Exponential estimates of solutions of pseudodifferential equations with operator-valued symbols. Applications to Schrödinger operators with operator-valued potentials

V. S. Rabinovich, S. Roch

Abstract

We consider a class of pseudodifferential operators with operator-valued symbols $a(x, \xi)$ under the assumption that $a(x, \xi)$ can be analytically extended with respect to ξ onto a tube domain $\mathbb{R}^n + i\mathcal{B}$ where $\mathcal{B}$ is a convex bounded domain in $\mathbb{R}^n$ containing the origin. The main result of the paper is exponential estimates at infinity of solutions of pseudodifferential equations $Op(a)u = f$. We apply this result to Schrödinger operators with operator-valued potentials and give applications to spectral properties of quantum waveguides. Our approach is based on the construction of the local inverse operator at infinity and on formulas for commutators of pseudodifferential operators with exponential weights.

MSC (2000): 35Sxx, 58Jxx, 81Q10
Key words: Pseudodifferential operators with operator-valued symbols, Fredholm property, exponential estimates of solutions

1 Introduction

We consider the class of pseudodifferential operators

$$(Op(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) u(y) e^{i(x-y)\cdot\xi} dy d\xi, \quad u \in S(\mathbb{R}^n, \mathcal{H}_1), \quad (1)$$

with symbols a having values in the space of bounded operators acting from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$ and satisfying additional estimates. In (1), $S(\mathbb{R}^n, \mathcal{H})$ is the space of $\mathcal{H}$-valued infinitely differentiable functions rapidly decreasing with all their derivatives.

We suppose that the symbol $a(x, \xi)$ can be analytically extended with respect to ξ onto a tube domain $\mathbb{R}^n + i\mathcal{B}$ where $\mathcal{B}$ is a convex bounded domain in $\mathbb{R}^n$ containing the origin. The main result of the paper is exponential estimates at infinity of solutions of pseudodifferential equations $Op(a)u = f$. We apply these estimates to Schrödinger operators with operator-valued potentials and discuss applications to quantum waveguides. Our approach is based on the construction of the local inverse operator at infinity and on results on commutators of pseudodifferential operators with exponential weights (see, for instance, [21, 29, 30]).

It turns out that many problems in mathematical physics can reduced to the study of associated pseudodifferential operators with operator-valued symbols. In particular, this happens for problems of wave propagation in acoustic, electromagnetic and quantum waveguides (see, for instance, [3, 32] and the references cited there).

Estimates of exponential decay are intensively studied in the literature. We would like to emphasize Agmon's monograph [1] where the exponential estimates of the behavior of solutions of second order elliptic operators have been obtained in terms of a special metric (now called the Agmon metric), but see also [4, 12, 13, 15, 21, 22, 23, 26, 29, 30]. In [33, 34], the authors established the relation between the essential spectrum of pseudodifferential operators and exponential decay of their solutions at infinity. The recent paper [31] by one of the authors is devoted to local exponential estimates of solutions of finite-dimensional h-pseudodifferential operators with applications to the tunnel effect for Schrödinger, Dirac and Klein-Gordon operators.

This paper is organized as follows. In Section 2 we present some auxiliary facts on operator-valued pseudodifferential operators. Some standard references for the theory of pseudodifferential operators are [18, 35, 37], whereas operator-valued pseudodifferential operators have been studied in [19, 20]. The approach in the latter books follows ideas by Hörmander and employs a special partition of unity connected with a metric defining the class of pseudodifferential operators. We will follow here the approach of [28], which based on the notion of a formal symbol. A main point is the representation of the symbol of a product of pseudodifferential operators and of a double pseudodifferential operators in form of an operator-valued double oscillatory integral. This approach allows us to extend the theory of scalar pseudodifferential operators to pseudodifferential operators with operator-valued symbols, and it provides us with an pseudodifferential operator calculus which is convenient for applications

In Section 3 we examine the local invertibility at infinity of operator-valued pseudodifferential operators in suitable functions spaces and discuss their Fredholm property. Section 4 is devoted to the exponential estimates at infinity of solutions of operator-valued pseudodifferential operators. In the concluding Section 5 we are going to study the Fredholm property of Schrödinger operators and derive exponential estimates at infinity of solutions of Schrödinger equations with operator-valued potentials. These general results are then applied to the Fredholm property of Schrödinger operators for quantum waveguides, for which we obtain exponential estimates of eigenfunctions of the discrete spectrum. Note that spectral problems for quantum waveguides have attracted many attention in the last time. See, for instance, [3, 8, 11, 7].

The work on this paper was supported by CONACYT Project 81615 and DFG Grant Ro 1100/8-1.

2 Calculus of pseudodifferential operators with operator-valued symbols

2.1 Notations

- Given Banach spaces X, Y, we denote the Banach space of all bounded linear operators acting from X in Y by $\mathcal{L}(X, Y)$. In case $X = Y$, we simply write $\mathcal{L}(X)$.

- Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then we denote by $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ the points of the dual space with respect to the scalar product $\langle x, \xi \rangle = x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$.

- For $j = 1, \ldots, n$, let $\partial_{x_j} := \frac{\partial}{\partial x_j}$ and $D_{x_j} := -i\frac{\partial}{\partial x_j}$. More generally, given a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, set $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \quad \text{and} \quad D_x^\alpha := D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}.$$

  For an operator-valued function $(x, \xi) \mapsto a(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$, we set $a_{(\alpha)}^{(\beta)} := D_x^\beta \partial_\xi^\alpha a$.

- Let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$.

- Let Ω be an open subset of $\mathbb{R}^n$ and X be a Banach space. We denote by

  (i) $C^\infty(\Omega, X)$ the set of all infinitely differentiable functions from Ω to X;

  (ii) $C_0^\infty(\Omega, X)$ the set of all functions in $C^\infty(\Omega, X)$ with a compact support in Ω;

  (iii) $C_b^\infty(\Omega, X)$ the set of all functions $a \in C^\infty(\Omega, X)$ such that

$$\sup_{x \in \Omega} \sum_{|\alpha| \le k} \|(\partial_x^\alpha a)(x)\|_X < \infty$$

  for every $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$;

  (iv) $S(\mathbb{R}^n, X)$ the set of all functions $a \in C^\infty(\mathbb{R}^n, X)$ such that

$$\sup_{x \in \mathbb{R}^n} \langle x \rangle^k \sum_{|\alpha| \le k} \|(\partial_x^\alpha a)(x)\|_X < \infty$$

  for every $k \in \mathbb{N}_0$.

  In each case, we omit X whenever $X = \mathbb{C}$.

- Let $\mathcal{H}$ be a Hilbert space and $u \in S(\mathbb{R}^n, \mathcal{H})$. Then we denote by

$$\hat{u}(\xi) = (Fu)(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx$$

the Fourier transform of u. Note that $F : S(\mathbb{R}^n, \mathcal{H}) \to S(\mathbb{R}^n, \mathcal{H})$ is an isomorphism with inverse

$$(F^{-1}\hat{u})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi} d\xi.$$

We write $S'(\mathbb{R}^n, \mathcal{H})$ for the space of distributions over $S(\mathbb{R}^n, \mathcal{H})$ and define the Fourier transform of distributions in $S'(\mathbb{R}^n, \mathcal{H})$ via duality. Note that $F : S'(\mathbb{R}^n, \mathcal{H}) \to S'(\mathbb{R}^n, \mathcal{H})$ is an isomorphism.

In what follows we consider separable Hilbert spaces $\mathcal{H}$ only.

2.2 Oscillatory vector-valued integrals

Let B be a Banach space, and let a be a function in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n, B)$ such that

$$|a|_{r,t} := \sum_{|\alpha| \le r,\, |\beta| \le t} \sup_{\mathbb{R}^n \times \mathbb{R}^n} \|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\|_B \, \langle \xi \rangle^{-m} < \infty \tag{2}$$

for all $r, l \in \mathbb{N}_0$. Further let $\chi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\chi(x, \xi) = 1$ for all points (x, ξ) in a neighborhood of the origin. Let $R > 0$. In what follows we let $\chi_R(x, \xi) := \chi(x/R, \xi/R)$.

