$ is densely embedded into $H^{-1}(\Omega)$. Moreover, $C_\mathrm{c}^\infty(\Omega)$ is densely embedded into $H^{-1}(\Omega)$. Denote the respective embeddings by $\iota\colon U \hookrightarrow H^{-1}(\Omega)$. We have shown that S_ι is Gâteaux differentiable at u if and only if $A(u) =_\mathrm{q} A_\mathrm{s}(u)$.

Example 4.6 For f in the generality of Assumption 4.2 the strict complementarity condition is not necessary for Gâteaux differentiability. As a counterexample consider the following setting. Let $p \in H^{-1}(\Omega)$ such that the strict complementarity condition is not satisfied in p. Now, consider the constant map

$$f\colon L^2(\Omega) \to H^{-1}(\Omega), v \mapsto p.$$

Then f fulfills the assumptions of Assumption 4.2 and S_f is constant and thus Gâteaux differentiable. Yet, the strict complementarity condition is not fulfilled in p.

4.2 Mosco convergence of admissible sets

The goal of this section is to introduce the notion of Mosco convergence as a tool to show convergence of solutions to varitional inequalities. In order to obtain elements of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$, the convergence analysis of Gâteaux derivatives $(S_f'(u_n))_{n\in\mathbb{N}}$ in sequences $(u_n)_{n\in\mathbb{N}}$ with $u_n \to u$ is indispensable. We have seen in Theorem 4.3 that the Gâteaux derivatives $S_f'(u_n)$ are solution operators of variational equations. Thus, a criterion on convergence of solutions to variational equations, or, more generally, variational inequalities, is of high interest for our purpose.

We introduce the notion of Mosco convergence in the following definition. The concept of this convergence for closed convex sets goes back to [Mos69]. In this form, the definition can be found in [Rod87, Ch. 4:4].

Definition 4.7 (Mosco convergence) We say that a sequence $(C_n)_{n\in\mathbb{N}}$ of nonempty, closed, convex subsets of a Banach space X converges to a set $C \subseteq X$ in the sense of Mosco if the following two conditions hold.

1. For each $x \in C$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in C_n$ holds for every $n \in \mathbb{N}$ and such that $x_n \to x$ in X as $n \to \infty$.
2. Assume $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ fulfilling $x_n \in C_n$ for all $n \in \mathbb{N}$. If for some $x \in X$ we have $x_{n_k} \rightharpoonup x$ in X, then the weak limit x is an element of C.

If $(C_n)_{n\in\mathbb{N}}$ converges towards C in the sense of Mosco, then we also write $C_n \to C$.

Note that the limit of a Mosco convergent sequence $(C_n)_{n\in\mathbb{N}}$ is unique. To see this, let $C, \tilde{C}$ be limits and $x \in C$. Now, choose $x_n \in C_n$ with $x_n \to x$. Then, in particular we find $x_n \rightharpoonup x$ and thus $x \in \tilde{C}$.

The following lemma characterizes the Mosco limit of increasing and decreasing sequences of sets, respectively. It can also be found in [Mos69, Lem. 1.2, 1.3].

Lemma 4.8 *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of nonempty, closed, convex subsets of a Banach space X.*

1. *If $(C_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets, it follows*

$$C_n \to \overline{\bigcup_{n\in\mathbb{N}} C_n}$$

in the sense of Mosco.

2. *If $(C_n)_{n\in\mathbb{N}}$ is decreasing, it holds*

$$C_n \to \bigcap_{n\in\mathbb{N}} C_n$$

in the sense of Mosco.

Proof. 1. Let $x \in \overline{\bigcup_{n\in\mathbb{N}} C_n}$. This means, x is the limit of some sequence $(y_m)_{m\in\mathbb{N}} \subseteq \bigcup_{n\in\mathbb{N}} C_n$. Since $(C_n)_{n\in\mathbb{N}}$ is increasing, there is a subsequence $(C_{n_m})_{m\in\mathbb{N}}$ with $y_m \in C_{n_m}$ for all $m \in \mathbb{N}$. Thus, for all $n \geq n_1$, we may set

$$x_n := y_{\max\{m\in\mathbb{N}|n_m\leq n\}} \in C_{n_{\max\{m\in\mathbb{N}|n_m\leq n\}}} \subseteq C_n.$$

For $n < n_1$, we choose $x_n \in C_n$ arbitrarily. It is clear that the constructed sequence $(x_n)_{n\in\mathbb{N}}$ converges to x, as well.

To show the second condition of Mosco convergence, let $(x_n)_{n\in\mathbb{N}}$ be a sequence satisfying $x_n \in C_n$ for all $n \in \mathbb{N}$ and assume $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence with $x_{n_k} \rightharpoonup x$ in X for some $x \in X$. From $C_m \subseteq \overline{\bigcup_{n\in\mathbb{N}} C_n}$ for all $m \in \mathbb{N}$ we conclude $(x_{n_k})_{k\in\mathbb{N}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} C_n}$. Now, $\bigcup_{n\in\mathbb{N}} C_n$ is convex. To see this, let $y_1, y_2 \in \bigcup_{n\in\mathbb{N}} C_n$. Then we find n_1 and n_2 such that $y_1 \in C_{n_1}$ and $y_2 \in C_{n_2}$. Since $(C_n)_{n\in\mathbb{N}}$ is increasing, we conclude $y_1, y_2 \in C_{\max(n_1,n_2)}$ and, by convexity of this set, any convex combination of y_1, y_2 is in $C_{\max(n_1,n_2)} \subseteq \bigcup_{n\in\mathbb{N}} C_n$. Thus, since the set $\bigcup_{n\in\mathbb{N}} C_n$ is convex, its closure is convex, too. By Mazur's lemma, we conclude that the weak limit of $(x_{n_k})_{k\in\mathbb{N}}$ is also in the closed convex set $\overline{\bigcup_{n\in\mathbb{N}} C_n}$.

2. Let $x \in \bigcap_{n\in\mathbb{N}} C_n$. From $\bigcap_{n\in\mathbb{N}} C_n \subseteq C_m$ for all $m \in \mathbb{N}$ we conclude that the sequence $x_n := x$ satisfies $x_n \in C_n$ for all $n \in \mathbb{N}$ as well as $x_n \to x$ in X as $n \to \infty$.

To show the second property of Mosco convergence, let $(x_n)_{n\in\mathbb{N}}$ be a sequence satisfying $x_n \in C_n$ for all $n \in \mathbb{N}$ and assume $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence with $x_{n_k} \rightharpoonup x$ in X for some $x \in X$. Recall that $(n_k)_{k\in\mathbb{N}}$ is strictly increasing. For all $m \in \mathbb{N}$, also the sequence $(x_{n_k})_{k=m}^{\infty}$ converges weakly to x and it holds $(x_{n_k})_{k=m}^{\infty} \subseteq C_{n_m}$ since $(C_n)_{n\in\mathbb{N}}$ is decreasing. By Mazur's lemma, we conclude that $x \in C_{n_m}$ for all $m \in \mathbb{N}$, i.e., $x \in \bigcap_{m\in\mathbb{N}} C_{n_m}$. Using once more that $(C_n)_{n\in\mathbb{N}}$ is decreasing, we have

$$\bigcap_{m\in\mathbb{N}} C_{n_m} = \bigcap_{n\in\mathbb{N}} C_n$$

and $x \in \bigcap_{n\in\mathbb{N}} C_n$ follows. Thus, we have shown the Mosco convergence of $(C_n)_{n\in\mathbb{N}}$ towards $\bigcap_{n\in\mathbb{N}} C_n$.

□

The following proposition establishes the connection between Mosco convergence of the admissible sets in variational inequalities and the convergence of the solutions to these variational inequalities. The statement and its proof is taken from [Rod87, Thm. 4.1], see also [Mos69, Prop. 3.5]. The formulation in Proposition 4.9 is adapted to our setting with $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$, but of course, it also holds in more general Hilbert space settings. Since this result is used frequently within this thesis, we also present the proof from [Rod87] here.

Proposition 4.9 *Let C_n, $n \in \mathbb{N}$, and C be nonempty, closed, convex subsets of $H_0^1(\Omega)$ such that $C_n \to C$ in the sense of Mosco. Furthermore, let $(p_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ and $p \in H^{-1}(\Omega)$ with $p_n \to p$. Then the solutions of*

$$\text{Find } x_n \in C_n: \quad \langle Lx_n - p_n, z_n - x_n\rangle \geq 0 \quad \forall\, z_n \in C_n \tag{4.8}$$

converge to the solution of

$$\text{Find } x \in C: \qquad \langle Lx - p, z - x\rangle \geq 0 \quad \forall\, z \in C. \tag{4.9}$$

Proof. As already mentioned, this proof is taken from [Rod87, Thm. 4.1]. Denote by x the solution of (4.9). Then, by the definition of Mosco convergence we find $\eta_n \in C_n$, $n \in \mathbb{N}$, such that $\eta_n \to x$ in $H_0^1(\Omega)$. Denote by $\alpha > 0$

the coercivity constant of L. Using the coercivity and (4.8) with $z_n = \eta_n$ we conclude

$$\begin{aligned} \alpha\|x_n - \eta_n\|^2 &\leq \langle Lx_n - L\eta_n, x_n - \eta_n\rangle \\ &= \langle Lx_n - p_n, x_n - \eta_n\rangle - \langle L\eta_n - p_n, x_n - \eta_n\rangle \\ &\leq -\langle L\eta_n - p_n, x_n - \eta_n\rangle \\ &\leq \|L\eta_n - p_n\|\|x_n - \eta_n\|. \end{aligned} \tag{4.10}$$

This implies

$$\begin{aligned} \|x_n\| &\leq \|x_n - \eta_n\| + \|\eta_n\| \\ &\leq \alpha^{-1}\|L\eta_n - p_n\| + \|\eta_n\| \\ &\leq \kappa \end{aligned}$$

for a constant $\kappa > 0$, since the convergent sequences $(\eta_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ and $(L\eta_n - p_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ are bounded. Now, the boundedness of $(x_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ implies the weak convergence of a subsequence $(x_{n_k})_{k\in\mathbb{N}}$. We denote the weak limit by η. By the Mosco convergence $C_n \to C$ we conclude that $\eta \in C$.

Let $v \in C$ be arbitrary and let $v_n \in C_n$, $n \in \mathbb{N}$, be elements with $v_n \to v$ in $H_0^1(\Omega)$. Then we have

$$\langle Lx_n - p_n, v_n - x_n\rangle \geq 0$$

and this implies

$$\begin{aligned} \langle Lv_n - p_n, v_n - x_n\rangle &= \langle Lv_n - Lx_n, v_n - x_n\rangle + \langle Lx_n - p_n, v_n - x_n\rangle \\ &\geq \alpha\|v_n - x_n\|^2 + \langle Lx_n - p_n, v_n - x_n\rangle \\ &\geq 0 \end{aligned}$$

for all $n \in \mathbb{N}$. Passing to the limit, we obtain

$$\langle Lv - p, v - \eta\rangle \geq 0 \tag{4.11}$$

since $Lv_{n_k} - p_{n_k} \to Lv - p$ in $H^{-1}(\Omega)$ and $v_{n_k} - x_{n_k} \rightharpoonup v - \eta$ in $H_0^1(\Omega)$

as $k \to \infty$. Since v was arbitrary, (4.11) holds for all $v \in C$. Let $z \in C$ be arbitrary and choose any $\theta \in (0, 1]$. Since C is convex, we can choose $v := \eta + \theta(z - \eta) \in C$ in (4.11) and obtain, after dividing by $\theta > 0$,

$$\langle L(\eta + \theta(z - \eta)) - p, z - \eta\rangle \geq 0.$$

Passing to the limit $\theta \to 0$ yields

$$\langle L\eta - p, z - \eta\rangle \geq 0. \tag{4.12}$$

Since $z \in C$ was arbitrary, (4.12) holds for all $z \in C$ which shows that η is the solution of (4.9), i.e., $\eta = x$.

Now, since $\eta_{n_k} \to x$ and $x_{n_k} \rightharpoonup x$ as $k \to \infty$, we conclude that $x_{n_k} - \eta_{n_k} \rightharpoonup 0$. Using this and recalling

$$\alpha\|x_{n_k} - \eta_{n_k}\|^2 \leq -\langle L\eta_{n_k} - p_{n_k}, x_{n_k} - \eta_{n_k}\rangle \to 0$$

as in (4.10), we conclude that the convergence $x_{n_k} - \eta_{n_k} \overset{k\to\infty}{\to} 0$ is strong in $H_0^1(\Omega)$. This shows that the whole sequence $(x_n)_{n\in\mathbb{N}}$ converges strongly to x. □

The result in Proposition 4.9 has the following implications for our aim to construct elements in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$.

Corollary 4.10 *Assume that U is a separable Banach space and let f satisfy Assumption 4.2(2.). Denote by $S_f\colon U \to H_0^1(\Omega)$ the solution operator of the variational inequality* (OP$_f$). *Let $u \in U$ be arbitrary and assume that*

$$(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_f} := \{v \in U \mid S_f \textit{ is Gâteaux differentiable at } v\}$$

is such that $u_n \overset{n\to\infty}{\to} u$ in U. For each $n \in \mathbb{N}$, let D_n be a quasi-open set with $I(f(u_n)) \subseteq_{\mathrm{q}} D_n \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u_n))$. If $H_0^1(D_n) \to H_0^1(D)$ in the sense of Mosco for some quasi-open set D, then it holds $S_f'(u_n) \to \Xi_D$ in the strong operator topology, where $\Xi_D(h)$ is the solution of the variational equation

$$\textit{Find } \delta \in H_0^1(D): \quad \langle L\delta - f'(u; h), z\rangle = 0 \quad \forall\, z \in H_0^1(D). \tag{4.13}$$

In particular, it holds $\Xi_D \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$.

Proof. Let $h \in U$ be arbitrary. In Theorem 4.3, we have seen that $S_f'(u_n; h)$ is the solution of

$$\text{Find } \delta_n \in H_0^1(D_n): \quad \langle L\delta_n - f'(u_n; h), z\rangle = 0 \quad \forall\, z \in H_0^1(D_n) \tag{4.14}$$

and (4.14) is of the form (4.8) with $C_n = H_0^1(D_n)$ and $p_n = f'(u_n; h)$ for each $n \in \mathbb{N}$. Note that the admissible sets are linear subspaces of $H_0^1(\Omega)$ and the variational inequalities reduce to variational equations. We have $f'(u_n; h) \to f'(u; h)$ since f is continuously differentiable. Now, Proposition 4.9 implies that $S_f'(u_n; h) \to \Xi_D(h)$, where $\Xi_D(h)$ is the solution of (4.13). In particular, $\Xi_D \in \mathcal{L}(U, H_0^1(\Omega))$ and $S_f'(u_n) \to \Xi_D$ in the strong operator topology. By definition of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$, see Definition 2.12, we conclude $\Xi_D \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$. □

As the preceding corollary suggests, our goal for the construction of elements in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$ for an arbitrary $u \in U$ is to find sequences $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_f}$ with $u_n \overset{n\to\infty}{\to} u$ such that $H_0^1(D_n) \to H_0^1(D)$ in the sense of Mosco can be verified for a quasi-open set D_n with $I(f(u_n)) \subseteq_{\mathrm{q}} D_n \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$ and some quasi-open set D.

4.3 Existence of monotone convergent sequences of points at which S_f is Gâteaux differentiable

In this section, we will show the existence of increasing and decreasing sequences of points in U where S_f, the solution operator of the variational inequality (OP_f), is Gâteaux differentiable converging to an arbitrary element $u \in U$. Once the existence is shown, then by the preceding section Section 4.2 and the analysis of the Mosco convergence of the sets $(D_n)_{n\in\mathbb{N}}$ for such a sequence $(u_n)_{n\in\mathbb{N}}$, the characterization of the limit will guide us to an element of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$.

The key assumption to show such a result is the condition that the positive cone in V has interior points and $V \hookrightarrow U$, cf. Assumption 4.2. Together with the generalization of Rademacher's theorem to infinite dimensions, cf. Theorem 2.8, the existence of monotone and convergent sequences in $\mathcal{D}_{S_f}$ can be shown.

Proposition 4.11 *Assume $T\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ is a locally Lipschitz continuous operator which is directionally differentiable. Let the conditions of Assumption 4.2 be satisfied and let $u \in U$ be arbitrary. Then there are sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ such that $T_f := T \circ f$ is Gâteaux differentiable at each u_n^+, u_n^- , $n \in \mathbb{N}$, as well as $u_n^+ \nearrow u$ and $u_n^- \searrow u$.*

Proof. Fix $u \in U$. Denote by $\mathcal{P}$ the positive cone in V and for $r > 0$ and $v \in V$ denote by $B_r(v)$ the open ball of radius r around v in V. For short, we write also B_r instead of $B_r(0)$. Let v^* be an interior point of $\mathcal{P}$. Note that this implies that for all $\lambda > 0$ the element λv^* is an interior point of $\mathcal{P}$. Without loss of generality, assume that $\|v^*\|_V = 1$.

We define $\tilde{T}_f\colon V \to H_0^1(\Omega)$ by

$$\tilde{T}_f(v) := T_f(u + v).$$

In this definition and in the rest of this proof, with a little abuse of notation, we do not write down the embedding $V \hookrightarrow U$ explicitly but regard V as a subset of U. Since V is continuously embedded into U, the operator $\tilde{T}_f$ is locally Lipschitz continuous on V. Therefore, by Theorem 2.8, the set of points in V in which $\tilde{T}_f$ is Gâteaux differentiable is dense in V. We construct a sequence

$$(v_n^+)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{\tilde{T}_f} = \{w \in V \mid \tilde{T}_f \text{ is Gâteaux differentiable at } w\}$$

with $v_n^+ \nearrow 0$ inductively. Then, in the last part of the proof, we will show that $(u_n^+)_{n\in\mathbb{N}}$ with $u_n^+ := u+v_n^+$, $n \in \mathbb{N}$, is a sequence in $\mathcal{D}_{T_f}$ with $u_n^+ \nearrow u$. A sketch visualizing the following inductive construction of $(v_n^+)_{n\in\mathbb{N}}$ is depicted in Fig. 4.1. For $n \in \mathbb{N}$ we define

$$\vartheta_n^+ := -2^{-n} v^*,$$

then each ϑ_n^+ is an element of $-\mathcal{P}$, i.e., $\vartheta_n^+ \leq_V 0$, and the sequence $(\vartheta_n^+)_{n\in\mathbb{N}}$

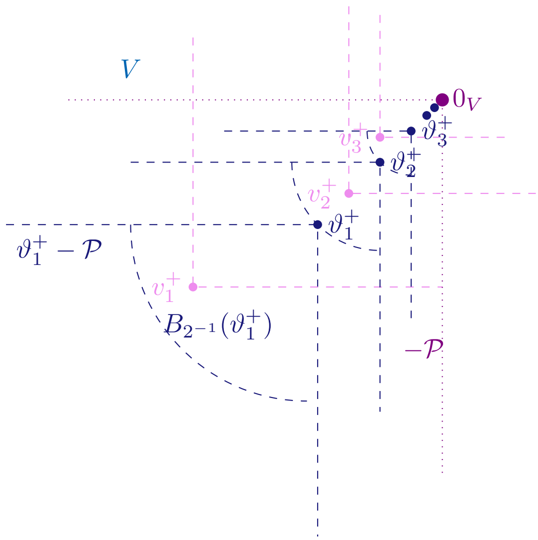


Figure 4.1. Sketch of the construction of the sequence $(v_n^+)_{n\in\mathbb{N}}$ in the proof of Proposition 4.11

is increasing. The element v_n^+ will be chosen in such a way that it satisfies

$$v_{n-1}^+ \leq_V v_n^+ \leq_V \vartheta_n^+.$$

This ensures, on the one hand, that the monotonicity of the sequence $(v_n)_{n\in\mathbb{N}}$ is satisfied up to the n-th member and, on the other hand, that choosing v_{m+1}^+ with $v_m \leq_V v_{m+1}^+ \leq_V 0$ is possible for all $m \geq n$ within the inductive construction. More precisely, the purpose of the elements $(\vartheta_n^+)_{n\in\mathbb{N}}$ is to ensure that nesting v_{n+1}^+ *between* v_n^+ and 0 is possible for all $n \in \mathbb{N}$, i.e., that the sets

$$\{w \in V \mid v_n^+ \leq_V w \leq_V 0\}$$

do all have interior points, so that Theorem 2.8 implies that they contain points in $\mathcal{D}_{\tilde{T}_f}$.

Starting with the first member of the sequence, we find and fix $v_1^+ \in$

$(\vartheta_1^+ - \mathcal{P}) \cap B_{2^{-1}}(\vartheta_1^+)$ where $\tilde{T}_f$ is Gâteaux differentiable. We have

$$\|v_1^+\|_V \leq \|\vartheta_1^+\|_V + \|v_1^+ - \vartheta_1^+\|_V \leq 2^{-1} + 2^{-1} = 1.$$

Now, for $n \geq 2$ assume that we have fixed $v_{n-1}^+ \in \vartheta_{n-1}^+ - \mathcal{P}$ with

$$\|v_{n-1}^+\|_V \leq 2^{-(n-2)}.$$

We argue that in V, the set

$$(\vartheta_n^+ - \mathcal{P}) \cap (v_{n-1}^+ + \mathcal{P}) \cap B_{2^{-n}}(\vartheta_n^+) \tag{4.15}$$

has nonempty interior:

Let $\varepsilon_n > 0$ be such that $B_{\varepsilon_n}(2^{-(n+1)}v^*) \subseteq \mathcal{P}$ holds and let y_n be an arbitrary element of B_{ε_n}. The following arguments show that the element $\vartheta_n^+ - 2^{-(n+1)}v^* + y_n$ is contained in all the sets that are intersected in (4.15) and thus, $\vartheta_n^+ - 2^{-(n+1)}v^* + B_{\varepsilon_n}$ is an open subset of the intersection.

It holds $2^{-(n+1)}v^* + y_n \in \mathcal{P}$, i.e.,

$$y_n \geq_V -2^{-(n+1)}v^* \tag{4.16}$$

and $\vartheta_n^+ - (2^{-(n+1)}v^* - y_n) \leq_V \vartheta_n^+$, therefore

$$\vartheta_n^+ - 2^{-(n+1)}v^* + y_n \in (\vartheta_n^+ - \mathcal{P}).$$

From (4.16), the definition of ϑ_n^+ and from $v_{n-1}^+ \in \vartheta_{n-1}^+ - \mathcal{P}$, i.e., $-v_{n-1}^+ \geq_V -\vartheta_{n-1}^+$, we conclude that

$$\begin{aligned}
\vartheta_n^+ - 2^{-(n+1)}v^* + y_n &= (v_{n-1}^+ - v_{n-1}^+) + \vartheta_n^+ - 2^{-(n+1)}v^* + y_n \\
&\geq_V v_{n-1}^+ - \vartheta_{n-1}^+ + \vartheta_n^+ - 2^{-(n+1)}v^* - 2^{-(n+1)}v^* \\
&= v_{n-1}^+ + 2^{-(n-1)}v^* - 2^{-n}v^* - 2 \cdot 2^{-(n+1)}v^* \\
&= v_{n-1}^+.
\end{aligned}$$

Furthermore, we estimate

$$\|\vartheta_n^+ - 2^{-(n+1)}v^* + y_n - \vartheta_n^+\|_V = \|2^{-(n+1)}v^* - y_n\|_V \leq 2^{-(n+1)} + \varepsilon_n \leq 2^{-n}.$$

Here, in the last step, we have used that $B_{\varepsilon_n}(2^{-(n+1)}v^*) \subseteq \mathcal{P}$ and $\|v^*\| = 1$ imply $\varepsilon_n \leq 2^{-(n+1)}$.

This shows that the set in (4.15) has nonempty interior. Thus, we can find a point in the intersection (4.15), denoted v_n^+, where $\tilde{T}_f$ is Gâteaux differentiable and which then fulfills

$$0 \geq_V \vartheta_n^+ \geq_V v_n^+ \geq_V v_{n-1}^+$$

as well as

$$\|v_n^+\|_V \leq \|\vartheta_n^+\|_V + \|\vartheta_n^+ - v_n^+\|_V \leq 2^{-n} + 2^{-n} = 2^{-(n-1)}.$$

In particular, we obtain a sequence $(v_n^+)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{\tilde{T}_f}$ with $v_n^+ \nearrow 0$ in V. To construct $(v_n^-) \subseteq \mathcal{D}_{\tilde{T}_f}$ with $v_n \searrow 0$, we set $\vartheta_n^- := -\vartheta_n^+ = 2^{-n}v^*$ and choose v_n^- in

$$(\vartheta_n^- + \mathcal{P}) \cap (\vartheta_{n-1}^- - \mathcal{P}) \cap B_{2^{-n}}(\vartheta_n^-)$$

instead of the set in (4.15). Then the remaining arguments are easily adapted to show the existence of $(v_n^-)_{n\in\mathbb{N}}$ with the mentioned properties.

Next, we argue that T_f is Gâteaux differentiable at $u_n^+ := u + v_n^+$, respectively $u_n^- := u + v_n^-$, for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. By the definition of $\tilde{T}_f$, it holds

$$\tilde{T}_f'(v_n^+; h) = T_f'(u_n^+; h) \quad \text{and} \quad \tilde{T}_f'(v_n^-; h) = T_f'(u_n^-; h)$$

for all $h \in V$. Now, it follows as in the proof of Lemma 2.7 that T_f is Gâteaux differentiable at u_n^+ and u_n^-.

By construction, $(u_n^+)_{n\in\mathbb{N}}$ is an increasing and $(u_n^-)_{n\in\mathbb{N}}$ is a decreasing sequence of Gâteaux differentiability points of T_f in U and both sequences converge to u. $\square$

The formulation in Proposition 4.11 is such that the statement can also be applied for applications where T_f is not the solution operator of (OP_f). We will apply the proposition also to the solution operator of the bilateral obstacle problem, see Chapter 6. Of course, we obtain the following corollary for the solution operator S_f of (OP_f).

Corollary 4.12 *Let Assumption 4.2 be satisfied and let $u \in U$ be arbitrary. Then there is an increasing sequence $(u_n^+)_{n\in\mathbb{N}}$ and a decreasing sequence $(u_n^-)_{n\in\mathbb{N}}$ such that the solution operator S_f of (OP_f) is Gâteaux differentiable at each u_n^+, u_n^- and it holds $u_n^+, u_n^- \to u$.*

4.4 Influence of Monotonicity

The purpose of this section is to analyze the monotonicity of the set-valued maps $u \mapsto H_0^1(D)$ for $D :=_{\mathrm{q}} I(f(u))$ and $D :=_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$. In the preceding section, we have seen that for arbitrary $u \in U$ there exists an increasing, respectively decreasing, sequence of points in U where S_f, the solution operator of (OP_f), is Gâteaux differentiable and which converges to u. If we can show the Mosco convergence of the sets $H_0^1(D_n)$ for an increasing or decreasing sequence $(u_n)_{n\in\mathbb{N}}$ and for D_n with $I(f(u_n)) \subseteq_{\mathrm{q}} D_n \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u_n))$ and characterize the limit $H_0^1(D)$, then Corollary 4.10 implies that Ξ as the solution operator of (4.13) is indeed an element of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$. We will motivate why monotonicity is a favorable property of the sequence $(u_n)_{n\in\mathbb{N}}$ when trying to show the Mosco convergence.

Therefore, in Section 4.4.1, we first show the monotonicity properties of the sets $H_0^1(D_n)$ for $D_n :=_{\mathrm{q}} I(f(u_n))$ and $D_n :=_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u_n))$. Afterwards, in Section 4.4.2, by considering an example, it is motivated that the Mosco convergence $H_0^1(D_n) \to H_0^1(D)$ can be shown when combining the suitable monotonicity of $(u_n)_{n\in\mathbb{N}}$ with the suitable characterization of D_n, D as either $I(f(u_n)), I(f(u))$ or $\Omega \setminus A_{\mathrm{s}}(f(u_n)), \Omega \setminus A_{\mathrm{s}}(f(u))$.

4.4.1 Monotonicity of the set-valued maps

Here, we show that $u \mapsto H_0^1(I(f(u)))$ and $u \mapsto H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ are monotone set-valued maps.

For the map $u \mapsto H_0^1(I(f(u)))$, this is an immediate consequence of

Lemma 3.6.

Lemma 4.13 *Suppose Assumption 4.1 is satisfied and let f, U fulfill the conditions of Assumption* 4.2(1.). *Then the set-valued map $U \ni u \mapsto H_0^1(I(f(u)))$ is increasing, i.e., $u_1 \geq_U u_2$ implies $H_0^1(I(f(u_1))) \supseteq H_0^1(I(f(u_2)))$.*

Proof. Assume $u_1, u_2 \in U$ satisfy $u_1 \geq_U u_2$. Since f is increasing, it holds $f(u_1) \geq f(u_2)$ in $H^{-1}(\Omega)$. By Lemma 3.6, it holds $I(f(u_1)) \supseteq_{\mathrm{q}} I(f(u_2))$. This shows $H_0^1(I(f(u_1))) \supseteq H_0^1(I(f(u_2)))$. $\square$

In contrast to the statement for the active sets in Lemma 3.6, it is not immediately clear from the monotonicity of the solution operator S_f of (OP_f) that the strictly active sets are also monotone. Nevertheless, such a result can be established. Before we can prove the statement, let us show the following proposition stating that perturbations in the obstacle outside the strictly active set leave the solution of (OP_f) unchanged.

The next results concerning the strictly active sets are modifications of the results in [RU20] for the bilateral obstacle problem.

Proposition 4.14 *Let $\zeta \in H^{-1}(\Omega)$ and let $v \in H_0^1(\Omega)_+$ such that $\{v > 0\} \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta)$. Then it holds $S_{\psi,\mathrm{id}}(\zeta) = S_{\psi-v,\mathrm{id}}(\zeta)$, where $S_{\phi,\mathrm{id}}\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ denotes the solution operator of* (OP_f) *with respect to the quasi upper-semicontinuous obstacle $\phi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ and w.r.t. the identity map* id *on $H^{-1}(\Omega)$.*

Proof. Obviously, $S_{\psi,\mathrm{id}}(\zeta) \geq \psi - v$, i.e., $S_{\psi,\mathrm{id}}(\zeta)$ is admissible. Now, let $z \in K_{\psi-v}$ be arbitrary. We have

$$
\begin{aligned}
&\langle LS_{\psi,\mathrm{id}}(\zeta) - \zeta, z - S_{\psi,\mathrm{id}}(\zeta)\rangle \\
&= \langle LS_{\psi,\mathrm{id}}(\zeta) - \zeta, z + v - S_{\psi,\mathrm{id}}(\zeta)\rangle + \langle LS_{\psi,\mathrm{id}}(\zeta) - \zeta, -v\rangle \geq 0
\end{aligned}
$$

by the variational inequality characterizing $S_{\psi,\mathrm{id}}(\zeta)$ since $z+v \geq \psi$ and using $v = 0$ q.e. on $A_{\mathrm{s}}(\zeta)$ which implies $\langle LS_{\psi,\mathrm{id}}(\zeta) - \zeta, -v\rangle = 0$, see Corollary 3.10. This shows $S_{\psi,\mathrm{id}}(\zeta) = S_{\psi-v,\mathrm{id}}(\zeta)$. $\square$

Lemma 4.15 *Suppose Assumption 4.1 is satisfied. Then the set-valued map* $H^{-1}(\Omega) \ni \zeta \mapsto H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$ *is increasing, i.e.,* $\zeta_1 \geq \zeta_2$ *implies* $H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta_1)) \supseteq H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta_2))$.

In particular, if f, U *fulfill the conditions of Assumption* 4.2(1.), *then the set-valued map* $U \ni u \mapsto H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ *is increasing.*

Proof. Let $\zeta_1, \zeta_2 \in H^{-1}(\Omega)$ and assume $\zeta_1 \geq \zeta_2$. We show that the strictly active sets satisfy $A_{\mathrm{s}}(\zeta_1) \subseteq_{\mathrm{q}} A_{\mathrm{s}}(\zeta_2)$, from which the conclusion of our lemma follows.

Assume $\Omega \setminus A_{\mathrm{s}}(\zeta_2) \neq_{\mathrm{q}} \emptyset$ (otherwise the assertion follows directly). Fix $v \in H_0^1(\Omega)_+$ satisfying $\{v > 0\} \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta_2)$, see Lemma 2.31.

As in Proposition 4.14, for quasi upper-semicontinuous obstacle $\phi \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ we denote by $S_{\phi,\mathrm{id}} \colon H^{-1}(\Omega) \to H_0^1(\Omega)$ the corresponding solution operator of (OP_f) for $f = \mathrm{id}$. Let $y_v(t) := S_{\psi - tv,\mathrm{id}}(\zeta_1)$, $t \in [0,1]$, and denote $\bar{y}_v(t) := y_v(t) + tv$. Note that $\bar{y}_v(t) \in K_\psi$ for each $t \in [0,1]$. Then it holds

$$\langle L y_v(t) - \zeta_1, z - y_v(t) \rangle \geq 0 \quad \forall\, z \in K_{\psi - tv}$$

which is equivalent to

$$\langle L \bar{y}_v(t) - \zeta_1 - tLv, \bar{z} - \bar{y}_v(t) \rangle \geq 0 \quad \forall\, \bar{z} \in K_\psi.$$

We conclude $y_v(t) = S_{\psi,\mathrm{id}}(T(tv)) - tv$ with $T \colon H_0^1(\Omega) \to H^{-1}(\Omega), v \mapsto \zeta_1 + Lv$. Since $S_{\psi,\mathrm{id}}$ is directionally differentiable in the Hadamard sense, see Proposition 4.1, we can apply the chain rule for the directional derivatives and obtain

$$y_v'(0;1) = S_{\psi,\mathrm{id}}'(T(0); T'(0;v)) - v = S_{\psi,\mathrm{id}}'(\zeta_1; Lv) - v.$$

Since $S_{\psi,\mathrm{id}}'(\zeta_1; Lv)$ is 0 q.e. on the strictly active set $A_{\mathrm{s}}(\zeta_1)$, compare Theorem 3.9, we have $y_v'(0;1) = -v < 0$ q.e. on $A_{\mathrm{s}}(\zeta_1) \cap \{v > 0\}$. Thus, by reducing the obstacle on a subset of $A_{\mathrm{s}}(\zeta_1)$ the solution with respect to the new obstacle will drop on this set.

Now, we show the statement of the lemma by contradiction. Therefore, assume the set $W \subseteq \Omega$ is a set of positive capacity which is weakly active for

ζ_2 and strictly active for ζ_1, i.e.,

$$W \subseteq_{\mathrm{q}} A_{\mathrm{s}}(\zeta_1) \subseteq_{\mathrm{q}} A(\zeta_2) \quad \text{and} \quad W \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta_2). \tag{4.17}$$

Let $v \in H_0^1(\Omega)_+$ satisfy $\{v > 0\} =_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta_2)$. Then, Proposition 4.14 yields

$$S_{\psi - v,\mathrm{id}}(\zeta_2) = S_{\psi,\mathrm{id}}(\zeta_2) \tag{4.18}$$

and on W we have

$$S_{\psi - v,\mathrm{id}}(\zeta_1)|_W < S_{\psi,\mathrm{id}}(\zeta_1)|_W = S_{\psi,\mathrm{id}}(\zeta_2)|_W \tag{4.19}$$

by the structure of the directional derivative with respect to the obstacle and by (4.17). Putting (4.18) and (4.19) together, we see that

$$S_{\psi - v,\mathrm{id}}(\zeta_2) > S_{\psi - v,\mathrm{id}}(\zeta_1)$$

on W. On the other hand, $S_{\psi - v,\mathrm{id}}(\zeta_1) \geq S_{\psi - v,\mathrm{id}}(\zeta_2)$ since $\zeta_1 \geq \zeta_2$, see Proposition 3.5. Thus, such a set W cannot exist and we conclude $A_{\mathrm{s}}(\zeta_1) \subseteq_{\mathrm{q}} A_{\mathrm{s}}(\zeta_2)$ and with that $H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta_1)) \supseteq H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta_2))$.

Now, the statement for the set-valued map $U \ni u \mapsto H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ follows since f is increasing. □

4.4.2 Monotonicity and Mosco convergence

In this subsection, we motivate the consideration of increasing and decreasing sequences $(u_n)_{n\in\mathbb{N}}$ for the analysis of the Mosco convergence $H_0^1(I(f(u_n))) \to H_0^1(I(f(u)))$, respectively $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))) \to H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$.

In the preceding Section 4.4.1, we have seen that the set-valued maps $u \mapsto H_0^1(I(f(u)))$ and $u \mapsto H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ are increasing. This property alone guarantees the existence of a Mosco limit for the sets $(H_0^1(I(f(u_n))))_{n\in\mathbb{N}}$ and $(H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))))_{n\in\mathbb{N}}$ when $(u_n)_{n\in\mathbb{N}}$ is increasing or decreasing, see

Lemma 4.8. The Mosco limit of the respective sequences is

$$\overline{\bigcup_{n\in\mathbb{N}} H_0^1(I(f(u_n)))}, \quad \text{respectively} \quad \overline{\bigcup_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n)))},$$

if $(u_n)_{n\in\mathbb{N}}$ is increasing and

$$\bigcap_{n\in\mathbb{N}} H_0^1(I(f(u_n))), \quad \text{respectively} \quad \bigcap_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n)))$$

if $(u_n)_{n\in\mathbb{N}}$ is decreasing. These characterizations of the limits depend on the sequence $(u_n)_{n\in\mathbb{N}}$ and is not an intrinsic characterization for given $u \in U$.

It is not clear that $H_0^1(I(f(u)))$ is the Mosco limit of $(H_0^1(I(f(u_n))))_{n\in\mathbb{N}}$ and that $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ is the Mosco limit of $(H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))))_{n\in\mathbb{N}}$, i.e., that we find a suitable intrinsic characterization for the limits. This is, as we will see, not true if the sequence $(u_n)_{n\in\mathbb{N}}$ does not have the appropriate direction of monotonicity.

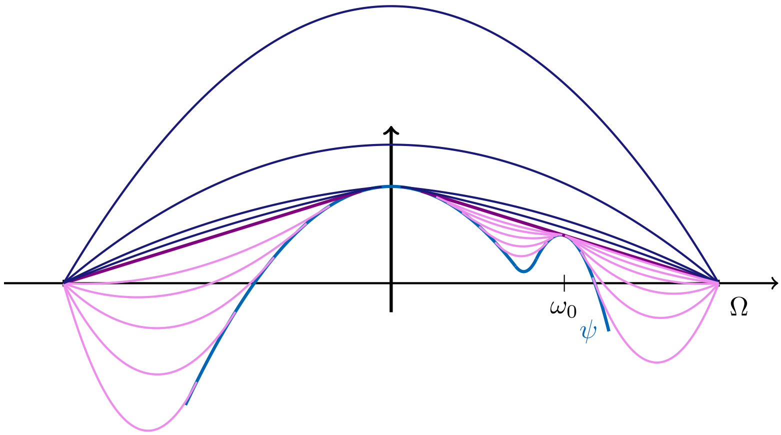
The example we will consider shows that the active and strictly active sets are stable only in one monotone direction and it is the opposite monotonicity for the active and for the strictly active sets, respectively.

Example 4.16 Solutions of the obstacle problem (OP_f) w.r.t. a family of different choices of $\zeta := f(u)$ are shown in Fig. 4.2. The associated values of ζ are chosen constant on Ω, some of the values are greater than zero, some of them are smaller than zero, and one of them is zero. We can also see the corresponding active sets $A(\zeta)$ and the strictly active sets $A_{\mathrm{s}}(\zeta)$ underneath.

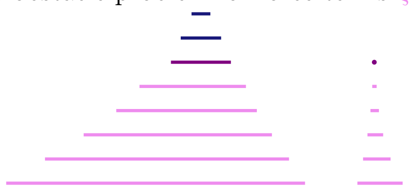
For $\zeta = 0$, the active set $A(0)$ is the union of an interval and an isolated point ω_0. The corresponding strictly active set $A_{\mathrm{s}}(0)$ does not contain ω_0, thus, the strict complementarity condition does not hold in $\zeta = 0$, i.e., $A(0) \neq A_{\mathrm{s}}(0)$. Note that a single point has capacity strictly positive in the one-dimensional case. Therefore, $u \in U$ with $f(u) = 0$ is a point where the respective solution operator is potentially non-Gâteaux differentiable.

Let us consider the sets $(H_0^1(I(f(u_n))))_{n\in\mathbb{N}}$. We argue that the Mosco limit will, in general, not be $H_0^1(I(f(u)))$ for a decreasing sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \to u$.

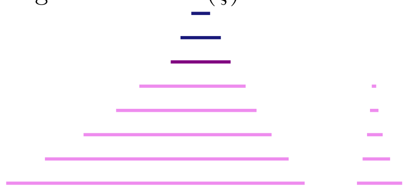
Exemplary, suppose $f(u) = 0$, $u_n \searrow u$, and assume the strict comple-



(a) An instance of the obstacle problem for force terms $\zeta \leq 0$, $\zeta = 0$ and $\zeta \geq 0$



(b) Corresponding active sets $A(\zeta)$ for the different choices of ζ



(c) Corresponding strictly active sets $A_{\mathrm{s}}(\zeta)$ for the different choices of ζ

Figure 4.2. Influence of monotonicity in the behavior of active and strictly active sets for the obstacle problem

mentary condition is satisfied in $f(u_n)$, $n \in \mathbb{N}$. In the situation of Fig. 4.2, choose an element $v \in H_0^1(\Omega)_+$ with $\{v > 0\} = \Omega \setminus A_{\mathrm{s}}(0)$, see Lemma 2.31, and define $v_n := v$ for all $n \in \mathbb{N}$. Then, it holds $v_n \in H_0^1(I(f(u_n)))$ for all $n \in \mathbb{N}$, since $I(f(u_n)) = \Omega \setminus A_{\mathrm{s}}(f(u_n)) \supseteq \Omega \setminus A_{\mathrm{s}}(0)$ for all $n \in \mathbb{N}$, as well as $v_n \to v$. Nevertheless, v is not an element of $H_0^1(I(0))$ since $v \neq 0$ on the isolated weakly active point. Therefore, recalling the second condition for Mosco convergence (see Definition 4.7), the Mosco limit of the sequence $(H_0^1(I(f(u_n))))_{n\in\mathbb{N}}$ is not $H_0^1(I(0))$ (but rather $H_0^1(\Omega \setminus A_{\mathrm{s}}(0))$).

Similarly, suppose $f(u) = 0$, $u_n \nearrow u$, and assume the strict complementary condition is satisfied in $f(u_n)$, $n \in \mathbb{N}$. In the situation of Fig. 4.2, we consider again $v \in H_0^1(\Omega)_+$ with $\{v > 0\} = \Omega \setminus A_{\mathrm{s}}(0)$, see Lemma 2.31. Now, any sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n)))$ for each $n \in \mathbb{N}$ is zero on $A_{\mathrm{s}}(f(u_n))$. By $A(0) \subseteq A(f(u_n)) = A_{\mathrm{s}}(f(u_n))$, the isolated weakly active point ω_0 of $A(0)$ is contained in $A_{\mathrm{s}}(f(u_n))$ and it holds $v_n(\omega_0) = 0$. On the other hand, by choice of v, it holds $v(\omega_0) > 0$. This shows that $v_n \to v$ cannot hold, since convergent sequences in $H_0^1(\Omega)$ possess pointwise q.e. convergent subsequences, and thus, in dimension $d = 1$, pointwise everywhere convergent subsequences. Recalling the first condition of Moscoc convergence, see Definition 4.7, the Mosco limit of the sequence $(H_0^1(\Omega \setminus A_{\mathrm{s}}(u_n)))_{n\in\mathbb{N}}$ is not $H_0^1(\Omega \setminus A_{\mathrm{s}}(0))$ (but rather $H_0^1(I(0))$).

