# Spatial discretization of $C^*$-algebras

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**Abstract**

Several classes of $C^*$-algebras (for instance, AF-algebras and quasidiagonal algebras) are distinguished by intrinsic finiteness properties. These properties can be used in principle to approximate the elements of the algebra by finite-dimensional (or discrete) objects. N. Brown has pointed out that this *intrinsic* or *algebraic* discretization works particularly well for irrational rotation algebras, in which case the the discrete approximations can not only be constructed effectively but also own excellent convergence properties. At the other end of the scale there are $C^*$-algebras of infinity type which resist any intrinsic discretization. This fact justifies to consider another kind of approximation by finite rank operators, which we call *spatial* discretization, and which is based on the finite sections method. We shall discuss spatial discretizations for the Toeplitz algebra, Cuntz algebras, and the algebra of band-dominated operators on $l^2(\mathbb{Z})$ (which appears as a special crossed product). Special attention is paid to the properties of stability, fractality and Fredholmness, which are borrowed from numerical analysis and which play a basic role in the analysis of the discretized algebras.

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# 1 Introduction

Several classes of $C^*$-algebras are distinguished by intrinsic finiteness properties. These properties can be used in principle to approximate the elements of the algebra by finite-dimensional (or discrete) objects and, thus, to discretize the algebra. Good candidates for a discretization in that sense are approximately finite algebras (which are inductive limits of finite-dimensional algebras) and quasi-diagonal algebras. N. Brown [7] has pointed out that the discretization procedure works particularly well for irrational rotation algebras in which case the discrete approximations can not only be constructed effectively but also own excellent convergence properties (for example, the sequence of the approximations is fractal in a sense which will be explained below). An excellent overview on algebras with finiteness properties is Brown's and Ozawa's recent monograph [8].

In these lectures, we will consider a completely different kind of discretization, called *spatial discretization*, the basic idea of which can be characterized as follows: We represent a given (abstract) $C^*$-algebra $\mathcal{A}$ *faithfully* as an algebra $\mathsf{A}$ of linear bounded operators on a separable Hilbert space. Then we choose a basis $\{e_i\}_{i \in \mathbb{N}}$ of that space, let $P_n$ stand for the orthogonal projection from $H$ onto the linear span of $e_1, \ldots, e_n$, and associate with each operator $A \in \mathsf{A}$ the sequence $(P_n A P_n)$ of its finite sections. The goal is to understand the $C^*$-algebra $\mathcal{S}(\mathsf{A})$ which is generated by all sequences $(P_n A P_n)$ with $A \in \mathsf{A}$.

The idea of spatial discretization has its origins in numerical analysis, where the numerical solution of an operator equation $Au = f$ is a basic problem. Numerical analysis provides a huge arsenal of methods to discretize this equation for several classes of operators. The perhaps simplest (from the conceptual point of view) and most universal (applicable to each operator) method is the finite sections method which replaces the equation $Au = f$ by the sequence of the finite-dimensional linear systems $P_n A P_n u_n = P_n f$, $n = 1, 2, \ldots$. The basic question is if these systems are uniquely solvable for sufficiently large $n$ and if their solutions $u_n$ tend to a solution of the original equation $Au = f$. The central aspect of this question is if the operators ($= n \times n$-matrices) $P_n A P_n$ are invertible for sufficiently large $n$ and if the norms of their inverses are uniformly bounded. In this case, the sequence $(P_n A P_n)$ is called *stable*.

A Neumann series argument shows that the sequence $(P_n A P_n)$ with $A \in \mathsf{A}$ is stable if and only if its coset is invertible in the quotient of the algebra $\mathcal{S}(\mathsf{A})$ by the ideal of all sequences which tend to zero *in the norm*. This observation due to Kozak brings numerical analysis into the realm of $C^*$-algebras (and conversely). It was soon realized that, for instance, Gelfand theory and its several non-commutative generalizations provide effective tools to study stability problems for the finite sections method for convolution type equations; see [13] for an overview. In the consequence, the algebras $\mathcal{S}(\mathsf{A})$ were examined for several classes of operator algebras $\mathsf{A}$. We will present some of these results in what follows. The pioneering example, which will also play a prominent role in these lectures, is the Toeplitz algebra $\mathsf{T}(C)$ generated all Toeplitz operators on $l^2(\mathbb{N})$ with continuous generating function. This algebra can be viewed as a faithful representation of the universal $C^*$-algebra generated by one isometry (Coburn's theorem, [9]). The algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections method is extremely well understood now; for several aspects of finite sections of Toeplitz operators as well as for the rich history of the field see [4, 5]. These results were later extended to algebras generated by Toeplitz operators with piecewise continuous (and even "more discontinuous") symbols and to algebras of singular integral operators, see [12]. The algebra $\mathcal{S}(\mathsf{BDO})$ of the finite sections of band-dominated operators was subject of [18, 22] (note that the algebra $\mathsf{BDO}$ of the band-dominated operators is a faithful representation of the reduced crossed product algebra $l^\infty(\mathbb{Z}) \times_{\alpha r} \mathbb{Z}$), and the algebra $\mathcal{S}(\mathsf{O}_N)$ where $\mathsf{O}_N$ is a concrete representation of the Cuntz algebra $\mathcal{O}_N$ was considered in [23].

Besides stability, these lectures will focus on two other basic notions, namely *fractality* and *Fredholmness*. Fractality is a property of sequence algebras under which the sequences in the algebra behave particulary well. For example, if $(A_n)$ is a bounded sequence of operators then the sequence $(\|A_n\|)$ is also bounded. But if $(A_n)$ is a bounded sequence in a fractal algebra then the sequence $(\|A_n\|)$ is convergent. Fredholmness can be considered as a adaption of the notion of a Fredholm operator to sequence algebras. Both notions will play a basic role in the analysis of the discretized algebras.

The textbooks and review papers [3, 5, 12, 13, 22, 28] provide both an introduction to the field and suggestions for further reading.

# 2 Stability

## 2.1 Algebras of matrix sequences

Let $\mathcal{F}$ denote the set of all bounded sequences $\mathbf{A} = (A_n)$ of matrices $A_n \in \mathbb{C}^{n \times n}$. Equipped with the operations

$$(A_n) + (B_n) := (A_n + B_n), \quad (A_n)(B_n) := (A_n B_n), \quad (A_n)^* := (A_n^*)$$

and the norm

$$\|\mathbf{A}\|_{\mathcal{F}} := \|A_n\|,$$

the set $\mathcal{F}$ becomes a $C^*$-algebra, and the set $\mathcal{G}$ of all sequences $(A_n) \in \mathcal{F}$ with $\lim \|A_n\| = 0$ forms a closed ideal of $\mathcal{F}$. The relevance of the algebra $\mathcal{F}$ and its ideal $\mathcal{G}$ in our context stems from the fact (following via a simple Neumann series argument which is left as an exercise) that a sequence $(A_n) \in \mathcal{F}$ is stable if, and only if, the coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$. This equivalence is also known as Kozak's theorem. Thus, every stability problem is equivalent to an invertibility problem in a suitably chosen $C^*$-algebra, and to understand stability means to understand subalgebras of the quotient algebra $\mathcal{F}/\mathcal{G}$. Note in this connection that

$$\limsup \|A_n\| = \|(A_n) + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} \tag{1}$$

for each sequence $(A_n)$ in $\mathcal{F}$ (a simple exercise again).

Let $H$ be a separable Hilbert space with basis $\{e_i\}_{i\in\mathbb{N}}$ and let $P_n$ stand for the orthogonal projection from $H$ onto the linear span of $e_1, \ldots, e_n$. Set $\mathcal{P} := (P_n)$. Then we let $\mathcal{F}^{\mathcal{P}}$ stand for the set of all sequences $\mathbf{A} = (A_n)$ of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ with the property that the sequences $(A_nP_n)$ and $(A_n^*P_n)$ converge strongly. By the uniform boundedness principle, the quantity $\sup \|A_nP_n\|$ is finite for every sequence $\mathbf{A}$ in $\mathcal{F}^C$. Thus, if we identify each operator $A_n$ on $\operatorname{im} P_n$ with its matrix representation with respect to the basis $e_0, \ldots, e_{n-1}$ of $\operatorname{im} P_n$, we can consider $\mathcal{F}^{\mathcal{P}}$ as a closed and symmetric subalgebra of $\mathcal{F}$ which contains $\mathcal{G}$ as its ideal. Note that the mapping

$$W : \mathcal{F}^{\mathcal{P}} \to L(H), \quad (A_n) \mapsto \text{s-lim}\, A_nP_n \tag{2}$$

is a $^*$-homomorphism.

## 2.2 Discretization of operator algebras

Let $\mathsf{A}$ be a $C^*$-subalgebra of the algebra $L(H)$. We write $D$ for the mapping of spatial (= finite sections) discretization, i.e.,

$$D : L(H) \to \mathcal{F}^{\mathcal{P}}, \quad A \mapsto (P_nAP_n), \tag{3}$$

and let $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ stand for the smallest closed subalgebra of the algebra $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $D(A)$ with $A \in \mathsf{A}$. Since $(P_nAP_n)^* = (P_nA^*P_n)$, $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ is a $C^*$-subalgebra of $\mathcal{F}^{\mathcal{P}}$, and the mapping $W$ is a $^*$-homomorphism from $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ to $\mathsf{A}$. Thus, the algebra $\mathsf{A}$ appears as a quotient of $\mathcal{S}(\mathsf{A})$ by the ideal of all sequences tending *strongly* to zero. On this level, one cannot say much about algebra $\mathcal{S}(\mathsf{A})$. The little one can say will follow easily from the following simple facts.

**Proposition 2.1** *Let $\mathsf{A}$ and $\mathcal{B}$ be $C^*$-algebras, $D : \mathsf{A} \to \mathcal{B}$ a linear contraction, and $W : \mathcal{B} \to \mathsf{A}$ a $C^*$-homomorphism such that $W(D(A)) = A$ for every $A \in \mathsf{A}$. Then*

*(a) $D$ is an isometry, $D(\mathsf{A})$ is a closed linear subspace of $\mathcal{B}$, and $\operatorname{alg} D(\mathsf{A})$, the smallest closed $C^*$-subalgebra of $\mathcal{B}$ which contains $D(\mathsf{A})$, splits into the direct sum*

$$\operatorname{alg} D(\mathsf{A}) = D(\mathsf{A}) \oplus (\ker W \cap \operatorname{alg} D(\mathsf{A})). \tag{4}$$

*Moreover, for every $A \in \mathsf{A}$,*

$$\|D(A)\| = \min_{K\in\ker W} \|D(A) + K\|. \tag{5}$$

(*b*) *If* $\mathcal{B} = \operatorname{alg} D(\mathsf{A})$, *then* $\ker W$ *coincides with the quasicommutator ideal of* $\mathcal{B}$, *i.e., with the smallest closed ideal of* $\mathcal{B}$ *which contains all quasicommutators* $D(A_1)D(A_2) - D(A_1A_2)$ *with* $A_1, A_2 \in \mathsf{A}$.

**Proof.** (*a*) Let $A \in \mathsf{A}$. The inequality

$$\|A\| = \|W(D(A))\| \leq \|D(A)\| \leq \|A\|$$

shows that $D$ is an isometry; hence, $D(\mathsf{A})$ is a closed subspace of $\mathcal{B}$. Let $B \in D(\mathsf{A}) \cap \ker W$. Write $B = D(A)$ with $A \in \mathsf{A}$. From $W(B) = 0$ we get $A = W(D(A)) = W(B) = 0$, whence $B = 0$. Thus, $D(\mathsf{A}) \cap \ker W = \{0\}$.

Let $B \in \operatorname{alg} D(\mathsf{A})$. Then $W(B - D(W(B))) = W(B) - W(B) = 0$, hence

$$B = D(W(B)) + (B - D(W(B)) \in D(\mathsf{A}) + \ker W,$$

whence $\operatorname{alg} D(\mathsf{A}) = D(\mathsf{A}) + (\ker W \cap \operatorname{alg} D(\mathsf{A}))$. This proves (4). To check (5), let $A \in \mathsf{A}$ and $K \in \ker W$. Then

$$\|A\| = \|W(D(A) + K)\| \leq \|D(A) + K\|$$

which implies that $\|D(A)\| \leq \|D(A) + K\|$ since $D$ is an isometry.

(*b*) Since $W$ is a homomorphism and $W \circ D$ is the identity on $\mathsf{A}$, one has $D(A_1)D(A_2) - D(A_1A_2) \in \ker W$ for all $A_1, A_2 \in \mathsf{A}$. Thus, $\ker W$ contains the quasicommutator ideal. For the reverse inclusion, let $K \in \ker W$ and $n$ a positive integer. Since $K \in \operatorname{alg} D(\mathsf{A})$, there are sums of products

$$K_n = \sum \prod D(A_{ij}^{(n)}) \quad \text{with} \quad A_{ij}^{(n)} \in \mathsf{A}$$

such that $\|K - K_n\| \leq 1/n$. Clearly, $K_n$ can be written as

$$\begin{aligned} K_n &= D(\sum \prod A_{ij}^{(n)}) + Q_n \\ &= D(W(K_n)) + Q_n = D(W(K_n - K)) + Q_n \end{aligned}$$

with $Q_n$ in the quasicommutator ideal. Then

$$\begin{aligned} \|K - Q_n\| &\leq \|K - K_n\| + \|K_n - Q_n\| \\ &\leq \|K - K_n\| + \|D(W(K_n - K))\| \\ &\leq 2\|K - K_n\| \leq 2/n. \end{aligned}$$

Thus, $K$ can be approximated as closely as desired by elements in the quasicommutator ideal. Since the quasicommutator ideal is closed, the assertion follows. ■

We apply the preceding proposition in the following context:

- $\mathsf{A}$ is a $C^*$-subalgebra of $L(H)$,
- $\mathcal{B} = \mathcal{S}^{\mathcal{P}}(\mathsf{A})$,
- $D$ is the restriction of the discretization (3) to $\mathsf{A}$, and
- $W$ is the restriction of the homomorphism (2) to $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$.

Then Proposition 2.1 specializes to the following.

**Proposition 2.2** *Let $\mathsf{A}$ be a $C^*$-subalgebra of $L(H)$. Then the finite sections discretization $D : \mathsf{A} \to \mathcal{F}^{\mathcal{P}}$ is an isometry, and $D(\mathsf{A})$ is a closed subspace of $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$. The algebra $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ splits into the direct sum*

$$\mathcal{S}^{\mathcal{P}}(\mathsf{A}) = D(\mathsf{A}) \oplus (\ker W \cap \mathcal{S}^{\mathcal{P}}(\mathsf{A})), \tag{6}$$

*and one has*

$$\|D(A)\| = \min_{K \in \ker W} \|D(A) + K\|$$

*for every operator $A \in \mathsf{A}$. Finally, $\ker W \cap \mathcal{S}^{\mathcal{P}}(\mathsf{A})$ is equal to the quasicommutator ideal of $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$, i.e., to the smallest closed ideal of $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ which contains all sequences $(P_nA_1P_nA_2P_n - P_nA_1A_2P_n)$ with operators $A_1,\, A_2 \in \mathsf{A}$.*

We denote the ideal $\ker W \cap \mathcal{S}^{\mathcal{P}}(\mathsf{A})$ of $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ by $\mathcal{J}^{\mathcal{P}}(\mathsf{A})$ and agree to omit the $\mathcal{P}$ in $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ and $\mathcal{J}^{\mathcal{P}}(\mathsf{A})$ if the dependence on the family $\mathcal{P}$ of projections is evident from the context. Since the first item in the decomposition $D(\mathsf{A}) \oplus \mathcal{J}(\mathsf{A})$ of $\mathcal{S}(\mathsf{A})$ is isomorphic (as a linear space) to $\mathsf{A}$, a main part of the description of the algebra $\mathcal{S}(\mathsf{A})$ is to identify the ideal $\mathcal{J}(\mathsf{A})$.