Proposition 1 *Let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, B)$ satisfy the estimates* (2). *Then the limit*

$$\mathcal{I}(a) := \lim_{R\to\infty} \iint_{\mathbb{R}^{2n}} \chi_R(x, \xi) a(x, \xi) e^{-ix\cdot\xi} d(x, \xi)$$

exists in the norm topology of B and

$$\mathcal{I}(a) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \langle \xi \rangle^{-2k_2} \langle D_x \rangle^{2k_2} \left\{ \langle x \rangle^{-2k_1} \langle D_\xi \rangle^{2k_1} a(x, \xi) \right\} e^{-ix\cdot\xi} d(x, \xi)$$

for all

$$2k_1 > n, \quad 2k_2 > n + m. \tag{3}$$

This limit is independent of k_1, k_2 satisfying (3) *and of the choice of χ. Moreover,*

$$\begin{aligned} \|\mathcal{I}(a)\|_B &\le C \sum_{|\alpha| \le 2k_1,\, |\beta| \le 2k_2} \sup_{\mathbb{R}^n \times \mathbb{R}^n} \|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\|_B \langle \xi \rangle^{-m} \\ &= C|a|_{2k_1,\, 2k_2}. \end{aligned} \tag{4}$$

The element $\mathcal{I}(a) \in B$ is called the *oscillatory integral* of a. In what follows, we use the notation

$$\text{osc} \iint_{\mathbb{R}^{2n}} a(x, \xi) e^{-ix\cdot\xi} dx d\xi$$

for the oscillatory integral $\mathcal{I}(a)$.

Proposition 2 *Let $a \in C_b^\infty(\mathbb{R}^n, B)$. Then, for each $x \in \mathbb{R}^n$,*

$$(2\pi)^{-n} \text{osc} \iint_{\mathbb{R}^{2n}} a(x+y) e^{-iy\cdot\xi} dy d\xi = a(x). \tag{5}$$

Propositions 1 and 2 are proved as in the scalar case by integrating by parts (see for instance [28]).

2.3 Pseudodifferential operators with operator-valued symbols

Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. A function $p : \mathbb{R}^n \to \mathcal{L}(\mathcal{H}', \mathcal{H})$ is said to be a *weight function* in the class $O(\mathcal{H}, \mathcal{H}')$ if the operator $p(\eta)$ is invertible for each $\eta \in \mathbb{R}^n$ and if there are constants $C > 0$ and $N \in \mathbb{R}$ such that

$$\max\left\{\|p(\eta)^{-1}p(\xi)\|_{\mathcal{L}(\mathcal{H}')}, \, \|p(\xi)p^{-1}(\eta)\|_{\mathcal{L}(\mathcal{H})}\right\} \le C(1 + |\xi - \eta|)^N \tag{6}$$

for arbitrary $\xi, \eta \in \mathbb{R}^n$. Let now $\mathcal{H}_1$, $\mathcal{H}_1'$, $\mathcal{H}_2$ and $\mathcal{H}_2'$ be Hilbert spaces and $p_1 \in O(\mathcal{H}_1, \mathcal{H}_1')$ and $p_2 \in O(\mathcal{H}_2, \mathcal{H}_2')$. We say that a function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ belongs to $S(p_1, p_2)$ if

$$|a|_l := \sum_{|\alpha+\beta| \le l} \sup_{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n} \|p_2^{-1}(\xi) \partial_x^\beta \partial_\xi^\alpha a(x, \xi) p_1(\xi)\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} < \infty \tag{7}$$

for every $l \in \mathbb{N}_0$. The seminorms $|a|_l$ define a Frechet topology on $S(p_1, p_2)$. The (operator-valued) functions in $S(p_1, p_2)$ are called symbols.

With each symbol $a \in S(p_1, p_2)$, we associate the pseudodifferential operator $Op(a)$ which acts at $u \in S(\mathbb{R}^n, \mathcal{H}_1)$ by

$$\begin{aligned} (Op(a)u)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} a(x, \xi) \hat{u}(\xi) e^{ix\cdot\xi} d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) u(y) e^{i(x-y)\cdot\xi} dy d\xi. \end{aligned} \tag{8}$$

We denote the set of all pseudodifferential operators with symbols in $S(p_1, p_2)$ by $OPS(p_1, p_2)$.

Besides these "common" pseudodifferential operators, we will also need double symbols and their associated double pseudodifferential operators. Let again $p_1 \in O(\mathcal{H}_1, \mathcal{H}_1')$ and $p_2 \in O(\mathcal{H}_2, \mathcal{H}_2')$. A function $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to belong to the class $S_d(p_1, p_2)$ of *double symbols* if

$$|a|_l := \sum_{|\alpha+\beta+\gamma| \le l} \sup_{(x,y,\xi) \in \mathbb{R}^{3n}} \|p_2(\xi)^{-1} \partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, y, \xi) p_1(\xi)\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} < \infty \tag{9}$$

for each $l \in \mathbb{N}_0$. We correspond to each double symbol $a \in S_d(p_1, p_2)$ the *double pseudodifferential operator*

$$(Op_d(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, y, \xi) u(y) e^{i(x-y)\cdot\xi} dy d\xi, \tag{10}$$

$u \in S(\mathbb{R}^n, \mathcal{H}_1)$, and denote the class of all double pseudodifferential operators by $OPS_d(p_1, p_2)$. Note that the estimates (6) and (7) imply that if $a \in S(p_1, p_2)$ or $S_d(p_1, p_2)$ there exist $M > 0$ and constants $C_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ such that

$$\|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \leq C_{\alpha\beta} \langle \xi \rangle^M \tag{11}$$

and

$$\|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, y, \xi)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \leq C_{\alpha\beta\gamma} \langle \xi \rangle^M \tag{12}$$

for all multi-indeces α, β, γ.

Integrating by parts one can prove as in the scalar case that the pseudodifferential operators (8) and (10) can be written of the form of double oscillatory integrals depending on the parameter $x \in \mathbb{R}^n$,

$$(Op(a)u)(x) = (2\pi)^{-n} \text{osc} \iint_{\mathbb{R}^{2n}} a(x, \xi) u(x + y) e^{-iy\cdot\xi} d\xi dy, \tag{13}$$

$$(Op_d(a)u)(x) = (2\pi)^{-n} \text{osc} \iint_{\mathbb{R}^{2n}} a(x, x + y, \xi) u(x + y) e^{-iy\cdot\xi} d\xi dy, \tag{14}$$

and that the operators $Op(a)$ and $Op_d(a)$ in (13) and (14) can be extended to bounded operators from $C_b^\infty(\mathbb{R}^n, \mathcal{H}_1)$ to $C_b^\infty(\mathbb{R}^n, \mathcal{H}_2)$.

The following definition of a formal symbol is crucial. For $\xi \in \mathbb{R}^n$, define $e_\xi : \mathbb{R}^n \to \mathbb{C}$ by $e_\xi(x) := e^{ix\cdot\xi}$. Let now A be a continuous linear operator from $C_b^\infty(\mathbb{R}^n, \mathcal{H}_1)$ to $C_b^\infty(\mathbb{R}^n, \mathcal{H}_2)$, and let $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then there is a bounded linear operator $\sigma_A(x, \xi) : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$e_{-\xi}(x)[A(e_\xi \otimes \varphi)](x) = \sigma_A(x, \xi)\varphi \tag{15}$$

for every $\varphi \in \mathcal{H}_1$. The function $\sigma_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is then called the *formal symbol* of A. It follows from this definition that there exists constants $C > 0$ and $N \in \mathbb{N}_0$ such that

$$\|\sigma_A(x, \xi)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \leq C \langle \xi \rangle^N. \tag{16}$$

Proposition 3 *Let $A : C_b^\infty(\mathbb{R}^n, \mathcal{H}_1) \to C_b^\infty(\mathbb{R}^n, \mathcal{H}_2)$ be a continuous linear operator with formal symbol σ_A. Then A acts at functions $u \in S(\mathbb{R}^n, \mathcal{H}_1)$ via*

$$(Au)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma_A(x, \xi) \hat{u}(\xi) d\xi. \tag{17}$$

Proof. Let $u \in S(\mathbb{R}^n, \mathcal{H}_1)$. Then

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) e_\xi(x) d\xi.$$

Let $\{\phi_j\}$ be an orthonormal basis of $\mathcal{H}_1$ and write $\hat{u}(\xi) = \sum_{j=1}^\infty \hat{u}_j(\xi)\phi_j$ with

Fourier coefficients $\hat{u}_j(\xi) = \langle \hat{u}(\xi),\, \phi_j \rangle_{\mathcal{H}_1}$. Hence,

$$
\begin{aligned}
(Au)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \hat{u}_j(\xi)(A(e_\xi \otimes \phi_j))(x) d\xi \qquad (18)\\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \hat{u}_j(\xi) e^{ix\cdot\xi} \sigma_A(x,\, \xi)\phi_j d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma_A(x,\, \xi)\hat{u}(\xi) d\xi.
\end{aligned}
$$

The last integral exists according to estimate (16). ■

Proposition 4 *Let $A = Op(a) \in OPS(p_1,\, p_2)$. Then A has a formal symbol σ_A which coincides with a.*

Proof. Let $\xi \in \mathbb{R}^n$ and $\varphi \in \mathcal{H}_1$. Then, by (13),

$$
\begin{aligned}
(A(e_\xi \otimes \varphi))(x) &= (2\pi)^{-n} \mathrm{osc} \iint_{\mathbb{R}^{2n}} a(x,\, \eta)\varphi e^{i(x+y)\cdot\xi} e^{-iy\cdot\eta} d\eta dy \\
&= e^{ix\cdot\xi} (2\pi)^{-n} \mathrm{osc} \iint_{\mathbb{R}^{2n}} a(x,\, \xi+\eta)\varphi e^{-iy\cdot\eta} d\eta dy. \qquad (19)
\end{aligned}
$$

Using equality (5) we obtain from (19)

$$
\sigma_A(x,\, \xi)\varphi = e^{-ix\cdot\xi} A(e_\xi \otimes \varphi)(x) = a(x,\, \xi)\varphi
$$

which gives the assertion. ■

The next propositions describe the main properties of pseudodifferential operators with operator-valued symbols.