4.5 Mosco convergence of the admissible sets

The purpose of this section is to show the Mosco convergence $H_0^1(I(f(u_n))) \to H_0^1(I(f(u)))$ for an arbitrary $u \in U$ and an increasing sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \to u$ and the Mosco convergence $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))) \to H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ for a decreasing sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \to u$.

In the preceding section we have motivated the influence of monotonicity of $(u_n)_{n\in\mathbb{N}}$ on the Mosco convergence of the sets $(H_0^1(I(f(u_n))))_{n\in\mathbb{N}}$ and $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n)))_{n\in\mathbb{N}}$ and the determination of the limit. These ideas are formalized in Theorem 4.17 and Theorem 4.18.

We start with the statement for the Sobolev spaces on the inactive sets.

Theorem 4.17 *Suppose Assumption 4.1 is satisfied and let f, U fulfill the conditions of Assumption* 4.2(1.). *Additionally, assume that f is continuous. Consider an arbitrary $u \in U$ and let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be an increasing sequence with $u_n \to u$. Then it holds*

$$H_0^1(I(f(u_n))) \to H_0^1(I(f(u)))$$

in the sense of Mosco.

Proof. By Lemma 4.13, the sequence of sets $(H_0^1(I(f(u_n)))_{n\in\mathbb{N}}$ is increasing. Thus, Lemma 4.8 implies

$$\lim_{n\to\infty} H_0^1(I(f(u_n))) = \overline{\bigcup_{n\in\mathbb{N}} H_0^1(I(f(u_n)))}.$$

Now, our goal is to show the equality $H_0^1(I(f(u))) = \overline{\bigcup_{n\in\mathbb{N}} H_0^1(I(f(u_n)))}$.

The inclusions $H_0^1(I(f(u_n))) \subseteq H_0^1(I(f(u)))$ for all $n \in \mathbb{N}$ imply

$$\bigcup_{n\in\mathbb{N}} H_0^1(I(f(u_n))) \subseteq H_0^1(I(f(u))).$$

Since the set on the right-hand side is closed, also

$$\overline{\bigcup_{n\in\mathbb{N}} H_0^1(I(f(u_n)))} \subseteq H_0^1(I(f(u)))$$

holds.

To show the opposite inclusion, assume $v \in H_0^1(I(f(u)))$. We will first show that the sequence $(I(f(u_n)))_{n\in\mathbb{N}}$ of quasi-open sets is an increasing quasi-covering of $I(f(u))$. By the monotonicity of $(u_n)_{n\in\mathbb{N}}$, it is clear from Lemma 4.13 that the sequence $(I(f(u_n)))_{n\in\mathbb{N}}$ is increasing. Since $u_n \to u$, it follows $S_f(u_n) \to S_f(u)$ in $H_0^1(\Omega)$ by continuity of $S_f = S_{\mathrm{id}} \circ f$, cf. Theorem 3.4, and therefore pointwise q.e. for a subsequence, see Lemma 2.28. This means for quasi-all $\omega \in \bigcap_{n\in\mathbb{N}} A(f(u_n))$ it holds $S_f(u)(\omega) = \psi(\omega)$, i.e., quasi-all such ω belong to $A(f(u))$ and

$$\operatorname{cap}\left(I(f(u)) \setminus \bigcup_{n\in\mathbb{N}} I(f(u_n))\right) = \operatorname{cap}\left(I(f(u)) \cap \bigcap_{n\in\mathbb{N}} A(f(u_n))\right) = 0$$

follows. Therefore, the family $(I(f(u_n)))_{n\in\mathbb{N}}$ is an increasing quasi-covering of $I(f(u))$. Now, by Lemma 2.29, there is a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \to v$ as well as

$$v_n \in H_0^1(I(f(u_n))) \subseteq \overline{\bigcup_{m\in\mathbb{N}} H_0^1(I(f(u_m)))}$$

for all $n \in \mathbb{N}$ and thus, the limit v is an element of the latter set. All in all, we deduce the Mosco convergence of the sequence $\big(H_0^1(I(f(u_n)))\big)_{n\in\mathbb{N}}$ towards $H_0^1(I(f(u)))$. ☐

Analogously, we also show the Mosco convergence of the Sobolev spaces on the complements of the strictly active sets for decreasing sequences $(u_n)_{n\in\mathbb{N}}$.

Theorem 4.18 *Suppose Assumption 4.1 is satisfied and let f, U fulfill the conditions of Assumption* 4.2(1.). *Additionally, assume that f is continuous. Consider an arbitrary $u \in U$ and let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be a decreasing sequence with $u_n \to u$. Then it holds*

$$H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))) \to H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$$

in the sense of Mosco.

Proof. By Lemma 4.15, the sequence of sets $(H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n)))_{n\in\mathbb{N}}$ is decreasing. Thus, Lemma 4.8 implies

$$\lim_{n\to\infty} H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))) = \bigcap_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n))).$$

Now, our goal is to show the equality $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u))) = \bigcap_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n)))$.

Since the sequence $H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n)))_{n\in\mathbb{N}}$ is decreasing, we conclude

$$H_0^1(\Omega \setminus A_\mathrm{s}(f(u))) \subseteq H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n)))$$

for all $n \in \mathbb{N}$, which implies

$$H_0^1(\Omega \setminus A_\mathrm{s}(f(u))) \subseteq \bigcap_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n))).$$

To show the opposite inclusion, assume $v \in \bigcap_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n)))$. Denote $\xi_n := LS_f(u_n) - f(u_n) \in H^{-1}(\Omega)_+$. Since $v = 0$ q.e. on $A_\mathrm{s}(f(u_n))$ for each $n \in \mathbb{N}$, we conclude

$$\langle \xi_n, |v| \rangle = 0$$

for all $n \in \mathbb{N}$, see Corollary 3.10. By $\xi_n \to \xi$ in $H^{-1}(\Omega)$, which follows from continuity of S_id, L and f, compare Theorem 3.4, we conclude

$$\langle \xi, |v| \rangle = \lim_{n\to\infty} \langle \xi_n, |v| \rangle = 0.$$

This implies $|v| = 0$ q.e. on $A_\mathrm{s}(f(u))$, in particular, $v \in H_0^1(\Omega \setminus A_\mathrm{s}(f(u)))$. We have shown

$$H_0^1(\Omega \setminus A_\mathrm{s}(f(u))) = \bigcap_{n\in\mathbb{N}} H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n))).$$

All in all, we deduce the Mosco convergence of the sequence $\big(H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n)))\big)_{n\in\mathbb{N}}$ towards $H_0^1(\Omega \setminus A_\mathrm{s}(f(u)))$. $\square$

Remark 4.19 The Mosco convergence $H_0^1(I(f(u_n))) \to H_0^1(I(f(\tilde{u})))$ for an arbitrary increasing sequence $(u_n)_{n\in\mathbb{N}}$ converging to $\tilde{u}$ and the Mosco convergence $H_0^1(\Omega \setminus A_\mathrm{s}(f(u_n))) \to H_0^1(\Omega \setminus A_\mathrm{s}(f(\tilde{u})))$ for a decreasing sequence can be interpreted as a continuity property of the set-valued maps $u \mapsto H_0^1(I(f(u)))$, respectively $u \mapsto H_0^1(\Omega \setminus A_\mathrm{s}(f(u)))$ in $\tilde{u}$.

4.6 Characterization of two generalized derivatives

In this section, we bring together the results of this chapter and formulate the main theorem of this chapter on generalized derivatives in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$ for the solution operator S_f of (OP_f).

Theorem 4.20 *Suppose the conditions of Assumption 4.1 and Assumption 4.2 are satisfied. Let $u \in U$ be arbitrary. For some $h \in U$, denote by $\Xi_{I(f(u))}(h)$ the solution of*

$$\textit{Find } \delta \in H_0^1(I(f(u))): \quad \langle L\delta - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(I(f(u))) \tag{4.20}$$

and by $\Xi_{\Omega \setminus A_{\mathrm{s}}(f(u))}(h)$ the solution of

$$\begin{aligned} &\textit{Find } \delta \in H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u))): \\ &\langle L\delta - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u))). \end{aligned} \tag{4.21}$$

Then it holds

$$\Xi_{I(f(u))}, \Xi_{\Omega \setminus A_{\mathrm{s}}(f(u))} \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u).$$

Proof. By Corollary 4.12, there is an increasing sequence $(u_n^+)_{n\in\mathbb{N}}$ and a decreasing sequence $(u_n^-)_{n\in\mathbb{N}}$ satisfying

$$(u_n^+)_{n\in\mathbb{N}}, (u_n^-)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_f} = \{v \in U \mid S_f \text{ is Gâteaux differentiable at } v\}$$

as well as $u_n^+ \to u$ and $u_n^- \to u$. Now, Theorem 4.17 implies that the sequence $(H_0^1(I(f(u_n^+))))_{n\in\mathbb{N}}$ converges to $H_0^1(I(f(u)))$ in the sense of Mosco. Moreover, by Theorem 4.18, the sequence $(H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u_n^-))))_{n\in\mathbb{N}}$ converges to $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ in the sense of Mosco. Using Corollary 4.10, we conclude that

$$\Xi_{I(f(u))}, \Xi_{\Omega \setminus A_{\mathrm{s}}(f(u))} \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u).$$

□

4.7 Adjoint representation of Clarke subgradients

Now, we derive Clarke subgradients for reduced objective functions in the context of optimization problems with the obstacle problem as a constraint. To this end, let $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ be a continuously differentiable objective function. We consider an optimization problem with respect to this objective function, which is constrained by our variational inequality

$$\begin{aligned} &\min_{y,u} J(y,u) \\ &\text{subject to} \quad y \in K_\psi, \\ &\qquad\qquad\quad \langle Ly - f(u), z - y\rangle \geq 0 \quad \forall\, z \in K_\psi. \end{aligned}$$

We present a formula for two generalized derivatives contained in Clarke's generalized differential $\partial_{\mathrm{C}} \hat{J}(u)$, see Definition 2.16, that can be obtained for the reduced objective function

$$\hat{J}(u) := J(S_f(u), u)$$

in an arbitrary point $u \in U$.

Theorem 4.21 *Suppose the conditions of Assumption 4.1 and Assumption 4.2 are satisfied and let $u \in U$ be arbitrary. Let $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ be a continuously differentiable objective function and denote by q be the unique solution of the variational equation*

$$\begin{aligned} &\textit{Find } q \in H_0^1(D): \\ &\langle L^* q, v\rangle = \langle J_y\left(S_f(u), u\right), v\rangle \quad \forall\, v \in H_0^1(D). \end{aligned} \tag{4.22}$$

Then the element

$$f'(u)^* q + J_u(S_f(u), u)$$

is in Clarke's generalized differential $\partial_{\mathrm{C}} \hat{J}(u)$. In (4.22), the respective sets

$$D :=_{\mathrm{q}} I(f(u)) \quad \textit{or} \quad D :=_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$$

can be chosen and result in a particular generalized derivative.

Here, J_y and J_u denote the continuous Fréchet derivatives of J with respect to y and u, respectively, $f'(u)^* \in \mathcal{L}\left(H_0^1(\Omega), U^*\right)$ is the (Banachian) adjoint operator of $f'(u) \in \mathcal{L}\left(U, H^{-1}(\Omega)\right)$ and $L^* \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ is the (Banachian) adjoint operator of $L \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$.

Proof. The coercivity of L^* follows from the coercivity of L, thus, the variational equation (4.22) has a unique solution, see Theorem 3.1. It holds

$$\partial_{\mathrm{C}}\hat{J}(u) \ni \Xi^* J_y\left(S_f(u), u\right) + J_u\left(S_f(u), u\right) \tag{4.23}$$

for all $\Xi \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$, see Lemma 2.17.

Assume that q solves (4.22) for $D :=_{\mathrm{q}} I(f(u))$, respectively $D :=_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$. For $h \in U$, denote by $\Xi_D(h)$ the solution of (4.20) or the solution of (4.21), respectively. Now, we have

$$\begin{aligned}
\langle f'(u)^* q, w\rangle_{U^*,U} &= \langle f'(u; w), q\rangle \\
&\overset{(4.20),(4.21)}{=} \langle L^* q, \Xi_D(w)\rangle \\
&\overset{(4.22)}{=} \langle J_y(S_f(u), u), \Xi_D(w)\rangle \\
&= \langle \Xi_D^* J_y(S_f(u), u), w\rangle_{U^*,U}
\end{aligned}$$

for all $w \in U$. Since $\Xi_D \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$, see Theorem 4.20, the statement follows from (4.23). $\square$

CHAPTER 5

Generalized differentials for the basic solution operator

In this section, we analyze generalized differentials for the solution operator S_{id} that maps $\zeta \in H^{-1}(\Omega)$ to the solution of the obstacle problem

$$\text{Find } y \in K_\psi : \quad \langle -\Delta y - \zeta, z - y \rangle \geq 0 \quad \forall\, z \in K_\psi. \tag{$\mathrm{OP}^{\Delta}_{\mathrm{id}}$}$$

Here, the closed, convex set K_ψ is defined as

$$K_\psi := \{z \in H^1_0(\Omega) \mid z \geq \psi \text{ q.e. in } \Omega\}.$$

This means, compared with the analysis of Chapter 4, we consider $L = -\Delta$ and f is the identity on $H^{-1}(\Omega)$.

Throughout this chapter, we assume that $\Omega \subseteq \mathbb{R}^d$ is an open and bounded domain, the obstacle $\psi \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is quasi upper-semicontinuous and chosen such that $K_\psi \neq \emptyset$ is guaranteed. Additionally, we suppose that ψ is Borel measurable, which, under the previous assumptions on ψ can always be achieved by a modification on a set of capacity zero, see Lemma 2.24(2.).

The goal is to find a specific representation of the generalized differentials $\partial^{\mathrm{ss}}_{\mathrm{B}} S_{\mathrm{id}}(\zeta)$, $\partial^{\mathrm{sw}}_{\mathrm{B}} S_{\mathrm{id}}(\zeta)$ and $\partial^{\mathrm{ws}}_{\mathrm{B}} S_{\mathrm{id}}(\zeta)$ in a point $\zeta \in H^{-1}(\Omega)$ independent from approximating sequences $(\zeta_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ with $\zeta_n \to \zeta$.

Again, the starting point is the characterization of the Gâteaux derivatives as solution operators of variational equations, or Dirichlet problems, on the inactive set $I(\zeta)$, see Theorem 4.3. This means, if $\zeta \in H^{-1}(\Omega)$ is a point at which S_{id} is Gâteaux differentiable and if $p \in H^{-1}(\Omega)$ is arbitrary, then $S'_{\text{id}}(\zeta; p)$ is the solution of the problem

$$\text{Find } \delta \in H_0^1(I(\zeta)) : \quad -\Delta\delta = p. \tag{5.1}$$

Recall that the strict complementarity condition is necessary and sufficient for the Gâteaux differentiability of S_{id} at $\zeta \in H^{-1}(\Omega)$, cf. Lemma 4.4. Defining a Borel measure $\infty_{A(\zeta)}$ which takes the value 0 for Borel sets $B \subseteq \Omega$ with $\operatorname{cap}(B \setminus I(\zeta)) = 0$ and the value ∞ otherwise, we can understand $S'_{\text{id}}(\zeta)$ as the solution operator of the *relaxed Dirichlet problem* involving the *capacitary measure* $\infty_{A(\zeta)}$ which maps $p \in H^{-1}(\Omega)$ to the solution of

$$\text{Find } \delta \in H_0^1(\Omega) \cap L^2_{\infty_{A(\zeta)}}(\Omega) : \quad -\Delta\delta + \infty_{A(\zeta)}(\delta\,\cdot) = p, \tag{5.2}$$

where $\mu(\delta\,\cdot)$ denotes the measure with density δ w.r.t. another measure μ. In other words, the two formulations in (5.1) and (5.2) are equivalent. Thus, the Gâteaux derivatives of S_{id} fall into the class of solution operators of relaxed Dirichlet problems and it is beneficial to study these objects.

A capacitary measure is a Borel measure which takes the value 0 on all Borel subsets of Ω with capacity zero. Additionally, capacitary measures fullfil a regularity condition. The rigorous definition of capacitary measures and relaxed Dirichlet problems will be given in Section 5.1.

On the set of capacitary measures the notion of γ-convergence is defined by the convergence of the solution operators of the respective relaxed Dirichlet problems in the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. It is known that the solution operator of a relaxed Dirichlet problem involving an arbitrary capacitary measure μ can be approximated (in the sense of γ-convergence) by a sequence of solution operators of Dirichlet problems on quasi-open subsets of Ω. Vice versa, each sequence of solution operators of Dirichlet problems on quasi-open sets has a subsequence converging to the solution operator of a relaxed Dirichlet problem relative to a capacitary measure.

In this framework, we will characterize the generalized differentials

$\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta)$, $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ and $\partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$ as solution operators of relaxed Dirichlet problems with particular conditions on the respective capacitary measures. These conditions depend on the behavior of the capacitary measures on the sets $I(\zeta)$ and $A_{\mathrm{s}}(\zeta)$. Since we assume that ψ is Borel measurable and since we consider Borel measurable quasi-continuous representatives of $H^1(\Omega)$-functions, these subsets of Ω are Borel measurable.

We will see that the generalized differentials involving the strong operator topology on $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ coincide with the set of solution operators of Dirichlet problems on quasi-open subsets which are supersets of the inactive set and subsets of the complement of the strictly active set. The generalized differential $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ contains additionally solution operators of relaxed Dirichlet problems with more general capacitary measures. We will give an example showing that $\partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta)$ can be very large.

When reviewing the concepts w.r.t. γ-convergence of capacitary measures we will also see the connection to the notion of Mosco convergence considered in Chapter 4. In fact, the convergence of solution operators of Dirichlet problems (or variational equations) on quasi-open sets to a solution operator of a Dirichlet problem on a quasi-open set is equivalent to the Mosco convergence of the respective H_0^1-spaces. Moreover, the limit of a sequence of solution operators of Dirichlet problems on quasi-open domains w.r.t. the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ is always a solution operator w.r.t. a quasi-open domain.

In the characterization of the generalized differentials of S_{id} we explicitly use that the operator is defined on the whole space $H^{-1}(\Omega)$ and not, as in Chapter 4 on a smaller subset as $L^2(\Omega)$ or even another space.

The outline of this chapter is as follows. The notions of capacitary measures and relaxed Dirichlet problems are introduced in Section 5.1. It is argued that the Gâteaux derivatives $S_{\mathrm{id}}'(\zeta)$ in points $\zeta \in H^{-1}(\Omega)$ at which S_{id} is Gâteaux differentiable are solution operators of relaxed Dirichlet problem involving capacitary measures. A concept of convergence for capacitary measures, or equivalently, for the solution operators of the respective relaxed Dirichlet problems, is introduced in Section 5.1.1. Additionally, we provide a characterization of this convergence by a metric and collect useful properties of the corresponding metric space. In Section 5.1.2, we prepare some more auxiliary results that will be used for the study of the generalized differentials of S_{id}. Without further assumptions, for arbitrary $\zeta \in H^{-1}(\Omega)$, we can obtain characterizations of the generalized differentials $\partial_{\text{B}}^{\text{ss}} S_{\text{id}}(\zeta)$ and $\partial_{\text{B}}^{\text{ws}} S_{\text{id}}(\zeta)$ in Section 5.2. Under continuity assumptions on ψ and $S_{\text{id}}(\zeta)$, the structure of $\partial_{\text{B}}^{\text{sw}} S_{\text{id}}$ is derived in Section 5.3. We give a short example to indicate that the generalized differential $\partial_{\text{B}}^{\text{ww}} S_{\text{id}}(\zeta)$ is very large even when S_{id} is Gâteaux differentiable at $\zeta \in H^{-1}(\Omega)$.

The results presented in this chapter originated in a cooperation with Gerd Wachsmuth and we present a collection of contents published in [RW20].

5.1 Capacitary measures and relaxed Dirichlet problems

In this section, we introduce the notions of capacitary measures and relaxed Dirichlet problems.

Assume that S_{id} is Gâteaux differentiable at $\zeta \in H^{-1}(\Omega)$. Recall that the strict complementarity condition $A(\zeta) =_{\text{q}} A_{\text{s}}(\zeta)$ holds at ζ, see Lemma 4.4, and that the Gâteaux derivative $S_{\text{id}}'(\zeta)$ is the solution operator $\Xi_{I(\zeta)} \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ that maps $p \in H^{-1}(\Omega)$ to the solution of the variational equation

$$\text{Find } \delta \in H_0^1(I(\zeta)) : \quad \langle -\Delta\delta, z\rangle = \langle p, z\rangle \quad \forall\, z \in H_0^1(I(\zeta)), \tag{5.3}$$

see Theorem 4.3. We will see that the operator $\Xi_{I(\zeta)}$ can be understood as the solution operator of a so-called *relaxed Dirichlet problem* involving a *capacitary measure* on Ω.

Let us now give the definition of capacitary measures.

Definition 5.1 (Capacitary measures) We denote by $\mathcal{M}_0(\Omega)$ the set of all Borel measures μ on Ω with the property that $\mu(B) = 0$ holds for every Borel set $B \subseteq \Omega$ with $\operatorname{cap}(B) = 0$. Additionally, we require that μ is regular in the sense that

$$\mu(B) = \inf\{\mu(O) : O \text{ quasi-open}, \ B \subseteq O\}$$

holds for every Borel set $B \subseteq \Omega$. The set $\mathcal{M}_0(\Omega)$ is called the set of capacitary measures on Ω.

Recall that by considering suitable representatives we can assume that quasi-open sets are Borel measurable, see Lemma 2.25, so that $\mu(O)$ is defined for every quasi-open set $O \subseteq \Omega$ and every $\mu \in \mathcal{M}_0(\Omega)$.

In fact, we already know a particular subclass of measures in $\mathcal{M}_0(\Omega)$, as the following lemma shows.

Lemma 5.2 *It holds $H^{-1}(\Omega)_+ \subseteq \mathcal{M}_0(\Omega)$.*

Proof. In Lemma 2.33, we have seen that each $\mu \in H^{-1}(\Omega)_+$ can be identified with a regular Borel measure $\tilde{\mu}$ and it holds $\tilde{\mu}(B) = 0$ for each Borel set B satisfying $\operatorname{cap}(B) = 0$. We argue that $\tilde{\mu}$ also fulfills the regularity property from Definition 5.1. Let $B \subseteq \Omega$ be an arbitrary Borel set. On the one hand,

$$\begin{aligned}\tilde{\mu}(B) &= \inf\{\tilde{\mu}(O) \mid O \text{ open}, \ B \subseteq O\} \\ &\geq \inf\{\tilde{\mu}(O) \mid O \text{ quasi-open}, \ B \subseteq O\}\end{aligned}$$

holds since every open set is quasi-open. On the other hand, we estimate

$$\tilde{\mu}(B) \leq \inf\{\tilde{\mu}(O) \mid O \text{ quasi-open}, \ B \subseteq O\}$$

by monotonicity of measures. Thus, $\tilde{\mu}$ is regular in the sense of Definition 5.1. □

Partially, the name capacitary measure is motivated by the property that capacitary measures, by definition, vanish on sets of capacity zero. Additionally, the following reverse statement holds.

Lemma 5.3 *Assume that B is a Borel set. If $\mu(B) = 0$ holds for all capacitary measures $\mu \in \mathcal{M}_0(\Omega)$, then $\operatorname{cap}(B) = 0$.*

Proof. Assume $\mu(B) = 0$ holds for all $\mu \in \mathcal{M}_0(\Omega)$. Then, in particular, $\tilde{\mu}(B) = 0$ for all $\tilde{\mu} \in H^{-1}(\Omega)_+$, see Lemma 5.2. Now, [BS00, Lem. 6.55] implies the statement. □

Let $\mu \in \mathcal{M}_0(\Omega)$. For $p \in [1, \infty)$ we define $L^p_\mu(\Omega)$ in the usual way. Note that any $v \in H^1_0(\Omega)$ has a quasi-continuous representative which is Borel measurable and uniquely determined up to a set of capacity zero, see Lemma 2.24(1.). Thus,

$$\int_\Omega |v|^p \, d\mu$$

is well-defined and if the integral is finite, we have $v \in L^p_\mu(\Omega)$.

It can be shown that equipped with the scalar product

$$(w, z)_\mu := \int_\Omega \nabla w^T \, \nabla z \, \mathrm{d}\lambda^d + \int_\Omega w \, z \, \mathrm{d}\mu = (w, z)_{H^1_0(\Omega)} + (w, z)_{L^2_\mu(\Omega)},$$

the space $H^1_0(\Omega) \cap L^2_\mu(\Omega)$ is a Hilbert space, see [BDM91, Prop. 2.1].

Now, for $p \in H^{-1}(\Omega)$, we consider the so-called *relaxed Dirichlet problem*

$$\begin{aligned} &\text{Find } \delta \in H^1_0(\Omega) \cap L^2_\mu(\Omega) : \\ &\langle -\Delta\delta, z\rangle + \int_\Omega \delta \, z \, \mathrm{d}\mu = \langle p, z\rangle \quad \forall\, z \in H^1_0(\Omega) \cap L^2_\mu(\Omega). \end{aligned} \tag{5.4}$$

Then a unique solution of (5.4) exists and coincides with the Fréchet-Riesz representative of $p \in H^{-1}(\Omega) \subseteq (H^1_0(\Omega) \cap L^2_\mu(\Omega))^*$. We denote the solution operator of (5.4) by Ξ_μ. For short, we also write

$$\text{Find } \delta \in H^1_0(\Omega) : \quad -\Delta\delta + \mu(\delta\,\cdot) = p$$

instead of (5.4).

The following proposition is essential in understanding the connection between relaxed Dirichlet problems and the Gâteaux derivatives operators of the obstacle problem.

Proposition 5.4 *Let $O \subseteq \Omega$ be a quasi-open set. Then the measure $\infty_{\Omega\setminus O}$*

defined by

$$\infty_{\Omega\setminus O}(B) := \begin{cases} 0, & \textit{if } \operatorname{cap}(B \setminus O) = 0, \\ +\infty, & \textit{otherwise,} \end{cases} \tag{5.5}$$

for all Borel sets $B \subseteq \Omega$ is a capacitary measure and the variational equation or Dirichlet problem

$$\textit{Find } \delta \in H_0^1(O): \quad \langle -\Delta\delta, z\rangle = \langle p, z\rangle \quad \forall\, z \in H_0^1(O) \tag{5.6}$$

is equivalent to the relaxed Dirichlet problem (5.4) *for* $\mu = \infty_{\Omega\setminus O}$.

Proof. From the definition it is clear that $\infty_{\Omega\setminus O}$ is a Borel measure which vanishes on Borel sets of capacity zero. It is easy to verify that $\infty_{\Omega\setminus O}$ is regular in the sense of Definition 5.1, see [DM87, Rem. 3.3]. Thus, $\infty_{\Omega\setminus O} \in \mathcal{M}_0(\Omega)$.

Next we argue that it holds $H_0^1(\Omega) \cap L^2_{\infty_{\Omega\setminus O}}(\Omega) = H_0^1(O)$. If v is in $H_0^1(O)$, then $\int_\Omega |v|^2 \, \mathrm{d}\infty_{\Omega\setminus O} = 0$, i.e., $v \in L^2_{\infty_{\Omega\setminus O}}(\Omega)$. Conversely, assume $\int_\Omega |v|^2 \, \mathrm{d}\infty_{\Omega\setminus O} < \infty$ for some $v \in H_0^1(\Omega)$. This implies $v = 0$ q.e. on $\Omega \setminus O$ by definition of $\infty_{\Omega\setminus O}$, i.e., $v \in H_0^1(O)$. These considerations also show that the solution of (5.6) is the solution of (5.4) with $\mu = \infty_{\Omega\setminus O}$ and vice versa. $\square$

The preceding proposition shows that the class of relaxed Dirichlet problems contains the class of Dirichlet problems of type (5.6). In particular, the variational equations (5.3), describing the Gâteaux derivatives are relaxed Dirichlet problems and it holds $S'_{\mathrm{id}}(\zeta) = \Xi_{\infty_{A(\zeta)}}$.

In the following, if $O \subseteq \Omega$ is quasi-open, we also use the notation $\Xi_O := \Xi_{\infty_{\Omega\setminus O}}$ for the solution operators of (5.6), in particular,

$$S'_{\mathrm{id}}(\zeta) = \Xi_{\infty_{A(\zeta)}} = \Xi_{I(\zeta)}.$$

5.1.1 The set of capacitary measures as a metric space

On $\mathcal{M}_0(\Omega)$ we will introduce the notion of γ-convergence. Later on we will see that this concept can be characterized by a metric space.

Definition 5.5 (γ-convergence of measures) Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_0(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$. We say that the sequence $(\mu_n)_{n\in\mathbb{N}}$ γ-converges to μ if and only if $(\Xi_{\mu_n})_{n\in\mathbb{N}}$ converges to Ξ_μ in the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. We also use the notation $\mu_n \xrightarrow{\gamma} \mu$ to express that $(\mu_n)_{n\in\mathbb{N}}$ γ-converges to μ.

Note that, by the following lemma, the γ-limit of a γ-convergent sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_0(\Omega)$ is unique.

Lemma 5.6 *Let $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ and assume that $\Xi_{\mu_1}(1) = \Xi_{\mu_2}(1)$. Then $\mu_1 = \mu_2$ follows. In particular, the map*

$$\mathcal{M}_0(\Omega) \to \Xi_\mu \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$$

is injective.

Proof. A proof can be found in [DMG94, Lem. 3.3]. □

Suppose $\zeta \in H^{-1}(\Omega)$. Then by the characterization of Gâteaux derivatives, see (5.3) and Proposition 5.4, the characterization of limits of sequences $(\infty_{\Omega\setminus I(\zeta_n)})_{n\in\mathbb{N}}$ for $(\zeta_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ with $\zeta_n \to \zeta$, respectively $\zeta_n \rightharpoonup \zeta$, w.r.t. the γ-convergence corresponds to the characterization of elements in $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$, respectively $\partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta)$.

The notion of convergence in Definition 5.5 is called γ-convergence to stress its relation to the Γ-convergence of suitable functionals. To this end, for $\mu \in \mathcal{M}_0(\Omega)$, we define $F_\mu\colon L^2(\Omega) \to [0,\infty]$ via

$$F_\mu(v) := \begin{cases} \int_\Omega |\nabla v|^2 \, \mathrm{d}\lambda^d + \int_\Omega v^2 \, \mathrm{d}\mu = (v,v)_\mu & \text{if } v \in H_0^1(\Omega) \cap L^2_\mu(\Omega), \\ +\infty & \text{else} \end{cases}$$

for $v \in L^2(\Omega)$.

Definition 5.7 (Γ-convergence of functionals) Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_0(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$ be given. We say that the functionals $(F_{\mu_n})_{n\in\mathbb{N}}$ Γ-converge towards F_μ in $L^2(\Omega)$ if and only if

$$\forall\,(w_n)_{n\in\mathbb{N}} \subseteq L^2(\Omega) \text{ with } w_n \to w \text{ in } L^2(\Omega): \qquad F_\mu(w) \le \liminf_{n\to\infty} F_{\mu_n}(w_n) \tag{5.7a}$$

$$\exists\,(w_n)_{n\in\mathbb{N}} \subseteq L^2(\Omega) \text{ with } w_n \to w \text{ in } L^2(\Omega): \qquad F_\mu(w) = \lim_{n\to\infty} F_{\mu_n}(w_n) \tag{5.7b}$$

hold for all $w \in L^2(\Omega)$. In this case, we write $F_{\mu_n} \xrightarrow{\Gamma} F_\mu$ in $L^2(\Omega)$.

Now, the following lemma shows equivalent conditions for γ-convergence.

Lemma 5.8 *Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_0(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$ be given. Then, the following statements are equivalent:*

(i) $\mu_n \xrightarrow{\gamma} \mu$,

(ii) $F_{\mu_n} \xrightarrow{\Gamma} F_\mu$ *in* $L^2(\Omega)$,

(iii) $\Xi_{\mu_n} \to \Xi_\mu$ *in the strong operator topology of* $\mathcal{L}(L^2(\Omega), L^2(\Omega))$,

(iv) $\Xi_{\mu_n} \to \Xi_\mu$ *in the weak operator topology of* $\mathcal{L}(L^2(\Omega), H_0^1(\Omega))$,

(v) $\Xi_{\mu_n}(1) \to \Xi_\mu(1)$ *in* $L^2(\Omega)$,

(vi) $\Xi_{\mu_n}(1) \rightharpoonup \Xi_\mu(1)$ *in* $H_0^1(\Omega)$.

Proof. Assume $\mu_n \xrightarrow{\gamma} \mu$. Then, for all $p \in H^{-1}(\Omega)$, in particular, for all $p \in L^2(\Omega)$, it holds $\Xi_{\mu_n}(p) \rightharpoonup \Xi_\mu(p)$ in $H_0^1(\Omega)$ by definition. Since $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, it follows $\Xi_{\mu_n}(p) \to \Xi_\mu(p)$ in $L^2(\Omega)$, and thus (iii) holds.

Now, suppose that $\Xi_{\mu_n} \to \Xi_\mu$ in the strong operator topology of $\mathcal{L}(L^2(\Omega), L^2(\Omega))$. Let $v \in L^2(\Omega)$. By [BDM91, (3.7)], there is a constant $c > 0$, such that $\|\Xi_{\mu_n}(v)\|_{H_0^1(\Omega)} \le c\|v\|$ holds. Thus there is a subsequence $(\Xi_{\mu_{n_k}}(v))_{k\in\mathbb{N}}$ that converges weakly in $H_0^1(\Omega)$. Hence $(\Xi_{\mu_{n_k}}(v))_{k\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ and the limit has to be $\Xi_\mu(v)$. Thus, the whole sequence $(\Xi_{\mu_n}(v))_{n\in\mathbb{N}}$ converges weakly to $\Xi_\mu(v)$ in $H_0^1(\Omega)$ and (iv) follows. The proof that (vi) follows from (v) is also contained in this argument.

(vi) is an immediate consequence of (iv) and (v) follows from (vi) by the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$.

The equivalence of (vi) and (i) has been shown, in a more general setting, in [DMM04, Thm. 5.1]. The equivalence between (iii) and (ii) can be checked as in [DMM87, Prop. 4.10]. □

As a corollary of the preceding Lemma 5.8 we conclude that $\mathcal{M}_0(\Omega)$ can be equipped with a metric and the resulting notion of convergence is equivalent to the γ-convergence. A different proof of this statement is given in [DMM87, Prop. 4.9].

Corollary 5.9 *The γ-convergence on $\mathcal{M}_0(\Omega)$ is metrizable.*

Proof. Positive definiteness of d defined by $d(\mu_1, \mu_2) := \|\Xi_{\mu_1}(1) - \Xi_{\mu_2}(1)\|_{L^2(\Omega)}$ for $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ follows from Lemma 5.6. Now, it is straightforward to check that d is a metric on $\mathcal{M}_0(\Omega)$. Using Lemma 5.8(i) and (v), convergence with respect to the metric d is equivalent to γ-convergence on $\mathcal{M}_0(\Omega)$. □

Lemma 5.8 and Corollary 5.9 illustrate the role of the so-called torsion function $\Xi_\mu(1)$ that will reappear in the next subsection. For now, let us collect some properties of the metric space $\mathcal{M}_0(\Omega)$. A proof of the following lemma on completeness of $\mathcal{M}_0(\Omega)$ can be found in [DMM87, Thm. 4.14] or in [DMG97, Thm. 4.5]. Recall that a metric space is complete if every Cauchy sequence converges and has a limit in the metric space.

Lemma 5.10 *$\mathcal{M}_0(\Omega)$ is a complete metric space.*

In particular, by the above lemma and by the characterization of Gâteaux derivatives of S_{id}, the generalized differentials $\partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta), \partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta), \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta), \partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta)$ are all contained in $\{\Xi_\mu : \mu \in \mathcal{M}_0(\Omega)\}$.

In the next lemma, it is shown that the class of measures $\infty_{\Omega \setminus O}$ for quasi-open sets $O \subseteq \Omega$, cf. (5.5), is a dense subset in $\mathcal{M}_0(\Omega)$. For a proof we refer to [DMM87, Thm. 4.16] or [DMM95].

Lemma 5.11 *For each $\mu \in \mathcal{M}_0(\Omega)$ there is a sequence $(O_n)_{n\in\mathbb{N}}$ of quasi-open subsets of Ω such that $\infty_{\Omega\setminus O_n} \xrightarrow{\gamma} \mu$.*

Remark 5.12 In shape optimization problems one is looking for the solutions of problems like, e.g.,

$$\begin{aligned} &\min_O J(\delta_O) \\ &\text{subject to} \quad \delta_O \in H_0^1(O), \\ &\qquad\qquad \langle -\Delta\delta_O, z\rangle = \langle p, z\rangle \quad \forall\, z \in H_0^1(O). \end{aligned}$$

Since, by Lemma 5.11, solutions of classical Dirichlet problems with varying (quasi-open) domains can converge to the solution of a relaxed Dirichlet problem with a capacitary measure involved, an optimal domain in shape optimization might not exist, see e.g. [BB05, Sect. 4.2] or [ABM14, Sect. 5.8.4].

The following theorem states that $\mathcal{M}_0(\Omega)$ is compact. By replacing $\mathbb{R}^n$ by Ω, a proof of this result can be found in [DMM87, Thm. 4.14].

Theorem 5.13 *For every sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_0(\Omega)$ there is a subequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that $\mu_{n_k} \xrightarrow{\gamma} \mu$.*

5.1.2 Useful properties

In this subsection we state some more results that will help us later on in the characterization of the Bouligand generalized differentials.

In the preceding chapters we have already used that for every quasi-open set $O \subseteq \Omega$ we can find an element $v \in H_0^1(\Omega)_+$ satisfying $\{v > 0\} = O$, see Lemma 2.31. The following theorem concretizes this result and states that the torsion function $\Xi_O(1)$ has exactly this property. Moreover, a dual statement showing the existence of an element $p \in H^{-1}(\Omega)_+$ with f-supp$(p) =_{\mathrm{q}} \Omega \setminus O$ is presented (recall Lemma 2.34), which also uses the torsion function.

Theorem 5.14 *Let $\mu \in \mathcal{M}_0(\Omega)$. Then it holds $\Xi_\mu(1) \geq 0$ q.e. in Ω. Let $O \subseteq \Omega$ be quasi-open and set $w := \Xi_O(1)$. Then, $w \geq 0$, $O =_{\mathrm{q}} \{w > 0\}$ and* *the element* $1 + \Delta w \in H^{-1}(\Omega)_+$ *satisfies* $\text{f-supp}(1 + \Delta w) =_{\mathrm{q}} \Omega \setminus O$.

Proof. For $\mu \in \mathcal{M}_0(\Omega)$ it holds $\Xi_\mu(1) \geq 0$ q.e. in Ω, by [DMG94, Proposition 2.4]. The assertions $O =_{\mathrm{q}} \{w > 0\}$ and $1 + \Delta w \in H^{-1}(\Omega)_+$ are well-known, see, e.g., [Vel15, Prop. 3.4.26] and [CDM92, Thm. 1].

It remains to check $C :=_{\mathrm{q}} \text{f-supp}(1 + \Delta w) =_{\mathrm{q}} \Omega \setminus O$. Using the third characterization in [HW18, Lem. 3.7],

we have

$$\langle 1 + \Delta w, v\rangle = 0 \quad \Leftrightarrow \quad v = 0 \text{ q.e. on } C \qquad \forall\, v \in H_0^1(\Omega)_+. \tag{5.8}$$

Using $w = \Xi_O(1)$, this directly implies that $C \subseteq_{\mathrm{q}} \Omega \setminus O$. Next, we define $\hat{w} = \Xi_{\Omega\setminus C}(1)$. Since $w \in H_0^1(O) \subseteq H_0^1(\Omega \setminus C)$, we have $\langle 1 + \Delta\hat{w}, w\rangle = 0$. Moreover, (5.8) implies $\langle 1 + \Delta w, \hat{w}\rangle = 0$. Using $\langle \Delta w, \hat{w}\rangle = \langle \Delta\hat{w}, w\rangle$, this yields

$$0 = \langle 1, \hat{w} - w\rangle = \int_\Omega \hat{w} - w \,\mathrm{d}\lambda^d.$$

Next, by the comparison principle from [DMM86, Thm. 2.10], see also [DMG94, Prop. 2.5], we find $\hat{w} \geq w$ and, therefore, $\hat{w} = w$. Finally, the first part of the proof yields $\Omega \setminus C =_{\mathrm{q}} \{\hat{w} > 0\} =_{\mathrm{q}} \{w > 0\} =_{\mathrm{q}} O$. Thus, $C =_{\mathrm{q}} \text{f-supp}(1 + \Delta w) =_{\mathrm{q}} \Omega \setminus O$. □

The next result shows that every capacitary measure can be approximated by Borel measures which are finite on all compact subsets of Ω.

Lemma 5.15 *Let* $\mu \in \mathcal{M}_0(\Omega)$. *Then there exists an increasing sequence of measures* $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_0(\Omega)$ *which are finite on compact subsets of* Ω *such that* $\mu_n \xrightarrow{\gamma} \mu$.

Proof. Let $w_0 := \Xi_\Omega(1)$ and for $\mu \in \mathcal{M}_0(\Omega)$ let $w := \Xi_\mu(1)$. In [DMG94, Prop. 4.7], it is shown that for the sequence $(w_n)_{n\in\mathbb{N}}$ defined by

$$w_n := \left(1 - \frac{1}{n}\right) w + \frac{1}{n} w_0,$$

the associated measures given by

$$\mu_n(B) := \begin{cases} \int_B \frac{\mathrm{d}(1+\Delta w_n)}{w_n} & \text{if } \operatorname{cap}(B \cap \{w_n = 0\}) = 0, \\ +\infty & \text{else,} \end{cases}$$

for each Borel set $B \subseteq \Omega$ are finite on compact subsets of Ω and γ-converge to μ. Thus, it remains to show the monotonicity of this sequence. Since $w_0 > 0$ q.e. on Ω by Theorem 5.14, it holds $\mu_n(B) = \int_B \frac{\mathrm{d}(1+\Delta w_n)}{w_n}$ for all $n \in \mathbb{N}$ and for all Borel sets B. By $-\Delta \Xi_\Omega(1) = 1$, the representation

$$\begin{aligned} \mu_n(B) &= \int_B \frac{\mathrm{d}(1+\Delta w_n)}{w_n} = \int_B \frac{\mathrm{d}(1+\Delta((1-1/n)w + 1/n\, w_0))}{(1-1/n)w + 1/n\, w_0} \\ &= \int_B \frac{\mathrm{d}(1 - 1/n + (1-1/n)\Delta w)}{(1-1/n)(w + 1/(n-1)\, w_0)} = \int_B \frac{\mathrm{d}(1+\Delta w)}{w + 1/(n-1)\, w_0} \end{aligned}$$

shows that $\mu_n \leq \mu_{n+1} \leq \mu$ holds for all $n \in \mathbb{N}$. □

The following lemma states that for every $\mu \in \mathcal{M}_0(\Omega)$, the image of Ξ_μ is dense in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$.