## 2.3 Discretization of the Toeplitz algebra

Unital $C^*$-algebras of infinite type typically contain non-unitary isometries, i.e. elements $s$ for which $s^*s$ is the identity element $e$, but $ss^* \neq e$. The perhaps simplest example is the universal algebra $C^*(s)$ generated by one isometry. The universal property of $C^*(s)$ means that whenever $S$ is an isometry in a $C^*$-algebra $\mathcal{A}$, then there is a $^*$-homomorphism from $C^*(s)$ onto the smallest $C^*$-subalgebra of $\mathcal{A}$ containing $S$ which sends $s$ to $S$. Coburn [9] showed that the algebra $C^*(s)$ is $^*$-isomorphic to the smallest closed $^*$-subalgebra $\mathsf{T}(C)$ of $L(l^2(\mathbb{Z}^+))$ which contains the (isometric) operator

$$V : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k \geq 0} \mapsto (0,\, x_0,\, x_1,\, \ldots)$$

of forward shift. The algebra $\mathsf{T}(C)$ is also known as the *Toeplitz algebra*, since each of its elements is of the form $T(c) + K$ where $T(c)$ is a Toeplitz operator and $K$ is a compact operator. To recall the definition of a Toeplitz operator, let $a$ be a function in $L^\infty(\mathbb{T})$ with $k$th Fourier coefficient

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta}\, d\theta, \quad k \in \mathbb{Z}.$$

Then the *Laurent operator* $L(a)$ on $l^2(\mathbb{Z})$, the *Toeplitz operator* $T(a)$ on $l^2(\mathbb{Z}^+)$, and the *Hankel operator* $H(a)$ on $l^2(\mathbb{Z}^+)$ *with generating function* $a$ are defined via their matrix representations with respect to the standard bases of $l^2(\mathbb{Z})$ and $l^2(\mathbb{Z}^+)$ by

$$L(a) = (a_{i-j})_{i,j=-\infty}^{\infty}, \quad T(a) = (a_{i-j})_{i,j=0}^{\infty}, \quad \text{and} \quad H(a) = (a_{i+j+1})_{i,j=0}^{\infty}.$$

These operators are bounded, and

$$\|H(a)\| \le \|T(a)\| = \|L(a)\| = \|a\|_\infty.$$

It is also useful to know that $L(a)$ and $T(a)$ are compact if and only if $a$ is the zero function, whereas $H(a)$ is compact for each continuous function $a$. With these notations, one has

**Theorem 2.3** $\mathsf{T}(C) = \{T(a) + K : a \in C(\mathbb{T})$ *and* $K \in K(l^2(\mathbb{Z}^+))\}$.

To discretize the Toeplitz algebra $\mathsf{T}(C)$, consider the orthogonal projections

$$P_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_0, x_1, x_2, \ldots, ) \mapsto (x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots)$$

which converge strongly to the identity operator. Evidently, $P_n$ projects onto the linear span of the first $n$ elements of the standard basis of $l^2(\mathbb{Z}^+)$. In accordance with the previous notation, we set $\mathcal{P} = (P_n)$ and let $\mathcal{S}(\mathsf{T}(C)) = \mathcal{S}^{\mathcal{P}}(\mathsf{T}(C))$ stand for the algebra of the finite sections discretization of the Toeplitz algebra. Thus, $\mathcal{S}(\mathsf{T}(C))$ is the smallest closed subalgebra of $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $(P_n(T(a) + K)P_n)$ with $a \in C(\mathbb{T})$ and $K$ compact. One can show that the sequences $(P_nT(a)P_n)$ with $a \in C(\mathbb{T})$ already generate $\mathcal{S}(\mathsf{T}(C))$.

It is a lucky circumstance that, as for the Toeplitz algebra $\mathsf{T}(C)$, the elements of $\mathcal{S}(\mathsf{T}(C))$ can be described explicitly. For this description, we will need the *reflection operators*

$$R_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_0, x_1, \ldots) \mapsto (x_{n-1}, x_{n-2}, \ldots, x_1, x_0, 0, 0, \ldots).$$

**Theorem 2.4** *The algebra* $\mathcal{S}(\mathsf{T}(C))$ *coincides with the set of all sequences*

$$(P_nT(a)P_n + P_nKP_n + R_nLR_n + G_n) \tag{7}$$

*where* $a \in C(\mathbb{T})$, $K$ *and* $L$ *are compact on* $l^2(\mathbb{Z}^+)$, *and* $(G_n) \in \mathcal{G}$.

**Proof.** Denote the set of all sequences of the form (7) by $\mathcal{S}_1$ for a moment. In a first step one shows that $\mathcal{S}_1$ is a symmetric algebra. This follows almost at once if Widom's identity

$$P_nT(ab)P_n = P_nT(a)P_nT(b)P_n + P_nH(a)H(\tilde{b})P_n + R_nH(\tilde{a})H(b)R_n, \quad (8)$$

where $\tilde{a}(t) := a(t^{-1})$, and the compactness of Hankel operators with continuous generating function are taken into account.

The proof that $\mathcal{S}_1$ is closed (hence, a $C^*$-algebra) proceeds in the standard way if one employs the fact that the strong limits $W(\mathbf{A}) := \text{s-lim}\, A_nP_n$ and $\widetilde{W}(\mathbf{A}) := \text{s-lim}\, R_nAR_n$ exist for each sequence $\mathbf{A} := (A_n) \in \mathcal{S}_1$ and that

$$W((P_nT(a)P_n + P_nKP_n + R_nLR_n + G_n)) = T(a) + K \quad (9)$$

and

$$\widetilde{W}((P_nT(a)P_n + P_nKP_n + R_nLR_n + G_n)) = T(\tilde{a}) + L. \quad (10)$$

Since the generating sequences of $\mathcal{S}(\mathsf{T}(C))$ belong to $\mathcal{S}_1$ and $\mathcal{S}_1$ is a closed algebra, we conclude that $\mathcal{S}(\mathsf{T}(C)) \subseteq \mathcal{S}_1$.

For the reverse inclusion we have to show that the sequence $(R_nLR_n)$ belongs to $\mathcal{S}(\mathsf{T}(C))$ for every compact operator $L$ and that $\mathcal{G} \subseteq \mathcal{S}(\mathsf{T}(C))$. Note that $V^*$ is the operator of backward shift and that all non-negative powers of $V$ and $V^*$ are Toeplitz operators with polynomial generating function. Hence, the identities

$$(R_nV^iP_1(V^*)^jR_n) = (P_n(V^*)^iP_n)(R_nP_1R_n)(P_nV^jP_n)$$

and $(R_nP_1R_n) = (P_n) - (P_nVP_n)(P_nV^*P_n)$ imply that $\mathcal{S}(\mathsf{T}(C))$ contains all sequences $(R_nLR_n)$ with $L$ a finite linear combination of operators of the form $V_iP_1V_{-j}$ with $i, j \geq 0$. Since these operators form a dense subset of $K(l^2(\mathbb{Z}^+))$, the first claim follows. The inclusion $\mathcal{G} \subseteq \mathcal{S}(\mathsf{T}(C))$ is a consequence of a more general result which we formulate as a separate proposition. ■

The following proposition shows a close symbiosis between sequences of the form $(P_nKP_n)$ with $K$ compact and sequences which tend to zero in the norm: each algebra which contains all sequences $(P_nKP_n)$ also contains all sequences tending to zero. The only (evidently necessary) obstruction is that no two of the $P_n$ coincide.

**Proposition 2.5** *Let $\mathcal{P} = (P_n)$ be a sequence of orthogonal projections of finite rank on a Hilbert space $H$. Suppose that the $P_n$ converge strongly to the identity operator and that $P_m \neq P_n$ whenever $m \neq n$. Then the ideal $\mathcal{G}^{\mathcal{P}}$ of all sequences which tend to zero in the norm is contained in the smallest closed subalgebra $\mathcal{J}$ of $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $(P_nKP_n)$ with $K$ compact.*

**Proof.** It is sufficient to show that, for each $n_0 \in \mathbb{N}$, there is a sequence $(G_n)$ in $\mathcal{J}$ such that $G_{n_0}$ is a projection of rank 1 and $G_n = 0$ for all $n \neq n_0$. Since the matrix algebras $\mathbb{C}^{k\times k}$ have no non-trivial ideals, this fact already implies that each sequence $(G_n)$ with arbitrarily prescribed $G_{n_0} \in L(\operatorname{im} P_{n_0})$ and $G_n = 0$ for $n \neq n_0$ belongs to $\mathcal{J}$. Since $\mathcal{G}^{\mathcal{P}}$ is generated (as a Banach space) by sequences of this special form, the assertion follows.

Let $n_0 \in \mathbb{N}$, put

$$\mathbb{N}_< := \{n \in \mathbb{N} : \operatorname{im} P_n \cap \operatorname{im} P_{n_0} \text{ is a proper subspace of } \operatorname{im} P_{n_0}\},$$

and set $\mathbb{N}_> := \mathbb{N} \setminus (\{n_0\} \cup \mathbb{N}_<)$. The set $\mathbb{N}_<$ is at most countable. If $n \in \mathbb{N}_<$, then none of the closed linear spaces $\operatorname{im} P_n \cap \operatorname{im} P_{n_0}$ has interior points relative to $\operatorname{im} P_{n_0}$. By the Baire category theorem, $\cup_{n\in\mathbb{N}_<}(\operatorname{im} P_n \cap \operatorname{im} P_{n_0})$ is a proper subset of $\operatorname{im} P_{n_0}$. Choose a unit vector

$$f \in \operatorname{im} P_{n_0} \setminus \cup_{n\in\mathbb{N}_<}(\operatorname{im} P_n \cap \operatorname{im} P_{n_0}).$$

Then $\|P_n f\| < 1$ for all $n \in \mathbb{N}_<$ by the Pythagoras theorem. (Indeed, otherwise $\|P_n f\| = 1$, and the equality $1 = \|f\|^2 = \|P_n f\|^2 + \|f - P_n f\|^2$ implies $f = P_n f$, whence $f \in \operatorname{im} P_n$.)

Let $Q_n := I - P_n$. If $n \in \mathbb{N}_>$, then $\operatorname{im} P_n \cap \operatorname{im} P_{n_0} = \operatorname{im} P_{n_0}$ by the definition of $\mathbb{N}_>$. Thus, $\operatorname{im} P_{n_0} \subseteq \operatorname{im} P_n$, and since no two of the projections $P_n$ coincide, this implies that $\operatorname{im} P_{n_0}$ is a proper subspace of $\operatorname{im} P_n$ and $\operatorname{im} Q_n$ is a proper subspace of $\operatorname{im} Q_{n_0}$ for $n \in \mathbb{N}_>$. Again by the Baire category theorem, $\cup_{n\in\mathbb{N}_>} \operatorname{im} Q_n$ is a proper subset of $\operatorname{im} Q_{n_0}$. Choose a unit vector

$$g \in \operatorname{im} Q_{n_0} \setminus \cup_{n\in\mathbb{N}_>} \operatorname{im} Q_n.$$

Then, as above, $\|Q_n g\| < 1$ for all $n \in \mathbb{N}_>$. Consider the operator $K : x \mapsto \langle x,\, g\rangle f$ on $H$. Its adjoint is $K^* : x \mapsto \langle x,\, f\rangle g$, and

$$P_n K Q_n K^* P_n x = \langle P_n x,\, f\rangle \, \langle Q_n g,\, g\rangle \, P_n f = \langle x,\, P_n f\rangle \, \|Q_n g\|^2 P_n f.$$

If $n \in \mathbb{N}_<$, then $\|P_n f\| < 1$, and if $n \in \mathbb{N}_>$, then $\|Q_n g\| < 1$ by construction. In both cases, $\|P_n K Q_n K^* P_n\| < 1$. In case $n = n_0$,

$$P_n K Q_n K^* P_n x = \langle x,\, f\rangle f$$

is an orthogonal projection of rank 1, which we call $P$. The sequence $\mathbf{K} := (P_n K Q_n K^* P_n)$ belongs to the algebra $\mathcal{J}$ since

$$(P_n K Q_n K^* P_n) = (P_n K K^* P_n) - (P_n K P_n)\,(P_n K^* P_n).$$

As $r \to \infty$, the powers $\mathbf{K}^r$ converge in the norm of $\mathcal{F}^{\mathcal{P}}$ to the sequence $(G_n)$ with $G_{n_0} = P \neq 0$ and $G_n = 0$ if $n \neq n_0$. Indeed, since $P_n \to I$ strongly, one has $\|Q_n g\| < 1/2$ for $n$ large enough, whence $\|P_n K Q_n K^* P_n\| < 1/2$ for these $n$, and for the remaining (finitely many) $n$ one has $\|P_n K Q_n K^* P_n\| < 1$ as we have seen above. Since $\mathbf{K}^r \in \mathcal{J}$ and $\mathcal{J}$ is closed, the sequence $(G_n)$ has the claimed properties. ∎

With the explicit description of $\mathcal{S}(\mathsf{T}(C))$ given by Theorem 2.4 one has full control on this algebra and its elements. From (9) it is evident that the kernel of the mapping $W$ (equivalently, the quasicommutator ideal of $\mathcal{S}(\mathsf{T}(C))$) is just the set of all sequences of the form $(R_nLR_n + G_n)$ with $L$ compact and $(G_n) \in \mathcal{G}$. Thus,

$$\begin{aligned} \mathcal{S}(\mathsf{T}(C)) &= \{(P_n(T(a)+K)P_n) : a \in C(\mathbb{T}),\, K \in K(l^2(\mathbb{Z}^+))\} \\ &\quad \oplus\{(R_nLR_n + G_n) : L \in K(l^2(\mathbb{Z}^+)),\, (G_n) \in \mathcal{G}\} \end{aligned}$$

is just the specification of (6) to the present context.

The stability of a sequence in $\mathcal{S}(\mathsf{T}(C))$ is related with its coset modulo $\mathcal{G}$. So let us see what Theorem 2.4 tells us about the quotient algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$. Since the ideal $\mathcal{G}$ lies in the kernel of the homomorphisms $W$ and $\widetilde{W}$, the mapping

$$\text{smb} : \mathcal{S}(\mathsf{T}(C))/\mathcal{G} \to L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)), \quad \mathbf{A} + \mathcal{G} \mapsto (W(\mathbf{A}), \widetilde{W}(\mathbf{A})) \quad (11)$$

is a well defined homomorphism. From Theorem 2.4 and from (9) and (10) we derive that the intersection of the kernels of $W$ and $\widetilde{W}$ is just the ideal $\mathcal{G}$, which implies the following.

**Theorem 2.6** *The mapping* smb *is a* $^*$*-isomorphism from* $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ *onto the* $C^*$*-subalgebra of* $L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$ *which consists of all pairs* $(W(\mathbf{A}), \widetilde{W}(\mathbf{A}))$ *with* $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$.