Proposition 5 *Every operator in $OPS(p_1,\, p_2)$ is bounded from $S(\mathbb{R}^n,\, \mathcal{H}_1)$ to $S(\mathbb{R}^n,\, \mathcal{H}_2)$.*

The proof makes use of estimates (11) and runs completely similar to the proof for scalar pseudodifferential operators (see, for instance, [28]). ■

Hence, the composition of pseudodifferential operators is well defined. Next we will see that the product of pseudodifferential operators is a pseudodifferential operator again.

Proposition 6 (*i*) *Let $A^1 = Op(a_1) \in OPS(p_1,\, p_2)$ and $A^2 = Op(a_2) \in OPS(p_2,\, p_3)$. Then $A^2A^1 \in OPS(p_1,\, p_3)$, and the symbol of A^2A^1 is given by*

$$
\sigma_{A^2A^1}(x,\, \xi) = (2\pi)^{-n} \mathrm{osc} \iint_{\mathbb{R}^{2n}} a_2(x,\, \xi+\eta) a_1(x+y,\, \xi) e^{i(x-y)\cdot\xi} dy d\xi. \qquad (20)
$$

(*ii*) *Let $A = Op_d(a) \in OPS_d(p_1,\, p_2)$. Then $A \in OPS(p_1,\, p_2)$, and the symbol of A is given by*

$$
\sigma_A(x,\, \xi) = (2\pi)^{-n} \mathrm{osc} \iint_{\mathbb{R}^{2n}} a(x,\, x+y,\, \xi+\eta) e^{i(x-y)\cdot\xi} dy d\xi. \qquad (21)
$$

Proof. The following proof mimics the proof for the scalar case (see [28]).
(i) Let $\varphi \in \mathcal{H}_1$. Then, applying formula (5) we obtain

$$\begin{aligned}
\sigma_{A^2A^1}(x,\,\xi)\phi &= e^{-ix\cdot\xi}A_2[A_1(e_\xi\phi)](x)\\
&= e^{-ix\cdot\xi}A_2(a_1(\cdot,\,\xi)e_\xi\phi)(x)\\
&= (2\pi)^{-n}\mathrm{osc}\iint_{\mathbb{R}^{2n}} a_2(x,\,\eta)a_1(y,\,\xi)e^{-i(x-y)\cdot(\xi-\eta)}\phi\,dyd\eta\\
&= (2\pi)^{-n}\mathrm{osc}\iint_{\mathbb{R}^{2n}} a_2(x,\,\xi+\eta)a_1(x+y,\,\xi)e^{-iy\cdot\eta}\phi\,dyd\eta.
\end{aligned}$$

Hence, formula (20) holds. Further we have to show that

$$\sup_{(x,\,\xi)\in\mathbb{R}^n\times\mathbb{R}^n}\|p_3^{-1}(\xi)\partial_\xi^\beta\partial_x^\alpha\sigma_{A^2A^1}(x,\,\xi)p_1(\xi)\|_{\mathcal{L}(\mathcal{H}_1',\,\mathcal{H}_3')}<\infty \tag{22}$$

for all multi-indices α, β. To prove these estimates, we use the representation for $\sigma_{A^2A^1}(x,\,\xi)$ as oscillatory operator-valued integral (20). Then the Leibnitz formula and property (6) of weight function indeed imply the estimate (22). Assertion (ii) can be proved in the same vein. ∎

An operator A^* is called the *formal adjoint* to the operator $A \in OPS(p_1,\,p_2)$ if, for arbitrary functions $u \in S(\mathbb{R}^n,\,\mathcal{H}_1)$ and $v \in S(\mathbb{R}^n,\,\mathcal{H}_2)$,

$$\langle Au,\,v\rangle_{L^2(\mathbb{R}^n,\,\mathcal{H}_2)} = \langle u,\,A^*v\rangle_{L^2(\mathbb{R}^n,\,\mathcal{H}_1)}. \tag{23}$$

Proposition 7 *Let $A = Op(a) \in OPS(p_1,\,p_2)$. Then $A^* \in OPS(p_2^*,\,p_1^*)$, and the symbol of A^* is given by*

$$\sigma_{A^*}(x,\,\xi) = (2\pi)^{-n}\mathrm{osc}\iint_{\mathbb{R}^{2n}} a^*(x+y,\,\xi+\eta)e^{i(x-y)\cdot\xi}dyd\xi \tag{24}$$

where

$$\langle a(x,\,\xi)u,\,v\rangle_{\mathcal{H}_2} = \langle u,\,a^*(x,\,\xi)v\rangle_{\mathcal{H}_1}$$

for all $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$.

By Proposition 7 and formula (23), one can think of operators in $OPS(p_1,p_2)$ as acting from $S'(\mathbb{R}^n,\,\mathcal{H}_1)$ to $S'(\mathbb{R}^n,\,\mathcal{H}_1)$.

Proposition 8 (Calderon-Villancourt) *If $A = Op(a) \in OPS(I_{\mathcal{H}_1},\,I_{\mathcal{H}_2})$, then A is bounded as operator from $L^2(\mathbb{R}^n,\,\mathcal{H}_1)$ to $L^2(\mathbb{R}^n,\,\mathcal{H}_2)$, and there exists constants $C > 0$ and $2k_1,\,2k_2 > n$ such that*

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n,\,\mathcal{H}_1),\,L^2(\mathbb{R}^n,\,\mathcal{H}_2))} \le C\sum_{|\alpha|\le 2k_1,\,|\beta|\le 2k_2}\sup_{(x,\xi)\in\mathbb{R}^{2n}}\|a_{(\alpha)}^{(\beta)}(x,\,\xi)\|_{\mathcal{L}(\mathcal{H}_1,\,\mathcal{H}_2)}.$$

Proposition 9 (Beals) *Let $A = Op(a) \in OPS(I_{\mathcal{H}_1},\,I_{\mathcal{H}_2})$ be invertible as operator from $L^2(\mathbb{R}^n,\,\mathcal{H}_1)$ to $L^2(\mathbb{R}^n,\,\mathcal{H}_2)$. Then $A^{-1} \in OPS(I_{\mathcal{H}_2},\,I_{\mathcal{H}_1})$.*

We want to extend the previous results to weights $p \in O(\mathcal{H}, \mathcal{H}')$ which satisfy the following additional condition: For every multi-index α there is a constant $C_\alpha > 0$ such that

$$\max\left\{\|\partial_\xi^\alpha p(\xi) p^{-1}(\xi)\|_{\mathcal{L}(\mathcal{H}',\,\mathcal{H}')},\, \|p^{-1}(\xi)\partial_\xi^\alpha p(\xi)\|_{\mathcal{L}(\mathcal{H},\,\mathcal{H})}\right\} \leq C_\alpha. \tag{25}$$

If p satisfies this condition, then $p \in S(p,\, I_{\mathcal{H}'})$ and $p \in S(I_{\mathcal{H}},\, p^{-1})$.

For p as in (25), we denote by $H(\mathbb{R}^n,\, p)$ the Banach space which is the closure of $S(\mathbb{R}^n,\, \mathcal{H})$ with respect to the norm

$$\|u\|_{H(\mathbb{R}^n,\, p)} := \|Op(p)u\|_{L^2(\mathbb{R}^n,\, \mathcal{H}')}.$$

It turns out that then $Op(p) : H(\mathbb{R}^n,\, p) \to L^2(\mathbb{R}^n,\, \mathcal{H}')$ is an isomorphism and $Op(p)^{-1} = Op(p^{-1})$. Using these facts one easily gets the following versions of Proposition 8 and 9, respectively.