Lemma 5.16 *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $\delta \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$. Then there is a sequence*

$$(\delta_n)_{n\in\mathbb{N}} \subseteq \{\Xi_\mu(p) : p \in H^{-1}(\Omega)\}$$

such that $\delta_n \to \delta$ in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$.

Proof. For every $n \in \mathbb{N}$ let $\delta_n \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of the problem

$$\begin{aligned} &\text{Find } \delta_n \in H_0^1(\Omega) \cap L_\mu^2(\Omega) : \\ &\langle -\Delta \delta_n, z\rangle + \int_\Omega \delta_n\, z \,\mathrm{d}\mu = -n \int_\Omega (\delta_n - \delta)\, z \,\mathrm{d}\lambda^d \quad \forall\, z \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{aligned}$$

We can write $\delta_n = \Xi_\mu(-n(\delta_n - \delta))$, thus $\delta_n \in \{\Xi_\mu(p) : p \in H^{-1}(\Omega)\}$. By [DMG94, Proposition 3.1], it holds $\delta_n \to \delta$ in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ and the conclusion follows. □

Let $\mu \in \mathcal{M}_0(\Omega)$. The following lemma gives insights into the pointwise q.e. behavior of elements in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ using the torsion function $w_\mu := \Xi_\mu(1)$.

Lemma 5.17 *Assume that $\mu \in \mathcal{M}_0(\Omega)$ and let z be an element of $H_0^1(\Omega) \cap L^2_\mu(\Omega)$. Then it holds $z \in H_0^1(\{w_\mu > 0\})$.*

Proof. By [DMG94, Prop. 3.4], it holds $\mu(B) = +\infty$ for all Borel sets $B \subseteq \Omega$ with $\operatorname{cap}(B \cap \{w_\mu = 0\}) > 0$. Thus $z = 0$ q.e. on $\{w_\mu = 0\}$ for all z in the image of Ξ_μ. By density of this set in $H_0^1(\Omega) \cap L^2_\mu(\Omega)$, see Lemma 5.16, it follows $z = 0$ q.e. on $\{w_\mu = 0\}$ for all $z \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$. It holds $w_\mu \geq 0$ on Ω by Theorem 5.14, therefore, each $z \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ is in $H_0^1(\{w_\mu > 0\})$. □

The next result characterizes the completion of $H_0^1(\Omega) \cap L^2_\mu(\Omega)$ in $H_0^1(\Omega)$. Note that for a quasi-open set $O \subseteq \Omega$ we already know that $H_0^1(\Omega) \cap L^2_{\infty_{\Omega\setminus O}}(\Omega) = H_0^1(O)$, see Proposition 5.4.

Lemma 5.18 *Let $\mu \in \mathcal{M}_0(\Omega)$ be given. Then,*

$$\overline{H_0^1(\Omega) \cap L^2_\mu(\Omega)}^{H_0^1(\Omega)} = H_0^1(\{w_\mu > 0\}).$$

Moreover, for any $v \in H_0^1(\{w_\mu > 0\})_+$, there exists a sequence $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega) \cap L^2_\mu(\Omega)$ such that $0 \leq v_n \leq v$ q.e. on Ω for all $n \in \mathbb{N}$ and $v_n \to v$ in $H_0^1(\Omega)$.

Proof. We set $Z := \overline{H_0^1(\Omega) \cap L^2_\mu(\Omega)}^{H_0^1(\Omega)}$. The inclusion $Z \subseteq H_0^1(\{w_\mu > 0\})$ is clear from Lemma 5.17.

Then, it can be checked that Z is a closed lattice ideal in $H_0^1(\Omega)$, i.e., it is a closed subspace with the property that $z \in Z$, $w \in H_0^1(\Omega)$ and $|w| \leq |z|$ imply $w \in Z$. Hence, [Sto93, Thm. 1] implies that $Z = H_0^1(O)$ for some quasi-open $O \subseteq \Omega$. Thus,

$$Z = H_0^1(O) \subseteq H_0^1(\{w_\mu > 0\})$$

and together with $w_\mu \in Z$ we get $O =_{\mathrm{q}} \{w_\mu > 0\}$. This shows the identity $\overline{H_0^1(\Omega) \cap L^2_\mu(\Omega)}^{H_0^1(\Omega)} = Z = H_0^1(O) = H_0^1(\{w_\mu > 0\})$.

The second assertion is clear since $w \mapsto \min(w, v)_+$ is continuous on $H_0^1(\Omega)$. □

The following lemma states that the solution operators associated to quasi-open sets are a sequentially closed set w.r.t. the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$.

Lemma 5.19 *Assume $(O_n)_{n\in\mathbb{N}}$ is a sequence of quasi-open subsets of Ω and suppose that $(\Xi_{O_n})_{n\in\mathbb{N}}$ converges in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ to some $\Xi \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. Then there is a quasi-open set $O \subseteq \Omega$ such that $\Xi = \Xi_O$.*

Proof. It holds $\Xi_{O_n}(1) \to \Xi(1)$ in $H_0^1(\Omega)$, therefore $(\infty_{\Omega\setminus O_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_0(\Omega)$, see the proof of Corollary 5.9. Thus, we know from Lemma 5.10 that $\Xi = \Xi_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$. For fixed $p \in H^{-1}(\Omega)$, we set $z_n := \Xi_{O_n}(p)$. Then, $z_n \to z := \Xi_\mu(p)$ in $H_0^1(\Omega)$ and this yields

$$\int_\Omega z^2 \,\mathrm{d}\mu = \langle \Delta z, z\rangle + \langle p, z\rangle = \lim_{n\to\infty} (\langle \Delta z_n, z_n\rangle + \langle p, z_n\rangle) = 0.$$

Hence, $\int_\Omega z^2 \,\mathrm{d}\mu = 0$ for all z in the range of Ξ_μ.

In order to find a quasi-open set $O \subseteq \Omega$ with $\Xi_\mu = \Xi_O$, we use the torsion function $w := \Xi_\mu(1)$ and set $O :=_{\mathrm{q}} \{w > 0\}$. From $\int_\Omega w^2 \,\mathrm{d}\mu = 0$ and $z \in H_0^1(O)$ for $z \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$, see Lemma 5.17, it follows that $w = \Xi_O(1)$. Thus, $\Xi_O = \Xi_\mu$ by Lemma 5.6. □

Recall that Gâteaux derivatives of S_{id} in a point $\zeta \in H^{-1}(\Omega)$ are of the structure $\Xi_{I(\zeta)}$, cf. (5.3). Now, from Lemma 5.19 we can deduce that the generalized differentials for the solution operator S_{id} defined by the strong operator topology such as $\partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$, $\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta)$ are subsets of $\{\Xi_O : O \subseteq \Omega \text{ quasi-open}\}$.

Remark 5.20 The following converse of the statement in Lemma 5.19 holds. Assume there are quasi-open sets $O_n, O \subseteq \Omega$, $n \in \mathbb{N}$, such that $\infty_{\Omega\setminus O_n} \xrightarrow{\gamma} \infty_{\Omega\setminus O}$. Then it holds $\Xi_{O_n} \to \Xi_O$ in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. In fact, in [ABM14, Prop. 5.8.6], it is shown that $\Xi_{O_n} \to \Xi_O$ in the strong operator topology of $\mathcal{L}(L^2(\Omega), H_0^1(\Omega))$. Now, the convergence in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ follows by using the theorem of Banach-Steinhaus.

For the proof of the following theorem explaining the connection between γ-convergence and Mosco convergence we refer to [BB05, Prop. 4.5.3, Rem. 4.5.5].

Theorem 5.21 *Let $O_n, O \subseteq \Omega$ be quasi-open sets. Then it holds $\infty_{\Omega\setminus O_n} \xrightarrow{\gamma} \infty_{\Omega\setminus O}$ if and only if $(H_0^1(O_n))_{n\in\mathbb{N}}$ converges to $H_0^1(O)$ in the sense of Mosco.*

Recall that the notion of Mosco convergence was introduced in Section 4.2. In Chapter 4 it was used to obtain two particular elements of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$. The result in Proposition 4.9 already gives one of the two implications of the statement in Theorem 5.21. Namely, the Mosco convergence $H_0^1(O_n) \to H_0^1(O)$ implies the convergence of solutions to variational equations and thus the convergence $\Xi_{O_n} \to \Xi_O$ in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ which in turn implies $\infty_{\Omega\setminus O_n} \xrightarrow{\gamma} \infty_{\Omega\setminus O}$. Now, Theorem 5.21 also gives the reverse statement.

While the two notions of convergence are equivalent when considering quasi-open sets, it is possible that a Mosco limit of a sequence $(H_0^1(O_n))$ does not exist if there is no quasi-open limit set O, even though the solutions of the corresponding variational equations converge weakly in $H_0^1(\Omega)$ to some limit. Then, by what we have seen so far, the limit operator is the solution operator of a relaxed Dirichlet problem involving a capacitary measure.

Now, we are going to analyze the convergence of a sum of two γ-convergent sequences. The following result is an auxiliary lemma.

Lemma 5.22 *Let $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ be sequences with $w_n \to w$ in $H_0^1(\Omega)$ and $v_n \rightharpoonup w$ in $H_0^1(\Omega)$ for some $w \in H_0^1(\Omega)$. Then, $z_n := \min(v_n, w_n)$ satisfies $z_n \rightharpoonup w$ in $H_0^1(\Omega)$ and*

$$\limsup_{n\to\infty} \left(\|z_n\|^2_{H_0^1(\Omega)} - \|v_n\|^2_{H_0^1(\Omega)} \right) \leq 0.$$

Proof. The weak convergence of $(z_n)_{n\in\mathbb{N}}$ follows from the weak sequential continuity of $\min(\cdot,\cdot)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$, see Proposition 2.19(3.). To obtain the desired inequality, we check

$$\|z_n\|^2_{H_0^1(\Omega)} - \|v_n\|^2_{H_0^1(\Omega)}$$

$$
\begin{aligned}
&= \|z_n - w_n\|^2_{H^1_0(\Omega)} - \|v_n - w_n\|^2_{H^1_0(\Omega)} + 2\,(z_n - v_n, w_n)_{H^1_0(\Omega)} \\
&= -\|\max(0, v_n - w_n)\|^2_{H^1_0(\Omega)} + 2\,(z_n - v_n, w_n)_{H^1_0(\Omega)} \\
&\le 2\,(z_n - v_n, w_n)_{H^1_0(\Omega)}.
\end{aligned}
$$

Now, the claim follows from $z_n - v_n \rightharpoonup 0$ and $w_n \to w$ in $H^1_0(\Omega)$. □

In the following proposition, the γ-convergence of the sum of two particular γ-convergent sequences is established.

Proposition 5.23 *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_0(\Omega)$ such that $\mu_n \xrightarrow{\gamma} \mu$ for some $\mu \in \mathcal{M}_0(\Omega)$ and let $(C_n)_{n\in\mathbb{N}}$ be a sequence of quasi-closed subsets of Ω such that $\infty_{C_n} \xrightarrow{\gamma} \infty_C$ for some quasi-closed set $C \subseteq \Omega$. Then, $\mu_n + \infty_{C_n} \xrightarrow{\gamma} \mu + \infty_C$.*

Proof. We use the characterization of γ-convergence via the Γ-convergence of the functionals $F_{\mu_n+\infty_{C_n}}$, see Lemma 5.8(i) and (ii). Therefore, we have to verify (5.7). Let $w \in L^2(\Omega)$ be given and consider an arbitrary sequence $(w_n)_{n\in\mathbb{N}} \subseteq L^2(\Omega)$ with $w_n \to w$ in $L^2(\Omega)$. We have to show

$$F_{\mu+\infty_C}(w) \le \liminf_{n\to\infty} F_{\mu_n+\infty_{C_n}}(w_n).$$

If the limes inferior is $+\infty$, there is nothing to show. Otherwise, we select a subsequence of $(w_n)_{n\in\mathbb{N}}$ (without relabeling), such that $(F_{\mu_n+\infty_{C_n}}(w_n))_{n\in\mathbb{N}}$ converges to this value and such that $F_{\mu_n+\infty_{C_n}}(w_n) < +\infty$ for all n.

By the definition of the functionals, this implies $w_n \in H^1_0(\Omega)$ as well as $\int_\Omega w_n^2 \,\mathrm{d}\infty_{C_n} < +\infty$, and these properties yield $w_n \in H^1_0(\Omega \setminus C_n)$. Consequently, we have

$$F_{\infty_C}(w) \le \liminf_{n\to\infty} F_{\infty_{C_n}}(w_n) \le \liminf_{n\to\infty} F_{\mu_n+\infty_{C_n}}(w_n) < +\infty.$$

Thus, $w \in H^1_0(\Omega \setminus C)$ and $\int_\Omega w^2 \,\mathrm{d}\infty_C = 0$. Now, the desired inequality follows from

$$F_{\mu+\infty_C}(w) = F_\mu(w) \le \liminf_{n\to\infty} F_{\mu_n}(w_n) = \liminf_{n\to\infty} F_{\mu_n+\infty_{C_n}}(w_n),$$

where we have used $F_{\mu_n} \xrightarrow{\Gamma} F_\mu$.

Further, we have to prove the existence of a sequence $(\tilde{w}_n)_{n\in\mathbb{N}} \subseteq L^2(\Omega)$ with $\tilde{w}_n \to w$ in $L^2(\Omega)$ and

$$F_{\mu+\infty_C}(w) = \lim_{n\to\infty} F_{\mu_n+\infty_{C_n}}(\tilde{w}_n).$$

It is enough to consider the case $w \geq 0$, otherwise apply the following arguments to w_+ and w_-. If $F_{\mu+\infty_C}(w) = \infty$, there is nothing left to show. Otherwise, we have $w \in H_0^1(\Omega \setminus C)$. From $F_{\mu_n} \xrightarrow{\Gamma} F_\mu$ and $F_{\infty_{C_n}} \xrightarrow{\Gamma} F_{\infty_C}$, we find sequences $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subseteq L^2(\Omega)$ with

$$\begin{aligned} v_n \to w \text{ in } L^2(\Omega) \quad &\text{and} \quad F_{\mu_n}(v_n) \to F_\mu(w), \\ w_n \to w \text{ in } L^2(\Omega) \quad &\text{and} \quad F_{\infty_{C_n}}(w_n) \to F_{\infty_C}(w). \end{aligned}$$

W.l.o.g., we can assume $v_n, w_n \geq 0$ (otherwise, replace v_n by $(v_n)_+$ and w_n by $(w_n)_+$). Moreover, we can assume that $F_{\mu_n}(v_n), F_{\infty_{C_n}}(w_n) < \infty$ for all $n \in \mathbb{N}$. Then we can infer $v_n \rightharpoonup w$ in $H_0^1(\Omega)$ and $w_n \to w$ in $H_0^1(\Omega)$. We define $z_n = \min(w_n, v_n)$ and already get $z_n \to w$ in $L^2(\Omega)$. To obtain the convergence of the function values, we use $z_n = 0$ q.e. on C_n to obtain

$$\begin{aligned} &F_{\mu_n+\infty_{C_n}}(z_n) = F_{\mu_n}(z_n) = \int_\Omega |\nabla z_n|^2 \,\mathrm{d}\lambda^d + \int_\Omega z_n^2 \,\mathrm{d}\mu_n \\ &\leq \int_\Omega |\nabla z_n|^2 \,\mathrm{d}\lambda^d + \int_\Omega v_n^2 \,\mathrm{d}\mu_n = F_{\mu_n}(v_n) + \left(\|z_n\|^2_{H_0^1(\Omega)} - \|v_n\|^2_{H_0^1(\Omega)}\right). \end{aligned}$$

Now, by using Lemma 5.22 and $w \in H_0^1(\Omega \setminus C)$ we obtain

$$\begin{aligned} F_{\mu+\infty_C}(w) &\leq \liminf_{n\to\infty} F_{\mu_n+\infty_{C_n}}(z_n) \\ &\leq \limsup_{n\to\infty} F_{\mu_n+\infty_{C_n}}(z_n) \leq \limsup_{n\to\infty} F_{\mu_n}(v_n) = F_\mu(w) = F_{\mu+\infty_C}(w). \end{aligned}$$

Thus, $F_{\mu+\infty_C}(w) = \lim_{n\to\infty} F_{\mu_n+\infty_{C_n}}(z_n)$. This finishes the proof of the convergence $F_{\mu_n+\infty_{C_n}} \xrightarrow{\Gamma} F_{\mu+\infty_C}$. $\square$

5.2 Generalized differentials involving the strong operator topology

Using the characterization of the Gâteaux derivative in a point $\zeta \in H^{-1}(\Omega)$ as $\Xi_{I(\zeta)}$ and the properties of solution operators of relaxed Dirichlet problems introduced in the previous section, we characterize the generalized differentials $\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta)$ and $\partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$ in this section.

We start with the following technique that is frequently used in the remaining parts of this chapter.

Proposition 5.24 *Assume $O \subseteq \Omega$ is quasi-open and let $v \in H_0^1(O)$. Then it holds $v = \Xi_O(-\Delta v)$.*

Proof. The statement follows directly from the definition of Ξ_O. □

In the following lemma, we find a superset of $\partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$.

Lemma 5.25 *Let $\zeta \in H^{-1}(\Omega)$. For each $\Xi \in \partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$ there is a quasi-open set $D \subseteq \Omega$ with $\operatorname{cap}(D \cap A_{\mathrm{s}}(\zeta)) = 0$ and $\Xi = \Xi_D$.*

Proof. By definition of $\partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$, see Definition 2.12, there is a sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_{\mathrm{id}}}$ such that $\zeta_n \rightharpoonup \zeta$ in $H^{-1}(\Omega)$, $S_{\mathrm{id}}(\zeta_n) \rightharpoonup S_{\mathrm{id}}(\zeta)$ in $H_0^1(\Omega)$ and $S_{\mathrm{id}}'(\zeta_n) \to \Xi$ in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. By the characterization of differentiability points of S_{id}, see Theorem 4.3, we have $S_{\mathrm{id}}'(\zeta_n) = \Xi_{I(\zeta_n)}$. From Lemma 5.19, we already know that $\Xi = \Xi_D$ for some quasi-open set $D \subseteq \Omega$.

It remains to check $\operatorname{cap}(D \cap A_{\mathrm{s}}(\zeta)) = 0$. From Lemma 2.31 (see also Theorem 5.14) we infer the existence of $v \in H_0^1(\Omega)_+$ with $\{v > 0\} =_{\mathrm{q}} D$. In particular, $v \in H_0^1(D)$ and this yields that $v = \Xi_D(-\Delta v)$, see Proposition 5.24, is the strong limit of $v_n := S_{\mathrm{id}}'(\zeta_n; -\Delta v)$. By the properties of $S_{\mathrm{id}}'(\zeta_n) = \Xi_{I(\zeta_n)}$, we have $v_n = 0$ q.e. on $A(\zeta_n) =_{\mathrm{q}} A_{\mathrm{s}}(\zeta_n)$. Thus,

$$\langle \xi_n, |v_n| \rangle = 0,$$

where $\xi_n := -\Delta S_{\mathrm{id}}(\zeta_n) - \zeta_n$, see Corollary 3.10. From $\xi_n \rightharpoonup \xi := -\Delta S_{\mathrm{id}}(\zeta) - \zeta$

and $|v_n| \to |v|$, we infer

$$\langle \xi, |v| \rangle = 0.$$

Thus, $v = 0$ q.e. on $A_{\mathrm{s}}(\zeta)$ by Corollary 3.10. Hence, by choice of v it holds $\operatorname{cap}(D \cap A_{\mathrm{s}}(\zeta)) = \operatorname{cap}(\{v \neq 0\} \cap A_{\mathrm{s}}(\zeta)) = 0$. □

The following auxiliary lemma shows that for arbitrary $\zeta \in H^{-1}(\Omega)$ and a sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ each element $v \in H_0^1(I(\zeta))$, $0 \le v \le 1$ is the limit of a sequence $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(I(\zeta_n))$. A related statement has been shown in the proof of Theorem 4.17 where the Mosco convergence of $H_0^1(I(f(\zeta_n)))$ was verified. Here, the sets $I(\zeta_n)$ are not necessarily increasing. However, the proof relies once more on increasing quasi-coverings, see Lemma 2.29.

Lemma 5.26 *Assume $(\zeta_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ is a sequence with $\zeta_n \to \zeta$ for some $\zeta \in H^{-1}(\Omega)$. Then for each $v \in H_0^1(I(\zeta))$ with $0 \le v \le 1$ there exists a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in H_0^1(I(\zeta_n))$ and $v_n \to v$ in $H_0^1(\Omega)$.*

Proof. We set $y := S_{\mathrm{id}}(\zeta)$ and $y_n := S_{\mathrm{id}}(\zeta_n)$. Let $t_n := \sup_{m=n,\dots,\infty} \|y_m - y\|_{H_0^1(\Omega)}^{1/2}$. Then, $(t_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative numbers with $t_n \ge \|y_n - y\|_{H_0^1(\Omega)}^{1/2}$ and $t_n \searrow 0$. We have $\{y > \psi\} =_{\mathrm{q}} \bigcup_{n=1}^{\infty} \{y > \psi + t_n\}$. Since the sets on the right-hand side are quasi-open and increasing in n, we can apply Lemma 2.29. This yields a sequence $(\tilde{v}_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ with $\tilde{v}_n \to v$ in $H_0^1(\Omega)$, $0 \le \tilde{v}_n \le 1$ and $\tilde{v}_n = 0$ q.e. on $\{y \le \psi + t_n\}$.

Next, using the definition and monotonicity of the set function cap, we observe

$$\begin{aligned}
\operatorname{cap}\left(\{y_n = \psi\} \cap \{y > \psi + t_n\}\right) &\le \operatorname{cap}(\{|y_n - y| > t_n\}) \\
&\le t_n^{-2} \|y_n - y\|_{H_0^1(\Omega)}^2 \to 0.
\end{aligned}$$

Thus, there exists a sequence $(w_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ with $w_n \to 0$ in $H_0^1(\Omega)$, $0 \le w_n \le 1$ and $w_n = 1$ q.e. on $\{y_n = \psi\} \cap \{y > \psi + t_n\}$, $n \in \mathbb{N}$, cf. Lemma 2.27. For $n \in \mathbb{N}$ we define $v_n := (\tilde{v}_n - w_n)_+$. By construction, $v_n \to v$ and $v_n = 0$ q.e. on $\{y_n = \psi\}$, i.e., $v_n \in H_0^1(I(\zeta_n))$. □

Next, we provide a characterization of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta)$.

Theorem 5.27 *Let $\zeta \in H^{-1}(\Omega)$ be given. Then it holds*

$$\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta) = \{\Xi_D \mid D \text{ is quasi-open and } I(\zeta) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta)\}.$$

Proof. "$\subseteq$": Let $\Xi \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta)$ be given. By definition, $S_{\mathrm{id}}'(\zeta_n) \to \Xi$ in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ for some sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_{\mathrm{id}}}$ with $\zeta_n \to \zeta$. From Lemma 5.25 and from $\Xi \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta) \subseteq \partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta)$, we already have $\Xi = \Xi_D$ for some quasi-open $D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta)$. It remains to check $I(\zeta) \subseteq_{\mathrm{q}} D$.

By Lemma 2.31 (see also Theorem 5.14), there is a function $v \in H_0^1(\Omega)$ with $0 \le v \le 1$ and $I(\zeta) =_{\mathrm{q}} \{v > 0\}$. From Lemma 5.26, we get a sequence $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ with $v_n \to v$ and $v_n \in H_0^1(I(\zeta_n))$. Together with Proposition 5.24 and Lemma 2.11(1.) we find

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \Xi_{I(\zeta_n)}(-\Delta v_n) = \lim_{n\to\infty} S_{\mathrm{id}}'(\zeta_n; -\Delta v_n) = \Xi_D(-\Delta v).$$

This gives $I(\zeta) =_{\mathrm{q}} \{v > 0\} \subseteq_{\mathrm{q}} D$.

"$\supseteq$": Let D be given as in the formulation of the theorem. From Lemma 2.31 (see also Theorem 5.14), we get a function $v \in H_0^1(\Omega)_+$ with $\{v > 0\} =_{\mathrm{q}} D$. Similarly, Theorem 5.14 gives $\nu \in H^{-1}(\Omega)_+$ with $\operatorname{f-supp}(\nu) =_{\mathrm{q}} \Omega \setminus D$. We define $\zeta_n := \zeta - (\Delta v + \nu)/n$, $y := S_{\mathrm{id}}(\zeta)$, $y_n := y + v/n$, $\xi_n := -\Delta y_n - \zeta_n$, $\xi := -\Delta y - \zeta$. Then

$$\xi_n = -\Delta y - \frac{1}{n}\Delta v - \zeta + \frac{1}{n}(\Delta v + \nu) = \xi + \frac{\nu}{n}$$

and we may check that $y_n = S_{\mathrm{id}}(\zeta_n)$. From $v \ge 0$, we infer $y_n \in K_\psi$. Further, for arbitrary $z \in K_\psi$ we have

$$\begin{aligned}\langle \xi_n, z - y_n\rangle &= \langle \xi, z - y\rangle + \left\langle \xi, -\frac{1}{n}v \right\rangle + \frac{1}{n}\left\langle \nu, z - y - \frac{v}{n}\right\rangle \\ &\ge 0 + 0 + 0 = 0.\end{aligned}$$

The second term is zero due to $v = 0$ q.e. on $\Omega \setminus D \supseteq_{\mathrm{q}} A_{\mathrm{s}}(\zeta)$, see Corollary 3.10. Similarly, the third term is nonnegative since $\operatorname{f-supp}(\nu) =_{\mathrm{q}} \Omega \setminus D$ and $z \ge \psi = y + v/n$ on $\Omega \setminus D$. Hence, $y_n = S_{\mathrm{id}}(\zeta_n)$. Thus, $I(\zeta_n) =_{\mathrm{q}} D =_{\mathrm{q}}$

$\Omega \setminus A_s(\zeta_n)$, i.e., $\zeta_n \in \mathcal{D}_{S_{\text{id}}}$, compare Lemma 4.4. Finally, $S'_{\text{id}}(\zeta_n) = \Xi_D$ for all $n \in \mathbb{N}$ and $\zeta_n \to \zeta$ ensure $\Xi_D \in \partial_{\text{B}}^{\text{ss}} S_{\text{id}}(\zeta)$.

□

Remark 5.28 In the proof of the inclusion "$\supseteq$" for Theorem 5.27, for arbitrary quasi-open D with $I(\zeta) \subseteq_{\text{q}} D \subseteq_{\text{q}} \Omega \setminus A_s(\zeta)$ a sequence $(\zeta_n)_{n\in\mathbb{N}}$ is constructed which converges to ζ such that the strict complementarity condition is fulfilled in ζ_n and such that D coincides with $I(\zeta_n)$ and thus also with $\Omega \setminus A_s(\zeta_n)$.

This proof technique is not transferable to the setting where f is an operator as in Chapter 4, since the sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ doesn't necessarily correspond to a suitable sequence in U.

We can also give a characterization of $\partial_{\text{B}}^{\text{ws}} S_{\text{id}}(\zeta)$. Indeed, for any $\zeta \in H^{-1}(\Omega)$, we will see that without any further assumptions it holds $\partial_{\text{B}}^{\text{ws}} S_{\text{id}}(\zeta) = \partial_{\text{B}}^{\text{ss}} S_{\text{id}}(\zeta)$.

Theorem 5.29 *Let $\zeta \in H^{-1}(\Omega)$ be given. Then,*

$$\partial_{\text{B}}^{\text{ws}} S_{\text{id}}(\zeta) = \{\Xi_D \mid D \textit{ is quasi-open and } I(\zeta) \subseteq_{\text{q}} D \subseteq_{\text{q}} \Omega \setminus A_s(\zeta)\}.$$

Proof. "$\supseteq$": This follows from $\partial_{\text{B}}^{\text{ws}} S_{\text{id}}(\zeta) \supseteq \partial_{\text{B}}^{\text{ss}} S_{\text{id}}(\zeta)$ and Theorem 5.27.
"$\subseteq$": Let $\Xi \in \partial_{\text{B}}^{\text{ws}} S_{\text{id}}(\zeta)$ be given. By definition, $S'_{\text{id}}(\zeta_n) \to \Xi$ in the strong operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ for some sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_{\text{id}}}$ with $\zeta_n \rightharpoonup \zeta$ and $S_{\text{id}}(\zeta_n) \rightharpoonup S_{\text{id}}(\zeta)$. From Lemma 5.25, we already have $\Xi = \Xi_D$ for some quasi-open $D \subseteq_{\text{q}} \Omega \setminus A_s(\zeta)$. It remains to check $I(\zeta) \subseteq_{\text{q}} D$. We set $w := \Xi_D(1)$ and $w_n := S'_{\text{id}}(\zeta_n; 1) = \Xi_{I(\zeta_n)}(1)$, cf. Theorem 4.3. From Theorem 5.14, we find $1 + \Delta w_n \geq 0$ and $\text{f-supp}(1 + \Delta w_n) =_{\text{q}} A(\zeta_n)$. Since $y_n := S_{\text{id}}(\zeta_n) = \psi$ q.e. on $A(\zeta_n)$ and since y_n and ψ are assumed to be Borel measurable, this gives

$$\int_\Omega (y_n - \psi) \, \mathrm{d}(1 + \Delta w_n) = 0.$$

The function $y - \psi$ is nonnegative and quasi lower-semicontinuous. Lemma 2.26 implies the existence of an increasing sequence $(v_m)_{m\in\mathbb{N}} \subseteq$

$H_0^1(\Omega)$ with $0 \le v_m$ and $v_m \nearrow y - \psi$ pointwise q.e. on Ω.
Let $m, n \in \mathbb{N}$. We have

$$\int_\Omega (v_m - y + y_n) \, \mathrm{d}(1 + \Delta w_n) \le \int_\Omega (y_n - \psi) \, \mathrm{d}(1 + \Delta w_n) = 0.$$

From $y_n \rightharpoonup y$ in $H_0^1(\Omega)$ and $w_n \to w$ in $H_0^1(\Omega)$, we infer

$$0 \le \int_\Omega v_m \, \mathrm{d}(1 + \Delta w) = \lim_{n\to\infty} \int_\Omega (v_m - y + y_n) \, \mathrm{d}(1 + \Delta w_n) \le 0.$$

Hence,

$$\int_\Omega v_m \, \mathrm{d}(1 + \Delta w) = 0.$$

Since $v_m \ge 0$ q.e. and $(1 + \Delta w)$-a.e. on Ω, we conclude $v_m = 0$ $(1 + \Delta w)$-a.e. on Ω. By monotone pointwise q.e. convergence of $(v_m)_{m\in\mathbb{N}}$ to $y - \psi$, we have

$$\bigcup_{m\in\mathbb{N}} \{v_m > 0\} =_{\mathrm{q}} I(\zeta).$$

Thus, since $(1 + \Delta w)(\{v_m > 0\}) = 0$ for all $m \in \mathbb{N}$ and since $(1 + \Delta w)$ vanishes on sets of capacity zero we conclude

$$(1 + \Delta w)(I(\zeta)) = (1 + \Delta w)\left(\bigcup_{m\in\mathbb{N}} \{v_m > 0\}\right) = 0$$

by σ-subadditivity of $(1 + \Delta w)$. This shows $I(\zeta) \subseteq_{\mathrm{q}} D$.

$\square$

5.3 Generalized differentials involving the weak operator topology

In this section, we will study the generalized differentials $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ and $\partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta)$. Since these differentials contain limits of Gâteaux derivatives in the weak operator topology, the corresponding sets are, in general, larger than the differentials using the strong operator topology, see Lemma 2.13. Moreover, since the completion of

$$\{\Xi_O \mid O \subseteq \Omega \text{ quasi-open}\}$$

w.r.t. the weak operator topology is $\mathcal{M}_0(\Omega)$, see Lemma 5.10 and Lemma 5.11, it is predictable that solution operators of certain relaxed Dirichlet problems w.r.t. capacitary measures are also contained in these generalized differentials.

We start with the analysis of $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$.

Lemma 5.30 *Let $\zeta \in H^{-1}(\Omega)$ be given. Then,*

$$\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta) \subseteq \{\Xi_\mu \mid \mu \in \mathcal{M}_0(\Omega), \mu(I(\zeta)) = 0 \text{ and } \mu = +\infty \text{ on } A_{\mathrm{s}}(\zeta)\}. \quad (5.9)$$

Here, $\mu = +\infty$ on $A_{\mathrm{s}}(\zeta)$ is to be understood as

$$\forall\, z \in H_0^1(\Omega) \cap L_\mu^2(\Omega): \qquad z = 0 \text{ q.e. on } A_{\mathrm{s}}(\zeta). \quad (5.10)$$

Proof. Let $\Xi \in \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ be given. By definition, there is a sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_{\mathrm{id}}}$ with $\zeta_n \to \zeta$ in $H^{-1}(\Omega)$ and $S_{\mathrm{id}}'(\zeta_n) \to \Xi$ in the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. From Lemma 5.10 we obtain $\Xi = \Xi_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$.

First, we show $\mu = +\infty$ on $A_{\mathrm{s}}(\zeta)$. Let $p \in H^{-1}(\Omega)$ be given. Then, $v_n := S_{\mathrm{id}}'(\zeta_n; p) \rightharpoonup \Xi_\mu(p) =: v$ and $|v_n| \rightharpoonup |v|$ in $H_0^1(\Omega)$, see Proposition 2.19(2.). For $\xi_n := -\Delta S_{\mathrm{id}}(\zeta_n) - \zeta_n$ and $\xi := -\Delta S_{\mathrm{id}}(\zeta) - \zeta$ we have $\xi_n \to \xi$ in $H^{-1}(\Omega)$. By $|v_n| = 0$ q.e. on $A_{\mathrm{s}}(\zeta_n)$, we find

$$0 = \lim_{n\to\infty} \langle \xi_n, |v_n| \rangle = \langle \xi, |v| \rangle.$$

Hence, $|v| = 0$ q.e. on $A_{\mathrm{s}}(\zeta)$, see Corollary 3.10. Since the range of Ξ_μ is dense in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$, see Lemma 5.16, we have $\mu = +\infty$ on $A_{\mathrm{s}}(\zeta)$.

It remains to show $\mu(I(\zeta)) = 0$. Let $v \in H_0^1(I(\zeta))$ with $0 \le v \le 1$ and $\{v > 0\} =_{\mathrm{q}} I(\zeta)$ be given, see Lemma 2.31 or Theorem 5.14. By Lemma 5.26, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \to v$ in $H_0^1(\Omega)$ and $v_n \in H_0^1(I(\zeta_n))$.

Therefore, $v_n = \Xi_{I(\zeta_n)}(-\Delta v_n) = S'_{\mathrm{id}}(\zeta_n; -\Delta v_n)$, see Proposition 5.24. Since $-\Delta v_n \to -\Delta v$ in $H^{-1}(\Omega)$, Lemma 2.11(2.) implies $v_n = S'_{\mathrm{id}}(\zeta_n; -\Delta v_n) \rightharpoonup \Xi_\mu(-\Delta v)$. Hence, $v = \Xi_\mu(-\Delta v)$ and therefore, $v \in L^2_\mu(\Omega)$. Testing the associated weak formulation with v, we infer

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}\lambda^d + \int_\Omega v^2 \, \mathrm{d}\mu = \langle -\Delta v, v \rangle = \int_\Omega |\nabla v|^2 \, \mathrm{d}\lambda^d.$$

Hence, $\int_\Omega v^2 \, \mathrm{d}\mu = 0$ and this means $v = 0$ μ-a.e. on Ω. Since $v > 0$ q.e. on $I(\zeta)$ and since μ vanishes on sets of capacity zero, we have $v > 0$ μ-a.e. on $I(\zeta)$. This implies $\mu(I(\zeta)) = 0$. □

Let us observe that if ζ is a point at which S_{id} is Gâteaux differentiable, then the strict complementarity condition holds at ζ, see Lemma 4.4, and consequently, the right-hand side in (5.9) reduces to $\{\Xi_{\infty_{A(\zeta)}}\} = \{\Xi_{I(\zeta)}\} = \{S'_{\mathrm{id}}(\zeta)\}$ and equality holds.

For arbitrary $\zeta \in H^{-1}(\Omega)$, the reverse inclusion in (5.9) is much harder to obtain, and we will prove it under some regularity assumption on ψ and $S_{\mathrm{id}}(\zeta)$.

However, in the special case that the entire set Ω is weakly active, i.e., $A(\zeta) =_{\mathrm{q}} \Omega$ and $A_{\mathrm{s}}(\zeta) =_{\mathrm{q}} \emptyset$, the equality in (5.9) just follows from the density result in Lemma 5.11.

Corollary 5.31 *Let $\zeta \in H^{-1}(\Omega)$ be given such that $A(\zeta) =_{\mathrm{q}} \Omega$ and $A_{\mathrm{s}}(\zeta) =_{\mathrm{q}} \emptyset$. Then,*

$$\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta) = \{\Xi_\mu \mid \mu \in \mathcal{M}_0(\Omega)\}.$$

In particular, (5.9) *holds with equality.*

Proof. "$\subseteq$": This inclusion is established in Lemma 5.30.
"$\supseteq$": From Theorem 5.27, we have

$$\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta) = \{\Xi_D \mid D \subseteq \Omega \text{ is quasi-open}\} \subseteq \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta).$$

Since the closure of the left-hand side w.r.t. the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H^1_0(\Omega))$ is $\{\Xi_\mu \mid \mu \in \mathcal{M}_0(\Omega)\}$, see Lemma 5.10, Lemma 5.11, and since $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ is closed in the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$, see Proposition 2.14(2.), this yields the claim.

□

The difficulty in proving the reverse implication in (5.9) in the general case lies in obtaining a suitable approximating sequence of quasi-open sets $(O_n)_{n\in\mathbb{N}}$ in the spirit of Lemma 5.11, which is typically proved in a rather abstract way.

In the following approach, the explicit construction from [DMM95] is used. This, however, needs that $A(\zeta_n)$ contains an open neighborhood of $A(\zeta)$ and, therefore, we assume continuity of $S_{\mathrm{id}}(\zeta)$ and ψ.

We first state two preparatory lemmata.

Lemma 5.32 *Let $\zeta \in H^{-1}(\Omega)$ be given and define $y := S_{\mathrm{id}}(\zeta)$. We assume that $y \in C(\overline{\Omega})$, $\psi \in C(\overline{\Omega}) \cap H^1(\Omega)$. Further, we assume that either $\psi \in H_0^1(\Omega)$ or $\psi < 0$ on $\partial\Omega$ holds.*

Then, there exists a sequence $(\zeta_n)_{n\in\mathbb{N}} \subseteq H^{-1}(\Omega)$ with $\zeta_n \to \zeta$ in $H^{-1}(\Omega)$ and such that $y_n := S_{\mathrm{id}}(\zeta_n)$ satisfies $y_n = \psi$ q.e. on an open neighborhood of $A(\zeta)$ as well as $\xi = -\Delta y - \zeta = -\Delta y_n - \zeta_n$.

Proof. Our strategy is to define y_n with the desired properties and to verify afterwards that y_n solves the obstacle problem with right-hand side $\zeta_n := -\Delta y_n - \xi$.

In the case that $\psi \in H_0^1(\Omega)$, we define $y_n := \max(y - 1/n, \psi)$. It is immediate that $y_n \in H_0^1(\Omega)$, $y_n \to y$ in $H_0^1(\Omega)$ and $y_n = \psi$ on $\{y < \psi + 1/n\}$, which is an open neighborhood of $A(\zeta)$.

In the case that $\psi < 0$ on $\partial\Omega$, we have $\psi \le c$ on $\partial\Omega$ for some constant $c < 0$. From $y = 0$ on $\partial\Omega$, we find that the set $\{y = \psi\}$ has a positive distance to the boundary of Ω. Thus, there exists a function $v \in C_c^\infty(\Omega)$ with $0 \le v \le 1$ and $v = 1$ on $\{y = \psi\}$. Now, we set $y_n := \max(y - v/n, \psi)$. Again, we find $y_n \in H_0^1(\Omega)$, $y_n \to y$ in $H_0^1(\Omega)$ and $y_n = \psi$ on $\{y < \psi + v/n\}$ which is also an open neighborhood of $A(\zeta)$.

Finally, we define $\zeta_n := -\Delta y_n - \xi$. Since $y_n \to y$ in $H_0^1(\Omega)$, it is immediate that $\zeta_n \to \zeta$ in $H^{-1}(\Omega)$ and we have to check that $y_n = S_{\mathrm{id}}(\zeta_n)$. The property

$y_n \in K_\psi$ is clear from the definition of y_n. From $A_{\mathrm{s}}(\zeta) \subseteq_{\mathrm{q}} \{y = \psi\} \subseteq \{y_n = \psi\}$, we infer $A_{\mathrm{s}}(\zeta) \subseteq_{\mathrm{q}} \{z \geq y_n\}$ for all $z \in K_\psi$. Hence,

$$\langle -\Delta y_n - \zeta_n, z - y_n \rangle = \langle \xi, z - y_n \rangle \geq 0.$$

This shows that $y_n = S_{\mathrm{id}}(\zeta_n)$. □

Let us state the following auxiliary result.

Lemma 5.33 *Let $\zeta \in H^{-1}(\Omega)$ be given such that the assumptions of Lemma 5.32 are satisfied. Then, for every measure $\mu \in \mathcal{M}_0(\Omega)$ which is finite on compact subsets of Ω with $\mu(I(\zeta)) = 0$, the measure $\nu = \mu + \infty_{A_{\mathrm{s}}(\zeta)}$ satisfies*

$$\Xi_\nu \in \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta).$$

Proof. Let μ be a given measure as in the formulation of the lemma. We can use the construction of [DMM95, Thm. 2.5] to obtain a sequence $(K_m)_{m\in\mathbb{N}}$ of compact subsets of Ω with the property that each K_m is contained in $\operatorname{supp}(\mu) + B_{1/m}$ and $\infty_{K_m} \xrightarrow{\gamma} \mu$. In particular, for all $n \in \mathbb{N}$, $K_m \subseteq \{y_n = \psi\}$ for m large enough with $y_n = S_{\mathrm{id}}(\zeta_n)$, where the sequence $(\zeta_n)_{n\in\mathbb{N}}$ is given by Lemma 5.32.

Now, we consider the sequence $(\nu_m)_{m\in\mathbb{N}}$ defined by

$$\nu_m := \infty_{K_m} + \infty_{A_{\mathrm{s}}(\zeta)} = \infty_{K_m \cup A_{\mathrm{s}}(\zeta)}$$

for $m \in \mathbb{N}$. By Proposition 5.23, we conclude that $\nu_m \xrightarrow{\gamma} \nu$ as $m \to \infty$. Fix $n \in \mathbb{N}$. Then, since

$$I(\zeta_n) \subseteq_{\mathrm{q}} K_m^{\complement} \cap (\Omega \setminus A_{\mathrm{s}}(\zeta)) =_{\mathrm{q}} K_m^{\complement} \cap (\Omega \setminus A_{\mathrm{s}}(\zeta_n)) \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(\zeta_n)$$

for m large enough, Theorem 5.27 implies that $\Xi_{\nu_m} \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta_n)$ for all but finitely many $m \in \mathbb{N}$. Thus, the set inclusion $\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta) \subseteq \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ (see Lemma 2.13) and Proposition 2.14(2.) imply that $\Xi_\nu \in \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta_n)$ for all $n \in \mathbb{N}$. Applying Proposition 2.14(2.) once more, we obtain that $\Xi_\nu \in \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ and the claim follows. □

Now, we are able to give the characterization of $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$ under the regularity assumptions of ψ and $S_{\mathrm{id}}(\zeta)$ considered in Lemma 5.32.