**Corollary 2.7** *A sequence* $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$ *is stable if and only if* $\text{smb}(\mathbf{A} + \mathcal{G})$ *is invertible in* $L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$.

Indeed, by the inverse closedness of $C^*$-algebras, the coset $\mathbf{A} + \mathcal{G} \in \mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$ if and only if it is invertible in $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$. So we arrived at a classical result:

**Corollary 2.8** *Let* $a \in C(\mathbb{T})$ *and* $K$ *compact. The finite sections sequence* $\mathbf{A} = (P_n(T(a) + K)P_n)$ *is stable if and only of the operator* $T(a) + K$ *is invertible.*

Indeed, using some special properties of Toeplitz operators it is easy to see that the invertibility of $W(\mathbf{A}) = T(a) + K$ already implies the invertibility of $\widetilde{W}(\mathbf{A}) = T(\tilde{a})$.

# 3 Fractality

## 3.1 Stability of subsequences

Clearly, a subsequence of a stable sequence is stable again. Does, conversely, the stability of a certain (infinite) subsequence of a sequence $\mathbf{A}$ imply the stability of the full sequence? In general certainly not; but this implication holds indeed if $\mathbf{A}$ belongs to the algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections method for Toeplitz operators. The argument is simple: The homomorphisms $W$ and $\widetilde{W}$ defined in the previous section are given by certain strong limits. Thus, the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ can be computed if only a subsequence of $\mathbf{A}$ is known. Moreover, if this subsequence is stable, then the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are already invertible. This implies the stability of the full sequence $\mathbf{A}$ via Corollary 2.7.

Employing Theorem 2.6 instead of Corollary 2.7 we can state this observation in a slightly different way: every sequence in $\mathcal{S}(\mathsf{T}(C))$ can be rediscovered from each of its (infinite) subsequences up to a sequence tending to zero in the norm. In that sense, the essential information on a sequence in $\mathcal{S}(\mathsf{T}(C))$ is stored in each of its subsequences. Subalgebras of $\mathcal{F}$ with this property were called *fractal* in [24] in order to emphasize exactly this self-similarity aspect. We will see some of the remarkable properties of fractal algebras in the following sections. We start with the general definition of fractal algebras.

## 3.2 Fractal algebras

Let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence. By $\mathcal{F}_\eta$ we denote the set of all subsequences $(A_{\eta(n)})$ of sequences $(A_n)$ in $\mathcal{F}$. As in Section 2.1, one can make $\mathcal{F}_\eta$ to a $C^*$-algebra in a natural way. The $^*$-homomorphism

$$R_\eta : \mathcal{F} \to \mathcal{F}_\eta, \quad (A_n) \mapsto (A_{\eta(n)})$$

is called the restriction of $\mathcal{F}$ onto $\mathcal{F}_\eta$. It maps the ideal $\mathcal{G}$ of $\mathcal{F}$ onto a closed ideal $\mathcal{G}_\eta$ of $\mathcal{F}_\eta$. For every subset $\mathcal{S}$ of $\mathcal{F}$, we abbreviate $R_\eta\mathcal{S}$ by $\mathcal{S}_\eta$.

Let $\mathcal{S}$ be a $C^*$-subalgebra of $\mathcal{F}$. A $^*$-homomorphism $W$ from $\mathcal{S}$ into a $C^*$-algebra $\mathcal{B}$ is called *fractal* if, for every strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, there is a mapping $W_\eta : \mathcal{S}_\eta \to \mathcal{B}$ such that $W = W_\eta R_\eta|_{\mathcal{S}}$. It is easy to see that $W_\eta$ is necessarily a $^*$-homomorphism again. A $C^*$-subalgebra $\mathcal{S}$ of $\mathcal{F}$ is called *fractal*, if the canonical homomorphism

$$\pi : \mathcal{S} \to \mathcal{S}/(\mathcal{S} \cap \mathcal{G}), \quad \mathbf{A} \mapsto \mathbf{A} + (\mathcal{S} \cap \mathcal{G})$$

is fractal. Thus, if $\mathcal{S}$ is a fractal algebra, then every sequence in $\mathcal{S}$ is uniquely determined by each of its subsequences up to a sequence in $\mathcal{G}$. Finally, a sequence $\mathbf{A} \in \mathcal{F}$ is *fractal* if the smallest $C^*$-subalgebra of $\mathcal{F}$ which contains the sequence $\mathbf{A}$ and the identity sequence is fractal. Note that is is not necessary for these definitions and for the following results that $\mathcal{F}$ is a product of finite-dimensional algebras.

The following result provides an equivalent characterization of fractal subalgebras of $\mathcal{F}$.

**Theorem 3.1** *A $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is fractal if and only if the implication*

$$R_\eta(\mathbf{A}) \in \mathcal{G}_\eta \;\Rightarrow\; \mathbf{A} \in \mathcal{G} \tag{12}$$

*holds for every sequence $\mathbf{A} \in \mathcal{A}$ and every strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$.*

**Proof.** Let $\mathcal{A}$ be fractal. Let $\mathbf{A} = (A_n) \in \mathcal{A}$ and $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence such that $R_\eta(\mathbf{A}) \in \mathcal{G}_\eta$. Then, for each $\varepsilon > 0$, there is an $n_0$ such that $\|A_{\eta(n)}\| \le \varepsilon$ for $n \ge n_0$. Set $\mu(n) := \eta(n + n_0)$. Then $\|R_\mu(\mathbf{A})\| \le \varepsilon$. The fractality of $\mathcal{A}$ implies that

$$\|\pi(\mathbf{A})\| \le \|\pi_\mu R_\mu(\mathbf{A})\| \le \|\pi_\mu\| \, \|R_\mu(\mathbf{A})\| = \|R_\mu(\mathbf{A})\| \le \varepsilon$$

for each $\varepsilon > 0$. Hence, $\mathbf{A} \in \mathcal{A} \cap \mathcal{G}$.

Conversely, suppose that the implication (12) holds for every sequence $\mathbf{A} \in \mathcal{A}$ and every strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$. Let $\mathbf{C}$ be a sequence in $R_\eta(\mathcal{A})$, and let $\mathbf{A}_1$ and $\mathbf{A}_2$ be sequences in $\mathcal{A}$ such that $R_\eta(\mathbf{A}_1) = R_\eta(\mathbf{A}_2) = \mathbf{C}$. Then $R_\eta(\mathbf{A}_1 - \mathbf{A}_2) \in \mathcal{G}_\eta$, whence $\mathbf{A}_1 - \mathbf{A}_2 \in \mathcal{G}$ by (12). One can thus define a mapping $\pi_\eta : R_\eta(\mathcal{A}) \to \mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ by $\pi_\eta(\mathbf{C}) := \mathbf{A} + (\mathcal{A} \cap \mathcal{G})$ where $\mathbf{A}$ is an arbitrarily chosen sequence in $\mathcal{A}$ with $R_\eta(\mathbf{A}) = \mathbf{C}$. Clearly, $\pi_\eta R_\eta = \pi$, whence the fractality of $\mathcal{A}$. ∎

**Corollary 3.2** (*a*) *If $\mathcal{A}$ is a fractal $C^*$-subalgebra of $\mathcal{F}$, then $\mathcal{A}_\eta \cap \mathcal{G}_\eta = (\mathcal{A} \cap \mathcal{G})_\eta$ for each strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$.*
(*b*) *$C^*$-subalgebras of fractal $C^*$-subalgebras of $\mathcal{F}$ are fractal.*
(*c*) *A $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is fractal if and only if the algebra $\mathcal{A} + \mathcal{G}$ is fractal.*

It is easy to see that the algebra $\mathcal{S}(\mathsf{T}(C))$ is a fractal subalgebra of $\mathcal{F}$. Indeed, the (evident) fractality of the homomorphisms $W$ and $\widetilde{W}$ implies that

$$\mathrm{smb}^{(o)} : \mathcal{S}(\mathsf{T}(C)) \to L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)), \quad \mathbf{A} \mapsto (W(\mathbf{A}), \widetilde{W}(\mathbf{A}))$$

is a fractal mapping. Hence, for each monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, there is a mapping $\mathrm{smb}^{(o)}_\eta$ such that $\mathrm{smb}^{(o)} = \mathrm{smb}^{(o)}_\eta \circ R_\eta$. Further, from Theorem 2.6 we know that the mapping

$$\mathrm{smb} : \mathcal{S}(\mathsf{T}(C))/\mathcal{G} \to L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)), \quad \mathbf{A} + \mathcal{G} \mapsto (W(\mathbf{A}), \widetilde{W}(\mathbf{A}))$$

is an isomorphism. Hence, $\mathrm{smb}^{-1} \circ \mathrm{smb}^{(o)}_\eta \circ R_\eta$ is the canonical homomorphism from $\mathcal{S}(\mathsf{T}(C))$ onto $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$.

## 3.3 Some consequences of fractality

The results in this section give a first impression of the power of fractality.

**Proposition 3.3** *Let $\mathcal{A}$ be a unital fractal $C^*$-subalgebra of $\mathcal{F}$. Then a sequence in $\mathcal{A}$ is stable if and only if it possesses a stable subsequence.*

**Proof.** Let $\mathbf{A} = (A_n) \in \mathcal{A}$, and let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence such that the sequence $R_\eta(\mathbf{A}) = (A_{\eta(n)})$ is stable. One can assume without loss that $A_{\eta(n)}$ is invertible for every $n \in \mathbb{N}$ (otherwise take a subsequence of $\eta$). Due to the inverse closedness of $\mathcal{A}_\eta$ in $\mathcal{F}_\eta$, there is a sequence $\mathbf{B} \in \mathcal{A}$ such that

$$R_\eta(\mathbf{A})\, R_\eta(\mathbf{B}) = R_\eta(\mathbf{B})\, R_\eta(\mathbf{A}) = R_\eta(\mathbf{I}) \tag{13}$$

with $\mathbf{I}$ the identity sequence. By hypothesis, the canonical homomorphism $\pi : \mathcal{A} \to \mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ factors into $\pi = \pi_\eta R_\eta$. Applying the homomorphism $\pi_\eta$ to (13), we thus get the invertibility of $\pi(\mathbf{A}) = \mathbf{A} + (\mathcal{A} \cap \mathcal{G})$ in $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$. Hence, $\mathbf{A}$ is a stable sequence. The reverse implication is obvious. ∎

**Proposition 3.4** *Let $\mathcal{A} \subset \mathcal{F}$ be a fractal $C^*$-algebra and $\mathbf{A} = (A_n) \in \mathcal{A}$. Then,*
*(a) for each strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$,*

$$\|\mathbf{A} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \|R_\eta(\mathbf{A}) + \mathcal{G}_\eta\|_{\mathcal{F}_\eta/\mathcal{G}_\eta}.$$

*(b) the limit $\lim_{n\to\infty} \|A_n\|$ exists and is equal to $\|\mathbf{A} + \mathcal{G}\|$.*

**Proof.** (*a*) By the third isomorphy theorem,

$$\|\mathbf{A} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \|\mathbf{A} + \mathcal{G}\|_{(\mathcal{A}+\mathcal{G})/\mathcal{G}} = \|\mathbf{A} + (\mathcal{A} \cap \mathcal{G})\|_{\mathcal{A}/(\mathcal{A}\cap\mathcal{G})} \tag{14}$$

for each sequence $\mathbf{A}$ in a (not necessarily fractal) $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{F}$. If $\mathcal{A}$ is fractal and $\mathbf{G} \in \mathcal{A} \cap \mathcal{G}$, then

$$\begin{aligned} \|\mathbf{A} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} &= \|\pi(\mathbf{A} + \mathbf{G})\|_{\mathcal{A}/(\mathcal{A}\cap\mathcal{G})} && \text{(by (14))} \\ &= \|\pi_\eta R_\eta(\mathbf{A} + \mathbf{G})\|_{\mathcal{A}/(\mathcal{A}\cap\mathcal{G})} && \text{(fractality of } \pi) \\ &\le \|R_\eta(\mathbf{A} + \mathbf{G})\|_{\mathcal{A}/(\mathcal{A}\cap\mathcal{G})}. \end{aligned}$$

Taking the infimum over all sequences $\mathbf{G} \in \mathcal{A} \cap \mathcal{G}$, and applying Corollary 3.2 (*a*), we obtain

$$\begin{aligned} \|\mathbf{A} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} &\le \|R_\eta(\mathbf{A}) + (\mathcal{A} \cap \mathcal{G})_\eta\|_{\mathcal{A}_\eta/(\mathcal{A}\cap\mathcal{G})_\eta} \\ &= \|R_\eta(\mathbf{A}) + (\mathcal{A}_\eta \cap \mathcal{G}_\eta)\|_{\mathcal{A}_\eta/(\mathcal{A}_\eta\cap\mathcal{G}_\eta)} \\ &= \|R_\eta(\mathbf{A}) + \mathcal{G}_\eta\|_{\mathcal{F}_\eta/\mathcal{G}_\eta} \end{aligned}$$

where we used (14) again. The reverse estimate is a consequence of the lim sup-formula (1):

$$\|(A_{\eta(n)}) + \mathcal{G}_\eta\|_{\mathcal{F}_\eta/\mathcal{G}_\eta} = \limsup \|A_{\eta(n)}\| \le \limsup \|A_n\| = \|(A_n) + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}}.$$

$(b)$ Choose a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that $\lim \|A_{\eta(n)}\| = \liminf \|A_n\|$. By part $(a)$ of this proposition and by (1),

$$\begin{aligned}\limsup \|A_n\| &= \|(A_n) + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \|(A_{\eta(n)}) + \mathcal{G}_\eta\|_{\mathcal{F}_\eta/\mathcal{G}_\eta} \\ &= \limsup \|A_{\eta(n)}\| = \lim \|A_{\eta(n)}\| = \liminf \|A_n\|\end{aligned}$$

which gives the assertion. ■

Our next goal is convergence properties of the spectra $\sigma(A_n)$ for fractal self-adjoint sequences $(A_n)$. The results will hold for fractal normal sequences as well, but there is no hope to say something substantial in case the sequence $(A_n)$ is not normal. We will need the following notions.

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of a metric space $X$. The *limes superior* $\limsup M_n$ (also called the *partial limiting set*) resp. the *limes inferior* $\liminf M_n$ (or the *uniform limiting set*) of the sequence $(M_n)$ consists of all points $x \in X$ which are a partial limit resp. the limit of a sequence $(m_n)$ of points $m_n \in M_n$. Observe that both $\limsup M_n$ and $\liminf M_n$ are closed sets. In case $X = \mathbb{C}$, the partial limiting set $\limsup M_n$ is never empty if $\cup_n M_n$ is bounded, whereas the uniform limiting set of a bounded set sequence can be empty.

The following is the analog of the limsup formula (1) for norms.