Proposition 10 *Let p_1, p_2 satisfy* (25) *and $A = Op(a) \in OPS(p_1, p_2)$. Then A is bounded as operator from $H(\mathbb{R}^n,\, p_1)$ to $H(\mathbb{R}^n,\, p_2)$, and*

$$\|A\|_{\mathcal{L}(H(\mathbb{R}^n,\, p_1),\, H(\mathbb{R}^n,\, p_2))} \leq C|a|_l$$

where $C > 0$ and $l \in \mathbb{N}$ are independent on A.

Proposition 11 *Let p_1, p_2 satisfy* (25)*, and let $A = Op(a) \in OPS(p_1, p_2)$ be invertible as operator from $H(\mathbb{R}^n,\, p_1)$ to $H(\mathbb{R}^n,\, p_2)$. Then $A^{-1} \in OPS(p_2,\, p_1)$.*

Let $a \in C_b^\infty(\mathbb{R}^n)$ and $\mathcal{H}$ a separable Hilbert space. In what follows we write $aI_{\mathcal{H}}$ for the operator of multiplication by a acting on $S'(\mathbb{R}^N,\, \mathcal{H})$. Note that this operator is bounded on $H(\mathbb{R}^n,\, p)$ for every weight function $p \in O(\mathcal{H},\, \mathcal{H}')$ which satisfies condition (25).

We note one more import property of operators in $OPS(p_1,\, p_2)$ which follows easily from Propositions 6 (i) and 10.

Proposition 12 *Let p_1, p_2 satisfy* (25) *and $A = Op(a) \in OPS(p_1,\, p_2)$. Further let $\varphi \in C_b^\infty(\mathbb{R}^n)$ and set $\varphi_R(x) := \varphi(x/R)$. Then, with $[A,\, \varphi_R] := A\varphi_R I_{\mathcal{H}_1} - \varphi_R I_{\mathcal{H}_2} A$,*

$$\lim_{R\to\infty} \|[A,\, \varphi_R]\|_{\mathcal{L}(H(\mathbb{R}^n,\, p_1),\, H(\mathbb{R}^n,\, p_2))} = 0. \tag{26}$$

2.4 Pseudodifferential operators with slowly oscillating symbols

We say that a symbol $a \in S(p_1, p_2)$ is *slowly oscillating* if, for all multi-indices α, β,

$$\|p_2^{-1}(\xi)\partial_x^\beta \partial_\xi^\alpha a(x,\, \xi) p_1(\xi)\|_{\mathcal{L}(\mathcal{H}_1',\, \mathcal{H}_2')} \leq C_{\alpha\beta}(x), \tag{27}$$

and if

$$\lim_{x\to\infty} C_{\alpha\beta}(x) = 0 \tag{28}$$

for all multi-indices α, β with $\beta \neq 0$. We denote this class of symbols by $S_{sl}(p_1, p_2)$ and write $OPS_{sl}(p_1, p_2)$ for the corresponding class of pseudodifferential operators. Furthermore, let $S^0(p_1, p_2)$ refer to the subset of $S_{sl}(p_1, p_2)$ of all symbols such that (28) holds for all multi-indices α, β.

Similarly, a double symbol $a \in S_d(p_1, p_2)$ is called *slowly oscillating* if, for all multi-indices α, β, γ,

$$\|p_2^{-1}(\xi)\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, y, \xi) p_1(\xi)\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \leq C_{\alpha\beta\gamma}(x, y)$$

where

$$\lim_{x\to\infty} \sup_{y\in\mathbb{R}^n} C_{\alpha\beta\gamma}(x, y) = 0$$

for all α, β, γ with $\beta \neq 0$ and

$$\lim_{y\to\infty} \sup_{x\in\mathbb{R}^n} C_{\alpha\beta\gamma}(x, y) = 0$$

for all α, β, γ with $\gamma \neq 0$. We denote the set of all slowly oscillating double symbols by $S_{d,sl}(p_1, p_2)$ and write $OPS_{d,sl}(p_1, p_2)$ for the corresponding class of double pseudodifferential operators.

The next proposition describes some properties of pseudodifferential operators with operator-valued slowly oscillating symbols which will be needed in what follows.

Proposition 13 *(i) Let* $A^1 = Op(a_1) \in OPS_{sl}(p_1, p_2)$ *and* $A^2 = Op(a_2) \in OPS_{sl}(p_2, p_3)$*. Then* $A^2A^1 \in OPS_{sl}(p_1, p_3)$*, and*

$$\sigma_{A^2A^1}(x, \xi) = a_2(x, \xi)a_1(x, \xi) + r(x, \xi)$$

where $r \in S^0(p_1, p_3)$*.*
(ii) Let $A = Op_d(a) \in OPS_{d,sl}(p_1, p_2)$*. Then* $A \in OPS_{sl}(p_1, p_2)$*, and*

$$\sigma_A(x, \xi) = a(x, x, \xi) + r(x, \xi)$$

where $r \in S^0(p_1, p_2)$*.*
(iii) Let $A = Op(a) \in OPS(p_1, p_2)$*. Then* $A^* \in OPS(p_2^*, p_1^*)$*, and*

$$\sigma_{A^*}(x, \xi) = a^*(x, x, \xi) + r(x, \xi)$$

where $r \in S^0(p_2^*, p_1^*)$*.*

2.5 Invertibility at infinity and Fredholm property of pseudodifferential operators

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Set $\phi := 1 - \chi$ and, for $R > 0$, $\chi_R(x) := \chi(x/R)$ and $\phi_R(x) := \phi(x/R)$. Further let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ and $B_R' := \{x \in \mathbb{R}^n : |x| > R\}$.

We say that an operator $A : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ is *locally invertible at infinity* if there is an $R_0 > 0$ such that, for every $R > R_0$, there are operators $\mathcal{L}_R$ and $\mathcal{R}_R$ such that

$$\mathcal{L}_R A \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1} \quad \text{and} \quad \phi_R A \mathcal{R}_R = \phi_R I_{\mathcal{H}_2}. \tag{29}$$

Operators $\mathcal{L}_R$ and $\mathcal{R}_R$ with these properties are called *locally left and right inverses of* A, respectively.

Theorem 14 *Let* $A = Op_d(a) \in OPS_{d,sl}(p_1, p_2)$. *Assume there is a constant* $R_0 > 0$ *such that the operator* $a(x, x, \xi) : \mathcal{H}_1 \to \mathcal{H}_2$ *is invertible for every* $(x, \xi) \in B'_{R_0} \times \mathbb{R}^n$ *and that*

$$\sup_{(x,\xi)\in B'_{R_0}\times\mathbb{R}^n} \left\| p_1^{-1}(x, \xi) a(x, x, \xi)^{-1} p_2(x, \xi) \right\|_{\mathcal{L}(\mathcal{H}'_2, \mathcal{H}'_1)} < \infty.$$

Then the operator $A : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ *is locally invertible at infinity.*

Proof. Given ϕ as above, choose $\varphi \in C_b^\infty(\mathbb{R}^n)$ such that $\varphi\phi = \phi$, and set $\varphi_R(x) := \varphi(x/R)$ for $R > R_0$. Condition (29) implies that the function $b_R(x, \xi) := \varphi_R(x) a(x, x, \xi)^{-1}$ belongs to $S(p_2, p_1)$. Hence, and by Proposition 13 (i),

$$Op(b_R) Op(a) \phi_R I_{\mathcal{H}_1} = (I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1}) \phi_R I_{\mathcal{H}_1}$$

where $q_R \in S^0(p_1, p_2)$. Moreover, one can prove that, for all multi-indices α, β,

$$\lim_{x\to\infty} \sup_{\xi\in\mathbb{R}^n} \| p_1^{-1}(\xi) \partial_x^\beta \partial_\xi^\alpha q_R(x, \xi) p_1(\xi) \|_{\mathcal{L}(\mathcal{H}'_1)} = 0$$

uniformly with respect to $R > R_0$. It follows from Proposition 10 that there exists an $R' > R_0$ such that

$$\| Op(q_R) \psi_R I_{\mathcal{H}_1} \|_{\mathcal{L}(H(\mathbb{R}^n, p_1))} < 1$$

for every $R > R'$. Hence,

$$(I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1})^{-1} Op(b_R) Op(a) \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1}, \tag{30}$$

and $Op(a)$ is locally invertible from the left at infinity, with a local left inverse operator given by

$$\mathcal{L}_R := (I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1})^{-1} Op(b_R) \in OPS(p_2, p_1).$$

In the same way, a local right inverse operator $\mathcal{R}_R \in OPS(p_2, p_1)$ can be constructed.