Theorem 5.34 *Let $\zeta \in H^{-1}(\Omega)$ be given such that the assumptions of Lemma 5.32 are satisfied. Then,* (5.9) *holds with equality, i.e.,*

$$\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta) = \{\Xi_\mu \mid \mu \in \mathcal{M}_0(\Omega), \mu(I(\zeta)) = 0 \textit{ and } \mu = +\infty \textit{ on } A_{\mathrm{s}}(\zeta)\}. \quad (5.11)$$

Proof. Let $\mu \in \mathcal{M}_0(\Omega)$ with $\mu(I(\zeta)) = 0$ and $\mu = \infty$ on $A_{\mathrm{s}}(\zeta)$. By Lemma 5.15, we find an increasing sequence $(\mu_m)_{m\in\mathbb{N}}$ of measures which are finite on compact subsets of Ω with $\mu_m \xrightarrow{\gamma} \mu$. Let $m \in \mathbb{N}$. Since $\mu_m \leq \mu$, it holds $\mu_m(I(\zeta)) = 0$. Thus, by Lemma 5.33, the measure $\nu_m := \mu_m + \infty_{A_{\mathrm{s}}(\zeta)}$ satisfies

$$\Xi_{\nu_m} \in \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta).$$

Furthermore, Proposition 5.23 implies that $\nu_m \xrightarrow{\gamma} \mu + \infty_{A_{\mathrm{s}}(\zeta)} = \mu$ as $m \to \infty$. The closedness property of $\partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}$, see Proposition 2.14(2.), implies that $L_\mu \in \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta)$. □

By means of an example, we indicate that the generalized differential $\partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta)$ can be surprisingly large. In fact, we have seen that for a point $\zeta \in \mathcal{D}_{S_{\mathrm{id}}}$ at which S_{id} is Gâteaux differentiable we have

$$\partial_{\mathrm{B}}^{\mathrm{ss}} S_{\mathrm{id}}(\zeta) = \partial_{\mathrm{B}}^{\mathrm{ws}} S_{\mathrm{id}}(\zeta) = \partial_{\mathrm{B}}^{\mathrm{sw}} S_{\mathrm{id}}(\zeta) = \{S_{\mathrm{id}}'(\zeta)\},$$

see Theorems 5.27 and 5.29 and Lemma 5.30. However, we will see that $\partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta)$ is not always a singleton for $\zeta \in \mathcal{D}_{S_{\mathrm{id}}}$.

Example 5.35 We use the classical construction of [CM97]. Therein, the authors construct a sequence $(O_n)_{n\in\mathbb{N}}$ of open subsets of Ω such that the solution operators Ξ_{O_n} of

$$\text{Find } \delta_n \in H_0^1(O_n): \quad -\Delta\delta_n = p$$

converge in the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ to the solution operator $\Xi_c := \Xi_{c\,\lambda^d}$ of

$$\text{Find } \delta \in H_0^1(\Omega): \quad -\Delta\delta + c\,\lambda^d(\delta\,\cdot) = p$$

for a positive constant $c > 0$.

We define $w = \Xi_c(1)$ and $w_n = \Xi_{O_n}(1)$. This yields $w_n \rightharpoonup w$. We fix the obstacle $\psi := 0$ and set $\zeta_n := -\Delta w_n - 2^{-n}\,\chi_{\Omega\setminus O_n}$ for $n \in \mathbb{N}$, $\zeta := -\Delta w$. Then, $w \in K_\psi$ by Theorem 5.14 and it is clear that $w = S_{\mathrm{id}}(\zeta)$. Moreover, again by Theorem 5.14, we have $w_n \in K_\psi$ and it holds $w_n = S_{\mathrm{id}}(\zeta_n)$ since $w_n \in H_0^1(O_n)$. Moreover, it holds $\zeta_n \rightharpoonup \zeta$.

Since $A(\zeta) =_{\mathrm{q}} \emptyset$, we have $\zeta \in \mathcal{D}_{S_{\mathrm{id}}}$. Similarly, we have $A(\zeta_n) =_{\mathrm{q}} \{w_n = 0\} =_{\mathrm{q}} \Omega \setminus O_n$, see Theorem 5.14. From $\xi_n := -\Delta w_n - \zeta_n = 2^{-n}\,\chi_{\Omega\setminus O_n}$, we have $A_{\mathrm{s}}(\zeta_n) =_{\mathrm{q}} \Omega\setminus O_n$, since $\Omega\setminus O_n$ is a finite union of balls (by construction). Thus, $\zeta_n \in \mathcal{D}_{S_{\mathrm{id}}}$ and $S'_{\mathrm{id}}(\zeta_n) = \Xi_{O_n}$. By construction, $\Xi_{O_n} \to \Xi_c$ in the weak operator topology of $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. Hence, $\Xi_c \in \partial_{\mathrm{B}}^{\mathrm{ww}} S_{\mathrm{id}}(\zeta)$ although $\zeta \in \mathcal{D}_{S_{\mathrm{id}}}$ and thus $S'_{\mathrm{id}}(\zeta) = \Xi_{\infty_{A(\zeta)}} \neq \Xi_c$, compare Lemma 5.6.

CHAPTER 6

Generalized derivatives for the solution operator of the bilateral obstacle problem

In this chapter, we derive generalized derivatives in $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$ for the composition S_f of the solution operator of the *bilateral* obstacle problem with a general monotone and continuously differentiable operator $f\colon U \to H^{-1}(\Omega)$ on a control Banach space U. For f and U we consider the assumptions from the previous analysis of the unilateral obstacle problem in Chapter 4. More precisely, $S_f := S_{\mathrm{id}} \circ f$ is the solution operator of the bilateral obstacle problem

$$\text{Find } y \in K_\psi^\varphi : \quad \langle Ly - f(u), z - y \rangle \geq 0 \quad \forall\, z \in K_\psi^\varphi. \tag{BOP$_f$}$$

Throughout this chapter, $\Omega \subseteq \mathbb{R}^d$ is an open, bounded set and $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ denotes a coercive operator. The admissible set K_ψ^φ in (BOP$_f$) is defined as

$$K_\psi^\varphi := \{z \in H_0^1(\Omega) \mid \psi \leq z \leq \varphi \text{ q.e. in } \Omega\}. \tag{6.1}$$

In general, we assume that the lower obstacle $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is quasi upper-semicontinuous and the upper obstacle $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ is quasi lower-semicontinuous. Moreover, we suppose that the resulting admissible set K_ψ^φ in (6.1) is nonempty. For the main results of this chapter, we additionally assume the following conditions to hold for L and ψ.

Assumption 6.1

1. *The operator L is assumed to be strictly T-monotone (see* (3.3)*).*
2. *We consider lower and upper obstacles $\psi, \varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and assume there is $c_\psi^\varphi > 0$ such that $\varphi - \psi \geq c_\psi^\varphi$ holds a.e. on Ω.*

Note that the assumption on strict T-monotonicity of L ensures that also the solution operator of the bilateral obstacle problem is increasing.

Throughout this chapter, U is a Banach space and $f\colon U \to H^{-1}(\Omega)$ is an operator. Similar to Chapter 4, for the main results in this chapter we additionally assume the following conditions.

Assumption 6.2

1. *We assume that the operator $f\colon U \to H^{-1}(\Omega)$ is defined on a partially ordered Banach space $(U, \geq_U)$. In addition, let f be increasing, i.e., $u_1 \geq_U u_2$ implies $f(u_1) \geq f(u_2)$ in $H^{-1}(\Omega)$.*
2. *The operator f is continuously differentiable.*
3. *U is separable and there is a partially ordered Banach space $(V, \geq_V)$ such that the positive cone $\mathcal{P} = \{v \in V : v \geq_V 0\}$ has nonempty interior and V is embedded into U. The order relation $\geq_V$ has the property that for all $v, w \in V$ with $v \geq_V w$ it holds $v + z \geq_V w + z$ for all $z \in V$ and $t\,v \geq_V t\,w$ for all $t \geq 0$. We assume that the linear embedding $\iota\colon V \to U$ is continuous, dense and increasing, i.e., compatible with the order structures in V and U. This means that $v \in V$ with $v \geq_V 0$ implies $\iota(v) \geq_U 0$ in U.*

Imposing these assumptions, the overall approach is similar to the strategy in Chapter 4. For arbitrary $u \in U$, we obtain the existence of increasing and decreasing sequences converging to u at which the locally Lipschitz continuous solution operator S_f of (BOP_f) is Gâteaux differentiable. Now, we characterize limits of the respective Gâteaux derivatives making use of the monotonicity structures.

Nevertheless, there is an essential difference in the analysis of the Gâteaux derivatives compared to the analysis for the unilateral obstacle problem. The multiplier $LS_f(u) - f(u) \in H^{-1}(\Omega)$ is no longer a nonnegative functional due to the bilateral constraints. Thus, the identification with a nonnegative measure, as in the unilateral case, is not possible. By representing the residual $LS_f(u) - f(u) \in H^{-1}(\Omega)$ as the difference $\tilde{\xi}_\psi - \tilde{\xi}^\varphi$ of two nonnegative measures (Theorem 6.5), we can define the strictly active set $A_{\mathrm{s}}(f(u))$ using the measures $\tilde{\xi}_\psi$ and $\tilde{\xi}^\varphi$. This gives us a description of the critical cone based on the pointwise q.e. behavior of elements in $H_0^1(\Omega)$. Recall that the critical cone is the admissible set in the variational inequality for the directional derivatives, cf. Theorem 3.3.

However, the representation of $LS_f(u) - f(u) \in H^{-1}(\Omega)$ as the difference of two measures requires care. We will see that, in general, not every element in $H_0^1(\Omega)$ is integrable with respect to the two measures and thus, the representation is not valid on the entire domain space $H_0^1(\Omega)$. In fact, it is valid only on $H_0^1(\Omega) \cap L^\infty(\Omega)$. We can also show that the representation holds true on the entire domain space $H_0^1(\Omega)$ if the active sets with respect to both obstacles have a positive distance. Subsequently, we give an example where the active sets do not have a positive distance and the representation fails on unbounded elements in $H_0^1(\Omega)$. Nonetheless, the pointwise description of the critical cone based on the strictly active set is given in any case.

Now, if the solution operator S_f is Gâteaux differentiable at u, then $S_f'(u)$ can be obtained as the solution operator of a variational equation on $H_0^1(D)$, where D can be chosen as any quasi-open subset of Ω satisfying $I(f(u)) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$, see Theorem 6.11.

When considering increasing, respectively decreasing, sequences $(u_n)_{n\in\mathbb{N}}$, we have to consider sets D_n which neither coincide with $I(f(u_n))$ nor with $\Omega \setminus A_{\mathrm{s}}(f(u_n))$ in the general case to show the Mosco convergence. The sets we study are complements of the combination of strictly active w.r.t. the lower obstacle and active set w.r.t. the upper obstacle and vice versa. These sets are non-monotone. This is due to the observation that strictly active and active sets are stable in opposite monotone directions and the behavior is exactly inverse for lower and upper obstacle. We are able to show the Mosco convergence of the corresponding sets $(H_0^1(D_n))_{n\in\mathbb{N}}$ towards $H_0^1(D)$ for D_n

and D as specified above. With this result, the convergence of the respective Gâteaux derivatives $S'_f(u_n)$ in the strong operator topology of $\mathcal{L}(U, H_0^1(\Omega))$ follows and by this strategy, we obtain elements of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$.

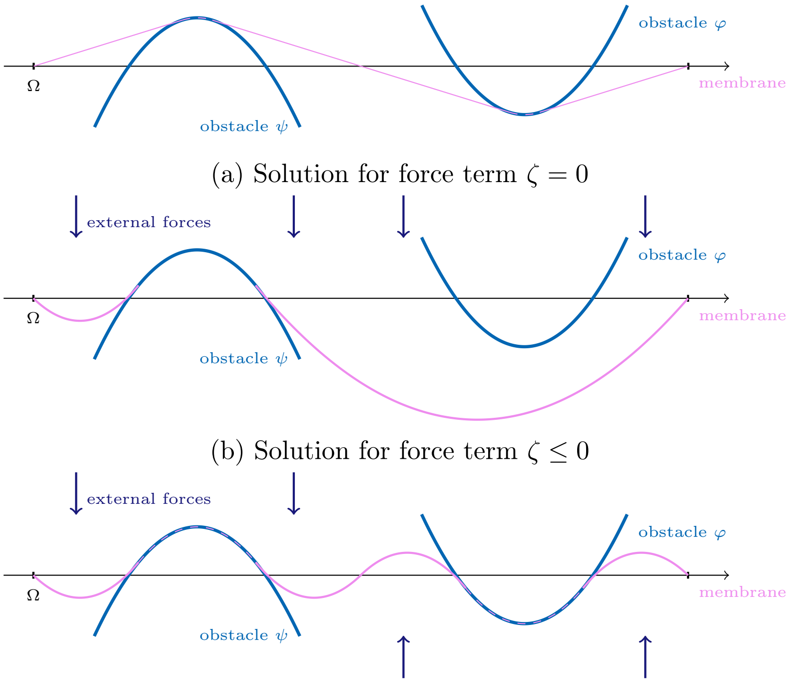
This chapter is organized as follows. In Section 6.1, we collect fundamental properties of the variational inequality (BOP_f) and of its solution operator S_f. In particular, we state the variational inequality for the directional derivative and the admissible set it the critical cone. The structure of the critical cone is analyzed in Section 6.2. To this end, a representation of the residual $LS_f(u) - f(u)$ as the difference $\tilde{\xi}_\psi - \tilde{\xi}^\varphi$ of two nonnegative measures is derived and the strictly active sets are defined based on these measures. A counterexample shows that this representation is, in general, only valid on $H_0^1(\Omega) \cap L^\infty(\Omega)$, which requires some care in the sequel. Moreover, a representation of $S'_f(u)$ is derived if u is a point at which S_f is Gâteaux differentiable. In Section 6.3, the monotonicity of the active and strictly active sets with respect to both obstacles is established. Using monotonicity and continuity properties, in Section 6.4, the Mosco convergence of $(H_0^1(D_n))_{n\in\mathbb{N}}$ to $H_0^1(D)$ is shown for two different choices of admissible sets $H_0^1(D_n)$, $H_0^1(D)$. in the variational equations for Gâteaux derivatives and for increasing and decreasing sequences $(u_n)_{n\in\mathbb{N}}$ converging to u, respectively. As a consequence, we can derive two elements of $\partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u)$ in Section 6.5. Finally, an adjoint representation of corresponding Clarke subgradients for an objective functional is derived in Section 6.6.

The results in this chapter are versions of the contents in [RU20].

6.1 Properties of the solution operator

In this section, we collect properties of the solution operator S_f of the bilateral obstacle problem (BOP_f). Let us first state that a unique solution of (BOP_f) exists for every $u \in U$ and that the corresponding solution operator is locally Lipschitz continuous.

Theorem 6.1 *Let $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be quasi upper-semicontinuous and let $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ be quasi lower-semicontinuous such that the admissible set*



(a) Solution for force term $\zeta = 0$

(b) Solution for force term $\zeta \leq 0$

(c) Solution for force term ζ

Figure 6.1. Bilateral obstacle problem for different force terms

K_ψ^φ in (6.1) is nonempty. Then, for each $u \in U$, the variational inequality (BOP_f) has a unique solution. Moreover, if f is locally Lipschitz continuous, then the solution operator $S_f \colon U \to H^{-1}(\Omega)$ of (BOP_f) is locally Lipschitz continuous.

Proof. By Theorem 3.1, the variational inequality (BOP_f) has a unique solution. Since $S_f = S_{\mathrm{id}} \circ f$ is Lipschitz continuous, see Theorem 3.1, the composition S_f is locally Lipschitz continuous if f is locally Lipschitz continuous. Here, S_{id} denotes the solution operator of (BOP_f) for $f = \mathrm{id} \colon H^{-1}(\Omega) \to H^{-1}(\Omega)$. $\square$

Figure 6.1 illustrates solutions of the bilateral obstacle problem for different force terms $\zeta \in H^{-1}(\Omega)$.

The next lemma summarizes the monotonicity of S_f with respect to the elements in U.

Lemma 6.2 *Assume that* $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ *satisfies Assumption* 6.1(1.)*. Let* $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ *be quasi upper-semicontinuous and let* $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ *be quasi lower-semicontinuous such that the admissible set* K_ψ^φ *in* (6.1) *is nonempty. Suppose that the conditions of Assumption* 6.2(1.) *on* f *and* U *are satisfied. Then the solution operator* $S_f\colon U \to H_0^1(\Omega)$ *of* (BOP$_f$) *is increasing, i.e., if* $u_1, u_2 \in U$ *satisfy* $u_1 \geq_U u_2$*, then it holds* $S_f(u_1) \geq S_f(u_2)$ *in* $H_0^1(\Omega)$*, i.e., a.e. and q.e. on* Ω*.*

Proof. It holds $S_f = S_{\mathrm{id}} \circ f$. Now, it follows from Proposition 3.2 that S_{id} is increasing. Since f is increasing, the composition S_f is increasing aswell. □

6.1.1 Differentiability properties of the solution operator

As for the unilateral obstacle problem, we can define the active and the inactive sets. Thus, we distinguish the following subsets of Ω for a fixed element $\zeta \in H^{-1}(\Omega)$ that result from the solution $S_{\mathrm{id}}(\zeta)$ of (BOP$_f$) for $f = \mathrm{id}$ being the identity operator on $H^{-1}(\Omega)$. Let $\zeta \in H^{-1}(\Omega)$. By

$$A(\zeta) :=_{\mathrm{q}} \{\omega \in \Omega \mid S_{\mathrm{id}}(\zeta)(\omega) = \psi(\omega) \text{ or } S_{\mathrm{id}}(\zeta)(\omega) = \varphi(\omega)\}$$

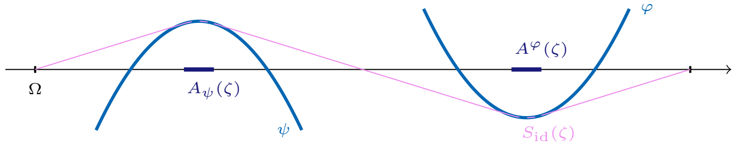
we denote the active set. We also distinguish the active sets with respect to ψ and φ, i.e., we define

$$A_\psi(\zeta) :=_{\mathrm{q}} \{\omega \in \Omega \mid S_{\mathrm{id}}(\zeta)(\omega) = \psi(\omega)\}$$

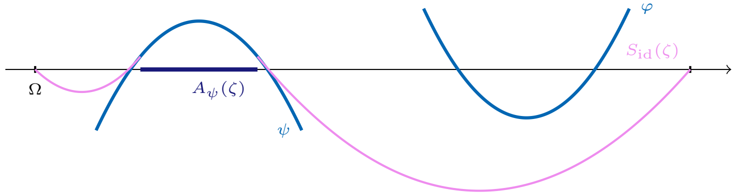
and

$$A^\varphi(\zeta) :=_{\mathrm{q}} \{\omega \in \Omega \mid S_{\mathrm{id}}(\zeta)(\omega) = \varphi(\omega)\}.$$

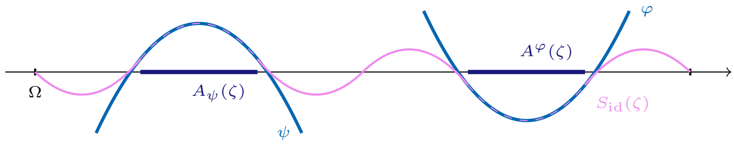
Note that $A(\zeta) =_{\mathrm{q}} A_\psi(\zeta) \cup A^\varphi(\zeta)$. As described in Lemma 2.22, by considering quasi-continuous representatives of $S_{\mathrm{id}}(\zeta) \in H_0^1(\Omega)$, the active sets are quasi-closed and defined up to a set of capacity zero. Moreover, by



(a) Active set for force term $\zeta = 0$



(b) Active set for force term $\zeta \leq 0$



(c) Active set for force term ζ

Figure 6.2. Active sets for different force terms, compare Fig. 6.1

Lemma 2.23, we can assume that they are Borel measurable.

We denote by $I(\zeta) :=_{\mathrm{q}} \Omega \setminus A(\zeta)$ the inactive set and by $I_\psi(\zeta) :=_{\mathrm{q}} \Omega \setminus A_\psi(\zeta)$, respectively $I^\varphi(\zeta) :=_{\mathrm{q}} \Omega \setminus A^\varphi(\zeta)$, the inactive sets with respect to the two obstacles.

Figure 6.2 illustrates the active sets with respect to the two obstacles in some exemplary scenarios.

Proposition 6.3 *Let $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be quasi upper-semicontinuous and $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ be quasi lower-semicontinuous such that the admissible set K_ψ^φ in* (6.1) *is nonempty. Assume that $f\colon U \to H^{-1}(\Omega)$ is directionally differentiable. Then the solution operator $S_f\colon U \to H_0^1(\Omega)$ of* (BOP_f) *is directionally differentiable. For given $u, h \in U$, the directional derivative $S_f'(u;h)$ is given by the solution of the variational inequality*

$$\textit{Find } \delta \in \mathcal{K}_{K_\psi^\varphi}(y,\xi): \quad \langle L\delta - f'(u;h), z - \delta\rangle \geq 0 \quad \forall\, z \in \mathcal{K}_{K_\psi^\varphi}(y,\xi). \tag{6.2}$$

Here, $y := S_f(u)$, $\xi := LS_f(u) - f(u)$ and $f'(u;h)$ denotes the directional derivative of f in $u \in U$ and in direction $h \in U$.

Moreover, $\mathcal{K}_{K_\psi^\varphi}(y,\xi) = \mathcal{T}_{K_\psi^\varphi}(y) \cap \xi^\perp$, where

$$\mathcal{T}_{K_\psi^\varphi}(y) = \{z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A_\psi(f(u)), z \leq 0 \text{ q.e. in } A^\varphi(f(u))\}. \tag{6.3}$$

Proof. By Lemma 3.8, K_ψ^φ is polyhedric at $(y, -\xi)$. Now, Theorem 3.3 implies that S_{id} is directionally differentiable. Observing that S_{id} is Lipschitz continuous, see Theorem 6.1, and thus, by Proposition 2.5, directionally differentiable in the Hadamard sense, Lemma 2.6 implies that the directional derivative $S_f'(u;h)$ exists and is given by the solution of (6.2). The characterization of (6.3) is established in [Mig76, Lem. 3.4]. □

6.2 Analysis of the critical cone

Let $u \in U$ and denote $y := S_f(u)$, $\xi := LS_f(u) - f(u)$. As in the case with a single obstacle, we want to find a suitable characterization of the critical cone

$$\mathcal{K}_{K_\psi^\varphi}(y,\xi) = \mathcal{T}_{K_\psi^\varphi}(y) \cap \xi^\perp.$$

Note that in the setting with a single lower obstacle such a characterization is given by

$$\{z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A(f(u)) \text{ and } z = 0 \text{ q.e. in } A_{\mathrm{s}}(f(u))\},$$

see (3.12).

A crucial difference to the case with only one obstacle is that ξ is not a nonnegative functional and thus cannot be identified with a nonnegative measure. Instead, we will see that, in some cases, it can be identified with the difference of two nonnegative measures. In general, i.e., when the active sets $A_\psi(f(u))$ and $A^\varphi(f(u))$ do not have a positive distance, $LS_f(u) - f(u)$

acts as the difference of two measures on all elements of $H_0^1(\Omega) \cap L^\infty(\Omega)$, but the characterization does not carry over to unbounded elements of $H_0^1(\Omega)$, see Example 6.8. Before we present the theorem on the representation, let us state an auxiliary lemma.

Lemma 6.4 *Assume that $v \in L^\infty(\Omega) \cap H^1(\Omega)$. Moreover, let $(w_n)_{n\in\mathbb{N}} \subseteq L^\infty(\Omega) \cap H_0^1(\Omega)$ be a sequence with $w_n \to 0$ in $H_0^1(\Omega)$ and $|w_n| \leq C$ a.e. for some $C > 0$ and all $n \in \mathbb{N}$. Then $v\,w_n \to 0$ in $H_0^1(\Omega)$.*

Proof. Let us recall that $v\,w_n \in H_0^1(\Omega)$ and $\nabla(v\,w_n) = w_n \nabla v + v \nabla w_n$, see Lemma 2.18. Thus,

$$\|v\,w_n\|_{H_0^1(\Omega)} \leq \|w_n \nabla v\|_{L^2(\Omega)} + \|v \nabla w_n\|_{L^2(\Omega)}$$

holds. The second term tends to zero since $v \in L^\infty(\Omega)$ and $\nabla w_n \to 0$ in $L^2(\Omega)$. Moreover, for a subsequence, the term $\|w_n \nabla v\|_{L^2(\Omega)}$ converges to zero aswell. To see this, pick any subsequence and choose a subsubsequence $(w_n)_n$ (without relabeling) that converges to 0 pointwise q.e., see Lemma 2.28, and thus pointwise λ^d-a.e. Now, the assertion follows from Lebesgue's dominated convergence theorem and $|w_n \nabla v| \leq C|\nabla v| \in L^2(\Omega)$. □

Now, we are looking for a characterization of the multiplier in terms of measures. Therefore, recall the definition of $\mathcal{M}_+(\Omega)$ in (2.8).

Theorem 6.5 *Assume that ψ, φ fulfill the conditions of Assumption* 6.1(2.). *Let $\zeta \in H^{-1}(\Omega)$ be arbitrary and set $y := S_{\mathrm{id}}(\zeta)$, $\xi := Ly - \zeta$. Then the following statements hold.*

1. *The functional $\xi \in H^{-1}(\Omega)$ acts as the difference $\tilde{\xi}_\psi - \tilde{\xi}^\varphi$ of nonnegative measures $\tilde{\xi}_\psi, \tilde{\xi}^\varphi \in \mathcal{M}_+(\Omega)$ on all elements of $H_0^1(\Omega) \cap C_{\mathrm{c}}(\Omega)$, i.e.,*

$$\langle \xi, w \rangle = \int_\Omega w \,\mathrm{d}\tilde{\xi}_\psi - \int_\Omega w \,\mathrm{d}\tilde{\xi}^\varphi \tag{6.4}$$

holds for all $w \in H_0^1(\Omega) \cap C_{\mathrm{c}}(\Omega)$.

2. *Let $A \subseteq \Omega$ be arbitrary. Then* $\mathrm{cap}(A) = 0$ *implies $\tilde{\xi}_\psi(A) = \tilde{\xi}^\varphi(A) = 0$.*
3. *The characterization* (6.4) *carries over to all $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In*

particular, the quasi-continuous representatives of w are $\tilde{\xi}_\psi$- and $\tilde{\xi}^\varphi$-integrable.

4. *Furthermore, it holds $y = \psi$ $\tilde{\xi}_\psi$-a.e. on Ω and $y = \varphi$ $\tilde{\xi}^\varphi$-a.e. on Ω, i.e., $\tilde{\xi}_\psi(I_\psi(\zeta)) = 0$ and $\tilde{\xi}^\varphi(I^\varphi(\zeta)) = 0$.*
5. *Assume $w \in H_0^1(\Omega) \cap L^1(\tilde{\xi}_\psi)$. Then we have $w \in L^1(\tilde{\xi}^\varphi)$ and (6.4) holds for w. The opposite statement with exchanged roles of $\tilde{\xi}_\psi$ and $\tilde{\xi}^\varphi$ is also true.*

Proof. 1. We define

$$v := \frac{y - \psi}{\varphi - \psi}.$$

By the assumptions on ψ and φ, we have $0 \le v \le 1$ and $v \in H^1(\Omega) \cap L^\infty(\Omega)$. Now, for $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have $v\,w$, $(1 - v)\,w \in H_0^1(\Omega)$, see Lemma 2.18, and we write

$$\langle \xi, w \rangle = \langle \xi, (1 - v)\,w \rangle + \langle \xi, v\,w \rangle.$$

Thus, we deduce

$$\langle \xi, w \rangle = \xi_\psi(w) - \xi^\varphi(w),$$

where ξ_ψ, ξ^φ are defined by

$$\xi_\psi \colon w \mapsto \langle \xi, (1 - v)\,w \rangle, \quad \xi^\varphi \colon w \mapsto \langle \xi, -v\,w \rangle.$$

Note that ξ_ψ, ξ^φ are nonnegative linear forms on $H_0^1(\Omega) \cap L^\infty(\Omega)$. To see this, assume $w \in H_0^1(\Omega)_+ \cap L^\infty(\Omega)$ and let first $\|w\|_{L^\infty} \le c_\psi^\varphi$. By definition of v, we have $-v\,w + y \in K_\psi^\varphi$ and therefore

$$\xi^\varphi(w) = \langle \xi, -v\,w + y - y \rangle \ge 0. \tag{6.5}$$

Since ξ^φ is linear, (6.5) holds for all $w \in H_0^1(\Omega)_+ \cap L^\infty(\Omega)$. In a similar fashion, we can show that ξ_ψ is nonnegative on $H_0^1(\Omega)_+ \cap L^\infty(\Omega)$.

In particular, ξ_ψ, ξ^φ are nonnegative linear forms on $H_0^1(\Omega) \cap C_c(\Omega)$. By [BS00, Lem. 6.53], ξ_ψ and ξ^φ have unique nonnegative continuous extensions over $C_{\mathrm{c}}(\Omega)$, also denoted by ξ_ψ, respectively ξ^φ. Moreover, by [BS00, Thm. 6.54], there are unique nonnegative, regular, locally finite Borel measures $\tilde{\xi}_\psi$, $\tilde{\xi}^\varphi$ such that

$$\xi_\psi(w) = \int_\Omega w \,\mathrm{d}\tilde{\xi}_\psi \quad \text{and} \quad \xi^\varphi(w) = \int_\Omega w \,\mathrm{d}\tilde{\xi}^\varphi$$

holds for all $w \in C_{\mathrm{c}}(\Omega)$. By completion, we obtain $\tilde{\xi}_\psi, \tilde{\xi}^\varphi \in \mathcal{M}_+(\Omega)$.

2. Now, we modify the proof of [BS00, Lem. 6.55] to show that for a set $A \subseteq \Omega$, $\operatorname{cap}(A) = 0$ implies $\tilde{\xi}_\psi(A) = \tilde{\xi}^\varphi(A) = 0$. W.l.o.g. we prove the statement for $\tilde{\xi}^\varphi$.

Let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ be a sequence with $\varepsilon_n \to 0$ as $n \to \infty$. Fix $n \in \mathbb{N}$. Then we find an open superset A_n of A in Ω with $\operatorname{cap}(A_n) < \varepsilon_n$. Furthermore, by Lemma 2.27, there is $u_n \in H^1_0(\Omega)_+$ satisfying $u_n = 1$ q.e. on A_n as well as $\|u_n\|^2_{H^1_0(\Omega)} = \operatorname{cap}(A_n) < \varepsilon_n$. Moreover, we can assume $u_n \in L^\infty(\Omega)$ and $0 \le u_n \le 1$, since $\min(z, 1) \in H^1_0(\Omega)$ satisfies $\|\min(z,1)\|_{H^1_0(\Omega)} \le \|z\|_{H^1_0(\Omega)}$ for $z \in H^1_0(\Omega)$. By regularity of $\tilde{\xi}^\varphi$, we can find a compact set $K_n \subseteq A_n$ satisfying $\tilde{\xi}^\varphi(A_n) \le \tilde{\xi}^\varphi(K_n) + \varepsilon_n$. Using a smooth version of Urysohn's lemma, there exists a function $g_n \in H^1_0(\Omega) \cap C_{\mathrm{c}}(\Omega)$ with values in $[0,1]$ satisfying $g_n = 1$ on K_n and having compact support in A_n. Then we have $1_{K_n} \le g_n \le u_n$ q.e. on Ω.

Now, we conclude

$$\begin{aligned}
\tilde{\xi}^\varphi(A_n) &\le \tilde{\xi}^\varphi(K_n) + \varepsilon_n \\
&\le \int_\Omega g_n \,\mathrm{d}\tilde{\xi}^\varphi + \varepsilon_n \\
&= \langle \xi, -v\, g_n \rangle + \varepsilon_n \\
&\le \langle \xi, -v\, u_n \rangle + \varepsilon_n \\
&\le \|\xi\|_{H^{-1}(\Omega)} \|v\, u_n\|_{H^1_0(\Omega)} + \varepsilon_n.
\end{aligned}$$

Using Lemma 6.4, we know that $\|v\, u_n\|_{H^1_0(\Omega)} \to 0$. Now, $\bigcap_{n\in\mathbb{N}} A_n$ is Borel measurable and

$$\tilde{\xi}^\varphi\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 0.$$

Since $A \subseteq \bigcap_{n\in\mathbb{N}} A_n$, we conclude $\tilde{\xi}^\varphi(A) = 0$.

3. Now, we argue in a similar fashion as in [BS00, Lem. 6.56] to show that each $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\langle \xi, w\rangle = \int_\Omega w \,\mathrm{d}\tilde{\xi}_\psi - \int_\Omega w \,\mathrm{d}\tilde{\xi}^\varphi.$$

Let $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we find $(\overline{w}_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ with $\overline{w}_n \to w$ in $H_0^1(\Omega)$. Definig $w_n := \max(-\|w\|_{L^\infty(\Omega)}, \min(\overline{w}_n, \|w\|_{L^\infty(\Omega)}))$ we have $w_n \in H_0^1(\Omega) \cap C_\mathrm{c}(\Omega)$ and $w_n \to w$ in $H_0^1(\Omega)$ as well as $|w_n| \le \|w\|_{L^\infty(\Omega)}$. Then, Lemma 6.4 yields $v\, w_n \to v\, w$ in $H_0^1(\Omega)$. Therefore, with

$$\| - v\,|w_n - w_m|\|_{H_0^1(\Omega)} = \big\||- v\,|w_n - w_m||\big\|_{H_0^1(\Omega)} = \|v\, w_n - v\, w_m\|_{H_0^1(\Omega)}$$

and

$$\begin{aligned}\|w_n - w_m\|_{L^1(\tilde{\xi}^\varphi)} &= \langle \xi, -v\,|w_n - w_m|\rangle \\ &\le \|\xi\|_{H^{-1}(\Omega)}\| - v\,|w_n - w_m|\|_{H_0^1(\Omega)}\end{aligned}$$

we find that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\tilde{\xi}^\varphi)$. Similarly, $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\tilde{\xi}_\psi)$. Thus, a subsequence converges pointwise $\tilde{\xi}^\varphi$-a.e. and $\tilde{\xi}_\psi$-a.e. to an element in $L^1(\tilde{\xi}^\varphi) \cap L^1(\tilde{\xi}_\psi)$. Now, pick a subsubsequence that converges pointwise q.e. to w. Then, by (2.), $w \in L^1(\tilde{\xi}^\varphi) \cap L^1(\tilde{\xi}_\psi)$ and Lebesgue's dominated convergence theorem implies

$$\langle \xi^\varphi, w\rangle = \underbrace{\langle \xi, -v(w - w_n)\rangle}_{\to 0 \text{ as } n\to\infty} + \underbrace{\int_\Omega w_n \,\mathrm{d}\tilde{\xi}^\varphi}_{\to \int_\Omega w \,\mathrm{d}\tilde{\xi}^\varphi \text{ along a subsequence}}.$$

This yields $\langle \xi^\varphi, w\rangle = \int_\Omega w \,\mathrm{d}\tilde{\xi}^\varphi$ and, similarly, $\langle \xi_\psi, w\rangle = \int_\Omega w \,\mathrm{d}\tilde{\xi}_\psi$.

4. We modify the proof of [Wac14, Prop. 2.5]. We consider a smooth cut-off function $\chi \in C_c^\infty(\Omega)$ with $0 \le \chi \le 1$ and $\chi = 1$ on a compact set $K \subseteq \Omega$.

We define $w := \chi\,[(1 - v)\,\psi + v\,y] + (1 - \chi)\,y$ and obtain $w \in K_\psi^\varphi$. This implies

$$0 \le \langle \xi, w - y\rangle$$

$$
\begin{aligned}
&= \langle \xi, \chi\,(1-v)\,\psi + \chi\,v\,y - \chi\,y \rangle \\
&= \langle \xi, (1-v)\,\chi\,(\psi - y) \rangle \\
&= \int_\Omega \chi\,(\psi - y)\ \mathrm{d}\tilde{\xi}_\psi.
\end{aligned}
$$

Since $\chi\,(\psi - y) \leq 0$ q.e. on Ω, and thus $\tilde{\xi}_\psi$-a.e., see (2.), we conclude $y = \psi$ $\tilde{\xi}_\psi$-a.e. on K. Covering Ω with countably many compact subsets, we infer $y - \psi = 0$ $\tilde{\xi}_\psi$-a.e. on Ω. Similarly, we can show $\varphi - y = 0$ $\tilde{\xi}^\varphi$-a.e. on Ω.

5. Assume $w \in H_0^1(\Omega) \cap L^1(\tilde{\xi}_\psi)$. We approximate w in $H_0^1(\Omega)$ by $(w_n)_{n\in\mathbb{N}}$ defined via $w_n := \max(-n, \min(n, w))$. Then we have $w_n \to w$ in $H_0^1(\Omega)$ and $w_n \to w$ pointwise $\tilde{\xi}_\psi$-a.e. (after choosing a subsequence). Since $|w_n| \leq |w|$ and since $w \in L^1(\tilde{\xi}_\psi)$, we apply Lebesgue's dominated convergence theorem and obtain $w_n \to w$ in $L^1(\tilde{\xi}_\psi)$.

From

$$
\begin{aligned}
&\|w_n - w_m\|_{L^1(\tilde{\xi}^\varphi)} = \int_\Omega |w_n - w_m|\ \mathrm{d}\tilde{\xi}^\varphi \\
&= \int_\Omega |w_n - w_m|\ \mathrm{d}\tilde{\xi}_\psi - \langle \xi, |w_n - w_m| \rangle \\
&\leq \|w_n - w_m\|_{L^1(\tilde{\xi}_\psi)} + \|\xi\|_{H^{-1}(\Omega)} \|w_n - w_m\|_{H_0^1(\Omega)}
\end{aligned}
$$

it follows that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\tilde{\xi}^\varphi)$ and we can again conclude that $w_n \to w$ in $L^1(\tilde{\xi}^\varphi)$.

From the representation

$$
\langle \xi, w_n \rangle = \int_\Omega w_n\ \mathrm{d}\tilde{\xi}_\psi - \int_\Omega w_n\ \mathrm{d}\tilde{\xi}^\varphi
$$

for all $n \in \mathbb{N}$, since $w_n \to w$ in $H_0^1(\Omega)$, $L^1(\tilde{\xi}_\psi)$ and $L^1(\tilde{\xi}^\varphi)$, we conclude

$$
\langle \xi, w \rangle = \int_\Omega w\ \mathrm{d}\tilde{\xi}_\psi - \int_\Omega w\ \mathrm{d}\tilde{\xi}^\varphi.
$$

The opposite statement follows similarly.

□

Remark 6.6 The results in Theorem 6.5 are generalizations of respective statements for the unilateral obstacle problem. Statements (1.) and (3.) correspond to the representation as a nonnegative measure in the unilateral case, cf. Lemma 3.7(1.). The unilateral counterpart for (2.) is given in Lemma 2.33 and the counterpart of (4.) is established in Lemma 3.7(2.).

In the subsequent lemma we assume that the active sets $A_\psi(\zeta)$ and $A^\varphi(\zeta)$ have a positive distance. With this condition we mean that there are quasi-closed sets $\tilde{A}_\psi(\zeta)$, $\tilde{A}^\varphi(\zeta)$ such that $\operatorname{dist}(\tilde{A}_\psi(\zeta), \tilde{A}^\varphi(\zeta)) > 0$ and the two sets coincide with $A_\psi(\zeta), A^\varphi(\zeta)$ up to a set of capacity zero.

Lemma 6.7 *Assume that ψ, φ fulfill the conditions of Assumption* 6.1(2.). *Let $\zeta \in H^{-1}(\Omega)$ be arbitrary and set $y := S_{\mathrm{id}}(\zeta)$ and $\xi := Ly - \zeta$. Suppose $A_\psi(\zeta)$ and $A^\varphi(\zeta)$ have a positive distance. Then, with $\tilde{\xi}_\psi, \tilde{\xi}^\varphi \in \mathcal{M}_+(\Omega)$ as in Theorem 6.5, it holds $H_0^1(\Omega) \subseteq L^1(\tilde{\xi}_\psi) \cap L^1(\tilde{\xi}^\varphi)$ and*

$$\langle \xi, w \rangle = \int_\Omega w \, \mathrm{d}\tilde{\xi}_\psi - \int_\Omega w \, \mathrm{d}\tilde{\xi}^\varphi$$

for all $w \in H_0^1(\Omega)$.

Proof. Since the active sets have a positive distance $C > 0$, we can find $v_2 \in C^\infty(\mathbb{R}^d)$ with $v_2 = 1$ q.e. on $A_\psi(\zeta)$ and $v_2 = 0$ q.e. outside $A_\psi(\zeta) + B_{C/2}$.

Since v_2 is smooth, we have $v_2\, w, (v_2 - 1)\, w \in H_0^1(\Omega)$ for all $w \in H_0^1(\Omega)$, see Lemma 2.18, and we define the functionals

$$\xi_\psi^2 \colon w \mapsto \langle \xi, v_2\, w \rangle, \quad \xi_2^\varphi \colon w \mapsto \langle \xi, (v_2 - 1)\, w \rangle$$

on $H_0^1(\Omega)$. Since v_2 is in $C^\infty(\mathbb{R}^d)$, it is easy to show that ξ_ψ^2 and ξ_2^φ are bounded linear functionals on $H_0^1(\Omega)$. Moreover, we have

$$\langle \xi, w \rangle = \langle \xi_\psi^2, w \rangle - \langle \xi_2^\varphi, w \rangle$$

for all $w \in H_0^1(\Omega)$.

Assume first w is in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we have

$$\int_\Omega w \, \mathrm{d}\tilde{\xi}_\psi = \int_\Omega v_2\, w \, \mathrm{d}\tilde{\xi}_\psi$$

$$
\begin{aligned}
&= \int_\Omega v_2\, w \; \mathrm{d}\tilde{\xi}_\psi - \int_\Omega v_2\, w \; \mathrm{d}\tilde{\xi}^\varphi \\
&= \langle \xi, v_2\, w \rangle \\
&= \langle \xi_\psi^2, w \rangle.
\end{aligned}
$$

Here, the first equation holds since $\tilde{\xi}_\psi(I_\psi(\zeta)) = 0$ and $w = v_2\, w$ q.e. and thus $\tilde{\xi}_\psi$-a.e. on $A_\psi(\zeta)$, see Theorem 6.5(2.) and (4.). Similarly, the second equation holds since $v_2\, w = 0$ $\tilde{\xi}^\varphi$-a.e. on Ω.