**Proposition 3.5** *Let $(A_n) \in \mathcal{F}$ be a normal sequence. Then*

$$\limsup \sigma(A_n) = \sigma_{\mathcal{F}/\mathcal{G}}((A_n) + \mathcal{G}).$$

**Proof.** Let $\lambda \in \sigma((A_n) + \mathcal{G})$. Then $(A_n - \lambda I_n)$ is a stable sequence. Since the norm $\|M\|$ and the spectral radius $\rho(M)$ of a normal matrix coincide, there is an $n_0 \in \mathbb{N}$ such that

$$\sup_{n\geq n_0} \rho((A_n - \lambda I_n)^{-1}) =: m < \infty.$$

Then, for all $n \geq n_0$,

$$m \geq \sup \{|t| : t \in \sigma((A_n - \lambda I_n)^{-1})\} = \sup \{|t|^{-1} : t \in \sigma(A_n - \lambda I_n)\}$$

whence

$$1/m \leq \inf \{|t| : t \in \sigma(A_n) - \lambda\} = \inf \{|t - \lambda| : t \in \sigma(A_n)\}$$

for all $n \geq n_0$. Hence, $\lambda$ cannot belong to $\limsup \sigma(A_n)$.

For the reverse inclusion assume the sequence $(A_n - \lambda I_n)$ fails to be stable. Then either there is an infinite subsequence $(A_{n_k} - \lambda I_{n_k})$ which consists of non-invertible matrices only, or all matrices $A_n - \lambda I_n$ with sufficiently large $n$ are invertible, but $\rho((A_{n_k} - \lambda I_{n_k})^{-1}) \to \infty$ as $k \to \infty$ for a certain subsequence. In the first case one has $\lambda \in \sigma(A_{n_k})$ for every $k$, whence $\lambda \in \limsup \sigma(A_n)$. In the second case one finds numbers $t_{n_k} \in \sigma(A_{n_k})$ such that $|t_{n_k} - \lambda|^{-1} \to \infty$ resp. $|t_{n_k} - \lambda| \to 0$ as $k \to \infty$ which implies $\lambda \in \limsup \sigma(A_n)$ also in this case. ∎

In the context of fractal algebras one can say again more.

**Proposition 3.6** *Let $\mathcal{A}$ be a fractal unital $C^*$-subalgebra of $\mathcal{F}$. If $(A_n) \in \mathcal{A}$ is normal, then*

$$\limsup \sigma(A_n) = \liminf \sigma(A_n) \quad (= \sigma_{\mathcal{F}/\mathcal{G}}((A_n) + \mathcal{G})). \tag{15}$$

**Proof.** Let $\lambda \in \mathbb{C} \setminus \liminf \sigma(A_n)$. Then there are a $\delta > 0$ and a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{dist}(\lambda, \sigma(A_{\eta(n)})) \geq \delta$ for all $n$. Thus, and since the $A_n$ are normal,

$$\sup_n \|(A_{\eta(n)} - \lambda I_{\eta(n)})^{-1}\| = \sup_n \rho((A_{\eta(n)} - \lambda I_{\eta(n)})^{-1}) < 1/\delta.$$

This shows that the sequence $(A_{\eta(n)} - \lambda I_{\eta(n)})$ is stable. Then, by Proposition 3.3, the sequence $(A_n - \lambda I_n)$ itself is stable. Hence, $\lambda \notin \sigma((A_n) + \mathcal{G}) = \limsup \sigma(A_n)$ by Proposition 3.5, which gives $\limsup \sigma(A_n) \subseteq \liminf \sigma(A_n)$. The reverse inclusion is evident. ∎

Results as in Propositions 3.4 and 3.6 can be derived also for other spectral quantities, for example for the sequences of the condition numbers, the sets of the singular values, the $\epsilon$-pseudospectra, and the numerical ranges of the $A_n$. For details see Chapter 3 in [13].

It is remarkable that for self-adjoint sequences, equality (15) is the only obstruction for being fractal.

**Theorem 3.7** *A self-adjoint sequence $(A_n) \in \mathcal{F}$ is fractal if and only if equality* (15) *holds.*

**Proof.** The 'only if'-part is Proposition 3.6. For the 'if'-part suppose that (15) holds. Let $\mathcal{A}$ denote the smallest closed subalgebra of $\mathcal{F}$ which contains the sequence $(A_n)$ and the identity sequence $(I_n)$. Further let $\eta : \mathbb{N} \to \mathbb{N}$ be a monotonically increasing sequence and write $\mathcal{A}_\eta$ for $R_\eta \mathcal{A}$.

The algebras $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ and $\mathcal{A}_\eta/(\mathcal{A}_\eta \cap \mathcal{G}_\eta)$ are isomorphic to $(\mathcal{A} + \mathcal{G})/\mathcal{G}$ and $(\mathcal{A}_\eta + \mathcal{G}_\eta)/\mathcal{G}_\eta$, respectively. The latter algebras are (as unital algebras) singly generated by their elements $(A_n) + \mathcal{G}$ and $(A_{\eta(n)}) + \mathcal{G}_\eta$, the spectra of which are $\limsup \sigma(A_n)$ and $\limsup \sigma(A_{\eta(n)})$, respectively, due to Proposition 3.5. Assumption (15) guarantees that these spectra coincide. Hence, by the Gelfand-Naimark theorem for singly generated $C^*$-algebras, the algebra $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ is $^*$-isomorphic to $\mathcal{A}_\eta/(\mathcal{A}_\eta \cap \mathcal{G}_\eta)$ with the isomorphism given by

$$(B_n) + (\mathcal{A} \cap \mathcal{G}) \mapsto (B_{\eta(n)}) + (\mathcal{A}_\eta \cap \mathcal{G}_\eta). \tag{16}$$

Let $\pi$ denote the canonical homomorphism from $\mathcal{A}$ onto $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$. Evidently, $\pi = \varphi_\eta \psi_\eta R_\eta$ where $\psi_\eta$ is the canonical homomorphism from $\mathcal{A}_\eta$ onto $\mathcal{A}_\eta/(\mathcal{A}_\eta \cap \mathcal{G}_\eta)$ and where $\varphi_\eta$ if the inverse of the isomorphism (16). Hence, $\pi$ is fractal. ∎

## 3.4 Fractal restrictions of separable algebras

We are now in a position to formulate and prove the main result of this section.

**Theorem 3.8 (Fractal restriction theorem)** *Let $\mathcal{A}$ be a separable unital $C^*$-subalgebra of $\mathcal{F}$. Then there exists a monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that the subalgebra $\mathcal{A}_\eta = R_\eta \mathcal{A}$ of $\mathcal{F}_\eta$ is fractal.*

We shall apply this result in case $\mathcal{A}$ is the algebra $\mathcal{S}(\mathsf{A})$ of the finite section method for a separable $C^*$-algebra $\mathsf{A}$. Recall that the mapping which associates with every subalgebra $\mathsf{A}$ of $L(H)$ the algebra $\mathcal{S}(\mathsf{A})$ generated by the finite sections sequences does not mind the individual properties of $\mathsf{A}$. If $\mathsf{A}$ is separable, this fact can be compensated to some extent by passing to a fractal restriction $\mathcal{S}_\eta(\mathsf{A})$ of $\mathcal{S}(\mathsf{A})$. Since the sequence $\eta$ depends on $\mathsf{A}$, the fractal restriction $\mathcal{S}_\eta(\mathcal{A})$ will reflect the structure of $\mathsf{A}$ in a much higher and more precise extent than the full algebra $\mathcal{S}(\mathcal{A})$ of the finite sections discretization. Enthusiastically formulated, the passage from $\mathcal{S}(\mathsf{A})$ to a fractal restriction gives the finite sections algebras a personality.

One cannot expect that Theorem 3.8 holds for arbitrary $C^*$-subalgebras of $\mathcal{F}$; for example it is certainly not valid for $l^\infty$ (which we identify with the algebra of all bounded sequences $(\alpha_n I_n)$ with complex numbers $\alpha_n$. On the other hand, non-separable $c^*$-subalgebras of $\mathcal{F}$ can possess fractal refinements as well; the algebra $\mathcal{S}(\mathsf{T}(PC))$ of the finite sections method for Toeplitz operators with piecewise continuous generating function can serve as an example.

We will first prove Theorem 3.8 in case the algebra $\mathcal{A}$ is singly generated by a self-adjoint sequence and the identity.

**Theorem 3.9** *Every self-adjoint sequence $(A_n) \in \mathcal{F}$ has a fractal subsequence.*

For the proof, we need some more facts on set sequences. Let $\mathbb{C}_{comp}$ denote the set of all non-empty and compact subsets of the complex plane. The *Hausdorff distance* of $L,\, M \in \mathbb{C}_{comp}$ is defined by

$$h(L,\, M) := \max\{\max_{l \in L} \operatorname{dist}(l,\, M),\, \max_{m \in M} \operatorname{dist}(m,\, L)\}$$

where $\operatorname{dist}(l,\, M) := \min_{m \in M} |l - m|$. The mapping $h : \mathbb{C}_{comp} \times \mathbb{C}_{comp} \to \mathbb{R}$ is called the *Hausdorff metric*. We denote the limit of a sequence $(M_n)$ with respect to $h$ by $\text{h-lim}\, M_n$. It is well known that $(\mathbb{C}_{comp},\, h)$ is a complete metric space. What we need in what follows is a compactness result which says that the relatively compact subsets of the metric space $(\mathbb{C}_{comp},\, h)$ are precisely its bounded subsets.

**Proposition 3.10** *(a) A set sequence $(M_n) \subset \mathbb{C}_{comp}$ converges with respect to the Hausdorff metric if and only if its partial and its uniform limiting sets coincide. In that case, $\limsup M_n = \liminf M_n = \text{h-lim}\, M_n$.*

*(b) Every bounded sequence in $(\mathbb{C}_{comp},\, h)$ possesses a convergent subsequence.*

**Proof of Theorem 3.9.** Consider the sets $M_n := \sigma(A_n)$. By Proposition 3.10 ($b$), there exists a subsequence $(M_{\eta(n)})_{n\in\mathbb{N}}$ of $(M_n)$ which converges with respect to the Hausdorff metric. Hence, by Proposition 3.10 ($a$),

$$\limsup_{n\to\infty} (M_{\eta(n)}) = \liminf_{n\to\infty} (M_{\eta(n)}).$$

Theorem 3.7 implies the fractality of the sequence $(A_{\eta(n)})_{n\geq 1}$. ■

**Proof of Theorem 3.8.** Let $\mathcal{A}$ be a separable $C^*$-subalgebra of $\mathcal{F}$ which contains the identity sequence. Let $(\mathbf{A}_k)_{k\in\mathbb{N}}$ be a countable dense subset of $\mathcal{A}$, and denote by $\mathbf{B}_{2k}$ and $\mathbf{B}_{2k-1}$ the real and the imaginary part of the sequence $\mathbf{A}_k$, respectively. Further, write $\mathcal{B} \subseteq \mathcal{A}$ for the set of all sequences $\mathbf{B}_k$ with $k \geq 1$ and $\mathcal{D} \subseteq \mathcal{A}$ for the set of all difference sequences $\mathbf{B}_k - \mathbf{B}_l$ with $k,\, l \geq 1$. The set $\mathcal{B} \cup \mathcal{D}$ is countable, and each of its elements is self-adjoint.

Let $(\mathbf{D}_k)_{k\in\mathbb{N}}$ be any enumeration of the elements of $\mathcal{B} \cup \mathcal{D}$. By Theorem 3.9, every sequence $\mathbf{D}_k$ possesses a fractal subsequence. We construct a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that the sequence $R_\eta \mathbf{D}_k$ is fractal for every $k \in \mathbb{N}$. This can be done by a standard diagonalization process as follows. Let $\eta_1 : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence such that $R_{\eta_1}\mathbf{D}_1$ is a fractal sequence. Then, for every $k \geq 2$, choose a subsequence $\eta_k$ of $\eta_{k-1}$ such that $R_{\eta_k}\mathbf{D}_k$ is a fractal sequence. Define $\eta$ by $\eta(n) := \eta_n(n)$. The sequence $\eta$ coincides (with possible exception of at most finitely many entries) with a subsequence of $\eta_k$ for every $k$. Hence, every sequence in $\mathcal{D}_\eta := R_\eta \mathcal{D}$ is fractal.

We claim that the algebra $\mathcal{A}_\eta := R_\eta \mathcal{A}$ is fractal. Thus, we have to verify that, given a subsequence $\mu$ of $\eta$, there is a homomorphism $\hat{\pi}_\mu$ such that

$$\hat{\pi}|_{\mathcal{A}_\eta} = \hat{\pi}_\mu R_\mu|_{\mathcal{A}_\eta}$$

where $\hat{\pi}$ is the canonical homomorphism from $\mathcal{F}_\eta$ onto $\mathcal{F}_\eta/\mathcal{G}_\eta$. Notice that the set of all sequences $R_\eta \mathbf{A}_k = R_\eta \mathbf{B}_{2k} + i\, R_\eta \mathbf{B}_{2k-1}$ with $k \in \mathbb{N}$ is dense in $\mathcal{A}_\eta$. Thus, without loss of generality and to simplify the notation, we assume in what follows that $\eta$ is the identity mapping and write $\mathcal{B}$ and $\mathcal{D}$ in place of $\mathcal{B}_\eta$ and $\mathcal{D}_\eta$ and $\pi$ in place of $\hat{\pi}$.

Let $\mu : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence. We start with defining the mapping $\pi_\mu$ on the set of the self-adjoint elements of $\mathcal{A}_\mu$. So let $\mathbf{A} \in \mathcal{A}$ and assume that $R_\mu \mathbf{A}$ is a self-adjoint sequence.

**Claim 1.** *There is a sequence $(\mathbf{C}_k)_{k\geq 1}$ in $\mathcal{B}$ such that*

$$\|R_\mu(\mathbf{A} - \mathbf{C}_k)\|_{\mathcal{F}_\mu} \to 0 \quad \textit{as } k \to \infty. \tag{17}$$

Indeed, decompose $\mathbf{A}$ into its real and imaginary part $\Re\mathbf{A} + i\Im\mathbf{A}$. Since $(\mathbf{A}_k)_{k\geq 1}$ is a dense subset of $\mathcal{A}$, one can approximate the sequence $\Re\mathbf{A}$ as closely as desired by sequences of the form $\Re\mathbf{A}_k = \mathbf{B}_{2k}$. Then, clearly, the sequence $\Re R_\mu \mathbf{A}$ can be approximated as closely as desired by sequences of the form $R_\mu \mathbf{B}_{2k} \in \mathcal{B}_\mu$. Since $\Re R_\mu \mathbf{A} = R_\mu \mathbf{A}$ by hypothesis, this gives the claim.

**Claim 2.** *The cosets $\mathbf{C}_k + \mathcal{G}$ are independent of the choice of the 'representative' $\mathbf{C}_k$ of the sequence $R_\mu \mathbf{C}_k$, i.e., if $\mathbf{C}, \mathbf{D} \in \mathcal{B}$ are sequences with $R_\mu \mathbf{C} = R_\mu \mathbf{D}$, then $\mathbf{C} + \mathcal{G} = \mathbf{D} + \mathcal{G}$.*

Indeed, the sequence $\mathbf{C} - \mathbf{D} =: (C_n - D_n)$ belongs to $\mathcal{D}$ and is, thus, fractal by construction. By Proposition 3.4 $(b)$, the limit $\lim \|C_n - D_n\|$ exists and is equal to $\|\mathbf{C} - \mathbf{D} + \mathcal{G}\|$. Since infinitely many of the differences $C_n - D_n$ are zero by assumption, this limit is zero, whence $\mathbf{C} - \mathbf{D} \in \mathcal{G}$.