It follows from the definition of the operators $\mathcal{L}_R$ and $\mathcal{R}_R$ that

$$\sup_{R>R'} \| \mathcal{L}_R \|_{\mathcal{L}(H(\mathbb{R}^n, p_2), H(\mathbb{R}^n, p_1))} < \infty, \tag{31}$$

$$\sup_{R>R'} \| \mathcal{R}_R \|_{\mathcal{L}(H(\mathbb{R}^n, p_2), H(\mathbb{R}^n, p_1))} < \infty$$

which finishes the proof. ∎

We say that a linear operator $A : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ is *locally Fredholm* if, for every $R > 0$, there exist bounded linear operators $\mathcal{B}_R, \mathcal{D}_R : H(\mathbb{R}^n, p_2) \to H(\mathbb{R}^n, p_1)$ and compact operators $T'_R : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_1)$ and $T''_R : H(\mathbb{R}^n, p_2) \to H(\mathbb{R}^n, p_2)$ such that

$$\mathcal{B}_R A \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1} + T'_R \quad \text{and} \quad \phi_R A \mathcal{D}_R = \phi_R I_{\mathcal{H}_2} + T''_R. \tag{32}$$

Theorem 15 *Let $A = Op_d(a) \in OPS_{d,sl}(p_1, p_2)$ an operator which satisfies the conditions of Theorem 14. If A is a locally Fredholm operator, then A is has the Fredholm property as operator from $H(\mathbb{R}^n, p_1)$ to $H(\mathbb{R}^n, p_2)$.*

Proof. Let R_0 be such that for every $R > R_0$ there exist local inverse operators $\mathcal{L}_R, \mathcal{R}_R \in OPS(p_2, p_1)$ of A. Set $\Lambda_R := \mathcal{B}_R \phi_R I_{\mathcal{H}_2} + \mathcal{L}_R \chi_R I_{\mathcal{H}_2}$. Then $\Lambda_R A = I_{\mathcal{H}_1} + T'_R + Q_R$ where $Q_R := \mathcal{B}_R[\phi_R, A] + \mathcal{B}_R[\chi_R, A]$ and where $T'_R : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_1)$ is compact. Proposition 6 implies that

$$\lim_{R\to 0} \|[\phi_R, A]\|_{\mathcal{L}(H(\mathbb{R}^n, p_1), H(\mathbb{R}^n, p_2))} = \lim_{R\to 0} \|[\chi_R, A]\|_{\mathcal{L}(H(\mathbb{R}^n, p_1), H(\mathbb{R}^n, p_2))} = 0. \tag{33}$$

From (33) and (31) we conclude that $\|Q_R\|_{\mathcal{L}(H(\mathbb{R}^n, p_1))} < 1$ for large enough $R > 0$. Hence, $\Lambda'_R := (I_{\mathcal{H}_1} + Q_R)^{-1}\Lambda_R$ is a left regularizator of A whenever R_0 is large enough. In the same way, a regularizator from the right-hand side can be found. ∎

3 Pseudodifferential operators with analytical symbols and local exponential estimates

3.1 Operators and weight spaces

Let $\mathcal{B} \subset \mathbb{R}^n$ be a convex bounded domain containing the origin. We say that a double symbol a belongs to $S_d(p_1, p_2, \mathcal{B})$ if

- the operator-valued function $\xi \mapsto a(x, y, \xi)$ can be extended analytically with respect to ξ into the tube domain $\mathbb{R}^n + i\mathcal{B}$ for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and
- for arbitrary multi-indices α, β, γ, there exists a constant $C_{\alpha\beta\gamma}$ such that
$$\|p_2^{-1}(\xi)\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, y, \xi + i\eta) p_1(\xi)\|_{\mathcal{L}(\mathcal{H}'_1, \mathcal{H}'_2)} \le C_{\alpha\beta\gamma}$$
for all $(x, y, \xi + i\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^n + i\mathcal{B})$.

We write $OPS_d(p_1, p_2, \mathcal{B})$ for the corresponding class of pseudodifferential operators with symbols in $S_d(p_1, p_2, \mathcal{B})$. Further we say that a positive C^∞-function $w(x) = e^{v(x)}$ is a weight in the class $\mathcal{R}(\mathcal{B})$ if

$$\frac{\partial v}{\partial x_j} \in C_b^\infty(\mathbb{R}^n), \qquad \lim_{x\to\infty} \frac{\partial^2 v(x)}{\partial x_i \partial x_j} = 0,$$

and $\nabla v(x) \in \mathcal{B}$ for every point $x \in \mathbb{R}^n$.

Proposition 16 *Let* $a \in S_{d,sl}(p_1, p_2, \mathcal{B}) := S_d(p_1, p_2, \mathcal{B}) \cap S_{d,sl}(p_1, p_2)$ *and* $w \in \mathcal{R}(\mathcal{B})$. *Then*

$$w^{-1}Op_d(a)wI = Op(\tilde{a}_w) + Q \tag{34}$$

where $Q \in OPS^0(p_1, p_2)$ *and the symbol* $\tilde{a}_w(x, \xi) = a(x, x, \xi + i\nabla v(x))$ *is in* $S_{sl}(p_1, p_2)$.

Proof. Let $a \in S_d(p_1, p_2, \mathcal{B})$ and $w = \exp v \in \mathcal{R}(\mathcal{B})$. It has been proved in [30] for the scalar case that $w^{-1}Op_d(a)wI = Op(a_w)$ where

$$a_w(x, y, \xi) = a(x, y, \xi + i\theta_w(x, y))$$

is a function in $S(p_1, p_2)$ and

$$\theta_w(x, y) = \int_0^1 (\nabla v)((1-t)x + ty)\, dt.$$

The proof for pseudodifferential operators with operator-valued symbols proceeds in the same way. Let now $a \in S_{sl}(p_1, p_2, \mathcal{B})$. Then a_w belongs to $S_{d,sl}(p_1, p_2)$. Hence, formula (34) is a consequence of Proposition 13 (ii). ■

3.2 Exponential estimates

For a C^∞-weight w, let $H(\mathbb{R}^n, p, w)$ denote the space of distributions with norm

$$\|u\|_{H(\mathbb{R}^n,\, p,\, w)} := \|wu\|_{H(\mathbb{R}^n,\, p)} < \infty. \tag{35}$$

In this section we are going to consider local exponential estimates for solutions of pseudodifferential equations with analytical symbols. Each equation $Au = f$ considered on a space with weight $w = e^v$ is equivalent to an equation $A_w\psi = \varphi$ where $\psi := wu \in H(\mathbb{R}^n, p_1)$, $\varphi := wf$, and $A_w := wAw^{-1}I$. Note that the main symbol of the operator A_w is $a(x, x, \xi + i\nabla v(x))$.

Theorem 17 *Let* $A = Op(a) \in OPS_{d,sl}(p_1, p_2, \mathcal{B})$ *and let* $w = \exp v \in \mathcal{R}(\mathcal{B})$ *be a weight with* $\lim_{x\to\infty} v(x) = \infty$. *Assume that there is an* R_0 *such that the operators* $a(x, x, \xi + it\nabla v(x))$ *are invertible for all* $(x, \xi) \in B'_{R_0} \times \mathbb{R}^n$ *and* $t \in [-1, 1]$ *and that*

$$\sup_{(x,\xi,t)\in B'_{R_0}\times\mathbb{R}^n\times[-1,1]} \|p_1^{-1}(\xi)a^{-1}(x, x, \xi + it\nabla v(x))p_2(\xi)\|_{\mathcal{L}(\mathcal{H}'_2, \mathcal{H}'_1)} < \infty. \tag{36}$$

Finally, let A *be locally Fredholm as operator from* $H(\mathbb{R}^n, p_1)$ *to* $H(\mathbb{R}^n, p_2)$. *If* $f \in H(\mathbb{R}^n, p_2, w)$, *then every solution of the equation* $Au = f$, *which a priori belongs to* $H(\mathbb{R}^n, p_1, w^{-1})$, *a posteriori belongs to* $H(\mathbb{R}^n, p_1, w)$.