Let $w \in H_0^1(\Omega)$. Now we have $\max(-n, \min(w, n)) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $w_n \overset{n\to\infty}{\to} w$ in $H_0^1(\Omega)$. Furthermore,

$$
\begin{aligned}
&\|w_n - w_m\|_{L^1(\tilde{\xi}_\psi)} \\
&= \int_\Omega |w_n - w_m| \; \mathrm{d}\tilde{\xi}_\psi = \langle \xi_\psi^2, |w_n - w_m| \rangle \le \|\xi_\psi^2\|_{H^{-1}(\Omega)} \|w_n - w_m\|_{H_0^1(\Omega)}.
\end{aligned}
$$

Thus, $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\tilde{\xi}_\psi)$. Since $w_n \to w$ pointwise q.e. and thus $\tilde{\xi}_\psi$-a.e., see Theorem 6.5(2.), we have $w_n \to w$ in $L^1(\tilde{\xi}_\psi)$.

Arguing for $\tilde{\xi}^\varphi$ in a similar fashion, we obtain that

$$
\langle \xi, w \rangle = \langle \xi_\psi^2, w \rangle - \langle \xi_2^\varphi, w \rangle = \int_\Omega w \; \mathrm{d}\tilde{\xi}_\psi - \int_\Omega w \; \mathrm{d}\tilde{\xi}^\varphi.
$$

□

The following example shows that, in general, i.e., when the active sets do not have a positive distance, the characterization of the functional $LS_{\mathrm{id}}(\zeta) - \zeta$ as the difference of the two measures $\tilde{\xi}_\psi$ and $\tilde{\xi}^\varphi$ does not need to apply for all possible arguments in $H_0^1(\Omega)$. See also [Wac18, App. 2] for a related example.

Example 6.8 For $d = 2$ and for $0 < \beta < \frac{1}{2}$, consider the function

$$
y(x) = \sin((-\ln(|x|))^\beta), \quad x \in \Omega := B_\rho(0), \quad \rho = \exp(-\pi^{1/\beta}) < 1. \tag{6.6}
$$

Then $y|_{\partial B_\rho(0)} = \sin(\pi) = 0$ and $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$, since $|y| \le 1$ as well as

$$
|\nabla y(x)|^2 = \frac{\cos^2((-\ln(|x|))^\beta)\beta^2(-\ln(|x|))^{2\beta-2}}{|x|^2} \frac{x_1^2 + x_2^2}{|x|^2}
$$

$$\leq \beta^2(-\ln(|x|))^{2\beta-2}|x|^{-2}$$

and thus

$$\begin{aligned}\|y\|^2_{H^1_0(\Omega)} = \|\nabla y\|^2_{L^2(\Omega)} &\leq 2\pi\beta^2 \int_0^\rho (-\ln(r))^{2\beta-2} r^{-2} r \,\mathrm{d}r \\ &= 2\pi\beta^2 \frac{(-\ln(r))^{2\beta-1}}{1-2\beta}\Big|_0^\rho < \infty\end{aligned}$$

since $\beta < \frac{1}{2}$. Now, we consider the obstacles given by $\psi(x) := \min\left(-\frac{1}{2}, y(x)\right)$, $\varphi(x) := \max\left(\frac{1}{2}, y(x)\right)$.

We have

$$(-\ln(r(t)))^\beta = t \quad \Leftrightarrow \quad r(t) = \exp(-t^{1/\beta})$$

and, for $k \in \mathbb{N}$, we set $r_k^\pm := r(2k\pi \pm \pi/2)$. This choice implies $\rho = r(\pi) > r_1^- > r_1^+ > r_2^- > r_2^+ > \ldots > 0$ and $y(r_k^\pm(\cos t, \sin t)) = \pm 1$ for all $t \in (0, 2\pi)$.

Now, let $\omega_k > 0$ be weights (that will be adjusted below), with $\sum_{k=1}^\infty \omega_k^2 < \infty$ and consider the functional

$$\langle \xi, w\rangle := \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi} (w(r_k^-(\cos t, \sin t)) - w(r_k^+(\cos t, \sin t)))\,\mathrm{d}t \tag{6.7}$$

for $w \in H^1_0(\Omega)$. Note that the integral in (6.7) is well-defined. To see this, observe first that the quasi-continuous representatives of w are unique up to a set of capacity zero. Let now $E \subseteq \Omega$ be a set of capacity zero. Then, for any radius $0 < R < \rho$, by [Hel75, Thm. 7.5], the surface measure σ_R on the sphere $S_R = \partial B_R$ satisfies $\sigma_R(E \cap S_R) = 0$.

We have $\xi \in H^{-1}(\Omega)$, since

$$\begin{aligned}|\langle \xi, w\rangle| &\leq \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi}\int_{r_k^+}^{r_k^-} |\nabla w(r(\cos t, \sin t))| \frac{1}{r} r \,\mathrm{d}r\,\mathrm{d}t \\ &\leq \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \|\nabla w\|_{L^2(B_{r_k^-}(0)\setminus B_{r_k^+}(0))} \left(\int_0^{2\pi}\int_{r_k^+}^{r_k^-} \frac{1}{r^2} r\,\mathrm{d}r\,\mathrm{d}t\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2\pi} \sum_{k=1}^{\infty} \omega_k \|\nabla w\|_{L^2(B_{r_k^-}(0)\setminus B_{r_k^+}(0))} \\
&\le \sqrt{2\pi} \left(\sum_{k=1}^{\infty} \omega_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \|\nabla w\|^2_{L^2(B_{r_k^-}(0)\setminus B_{r_k^+}(0))} \right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi} \left(\sum_{k=1}^{\infty} \omega_k^2 \right)^{\frac{1}{2}} \|\nabla w\|_{L^2(\Omega)}.
\end{aligned}
$$

Here, we have used that, for all $k \in \mathbb{N}$, the sets $B_{r_k^-}(0) \setminus B_{r_k^+}(0)$ are disjoint subsets of Ω.

Set $\zeta := Ly - \xi$. We argue that the function $y \in K_\psi^\varphi$ as defined in (6.6) satisfies the bilateral obstacle problem (BOP_f) for $f = \mathrm{id}$, i.e., $y = S_{\mathrm{id}}(\zeta)$. First note that $y \in K_\psi^\varphi \subseteq H_0^1(\Omega) \cap L^\infty(\Omega)$ by the choice of y, ψ and φ. Now, let $z \in K_\psi^\varphi$ be arbitrary. Then $z - y \ge 0$ on $\{y = \psi\}$ and $z - y \le 0$ on $\{y = \varphi\}$. Additionally, we have

$$
\operatorname{supp}(\tilde{\xi}_\psi) \setminus \{0\} = \bigcup_{k=1}^{\infty} \partial B_{r_k^-} \subseteq \{y = \psi\} \tag{6.8}
$$

and

$$
\operatorname{supp}(\tilde{\xi}^\varphi) \setminus \{0\} = \bigcup_{k=1}^{\infty} \partial B_{r_k^+} \subseteq \{y = \varphi\}. \tag{6.9}
$$

This yields

$$
\begin{aligned}
\langle \xi, z - y \rangle &\overset{(6.4)}{=} \int_\Omega z - y \,\mathrm{d}\tilde{\xi}_\psi - \int_\Omega z - y \,\mathrm{d}\tilde{\xi}^\varphi \\
&= \int_{\{y=\psi\}} z - y \,\mathrm{d}\tilde{\xi}_\psi - \int_{\{y=\varphi\}} z - y \,\mathrm{d}\tilde{\xi}^\varphi \ge 0
\end{aligned}
$$

and we obtain $y = S_{\mathrm{id}}(\zeta)$.

Note that by (6.8) and (6.9) we find

$$
\operatorname{dist}(A_\psi, A^\varphi) = \operatorname{dist}(\{y = \psi\}, \{y = \varphi\}) = 0
$$

since $r_k^{\pm} \searrow 0$. Thus, Lemma 6.7 does not apply.

For $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}\langle \xi, w\rangle &= \int_\Omega w \, \mathrm{d}\tilde{\xi}_\psi - \int_\Omega w \, \mathrm{d}\tilde{\xi}^\varphi \\ &= \int_\Omega w \, \mathrm{d}\left(\sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+) r_k^-}} \sigma_{r_k^-}\right) - \int_\Omega w \, \mathrm{d}\left(\sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+) r_k^+}} \sigma_{r_k^+}\right)\end{aligned} \tag{6.10}$$

where $\tilde{\xi}_\psi, \tilde{\xi}^\varphi$ are nonnegative finite measures with support in A_ψ and A^φ, respectively. In fact, to show that $\tilde{\xi}^\varphi$ is a finite measure, we observe that

$$\ln(r_k^-/r_k^+) = (2k\pi + \pi/2)^{1/\beta} - (2k\pi - \pi/2)^{1/\beta} \begin{cases} \geq (2k\pi - \pi/2)^{1/\beta - 1}\pi/\beta, \\ \leq (2k\pi + \pi/2)^{1/\beta-1}\pi/\beta. \end{cases} \tag{6.11}$$

Hence, for any $w \in C(\overline{\Omega})$ with $0 \leq w \leq 1$

$$\begin{aligned} 0 \leq \int_\Omega w \, \mathrm{d}\tilde{\xi}^\varphi &\leq \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi} 1 \, \mathrm{d}t \\ &\leq 2\pi \left(\sum_{k=1}^\infty \omega_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \frac{1}{\ln(r_k^-/r_k^+)}\right)^{\frac{1}{2}} \\ &\leq 2\pi \left(\sum_{k=1}^\infty \omega_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \frac{1}{\pi/\beta(2k\pi - \pi/2)^{1/\beta-1}}\right)^{\frac{1}{2}} \leq C\end{aligned}$$

with a constant $C > 0$, since $\beta < 1/2$. The same argument shows that ξ^φ is a bounded functional on $H_0^1(\Omega) \cap L^\infty(\Omega)$ w.r.t. $\|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^\infty(\Omega)}$.

Now, consider the unbounded function $w(x) = (-\ln(|x|))^\beta - \pi$. Then since

$$|\nabla w(x)|^2 = \frac{\beta^2(-\ln(|x|))^{2\beta-2}}{|x|^2} \frac{x_1^2 + x_2^2}{|x|^2} = \beta^2(-\ln(|x|))^{2\beta-2}|x|^{-2},$$

we have $w \in H_0^1(\Omega)$ as above. With $\omega_k = k^{-1}$, $\beta = \frac{1}{3}$ and by using (6.11), we obtain the estimate

$$\begin{aligned}\int_\Omega w \, \mathrm{d}\tilde{\xi}^\varphi &= \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi} (2k\pi + \tfrac{\pi}{2} - \pi) \, \mathrm{d}t \\ &\geq \sum_{k=1}^\infty \frac{2\pi(2k\pi - \frac{\pi}{2})}{k(2k\pi + \frac{\pi}{2})\sqrt{3\pi}} \geq \sum_{k=1}^\infty \frac{2\pi \, \frac{3}{2} k\pi}{k \, \frac{5}{2} k\pi \, \sqrt{3\pi}} = \infty.\end{aligned}$$

Similarly, we find

$$\int_\Omega w \, \mathrm{d}\tilde{\xi}_\psi \geq \sum_{k=1}^\infty \frac{2\pi \, \frac{1}{2} k\pi}{k \, \frac{5}{2} k\pi \, \sqrt{3\pi}} = \infty.$$

This shows that, in general, the representation (6.4) does not hold for all $w \in H_0^1(\Omega)$.

In the following lemma, we find a characterization of the critical cone. This result is a counterpart to the characterization of the critical cone for the unilateral obstacle problem, see Theorem 3.9. Again, the proof is based on the proof of [Wac14, Lem. 3.1] with slight modifications.

Lemma 6.9 *Assume that ψ, φ fulfill the conditions of Assumption 6.1(2.). Let $\zeta \in H^{-1}(\Omega)$ be arbitrary and set $y := S_{\mathrm{id}}(\zeta)$, $\xi := Ly - \zeta$. Then the critical cone*

$$\mathcal{K}_{K_\psi^\varphi}(y, \xi) = \mathcal{T}_{K_\psi^\varphi}(y) \cap \xi^\perp$$

has the following structure. There exist quasi-closed sets $A_\psi^{\mathrm{s}}(\zeta) \subseteq_{\mathrm{q}} A_\psi(\zeta)$ and $A_{\mathrm{s}}^\varphi(\zeta) \subseteq_{\mathrm{q}} A^\varphi(\zeta)$ which are unique up to sets of capacity zero such that

$$\begin{aligned}\mathcal{K}_{K_\psi^\varphi}(y, \xi) &= \big\{ z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A_\psi(\zeta), z \leq 0 \text{ q.e. in } A^\varphi(\zeta) \\ &\qquad \text{and } \langle \xi, z \rangle = 0 \} \\ &= \big\{ z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A_\psi(\zeta), z \leq 0 \text{ q.e. in } A^\varphi(\zeta) \\ &\qquad \text{and } z = 0 \text{ q.e. in } A_\psi^{\mathrm{s}}(\zeta) \cup A_{\mathrm{s}}^\varphi(\zeta) \big\} .\end{aligned} \tag{6.12}$$

Proof. Recalling (6.3) from Proposition 6.3, we see that the first equation in (6.12) holds. Assume z is an element of $\mathcal{K}_{K_\psi^\varphi}(y,\xi)$. By polyhedricity, see Lemma 3.8, there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}_{K_\psi^\varphi}(y) \cap \xi^\perp$ with $z_n \to z$ in $H_0^1(\Omega)$. Since ψ, φ are elements of $L^\infty(\Omega)$, and by the structure of $\mathcal{R}_{K_\psi^\varphi}(y)$, see (3.4), we conclude $(z_n)_{n\in\mathbb{N}} \subseteq L^\infty(\Omega)$. Using Theorem 6.5, each z_n is integrable with respect to $\tilde{\xi}_\psi$ and $\tilde{\xi}^\varphi$. Let $n \in \mathbb{N}$ be fixed. By $\mathcal{R}_{K_\psi^\varphi}(y) \subseteq \mathcal{T}_{K_\psi^\varphi}(y)$ we have $z_n \geq 0$ q.e. in $A_\psi(\zeta)$ and $z_n \leq 0$ q.e. in $A^\varphi(\zeta)$. We have

$$0 = \langle \xi, z_n \rangle = \int_\Omega z_n \, \mathrm{d}\tilde{\xi}_\psi - \int_\Omega z_n \, \mathrm{d}\tilde{\xi}^\varphi = \int_{A_\psi(\zeta)} z_n \, \mathrm{d}\tilde{\xi}_\psi - \int_{A^\varphi(\zeta)} z_n \, \mathrm{d}\tilde{\xi}^\varphi, \tag{6.13}$$

as $\tilde{\xi}_\psi(I_\psi(\zeta))) = 0$ and $\tilde{\xi}^\varphi(I^\varphi(\zeta)) = 0$, see Theorem 6.5(4.). Since $z_n \geq 0$ $\tilde{\xi}_\psi$-a.e. on $A_\psi(\zeta)$ and $z_n \leq 0$ $\tilde{\xi}^\varphi$-a.e. on $A^\varphi(\zeta)$, cf. Theorem 6.5(2.), we conclude that $z_n = 0$ $\tilde{\xi}_\psi$-a.e. on $A_\psi(\zeta)$ and $z_n = 0$ $\tilde{\xi}^\varphi$-a.e. on $A^\varphi(\zeta)$. Using once more that $\tilde{\xi}_\psi(I_\psi(\zeta)) = 0$ and $\tilde{\xi}^\varphi(I^\varphi(\zeta)) = 0$, we can see that this means $z_n = 0$ $\tilde{\xi}_\psi$- and $\tilde{\xi}^\varphi$-a.e. on Ω. Since $z_n \to z$ for a subsequence pointwise q.e. and thus $\tilde{\xi}_\psi$- and $\tilde{\xi}^\varphi$-a.e., see Theorem 6.5(2.), we conclude $z = 0$ $\tilde{\xi}_\psi$- and $\tilde{\xi}^\varphi$-a.e.

Vice versa, assume $z \in \mathcal{T}_{K_\psi^\varphi}(y)$ and $z = 0$ $\tilde{\xi}_\psi$- and $\tilde{\xi}^\varphi$-a.e. Using Theorem 6.5(5.), we see that (6.13) holds (with z_n replaced by z) and thus $z \in \mathcal{T}_{K_\psi^\varphi}(y) \cap \xi^\perp$ follows.

Thus, we have shown that

$$\mathcal{K}_{K_\psi^\varphi}(y,\xi) = \big\{ z \in H_0^1(\Omega) \mid z \geq 0 \text{ q.e. in } A_\psi(\zeta), z \leq 0 \text{ q.e. in } A^\varphi(\zeta), \\ z = 0 \ \tilde{\xi}_\psi\text{- and } \tilde{\xi}^\varphi\text{-a.e.} \big\}.$$

By [Sto93, Thm. 1], there exist quasi-closed sets $A_\psi^{\mathrm{s}}(\zeta)$ and $A_{\mathrm{s}}^\varphi(\zeta)$ such that

$$\{z \in H_0^1(\Omega) \mid z = 0 \ \tilde{\xi}_\psi\text{-a.e.}\} = \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } A_\psi^{\mathrm{s}}(\zeta)\} \tag{6.14}$$

and

$$\{z \in H_0^1(\Omega) \mid z = 0 \ \tilde{\xi}^\varphi\text{-a.e.}\} = \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } A_{\mathrm{s}}^\varphi(\zeta)\}. \tag{6.15}$$

We have $y - \psi = 0$ $\tilde{\xi}_\psi$-a.e. and thus $y - \psi = 0$ q.e. on $A_\psi^{\mathrm{s}}(\zeta)$, see (6.14), which implies $A_\psi^{\mathrm{s}}(\zeta) \subseteq_{\mathrm{q}} A_\psi(\zeta)$. The same arguments apply to show $A_{\mathrm{s}}^\varphi(\zeta) \subseteq_{\mathrm{q}}$

$A^{\varphi}(\zeta)$. $\square$

As a bilateral counterpart to the result in Corollary 3.10, we obtain the following corollary.

Corollary 6.10 *Assume that ψ, φ fulfill the conditions of Assumption* 6.1(2.). *Let $\zeta \in H^{-1}(\Omega)$ be arbitrary and denote $y := S_{\mathrm{id}}(\zeta)$, $\xi := Ly - \zeta$. Let $z \in H_0^1(\Omega)$. Then the following statements are equivalent.*

(i) $z = 0$ *q.e. in* $A_{\psi}^{\mathrm{s}}(\zeta)$.

(ii) $z = 0$ $\tilde{\xi}_{\psi}$*-a.e. in* Ω.

The statements imply $z \in L^1(\tilde{\xi}_{\psi}) \cap L^1(\tilde{\xi}^{\varphi})$ and $\langle \xi, z\rangle = -\int_{\Omega} z \, d\tilde{\xi}^{\varphi}$. Similarly, the following statements are equivalent.

(i) $z = 0$ *q.e. in* $A_{\mathrm{s}}^{\varphi}(\zeta)$.

(ii) $z = 0$ $\tilde{\xi}^{\varphi}$*-a.e. in* Ω.

The statements imply $z \in L^1(\tilde{\xi}_{\psi}) \cap L^1(\tilde{\xi}^{\varphi})$ and $\langle \xi, z\rangle = \int_{\Omega} z \, d\tilde{\xi}_{\psi}$. Here, $\tilde{\xi}_{\psi}$, $\tilde{\xi}^{\varphi} \in \mathcal{M}_+(\Omega)$ are defined as in Theorem 6.5.

Proof. The equivalences are implied by the proof of Lemma 6.9, see (6.14) and (6.15). The statements $z \in L^1(\tilde{\xi}_{\psi}) \cap L^1(\tilde{\xi}^{\varphi})$ and $\langle \xi, z\rangle = \int_{\Omega} w \,\mathrm{d}\tilde{\xi}_{\psi} - \int_{\Omega} w \,\mathrm{d}\tilde{\xi}^{\varphi}$ follow from Theorem 6.5(5.). This shows the statement of the corollary. $\square$

In the following sections, for $\zeta \in H^{-1}(\Omega)$, we also write

$$A_{\mathrm{s}}(\zeta) :=_{\mathrm{q}} A_{\psi}^{\mathrm{s}}(\zeta) \cup A_{\mathrm{s}}^{\varphi}(\zeta)$$

for the strictly active set with respect to both obstacles, we have $A_{\mathrm{s}}(\zeta) \subseteq_{\mathrm{q}} A(\zeta)$. Moreover, we will use the notation

$$A_{\psi}^{\mathrm{w}}(\zeta) :=_{\mathrm{q}} A_{\psi}(\zeta) \setminus A_{\psi}^{\mathrm{s}}(\zeta)$$

for the weakly active set with respect to the lower obstacle ψ and

$$A_{\mathrm{w}}^{\varphi}(\zeta) :=_{\mathrm{q}} A^{\varphi}(\zeta) \setminus A_{\mathrm{s}}^{\varphi}(\zeta)$$

for the weakly active set with respect to the upper obstacle φ. For the sake

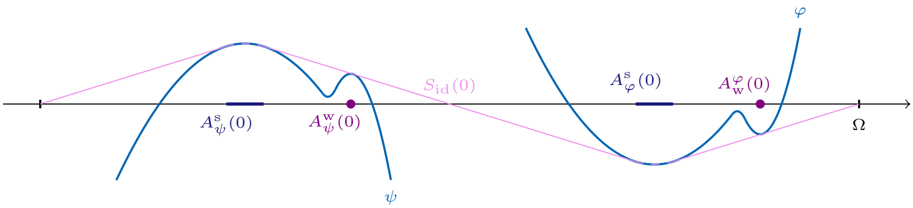


Figure 6.3. Strictly and weakly active sets for a bilateral obstacle problem with force term $\zeta = 0$

of completeness, we introduce the notation

$$A_{\mathrm{w}}(\zeta) :=_{\mathrm{q}} A^{\mathrm{w}}_{\psi}(\zeta) \cup A^{\varphi}_{\mathrm{w}}(\zeta)$$

for the weakly active set with respect to upper and lower obstacle.

Figure 6.3 illustrates an example where the strictly and weakly active sets are shown for fixed $\zeta = 0 \in H^{-1}(\Omega)$.

6.2.1 Gâteaux differentiability of the solution operator

As in the case of unilateral obstacle problems, in points u where S_f is Gâteaux differentiable, we can replace the critical cone in the characterization of the directional derivative by the largest linear subset contained in the critical cone, and by the linear hull of the critical cone, respectively. Both versions yield a characterization of the Gâteaux derivative. The result in the case of the unilateral obstacle problem can be found in Theorem 4.3.

For the bilateral case, characterizations of the Gâteaux derivative are summarized in the following theorem. Note that despite the difference in the analysis of the critical cone the characterization of Gâteaux derivatives is similar to the unilateral case.

Theorem 6.11 *Assume that ψ, φ fulfill the conditions of Assumption* 6.1(2.)*. Moreover, assume that $f\colon U \to H^{-1}(\Omega)$ is directionally differentiable. Suppose that the solution operator S_f of* (BOP$_f$) *is Gâteaux differentiable at $u \in U$ and let $h \in U$ be arbitrary. Then the directional derivative $S_f'(u; h)$ is* *determined by the solution of the variational equation*

$$\text{Find } \delta \in H_0^1(D): \quad \langle L\delta - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(D). \tag{6.16}$$

Here, any quasi-open set D with $I(f(u)) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$ is admissible and provides the same solution δ.

Proof. The assumption that u is a point where S_f is Gâteaux differentiable implies that $S_f'(u;\cdot)$ is linear and the image is a linear subspace of $H_0^1(\Omega)$. By the characterization (6.2) in Proposition 6.3, the image of $S_f(u;\cdot)$ lies in a linear subspace of the critical cone $\mathcal{K}_{K_\psi}(y,\xi)$ for $y = S_f(u)$, $\xi = Ly - f(u)$. The structure of the critical cone, cf. (6.12), implies that $S_f'(u;h) \in H_0^1(I(f(u)))$ for all $h \in U$, since $H_0^1(I(f(u)))$ is the largest linear subset contained in the critical cone. Thus, for all $h \in U$ it holds $S_f'(u;h) \in H_0^1(I(f(u)))$ and

$$\langle LS_f'(u;h) - f'(u;h), z - S_f'(u;h)\rangle \geq 0 \quad \forall\, z \in H_0^1(I(f(u))) \subset \mathcal{K}_{K_\psi}(y,\xi).$$

Since $H_0^1(I(f(u)))$ is a linear subspace, the variational inequality becomes a variational equation and thus $S_f'(u;h)$ is determined by the unique solution of (6.16) for $D :=_{\mathrm{q}} I(f(u))$.

On the other hand, the image of $S_f(u;\cdot)$ is also contained in the linear hull of the critical cone $\mathcal{K}_{K_\psi}(y,\xi)$, and this is the set $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$. We argue that the inequality

$$\langle LS_f'(u;h) - f'(u;h), z - S_f'(u;h)\rangle \geq 0$$

is fulfilled for all test functions z from $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$, and not only from the subset $\mathcal{K}_{K_\psi}(y,\xi)$.

Adopting the linearity arguments from the proof of Theorem 4.3, one can also use test functions from the negative critical cone. Let $z \in H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ be arbitrary. Since the two sets $\Omega \setminus (A_{\mathrm{s}}(f(u)) \cup A_\psi(f(u)))$ and $\Omega \setminus (A_{\mathrm{s}}(f(u)) \cup A^\varphi(f(u)))$ are a quasi-covering of $\Omega \setminus A_{\mathrm{s}}(f(u))$, we can find a sequence $(z_\psi^n + z_n^\varphi)_{n\in\mathbb{N}}$ converging to z and fulfilling $z_\psi^n \in H_0^1(\Omega \setminus (A_{\mathrm{s}}(f(u)) \cup A^\varphi(f(u))))$ and $z_n^\varphi \in H_0^1(\Omega \setminus (A_{\mathrm{s}}(f(u)) \cup A_\psi(f(u))))$, see the proof of Lemma 2.29. Considering positive and negative parts, we write

$$z_\psi^n = (z_\psi^n)_+ - (z_\psi^n)_- \quad \text{and } z_n^\varphi = (z_n^\varphi)_+ - (z_n^\varphi)_-.$$

The representation in (6.12) implies that $(z_\psi^n)_+$, $(z_\psi^n)_-$, $-(z_n^\varphi)_+$ and $-(z_n^\varphi)_-$ are elements of the critical cone. This shows

$$\langle LS_f'(u;h) - f'(u;h), z_\psi^n + z_n^\varphi - S_f'(u;h)\rangle \geq 0$$

for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ and observing that $H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ is a linear subspace we obtain

$$\langle LS_f'(u;h) - f'(u;h), z\rangle = 0$$

for all $z \in H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$.

Consider now an arbitrary quasi-open set D with $I(u) \subseteq_{\mathrm{q}} D \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u))$. Then we have $H_0^1(I(f(u))) \subseteq H_0^1(D) \subseteq H_0^1(\Omega \setminus A_{\mathrm{s}}(f(u)))$ and together with the previous observations this implies that for arbitrary $h \in U$, $S_f'(u;h)$ is the solution of (6.16). □

6.3 Monotonicity of the active and strictly active sets

Now, we study the monotonicity of the active and strictly active sets. Within this section, we specify our notation and write $S_{\psi,f}^\varphi$ depending on ψ and φ instead of S_f for the solution operator of (BOP_f).

The monotonicity of the active sets is a direct consequence of Lemma 6.2.

Lemma 6.12 *Assume that $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ satisfies Assumption 6.1(1.). Let $\psi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be quasi upper-semicontinuous and let $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ be quasi lower-semicontinuous such that the admissible set K_ψ^φ in (6.1) is nonempty. Suppose the conditions of Assumption 6.2(1.) on f and U are satisfied. Let $u_1, u_2 \in U$ satisfy $u_1 \geq_U u_2$. Then*

1. $A_\psi(f(u_1)) \subseteq_{\mathrm{q}} A_\psi(f(u_2))$,
2. $A^\varphi(f(u_1)) \supseteq_{\mathrm{q}} A^\varphi(f(u_2))$.

The following lemma is an auxiliary result and presents monotonicity properties of the variational inequality (BOP_f) with respect to one of the obstacles. The proof is very similar to the one of Proposition 3.2 on monotonicity with respect to the force terms.

Lemma 6.13 *Assume that* $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ *satisfies Assumption* 6.1(1.). *For* $i = 1, 2$, *let* $\psi_i \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ *be quasi upper-semicontinuous and let* $\varphi\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ *be quasi lower-semicontinuous such that the admissible sets* $K_{\psi_i}^\varphi$ *in* (6.1) *are nonempty. Let* $\zeta \in H^{-1}(\Omega)$. *Then* $\psi_1 \geq \psi_2$ *q.e. in* Ω *implies* $S_{\psi_1,\mathrm{id}}^\varphi(\zeta) \geq S_{\psi_2,\mathrm{id}}^\varphi(\zeta)$ *a.e. and q.e. in* Ω.

Proof. Set $y_i := S_{\psi_i,\mathrm{id}}^\varphi(\zeta)$. We test the variational inequality characterizing y_1 with $z_1 = \max(y_1, y_2) = y_1 + (y_2 - y_1)_+ \in K_{\psi_1}^\varphi$ and the variational inequality characterizing y_2 with $z_2 = \min(y_1, y_2) = y_2 - (y_2 - y_1)_+ \in K_{\psi_2}^\varphi$, respectively, and obtain

$$0 \leq \langle Ly_1 - \zeta, z_1 - y_1\rangle = \langle Ly_1 - \zeta, (y_2 - y_1)_+\rangle$$

and

$$0 \leq \langle Ly_2 - \zeta, z_2 - y_2\rangle = \langle Ly_2 - \zeta, -(y_2 - y_1)_+\rangle.$$

Summing up both inequalities we obtain

$$\langle Ly_1 - Ly_2, (y_2 - y_1)_+\rangle \geq 0.$$

By strict T-monotonicity, see (3.3), we have $(y_2 - y_1)_+ = 0$, i.e., $y_1 \geq y_2$ a.e. and q.e. in Ω. □

The following proposition is a counterpart to the result in Proposition 4.14. The proof is slightly different due to the presence of the upper obstacle and since the obstacles are assumed to be in $H^1(\Omega)$, and not merely quasi upper-/lower-semicontinuous functions.

Proposition 6.14 *Suppose ψ, φ satisfy the conditions of Assumption* 6.1(2.)*. Let $\zeta \in H^{-1}(\Omega)$ and let $v \in H_0^1(\Omega)_+$ such that $\{v > 0\} \subseteq_{\mathrm{q}} \Omega \setminus A_\psi^{\mathrm{s}}(\zeta)$. Then it holds $S_{\psi,\mathrm{id}}^\varphi(\zeta) = S_{\psi - v,\mathrm{id}}^\varphi(\zeta)$.*

Proof. Obviously, $\psi - v \leq S_{\psi,\mathrm{id}}^\varphi(\zeta) \leq \varphi$, i.e., $S_{\psi,\mathrm{id}}^\varphi(\zeta) \in K_{\psi-v}^\varphi$. Now, let $z \in K_{\psi-v}^\varphi$ be arbitrary. Using $(\psi - z)_+ - = 0$ q.e. on $A_\psi^{\mathrm{s}}(\zeta)$ which implies

$$\langle LS_{\psi,\mathrm{id}}(\zeta) - \zeta, -(\psi - z)_+\rangle = -\int_\Omega -(\psi - z)_+ \,\mathrm{d}\tilde{\xi}^\varphi \geq 0,$$

see Corollary 6.10, we obtain

$$\begin{aligned}&\langle LS_{\psi,\mathrm{id}}^\varphi(\zeta) - \zeta, z - S_{\psi,\mathrm{id}}^\varphi(\zeta)\rangle\\ &= \langle LS_{\psi,\mathrm{id}}^\varphi(\zeta) - \zeta, \max(z,\psi) - S_{\psi,\mathrm{id}}^\varphi(\zeta)\rangle + \langle LS_{\psi,\mathrm{id}}^\varphi(\zeta) - \zeta, -(\psi - z)_+\rangle\\ &\geq 0.\end{aligned}$$

Here, we have used the variational inequality characterizing $S_{\psi,\mathrm{id}}^\varphi(\zeta)$ since $\psi \leq \max(z,\psi) \leq \varphi$. This shows $S_{\psi,\mathrm{id}}^\varphi(\zeta) = S_{\psi-v,\mathrm{id}}^\varphi(\zeta)$. □

Before we are able to derive montonicity properties of the strictly active sets let us state the following auxiliary result.

Lemma 6.15 *Let $\zeta \in H^{-1}(\Omega)$. Then we have $-S_{\psi,\mathrm{id}}^\varphi(\zeta) = S_{-\varphi,\mathrm{id}}^{-\psi}(-\zeta)$. Moreover, $A_\psi(\zeta) =_{\mathrm{q}} \tilde{A}^{-\psi}(-\zeta)$ and $A^\varphi(\zeta) =_{\mathrm{q}} \tilde{A}_{-\varphi}(-\zeta)$. Here, $\tilde{A}_{-\varphi}(\zeta) =_{\mathrm{q}} \{\omega \in \Omega \mid S_{-\varphi,\mathrm{id}}^{-\psi}(\zeta)(\omega) = -\varphi(\omega)\}$ and $\tilde{A}^{-\psi}(\zeta) =_{\mathrm{q}} \{\omega \in \Omega \mid S_{-\varphi,\mathrm{id}}^{-\psi}(\zeta)(\omega) = -\psi(\omega)\}$ denote the respective active sets for $S_{-\varphi,\mathrm{id}}^{-\psi}(\zeta)$.*
Furthermore, if the conditions of Assumption 6.1 are fulfilled, we have $A_\psi^{\mathrm{s}}(\zeta) =_{\mathrm{q}} \tilde{A}_{\mathrm{s}}^{-\psi}(-\zeta)$ and $A_{\mathrm{s}}^\varphi(\zeta) =_{\mathrm{q}} \tilde{A}_{-\varphi}^{\mathrm{s}}(-\zeta)$. Here, $\tilde{A}_{\mathrm{s}}^{-\psi}(\zeta), \tilde{A}_{-\varphi}^{\mathrm{s}}(\zeta)$ denote the strictly active sets for $S_{-\varphi,\mathrm{id}}^{-\psi}(\zeta)$.

Proof. First, let us note that for $z \in H_0^1(\Omega)$ the inequalities $\psi \leq z \leq \varphi$ are valid if and only if $-\varphi \leq -z \leq -\psi$. This implies

$$-K_\psi^\varphi = K_{-\varphi}^{-\psi}.$$

Thus, $-S^{\varphi}_{\psi,\mathrm{id}}(\zeta) \in K^{-\psi}_{-\varphi}$. Let now $z \in K^{-\psi}_{-\varphi}$ be arbitrary. Then we have

$$\begin{aligned}
&\langle L(-S^{\varphi}_{\psi,\mathrm{id}}(\zeta)) + \zeta, z - (-S^{\varphi}_{\psi,\mathrm{id}}(\zeta))\rangle \\
&= -\langle LS^{\varphi}_{\psi,\mathrm{id}}(\zeta) - \zeta, z - (-S^{\varphi}_{\psi,\mathrm{id}}(\zeta))\rangle \\
&= \langle LS^{\varphi}_{\psi,\mathrm{id}}(\zeta) - \zeta, -z - S^{\varphi}_{\psi,\mathrm{id}}(\zeta)\rangle \\
&\geq 0.
\end{aligned}$$

This yields $-S^{\varphi}_{\psi,\mathrm{id}}(\zeta) = S^{-\psi}_{-\varphi,\mathrm{id}}(-\zeta)$.

Now, it holds $S^{-\psi}_{-\varphi,\mathrm{id}}(-\zeta)(\omega) = -\psi(\omega)$ if and only if $S^{\varphi}_{\psi,\mathrm{id}}(\zeta)(\omega) = \psi(\omega)$ and we have $S^{-\psi}_{-\varphi,\mathrm{id}}(-\zeta)(\omega) = -\varphi(\omega)$ if and only if $S^{\varphi}_{\psi,\mathrm{id}}(\zeta)(\omega) = \varphi(\omega)$, thus $A_{\psi}(\zeta) =_{\mathrm{q}} \tilde{A}^{-\psi}(-\zeta)$ and $A^{\varphi}(\zeta) =_{\mathrm{q}} \tilde{A}_{-\varphi}(-\zeta)$.

In addition, we have

$$LS^{\varphi}_{\psi,\mathrm{id}}(\zeta) - \zeta = -(LS^{-\psi}_{-\varphi,\mathrm{id}}(-\zeta) - (-\zeta)).$$

This shows the statements for the strictly active sets. □

Remark 6.16 If $f \colon U \to H^{-1}(\Omega)$ satisfies $f(-u) = -f(u)$ for all $u \in U$, then we obtain $-S^{\varphi}_{\psi,f}(u) = S^{-\varphi}_{-\psi,f}(-u)$.

Now, we check the monotonicity of the strictly active sets. The proof is related to the proof of Lemma 4.15.

Proposition 6.17 *Suppose Assumption 6.1 is satisfied. Assume ζ_1, ζ_2 are elements of $H^{-1}(\Omega)$ and suppose $\zeta_1 \geq \zeta_2$. Then it holds*

1. $A^{\mathrm{s}}_{\psi}(\zeta_1) \subseteq_{\mathrm{q}} A^{\mathrm{s}}_{\psi}(\zeta_2)$,
2. $A^{\varphi}_{\mathrm{s}}(\zeta_1) \supseteq_{\mathrm{q}} A^{\varphi}_{\mathrm{s}}(\zeta_2)$.

In particular, let f, U fulfill the conditions of Assumption 6.2(1.). Assume u_1, u_2 are elements of U and suppose $u_1 \geq_U u_2$. Then it holds

1. $A^{\mathrm{s}}_{\psi}(f(u_1)) \subseteq_{\mathrm{q}} A^{\mathrm{s}}_{\psi}(f(u_2))$,
2. $A^{\varphi}_{\mathrm{s}}(f(u_1)) \supseteq_{\mathrm{q}} A^{\varphi}_{\mathrm{s}}(f(u_2))$.

Proof. 1. Define

$$\mathcal{U} :=_{\mathrm{q}} \{S^{\varphi}_{\psi,\mathrm{id}}(\zeta_1) - \psi < (\varphi - \psi)/2\}. \tag{6.17}$$

Then $\mathcal{U}$ is quasi-open, see Lemma 2.22, and $A^{\mathrm{s}}_{\psi}(\zeta_1) \subseteq_{\mathrm{q}} A_{\psi}(\zeta_1) \subseteq_{\mathrm{q}} \mathcal{U} \subseteq_{\mathrm{q}} I^{\varphi}(\zeta_1)$ holds.

Assume $\mathcal{U} \setminus A^{\mathrm{s}}_{\psi}(\zeta_2) \neq_{\mathrm{q}} \emptyset$ (otherwise the assertion follows directly). Fix $v \in H^1_0(\mathcal{U})_+$ satisfying $\{v > 0\} =_{\mathrm{q}} \mathcal{U} \setminus A^{\mathrm{s}}_{\psi}(\zeta_2)$, $v \leq (\varphi - \psi)/2$, see Lemma 2.31.

Let $y_v(t) = S^{\varphi}_{\psi - tv, \mathrm{id}}(\zeta_1)$, $t \in [0,1]$, and set $\bar{y}_v(t) := y_v(t) + tv$. Note that $\bar{y}_v(t) \in K^{\varphi}_{\psi}$ for each $t \in [0,1]$ since $\psi \leq \bar{y}_v(t) = y_v(t) + tv \leq y_v(0) + tv \leq \varphi$ by Lemma 6.13 and the definition of $\mathcal{U}$ and v.

Then it holds

$$\langle L y_v(t) - \zeta_1, z - y_v(t) \rangle \geq 0 \quad \forall\, z \in K^{\varphi}_{\psi - tv}$$

which is equivalent to

$$\langle L \bar{y}_v(t) - \zeta_1 - tLv, \bar{z} - \bar{y}_v(t) \rangle \geq 0 \quad \forall\, \bar{z} \in K^{\varphi + tv}_{\psi}.$$

This in turn implies

$$\langle L \bar{y}_v(t) - \zeta_1 - tLv, \bar{z} - \bar{y}_v(t) \rangle \geq 0 \quad \forall\, \bar{z} \in K^{\varphi}_{\psi}.$$

We conclude $y_v(t) = S^{\varphi}_{\psi, \mathrm{id}}(T(tv)) - tv$ with $T \colon H^1_0(\Omega) \to H^{-1}(\Omega), v \mapsto \zeta_1 + Lv$. Since $S^{\varphi}_{\psi, \mathrm{id}}$ is directionally differentiable in the Hadamard sense, we can apply the chain rule for the directional derivatives and obtain

$$y_v'(0;1) = (S^{\varphi}_{\psi, \mathrm{id}})'(T(0); T'(0;v)) - v = (S^{\varphi}_{\psi, \mathrm{id}})'(\zeta_1; Lv) - v.$$

Since $(S^{\varphi}_{\psi, \mathrm{id}})'(\zeta_1; Lv)$ is 0 q.e. on the strictly active set $A_{\mathrm{s}}(\zeta_1)$, compare Proposition 6.3 and, in particular, Lemma 6.9, we have $y_v'(0;1) = -v < 0$ q.e. on $A_{\mathrm{s}}(\zeta_1) \cap \{v > 0\}$.

Thus, by reducing the lower obstacle on a subset of $A^{\mathrm{s}}_{\psi}(\zeta_1)$ the solution with respect to the new obstacle will drop on this set.

Now, we show the statement by contradiction. Therefore, assume the set $W \subseteq \Omega$ is a set of positive capacity which is (lower) weakly active for ζ_2 and (lower) strictly active for ζ_1, more precisely,

$$W \subseteq_{\mathrm{q}} A^{\mathrm{s}}_{\psi}(\zeta_1) \subseteq_{\mathrm{q}} A_{\psi}(\zeta_1) \subseteq_{\mathrm{q}} A_{\psi}(\zeta_2) \quad \text{and} \quad W \subseteq_{\mathrm{q}} \Omega \setminus A^{\mathrm{s}}_{\psi}(\zeta_2), \tag{6.18}$$

where we have used the inclusion of the active sets, see Lemma 6.12. Then $\mathcal{U}$ as in (6.17) is a quasi-open neighborhood of W contained in $I^{\varphi}(\zeta_1)$.
As above, let $v \in H_0^1(\mathcal{U})_+$ satisfy $\{v > 0\} =_{\mathrm{q}} \mathcal{U} \setminus A^{\mathrm{s}}_{\psi}(\zeta_2)$. Then, Proposition 6.14 yields

$$S^{\varphi}_{\psi-v,\mathrm{id}}(\zeta_2) = S^{\varphi}_{\psi,\mathrm{id}}(\zeta_2) \tag{6.19}$$

and on W we have

$$S^{\varphi}_{\psi-v,\mathrm{id}}(\zeta_1)|_W < S^{\varphi}_{\psi,\mathrm{id}}(\zeta_1)|_W = S^{\varphi}_{\psi,\mathrm{id}}(\zeta_2)|_W \tag{6.20}$$

by the structure of the directional derivative with respect to the obstacle and by (6.18). Putting (6.19) and (6.20) together, we see that

$$S^{\varphi}_{\psi-v,\mathrm{id}}(\zeta_2) > S^{\varphi}_{\psi-v,\mathrm{id}}(\zeta_1)$$

on W. On the other hand, $S^{\varphi}_{\psi-v,\mathrm{id}}(\zeta_1) \geq S^{\varphi}_{\psi-v,\mathrm{id}}(\zeta_2)$ since $\zeta_1 \geq \zeta_2$, see Lemma 6.2. Thus, such a set W cannot exist and we conclude $A^{\mathrm{s}}_{\psi}(\zeta_1) \subseteq_{\mathrm{q}} A^{\mathrm{s}}_{\psi}(\zeta_2)$.
2. By Lemma 6.15, we have $A^{\varphi}_{\mathrm{s}}(\zeta_i) =_{\mathrm{q}} \tilde{A}^{\mathrm{s}}_{-\varphi}(-\zeta_i)$ for $i = 1, 2$, where we use the same notation as in Lemma 6.15. Now, the first part of the lemma implies the statement, since

$$A^{\varphi}_{\mathrm{s}}(\zeta_1) =_{\mathrm{q}} \tilde{A}^{\mathrm{s}}_{-\varphi}(-\zeta_1) \supseteq_{\mathrm{q}} \tilde{A}^{\mathrm{s}}_{-\varphi}(-\zeta_2) =_{\mathrm{q}} A^{\varphi}_{\mathrm{s}}(\zeta_2).$$

Now, for f and U fulfilling the conditions of Assumption 6.2(1.), the statements $A^{\mathrm{s}}_{\psi}(f(u_1)) \subseteq_{\mathrm{q}} A^{\mathrm{s}}_{\psi}(f(u_2))$ and $A^{\varphi}_{\mathrm{s}}(f(u_1)) \supseteq_{\mathrm{q}} A^{\varphi}_{\mathrm{s}}(f(u_2))$ follow since f is increasing. $\square$

6.4 Mosco convergence

For the rest of the chapter we use again the notation S_f for the solution operator of (BOP_f).