**Claim 3.** *The cosets $\mathbf{C}_k + \mathcal{G}$ converge in $\mathcal{F}/\mathcal{G}$ as $k \to \infty$.*

Indeed, the fractality of the sequences $\mathbf{C}_k - \mathbf{C}_l \in \mathcal{D}$ and Proposition 3.4 $(a)$ imply

$$\begin{aligned} \|\mathbf{C}_k - \mathbf{C}_l + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} &= \|R_\mu \mathbf{C}_k - R_\mu \mathbf{C}_l + \mathcal{G}_\mu\|_{\mathcal{F}_\mu/\mathcal{G}\mu} \\ &\le \|R_\mu \mathbf{C}_k - R_\mu \mathbf{C}_l\|_{\mathcal{F}_\mu}. \end{aligned} \tag{18}$$

Since the sequences $R_\mu \mathbf{C}_k$ converge to $R_\mu \mathbf{A}$ as $k \to \infty$, $(\mathbf{C}_k + \mathcal{G})_{k \ge 1}$ is a Cauchy sequence and thus convergent. This settles Claim 3. We denote the limit of the cosets $\mathbf{C}_k + \mathcal{G}$ by $\mathbf{C} + \mathcal{G}$.

**Claim 4.** *The coset $\mathbf{C} + \mathcal{G}$ does not depend on the choice of the sequence $(R_\mu \mathbf{C}_k)_{k \ge 1}$ which approximates $R_\mu \mathbf{A}$.*

Indeed, let $(R_\mu \mathbf{D}_k)_{k \ge 1} \subseteq \mathcal{B}_\mu$ a sequence which also converges to $R_\mu \mathbf{A}$ and which determines a coset $\mathbf{D} + \mathcal{G}$ (in the same way as the sequence $(R_\mu \mathbf{C}_k)$ determines the coset $\mathbf{C} + \mathcal{G}$). As in (18) one then has

$$\begin{aligned} \|\mathbf{C} - \mathbf{D} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} &= \lim_{k \to \infty} \|\mathbf{C}_k - \mathbf{D}_k + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} \\ &\le \lim_{k \to \infty} \|R_\mu(\mathbf{C}_k - \mathbf{D}_k) + \mathcal{G}_\mu\|_{\mathcal{F}_\mu/\mathcal{G}_\mu} \\ &\le \limsup_{k \to \infty} \|R_\mu \mathbf{C}_k - R_\mu \mathbf{D}_k\|_{\mathcal{F}_\mu} \end{aligned} \tag{19}$$

(recall that the sequence $\mathbf{C}_k - \mathbf{D}_k$ is fractal by construction). Since both sequences $(R_\mu \mathbf{C}_k)_{k \ge 1}$ and $(R_\mu \mathbf{C}_k)_{k \ge 1}$ have the same limit as $k \to \infty$, the right hand side of (19) tends to zero which gives the claim. Thus, every self-adjoint sequence $R_\mu \mathbf{A} \in \mathcal{A}_\mu$ determines a unique coset $\mathbf{C} + \mathcal{G}$ which we denote by $\pi_\mu(R_\mu \mathbf{A})$.

**Claim 5.** *For every self-adjoint sequence $\mathbf{A} \in \mathcal{A}$ and every strongly monotonically increasing sequence $\mu : \mathbb{N} \to \mathbb{N}$,*

$$\pi_\mu(R_\mu \mathbf{A}) = \mathbf{A} + \mathcal{G}. \tag{20}$$

Indeed, there might be several sequences $(R_\mu \mathbf{C}_k)_{k \ge 1}$ which converge to $R_\mu \mathbf{A}$. Among these sequences there is (by assumption) at least one such that $\mathbf{C}_k \to \mathbf{A}$ in $\mathcal{F}$. For this sequence, one has $\mathbf{C}_k + \mathcal{G} \to \mathbf{A} + \mathcal{G}$ in $\mathcal{F}/\mathcal{G}$. The limit $\lim_{k \to \infty}(\mathbf{C}_k + \mathcal{G})$

is independent of the choice of the sequence $(\mathbf{C}_k)$ as we have seen in Claim 4. Hence, $\pi_\mu(R_\mu \mathbf{A}) = \mathbf{A} + \mathcal{G}$, which settles the construction of $\pi_\mu$ on the set of the self-adjoint sequences in $\mathcal{A}_\mu$.

To finish the proof, let $R_\mu \mathbf{A}$ be an arbitrary (not necessarily self-adjoint) sequence in $\mathcal{A}_\mu$. Then define $\pi_\mu(R_\mu \mathbf{A}) := \pi_\mu(\Re R_\mu \mathbf{A}) + i\, \pi_\mu(\Im R_\mu \mathbf{A})$. By (20),

$$\pi_\mu(R_\mu \mathbf{A}) = (\Re \mathbf{A}) + \mathcal{G} + i\,((\Im \mathbf{A}) + \mathcal{G}) = \mathbf{A} + \mathcal{G},$$

whence $\pi_\mu R_\mu|_{\mathcal{A}} = \pi|_{\mathcal{A}}$ as desired. ■

# 4 Fredholmness

## 4.1 A distinguished ideal of $\mathcal{S}(\mathsf{T}(C))$

We come back to our running example, the algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections method for Toeplitz operators. There is an ideal hidden in the algebra $\mathcal{S}(\mathsf{T}(C))$ which did not appear explicitly in the previous considerations but which always acted as a player in the background and which will play (together with its relatives) an outstanding role in what follows. Let

$$\mathcal{J} = \{(P_n K P_n + R_n L R_n + G_n) : K,\, L \text{ compact},\ (G_n) \in \mathcal{G}\}. \tag{21}$$

**Theorem 4.1** *(a) $\mathcal{J}$ is a closed ideal of $\mathcal{S}(\mathsf{T}(C))$.*

*(b) The quotient algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{J}$ is ${}^*$-isomorphic to $C(\mathbb{T})$, and the mapping $(P_n T(a) P_n) + \mathcal{J} \mapsto a$ is a ${}^*$-isomorphism between these algebras.*

**Proof.** $(a)$ First note that $\mathcal{J} \subset \mathcal{S}(\mathsf{T}(C))$ by Theorem 2.4. The closedness of $\mathcal{J}$ in $\mathcal{S}(\mathsf{T}(C))$ follows by standard arguments. To check that $\mathcal{J}$ is a left ideal, let $a \in C(\mathbb{T})$ and let $K$ and $L$ be compact. Then

$$\begin{aligned} &P_n T(a) P_n (P_n K P_n + R_n L R_n) \\ &\quad = P_n T(a) P_n K P_n + R_n (R_n T(a) R_n) L R_n \\ &\quad = P_n T(a) P_n K P_n + R_n T(\tilde{a}) P_n L R_n \\ &\quad = P_n T(a) K P_n + R_n T(\tilde{a}) L R_n - P_n T(a) Q_n K P_n - R_n T(\tilde{a}) Q_n L R_n \end{aligned}$$

with $Q_n := I - P_n$. The operators $T(a)K$ and $T(\tilde{a})L$ are compact. Since the operators $Q_n$ converge strongly to zero and $K$ and $L$ are compact, the last two operators converge to zero in the norm. Hence, $(P_n T(a) P_n)\,(P_n K P_n + R_n L R_n) \in \mathcal{J}$. Similarly one checks that $\mathcal{J}$ is a right ideal.

$(b)$ Widom's identity (8) together with the compactness of Hankel operators with continuous generating function imply that the mapping $a \mapsto (P_n T(a) P_n) + \mathcal{J}$ is a ${}^*$-homomorphism from $C(\mathbb{T})$ into $\mathcal{S}(\mathsf{T}(C))/\mathcal{J}$. This homomorphism is surjective by Theorem 2.4. To get its injectivity, let $(P_n T(a) P_n) \in \mathcal{J}$ for a continuous function $a$. Then there are compact operators $K$ and $L$ and a zero sequence $(G_n)$ such that

$$P_nT(a)P_n = P_nKP_n + R_nLR_n + G_n \quad \text{for all}\, n \in \mathbb{N}.$$

Letting $n$ go to infinity yields the compactness of $T(a)$. But then $a$ is the zero function. ∎

The importance of the ideal $\mathcal{J}$ results from several facts:

- The algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{J}$ is commutative, hence subject to Gelfand-Naimark theory. Similarly, factorization of a subalgebra $\mathcal{A}$ of $\mathcal{F}$ by $\mathcal{J}$ (or by an ideal with similar properties; see below) often yields quotient algebras $\mathcal{A}/\mathcal{J}$ which can be effectively studied by tools like central localization or other non-commutative generalizations of Gelfand theory.

- The algebra $\mathcal{J}/\mathcal{G}$ has exactly two non-equivalent irreducible representations which are given by the homomorphisms $W$ and $\widetilde{W}$. These representations extend to representations of $\mathcal{S}(\mathsf{T}(C))$ (of course, also given by $W$ and $\widetilde{W}$) which the property that a sequence $\mathbf{A}$ in $\mathcal{S}(\mathsf{T}(C))$ is stable if and only if the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are invertible. In this sense, the irreducible representations of $\mathcal{J}$ yield a sufficient family of irreducible representations of $\mathcal{S}(\mathsf{T}(C))$. Similar effects can be observed in numerous instances.

- Invertibility modulo $\mathcal{J}$ can be lifted in the following sense. Let $\mathcal{F}^{\mathcal{J}}$ stand for the largest subalgebra of $\mathcal{F}$ for which $\mathcal{J}$ is an ideal. Then the mappings $W$ and $\widetilde{W}$ extend to irreducible representations of $\mathcal{F}^{\mathcal{J}}$, and a sequence $\mathbf{A} \in \mathcal{F}^{\mathcal{J}}$ is stable if and only if the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are invertible and if the coset $\mathbf{A} + \mathcal{J}$ is invertible in the quotient $\mathcal{F}^{\mathcal{J}}/\mathcal{J}$. Again, such a lifting result holds in a much more general context.

The ideal $\mathcal{J}$ is clearly related with compact operators. We will now introduce a larger ideal $\mathcal{K}$ of sequences of compact type.

## 4.2 Compact sequences

A sequence $(K_n)$ in the $C^*$-algebra $\mathcal{F}$ is a *sequence of rank one matrices* if every matrix $K_n$ has range dimension less than or equal to one. The smallest closed ideal of $\mathcal{F}$ which contains all sequences of rank one matrices will be denoted by $\mathcal{K}$. Thus, a sequence $(A_n) \in \mathcal{F}$ belongs to $\mathcal{K}$ if and only if, for every $\varepsilon > 0$, there is a sequence $(K_n) \in \mathcal{F}$ such that

$$\sup_n \|A_n - K_n\| < \varepsilon \quad \text{and} \quad \sup_n \operatorname{rank} K_n < \infty. \tag{22}$$

We refer to the elements of $\mathcal{K}$ as *compact sequences*. The role of the ideal $\mathcal{K}$ in numerical analysis can be compared with the role of the ideal of the compact operators in operator theory.

Notice that $\mathcal{G} \subseteq \mathcal{K}$. Indeed, given a sequence $(G_n) \in \mathcal{G}$ and an $\varepsilon > 0$, set $K_n := G_n$ if $\|G_n\| \geq \varepsilon$ and $K_n := 0$ otherwise. Then (22) is satisfied since there are only finitely many operators $K_n$ which are not zero.

An appropriate notion of the rank of a sequence in $\mathcal{F}$ can be introduced as follows. We say that a sequence $\mathbf{A} \in \mathcal{F}$ has *finite essential rank* if it is the sum of a sequence $(G_n)$ in $\mathcal{G}$ and of a sequence $(K_n)$ with $\sup_n \operatorname{rank} K_n < \infty$. If $\mathbf{A}$ is of finite essential rank, then there is a smallest integer $r \geq 0$ such that $\mathbf{A}$ can be written as $(G_n)+(K_n)$ with $(G_n) \in \mathcal{G}$ and $\sup_n \operatorname{rank} K_n \leq r$. We call this integer the *essential rank* of $\mathbf{A}$ and write $\operatorname{ess\,rank} \mathbf{A} = r$. If $\mathbf{A}$ is not of finite essential rank, then we put $\operatorname{ess\,rank} \mathbf{A} = \infty$. Thus, the sequences of essential rank 0 are just the sequences in $\mathcal{G}$. Clearly, the sequences of finite essential rank form an ideal of $\mathcal{F}$ which is dense in $\mathcal{K}$, and

$$\operatorname{ess\,rank}(\mathbf{A}+\mathbf{B}) \leq \operatorname{ess\,rank} \mathbf{A} + \operatorname{ess\,rank} \mathbf{B},$$

$$\operatorname{ess\,rank}(\mathbf{AB}) \leq \min\{\operatorname{ess\,rank} \mathbf{A},\, \operatorname{ess\,rank} \mathbf{B}\}$$

for arbitrary sequences $\mathbf{A}, \mathbf{B} \in \mathcal{F}$.

Consider our running example. It is not hard to see that the intersection of the algebra $\mathcal{S}(\mathsf{T}(C))$ with the ideal $\mathcal{K}$ is just the distinguished ideal $\mathcal{J}$ which we examined in the preceding section. Moreover, the essential rank of the sequence $(P_nKP_n + R_nLR_n + G_n)$ turns out to be $\operatorname{rank} K + \operatorname{rank} L$.

There are several equivalent characterizations of compact sequences. Since the entries $A_n$ of the sequences are $n \times n$-matrices, a characterization of compactness and of the essential rank via the singular values of the $A_n$ will be particularly useful for our purposes. Recall from linear algebra that the singular values of an $n \times n$ matrix $A$ are the non-negative square roots of the eigenvalues of $A^*A$. We denote them by

$$\|A\| = \Sigma_1(A) \geq \Sigma_2(A) \geq \ldots \geq \Sigma_n(A) \geq 0 \tag{23}$$

if they are ordered decreasingly and by

$$0 \leq \sigma_1(A) \leq \sigma_2(A) \leq \ldots \leq \sigma_n(A) = \|A\| \tag{24}$$

in case of increasing order. Thus, $\sigma_k(A) = \Sigma_{n-k+1}(A)$. Notice that the matrices $A^*A$ and $AA^*$ are unitarily equivalent, whence $\Sigma_k(A) = \Sigma_k(A^*)$ for every $k$. We will also need the fact that every $n\times n$ matrix $A$ has a *singular value decomposition*

$$A = E^* \operatorname{diag}(\Sigma_1(A),\, \ldots,\, \Sigma_n(A))F$$

with unitary matrices $E$ and $F$. A second simple fact from linear algebra which we will use several times is the following.

**Lemma 4.2** *Let $A$ be an $n \times n$ matrix with* $\operatorname{rank} A = r$ *for some $r \in \{1, \ldots, n\}$. Then* $\operatorname{rank} A' \geq r$ *for each $n \times n$ matrix $A'$ with $\|A - A'\| < \Sigma_r(A)$.*

The announced characterization of compact sequences in terms of singular values reads as follows.

**Theorem 4.3** *The following conditions are equivalent for a sequence* $(K_n) \in \mathcal{F}$*:*
$(a)$ $\lim_{k\to\infty} \sup_{n\geq k} \Sigma_k(K_n) = 0$*;*
$(b)$ $\lim_{k\to\infty} \limsup_{n\to\infty} \Sigma_k(K_n) = 0$*;*
$(c)$ *the sequence* $(K_n)$ *is compact.*

Since the sequence $k \mapsto \sup_{n\geq k} \Sigma_k(K_n)$ is monotonically decreasing, the lim in $(a)$ and $(b)$ can be replaced by an inf.