Proof. Condition (36) implies that the operators A_{w^t} are locally invertible at infinity, and the local Fredholm property of A moreover implies that these operators are locally Fredholm for each $t \in [-1, 1]$. Hence, by Theorem 15, each operator $A_{w^t} : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ has the Fredholm property. Note that the symbol of A_{w^t} is given by

$$\sigma_{A_{w^t}}(x, \xi) = (2\pi)^{-n} \int \int_{\mathbb{R}^{2n}} a(x, y, \xi + it\theta_w(x, y)) e^{-iy\cdot\xi} dy d\xi. \tag{37}$$

This formula shows that the mapping $[-1, 1] \to S(p_1, p_2)$, $t \mapsto \sigma_{A_{w^t}}$ is continuous. Thus, and by Proposition 10, the mapping

$$[-1, 1] \to \mathcal{L}(H(\mathbb{R}^n, p_1), H(\mathbb{R}^n, p_2)), \quad t \mapsto A_{w^t}$$

is continuous. This shows that the Fredholm index of the operator $A_{w^t} : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ does not depend on $t \in [-1, 1]$. Hence, the operator A, considered as operator from $H(\mathbb{R}^n, p_1, w)$ to $H(\mathbb{R}^n, p_2, w)$, and the same operator A, but now considered as operator from $H(\mathbb{R}^n, p_1, w^{-1})$ to $H(\mathbb{R}^n, p_2, w^{-1})$, are Fredholm with the same Fredholm indices.

Further, since $H(\mathbb{R}^n, p_j, w)$ is a dense subset of $H(\mathbb{R}^n, p_j, w^{-1})$ for $j = 1, 2$, we conclude that the kernel of A, considered as operator from $H(\mathbb{R}^n, p_1, w)$ to $H(\mathbb{R}^n, p_2, w))$, coincides with the kernel of A, now considered as operator from $H(\mathbb{R}^n, p_1, w^{-1})$ to $H(\mathbb{R}^n, p_2, w^{-1}))$. Finally, if $u \in H(\mathbb{R}^n, p_1, w^{-1})$ is a solution of the equation $Au = f$ with $f \in H(\mathbb{R}^n, p_2, w)$, then $u \in H(\mathbb{R}^n, p_1, w)$ (see, for instance, [14], p. 308). ■

4 Schrödinger operators with operator-valued potentials

4.1 Fredholm property

Let T be a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with dense domain D_T. We suppose that

$$\langle T\varphi, \varphi\rangle \geq \varepsilon \|\varphi\|^2 \quad \text{for } \varphi \in \mathcal{H}$$

with a certain $\varepsilon > 0$. Let $T = \int_\varepsilon^\infty \lambda dE_\lambda$ refer to the spectral decomposition of T. For $s \in \mathbb{R}$, define the fractional powers T^s of T by $T^s = \int_\varepsilon^\infty \lambda^s dE_\lambda$ and consider T^s as unbounded operator with domain

$$D_{T^s} = \{f \in \mathcal{H} : \int_\varepsilon^\infty \lambda^{2s} d\|E_\lambda f\|^2 < \infty\}.$$

We provide D_{T^s} with the scalar product

$$\langle u, v\rangle_{D_{T^s}} := \int_\varepsilon^\infty \lambda^{2s}\, d\langle E_\lambda u, v\rangle_{\mathcal{H}}$$

which makes D_{T^s} to a Hilbert space.

Suppose that, for each $x \in \mathbb{R}^n$, we are given a bounded linear operator $L(x) : D_{T^{1/2}} \to D_{T^{-1/2}}$ which is symmetric on $D_{T^{1/2}}$, i.e.,

$$\langle L(x)\varphi,\, \psi\rangle_{\mathcal{H}} = \langle \varphi, L(x)\psi\rangle_{\mathcal{H}} \quad \text{for all } \varphi,\, \psi \in D_{T^{1/2}}.$$

We assume that the function $x \mapsto L(x)$ is strongly differentiable and that

$$\sup_{x\in\mathbb{R}^n} \|T^{-1/2}\partial_x^\beta L(x)T^{-1/2}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{38}$$

for every multi-index β. Moreover, we suppose that

$$\lim_{x\to\infty} \|T^{-1/2}\partial_{x_j} L(x)T^{-1/2}\|_{\mathcal{L}(\mathcal{H})} = 0 \quad \text{for } j = 1,\, \ldots,\, n. \tag{39}$$

We consider the Schrödinger operator

$$(\mathbb{H}u)(x) := -\partial_{x_j}\rho^{jk}(x)\partial_{x_k}u(x) + L(x)u(x), \quad x \in \mathbb{R}^n, \tag{40}$$

on the Hilbert space $L^2(\mathbb{R}^n,\, \mathcal{H})$ of vector-functions with values in $\mathcal{H}$. In (40) and in what follows, we make use of the Einstein summation convention. We will assume that $\rho^{jk} \in C_b^\infty(\mathbb{R}^n,\, \mathcal{L}(\mathcal{H}))$ and

$$\lim_{x\to\infty} \partial_{x_l}\rho^{jk}(x) = 0 \quad \text{for } i = 1,\, \ldots,\, n, \tag{41}$$

and that $\rho^{kj} = (\rho^{jk})^*$, and there is a $C > 0$ such that, for every $\varphi \in \mathcal{H}$,

$$\langle \rho^{jk}(x)\xi_j\xi_k\varphi,\, \varphi\rangle_{\mathcal{H}} \geq C|\xi|^2\|\varphi\|^2_{\mathcal{H}}. \tag{42}$$

Let $p(\xi) := (|\xi|^2 + T)^{1/2}$, and write $H(\mathbb{R}^n,\, p)$ for the Hilbert space with norm

$$\|u\|_{H(\mathbb{R}^n,\, p)} := \|(-\Delta + T)^{1/2}u\|_{L^2(\mathbb{R}^n,\, \mathcal{H})}.$$

The estimates (38), (39) and (41) imply that $\mathbb{H}$ is a pseudodifferential operator in the class $OPS_{sl}(p,\, p)$ with symbol

$$\sigma_{\mathbb{H}}(x,\, \xi) = \rho^{jk}(x)\xi_j\xi_k + i\frac{\partial\rho^{jk}(x)}{\partial x_j}\xi_k + L(x).$$

The following theorem states conditions for the Fredholmness of the operator $\mathbb{H} : H(\mathbb{R}^n,\, p) \to H(\mathbb{R}^n,\, p^{-1})$.

Theorem 18 *Let conditions* (38) $-$ (42) *hold, and assume there are constants* $R > 0$ *and* $C > 0$ *such that*

$$\Re\langle L(x)\varphi,\, \varphi\rangle_{\mathcal{H}} \geq C\langle T\varphi,\, \varphi\rangle_{\mathcal{H}} \tag{43}$$

for every $x \in B_R'$ *and every vector* $\varphi \in D_{T^{1/2}}$. *If the operator* $\mathbb{H} : H(\mathbb{R}^n,\, p) \to H(\mathbb{R}^n,\, p^{-1})$ *is locally Fredholm, then it is already a Fredholm operator.*

Proof. Conditions (42) and (43) imply that there exist $C > 0$ and $R > 0$ such that, for every $x \in B'_R$ and every $\varphi \in D_{T^{1/2}}$,

$$\Re\langle\sigma_{\mathbb{H}}(x,\,\xi)\varphi,\,\varphi\rangle_{\mathcal{H}} \geq C\langle(|\xi|^2 I + T)\varphi,\,\varphi\rangle_{\mathcal{H}}. \tag{44}$$

It follows from estimate (44) that, for every $x \in B'_R$ and every $\psi \in \mathcal{H}$,

$$\Re\langle(|\xi|^2 I + T)^{-1/2}\sigma_{\mathbb{H}}(x,\,\xi)(|\xi|^2 I + T)^{-1/2}\psi,\,\psi\rangle_{\mathcal{H}} \geq C\|\psi\|^2_{\mathcal{H}}. \tag{45}$$

This estimate yields that the operator $(|\xi|^2 I + T)^{-1/2}\sigma_{\mathbb{H}}(x,\,\xi)(|\xi|^2 I + T)^{-1/2}$ is invertible on $\mathcal{H}$ for every $x \in B'_R$ and every $\xi \in \mathbb{R}^n$ and that

$$\sup_{(x,\xi)\in B'_R\times\mathbb{R}^n} \|(|\xi|^2 I + T)^{1/2}\sigma^{-1}_{\mathbb{H}}(x,\,\xi)(|\xi|^2 I + T)^{1/2}\|_{\mathcal{L}(\mathcal{H})} \leq C^{-1}. \tag{46}$$