The goal of this section is to choose a suitable characterization of the

Gâteaux derivatives such that for arbitrary $u \in U$ we can show the convergence $S'_f(u_n) \to \Xi$ in the strong operator topology for a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_f} = \{v \in U \mid S_f \text{ is Gâteaux differentiable at } v\}$ converging to u and characterize the limit Ξ. Recall that the Gâteaux derivatives are solution operators of variational equations on spaces $H^1_0(D_n)$ where the quasi-open sets D_n satisfy $I(f(u_n)) \subseteq_{\mathrm{q}} D_n \subseteq_{\mathrm{q}} \Omega \setminus A_{\mathrm{s}}(f(u_n))$, see Theorem 6.11,

As in the analysis of the unilateral obstacle problem, cf. Sections 4.4.2 and 4.5, the individual parts of the active sets show a stable behavior in a point $u \in U$ when either an increasing or decreasing sequence $(u_n)_{n\in\mathbb{N}}$ converging to $u \in U$ is considered. Thus, we consider complements of suitable compositions of active and strictly active sets with respect to both obstacles as candidates for the sequences $(D_n)_{n\in\mathbb{N}}$ for increasing and decreasing $(u_n)_{n\in\mathbb{N}}$. Note that such sets D_n, $n \in \mathbb{N}$, satisfy $I(f(u_n)) \subseteq_{\mathrm{q}} D_n \subseteq_{\mathrm{q}} \Omega\setminus A_{\mathrm{s}}(f(u_n))$. This means that if the elements u_n are points at which S_f is Gâteaux differentiable, the Mosco convergence implies the convergence of the Gâteaux derivatives, see Proposition 4.9.

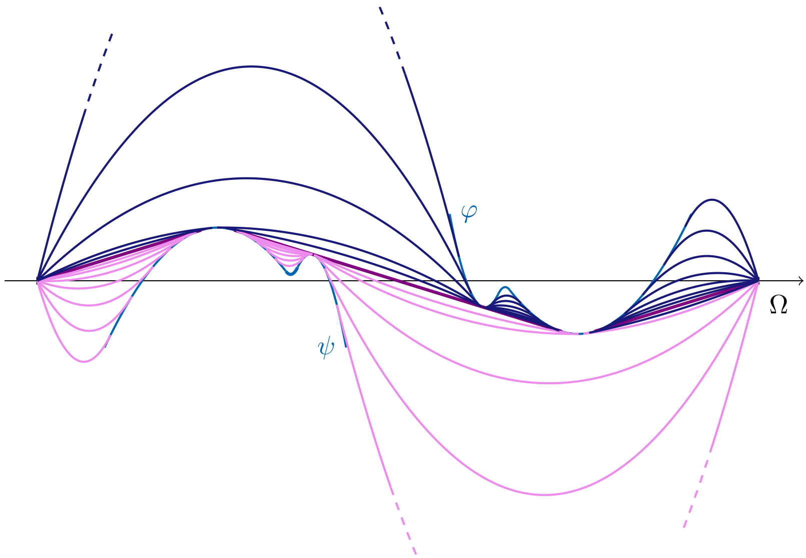
Already in the case of the unilateral obstacle problem, the active sets showed a stable behavior for increasing sequences $(u_n)_{n\in\mathbb{N}}$ while the strictly active sets showed a stable behavior for decreasing sequences $(u_n)_{n\in\mathbb{N}}$, compare Sections 4.4.2 and 4.5. This behavior carries over to the active and strictly active sets with respect to the lower obstacle for the bilateral obstacle problem. The behavior of the active and strictly active sets with respect to the upper obstacle problem is reverse. Thus, for arbitrary $u \in U$, this motivates us to show the Mosco convergence

$$H^1_0(I(f(u_n)) \cup A^{\varphi}_{\mathrm{w}}(f(u_n))) \overset{n\to\infty}{\to} H^1_0(I(f(u)) \cup A^{\varphi}_{\mathrm{w}}(f(u)))$$

for an increasing sequence $(u_n)_{n\in\mathbb{N}}$ converging to u and the Mosco convergence

$$H^1_0(I(f(u_n)) \cup A^{\mathrm{w}}_{\psi}(f(u_n))) \overset{n\to\infty}{\to} H^1_0(I(f(u)) \cup A^{\mathrm{w}}_{\psi}(f(u)))$$

for a decreasing sequence $(u_n)_{n\in\mathbb{N}}$ converging to u. Note that in general, none of the sequences $(H^1_0(I(f(u_n)) \cup A^{\varphi}_{\mathrm{w}}(f(u_n))))_{n\in\mathbb{N}}$, $(H^1_0(I(f(u_n)) \cup A^{\mathrm{w}}_{\psi}(f(u_n))))_{n\in\mathbb{N}}$ is monotone.



(a) An instance of the bilateral obstacle problem for force terms $\zeta \leq 0$, $\zeta = 0$ and $\zeta \geq 0$

(b) Corresponding sets $A_\psi(\zeta) \cup A_{\mathrm{s}}^{\varphi}(\zeta) =_{\mathrm{q}} (I(\zeta) \cup A_{\mathrm{w}}^{\varphi}(\zeta))^{\complement}$ for the different choices of ζ

(c) Corresponding sets $A_\psi^{\mathrm{s}}(\zeta) \cup A^{\varphi}(\zeta) =_{\mathrm{q}} (I(\zeta) \cup A_\psi^{\mathrm{w}}(\zeta))^{\complement}$ for the different choices of ζ

Figure 6.4. Influence of monotonicity in the behavior of active and strictly active sets for the bilateral obstacle problem

In Fig. 6.4, solutions of the bilateral obstacle problem for a family of values for $\zeta \in H^{-1}(\Omega)$ are illustrated and the respective parts of the active set considered here are depicted.

The following result establishes the Mosco convergence.

Theorem 6.18 *Suppose Assumption 6.1 is satisfied and let f, U fulfill the conditions of Assumption* 6.2(1.)*. Additionally, assume that f is continuous. Consider an arbitrary $u \in U$.*

1. *Let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be an increasing sequence with $u_n \to u$ in U. Then $H_0^1(I(f(u_n)) \cup A_{\mathrm{w}}^{\varphi}(f(u_n))) \to H_0^1(I(f(u)) \cup A_{\mathrm{w}}^{\varphi}(f(u)))$ in the sense of Mosco.*
2. *Let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be a decreasing sequence with $u_n \to u$ in U. Then $H_0^1(I(f(u_n)) \cup A_{\psi}^{\mathrm{w}}(f(u_n))) \to H_0^1(I(f(u)) \cup A_{\psi}^{\mathrm{w}}(f(u)))$ in the sense of Mosco.*

Proof. 1. Initially, we show the first condition of Mosco convergence stated in Definition 4.7. Therefore, let $v \in H_0^1(I(f(u)) \cup A_{\mathrm{w}}^{\varphi}(f(u)))$. Since we can consider positive and negative parts separately, we can assume w.l.o.g. $v \geq 0$. We can rewrite the function space as

$$\begin{aligned}&H_0^1(I(f(u)) \cup A_{\mathrm{w}}^{\varphi}(f(u)))\\&= \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } A_{\psi}(f(u)) \text{ and } z = 0 \text{ q.e. on } A_{\mathrm{s}}^{\varphi}(f(u))\}.\end{aligned}$$

Since $A_{\mathrm{s}}^{\varphi}(f(u_n)) \subseteq_{\mathrm{q}} A_{\mathrm{s}}^{\varphi}(f(u))$ for all $n \in \mathbb{N}$, see Proposition 6.17, it holds $v = 0$ q.e. on $A_{\mathrm{s}}^{\varphi}(f(u_n))$ for all $n \in \mathbb{N}$.

Since S_{id} is Lipschitz continuous, see Theorem 6.1, and f is assumed to be continuous, we have $S_f(u_n) \to S_f(u)$ in $H_0^1(\Omega)$. Thus, we conclude $S_f(u_n) \to S_f(u)$ for a subsequence pointwise quasi-everywhere, see Lemma 2.28. This means

$$\operatorname{cap}\left(I_{\psi}(f(u)) \setminus \bigcup_{n\in\mathbb{N}} I_{\psi}(f(u_n))\right) = 0,$$

i.e., $(I_{\psi}(f(u_n)))_{n\in\mathbb{N}}$ is a quasi-covering of $I_{\psi}(f(u))$, which is increasing in n, see Lemma 6.12. We can therefore find a sequence $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ with

$v_n \to v$ in $H_0^1(\Omega)$ and $v_n = 0$ q.e. on $A_\psi(f(u_n))$, see Lemma 2.29. Since $v \geq 0$, by considering positive parts, we can assume that $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)_+$. By setting

$$z_n := \min(v_n, v)$$

we have $z_n \in H_0^1(I(f(u_n)) \cup A_{\mathrm{w}}^\varphi(f(u_n)))$ for all $n \in \mathbb{N}$ as well as $z_n \to v$ in $H_0^1(\Omega)$.

Now, let us verify the second condition for Mosco convergence, cf. Definition 4.7. To this end, let $(v_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ be a sequence with $v_n \in H_0^1(I(f(u_n)) \cup A_{\mathrm{w}}^\varphi(f(u_n)))$ for all $n \in \mathbb{N}$. Assume there is a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ with $v_{n_k} \rightharpoonup v$ for some $v \in H_0^1(\Omega)$ as $k \to \infty$. Since $A_\psi(f(u)) \subseteq_{\mathrm{q}} A_\psi(f(u_{n_k}))$ for all $k \in \mathbb{N}$, see Lemma 6.12, we conclude $v \in H_0^1(I_\psi(f(u)))$ by Mazur's lemma. Using Corollary 6.10, from $v_n = 0$ q.e. on $A_{\mathrm{s}}^\varphi(f(u_n))$ and $v_n = 0$ q.e. on $A_\psi(f(u_n)) \supseteq_{\mathrm{q}} A_\psi^{\mathrm{s}}(f(u_n))$ we conclude

$$\langle LS_f(u_n) - f(u_n), |v_n|\rangle = \int_\Omega |v_n| \,\mathrm{d}\tilde{\xi}_\psi^n - \int_\Omega |v_n| \,\mathrm{d}\tilde{\xi}_n^\varphi = 0$$

for all $n \in \mathbb{N}$. From $v_{n_k} \rightharpoonup v$ in $H_0^1(\Omega)$ we conclude $|v_{n_k}| \rightharpoonup |v|$ in $H_0^1(\Omega)$, see Proposition 2.19(2.). Since also $LS_f(u_{n_k}) - f(u_{n_k}) \to LS_f(u) - f(u)$ in $H^{-1}(\Omega)$ we deduce

$$0 = \langle LS_f(u) - f(u), |v|\rangle = -\int_\Omega |v| \,\mathrm{d}\xi^\varphi$$

using $v = 0$ q.e. on $A_\psi(f(u)) \supseteq_{\mathrm{q}} A_\psi^{\mathrm{s}}(f(u))$ as derived above, cf. Corollary 6.10. Finally, this implies $v = 0$ ξ^φ-a.e. on Ω and thus, using again Corollary 6.10, $v = 0$ q.e. on $A_{\mathrm{s}}^\varphi(f(u))$. We have shown that v is an element of $H_0^1(I(f(u)) \cup A_{\mathrm{w}}^\varphi(f(u)))$, and thus, the second condition for Mosco convergence is verified.

2. Again, this part of the lemma follows from the first part of the lemma combined with Lemma 6.15.

□

6.5 Generalized derivatives for the bilateral obstacle problem

In this section, we will present a characterization of two generalized derivatives for the solution operator S_f of (BOP_f).

Theorem 6.19 *Suppose the conditions of Assumption 6.1 and Assumption 6.2 are satisfied. Let $u \in U$ be arbitrary. For $h \in U$, denote by $\Xi_{I(f(u))\cup A_{\mathrm{w}}^{\varphi}(f(u))}(h)$ the solution of*

$$\begin{aligned}&\textit{Find } \delta \in H_0^1(I(f(u)) \cup A_{\mathrm{w}}^{\varphi}(f(u))) :\\ &\langle L\delta - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(I(f(u)) \cup A_{\mathrm{w}}^{\varphi}(f(u)))\end{aligned}$$

and by $\Xi_{I(f(u))\cup A_{\psi}^{\mathrm{w}}(f(u))}(h)$ the solution of

$$\begin{aligned}&\textit{Find } \delta \in H_0^1(I(f(u)) \cup A_{\psi}^{\mathrm{w}}(f(u))) :\\ &\langle L\delta - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(I(f(u)) \cup A_{\psi}^{\mathrm{w}}(f(u))).\end{aligned}$$

Then it holds

$$\Xi_{I(f(u))\cup A_{\mathrm{w}}^{\varphi}(f(u))}, \Xi_{I(f(u))\cup A_{\psi}^{\mathrm{w}}(f(u))} \in \partial_{\mathrm{B}}^{\mathrm{ss}} S_f(u). \tag{6.21}$$

Proof. By Proposition 4.11, there is an increasing sequence $(u_n^+)_{n\in\mathbb{N}}$ and a decreasing sequence $(u_n^-)_{n\in\mathbb{N}}$ satisfying

$$(u_n^+)_{n\in\mathbb{N}}, (u_n^-)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{S_f} = \{v \in U \mid S_f \text{ is Gâteaux differentiable at } v\}$$

Theorem 6.18 implies that the sequence $(H_0^1(I(f(u_n^+)) \cup A_{\mathrm{w}}^{\varphi}(f(u_n^+))))_{n\in\mathbb{N}}$ converges to $H_0^1(I(f(u)) \cup A_{\mathrm{w}}^{\varphi}(f(u)))$ and the sequence $(H_0^1(I(f(u_n^-)) \cup A_{\psi}^{\mathrm{w}}(f(u_n^-))))_{n\in\mathbb{N}}$ converges to $H_0^1(I(f(u)) \cup A_{\psi}^{\mathrm{w}}(f(u)))$ in the sense of Mosco. Thus, recalling the characterization of the Gâteaux derivatives in Theorem 6.11 and using Proposition 4.9, we conclude that the sequences of Gâteaux derivatives $(S_f'(u_n^+))_{n\in\mathbb{N}}$, respectively $(S_f'(u_n^-))_{n\in\mathbb{N}}$ converge to $\Xi_{I(f(u))\cup A_{\mathrm{w}}^{\varphi}(f(u))}$, respectively $\Xi_{I(f(u))\cup A_{\psi}^{\mathrm{w}}(f(u))}$ in the strong operator topology of $\mathcal{L}(U, H_0^1(\Omega))$. This shows (6.21). □

6.6 Adjoint representation of Clarke subgradients

As in the unilateral case, see Section 4.7, we can find an adjoint representation for the Clarke subgradient of a reduced objective function.

Therefore, let $J\colon H_0^1(\Omega)\times U \to \mathbb{R}$ be a continuously differentiable objective function. We consider an optimization problem minimizing this objective function, which is constrained by the bilateral obstacle problem

$$\begin{aligned}
&\min_{y,u} J(y,u) \\
&\text{subject to} \quad y \in K_\psi^\varphi, \\
&\qquad\qquad \langle Ly - f(u), z - y\rangle \geq 0 \quad \forall\, z \in K_\psi^\varphi.
\end{aligned}$$

We present a formula for two generalized derivatives contained in Clarke's generalized differential $\partial_{\mathrm{C}}\hat{J}(u)$, see Definition 2.16, that can be obtained for the reduced objective function

$$\hat{J}(u) := J(S_f(u), u)$$

in an arbitrary point $u \in U$.

Corollary 6.20 *Suppose that the conditions of Assumption 6.1 and Assumption 6.2 are satisfied and let $u \in U$ be arbitrary. Let $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ be a continuously differentiable objective function and denote by q be the unique solution of the variational equation*

$$\begin{aligned}
&\text{Find } q \in H_0^1(D): \\
&\langle L^* q, v\rangle = \langle J_y\left(S_f(u), u\right), v\rangle \quad \forall\, v \in H_0^1(D).
\end{aligned} \tag{6.22}$$

Then the element

$$f'(u)^* q + J_u(S_f(u), u)$$

is contained in Clarke's generalized differential $\partial_{\mathrm{C}}\hat{J}(u)$. In (6.22), the respec-

tive sets

$$D :=_{\mathrm{q}} I(f(u)) \cup A^{\varphi}_{\mathrm{w}}(f(u)) \qquad or \qquad D :=_{\mathrm{q}} I(f(u)) \cup A^{\mathrm{w}}_{\psi}(f(u))$$

can be chosen and result in a particular generalized derivative.

Here, J_y and J_u denote the continuous Fréchet derivatives of J with respect to y and u, respectively, $f'(u)^* \in \mathcal{L}\left(H^1_0(\Omega), U^*\right)$ is the (Banachian) adjoint operator of $f'(u) \in \mathcal{L}\left(U, H^{-1}(\Omega)\right)$ and $L^* \in \mathcal{L}\left(H^1_0(\Omega), H^{-1}(\Omega)\right)$ is the (Banachian) adjoint operator of $L \in \mathcal{L}\left(H^1_0(\Omega), H^{-1}(\Omega)\right)$.

Proof. Using the elements of $\partial^{\mathrm{ss}}_{\mathrm{B}} S_f(u)$ for the solution operator S_f of the bileratal obstacle problem (BOP_f) derived in Theorem 6.19, the proof is similar to the proof of Theorem 4.21. $\square$

CHAPTER 7

Error estimates for generalized derivatives

This chapter treats the numerical challenges arising when computing inexact subgradients for the solution operator of a reduced objective function. This task arises when, e.g., computing a subgradient in a bundle method, see [HU19].

There are mainly two aspects where inexactness and discretization errors appear in this context. First of all, the solution of the obstacle problem itself (which gives the state in the optimization problem) is determined only inexactly and one obtains discrete solutions. This means that the corresponding exact active and strictly active sets are not available and one has to compute the generalized derivative or the subgradient on an inexact domain, see (7.1). We focus on this difficulty in the present chapter. Secondly, the system for the generalized derivative is discretized itself, which induces inexactness due to discretization errors.

We proceed as follows. In Section 7.1, we introduce the continuous and discrete obstacle problem that we consider in this chapter. In particular, we will see how the discrete solution of the obstacle problem affects the continuous systems for the generalized derivatives. An error estimate for the generalized derivative and the inexact generalized derivative based on the discrete solution of the obstacle problem is derived in Section 7.2. It relies on suitable approximations of the respective domain on which the continuous generalized derivative is computed from the in- and outside. These approximations are constructed in the upcoming parts of the chapter. To find appropriate approximations, it is not possible to work with arbitrarily irregular data. Therefore, we review some fundamental regularity results for the obstacle problem in Section 7.3. Section 7.4 follows up with a survey on L^∞-error estimates for the obstacle problem available in the literature, which give an upper bound for the maximum pointwise error between the continuous and discrete solutions of the obstacle problem. These error estimates are the basis for constructing convergent discrete subset approximations of the inactive set. The definition is shown in Section 7.5 and properties of the approximations are investigated. It is motivated in Section 7.6 that the inactive set is very hard to approximate from the outside. Therefore, the approximations of the inactive set found in Section 7.5 are used to construct both sub- and superset approximations of the complement of the strictly active set. For the construction, error estimates for the distances of the discrete and continuous free boundaries, the boundaries seperating the (inexact) active and inactive sets, are necessary and a version is presented in Section 7.6. These types of estimates for free boundaries usually rely on a so-called nondegeneracy condition, a regularity condition for the continuous problem that ensures a minimum growth of the solution away from the obstacle outside the active set and which originally goes back to Caffarelli. Based on this regularity assumption, the structure of the weakly and strictly active set is investigated in Section 7.6.2. Using the structure of the weakly and strictly active sets, in Section 7.6.3, sub- and superset approximations of $\Omega \setminus A_{\mathrm{s}}(\zeta)$ are constructed and the requirements for the usage in the error estimates for the generalized derivatives are verified. In Section 7.7, an alternative discrete subset approximation of the inactive set is presented that can be used instead of the set introduced in Section 7.5. Finally, our findings are summarized in Section 7.8. The chapter concludes with some numerical examples in Section 7.9.

7.1 Inexact generalized derivative on inexact domain

In this subsection, we introduce the overall setting, i.e., the continuous and discrete obstacle problem and the resulting continuous and inexact systems for generalized derivatives.

We assume that $\Omega \subseteq \mathbb{R}^d$ is an open, bounded domain. Let us consider the following continuous problem for a given $\zeta := f(u) \in L^2(\Omega)$ and $\psi \in H^1(\Omega)$

$$\text{Find } y \in K_\psi : \quad \langle -\Delta y - \zeta, z - y \rangle \geq 0 \quad \forall\, z \in K_\psi. \tag{COP}$$

As usual, the set K_ψ is defined as

$$K_\psi := \{z \in H_0^1(\Omega) : z \geq \psi \text{ a.e. in } \Omega\}.$$

The variational inequality (COP) coincides with the variational inequality ($\mathrm{OP}_{\mathrm{id}}$) for $L = -\Delta$.

Let $(\mathcal{T}_h)_{h>0}$ be a family of triangulations of Ω, i.e., a family of partitionings of Ω into triangles. We assume that the triangulations are regular and quasi-uniform and denote by $h > 0$ the mesh size, see [BS08] for details.

Now, for some $h > 0$, the solution of the obstacle problem is computed on a discrete level as the solution of

$$\text{Find } y_h \in K_h : \quad \langle -\Delta y_h - \zeta, z_h - y_h \rangle \geq 0 \quad \forall\, z_h \in K_h \tag{DOP}$$

with

$$K_h := \{z_h \in \mathbb{V}_h^0 \mid z_h \geq \psi_h \text{ a.e. in } \Omega\}$$

as well as

$$\mathbb{V}_h := \{v \in C(\bar{\Omega}) \mid v|_T \text{ is affine for all } T \in \mathcal{T}_h\} \quad \text{and} \quad \mathbb{V}_h^0 := \mathbb{V}_h \cap H_0^1(\Omega).$$

Here, $\psi_h = L_h \psi$ is the discrete obstacle and L_h is the Lagrange interpolation operator onto $\mathbb{V}_h$. This is the formulation of a discrete obstacle problem, which can also be found, e.g., in [NSV05]. The discrete variational inequality (DOP) admits a unique solution, see [Fri88], [KS00], [NSV05]. For a fixed

$\zeta \in L^2(\Omega)$ we denote the solution by y_h. Moreover, it holds $y_h \to y$ in $H_0^1(\Omega)$ as $h \to 0$, where y denotes the solution of (COP), see e.g., [Cia75, Thm. 9.2].

Note that, if Ω is not polyhedral, we could consider triangulations of Ω_h instead, where $\Omega_h \subseteq \Omega$ is polyhedral. Then $z_h \geq \psi_h$ is prescribed only on Ω_h and for the elements in $\mathbb{V}_h$ we demand $v|_{\bar{\Omega}\setminus\Omega_h} = 0$.

Let $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ be a continuously differentiable objective function. Based on the discrete solution y_h the goal is to find the solution q_h to

$$\text{Find } q_h \in H_0^1(D_h): \quad \langle -\Delta q_h, v_h \rangle = \langle J_y\left(y_h, u\right), v_h \rangle \quad \forall\, v_h \in H_0^1(D_h) \tag{7.1}$$

and to estimate the error $\|q - q_h\|_{H_0^1(\Omega)}$ relative to the solution q to

$$\text{Find } q \in H_0^1(D): \quad \langle -\Delta q, v \rangle = \langle J_y\left(y, u\right), v \rangle \quad \forall\, v \in H_0^1(D). \tag{7.2}$$

Here, D is a quasi-open set with $I(\zeta) \subseteq D \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta)$ admissible for the computation of a generalized derivative for the reduced objective function $\hat{J} = J(S_f(\cdot), \cdot)$, see Theorem 4.21 and Section 5.2 and D_h is an approximation thereof, based on the discrete solution y_h. We will discuss later on how D_h can be defined. Then the estimate for the error $\|q - q_h\|_{H_0^1(\Omega)}$ yields an estimate for the error of the Clarke subgradients

$$\xi(u) := f'(u)^* q + J_u(y, u) \in \partial_{\mathrm{C}} \hat{J}(u) \quad \text{and} \quad \xi_h(u) := f'(u)^* q_h + J_u(y_h, u) \tag{7.3}$$

for the reduced objective function $\hat{J}(\cdot) := J(S_f(\cdot), \cdot)$ via

$$\begin{aligned} &\|\xi(u) - \xi_h(u)\|_{U^*} \\ &\leq \|f'(u)^*\|_{\mathcal{L}(H_0^1(\Omega), U^*)} \|q - q_h\|_{H_0^1(\Omega)} + \|J_u(y, u) - J_u(y_h, u)\|_{U^*}, \end{aligned}$$

see Theorem 4.21. In this chapter, we always assume that the requirements on U and $f\colon U \to H^{-1}(\Omega)$ from Assumption 4.2 are fulfilled so that the obtained results in Chapter 4 are applicable. With a little abuse of language, we often call the solutions of (7.1) and (7.2) (inexact) generalized derivatives since they give the respective Clarke subgradients as in (7.3).

7.2 Error estimates for approximate Clarke subgradients

In this section we develop an error estimate for the inexact generalized derivatives of (7.1). Here, we consider decreasing sequences $(h_n)_{n\in\mathbb{N}}$ of positive numbers with $h_n \to 0$ in the context of triangulations with mesh size h_n. This describes the situation of successively refined meshes with mesh sizes $h_n \to 0$. Let us also fix the notation $y_n := y_{h_n}$ for the solution of (DOP), $q_n := q_{h_n}$ for the solution of (7.1) and $D_n := D_{h_n}$ for a discrete approximation of the quasi-open set D with $I(\zeta) \subseteq D \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta)$. An appropriate choice of the set D and the approximations D_n will be given later on. In the following sections, we always assume $D_n \subseteq D$.

Note that the mere Mosco convergence of $(H_0^1(D_n))_{n\in\mathbb{N}}$ to $H_0^1(D)$ yields that the solutions $(q_n)_{n\in\mathbb{N}}$ of (7.1) converge to the solution q of (7.2), see Proposition 4.9. For the usage of such approximate subgradients in, e.g., bundle methods, compare [HU19], it would be useful to know, a priori or a posteriori, a computable upper bound for the error $\|q_n - q\|_{H_0^1(\Omega)}$ which can be computed in each iteration of the bundle method and which gets arbitrarily small as $n \to \infty$. This helps in estimating the rate of convergence and in designing stopping criteria.

We will first find an upper bound for the error $\|q - q_n\|_{H_0^1(\Omega)}$ which cannot immediately be evaluated with knowledge only of the discrete solution and not the continuous one. Afterwards, in Corollary 7.3, an error estimate will be proposed that necessitates the existence of an additional Mosco convergent superset approximation $\tilde{D}_n$ of D.

Implicitly, we can estimate the error $\|q - q_n\|_{H_0^1(\Omega)}$ for the solutions q_n, q of (7.1) and (7.2) in the following way.

Lemma 7.1 *Let D be a quasi-open set with $I(\zeta) \subseteq D \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta)$ and, for $n \in \mathbb{N}$, let $D_n \subseteq D$ be quasi-open subsets. Denote by $q \in H_0^1(\Omega)$ the solution of* (7.2) *and for $n \in \mathbb{N}$ denote by q_n the solution of* (7.1)*. Then it holds*

$$\begin{aligned} &\|q - q_n\|_{H_0^1(\Omega)} \\ &\leq \| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(D)} + \|J_y(y_n, u) - J_y(y, u)\|_{H^{-1}(\Omega)}. \end{aligned} \tag{7.4}$$

If $H_0^1(D_n) \to H_0^1(D)$ in the sense of Mosco, we additionally have $\| -\Delta q_n - J_y(y_n, u)\|_{H^{-1}(D)} \to 0$ as $n \to \infty$.

Proof. Let us note that $q - q_n \in H_0^1(D)$, but not necessarily $q - q_n \in H_0^1(D_n)$. We observe

$$\begin{aligned}
&\|q - q_n\|^2_{H_0^1(\Omega)} \\
=&(\nabla(q - q_n), \nabla(q - q_n)) \\
=&(\nabla q, \nabla(q - q_n)) - \langle J_y(y, u), q - q_n\rangle - (\nabla q_n, \nabla(q - q_n)) \\
&+ \langle J_y(y, u), q - q_n\rangle \\
=&-(\nabla q_n, \nabla(q - q_n)) + \langle J_y(y, u), q - q_n\rangle \\
=&-(\nabla q_n, \nabla(q - q_n)) + \langle J_y(y_n, u), q - q_n\rangle + \langle J_y(y, u) - J_y(y_n, u), q - q_n\rangle \\
=&-\langle -\Delta q_n - J_y(y_n, u), q - q_n\rangle_{H^{-1}(D), H_0^1(D)} + \langle J_y(y, u) - J_y(y_n, u), q - q_n\rangle \\
\le&\big(\| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(D)} + \|J_y(y_n, u) - J_y(y, u)\|_{H^{-1}(\Omega)}\big) \\
&\cdot \|q - q_n\|_{H_0^1(\Omega)}.
\end{aligned}$$

This implies

$$\|q - q_n\|_{H_0^1(\Omega)} \le \| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(D)} + \|J_y(y_n, u) - J_y(y, u)\|_{H^{-1}(\Omega)}.$$

Assume the sequence $(H_0^1(D_n))_{n\in\mathbb{N}}$ converges to $H_0^1(D)$ in the sense of Mosco. Then, by Proposition 4.9, it holds $q_n \to q$ in $H_0^1(\Omega)$ as $n \to \infty$. Since also $y_n \to y$ in $H_0^1(\Omega)$ as $n \to \infty$ and since J is continuously differentiable we conclude

$$(-\Delta q_n - J_y(y_n, u)) \to (-\Delta q - J_y(y, u))$$

in $H^{-1}(\Omega)$. This implies $(-\Delta q_n - J_y(y_n, u)) \to (-\Delta q - J_y(y, u)) = 0$ in $H^{-1}(D)$ by (7.2). □

The error estimate (7.4) is not yet suitable, since the $H^{-1}(D)$ norm cannot be computed without knowledge of the exact set D. Let us establish the following tool.

Lemma 7.2 *Let $p_n, p \in H^{-1}(\Omega)$ and let $D \subseteq \Omega$ be a quasi-open set. Assume that $p_n \to p$ in $H^{-1}(\Omega)$ and $p_n \to 0$ in $H^{-1}(D)$ as $n \to \infty$. If there is a sequence $(\tilde{D}_n)_{n\in\mathbb{N}}$ of quasi-open supersets of D with the property that $H_0^1(\tilde{D}_n) \to H_0^1(D)$ in the sense of Mosco, then $\|p_n\|_{H^{-1}(\tilde{D}_n)} \to 0$.*

Proof. For $p \in H^{-1}(\Omega)$ we fix $v_n \in H_0^1(\tilde{D}_n)$, $\|v_n\|_{H_0^1(\tilde{D}_n)} \le 1$, $n \in \mathbb{N}$, with $\langle p, v_n\rangle_{H^{-1}(\tilde{D}_n),H_0^1(\tilde{D}_n)} \ge \|p\|_{H^{-1}(\tilde{D}_n)} - \frac{1}{n}$. Then, the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and we can extract a weakly convergent subsequence $(v_{n_k})_{k\in\mathbb{N}}$. We denote the weak limit by v. Note that $\|v\|_{H_0^1(\Omega)} \le 1$ by Mazur's lemma. By Mosco convergence of $H_0^1(\tilde{D}_n)$ to $H_0^1(D)$, the weak limit v is in $H_0^1(D)$. We have

$$\begin{aligned}
\|p\|_{H^{-1}(D)} \ge \langle p, v\rangle_{H^{-1}(D),H_0^1(D)} &= \langle p, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\
&= \lim_{k\to\infty} \langle p, v_{n_k}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\
&= \lim_{k\to\infty} \langle p, v_{n_k}\rangle_{H^{-1}(\tilde{D}_{n_k}),H_0^1(\tilde{D}_{n_k})} \\
&\ge \lim_{k\to\infty} \left(\|p\|_{H^{-1}(\tilde{D}_{n_k})} - \frac{1}{n_k} \right).
\end{aligned}$$

This, together with $\|p\|_{H^{-1}(D)} \le \|p\|_{H^{-1}(\tilde{D}_{n_k})}$ for all $k \in \mathbb{N}$, which follows from the inclusion $H_0^1(D) \subseteq H_0^1(\tilde{D}_{n_k})$, implies that $\langle p, v\rangle_{H^{-1}(D),H_0^1(D)} = \|p\|_{H^{-1}(D)}$. By a subsequence-subsequence argument, we can conclude that $\left(\|p\|_{H^{-1}(\tilde{D}_n)}\right)_{n\in\mathbb{N}}$ converges to $\|p\|_{H^{-1}(D)}$.

Now, we can estimate

$$\begin{aligned}
\|p_n\|_{H^{-1}(\tilde{D}_n)} &\le \|p_n - p\|_{H^{-1}(\tilde{D}_n)} + \|p\|_{H^{-1}(\tilde{D}_n)} \\
&\le \|p_n - p\|_{H^{-1}(\Omega)} + \|p\|_{H^{-1}(\tilde{D}_n)} \\
&\to \|p\|_{H^{-1}(D)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|p_n\|_{H^{-1}(\tilde{D}_n)} &\ge \|p\|_{H^{-1}(\tilde{D}_n)} - \|p - p_n\|_{H^{-1}(\tilde{D}_n)} \\
&\ge \|p\|_{H^{-1}(\tilde{D}_n)} - \|p - p_n\|_{H^{-1}(\Omega)}
\end{aligned}$$

$$\to \|p\|_{H^{-1}(D)}$$

and we conclude that

$$\|p_n\|_{H^{-1}(\tilde{D}_n)} \to \|p\|_{H^{-1}(D)} = 0.$$

□

As stated in the following corollary, the construction of an outer approximation of D, which is computed based on y_n, yields an error estimate for $\|q - q_n\|_{H_0^1(\Omega)}$.

Corollary 7.3 *Let D be a quasi-open set with $I(\zeta) \subseteq D \subseteq \Omega \setminus A_\mathrm{s}(\zeta)$ and, for $n \in \mathbb{N}$, assume D_n, $\tilde{D}_n$ are quasi-open sets with*

$$D_n \subseteq D \subseteq \tilde{D}_n \tag{7.5}$$

and

$$H_0^1(D_n) \to H_0^1(D) \quad \text{and} \quad H_0^1(\tilde{D}_n) \to H_0^1(D) \tag{7.6}$$

in the sense of Mosco. Let q_n, q be the solutions of (7.1), (7.2), *respectively. Then the error estimate*

$$\begin{aligned}&\|q - q_n\|_{H_0^1(\Omega)}\\ &\leq \| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)} + \|J_y(y_n, u) - J_y(y, u)\|_{H^{-1}(\Omega)} \overset{n\to\infty}{\to} 0\end{aligned}$$

is valid.

Proof. Combining (7.4) with the inclusion $H_0^1(D) \subseteq H_0^1(\tilde{D}_n)$ for all $n \in \mathbb{N}$, we derive

$$\begin{aligned}&\|q - q_n\|_{H_0^1(\Omega)}\\ &\leq \| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)} + \|J_y(y_n, u) - J_y(y, u)\|_{H^{-1}(\Omega)}.\end{aligned}$$

As argued in the proof of Lemma 7.1, we have $(-\Delta q_n - J_y(y_n, u)) \to (-\Delta q - J_y(y, u))$ in $H^{-1}(\Omega)$. By Lemma 7.1, it holds $\|-\Delta q_n - J_y(y_n, u)\|_{H^{-1}(D)} \overset{n\to\infty}{\to}$

0. Now, Lemma 7.2 implies that $\| -\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)} \stackrel{n\to\infty}{\to} 0$ and the conclusion follows. $\square$

7.3 Short survey on regularity results for the solution of the obstacle problem

The derivation of error estimates for a generalized derivative that is computed based on discrete solutions of the obstacle problem will require restrictions on the data guaranteeing a certain regularity of the solution y of (COP). Therefore, the purpose of this short section is to collect some established smoothness results for the solution of the obstacle problem under regularity assumptions on the data ζ, ψ and Ω.

In the literature, many authors are concerned with regularity results for the solutions of obstacle problems. Among many others, we mention [CK80], [Fre72], [Fri88], [KS00] and [Rod87].

In the remainder of this chapter, we always assume that ψ and y are continuous. However, the assumption on y can be verified under mild assumptions on ζ and ψ, since $H^2(\Omega)$ elements have continuous representatives in dimension $d = 2, 3$. The result is taken from [Rod87, Cor. 5:2.3] and the proof is based on dual estimates.

Lemma 7.4 *Assume $\zeta \in L^2(\Omega)$ and $\psi \in H^2(\Omega)$. If Ω is a convex domain or if $\partial\Omega$ is sufficiently smooth, the solution y of* (COP) *satisfies $y \in H^2(\Omega)$.*

Moreover, the following statement concerning the continuity of the solution of (COP) is true and can be found in [Rod87, Thm. 5:2.7].

Lemma 7.5 *Assume $\zeta \in W^{-1,s}(\Omega)$ for some $s > d \geq 2$. and let $\psi \in C(\overline{\Omega})$. If Ω has a Lipschitz boundary, then the solution y of* (COP) *satisfies $y \in C(\bar{\Omega}) \cap H_0^1(\Omega)$.*

We also mention the following result on solutions in spaces of higher regularity which is taken from [KS00, Thm. IV 2.3].

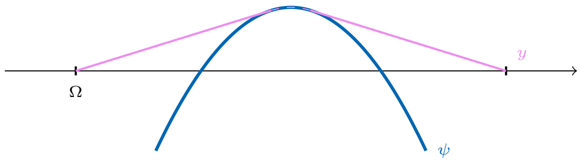


Figure 7.1. One-dimensional solution $y = S_{\mathrm{id}}(0)$ of the obstacle problem with discontinuous second derivatives at $\partial A(0)$

Lemma 7.6 *Let Ω be an open connected domain with suffiently smooth boundary $\partial\Omega$. Assume there is a number s with $d < s < \infty$ and suppose $\zeta \in L^s(\Omega)$ and* $\max(-\Delta\psi - \zeta, 0) \in L^s(\Omega)$. *Then, the solution y of* (COP) *is in $H^{2,s}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for $\beta = 1 - \frac{d}{s}$.*

In general, the solution y inherits the smoothness of the obstacle. It is clear that at least on the interior of $A(\zeta)$, y will be as regular as ψ. On $I(\zeta)$, regularity theory for the Poisson problem will apply, when $\zeta \in L^2(\Omega)$. Nevertheless, the presence of the obstacle implies that the regularity of the solution y is limited, regardless of how smooth ψ and ζ are.

This can already be seen in dimension $d = 1$. Consider a parabolic obstacle ψ with $\psi > 0$ somewhere in Ω and $\psi < 0$ on $\partial\Omega$. We consider the solution of the obstacle problem for $\zeta = 0$. Note that $\psi, \zeta \in C^\infty(\overline{\Omega})$. The solution of the obstacle problem is piecewise affine (on the inactive set), satisfies $y = 0$ on $\partial\Omega$ and strikes the obstacle tangentially. The situation is illustrated in Fig. 7.1. Here, the second derivatives of y are discontinuous on $\partial A(0)$.

In fact, it is shown that the optimal regularity of the obstacle problem is $C^{1,1}$, as the following lemma by [BK74, Thm. 1] demonstrates. We mention also the references [Fre72] and [Fri88].

Lemma 7.7 *Assume $\zeta \in C^1(\overline{\Omega})$ and $\psi \in C^2(\overline{\Omega})$. Suppose Ω has a smooth boundary $\partial\Omega$. Then the solution y of* (COP) *satisfies $y \in C^{1,1}(\Omega)$.*

In the formulation in [BK74], the admissible set K_ψ in (COP) is replaced by the set of *Lipschitz functions* v on Ω which satisfy $v \geq \psi$ on Ω and $v = 0$ on $\partial\Omega$. This is a convex subset of K_ψ. Since under the regularity assumptions in Lemma 7.7 the solution y of (COP) is also Lipschitz continuous, see

Lemma 7.6, y is a solution of the problem considered in [BK74].

Note that this is only a short selection of results concerning the regularity of solutions to (COP) that show that the assumptions on the solution we have in this chapter can be fulfilled and can be guaranteed a priori by requiring the appropriate regularity of the data.

7.4 Short survey on L^∞-estimates for the solution of the discrete obstacle problem

In this section, we will review various types of L^∞-error estimates for the obstacle problems available in the literature.

We aim to obtain inexact generalized derivatives in a point $u \in U$ based on the inexact finite element solution of the obstacle problem depending on the mesh size h. Since the systems for the generalized derivatives (7.1) and (7.2) depend on the solution of the obstacle problem $y = S_f(u)$, respectively y_h, itself, or, more precisely, on the complements of the respective active and strictly active set, the error $\|y - y_h\|$ is relevant also for the generalized derivatives. Since the active set $A(\zeta) = \{y = \psi\}$ is characterized by the *pointwise behavior* of the solution y, the pointwise error of $y - y_h$ is of interest and is best estimated via the error $\|y - y_h\|_{L^\infty(\Omega)}$ in the $L^\infty(\Omega)$-norm. Note that we assume that y is continuous here. Thus, the knowledge of an upper bound ε_h for $\|y - y_h\|_{L^\infty(\Omega)}$ helps to control and analyze the error in the active sets and in their boundaries, the *free boundaries*, see Section 7.6.1.

A priori L^∞-error estimates for the obstacle problem have been derived in the literature. They are based on the discrete maximum principle of Raviart-Ciarlet, see [Cia70] and [CR73]. However, the validity of the discrete maximum principle requires several assumptions on the triangulations.