**Proof.** The implication $(a) \Rightarrow (b)$ is evident. Let $(K_n) \in \mathcal{F}$ be a sequence which satisfies condition $(b)$, and let

$$K_n = E_n^* \operatorname{diag}(\Sigma_1(K_n), \ldots, \Sigma_n(K_n))F_n$$

be the singular value decomposition of $K_n$. For every $n \in \mathbb{N}$ and $k \geq 1$, set

$$K_n^{(k)} := \begin{cases} E_n^* \operatorname{diag}(\Sigma_1(K_n) \ldots, \Sigma_{k-1}(K_n), 0, \ldots, 0)F_n & \text{if } \ 1 < k \leq n, \\ 0 & \text{if } \ 1 = k \leq n, \\ K_n & \text{if } \ n < k. \end{cases}$$

Then, for $n > k$,

$$\|K_n - K_n^{(k)}\| = \|E_n^* \operatorname{diag}(0, \ldots, 0, \Sigma_k(K_n), \ldots, \Sigma_n(K_n))F_n\| = \Sigma_k(K_n),$$

and the limsup formula (1) for the norm of a coset in $\mathcal{F}/\mathcal{G}$ yields

$$\|(K_n) - (K_n^{(k)}) + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \limsup_{n\to\infty} \Sigma_k(K_n).$$

Together with property $(b)$, this implies that

$$\lim_{k\to\infty} \|(K_n) - (K_n^{(k)}) + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \lim_{k\to\infty} \limsup_{n\to\infty} \Sigma_k(K_n) = 0.$$

Thus, for each $k \in \mathbb{N}$, there is a sequence $(C_n^{(k)})$ in $\mathcal{G}$ such that

$$\lim_{k\to\infty} \|(K_n) - (K_n^{(k)}) - (C_n^{(k)})\|_{\mathcal{F}} = 0,$$

i.e., the sequence $(K_n)$ is the limit as $k \to \infty$ of the sequences $(K_n^{(k)} + C_n^{(k)})_{n\in\mathbb{N}}$. Since $\operatorname{rank} K_n^{(k)} \leq k - 1$ by definition, each of these sequences belongs to $\mathcal{K}$. Hence, $(K_n)$ is a compact sequence.

For the implication $(c) \Rightarrow (a)$, take a compact sequence $(K_n)$. The sequence $(\sup_{n\geq k} \Sigma_k(K_n))_{k\geq 1}$ is monotonically decreasing and bounded below (by zero) and, hence, convergent. Assume that the limit of this sequence is positive. Then there is a $C > 0$ such that $\sup_{n\geq k} \Sigma_k(K_n) > C$ for all $k \geq 1$. Thus, there are numbers $n_k \geq k$ such that

$$\Sigma_k(K_{n_k}) > C \quad \text{for all } k \geq 1. \tag{25}$$

On the other hand, since the sequence $(K_n)$ is compact, there is a sequence $(R_n) \in \mathcal{F}$ with

$$\sup_n \operatorname{rank} R_n < \infty \quad \text{and} \quad \sup_n \|K_n - R_n\| < C. \tag{26}$$

In particular, for each $k$ one has $\|K_{n_k} - R_{n_k}\| < C$, which implies via Lemma 4.2 and (25) that $\operatorname{rank} R_{n_k} \geq k$. Since $k$ can be chosen arbitrarily large, this contradicts the first condition in (26). Hence, the sequence $(\sup_{n\geq k} \Sigma_k(K_n))_{k\geq 1}$ cannot have a positive limit, whence condition $(a)$. ■

In the same vein one can prove the following characterization of sequences of essential rank $r$.

**Corollary 4.4** *A sequence $(K_n) \in \mathcal{F}$ is of essential rank $r$ if and only if*

$$\limsup_{n\to\infty} \Sigma_r(K_n) > 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \Sigma_{r+1}(K_n) = 0.$$

One consequence is the *lower semi-continuity* of the essential rank function.

**Corollary 4.5** *If* $\operatorname{ess\,rank}(K_n) = r$, *then* $\operatorname{ess\,rank}(K'_n) \geq r$ *for all sequences $(K'_n)$ which are sufficiently close to $(K_n)$.*

Another corollary concerns the behavior of the small singular values of $K_n$.

**Corollary 4.6** *Let $(K_n) \in \mathcal{K}$. Then the limit $\lim_{n\to\infty} \sigma_k(K_n)$ exists and is equal to 0 for every $k$.*

**Proof.** Let $\varepsilon > 0$. By Theorem 4.3, there is a $k_0$ such that $\sup_{n\geq k_0} \Sigma_{k_0}(K_n) < \varepsilon$. Then, for all $n \geq n_0 := k_0 + k - 1$,

$$\sigma_k(K_n) = \Sigma_{n-k+1}(K_n) \leq \Sigma_{k_0}^{(n)} \leq \sup_{n\geq k_0} \Sigma_{k_0}(K_n) < \varepsilon,$$

which gives the assertion. ■

When thinking about how to introduce an appropriate ideal of compact sequences, one certainly has in mind that at least the constant sequence $(P_1)$ should be compact. So one is lead to consider the smallest closed ideal of $\mathcal{F}$ which contains this sequence, and this ideal is a minimal candidate of what might be called an ideal of compact operators. At the other end of the scale, one might call a sequence $\mathbf{K} = (K_n) \in \mathcal{F}$ compact if $W(\mathbf{K})$ is a compact operator for each irreducible representation of $\mathcal{F}$, yielding a maximal version of an ideal of compact operators. Both versions coincide and lead to the ideal $\mathcal{K}$.

**Theorem 4.7** *$\mathcal{K}$ is the smallest closed ideal of $\mathcal{F}$ which contains the constant sequence $(P_1)$.*

**Theorem 4.8** *(a) A sequence $\mathbf{K} \in \mathcal{F}$ belongs to the ideal $\mathcal{K}$ if and only if $W(\mathbf{K})$ is compact for every irreducible representation $W$ of $\mathcal{F}$.*
*(b) A coset $\mathbf{K} + \mathcal{G} \in \mathcal{F}/\mathcal{G}$ belongs to the ideal $\mathcal{K}/\mathcal{G}$ if and only if $W(\mathbf{K} + \mathcal{G})$ is compact for every irreducible representation $W$ of $\mathcal{F}/\mathcal{G}$.*

A crucial step in the proof is to show that rank $W(\mathbf{K}) \leq 1$ for each sequence $\mathbf{K} \in \mathcal{F}$ of rank one matrices and each irreducible representation $W$ of $\mathcal{F}$. For details of the proofs of the preceding theorems as well as for further characterizations of the preceding theorems, see [22]. Let me also mention that $\mathcal{K}/\mathcal{G}$ is an essential ideal of $\mathcal{F}/\mathcal{G}$.

## 4.3 Fredholm sequences

Corresponding to the ideal $\mathcal{K}$ we introduce an appropriate class of Fredholm sequences by calling a sequence $(A_n) \in \mathcal{F}$ *Fredholm* if it is invertible modulo the ideal $\mathcal{K}$ of the compact sequences. The following properties of Fredholm sequences are obvious.
– Stable sequences are Fredholm.
– Adjoints of Fredholm sequences are Fredholm.
– Products of Fredholm sequences are Fredholm.
– The sum of a Fredholm and a compact sequence is Fredholm.
– The set of all Fredholm sequences is open in $\mathcal{F}$.
For alternate characterizations of Fredholm sequences, let $\sigma_1(A) \leq \ldots \leq \sigma_n(A)$ denote the singular values of an $n \times n$ matrix $A$.

**Theorem 4.9** *The following conditions are equivalent for a sequence $(A_n) \in \mathcal{F}$:*
*(a) The sequence $(A_n)$ is Fredholm.*
*(b) There are sequences $(B_n) \in \mathcal{F}$ and $(J_n) \in \mathcal{K}$ with $\sup_n \operatorname{rank} J_n < \infty$ such that*

$$B_n A_n = I_n + J_n \quad \text{for all } n \in \mathbb{N}. \tag{27}$$

*(c) There is a $k \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0. \tag{28}$$

**Proof.** $(a) \Rightarrow (b)$: Let $(A_n) \in \mathcal{F}$ be a Fredholm sequence. Then there are sequences $(C_n) \in \mathcal{F}$ and $(K_n) \in \mathcal{K}$ such that $(C_n)(A_n) = (I_n) + (K_n)$. Choose a sequence $(L_n) \in \mathcal{K}$ with $\|(L_n) - (K_n)\|_{\mathcal{F}} < 1/2$ and $\sup \operatorname{rank} L_n < \infty$. Then

$$(C_n)(A_n) = (I_n) + (K_n - L_n) + (L_n).$$

Since $(I_n) + (K_n - L_n)$ is invertible in $\mathcal{F}$, we obtain (27) with

$$B_n := (I_n + K_n - L_n)^{-1} C_n \quad \text{and} \quad J_n := (I_n + K_n - L_n)^{-1} L_n.$$

$(b) \Rightarrow (c)$: Let the singular value decomposition of $A_n$ be given by

$$A_n = E_n^* \Sigma_n F_n := E_n^* \operatorname{diag}(\sigma_1(A_n), \ldots, \sigma_n(A_n)) F_n.$$

After multiplication by $F_n$ and $F_n^*$, the identity (27) becomes

$$(F_n B_n E_n^*)(\Sigma_n) = (I_n) + (F_n J_n F_n^*).$$

Abbreviating $C_n := F_n B_n E_n^*$ and $K_n := F_n J_n F_n^*$ we get

$$C_n \Sigma_n = C_n \operatorname{diag}(\sigma_1(A_n), \ldots, \sigma_n(A_n)) = I_n + K_n \quad \text{for all } n \in \mathbb{N} \tag{29}$$

where still $\sup_n \operatorname{rank} K_n < \infty$. Let $k := \limsup_{n\to\infty} \operatorname{rank} K_n$. We claim that $\liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0$. Contrary to what we want to show, assume that there is an infinite subsequence $(n_l)_{l\geq 1}$ of $\mathbb{N}$ with $\lim_{l\to\infty} \sigma_{k+1}(A_{n_l}) = 0$. Multiplying (29) from both sides by $P_{k+1}$, we get

$$P_{k+1} C_{n_l} \Sigma_{n_l} P_{k+1} = P_{k+1} + P_{k+1} K_{n_l} P_{k+1}.$$

Since

$$\|\Sigma_{n_l} P_{k+1}\| = \|\operatorname{diag}(\sigma_1(A_{n_l}), \ldots, \sigma_{k+1}(A_{n_l}), 0, \ldots, 0)\| = \sigma_{k+1}(A_{n_l}) \to 0,$$

one has

$$\lim_{l\to\infty} \|P_{k+1} + P_{k+1} K_{n_l} P_{k+1}\| = 0.$$

Thus, the matrices $P_{k+1} K_{n_l} P_{k+1} \in \mathbb{C}^{(k+1)\times(k+1)}$ are invertible for all sufficiently large $n_l$. But this is impossible since $P_{k+1}$ has rank $k+1$, whereas $\operatorname{rank} K_{n_l} \leq k$. This proves the claim which, on its hand, implies assertion $(c)$.

$(c) \Rightarrow (a)$: As in the previous part of the proof, let $A_n = E_n^* \Sigma_n F_n$ refer to the singular value decomposition of $A_n$, and let $k$ be a non-negative integer such that

$$\liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0.$$

Then the sequence $(\Sigma_n + P_k)_{n\geq 1}$ (with $P_0 := 0$) is stable, and so is the sequence $(A_n + E_n^* P_k F_n)_{n\in\mathbb{N}}$. Thus, there are sequences $(C_n) \in \mathcal{F}$ and $(G_n), (H_n) \in \mathcal{G}$ such that

$$(C_n)(A_n + E_n^* P_k F_n) = (I_n) + (G_n) \quad \text{and} \quad (A_n + E_n^* P_k F_n)(C_n) = (I_n) + (H_n),$$

whence

$$(C_n)(A_n) = (I_n) + (G_n) - (C_n E_n^* P_k F_n)$$

and

$$(A_n)(C_n) = (I_n) + (H_n) - (E_n^* P_k F_n C_n).$$

The sequences $(G_n) - (C_n E_n^* P_k F_n)$ and $(H_n) - (E_n^* P_k F_n C_n)$ are of finite essential rank. Hence, $(A_n)$ is invertible modulo $\mathcal{K}$. ■

The preceding theorem suggests to introduce the *$\alpha$-number* $\alpha(\mathbf{A})$ of a Fredholm sequence $\mathbf{A} = (A_n)$ which corresponds to the kernel dimension of a Fredholm operator. By definition, $\alpha(\mathbf{A})$ is the smallest non-negative integer $k$ for which (28) is true. Equivalently, $\alpha(\mathbf{A})$ is the smallest non-negative integer $k$ for which there exist a sequence $(B_n) \in \mathcal{F}$ and a sequence $(J_n) \in \mathcal{K}$ of essential rank $k$ such that $B_n A_n^* A_n = I_n + J_n$ for all $n \in \mathbb{N}$. The latter fact follows easily from the proof of the preceding theorem.

The *index* of a Fredholm sequence $\mathbf{A}$ is the integer

$$\operatorname{ind}(\mathbf{A}) := \alpha(\mathbf{A}) - \alpha(\mathbf{A}^*).$$

Observe that, in the case at hand, the index of a Fredholm sequence always zero. This is a consequence of the fact that the operators $A_n$ act on finite dimensional spaces which implies that $A_n^* A_n$ and $A_n A_n^*$ have the same eigenvalues, even with respect to their multiplicity. So the more interesting quantity associated with a Fredholm sequence seems to be its $\alpha$-number. On the other hand, the vanishing of the index of a Fredholm sequence allows one to make use of the index as a conservation quantity.

If $A$ is a Fredholm operator with index 0, then there is an operator $K$ with finite rank such that $A + K$ is invertible. The analog for Fredholm sequences reads as follows. Notice that there is no index obstruction since the index of a Fredholm sequence is always 0.

**Theorem 4.10** *If $\mathbf{A} \in \mathcal{F}$ is a Fredholm sequence, then there is a sequence $\mathbf{K} \in \mathcal{K}$ with* ess rank $\mathbf{K} \le \alpha(\mathbf{A})$ *such that $\mathbf{A} + \mathbf{K}$ is a stable sequence.*

**Proof.** Let $k$ denote the $\alpha$-number of $\mathbf{A} =: (A_n)$, and let

$$A_n = E_n^* \operatorname{diag}(\sigma_1(A_n), \ldots, \sigma_n(A_n)) F_n$$

be the singular value decomposition of $A_n$. Set

$$K_n := E_n^* \operatorname{diag}(\sigma_{k+1}(A_n) - \sigma_1(A_n), \ldots, \sigma_{k+1}(A_n) - \sigma_k(A_n), 0, \ldots, 0) F_n$$

and $\mathbf{K} := (K_n)$. Then $\operatorname{rank} K_n \le k$ for each $n$, hence ess rank $\mathbf{K} \le k$, and the sequence $\mathbf{A} + \mathbf{K}$ is stable. ■

# 5 Fractal algebras of compact sequences

In this section we consider compact and Fredholm sequences in fractal algebras. The property of fractality has some striking consequences. For example, fractal ideals in $\mathcal{K}$ are constituted of blocks which are isomorphic to the ideal of the compact operators on a Hilbert space. There will be also a nice formula for the alpha-number of a Fredholm sequence. We consider these results as interesting and beautiful, and they are certainly worth to be mentioned here. On the other hand, the main examples which we shall examine later are either not fractal (discretizations of band-dominated operators) or do not contain non-trivial compact sequences (discretizations of Cuntz algebras). So we shall be very brief in this section and omit many details and almost all proofs.