Hence, the conditions of Theorem 15 are satisfied, and $\mathbb{H}$ has the Fredholm property as operator from $H(\mathbb{R}^n,\,p)$ to $H(\mathbb{R}^n,\,p^{-1})$. ■

4.2 Exponential estimates

The following theorem gives exponential estimates at infinity for solution of the Schrödinger equation $\mathbb{H}u = f$. These estimates van be viewed as an operator-valued analog of Agmon's estimates ([1]). We will use the notation

$$|\nabla v(x)|^2_{\rho(x)} := \rho^{jk}(x)\frac{\partial v(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_k}.$$

Theorem 19 *Let the ρ^{jk} be scalar real-valued functions such that*

$$\rho^{jk}(x)\xi_j\xi_k \geq C|\xi|2 \quad \textit{for all}\, \xi \in \mathbb{R}^n$$

and conditions (38)−(41) hold. Let $w = \exp v$ be a weight in $\mathcal{R}(\mathcal{B})$ (with $\mathcal{B} \subset \mathbb{R}^n$ a convex bounded domain containing the origin) such that $\lim_{x\to\infty} v(x) = +\infty$ and there exist $R > 0$ and $C > 0$ with

$$\Re\langle(-|\nabla v(x)|^2_{\rho(x)} I_{\mathcal{H}} - L(x))\varphi,\,\varphi\rangle_{\mathcal{H}} \geq C\langle T\varphi,\,\varphi\rangle_{\mathcal{H}} \tag{47}$$

for every $x \in B'_R$ and every $\varphi \in D_{T^{1/2}}$. If $\mathbb{H} : H(\mathbb{R}^n,\,p) \to H(\mathbb{R}^n,\,p^{-1})$ is a locally Fredholm operator, then every solution of the equation $\mathbb{H}u = f$ with right-hand side $f \in H(\mathbb{R}^n,\,p^{-1}, w)$, which a priori belongs to the space $H(\mathbb{R}^n,\,p,\,w^{-1})$, a posteriori belongs to the space $H(\mathbb{R}^n,\,p,\,w)$.

Proof. We have

$$\Re\langle\sigma_{\mathbb{H}}(x,\,\xi + it\nabla v(x))\varphi,\,\varphi\rangle = \Re\left\langle(\rho^{jk}(x)\xi_j\xi_k - t^2|\nabla v(x)|^2_{\rho(x)} + L(x))\varphi,\,\varphi\right\rangle.$$

Condition (47) implies that, for every $x \in B'_R$, $\varphi \in D_{T^{1/2}}$, and $t \in [-1,\,1]$,

$$\Re\,\langle\sigma_{\mathbb{H}}(x,\,\xi + it\nabla v(x))\varphi,\,\varphi\rangle_{\mathcal{H}} \geq C\langle(|\xi|^2 I + T)\varphi,\,\varphi\rangle_{\mathcal{H}}. \tag{48}$$

As in the proof of Theorem 18, we conclude from (48) that

$$\sup_{(x,\xi,t)\in B'_R\times\mathbb{R}^n\times[-1,1]} \left\|(|\xi|^2 I + T)^{1/2}\sigma^{-1}_{\mathbb{H}}(x,\,\xi + it\nabla v(x))(|\xi|^2 I + T)^{1/2}\right\|_{\mathcal{L}(\mathcal{H})}$$

is finite. Thus, all conditions of Theorem 17 are satisfied. ■

4.3 Quantum waveguides

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with a sufficiently regular boundary, and let $\Phi \in C^{(1)}(\bar{\mathcal{D}}) \otimes C_b^\infty(\mathbb{R})$ be real-valued potential slowly oscillating with respect to the second variable, i.e., $\lim_{y\to\infty} \frac{\partial \Phi(x,y)}{\partial y} = 0$ uniformly with respect to $x \in \bar{\mathcal{D}}$.

We consider the spectral problem for the Schrödinger equation in the quantum waveguide, i.e.,

$$((\mathbb{H} - \lambda I)u)(x, y) = \left(-\frac{\partial^2}{\partial y^2} - \Delta_x + \Phi(x, y) - \lambda I \right) u(x, y) = 0 \tag{49}$$

for $(x, y) \in \mathcal{D} \times \mathbb{R} =: \Pi$, and

$$u|_{\partial \mathcal{D}} = 0. \tag{50}$$

This equation describes the bound states of the quantum system on the configurarion space Π.

The operator $\mathbb{H} - \lambda I$ can be realized as a pseudodifferential operator with operator-valued symbol $\sigma_{\mathbb{H}-\lambda I}(y, \xi) = \xi^2 + L_\lambda(y)$ where

$$\begin{aligned} (L_\lambda(y)\varphi)(x) &= (-\Delta_x + \Phi(x, y) - \lambda I)\varphi(x) \quad \text{for } x \in \mathcal{D}, \\ \varphi|_{\partial\mathcal{D}} &= 0 \end{aligned}$$

is the operator of the Dirichlet problem in $\mathcal{D}$ depending on the parameter $y \in \mathbb{R}$.

Let T be the operator of the Dirichlet problem for the Laplacian $-\Delta_x$ in the domain $\mathcal{D}$, considered as an unbounded operator on $\mathcal{H} = L^2(\mathcal{D})$ with domain $D_T = \{\varphi \in H^2(\mathcal{D}) : \varphi|_{\partial\mathcal{D}} = 0\}$ where $H^2(\mathcal{D})$ is the standard Sobolev space on $\mathcal{D}$. It is well-known that T is a positive definit operator. Then we can write $L_\lambda(y) = T + \tilde{\Phi}(y) - \lambda I$ with $(\tilde{\Phi}(y)u)(x) := \Phi(x, y)u(x)$ for $x \in \mathcal{D}$. One can show that condition (38) is satisfied.

As above we set $p(\xi) = \xi^2 I + T$ for $\xi \in \mathbb{R}$, and we denote by $H(\mathbb{R}, p)$ the closure of $C_0^\infty(\Pi)$ in the norm

$$\|u\|_{H(\mathbb{R},\, p)} := \left\| \left(-\frac{d^2}{dy^2} + T \right)^{1/2} u \right\|_{L^2(\Pi)}.$$

Then, with the standard Sobolov space notation,

$$H(\mathbb{R}, p) = \overset{\circ}{H}{}^1(\Pi) = \{u \in H^1(\Pi) : u|_{\partial\Pi} = 0\}$$

and $H(\mathbb{R}, p^{-1}) = (\overset{\circ}{H}{}^1(\Pi))^* = H^{-1}(\Pi)$. First we consider the problem of Fredholmness of the operator

$$\mathbb{H} - \lambda I_{\mathcal{H}} : \overset{\circ}{H}{}^1(\Pi) \to H^{-1}(\Pi).$$

Set $\Phi^{\inf} := \lim_{R\to\infty} \inf_{(x,y)\in\mathcal{D}\times B_R'} \Phi(x, y)$.

Theorem 20 *If $\lambda < \Phi^{\inf}$, then $\mathbb{H} - \lambda I_{\mathcal{H}} : \overset{\circ}{H}{}^1(\Pi) \to H^{-1}(\Pi)$ is a Fredholm operator.*

Proof. It follows from standard locally elliptic estimates for the Dirichlet problem in bounded domains that the operator $\mathbb{H}-\lambda I_{\mathcal{H}} : \mathring{H}^1(\Pi) \to H^{-1}(\Pi)$ is locally Fredholm. The assumption $\lambda < \Phi^{\inf}$ implies that condition (43) of Theorem 18 is satisfied. Hence, $\mathbb{H} - \lambda I_{\mathcal{H}}$ is a Fredholm operator. ∎

Corollary 21 *The part of the spectrum of* $\mathbb{H}$ *which lies in* $(-\infty, \Phi^{\inf})$ *is discrete.*

Proof. It is clear that if $\lambda < \inf_{(x,\, y)\in\Pi} \Phi(x, y) < \Phi^{\inf}$, then

$$\langle(\mathbb{H} - \lambda I_{\mathcal{H}})\varphi,\, \varphi\rangle_{\mathcal{H}} \geq C\|\varphi\|^2_{\mathcal{H}}$$

for every $\varphi \in H^2(\Pi) \cap \mathring{H}^1(\Pi)$. Since every generalized solution of the Dirichlet problem belongs to $H^2(\Pi) \cap \mathring{H}^1(\Pi)$, we obtain that the kernel of $\mathbb{H} - \lambda I_{\mathcal{H}}$ in $\mathring{H}^1(\Pi)$ is trivial. Because $(\mathbb{H} - \lambda I_{\mathcal{H}})^* = \mathbb{H} - \lambda I_{\mathcal{H}}$, the cokernel of $\mathbb{H} - \lambda I_{\mathcal{H}}$ is trivial, too. Hence, if $\lambda < \inf_{(x,y)\in\Pi} \Phi(x, y)$, then the operator $\mathbb{H} - \lambda I_{\mathcal{H}}$ is invertible. Thus, for all $\lambda < \Phi^{\inf}$, the operator $\mathbb{H} - \lambda I_{\mathcal{H}}$ is Fredholm with index zero. But then $\mathbb{H} - \lambda I_{\mathcal{H}}$ is invertible for all $\lambda < \Phi^{\inf}$, with possible exception of a discrete set of eigenvalues of finite multiplicity. ∎