In [Nit77], for space dimension $d = 2$, obstacle $\psi \in W^{2,\infty}(\Omega)$ and for bounded $\Omega \subseteq \mathbb{R}^d$ with sufficiently smooth boundary, an estimate of the form

$$\|y - y_h\|_{L^\infty(\Omega)} \leq Ch^2 |\ln h| \|y\|_{W^{2,\infty}(\Omega)} \tag{7.7}$$

is derived. From this, for sufficiently smooth data, we can derive the inequality

$$\|y - y_h\|_{L^\infty(\Omega)} \leq \tilde{C} h^2 |\ln h| (\|\zeta\|_{W^{1,\infty}(\Omega)} + \|\psi\|_{W^{2,\infty}(\Omega)}). \tag{7.8}$$

The estimate (7.8) can be obtained from (7.7) as follows. We can combine the proof of [KS00, Thm. IV 6.3] with [KS00, Lem. IV 5.1, Lem. IV 6.2] to obtain the inequality (7.8) with Ω replaced by a compact subset K. If $\psi < 0$ on $\partial\Omega$, we can use [KS00, Cor. IV 6.4] and elliptic regularity theory, see, e.g., [GT01], to obtain (7.8).

Moreover, using stability arguments, [Chr17] establishes the estimate

$$\|y - y_h\|_{L^\infty(\Omega_h)} \leq C |\ln h| h^{2-2/q} (\|\zeta\|_{L^q(\Omega)} + \|\psi\|_{W^{2,q}(\Omega)}) \tag{7.9}$$

for a bounded domain Ω with sufficiently smooth boundary, $\zeta \in L^q(\Omega)$, $\psi \in W^{2,q}(\Omega)$ and $2 < q < \infty$.

Related results are obtained in [Bai77], [FV82], [MT13], see also [NOS15] for an overview.

The application of the discrete maximum principle imposes constraints on the triangulations. To circumvent such assumptions, other approaches yield a priori L^∞-error estimates for the obstacle problem. It is possible to obtain error estimates in the energy norm, i.e., error estimates for $\|y - y_h\|_{H^1(\Omega)}$, see [BHR77], [Mos77]. By using inverse inequalities like

$$\|v_h\|_{L^\infty(\Omega)} \leq C |\ln h| \|v_h\|_{H^1(\Omega)},$$

in dimension $d = 2$, see, e.g., [Noc86], a bound for $\|y - y_h\|_{L^\infty(\Omega)}$ can be obtained. Note that this bound is, in general, not sharp. This strategy is also mentioned in [Noc86].

In [NSV05], a posteriori $L^\infty(\Omega)$-error estimates for the obstacle problem are established. Unlike in a priori analysis, no restrictions in the choice of triangulations are required, since the *continuous* maximum principle is applied within the derivation. Moreover, these error estimates in the $L^\infty(\Omega)$ norm are used to control the error in the respective active sets. We will focus more on this detail in Section 7.6.1.

For this thesis, we do not rely on one specific bound ε_h for the error and we do not fix any assumptions on the mesh. By requiring the appropriate assumptions, any of the $L^\infty(\Omega)$-error estimates can be used. We use the notation ε_h for any upper bound for $\|y - y_h\|_{L^\infty(\Omega)}$ converging to 0 as $h \to 0$.

7.5 Discrete inner approximation of the inactive set

The goal of this section is to introduce and to analyze the sets I_n from the literature, which are subset approximations of the inactive set $I(\zeta)$. These approximations are based on an upper bound for the L^∞-error $\|y - y_{h_n}\|_{L^\infty(\Omega)}$. We will see that the Mosco convergence $H_0^1(I_n) \to H_0^1(I(\zeta))$ can be shown. The sets I_n will be the basis to find approximations of $D := \Omega \setminus A_{\mathrm{s}}(\zeta)$ in the upcoming sections.

In the following sections, we again consider sequences $(h_n)_{n\in\mathbb{N}}$ of mesh sizes with $h_n \to 0$ and use again the notation with subscript n instead of h as in Section 7.2. In particular, we denote by $\varepsilon_n := \varepsilon_{h_n}$ a convergent upper bound for $\|y - y_n\|_{L^\infty(\Omega)}$, see Section 7.4.

From the numerical solutions $(y_n)_{n\in\mathbb{N}}$ of (DOP), we want to obtain a suitable approximation of the inactive set $I(\zeta)$. Note that the sets $\{y_n > \psi\}$ can be very different from $I(\zeta)$, even for large $n \in \mathbb{N}$ and when $y_n \to y$ in $L^\infty(\Omega)$. In [BP77, Thm. 1.1.], a suitable definition for the approximation I_n using the L^∞-error ε_n is proposed. The following result is a slight modification thereof.

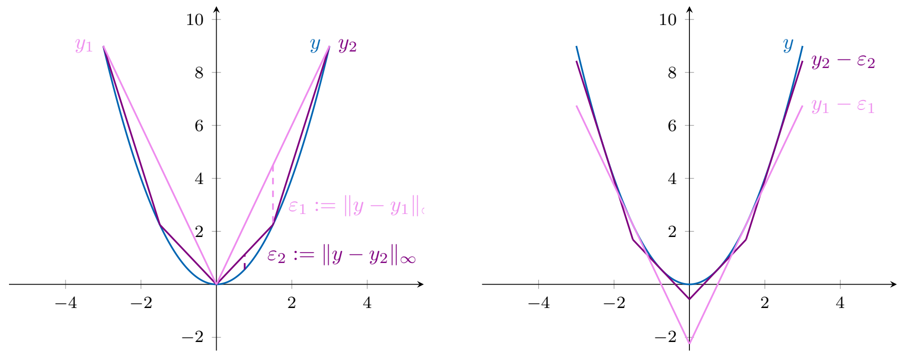
Lemma 7.8 *Let $\zeta := f(u) \in L^\infty(\Omega)$ be fixed. Assume $y, \psi \in C(\overline{\Omega})$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be such that*

$$\|y - y_n\|_{L^\infty(\Omega)} \le \varepsilon_n \overset{n\to\infty}{\to} 0$$

holds for the solutions y, y_n of (COP) *and* (DOP)*. Define*

$$I_n := I_n(\zeta) := \{\omega \in \Omega \mid y_n(\omega) > \psi(\omega) + \varepsilon_n\}. \tag{7.10}$$

Then it holds $I_n \subseteq I(\zeta)$ for all $n \in \mathbb{N}$ and $\liminf_{n\to\infty} I_n = I(\zeta)$, i.e.,



(a) $y(x) = x^2$ and interpolation approximations y_i with maximal error ε_i, $i = 1, 2$

(b) Shifted functions $y_i - \varepsilon_i$ as underestimators for y

Figure 7.2. Subset approximations $\{y_i - \varepsilon_i > 0\}$ of $\{y > 0\}$ using L^∞-error estimates $\varepsilon_i := \|y - y_i\|_{L^\infty(\Omega)}$ for approximations y_i of y, $i = 1, 2$

$\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} I_k = I(\zeta)$.

Proof. The first part of the proof can be found in the proof of [NSV05, Thm. 4.1]. For convenience, we also state it here. Assume $\omega \in A(\zeta)$. Then it holds

$$y_n(\omega) = y(\omega) + (y_n(\omega) - y(\omega)) \leq \psi(\omega) + \varepsilon_n.$$

This shows $\omega \in I_n^\complement$.

In a similar form, the subsequent part of the proof can be found in [BP77, Thm. 1.1]. Here, the set I_n is defined slightly different. Again, we also state this part of the proof. Assume $\omega \in I(\zeta)$, i.e., $y(\omega) - \psi(\omega) = c$ for some $c > 0$. Let $n \in \mathbb{N}$ be such that $\varepsilon_k < c/2$ for all $k \geq n$. Then,

$$y_k(\omega) - \psi(\omega) = c + y_k(\omega) - y(\omega) \geq c - \varepsilon_k > \varepsilon_k$$

for all $k \geq n$. This shows $\omega \in \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} I_k$. All in all, we conclude $\liminf\limits_{n\to\infty} I_n = I(\zeta)$. $\square$

Exemplary, Fig. 7.2 shows the function $y(x) = x^2$ and corresponding L^∞-errors ε_i for interpolated approximations y_i, $i = 1, 2$. We can see that $y_i - \varepsilon_i$

are underestimators for y and that the sets $\{y_i - \varepsilon_i > 0\}$ are subsets of $\{y > 0\}$.

In Section 7.7, we will propose an analogue J_n of I_n using the discrete obstacle ψ_n instead of the continuous one ψ. This choice can have advantages for the implementation compared to the choice I_n. Let us note that we can always use J_n instead of I_n in the following analysis. This will be justified in Section 7.7. For a clear and simpler presentation, we focus on I_n now and introduce the alternative J_n later on in Section 7.7.

Now, we will see that with the choices I_n from (7.10) we can show the Mosco convergence $H_0^1(I_n) \to H_0^1(I(\zeta))$. Recall that with Lemma 7.1, we obtain the convergence of the solutions of (7.1) with $D_n := I_n$ to the solution of (7.2) with $D := I(\zeta)$. Moreover, the error estimate (7.4) holds with $D = I(\zeta)$. To establish the Mosco convergence, we first show that the sets $\left(\bigcap_{k\geq n} I_k\right)_{n\in\mathbb{N}}$ are open, which implies that they constitute an open covering of $I(\zeta)$, see Lemma 7.8.

Lemma 7.9 *Assume the conditions of Lemma 7.8 are fulfilled. Then the sets* $\bigcap_{k\geq n} I_k$ *are open for all* $n \in \mathbb{N}$.

Proof. Assume $n \in \mathbb{N}$ and $\omega \in \bigcap_{k\geq n} I_k \subseteq I(\zeta)$, compare Lemma 7.8. Then it holds

$$c := y(\omega) - \psi(\omega) > 0.$$

Since y and ψ are continuous, there is $C > 0$ such that

$$y(z) - \psi(z) > c/2$$

for all $z \in B_C(\omega)$. Let $n_0 \in \mathbb{N}$ be such that $\varepsilon_k \leq c/4$ for all $k \geq n_0$. Then it holds for all $k \geq n_0$ and for all $z \in B_C(\omega)$

$$\begin{aligned} y_k(z) - \psi(z) &= y_k(z) - y(z) + y(z) - \psi(z) \\ &> -\varepsilon_k + c/2 \\ &\geq \varepsilon_k. \end{aligned}$$

We conclude $B_C(\omega) \subseteq \bigcap_{k\geq n_0} I_k$. If $n < n_0$, $\bigcap_{k\geq n} I_k$ is the finite intersection of $\bigcap_{k\geq n_0} I_k$ with open sets and thus contains a ball $B_{\tilde{C}}(\omega)$. Since $\omega \in \bigcap_{k\geq n} I_k$ was arbitrary, we conclude that $\bigcap_{k\geq n} I_k$ is open. □

Now, we are able to prove the Mosco convergence.

Theorem 7.10 *Assume the conditions of Lemma 7.8 are satisfied. Then it holds $H_0^1(I_n) \to H_0^1(I(\zeta))$ in the sense of Mosco.*

Proof. Let $v \in H_0^1(I(\zeta))$ be arbitrary. The family of sets $(\bigcap_{k\geq n} I_k)_{n\in\mathbb{N}}$ is an increasing covering of $I(\zeta)$, see Lemma 7.9 and Lemma 7.8. Thus, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \to v$ in $H_0^1(I(\zeta))$ as $n \to \infty$ and such that $v_n \in H_0^1(\bigcap_{k\geq n} I_k)$ for each $n \in \mathbb{N}$, see Lemma 2.29. In particular, $v_n \in H_0^1(I_n)$ since $\bigcap_{k\geq n} I_k \subseteq I_n(\zeta)$.

Suppose there is a sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in H_0^1(I_n)$ and $w_{n_k} \rightharpoonup w$ in $H_0^1(\Omega)$ for some $w \in H_0^1(\Omega)$ and for a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$. Since $I_n \subseteq I(\zeta)$ holds, see Lemma 7.8, we have $w_n \in H_0^1(I(\zeta))$ for all $n \in \mathbb{N}$. Then the weak limit w is also in $H_0^1(I(\zeta))$ by Mazur's lemma. □

7.6 Discrete approximations of the complement of the strictly active set

In this section, we will derive and analyze approximations D_n and $\tilde{D}_n$ of the set $D := \Omega \setminus A_{\mathrm{s}}(\zeta)$ in order to obtain an error estimate for the generalized derivative on the domain $D = \Omega \setminus A_{\mathrm{s}}(\zeta)$ as proposed in Corollary 7.3.

We have seen in the preceding section that the sets $(I_n)_{n\in\mathbb{N}}$ as defined in (7.10) satisfy $I_n \subseteq I(\zeta)$ for all $n \in \mathbb{N}$ as well as $H_0^1(I_n) \to H_0^1(I(\zeta))$ in the sense of Mosco, see Lemma 7.8 and Theorem 7.10. Using Lemma 7.1, we can obtain a convergent sequence of inexact Clarke subgradients $(q_n)_{n\in\mathbb{N}}$ by solving (7.1) with $D_n := I_n$ for all $n \in \mathbb{N}$ and the limit is the Clarke subgradient q based on $D := I(\zeta)$, compare (7.2). Motivated by Corollary 7.3, to obtain a computable error estimate for the term $\|q - q_n\|_{H_0^1(\Omega)}$ a sequence $(\tilde{I}_n)_{n\in\mathbb{N}}$ of quasi-open supersets of $I(\zeta)$, such that $(H_0^1(\tilde{I}_n))_{n\in\mathbb{N}}$ converges to $H_0^1(I(\zeta))$, would need to be constructed. In general, the weakly active set

$A_{\mathrm{w}}(\zeta)$ can be thin and irregular, see also the discussion Section 7.6.2. That makes it hard to approximate or predict the inactive set from the outside, or, equivalently, the active set from the inside, based on the numerics.

Instead, in this section, we try to obtain approximations D_n and $\tilde{D}_n$ as in (7.5) for the quasi-open set $D := \Omega \setminus A_{\mathrm{s}}(\zeta)$. In Theorem 4.20 we have seen that this choice indeed yields a generalized derivative for a general monotone and continuously differentiable operator $f : U \to H^{-1}(\Omega)$. The tools we use are estimates for the distance of the free boundaries $\partial I(\zeta)$ and ∂I_n, which are available in the literature. The validity of these estimates is ensured under the so-called *nondegeneracy condition* (ND_η). Based on this condition, it can also be shown that the weakly active set does not have interior points, see Lemma 7.15, and that the strictly active set is the (fine) closure of the interior of the active set, see Corollary 7.21. This structure highlights that the strictly active set is better suited to be approximated from both sides, from in- and outside. The construction of these discrete approximations based on the estimates for the distances of the free boundaries is the purpose of this section.

In Section 7.6.1, we will see that an error estimate for the distance of the free boundaries $\partial I(\zeta)$ and ∂I_n has been obtained under the nondegeneracy condition in the literature. Assuming this condition, we analyze the topological structure of the weakly and the strictly active set in Section 7.6.2. We make use of this structure and suggest approximations of the complement of the strictly active set in Section 7.6.3. We show that these approximations fulfill the conditions of Corollary 7.3 so that an error estimate for the generalized derivative as in (7.1) is available.

7.6.1 Error estimate for the discrete and continuous free boundaries

In this subsection, we state a result concerning error estimates for the free boundaries $\partial I(\zeta)$ and ∂I_n. Here, for some $n \in \mathbb{N}$, I_n is the approximate inactive set defined in (7.10).

In the literature, error estimates for the distance of the boundaries of $I(\zeta)$

and of I_n have been discussed. As references, we mention, among other related references, [BC83], [DS00], [Noc86], [NSV05].

First, we show that the nondegeneracy condition implies a quadratic growth property of $y - \psi$ away from the active set. The formulation is taken from [NSV05] and the proof is based on [Fri88, Ch. 2, Lem. 3.1].

Lemma 7.11 *Let $\zeta \in L^2(\Omega)$ and let y, ψ be continuous. Suppose there is an open neighborhood $\mathcal{U} \subseteq \Omega$ of the active set $A(\zeta)$ and a positive number $\eta > 0$ such that the nondegeneracy condition*

$$\langle -\Delta\psi - \zeta, v\rangle \geq \eta \int_{\mathcal{U}} v \,\mathrm{d}\lambda^d \quad \forall\, v \in H_0^1(\mathcal{U})_+ \qquad (\mathrm{ND}_\eta)$$

holds on $\mathcal{U}$. Then, for any $\omega_0 \in \overline{I(\zeta)}$ and any $r > 0$ such that $B_r(\omega_0) \subseteq \mathcal{U}$ it holds

$$\sup_{\omega \in B_r(\omega_0)} y(\omega) - \psi(\omega) \geq y(\omega_0) - \psi(\omega_0) + \frac{\eta r^2}{2d}. \qquad (\mathrm{QG})$$

Proof. Let us first consider the case $\omega_0 \in I(\zeta)$ and assume $B_r(\omega_0) \subseteq \mathcal{U}$. The function

$$w(x) = y(x) - \psi(x) - y(\omega_0) + \psi(\omega_0) - \frac{\eta}{2d}|x - \omega_0|^2$$

satisfies $w(\omega_0) = 0$ and also

$$\begin{aligned}\langle -\Delta w, v\rangle &= \langle -\Delta y + \Delta\psi, v\rangle + \eta \frac{2d}{2d} \int_{\mathcal{U}} v \,\mathrm{d}\lambda^d \\ &= \langle \Delta\psi + \zeta, v\rangle + \eta \int_{\mathcal{U}} v \,\mathrm{d}\lambda^d \\ &\leq 0\end{aligned}$$

for all $v \in H_0^1(\mathcal{U})_+ \cap H_0^1(I(\zeta))$. Here, we have used (ND_η) and $\langle -\Delta y - \zeta, v\rangle = 0$ due to $v = 0$ q.e. on $A_{\mathrm{s}}(\zeta)$, see Corollary 3.10. In particular, $\langle -\Delta w, v\rangle \leq 0$ for all $v \in H_0^1(B_r(\omega_0) \cap I(\zeta))_+$. By the maximum principle

[KS00, Thm. II 5.7], it holds

$$w(x) \leq \sup_{\omega \in \partial(B_r(\omega_0) \cap I(\zeta))} w(\omega)$$

in $B_r(\omega_0) \cap I(\zeta)$. Since $w < 0$ holds on $\partial I(\zeta)$, there must exist a point $\omega_1 \in \partial B_r(\omega_0) \cap I(\zeta)$ such that $w(x_1) \geq 0$ and (QG) follows.

For $\omega_0 \in \partial I(\zeta)$, let $(\omega^n)_{n\in\mathbb{N}} \subseteq I(\zeta)$ be a sequence with $\omega^n \to \omega_0$. Then (QG) is already shown for ω^n, $n \in \mathbb{N}$, and, using continuity arguments, we obtain the statement for ω_0. □

Remark 7.12 The original strong formulation of nondegeneracy goes back to Caffarelli, see also [Caf81]. The size of η in (ND$_\eta$) determines the growth rate of $y - \psi$ and the larger η is, the more stable are the (discete) free boundaries, see [NSV05, Rem. 4.2].

Based on the nondegeneracy condition (ND$_\eta$) and the resulting quadratic growth property (QG) the following result on the distance of the free boundaries $\partial I(\zeta)$ and ∂I_n can be obtained. It is based on [NSV05].

Lemma 7.13 *We assume that $\zeta \in L^2(\Omega)$, $y, \psi \in C(\bar{\Omega})$ and $\psi < 0$ on $\partial\Omega$. Suppose there exists a neighborhood $\mathcal{U}$ of $A(\zeta)$ and a positive number $\eta > 0$ such that the nondegeneracy condition* (ND$_\eta$) *holds on $\mathcal{U}$. Assume $(\varepsilon_n)_{n\in\mathbb{N}}$ is a family of upper bounds for the errors $\|y - y_n\|_{L^\infty(\Omega)}$ for the solutions $y, y_n := y_{h_n}$ of* (COP) *and* (DOP)*, which converges to* 0 *as $n \to \infty$. Set*

$$r_n := 2\sqrt{\frac{d\,\varepsilon_n}{\eta}}. \tag{7.11}$$

Then, if n is large enough, it holds

$$\{\omega \in \Omega \mid \operatorname{dist}(\omega, I_n) \geq r_n\} \subseteq A(\zeta).$$

Proof. Since $A(\zeta)$ and $\partial\Omega$ are compact and disjoint sets, it holds $\operatorname{dist}(\partial\Omega, A(\zeta)) > 0$. Thus, w.l.o.g., we can assume $\operatorname{dist}(\mathcal{U}, \partial\Omega) > 0$. Since y and ψ are continuous and since $\Omega \backslash \mathcal{U}$ is relatively compact, we find a constant

$c > 0$ such that $y - \psi > c$ holds outside $\mathcal{U}$. By definition of I_n, this implies

$$I_n^\complement \subseteq \mathcal{U}$$

if $n \in \mathbb{N}$ is sufficiently large. Moreover, if n is large enough, we also have $r_n < \operatorname{dist}(\mathcal{U}, \partial\Omega)$. Let us assume that $n \in \mathbb{N}$ is sufficiently large in the sense of these two conditions.

Let $\omega \in \Omega$ with $\operatorname{dist}(\omega, I_n) \geq r_n$ and assume $\omega \in I(\zeta)$, i.e.,

$$y(\omega) > \psi(\omega).$$

In particular, we have $\omega \in I_n^\complement \subseteq \mathcal{U}$ and thus, $\overline{B_{r_n}(\omega)} \subseteq \Omega$. We even observe $\overline{B_{r_n}(\omega)} \subseteq I_n^\complement \subseteq \mathcal{U}$.

Now, Lemma 7.11 implies

$$\sup_{x \in B_{r_n}(\omega)} y(x) - \psi(x) > \frac{\eta r_n^2}{2d}.$$

This means, for some $x \in B_{r_n}(\omega)$ it holds

$$y_n(x) = y(x) + (y_n(x) - y(x)) > \psi(x) + \frac{\eta r_n^2}{2d} - \varepsilon_n = \psi(x) + \varepsilon_n$$

and this contradicts $\overline{B_{r_n}(\omega)} \subseteq I_n^\complement$. We conclude $\omega \in A(\zeta)$. □

Remark 7.14

1. In the classical literature concerning the analysis of the free boundaries in the context of obstacle problems, the obstacle problem *for a plate* is investigated, see, e.g., [BC83], [Caf81], [DS00], [Noc86]. This means, the obstacle problem for $\tilde{\psi} = 0$ is considered, while the admissible displacements satisfy $\tilde{z} = g$ on the boundary of Ω for some $H^1(\Omega)$ function g satisfying $g > 0$ q.e. on $\partial\Omega$. For the variational inequality with general obstacle $\psi \in H^1(\Omega)$ and zero boundary constraints as in (COP), $S_f(u) - \psi$ solves the obstacle problem for a plate with $g = -\psi$.
2. Several results analyzing the Hausdorff distance between the exact and the approximate free boundary require regularity of $\partial A(\zeta)$, see, e.g., [BC83], [DS00], [Noc86]. As stressed in [NSV05, Rem. 4.4], the deriva-

tion of the error estimate in Lemma 7.13 does not require any regularity assumptions on the exact free boundary $\partial A(\zeta)$.

7.6.2 Structure of the weakly and strictly active set under regularity assumptions

Assuming the nondegeneracy condition, we analyze the topological structures of the weakly and the strictly active set. Our findings establish properties seperately for the weakly and the strictly active set. This approach is not common in the classical literature, where the free boundary analysis and the investigation of topological structures is usually performed for the overall active set and its boundary. The distinction between strictly active and weakly active set is not drawn. Yet, as we will see, there are fundamental differences and the strictly active set often exhibits a considerably more regular structure than the overall active set.

The following lemma investigates the behavior of the weakly active set and shows that it has empty interior.

Lemma 7.15 *Let $\zeta \in L^2(\Omega)$. Assume the nondegeneracy condition* (ND_η) *holds on a neighborhood $\mathcal{U}$ of the active set. Then the weakly active set does not have interior points.*

Proof. Let $v \in H_0^1(\Omega)_+$ with

$$\{v > 0\} = \mathrm{int}(A_{\mathrm{w}}(\zeta)), \tag{7.12}$$

see Lemma 2.31. This implies

$$\begin{aligned} \langle -\Delta\psi - \zeta, v\rangle &= \int_\Omega \nabla\psi^T \nabla v \,\mathrm{d}\lambda^d - (\zeta, v) \\ &= \int_{A_{\mathrm{w}}(\zeta)} \nabla y^T \nabla v \,\mathrm{d}\lambda^d - (\zeta, v) \\ &= \int_\Omega \nabla y^T \nabla v \,\mathrm{d}\lambda^d - (\zeta, v) \\ &= \langle -\Delta y - \zeta, v\rangle = 0. \end{aligned}$$

Here, we have used that $\nabla v = 0$ a.e. on $\Omega \setminus A_\mathrm{w}(\zeta)$ and $\nabla \psi = \nabla y$ a.e. on $A_\mathrm{w}(\zeta)$, see [ABM14, Prop. 5.8.2]. In the last step, we use $v = 0$ q.e. on $A_\mathrm{s}(\zeta)$, see Corollary 3.10. On the other hand, the nondegeneracy condition (ND_η) yields

$$\langle -\Delta\psi - \zeta, v\rangle \ge \eta \int_{\mathcal{U}} v \,\mathrm{d}\lambda^d.$$

Combining the two arguments, we conclude $\int_{\mathcal{U}} v \,\mathrm{d}\lambda^d = 0$ and thus $v = 0$ a.e. on $\mathcal{U}$. Observing $\operatorname{int}(A_\mathrm{w}(\zeta)) \subseteq \mathcal{U}$ and recalling (7.12), we conclude $\lambda^d(\operatorname{int}(A_\mathrm{w}(\zeta))) = 0$ and since this set is open we derive $\operatorname{int}(A_\mathrm{w}(\zeta)) = \emptyset$. $\square$

Before we analyze the structure of the strictly active set, we recall that the strictly active set $A_\mathrm{s}(\zeta)$ is defined as f-supp$(\tilde{\xi})$, cf. Lemma 2.34. In Remark 2.35 we have already indicated that f-supp$(\tilde{\xi})$ is the support of $\tilde{\xi}$ w.r.t. the fine topology on $\mathbb{R}^d$ and we refer again to [Wac14, App. A].

Here, we will explicitly use this characterization. We do not introduce the fine topology, but let us collect the following properties. The fine topology is finer than the usual topology on $\mathbb{R}^d$, i.e., every open set is *finely open*. For a set $E \subseteq \mathbb{R}^d$ we denote the closure w.r.t. the fine topology, the *fine closure*, by f-cl(E) and the interior w.r.t. the fine topology, the *fine interior*, by f-int(E).

The following result holds, see [Wac14, App. A].

Lemma 7.16 *Let $\zeta \in H^{-1}(\Omega)$ and denote $y := S_\mathrm{id}(\zeta)$, $\xi := -\Delta y - \zeta \in H^{-1}(\Omega)_+$. Then there exists a largest finely open set $O \subseteq \Omega$ with $\tilde{\xi}(O) = 0$ and $A_\mathrm{s}(\zeta) = $ f-supp$(\tilde{\xi}) = O^\complement$ holds.*

The next result states that quasi-continuous functions are finely continuous quasi-everywhere. For a proof we refer to [AH96, Thm. 6.4.5].

Proposition 7.17 *Every quasi-continuous function is finely continuous quasi-everywhere in Ω.*

Remark 7.18 The fine topology is the coarsest topology such that all subharmonic functions are continuous. Since the quasi-open sets do not constitute a topology, it is often helpful to work with the fine topology which is compatible with most of the concepts related to quasi-open sets and quasi-continuous functions. For details, we refer to [AH96] and [HKM93].

Before we state the subsequent auxiliary lemma, let us recall that any $\xi \in H^{-1}(\Omega)_+$ can be identified with a regular Borel measure $\tilde{\xi}$ and that the quasi-continuous representative of some $v \in H_0^1(\Omega)$ is integrable with respect to $\tilde{\xi}$ and it holds $\langle \xi, v\rangle = \int_\Omega v \, \mathrm{d}\tilde{\xi}$, see Lemma 2.33.

Lemma 7.19 *Let $\xi \in H^{-1}(\Omega)_+$ and let $B \subseteq \Omega$ be a Borel measurable set. We consider the restriction $\xi|_B$ defined by*

$$\langle \xi|_B, v\rangle := \int_B v \, \mathrm{d}\tilde{\xi} = \int_\Omega v \, \mathrm{d}\tilde{\xi}|_B,$$

where $\tilde{\xi}$ is the Borel measure associated with ξ and $\tilde{\xi}|_B$ the trace measure or restricted measure w.r.t. B.

1. *We have $\xi|_B \in H^{-1}(\Omega)_+$.*
2. *Assume that ∂B has Lebesgue measure zero and let $\xi \in L^2(\Omega)_+ \subseteq H^{-1}(\Omega)_+$. Then* $\text{f-supp}(\tilde{\xi}|_B) = \text{f-cl}(\text{int}(B) \cap \text{f-supp}(\tilde{\xi}))$.

Proof. 1. The operator $\xi|_B$ is well-defined and linear on $H_0^1(\Omega)$. Furthermore, we have

$$|\langle \xi|_B, v\rangle| \leq \int_B |v| \, \mathrm{d}\tilde{\xi} \leq \int_\Omega |v| \, \mathrm{d}\tilde{\xi} = \langle \xi, |v|\rangle,$$

thus, using $\||v|\|_{H_0^1(\Omega)} = \|v\|_{H_0^1(\Omega)}$, see Proposition 2.19(1.), we see that $\xi|_B$ is bounded. Since $\xi|_B$ can be identified with the trace measure $\tilde{\xi}|_B$, it is clear that $\xi|_B$ is nonnegative and we conclude $\xi|_B \in H^{-1}(\Omega)_+$.
2. Now, assume ∂B has Lebesgue measure zero and that $\xi \in L^2(\Omega)_+$. Let

$$O := \text{f-int}(\overline{B^\complement} \cup \text{f-supp}(\tilde{\xi})^\complement).$$

Using $\tilde{\xi}(E) = 0$ for every Borel set $E \subseteq \text{f-supp}(\tilde{\xi})^\complement$, cf. Lemma 7.16, we derive

$$\tilde{\xi}|_B(O) = \tilde{\xi}(B \cap O) \leq \tilde{\xi}(B \cap \overline{B^\complement}) + \tilde{\xi}(B \cap \text{f-supp}(\tilde{\xi})^\complement) = 0$$

since $\tilde{\xi}$ is absolutely continuous w.r.t. the Lebesgue measure and since $\lambda^d(\partial B) = 0$. Thus, $\text{f-supp}(\tilde{\xi}|_B) \subseteq \text{f-cl}(\text{int}(B) \cap \text{f-supp}(\tilde{\xi}))$.
On the other hand, let $O := \text{f-supp}(\tilde{\xi}|_B)^{\complement}$, i.e., O is the largest finely open set with $\tilde{\xi}|_B(O) = 0$. Then $O \cap \text{int}(B)$ is finely open and $\tilde{\xi}(O \cap \text{int}(B)) = 0$. This implies $O \cap \text{int}(B) \subseteq \text{f-supp}(\tilde{\xi})^{\complement}$. We conclude $O \subseteq \overline{B^{\complement}} \cup \text{f-supp}(\tilde{\xi})^{\complement}$ and obtain $\text{int}(B) \cap \text{f-supp}(\tilde{\xi}) \subseteq \text{f-supp}(\tilde{\xi}|_B)$. Since the set on the right-hand side is finely closed, we even have $\text{f-cl}(\text{int}(B) \cap \text{f-supp}(\tilde{\xi})) \subseteq \text{f-supp}(\tilde{\xi}|_B)$.

□

In the following analysis of the strictly active set, we additionally assume $\psi \in H^2(\Omega)$. Note that this implies $(-\Delta\psi - \zeta) \in L^2(\Omega)$ if $\zeta \in L^2(\Omega)$. Let us observe the following structure.

Lemma 7.20 *Let $\zeta \in L^2(\Omega)$. Assume the nondegeneracy condition* (ND$_\eta$) *holds on a neighborhood $\mathcal{U}$ of the active set and, in addition, let $\psi \in H^2(\Omega)$. Denote $\xi_\psi := (-\Delta\psi - \zeta)_+ \in L^2(\Omega)_+$. Then* $\text{f-supp}(\tilde{\xi}_\psi) \supseteq A(\zeta)$ *holds.*

Proof. Assume $\emptyset \neq O \subseteq \mathcal{U}$ is finely open. Let $v \in H_0^1(O)_+$ with $\{v > 0\} = O$, compare Lemma 2.31. We conclude

$$\int_\Omega (-\Delta\psi - \zeta)\, v \,\mathrm{d}\lambda^d = \langle -\Delta\psi - \zeta, v\rangle \geq \eta \int_{\mathcal{U}} v \,\mathrm{d}\lambda^d$$

by the nondegeneracy condition (ND$_\eta$). This means $\int_\Omega (-\Delta\psi - \zeta)\, v \,\mathrm{d}\lambda^d > 0$, since $v \in H_0^1(O)_+$ and since O has positive Lebesgue measure, see [AG01, Thm. 7.3.11, Cor. 7.2.4]. We have

$$\int_\Omega v \,\mathrm{d}\tilde{\xi}_\psi = \int_\Omega (-\Delta\psi - \zeta)_+\, v \,\mathrm{d}\lambda^d \geq \int_\Omega (-\Delta\psi - \zeta)\, v \,\mathrm{d}\lambda^d > 0,$$

which implies

$$\tilde{\xi}_\psi(O) > 0.$$

Recalling Lemma 7.16, this shows

$$A(\zeta) \subseteq \mathcal{U} \subseteq \text{f-supp}(\tilde{\xi}_\psi).$$

□

Now, we are in the position to show that the strictly active set is the fine closure of its interior points.

Corollary 7.21 *Suppose* Ω *is convex or has a sufficiently regular boundary. Let* $\zeta \in L^2(\Omega)$. *Assume the nondegeneracy condition* (ND_η) *holds on a neighborhood* $\mathcal{U}$ *of the active set and additionally assume* $\psi \in H^2(\Omega)$ *and* $\lambda^d(\partial A(\zeta)) = 0$. *Denote* $\xi := -\Delta y - \zeta$. *Then it holds* $A_{\mathrm{s}}(\zeta) = \text{f-supp}(\tilde{\xi}) = \text{f-cl}(\mathrm{int}(A(\zeta)))$.

Proof. Under the regularity assumptions we have $y \in H^2(\Omega)$, see Lemma 7.4. Moreover,

$$-\Delta y - \zeta = \begin{cases} -\Delta\psi - \zeta & \text{a.e. on } A(\zeta), \\ 0 & \text{a.e. on } I(\zeta), \end{cases} \tag{7.13}$$

cf. [CW20, Thm. 2.2], which is a consequence of the regularity and of $\nabla\psi = \nabla y$ a.e. on $A(\zeta)$, see [ABM14, Prop. 5.8.2]. Since ξ is nonnegative, we have $-\Delta y - \zeta \geq 0$ a.e. on Ω and conclude that $-\Delta\psi - \zeta \geq 0$ a.e. on $A(\zeta)$. By (7.13), we have

$$\xi = \xi_\psi|_{A(\zeta)},$$

a.e. in Ω, where $\xi_\psi := (-\Delta\psi - \zeta)_+$ as in Lemma 7.20. We apply Lemma 7.19 and Lemma 7.20 and obtain

$$A_{\mathrm{s}}(\zeta) = \text{f-supp}(\tilde{\xi}) = \text{f-cl}(\mathrm{int}(A(\zeta)) \cap \text{f-supp}(\tilde{\xi}_\psi)) = \text{f-cl}(\mathrm{int}(A(\zeta))),$$

□

In the following, we will always assume that $\lambda^d(\partial A(\zeta)) = 0$, so that we can use the result in Corollary 7.21. Using the nondegeneracy condition (ND_η) and regularity assumptions on ζ and ψ, it is possible to guarantee a priori the condition $\lambda^d(\partial A(\zeta)) = 0$, as the following lemma states. The results is taken from [Fri88, Ch. 2, Thm. 3.5].

Lemma 7.22 *Let Ω have a smooth boundary $\partial\Omega$. Assume $\zeta \in C^1(\overline{\Omega})$ and $\psi \in C^2(\overline{\Omega})$. Suppose the nondegeneracy condition* (ND_η) *holds. Then this implies $\lambda^d(\partial A(\zeta)) = 0$.*

In [Fri88, Ch. 2, Thm. 3.5], the proof of the above lemma is based on the $C^{1,1}(\Omega)$ regularity of y, see Lemma 7.7 and the property (QG) implied by (ND_η). It is shown that points in $\partial A(\zeta)$ do not have density one, which in turn implies $\lambda^d(\partial A(\zeta)) = 0$.

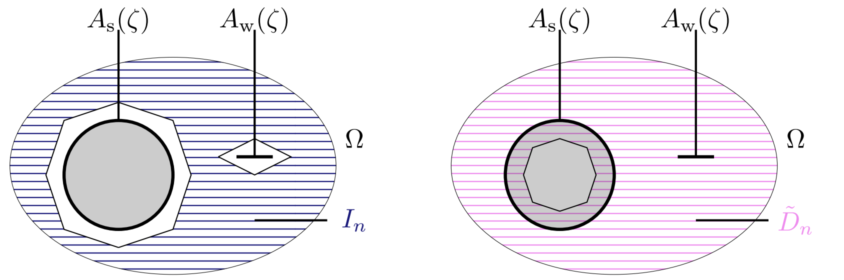
7.6.3 Construction of discrete inner and outer approximations of the complement of the strictly active set

As highlighted at the beginning of Section 7.6, we want to construct discrete approximations $D_n, \tilde{D}_n$ of $D := \Omega \setminus A_{\mathrm{s}}(\zeta)$ satisfying the inclusions $D_n \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta) \subseteq \tilde{D}_n$. The purpose of this subsection is the formulation of the construction and the verification, that the constructed sets satisfy (7.5) and (7.6), cf. Corollary 7.3.

Here, we will make use of the structures of the weakly and strictly active set established in Section 7.6.2. Therefore, within this section, we fix the following assumptions that have been established consecutively within the preceding subsections.

Assumption 7.1 *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of mesh size parameters with $h_n \to 0$ as $n \to \infty$. Let Ω be convex or assume it has a sufficiently regular boundary. We assume that $\zeta \in L^2(\Omega)$, $y, \psi \in C(\bar{\Omega})$, $\psi \in H^2(\Omega)$ and $\psi < 0$ on $\partial\Omega$. Suppose there exists a neighborhood $\mathcal{U}$ of $A(\zeta)$ and a positive number $\eta > 0$ such that the nondegeneracy condition* (ND_η) *holds on $\mathcal{U}$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence satisfying $\varepsilon_n \to 0$ as $n \to \infty$ as well as $\|y - y_n\|_{L^\infty(\Omega)} \le \varepsilon_n$ for all $n \in \mathbb{N}$. Moreover, we assume $\lambda^d(\partial A(\zeta)) = 0$.*

Now, suppose that the conditions of Assumption 7.1 are satisfied and let us begin with the construction of a superset approximation of $\Omega \setminus A_{\mathrm{s}}(\zeta)$. We



(a) Exemplary presentation of I_n (b) Exemplary presentation of $\tilde{D}_n$

Figure 7.3. Construction of $\tilde{D}_n$ from I_n using the upper bound for the error $\tilde{r}_n$

define

$$\tilde{D}_n := \{\omega \in \Omega \mid \operatorname{dist}(\omega, I_n) < \tilde{r}_n\}, \tag{7.14}$$

with $I_n = \{y_n - \psi > \varepsilon_n\}$ as in (7.10) and $\tilde{r}_n$ satisfying

$$\tilde{r}_n > r_n \quad \text{and} \quad \tilde{r}_n \overset{n\to\infty}{\to} 0$$

for $r_n = 2\sqrt{\frac{d\,\varepsilon_n}{\eta}}$ as in (7.11).

Lemma 7.23 *Suppose the conditions in Assumption 7.1 are fulfilled. Let $\tilde{D}_n$ be defined as in* (7.14). *Then it holds*

$$\Omega \setminus A_s(\zeta) \subseteq \tilde{D}_n$$

for $n \in \mathbb{N}$ large enough and

$$H_0^1(\tilde{D}_n) \to H_0^1(\Omega \setminus A_s(\zeta))$$

in the sense of Mosco.

Proof. Using Corollary 7.21 and Lemma 7.13 for large enough $n \in \mathbb{N}$ we observe

$$\Omega \setminus A_s(\zeta) = (\text{f-cl}(\operatorname{int}(A(\zeta))))^{\complement} \subseteq \overline{I(\zeta)} \subseteq \{\omega \in \Omega \mid \operatorname{dist}(\omega, I_n) \le r_n\} \subseteq \tilde{D}_n$$

Assume $v \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$. Then $v_n := v \in H_0^1(\tilde{D}_n)$ for $n \in \mathbb{N}$ large enough and $v_n \to v$ in $H_0^1(\Omega)$, which shows the first condition for Mosco convergence.

Suppose there is a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in H_0^1(\tilde{D}_n)$ and $v_{n_k} \rightharpoonup v$ for a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$.

We can write

$$\operatorname{int}(A(\zeta)) = \bigcup_{m\in\mathbb{N}} A(\zeta)_m := \bigcup_{m\in\mathbb{N}} \{\omega \in A(\zeta) \mid \operatorname{dist}(\omega, \partial A(\zeta)) \geq m^{-1}\}.$$

By the inclusion $I_n \subseteq I(\zeta)$, see Lemma 7.8, and since $\tilde{r}_n \to 0$ as $n \to \infty$, there is $N_m \in \mathbb{N}$ such that $v_n(x) = 0$ for all $n \geq N_m$ and for quasi-all $x \in A(\zeta)_m$. Let $\xi_m \in H^{-1}(\Omega)_+$ with f-supp$(\tilde{\xi}_m) = A(\zeta)_m$, see Theorem 5.14. Since $v_n = 0$ q.e. on f-supp$(\tilde{\xi}_m)$, we conclude $\langle \xi_m, |v_n| \rangle = 0$ for all $n \geq N_m$, see Lemma 2.34. This implies $\langle \xi_m, |v| \rangle = 0$ by the weak convergence $|v_n| \rightharpoonup |v|$, see Proposition 2.19(2.). We thus have $v = 0$ q.e. on $A(\zeta)_m$, see Lemma 2.34. Repeating this argument for all $m \in \mathbb{N}$, we conclude that $v = 0$ q.e. on $\operatorname{int}(A(\zeta))$.

Since (a representative of) v is finely continuous quasi-everywhere, see Proposition 7.17, $v = 0$ q.e. on f-cl$(\operatorname{int}(A(\zeta)))$ follows. Thus, $v = 0$ q.e. on $A_{\mathrm{s}}(\zeta)$, see Corollary 7.21, and we obtain $v \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$. □

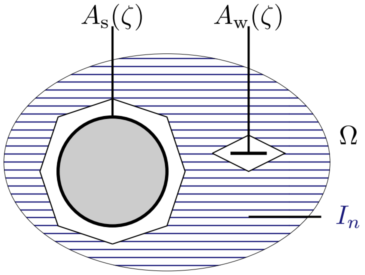
To obtain a suitable subset approximation of $\Omega \setminus A_{\mathrm{s}}(\zeta)$, we define and analyze the following sets

$$D_n := \left(\bigcup \{ C \mid C \text{ connected component of } I_n^{\complement} \text{ with } C \cap \tilde{D}_n^{\complement} \neq \emptyset \} \right)^{\complement}. \quad (7.15)$$

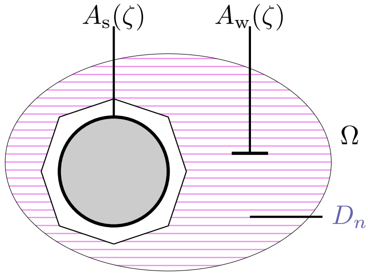
Here, $\tilde{D}_n$ is defined as in (7.14).