## 5.1 Fractality and large singular values

First we will see that the singular values of fractal compact sequences behave as the singular values of compact operators on Hilbert space, i.e., the set of the singular values is countable and has 0 as its only possible accumulation point. Note that this does not hold for general compact sequences. For example, take an enumeration $(a_n)$ of the rational numbers in [0, 1] and set

$$K_n := a_n P_n P_1 P_n = \operatorname{diag}(a_n, 0, \ldots, 0).$$

Then the sequence $(K_n)$ is compact (it consists of rank one matrices), but the spectrum of its coset $(K_n) + \mathcal{G}$ is the closed interval [0, 1].

Given an $n \times n$-matrix $A$, let again $\Sigma_1(A) \geq \ldots \geq \Sigma_n(A) \geq 0$ denote the singular values of $A$, and write $\sigma_2(A)$ for the set of the singular values of $A$. Since the singular values of $A$ are the eigenvalues of self-adjoint matrix $(A^*A)^{1/2}$, it is an immediate consequence of Proposition 3.6 that, for each sequence $(A_n)$ in a fractal algebra, the sets $\sigma_2(A_n)$ converge with respect to the Hausdorff metric. In particular,

$$\limsup \sigma_2(A_n) = \liminf \sigma_2(A_n) = \sigma_2\left((A_n) + \mathcal{G}\right). \tag{30}$$

If $(A_n) \in \mathcal{F}$ is a fractal sequence, then the sequence $(\Sigma_1(A_n))$ of the largest singular values of $A_n$ converges. This fact follows immediately from Proposition 3.4 $(b)$ and the identity $\Sigma_1(A_n) = \|A_n\|$. One cannot expect that the sequence of the second singular values $\Sigma_2(A_n)$ converges, too. Indeed, the sequence defined by

$$A_n := \begin{cases} \operatorname{diag}(1, 0, 0, \ldots, 0) & \text{if } n \text{ is odd} \\ \operatorname{diag}(1, 1, 0, \ldots, 0) & \text{if } n \text{ is even} \end{cases}$$

is fractal by Theorem 3.7, but the sequence of its second singular values alternates between 0 and 1 and has thus two accumulation points. In fact, one can show that the sequence $(\Sigma_2(A_n))$ can possess *at most two* limiting points, at most one of which is different from $\lim \Sigma_1(A_n)$. This fact holds more general.

**Proposition 5.1** *If the sequence* $(A_n) \in \mathcal{F}$ *is fractal, then the set*

$$\limsup_{n\to\infty} \{\Sigma_1(A_n),\, \ldots,\, \Sigma_k(A_n)\}$$

*contains at most* $k$ *elements.*

**Proof.** Write $\Pi_j$ for the set of all partial limits of the sequence $(\Sigma_j(A_n))_{n\in\mathbb{N}}$. We first verify that

$$\Pi_1 \cup \ldots \cup \Pi_k = \limsup_{n\to\infty} \{\Sigma_1(A_n),\, \ldots,\, \Sigma_k(A_n)\} \quad \text{for every } k \in \mathbb{N}. \tag{31}$$

The inclusion $\subseteq$ is evident. Conversely, if $\lambda$ belongs to the right-hand side of (31), then there are strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ and numbers $k_n$ in $\{1, \ldots, k\}$ such that $\lambda = \lim_{n\to\infty} \Sigma_{k_n}(A_{\eta(n)})$. Since $k_n$ can take only finitely many values, there is a $k_0$ between 1 and $k$ and a subsequence $\mu$ of $\eta$ such that $\lambda = \lim_{n\to\infty} \Sigma_{k_0}(A_{\mu(n)})$. Hence, $\lambda \in \Pi_{k_0}$.

We have already mentioned that $\Pi_1$ is a singleton. Next we show that for each $j \geq 1$ the difference $\Pi_{j+1} \setminus (\Pi_1 \cup \ldots \cup \Pi_j)$ contains at most one element. Assume there are points $\alpha$ and $\beta$ in $\Pi_{j+1} \setminus (\Pi_1 \cup \ldots \cup \Pi_j)$ with $\alpha > \beta$. Choose a subsequence $(\Sigma_{j+1}(A_{\eta(n)}))$ of $(\Sigma_{j+1}(A_n))$ which converges to $\beta$ as $n \to \infty$. Then $\alpha$ cannot belong to the partial limiting set $\limsup \sigma_2(A_{\eta(n)})$. Indeed, if $\alpha \in \limsup \sigma_2(A_{\eta(n)})$ then

$$\alpha \in \limsup_{n\to\infty} \{\Sigma_1(A_{\eta(n)}),\, \ldots,\, \Sigma_j(A_{\eta(n)})\}$$

due to monotonicity reasons. Then

$$\alpha \in \limsup_{n\to\infty} \{\Sigma_1(A_n),\, \ldots,\, \Sigma_j(A_n)\} = \Pi_1 \cup \ldots \cup \Pi_j$$

which was excluded. Hence, $\alpha \in \limsup \sigma_2(A_n) \setminus \liminf \sigma_2(A_n)$, which contradicts (30). ∎

Here is the announced result on singular values of fractal compact sequences.

**Theorem 5.2** *Let* $(K_n) \in \mathcal{K}$ *be a fractal sequence. Then the set* $\text{h-lim}\, \sigma_2(K_n) = \sigma_2((K_n) + \mathcal{G})$ *is at most countable, it contains the point* 0*, and* 0 *is the only accumulation point of this set.*

**Proof.** Let $\varepsilon > 0$. By Theorem 4.3 $(a)$, $\lim_{k\to\infty} \sup_{n\geq k} \Sigma_k(K_n) = 0$. Thus, there is a $k_0$ such that $\sup_{n\geq k} \Sigma_k(K_n) \leq \varepsilon$ for each $k \geq k_0$, whence

$$\limsup_{n\to\infty} \{\Sigma_{k_0}(K_n),\, \ldots,\, \Sigma_n(K_n)\} \subseteq [0,\, \varepsilon].$$

Hence, every point in $\text{h-lim}\, \sigma_2(K_n) \setminus [0, \varepsilon]$ must lie in

$$\limsup_{n\to\infty} \{\Sigma_1(K_n),\, \ldots,\, \Sigma_{k_0-1}(K_n)\}$$

which is a finite set by Proposition 5.1. Consequently, $\text{h-lim}\, \sigma_2(K_n)$ is at most countable and has 0 as only possible accumulation point. That 0 indeed belongs to this set is a consequence of Corollary 4.6. ∎

## 5.2 Compact elements in $C^*$-algebras

There is a notion of a compact element in a general $C^*$-algebra $\mathcal{A}$. A non-zero element $k$ of $\mathcal{A}$ is said to be *of rank one* if, for each $a \in \mathcal{A}$, there is a complex number $\mu$ such that $kak = \mu k$. We let $\mathcal{C}(\mathcal{A})$ stand for the smallest closed subalgebra of $\mathcal{A}$ which contains all elements of rank one. If such elements do not exist, we set $\mathcal{C}(\mathcal{A}) = \{0\}$. Since the product of a rank one element with an arbitrary element of $\mathcal{A}$ is zero or rank one again, $\mathcal{C}(\mathcal{A})$ is a closed ideal of $\mathcal{A}$. There are several equivalent descriptions of the ideal $\mathcal{C}(\mathcal{A})$. To state the descriptions which are important in what follows, we need some more notation.

A $C^*$-algebra is called *elementary* if it is $^*$-isomorphic to the ideal $K(H)$ of the compact operators on some Hilbert space $H$. A $C^*$-algebra $\mathcal{J}$ is called *dual* if it is $^*$-isomorphic to a direct sum of elementary algebras. Thus, there is an index set $T$, for each $t \in T$ there is an elementary algebra $\mathcal{J}_t$, and $\mathcal{J}$ is $^*$-isomorphic to the $C^*$-algebra of all bounded functions $a$ which are defined on $T$, take a value $a(t)$ in $\mathcal{J}_t$ at $t \in T$, and which are such that for each $\varepsilon > 0$, there are only finitely many $t \in T$ with $\|a(t)\| > \varepsilon$. An alternate way to think of dual algebras is the following. Let $\{\mathcal{J}_t\}_{t \in T}$ be a family of elementary ideals of a $C^*$-algebra $\mathcal{A}$ with the property that $\mathcal{J}_s \mathcal{J}_t$ is the zero ideal whenever $s \neq t$. Then the smallest closed subalgebra of $\mathcal{A}$ which contains all algebras $\mathcal{J}_t$ is a dual algebra, and each dual algebra is of this form.

**Theorem 5.3** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{J}$ a closed ideal of $\mathcal{A}$. The following assertions are equivalent:*
*(a) $\mathcal{J} = \mathcal{C}(\mathcal{J})$.*
*(b) $\mathcal{J}$ is a dual algebra.*
*(c) The spectrum of every self-adjoint element of $\mathcal{J}$ is at most countable and has 0 as only possible accumulation point.*

For each dual ideal of a $C^*$-algebra there is a lifting theorem as follows. For a proof, see [13].

**Theorem 5.4 (Lifting theorem for dual ideals)** *Let $\mathcal{A}$ be a unital $C^*$- algebra. For every element $t$ of a set $T$, let $\mathcal{J}_t$ be an elementary ideal of $\mathcal{A}$ such that $\mathcal{J}_s \mathcal{J}_t = \{0\}$ whenever $s \neq t$, and let $W_t : \mathcal{A} \to L(H_t)$ denote the irreducible representation of $\mathcal{A}$ which lifts $\mathcal{J}_t$. Let further $\mathcal{J}$ stand for the smallest closed ideal of $\mathcal{A}$ which contains all ideals $\mathcal{J}_t$.*

*(a) An element $a \in \mathcal{A}$ is invertible if and only if the coset $a + \mathcal{J}$ is invertible in $\mathcal{A}/\mathcal{J}$ and if all elements $W_t(a)$ are invertible in $\mathcal{B}_t$.*

*(b) The separation property holds, i.e. $W_s(\mathcal{J}_t) = \{0\}$ whenever $s \neq t$.*

*(c) If $j \in \mathcal{J}$, then $W_t(j)$ is compact for every $t \in T$.*

*(d) If the coset $a + \mathcal{J}$ is invertible, then all operators $W_t(a) \in L(H_t)$ are Fredholm, and there are at most finitely many of these operators which are not invertible.*

**Corollary 5.5** *Let $\mathcal{A}$ be a unital and fractal $C^*$-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$. Then the ideal $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ of $\mathcal{F}/\mathcal{G}$ is a dual algebra.*

Indeed, this follows immediately from Theorem 5.2 and Theorem 5.3. As a consequence we obtain that the algebra $\mathcal{A}/\mathcal{G}$ and its ideal $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ are subject to the lifting theorem.

Next we are going to examine the consequences of the lifting theorem for sequence algebras. We will do this in the slightly more general context of Silbermann pairs. A *Silbermann pair* $(\mathcal{A}, \mathcal{J})$ consists of a unital $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{F}$ which contains the ideal $\mathcal{G}$ and of a closed ideal $\mathcal{J}$ of $\mathcal{A}$ which contains $\mathcal{G}$ properly and which consists of compact sequences only, and for which $\mathcal{J}/\mathcal{G}$ is a dual subalgebra of $\mathcal{K}/\mathcal{G}$. This property ensures that the lifting theorem applies to Silbermann pairs. Every sequence in $\mathcal{A}$ which is invertible modulo $\mathcal{J}$ is called an *$\mathcal{J}$-Fredholm* sequence. Note that each $\mathcal{J}$-Fredholm sequence is Fredholm in sense of Section 4.3 (but, of course, a Fredholm sequence in $\mathcal{A}$ is not necessarily $\mathcal{J}$-Fredholm). Under the conditions of Corollary 5.5, $(\mathcal{A}, \mathcal{A} \cap \mathcal{K})$ is a Silbermann pair, and in this setting a sequence in $\mathcal{A}$ is $(\mathcal{A} \cap \mathcal{K})$-Fredholm if and only if it is Fredholm.

## 5.3 Weights of elementary algebras of sequences

Recall that a projection in a $C^*$-algebra is a self-adjoint element $p$ with $p^2 = p$. Our starting point is a result on liftings of rank one projections. Let $\mathcal{J}$ be an elementary $C^*$-subalgebra of $\mathcal{F}/\mathcal{G}$, i.e., $\mathcal{J}$ is $^*$-isomorphic to the $C^*$-algebra $K(H)$ of the compact operators on a Hilbert space $H$.

**Proposition 5.6** $(a)$ *Every projection $p \in \mathcal{J}$ lifts to a sequence $(\Pi_n) \in \mathcal{F}$ of orthogonal projections, i.e., $(\Pi_n) + \mathcal{G} = p$.*

$(b)$ *If $p$ and $q$ are rank one projections in $\mathcal{J}$ which lift to sequences of projections $(\Pi_n^p)$ and $(\Pi_n^q)$, respectively, then*

$$\dim \operatorname{im} \Pi_n^p = \dim \operatorname{im} \Pi_n^q$$

*for all sufficiently large $n$.*

Thus, the entries of the sequence $(\dim \operatorname{im} \Pi_n^p)_{n \geq 1}$ *for large $n$* are uniquely determined by the algebra $\mathcal{J}$; they do neither depend on the choice of the rank one projection $p$ nor on its lifting.

For a precise formulation, we define an equivalence relation $\sim$ in the set of all sequences of non-negative integers by calling two sequences $(\alpha_n)$, $(\beta_n)$ *equivalent* if $\alpha_n = \beta_n$ for all sufficiently large $n$. Then Proposition 5.6 states that the equivalence class which contains the sequence $(\dim \operatorname{im} \Pi_n^p)_{n \geq 1}$ is uniquely determined by the algebra $\mathcal{J}$. We denote this equivalence class by $\alpha^{\mathcal{J}}$ and call it the *weight* of the elementary algebra $\mathcal{J}$. The algebra $\mathcal{J}$ is said to be an *algebra of positive* *weight* if the equivalence class $\alpha^{\mathcal{J}}$ contains a sequence consisting of positive numbers only, and $\mathcal{J}$ is an *algebra of weight one* if the equivalence class $\alpha^{\mathcal{J}}$ contains the constant sequence $(1, 1, \ldots)$. Note that the weight is *bounded* if $\mathcal{J}$ is in $\mathcal{K}/\mathcal{G}$, since then $(\Pi_n^p)$ is a compact sequence and has finite essential rank.

## 5.4 Silbermann pairs and $\mathcal{J}$-Fredholm sequences

Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair. Being dual, the algebra $\mathcal{J}/\mathcal{G}$ is the direct sum of a family $(I_t)_{t\in T}$ of elementary algebras with associated bijective representations $W_t : I_t \to K(H_t)$. These representations extent to irreducible representations of $\mathcal{A}$ into $L(H_t)$ which we denote by $W_t$ again. In this context, the Lifting theorem 5.4 specifies as follows.