Theorem 22 *Let* $\lambda < \Phi^{\inf}$ *and* $w = \exp v \in \mathcal{R}(\mathcal{B})$ *for a convex bounded domain* $\mathcal{B} \subset \mathbb{R}^n$ *containing the origin. Suppose that* $\lim_{y\to\infty} v(y) = +\infty$ *and that*

$$\lim_{R\to\infty} \inf_{(x,y)\in\mathcal{D}\times B'_R} \left(\Phi(x,y) - \lambda I - \left(\frac{dv(y)}{dy}\right)^2\right) > 0. \tag{51}$$

Then every solution $u_\lambda \in \mathring{H}^1(\Pi) = H(\mathbb{R}, p)$ *of the equation* $(\mathbb{H} - \lambda I_{\mathcal{H}})u_\lambda = 0$ *belongs to the space* $H(\mathbb{R}, p, w) = \mathring{H}^1(\Pi, w)$.

Proof. It is easy to check that condition (51) provides the fulfillment of condition (47) of Theorem 19. ∎

References

[1] S. Agmon, *Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations.* Princeton University Press, Princeton, 1982.

[2] R. Beals, *Characterization of pseudodifferential operators and applications.* Duke Math. J. **44** (1977), 45-57.

[3] V.V. Belov, S.Yu. Dobrokhotov, T.Ya. Tudorovskiy, *Operator separation of variables for adiabatic problems in quantum and wave mechanics.* J. Engineering Math. **55** (2006), Nr. 1-4, 183-237.

[4] M. Dimassi, J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit.* London Math. Soc. Lecture Notes **268**, Cambridge Univesity Press, Cambridge 1999.

[5] J. Dittrich, J. Kříž, *Bound states in straight quantum waveguides with combined boundary conditions.* J. Math. Phys. **43** (2002), 3892-3915.

[6] J. Dittrich, J. Kříž, *Curved planar quantum wires with Dirichlet and Neumann boundary conditions.* J. Phys. A: Math. Gen. **35** (2002), L269-L275.

[7] P. Duclos, P. Exner, *Curvature-induced bound states in quantum waveguides in two and three dimensions.* Rev. Math. Phys. **7** (1995), 73-102.

[8] P. Duclos, P. Exner, B. Meller, *Open quantum dots: Resonances from perturbed symmetry and bound states in strong magnetic fields.* Rep. Math. Phys. **47** (2001), 253-267.

[9] P. Duclos, P. Exner, B. Meller, *Exponential bounds on curvature induced resonances in a two-dimensional Dirichlet tube.* Helv. Phys. Acta **71** (1998), 477-492.

[10] P. Duclos, P. Exner, P. Stovıcek, *Curvature induced resonances in a two-dimensional Dirichlet tube*, Ann. Inst. Henri Poincare Vol. **62** (1995), 1, 81-101.

[11] T. Ekholm and H. Kovarık, *Stability of the magnetic Schrödinger operator in a waveguide.* Comm. Part. Diff. Eq. **30** (2005), 539-565.

[12] R. Froese, I. Herbst, *Exponential bound and absence of positive eigenvalue for N-body Schrödinger operators.* Comm. Math. Phys. **87** (1982), 429-447.

[13] R. Froese, I. Herbst, M. Hoffman-Ostenhof, T. Hoffman-Ostenhof, *L^2-exponential lower bound of the solutions of the Schrödinger equation.* Comm. Math. Phys. **87** (1982), 265-286.

[14] I. Gohberg, I. Feldman, *Convolutions Equations and Projection Method for Their Solution.* Nauka, Moskva 1971 (Russian, Engl. transl.: Amer. Math. Soc. Transl. Math. Monogr. **41**, Providence, R.I., 1974).

[15] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications.* Lect. Notes Math. **1336**, Springer, Berlin 1998.

[16] B. Helffer, J. Sjöstrand, *Effect tunnel pour l'équation de Schrödinger avec champ magnétique*, Annali Scuola Normale Superiore di Pisa, Classe di Scienze 4 serie, t. 14, no. 4 (1987), 625-657.

[17] B. Helffer, J. Sjöstrand, *Multiple wells in the semiclassical limit.* Math. Nachr. **124** (1985), 263-313.

[18] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo- Differential Operators.* Springer, New York, Berlin, Heidelberg 2007.

[19] S. Levendorskii, *Asymptotic Distribution of Eigenvalues of Differential Operators.* Kluwer, Dordrecht 1990.

[20] S. Levendorskii, *Degenerate Elliptic Equations*. Kluwer, Dordrecht 1994.

[21] Ya.A. Luckiy, V.S. Rabinovich, *Pseudodifferential operators on spaces of functions of exponential behavior at infinity*. Funkts. Anal. Prilozh. **4** (1977), 79-80 (Russian).

[22] A. Martinez, *Microlocal exponential estimates and application to tunneling*. In: Microlocal Analysis and Spectral Theory, L. Rodino (Editor), NATO ASI Series, Ser. C: Math. Phys. Sc. **490** (1996), 349-376.

[23] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis*. Springer, New York 2002.

[24] A. Martinez, S. Nakamura, V. Sordoni, *Phase space tunneling in multistate scattering*. Preprint UTMS 2001–1, January 15, 2001.

[25] S. Nakamura, *Tunneling Estimates for Magnetic Schrödinger Operators*. Comm. Math. Phys. **200** (1999), 25-34.

[26] S. Nakamura, *Agmon-type exponential decay estimates for pseudodifferential operators*. J. Math. Sci. Univ. Tokyo **5** (1998), 693-712.

[27] G. Nenciu, *On exponential decay of solutions of Schrödinger and Dirac equations: bounds of eigenfunctions corresponding to energies in the gaps of essential spectrum*. Journées Équations aux Dérivées Partielles 1994, Art. No. 7, 10 p., full text djvu Reviews Zbl 0948.35506.

[28] V.S. Rabinovich, *An Introductionary Course on Pseudodifferential Operators*. Instituto Superior Técnico. Textos de Matemática. Centro de Matemática Aplicada **1**, Lisbon 1998, 79 p.

[29] V.S. Rabinovich, *Pseudodifferential operators with analytic aymbols and some of its applications*. Linear Topological Spaces and Complex Analysis **2**, Metu-Tübitak, Ankara 1995, 79-98.

[30] V. Rabinovich, *Pseudodifferential operators with analytic symbols and estimates for eigenfunctions of Schrödinger operators*. Z. Anal. Anwend. (J. Anal. Appl.) **21** (2002), 2, 351-370.

[31] V.S. Rabinovich, *Local Exponential Estimates for h-Pseudodifferential Operators and Tunneling for Schrödinger, Dirac, and Square Root Klein-Gordon Operators*. Russian J. Math. Phys. **16** (2009), 2, 300-308.

[32] V. Rabinovich, M.Q. Cerdan, *Invertibility of Helmholtz operators for nonhomogeneous medias*. Math. Meth. in Applied Science 2009, (www.interscience.wiley.com) DOI: 10.1002/mma.1215.

[33] V. Rabinovich, S. Roch, *Essential spectrum and exponential decay estimates of elliptic systems of partial differential equations, Applications to Schrödinger and Dirac operators*. Georgian Math. J.**15** (2008), 2, 1-19.

[34] V.S. Rabinovich, S. Roch, *Essential Spectra of Pseudodifferential Operators and Exponential Decay of their Solutions. Applications to Schrödinger Operators*. In: Operator Algebras, Operator Theory and Applications, Ser. Oper. Theory: Adv. Appl. **181**, Birkhäuser, Basel, Boston, Berlin 2008, 335-384.

[35] M.A. Shubin, *Pseudodifferential Operators and Spectral Theory*. Springer, 2001 (second ed.).

[36] B. Simon, *Semiclassical analysis of low lying eigenvalues II. Tunneling*. Ann. Math. **120** (1984), 89-118.

[37] M.E. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, New Jersey, 1981.