Lemma 7.24 *Suppose the conditions in Assumption 7.1 are fulfilled. Furthermore, assume that* $\operatorname{dist}(A_{\mathrm{w}}(\zeta), A_{\mathrm{s}}(\zeta)) > 0$ *and that we find a positive number* $\kappa > 0$ *such that* $B_\kappa(x) \subseteq A_{\mathrm{s}}(\zeta)$ *holds for at least one* x *in every connected component of* $A_{\mathrm{s}}(\zeta)$. *Let* D_n *be defined as in* (7.15). *Then it holds*

$$D_n \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta)$$



(a) Exemplary presentation of I_n and $\tilde{D}_n$



(b) Exemplary presentation of D_n

Figure 7.4. Construction of D_n using connected components of $I_n^{\complement}$ and $\tilde{D}_n^{\complement}$

for $n \in \mathbb{N}$ large enough and

$$H_0^1(D_n) \to H_0^1(\Omega \setminus A_s(\zeta))$$

in the sense of Mosco.

Proof. Let $\omega_0 \in A_s(\zeta)$. First of all, we have $\omega_0 \in I_n^{\complement}$ since $I_n \subseteq I(\zeta)$, see Lemma 7.8. We show that the connected component C of $I_n^{\complement}$ containing ω_0 fulfills $C \cap \tilde{D}_n^{\complement} \neq \emptyset$.

Recall that by Corollary 7.21 it holds $A_s(\zeta) = \text{f-cl}(\text{int}(A(\zeta)))$. Since the connected component of $A_s(\zeta)$ including ω_0 and thus the component C contains a ball of radius κ, the center of this ball is contained in $\tilde{D}_n^{\complement}$ if $\tilde{r}_n < \kappa$, i.e., if n is large enough. This shows $\omega_0 \in D_n^{\complement}$ and we conclude $D_n \subseteq \Omega \setminus A_s(\zeta)$ for large enough $n \in \mathbb{N}$.

The family of sets $\left(\text{int}\left(\bigcap_{k \geq n} D_k\right)\right)_{n \in \mathbb{N}}$ is increasing. We want to argue, that it is also a covering of $\Omega \setminus A_s(\zeta)$. The inclusion $\bigcap_{k \geq n} D_k \supseteq \bigcap_{k \geq n} I_k$ implies $\text{int}\left(\bigcap_{k \geq n} D_k\right) \supseteq \bigcap_{k \geq n} I_k$ since the sets $\bigcap_{k \geq n} I_k$, $n \in \mathbb{N}$, are open, see Lemma 7.9. From Lemma 7.8 we know that $\left(\bigcap_{k \geq n} I_k\right)_{n \in \mathbb{N}}$ covers $I(\zeta)$. Thus, it is clear that $\left(\text{int}\left(\bigcap_{k \geq n} D_k\right)\right)_{n \in \mathbb{N}}$ covers $I(\zeta)$.

Assume $P \subseteq A_w(\zeta)$ is of positive capacity and contained in one connected component of $A_w(\zeta)$. Let $\mathcal{V}_s$ and $\mathcal{V}_w$ be open neighborhoods of $A_s(\zeta)$ and $A_w(\zeta)$, respectively, and assume $\mathcal{V}_s \cap \mathcal{V}_w = \emptyset$. This is possible, since

$\operatorname{dist}(A_{\mathrm{w}}(\zeta), A_{\mathrm{s}}(\zeta)) > 0$ by assumption. Note that $\mathcal{V}_{\mathrm{s}}$ and $\mathcal{V}_{\mathrm{w}}$ have disjoint connected components.

Since y and ψ are continuous and since $\psi < 0$ on $\partial\Omega$, we have $y - \psi \geq c > 0$ outside $\mathcal{V} := \mathcal{V}_{\mathrm{s}} \cup \mathcal{V}_{\mathrm{w}}$ for some $c > 0$. This means, if n is big enough, it holds $I_n^{\complement} \subseteq \mathcal{V}$. In particular, any conncected component of $I_n^{\complement}$ is either contained in $\mathcal{V}_{\mathrm{s}}$ or in $\mathcal{V}_{\mathrm{w}}$.

Since $\tilde{D}_n^{\complement} \subseteq A_{\mathrm{s}}(\zeta)$ if n is large enough, this shows that $\mathcal{V}_{\mathrm{w}}$ contains only connected components C of $I_n^{\complement}$ with $C \cap \tilde{D}_n^{\complement} = \emptyset$. We conclude $\mathcal{V}_{\mathrm{w}} \subseteq \bigcap_{k \geq n} D_k$ and from this

$$P \subseteq \operatorname{int}\left(\bigcap_{k \geq n} D_k\right).$$

Thus, $\left(\operatorname{int}\left(\bigcap_{k \geq n} D_k\right)\right)_{n \in \mathbb{N}}$ is a quasi-covering of $\Omega \setminus A_{\mathrm{s}}(\zeta)$.

Let $v \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$. By Lemma 2.29, there exists a sequence $(v_n)_{n \in \mathbb{N}}$ with $v_n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$ and such that $v_n \in H_0^1\left(\operatorname{int}\left(\bigcap_{k \geq n} D_k\right)\right)$ for each $n \in \mathbb{N}$. In particular, $v_n \in H_0^1(D_n)$ since $\operatorname{int}\left(\bigcap_{k \geq n} D_k\right) \subseteq D_n$.

It remains to show the weak limit property for the Mosco convergence. Suppose there is a sequence $(w_n)_{n \in \mathbb{N}}$ with $w_n \in H_0^1(D_n)$ and $w_{n_k} \rightharpoonup w$ in $H_0^1(\Omega)$ for a subsequence $(w_{n_k})_{k \in \mathbb{N}}$ of $(w_n)_{n \in \mathbb{N}}$ and some $w \in H_0^1(\Omega)$. Since

$$D_n \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta) \subseteq \tilde{D}_n$$

holds for all $n \in \mathbb{N}$ large enough, cf. Lemma 7.23, we have $w_n \in H_0^1(\tilde{D}_n)$ if $n \in \mathbb{N}$ is large enough. By the Mosco convergence of $H_0^1(\tilde{D}_n)$ to $H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$, see Lemma 7.23, we conclude $w \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$. □

Remark 7.25 The assumption in Lemma 7.24 that there is $\kappa > 0$ such that $B_\kappa(x) \subseteq A_{\mathrm{s}}(\zeta)$ holds for some x in any connected component of $A_{\mathrm{s}}(\zeta)$ is fulfilled if the strictly active set has only finitely many connected components, since $A_{\mathrm{s}}(\zeta) = \text{f-cl}(\operatorname{int}(A(\zeta)))$, see Corollary 7.21.

7.6.4 Alternative discrete inner approximation of the complement of the strictly active set

In addition, we propose another possible choice for the discrete approximations D_n of $\Omega \setminus A_{\mathrm{s}}(\zeta)$. To distinguish it from the choice in (7.15), we use the notation E_n. The construction relies on the Lipschitz domain structure of the strictly active set. Indeed, in Lemma 7.29, we will assume that the strictly active set is a Lipschitz domain. In contrast to Lemma 7.24, we do not need to assume that the weakly and strictly active set have a positive distance from each other.

First, we give some definitions and properties connected with cones and Lipschitz domains.

The first two points in the following definition are based on [Wlo87, Def. 2.3].

Definition 7.26 (Cone property)

1. For $x \in \mathbb{R}^d$, $\rho > 0$ and an open nonempty subset Σ of $S_\rho(x) := \{y \mid \|y - x\| = \rho\}$ we call the set

$$C(x, \rho, \Sigma) = B_\rho(x) \cap \{\beta(y - x) \mid y \in \Sigma, \beta > 0\}$$

   a cone with vertex in x.
2. An open and set E of $\mathbb{R}^d$ has the cone property, if for each $x \in \overline{E}$ there is a cone C_x with vertex at x which is congruent to a fixed cone C_0 such that the subset C_x is contained in E. Here, the statement that C_x is congruent to C_0 means that C_x is a possibly translated and rotated copy of C_0.
3. Let $E \subseteq \mathbb{R}^d$ be a Lipschitz domain. Denote by $C_0 := C_0(x, \rho, \Sigma)$ a cone such that E has the cone property with cone C_0. Let $z \in \Sigma$ and let $B_r(z)$ be a ball such that $B_r(z) \cap S_\rho(x) \subseteq \Sigma$. We say that E has at least the interior angle $\alpha > 0$, if the convex cone induced by $B_r(z) \cap S_\rho(x)$, i.e., the cone

$$\{\beta(y - x) \mid y \in B_r(z) \cap S_\rho(x), \beta > 0\}$$

has the angle α.

The next lemma is taken from [Wlo87, Thm. 2.1].

Lemma 7.27 *Let E be a bounded Lipschitz domain. Then E has the cone property.*

The subsequent lemma gives a sufficient condition for a point in the boundary $\partial A(\zeta)$ of the active set to be a point where $\partial A(\zeta)$ is locally Lipschitz. It is taken from [Caf77].

Lemma 7.28 *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with sufficiently regular boundary and assume $\psi \in C^2(\Omega)$. Suppose that $\zeta \in L^2(\Omega)$ is bounded and locally Hölder continuous. If ω_0 is a point of positive Lebesgue density for $A(\zeta)$, i.e.,*

$$\lim_{r\to 0} \frac{\lambda^d(B_r(\omega_0)\cap A(\zeta))}{\lambda^d(B_r(\omega_0))} > 0,$$

then in a neighborhood of w_0, $\partial A(\zeta)$ can be represented as the graph of a Lipschitz function.

Proof. By [GT01, Thm. 4.3], the Poisson equation

$$-\Delta v = \zeta, \quad v = 0 \text{ on } \partial\Omega$$

has a unique solution $v \in C^2(\Omega) \cap C(\overline{\Omega})$. The obstacle problem with right hand side ζ as in (COP) is then equivalent to the problem

$$\text{Find } w \in K_{\psi-v} : \quad (\nabla w, \nabla(z-w)) \geq 0 \quad \forall\, z \in K_{\psi-v}$$

and it holds $w = y - v$. By Lemma 7.6, the solution w is in $C^{1,\beta}(\overline{\Omega})$ for any $\beta \in (0,1)$. In particular, w is Lipschitz continuous. This shows that w is the solution to the problem considered in [Caf77]. Noting that $\{w = \psi - v\} = \{y = \psi\}$, we can apply [Caf77, Thm. 2] to deduce the statement. $\square$

We define

$$E_n := \{\omega \in \Omega \mid \operatorname{dist}(\omega, \tilde{D}_n^{\complement}) > \gamma_n\} \tag{7.16}$$

for some γ_n specified in Lemma 7.29.

Lemma 7.29 *Suppose the conditions in Assumption 7.1 are fulfilled. Assume that $A_{\mathrm{s}}(\zeta)$ is a closed set with Lipschitz boundary that has at least the interior angle $\alpha > 0$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence with $\gamma_n \to 0$ as $n \to \infty$ and*

$$\gamma_n \geq \frac{\tilde{r}_n}{\sin(\alpha/2)}.$$

Let E_n be defined as in (7.16)*. Then it holds*

$$E_n \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta)$$

for $n \in \mathbb{N}$ large enough and

$$H_0^1(E_n) \to H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$$

in the sense of Mosco.

Proof. Since $\operatorname{int}(A_{\mathrm{s}}(\zeta))$ has the cone property, see Lemma 7.27, we find a fixed cone C_0 with radius $\alpha > 0$, such that for each $x \in A_{\mathrm{s}}(\zeta)$ the set C_x is contained in $A_s(\zeta)$.

Let $n \in \mathbb{N}$ be fixed and let $\omega \in A_{\mathrm{s}}(\zeta)$ be arbitrary.

For $x \in A_{\mathrm{s}}(\zeta)$ we have

$$C_x \subseteq A_{\mathrm{s}}(\zeta) \subseteq I_n^{\complement},$$

i.e.,

$$I_n \subseteq C_x^{\complement}.$$

This shows

$$\{x \in \Omega \mid \operatorname{dist}(x, I_n) \geq \tilde{r}_n\} \supseteq \{x \in \Omega \mid \operatorname{dist}(x, C_\omega^{\complement}) \geq \tilde{r}_n\}.$$

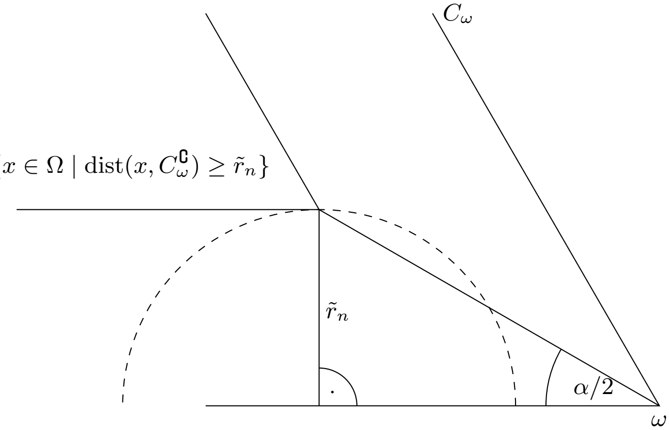


Figure 7.5. The cone C_ω and the set of points within C_ω that have a distance of at least $\tilde{r}_n$ to $\Omega \setminus C_\omega$

Using this and the trigonometry indicated in Fig. 7.5, we estimate

$$\begin{aligned}\operatorname{dist}(\omega, \tilde{D}_n^\complement) &= \operatorname{dist}(\omega, \{x \in \Omega \mid \operatorname{dist}(x, I_n) \geq \tilde{r}_n\}) \\ &\leq \operatorname{dist}(\omega, \{x \in \Omega \mid \operatorname{dist}(x, C_\omega^\complement) \geq \tilde{r}_n\}) \\ &\leq \frac{\tilde{r}_n}{\sin(\alpha/2)} \leq \gamma_n.\end{aligned}$$

This shows $\omega \in E_n^\complement$. Since $\omega \in A_\mathrm{s}(\zeta)$ was arbitrary, we have shown $E_n \subseteq \Omega \setminus A_\mathrm{s}(\zeta)$.

We want to show that the sets $\left(\operatorname{int}\left(\bigcap_{k\geq n} E_k\right)\right)_{n\in\mathbb{N}}$ are a quasi-covering of $\Omega \setminus A_\mathrm{s}(\zeta)$. Let $\omega \in \Omega \setminus A_\mathrm{s}(\zeta)$. Since $A_\mathrm{s}(\zeta)$ is closed, ω has a fixed distance > 0 to the set $A_\mathrm{s}(\zeta)$. If $n \in \mathbb{N}$ is sufficiently large, we find $c(n) > 0$ such that

$$B_{c(n)}(\omega) \subseteq \{x \in \Omega \mid \operatorname{dist}(x, A_\mathrm{s}(\zeta)) > \gamma_n\}$$

and thus, by $\tilde{D}_n^\complement \subseteq A_\mathrm{s}(\zeta)$, see Lemma 7.23,

$$B_{c(n)}(\omega) \subseteq \{x \in \Omega \mid \operatorname{dist}(x, \tilde{D}_n^\complement) > \gamma_n\} = E_n.$$

In particular, $\omega \in \operatorname{int}\left(\bigcap_{k\geq n} E_k\right)$ for some $n \in \mathbb{N}$ large enough. Thus, the family $\left(\operatorname{int}\left(\bigcap_{k\geq n} E_k\right)\right)_{n\in\mathbb{N}}$ is an increasing covering of $\Omega \setminus A_{\mathrm{s}}(\zeta)$.

To prove Mosco convergence, we proceed analogously to the proof of Lemma 7.24. Let $v \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$. By Lemma 2.29, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$ and such that $v_n \in H_0^1\left(\operatorname{int}\left(\bigcap_{k\geq n} E_k\right)\right)$ for each $n \in \mathbb{N}$. In particular, $v_n \in H_0^1(E_n)$ since $\operatorname{int}\left(\bigcap_{k\geq n} E_k\right) \subseteq E_n$.

Now, we verify the second condition for Mosco convergence. Suppose that there is a sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in H_0^1(E_n)$ and $w_{n_k} \rightharpoonup w$ in $H_0^1(\Omega)$ for a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$. Since

$$E_n \subseteq \Omega \setminus A_{\mathrm{s}}(\zeta) \subseteq \tilde{D}_n$$

holds for all $n \in \mathbb{N}$ large enough, cf. Lemma 7.23, we have $w_n \in H_0^1(\tilde{D}_n)$ if $n \in \mathbb{N}$ is large enough. By the Mosco convergence of $H_0^1(\tilde{D}_n)$ to $H_0^1(\Omega\setminus A_{\mathrm{s}}(\zeta))$, see Lemma 7.23, we conclude $w \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$. $\square$

Remark 7.30 A downside of this approach is that, in practice, we do not know the interior angle in the Lipschitz domain $A_{\mathrm{s}}(\zeta)$. This unknown constant leads to a worse approximation of $\Omega \setminus A_{\mathrm{s}}(\zeta)$ and a slower Mosco convergence of $H_0^1(E_n) \to H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta))$ compared to the choice D_n in (7.15). In this case, one cannot set $\gamma_n := \frac{\tilde{r}_n}{\sin(\alpha/2)}$ in (7.16), but has to use, e.g.,

$$\gamma_n := \tilde{r}_n^{1-\kappa} \geq \frac{\tilde{r}_n}{\sin(\alpha/2)}$$

for $\kappa > 0$ small. The last inequality holds if n is large enough.

7.7 Alternative inner approximation of the inactive set

As an alternative for the sets I_n introduced in (7.10), we can consider the discrete inactive sets

$$J_n := \left\{ y_n > \psi_n + \varepsilon_n + \|(\psi - \psi_n)_+\|_{L^\infty(\{y_n \le \psi_n + \varepsilon_n + \|(\psi-\psi_n)_+\|_{L^\infty(\Omega)}\})} \right\}. \tag{7.17}$$

Note that $\psi_n \to \psi$ uniformly. Then, the analogous statements in Lemma 7.8 also hold for J_n. To see this, assume ω is an element of $A(\zeta)$. Then it holds

$$\begin{aligned} y_n(\omega) &= y(\omega) + (y_n(\omega) - y(\omega)) \\ &= \psi(\omega) + \psi_n(\omega) - \psi_n(\omega) + (y_n(\omega) - y(\omega)) \\ &\le \psi_n(\omega) + \varepsilon_n + \|(\psi - \psi_n)_+\|_{L^\infty(\Omega)}. \end{aligned}$$

This shows $\omega \in \{y_n \le \psi_n + \varepsilon_n + \|(\psi - \psi_n)_+\|_{L^\infty(\Omega)}\}$ and thus

$$y_n(\omega) \le \psi_n(\omega) + \varepsilon_n + \|(\psi - \psi_n)_+\|_{L^\infty(\{y_n \le \psi_n + \varepsilon_n + \|(\psi-\psi_n)_+\|_{L^\infty(\Omega)}\})}.$$

We conclude $J_n \subseteq I(\zeta)$ for all $n \in \mathbb{N}$.

To verify the property $\liminf_{n\to\infty} J_n = I(\zeta)$, we suppose $\omega \in I(\zeta)$ and $y(\omega) - \psi(\omega) =: c > 0$. Let $n \in \mathbb{N}$ be such that

$$2\varepsilon_k + \|(\psi - \psi_k)_+\|_{L^\infty(\{y_k \le \psi_k + \varepsilon_k + \|(\psi-\psi_k)_+\|_{L^\infty(\Omega)}\})} - \psi(\omega) + \psi_k(\omega) < c$$

holds for all $k \ge n$. Then we conclude

$$\begin{aligned} y_k(\omega) - \psi_k(\omega) &= y(\omega) - \psi(\omega) + \psi(\omega) - \psi_k(\omega) - y(\omega) + y_k(\omega) \\ &\ge c + \psi(\omega) - \psi_k(\omega) - \varepsilon_k \\ &> \varepsilon_k + \|(\psi - \psi_k)_+\|_{L^\infty(\{y_k \le \psi_k + \varepsilon_k + \|(\psi-\psi_k)_+\|_{L^\infty(\Omega)}\})}. \end{aligned}$$

This shows that the sets $(\bigcap_{k\ge n} J_k)_{n\in\mathbb{N}}$ cover $I(\zeta)$.

In particular, the Mosco convergence $H_0^1(J_n) \to H_0^1(I(\zeta))$ can be shown as in Theorem 7.10.

To obtain a similar statement to the one in Lemma 7.13, we set

$$(r_n^J)^2 := \frac{2d}{\eta}\big(2\varepsilon_n + \|(\psi - \psi_n)_+\|_{L^\infty(\{y_n \le \psi_n + \varepsilon_n + \|(\psi-\psi_n)_+\|_{L^\infty(\Omega)}\})} \\ + \|(\psi_n - \psi)_+\|_{L^\infty(\{y_n \le \psi_n + \varepsilon_n + \|(\psi-\psi_n)_+\|_{L^\infty(\Omega)}\})}\big).$$

Now, a slight modification of the proof of Lemma 7.13 shows that

$$\{\omega \in \Omega \mid \operatorname{dist}(\omega, J_n) \ge r_n^J\} \subseteq A(\zeta).$$

Choosing $\tilde r_n^J > r_n^J$ with $\tilde r_n^J \to 0$, we can define $\tilde D_n^J$ as in (7.14) with $\tilde r_n$ replaced by $\tilde r_n^J$, use $\tilde D_n^J$ instead of $\tilde D_n$ in the definitions of D_n in (7.15) or E_n in (7.16) and all results established in Section 7.6.3 carry over using the properties of J_n that are similar to the properties of I_n.

7.8 Summary of the results

Collecting the results in this chapter, we can formulate the following theorem on error estimates for a generalized derivative for the obstacle problem.

Theorem 7.31 *Assume the conditions in Assumption 7.1 are satisfied and let $\tilde D_n$ be defined as in (7.14). Denote by q the solution of*

$$\textit{Find } q \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta)): \quad \langle -\Delta q, v\rangle = \langle J_y(y,u), v\rangle \quad \forall\, v \in H_0^1(\Omega \setminus A_{\mathrm{s}}(\zeta)),$$

i.e., $f'(u)^*q + J_u(y,u)$ is a Clarke generalized derivative for $\hat J$. In addition, consider one of the following cases.

1. *Suppose $\operatorname{dist}(A_{\mathrm{w}}(\zeta), A_{\mathrm{s}}(\zeta)) > 0$ and let $\kappa > 0$ be a positive number such that $B_\kappa(x) \subseteq A_{\mathrm{s}}(\zeta)$ holds for some x in every connected component of $A_{\mathrm{s}}(\zeta)$. Let D_n be defined as in (7.15) and denote by $(q_n)_{n\in\mathbb{N}}$ the solutions of*

$$\textit{Find } q_n \in H_0^1(D_n): \quad \langle -\Delta q_n, v_n\rangle = \langle J_y(y_n,u), v_n\rangle \quad \forall\, v_n \in H_0^1(D_n).$$

2. *Suppose $A_{\mathrm{s}}(\zeta)$ is a closed set with Lipschitz boundary and let E_n be defined as in (7.16) with γ_n as specified in Lemma 7.29. Denote by $(q_n)_{n\in\mathbb{N}}$ the solutions of*

$$\text{Find } q_n \in H_0^1(E_n): \quad \langle -\Delta q_n, v_n\rangle = \langle J_y(y_n, u), v_n\rangle \quad \forall\, v_n \in H_0^1(E_n).$$

In both cases, $q_n \to q$ holds as $n \to \infty$ as well as

$$\begin{aligned} &\|q - q_n\|_{H_0^1(\Omega)} \\ &\leq \|-\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)} + \|J_y(y_n, u) - J_y(y, u)\|_{H^{-1}(\Omega)} \overset{n\to\infty}{\to} 0. \end{aligned} \tag{7.18}$$

7.9 Numerical examples

In this section, we test our considerations on error estimates for the generalized derivatives.

We deal with the following test setting. In the domain $\Omega := B_1(0) := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ we consider the radial symmetric obstacle

$$\tilde{\psi}(x) := -x_1^2 - x_2^2 + \frac{1}{2} =: \tilde{\psi}_{\mathrm{c}}(|x|)$$

with $\tilde{\psi}_{\mathrm{c}}\colon \mathbb{R} \to \mathbb{R}, s \mapsto \frac{1}{2} - s^2$. We are looking for the radial symmetric solution y of the obstacle problem (COP) for $\zeta = 0$ in this setting.

We show that for $r \in (0,1)$ solving the equation

$$\ln(r) = -\frac{1}{4r^2} + \frac{1}{2}$$

the function

$$y(x) = \begin{cases} \tilde{\psi}(x) & \text{for } |x| \leq r, \\ -2r^2 \ln(|x|) & \text{for } r < |x| \leq 1 \end{cases} =: y_{\mathrm{c}}(|x|)$$

solves the obstacle problem in the given setting.

It is easy to check that y is continuously differentiable and that $y(x) = 0$ for all x with $|x| = 1$, i.e., $y \in H_0^1(\Omega)$. Moreover, for $t \in [r, 1]$ it holds

$$y_{\mathrm{c}}(t) - \tilde{\psi}_{\mathrm{c}}(t) = -2r^2 \ln(t) - \frac{1}{2} + t^2.$$

Using $t \geq r$ we can estimate the derivative on $(r, 1)$ and obtain

$$y_{\mathrm{c}}'(t) - \tilde{\psi}_{\mathrm{c}}'(t) = \frac{-2r^2}{t} + 2t \geq \frac{-2r^2}{r} + 2t = -2r + 2t \geq 0.$$

Thus, since $y(x) = \tilde{\psi}(x)$ for $|x| = r$, we conclude that $y \geq \tilde{\psi}$ holds on Ω. This shows $y \in K_\psi$.

For $|x| < r$ we have

$$-\Delta y(x) = -\Delta \psi(x) = 2 + 2 > 0$$

and for $|x| > r$ we compute

$$-\Delta y(x) = \frac{4x_1^2 r^2}{|x|^4} + \frac{4x_2^2 r^2}{|x|^4} - \frac{4r^2}{|x|^2} = 0.$$

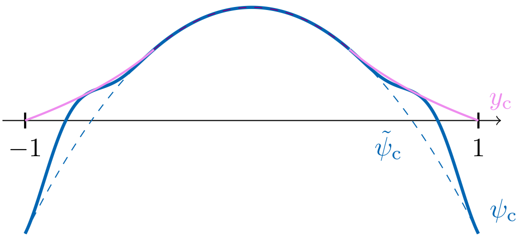
Thus, for $z \in K_\psi$

$$\begin{aligned}
&\langle -\Delta y, z - y\rangle \\
&= \int_\Omega -\Delta y\,(z - y)\,\mathrm{d}\lambda^2 = \int_{B_r(0)} 4(z - y)\,\mathrm{d}\lambda^2 = \int_{B_r(0)} 4(z - \psi)\,\mathrm{d}\lambda^2 \geq 0
\end{aligned} \tag{7.19}$$

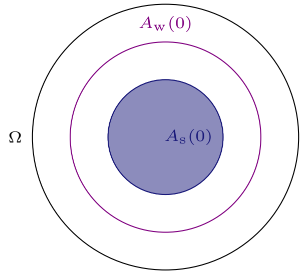
holds.

The active set for this example is strictly active. Therefore we perturb the obstacle inside of the inactive set to obtain a problem where the strict complementarity condition does not hold and where the solution y we have just derived does not change. Therefore, we set

$$\vartheta(x) := \begin{cases} 0 & \text{for } 0 \leq |x| \leq r, \\ \frac{1}{2}\left(1 - \cos\left(\frac{|x| - r}{\frac{1+r}{2} - r}\pi\right)\right) & \text{for } r < |x| \leq 1 \end{cases}$$



(a) Cross section of radial symmetric obstacle problem



(b) Corresponding active sets in Ω

Figure 7.6. Visualisation of the obstacle problem considered in Section 7.9

and consider the perturbed obstacle

$$\psi(x) := \vartheta(x)\, y(x) + (1 - \vartheta(x))\, \tilde{\psi}(x) \quad (x \in B_1(0)).$$

We have $0 \leq \vartheta(x) \leq 1$ for all $x \in B_1(0)$, thus $y(x) \geq \psi(x)$. Moreover, it holds $\vartheta(x) = 1$ if and only if $x = \frac{1+r}{2}$. This implies that the set $S_{\frac{1+r}{2}}(0)$ is weakly active.

Note that solution y and the multiplier $-\nabla y$ are not affected by the perturbation of the obstacle. In fact, since $z \geq \psi$ is only important in the set $B_r(0)$ in (7.19), y is the solution of the perturbed obstacle problem.

The cross section of the obstacle and the corresponding solution of this problem as well as the respective strictly and weakly active set are shown in Fig. 7.6.

For the implementation in MATLAB we consider the continuously differentiable objective function $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ given by

$$J(y, u) := \frac{1}{2} \|y - y_d\|^2_{L^2(\Omega)}$$

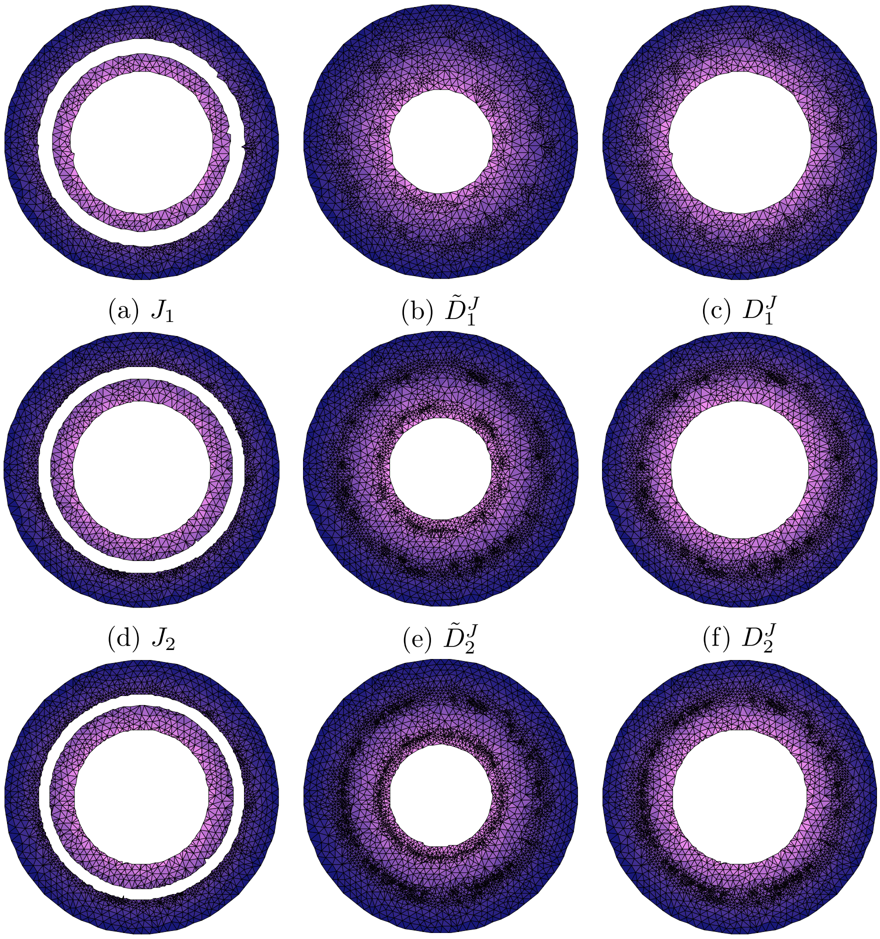
and we choose the constant desired state $y_d := \frac{1}{4}$. Note that the exact choice of U and f are not relevant for our computation since we assume that the conditions of Assumption 4.2 are fulfilled and since we already know how a generalized derivative can be obtained.

We verify the conditions of Assumption 7.1. It is straightforward to verify the regularity assumptions on ζ, y and ψ. Moreover, $\lambda^2(\partial A(0)) = 0$ holds. In addition, we have $-\Delta\psi \geq 4$ on $A(0)$ and $-\Delta\psi \geq 3.8$ holds a.e. in a neighborhood of the active set. Thus, the nondegeneracy condition (ND$_\eta$) is satisfied.

By the structure of $A(0)$ we have $\operatorname{dist}(A_{\mathrm{w}}(0), A_{\mathrm{s}}(0)) > 0$ and also $B_r(0) \subseteq A_{\mathrm{s}}(0)$. Thus, the results from Theorem 7.31(1.) are applicable.

In a first computation, we use the sets J_n introduced in (7.17). The boundaries of these sets are level sets of piecewise affine functions, thus, in each triangle the boundary can be determined accurately and the mesh is adjusted accordingly in each iteration. This leads to a less serrated appearance of the boundary of the constructed sets J_n and consequently also $\tilde{D}_n^J$ and D_n^J. The resulting sets $\tilde{D}_n^J$ and D_n^J in the first iterations are shown in Fig. 7.7. Here, we use the a posteriori $L^\infty(\Omega)$-error estimates from [NSV05] to construct the sets J_n. The mesh is refined adaptively taking into account the error contribution for the $L^\infty(\Omega)$-error estimate ε_n and the error $\|(\psi - \psi_n)_+\|_{L^\infty(\{y_n \leq \psi_n + \varepsilon_n + \|(\psi - \psi_n)_+\|_{L^\infty(\Omega)}\})}$ in each triangle separately. These quantities determine the quality of the approximation J_n, see (7.17). Exemplary, the resulting generalized derivative q_2 in iteration $n = 2$ is shown in Fig. 7.8. Table 7.1 shows the computed contributions $\| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n^J)}$ to the upper bounds for the error $\|q - q_n\|_{H_0^1(\Omega)}$, see (7.18). Moreover, the used radii for the construction of $\tilde{D}_n^J$ from J_n, see (7.14), are recorded as well as the ratio

$$\frac{\| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n^J)}}{\sqrt{\tilde{r}_n^J}}.$$



(a) J_1 (b) $\tilde{D}_1^J$ (c) D_1^J

(d) J_2 (e) $\tilde{D}_2^J$ (f) D_2^J

(g) J_3 (h) $\tilde{D}_3^J$ (i) D_3^J

Figure 7.7. Construction of the sets J_n, $\tilde{D}_n^J$ and D_n^J in the numerical example in iterations $n = 1, 2, 3$

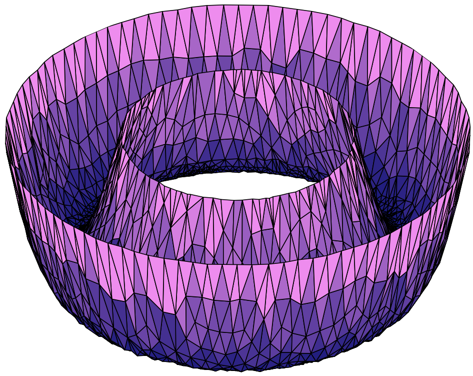


Figure 7.8. Numerical subgradient in iteration $n = 2$ for the first computation

Table 7.1. Terms $\|-\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n^J)}$, $\tilde{r}_n^J$ and rate $\|-\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n^J)}/\sqrt{\tilde{r}_n^J}$ in iteration n

Iteration	Error term for $\|q - q_n\|_{H_0^1(\Omega)}$	$\tilde{r}_n^J$	Ratio
0	0.0260	0.2225	0.0551
1	0.0164	0.1394	0.0439
2	0.0150	0.1292	0.0417
3	0.0143	0.1183	0.0416
4	0.0126	0.0955	0.0408
5	0.0112	0.0736	0.0413
6	0.0105	0.0650	0.0412
7	0.0096	0.0539	0.0414
8	0.0085	0.0414	0.0418

One can see that this ratio is approximately constant. Indeed, if the considered sets are sufficiently regular, one can theoretically derive an estimate of the form

$$\| -\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n^J)} \leq C\sqrt{\tilde{r}_n^J}$$

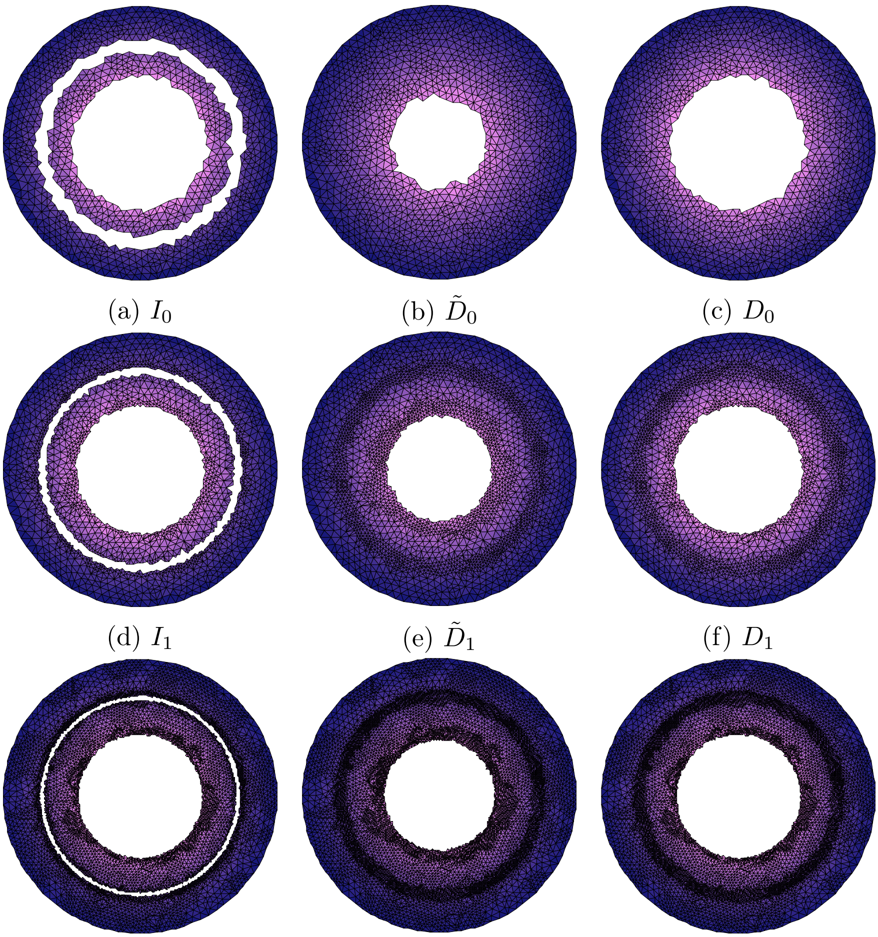
for a constant C.

In a second computation, we use the sets I_n as in (7.10). The constructed sets I_n, $\tilde{D}_n$ and D_n are shown in Fig. 7.9. This time, the mesh is not adjusted when the allocation of triangles to either I_n or $I_n^{\complement}$ is performed. Instead, we use a possibly larger set $\tilde{D}_n$ for the error estimate and a possibly smaller set D_n as a domain for the computation of the subgradient. As a result, the boundaries of the constructed sets are more serrated compared to the sets in the first experiment. Again, the a posteriori $L^\infty(\Omega)$-error estimates from [NSV05] are used. Since we know the exact solution y, we can tighten the error estimate from [NSV05] for this particular example leading to a more accurate approximation of the complement of the strictly active set and speeding up the convergence process. The resulting generalized derivative q_2 in iteration $n = 3$ is shown in Fig. 7.10.

Table 7.2 shows again the computed contributions $\| -\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)}$ to the upper bounds for the error $\|q - q_n\|_{H_0^1(\Omega)}$, compare (7.18), as well as the considered radii $\tilde{r}_n$ and the ratio

$$\frac{\| -\Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)}}{\sqrt{\tilde{r}_n}}.$$

By using the previously established error estimates in the two computations, we have shown the applicability of our results, which ends this chapter.



(a) I_0 (b) $\tilde{D}_0$ (c) D_0

(d) I_1 (e) $\tilde{D}_1$ (f) D_1

(g) I_2 (h) $\tilde{D}_2$ (i) D_2

Figure 7.9. Construction of the sets I_n, $\tilde{D}_n$ and D_n in the numerical example in iterations $n = 0, 1, 2$

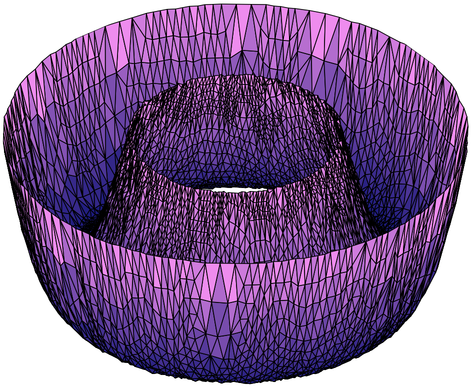


Figure 7.10. Numerical subgradient in iteration $n = 3$ for the second computation

Table 7.2. Terms $\| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)}$, $\tilde{r}_n$ and rate $\| - \Delta q_n - J_y(y_n, u)\|_{H^{-1}(\tilde{D}_n)}/\sqrt{\tilde{r}_n}$ in iteration n

Iteration	Error term for $\|q - q_n\|_{H_0^1(\Omega)}$	$\tilde{r}_n$	Ratio
0	0.0152	0.0649	0.0597
1	0.0118	0.0459	0.0551
2	0.0090	0.0271	0.0547
3	0.0074	0.0174	0.0561
4	0.0062	0.0130	0.0544
5	0.0049	0.0079	0.0551
6	0.0041	0.0052	0.0569
7	0.0032	0.0034	0.0549
8	0.0027	0.0024	0.0551

Conclusion and outlook

In this thesis we have derived generalized derivatives for solution operators of obstacle problems.

Considering a monotone operator with suitable properties, we have shown that solution operators of variational equations on the inactive set and on the complement of the strictly active set are elements of the generalized differential for the composition of the solution operator of the obstacle problem with the monotone operator.

Additionally, we studied the solution operator on $H^{-1}(\Omega)$ and we were able to present a complete representation of the generalized differentials defined by the strong operator topology for the convergence of the Gâteaux derivatives. These differentials consist exactly of all solution operators relative to quasi-open domains which are located between the inactive set and the complement of the strictly active set.

If we consider the generalized differentials using the weak operator topology for the convergence of the Gâteaux derivatives, then solution operators of relaxed Dirichlet problems with respect to capacitary measures are also contained in the generalized differentials. These measures vanish on the inactive set and have infinite values on the strictly active set.

Moreover, we have analyzed the bilateral obstacle problem and provided generalized derivatives for its solution operator. Here, we have also considered the composition with a monotone, possibly nonlinear operator.

Finally, we have seen how inexact generalized derivatives can be computed without knowledge of the exact active and strictly active sets. We have developed an error estimate for this particular inexact generalized derivatives and have tested our results in a numerical example.

Of course, there are related problems with a similar structure that have not been addressed in this thesis. For example, the directional derivative of the solution operator of a certain parabolic variational inequality is derived in [JKRSo03]. Similar to the result by [Mig76], this directional derivative is given by a variational inequality. Based on this result, it might also be possible to obtain generalized derivatives in this setting. This is still work in progress.

Recently, the directional differentiability of quasi-variational inequalities of obstacle type have been studied in [AHR19]. This might also be a starting point to obtain generalized derivatives.

In addition, in [HRUU21], stochastic obstacle problems are considered and generalized derivatives are obtained. A short remark on generalized derivatives in the context of a shape optimization problem for the obstacle problem is contained in [RU19].

Finally, the implementation of a Bundle method as in [HU19] that applies the error estimates derived in Chapter 7 remains to be realized.

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