**Theorem 5.7 (Lifting theorem for Silbermann pairs)** *Let $(\mathcal{A}, \mathcal{J})$ form a Silbermann pair.*

*(a) A sequence $\mathbf{A} \in \mathcal{A}$ is stable if and only if it is $\mathcal{J}$-Fredholm and if the operators $W_t(\mathbf{A})$ are invertible for each $t \in T$.*

*(b) The separation property holds, i.e., $W_s(I_t) = \{0\}$ whenever $s \neq t$.*

*(c) If $\mathbf{J} \in \mathcal{J}$, then $W_t(\mathbf{J})$ is a compact operator for every $t \in T$.*

*(d) If the sequence $\mathbf{A} \in \mathcal{A}$ is $\mathcal{J}$-Fredholm, then all operators $W_t(\mathbf{A})$ are Fredholm, and there are at most finitely many of these operators which are not invertible.*

For each $t \in T$, we choose and fix a representative $(\alpha_n^t)$ of the weight $\alpha^{I_t}$ of the elementary ideal $I_t$. Let the sequence $\mathbf{A} := (A_n) \in \mathcal{A}$ be $\mathcal{J}$-Fredholm. Assertion $(c)$ of the Lifting theorem 5.7 implies that the sum

$$\alpha_n(\mathbf{A}) := \sum_{t\in T} \alpha_n^t \dim\ker W_t(\mathbf{A}) \tag{32}$$

is finite. Evidently, this definition depends on the choice of the representatives of the weight functions. But since only a finite number of items in the sum (32) is not zero, the equivalence class of the sequence $(\alpha_n(\mathbf{A}))$ modulo $\sim$ is uniquely determined. Thus, the entries of that sequence are uniquely determined for sufficiently large $n$.

The main result of the present section is the following splitting property of the singular values of a $\mathcal{J}$-Fredholm sequence. The numbers $\sigma_k(A_n)$ with $1 \leq k \leq n$ denote again the increasingly ordered singular values of $A_n$.

**Theorem 5.8** *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair, and let the sequence $\mathbf{A} = (A_n)$ be $\mathcal{J}$-Fredholm. Then $\mathbf{A}$ is a Fredholm sequence, and*

$$\lim_{n\to\infty} \sigma_{\alpha_n(\mathbf{A})}(A_n) = 0 \quad \textit{whereas} \quad \liminf_{n\to\infty} \sigma_{\alpha_n(\mathbf{A})+1}(A_n) > 0. \tag{33}$$

The proof makes use of results on lifting of families of mutually orthogonal projections and on generalized (or Moore-Penrose) invertibility. For details see [21].

Theorem 5.8 has some remarkable consequences. First note that the number

$$\alpha(\mathbf{A}) := \limsup_{n\to\infty} \alpha_n(\mathbf{A}) \tag{34}$$

is well defined and finite for every $\mathcal{J}$-Fredholm sequence $\mathbf{A} \in \mathcal{A}$. Since $(\alpha_n(\mathbf{A}))$ is a sequence of non-negative integers, it possesses a constant subsequence the entries of which are equal to $\alpha(\mathbf{A})$ given by (34). Together with (33), this shows that

$$\liminf_{n\to\infty} \sigma_{\alpha(\mathbf{A})}(A_n) = 0 \quad \text{and} \quad \liminf_{n\to\infty} \sigma_{\alpha(\mathbf{A})+1}(A_n) > 0. \tag{35}$$

**Corollary 5.9** *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair and $\mathbf{A} \in \mathcal{A}$ a $\mathcal{J}$-Fredholm sequence. Then the $\alpha$-number of the Fredholm sequence $\mathbf{A}$ is given by* (34)*.*

Let again $\mathbf{A} = (A_n)$ be $\mathcal{J}$-Fredholm. Evidently, for large $n$, the singular values of $A_n$ are located in the union $[0, \varepsilon_n] \cup [d, \infty)$ where

$$\varepsilon_n := \sigma_{\alpha_n(\mathbf{A})}(A_n) \quad \text{and} \quad d := \liminf_{n\to\infty} \sigma_{\alpha_n(\mathbf{A})+1}(A_n)/2.$$

From (33) one concludes that $\varepsilon_n \to 0$ as $n \to \infty$ and $d > 0$. Thus, the singular values of the entries of $\mathcal{J}$-Fredholm sequence own the splitting property.

Note that the number of the singular values of $A_n$ which lie in $[0, \varepsilon_n]$ depends on $n$ in general (it is just given by the quantity $\alpha_n(\mathbf{A})$ in (32)). A concrete instance where this dependence on $n$ can be observed occurs will be examined in Example 5.13 below. The idea used there allows one to construct Silbermann pairs with arbitrarily prescribed weight sequences $(\alpha_n^t)$. On the other hand, many of the approximation methods used in practice have the property that every rank one projection in $\mathcal{J}/\mathcal{G}$ lifts to a sequence of projections of rank one. Thus, in this case, the numbers $\alpha_n^t$ are independent on $n$ and can be chosen to be 1 for all $n$. For Silbermann pairs with this property, Theorem 5.8 and its Corollary 5.9 specify as follows.

**Corollary 5.10** *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair where all weight sequences $(\alpha_n^t)$ are identically equal to one, and let $\mathbf{A} \in \mathcal{A}$ be a $\mathcal{J}$-Fredholm sequence. Then*

$$\alpha(\mathbf{A}) = \sum_{t\in T} \dim\ker W_t(\mathbf{A}), \tag{36}$$

*and the sequence $\mathbf{A}$ has the $\alpha(\mathbf{A})$-splitting property, i.e., the number of the singular values of $A_n$ which tend to zero is $\alpha(\mathbf{A})$.*

Let us consider a few examples.

**Example 5.11** The simplest Silbermann pairs $(\mathcal{A}, \mathcal{J})$ arise when $\mathcal{J}/\mathcal{G}$ is an elementary algebra. For a concrete model, let $\mathcal{P} = (P_n)$ be a sequence of orthogonal projections of finite rank on a Hilbert space $H$ which converge strongly to the identity operator. Let $\mathcal{F}^{\mathcal{P}}$ denote the $C^*$-algebra of all sequences $\mathbf{A} = (A_n)$ of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ which converge $C^*$-strongly to an operators $W(\mathbf{A})$. The set

$$\mathcal{J}^{\mathcal{P}} := \{(P_n K P_n + G_n) : K \in K(H), \, (G_n) \in \mathcal{G}\}$$

forms a closed ideal of the algebra $\mathcal{F}^{\mathcal{P}}$, and $(\mathcal{F}^{\mathcal{P}}, \mathcal{J}^{\mathcal{P}})$ is a Silbermann pair for which $\mathcal{J}^{\mathcal{P}}/\mathcal{G}^{\mathcal{P}}$ is $^*$-isomorphic to $K(H)$. Moreover, $\mathcal{J}^{\mathcal{P}}$ is an algebra of weight one. In this setting, Theorems 5.7 and 5.8 and Corollary 5.10 specify as follows.

**Corollary 5.12** *Every $\mathcal{J}^{\mathcal{P}}$-Fredholm sequence $\mathbf{A} \in \mathcal{F}^{\mathcal{P}}$ owns the finite splitting property, and its splitting number $\alpha(\mathbf{A})$ is equal to* $\dim \ker W(\mathbf{A})$ *where $W(\mathbf{A})$ refers to the strong limit of the sequence $\mathbf{A}$.*

**Example 5.13** Define sequences $\mathcal{P} = (P_n)$ and $(R_n)$ as in Section 2.3. Consider the set $\mathcal{A}$ of all sequences $\mathbf{A} = (A_n)$ in $\mathcal{F}^{\mathcal{P}}$ for which the strong limits

$$\text{s-lim}\, A_n P_n, \quad \text{s-lim}\, A_n^* P_n, \quad \text{s-lim}\, R_n A_n R_n, \quad \text{s-lim}\, R_n A_n^* R_n$$

exist. We denote the first and third of these strong limits by $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$, respectively. One can straightforwardly check that $\mathcal{A}$ is a $C^*$-subalgebra of $\mathcal{F}^{\mathcal{P}}$, that $W$ and $\widetilde{W}$ are $^*$-homomorphisms on $\mathcal{A}$, and that

$$\mathcal{J} := \{(P_n K P_n + R_n L R_n + G_n) : K, \, L \in K(l^2(\mathbb{Z}^+)), \, (G_n) \in \mathcal{G}^{\mathcal{P}}\}$$

is a closed ideal of $\mathcal{A}$ for which $(\mathcal{A}, \mathcal{J})$ becomes a Silbermann pair. Moreover, the two involved weight sequences can be chosen to be identically one.

The algebra $\mathcal{A}$ contains all sequences $(P_n T(a) P_n)$ of the finite sections of Toeplitz operators $T(a)$ with generating function $a \in L^\infty(\mathbb{T})$. Thus, Corollary 5.10 implies the following result which holds for arbitrary bounded Toeplitz operators.

**Corollary 5.14** *Let $a \in L^\infty(\mathbb{T})$. If the sequence $\mathbf{A} := (P_n T(a) P_n)$ is invertible modulo the ideal $\mathcal{J}$, then $\mathbf{A}$ is a Fredholm sequence with $\alpha$-number*

$$\alpha(\mathbf{A}) = \dim \ker T(a) + \dim \ker T(\tilde{a})$$

*where $\tilde{a}(t) = a(t^{-1})$.*

Note that neither a criterion for the Fredholmness of the sequence $(P_n T(a) P_n)$ with general $a \in L^\infty(\mathbb{T})$ nor an explicit formula for their $\alpha$-number is known. We also do not know anything on the fractality of such sequences. Recall in this connection that Treil constructed an invertible Toeplitz operator for which the finite sections sequence $(P_n T(a) P_n)$ fails to be stable. It is not known if Treil's sequence is Fredholm. ∎

**Example 5.15** Here we construct an example with non-constant weight. Let the operators $P_n$ and $R_n$ be as in Example 5.13, and set $\mathcal{P} = (P_n)_{n\geq 1}$. Consider the smallest closed subalgebra $\mathcal{A}$ of $\mathcal{F}^{\mathcal{P}}$ which contains the identity sequence $(I_n)$, the ideal $\mathcal{G}$ of the zero sequences and all sequences $(K_n)$ of the form

$$K_n := \begin{cases} P_n K P_n & \text{if } n \text{ is odd} \\ P_n K P_n + R_n K R_n & \text{if } n \text{ is even} \end{cases}$$

where $K \in K(l^2(\mathbb{Z}^+))$. One easily checks that if $(A_n)$ is a sequence in $\mathcal{A}$ then every entry $A_n$ is of the form

$$A_n := \begin{cases} \gamma I_n + P_n K P_n + G_n & \text{if } n \text{ is odd} \\ \gamma I_n + P_n K P_n + R_n K R_n + G_n & \text{if } n \text{ is even} \end{cases} \tag{37}$$

where $\gamma \in \mathbb{C}$, $K$ is compact, and $(G_n) \in \mathcal{G}$. Clearly, $\mathcal{A}$ is a unital $C^*$-subalgebra of $\mathcal{F}^{\mathcal{P}}$, and the mapping

$$W : \mathcal{A} \to L(l^2(\mathbb{Z}^+)), \quad (A_n) \mapsto \text{s-lim}\, A_n P_n$$

is a representation of $\mathcal{A}$ which maps the sequence $(A_n)$ given by (37) to the operator $\gamma I + K$. We show that the invertibility of the operator $\gamma I + K$ implies the stability of the sequence $\mathbf{A}$ defined by (37). Indeed, consider the strongly monotonically increasing sequences $\mu, \eta : \mathbb{N} \to \mathbb{N}$ given by $\mu(n) = 2n$ and $\eta(n) = 2n + 1$. By Theorem 2.4, the restricted sequences $\mathbf{A}_\mu$ and $\mathbf{A}_\eta$ belong to the corresponding restricted algebras $\mathcal{S}(\mathsf{T}(C))_\mu$ and $\mathcal{S}(\mathsf{T}(C))_\eta$ of the finite sections method for Toeplitz operators, respectively. Since

$$R_{2n} A_{2n} R_{2n} \to \gamma I + K = W(\mathbf{A}) \quad \text{strongly as } n \to \infty,$$

we conclude from Corollary 2.8 that the invertibility of $W(\mathbf{A}) = \gamma I + K$ implies the stability of both $\mathbf{A}_\mu$ and $\mathbf{A}_\eta$ and, hence of the sequence $\mathbf{A}$.

It is further easy to check that the set $\mathcal{J}$ of all sequences of the form (37) with $\gamma = 0$ forms a closed ideal of $\mathcal{A}$ and that the quotient algebra $\mathcal{J}/\mathcal{G}$ is $^*$-isometric (via $W$) to $K(l^2(\mathbb{Z}^+))$. Set $\Pi_n := P_n P_1 P_n$ if $n$ is odd and $\Pi_n := P_n P_1 P_n + R_n P_1 R_n$ if $n$ is even. Then the sequence $(\Pi_n)$ belongs to $\mathcal{J}$, the coset $p := (\Pi_n) + \mathcal{G}$ is a non-trivial minimal projection in $\mathcal{J}/\mathcal{G}$, and one has

$$\dim \operatorname{im} \Pi_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Thus, the alternating sequence $(1, 2, 1, 2, \ldots)$ is a representative of the (only) weight related with $\mathcal{J}$, and the identity (32) specifies to

$$\alpha_n(\mathbf{A}) = \begin{cases} \dim \ker W_t(\mathbf{A}) & \text{if } n \text{ is odd} \\ 2 \dim \ker W_t(\mathbf{A}) & \text{if } n \text{ is even} \end{cases}$$

(which also could have been verified directly without effort). ∎

## 5.5 Complete Silbermann pairs

Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair. We call this pair *complete* if the ideal $\mathcal{G}$ is properly contained in $\mathcal{J}$ and if the family $\{W_t\}_{t\in T}$ of the lifting homomorphisms of $(\mathcal{A}, \mathcal{J})$ is sufficient in the sense that a sequence $\mathbf{A} \in \mathcal{A}$ is stable if and only if the operators $W_t(\mathbf{A})$ are invertible for every $t \in T$.

**Theorem 5.16** *Let $(\mathcal{A}, \mathcal{J})$ be a complete Silbermann pair and let $\mathbf{A} \in \mathcal{A}$. Then*
*$(a)$ $\mathbf{A}$ is stable if and only if all operators $W_t(\mathbf{A})$ are invertible;*
*$(b)$ $\|\mathbf{A} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \max_{t\in T} \|W_t(\mathbf{A})\|$.*
*$(c)$ $\mathbf{A}$ is $\mathcal{J}$-Fredholm if and only if all operators $W_t(\mathbf{A})$ are Fredholm and if there are only finitely many of them which are not invertible;*
*$(d)$ $\mathbf{A} \in \mathcal{J}$ if and only if all operators $W_t(\mathbf{A})$ are compact and if, for each $\varepsilon > 0$, there are only finitely many of them with $\|W_t(\mathbf{A})\| > \varepsilon$.*

**Proof.** Assertion $(a)$ is a re-formulation of the sufficiency condition. assertion $(b)$ is an immediate consequence of $(a)$ since every homomorphism between between $C^*$-algebras which preserves spectra also preserves spectral radii, hence norms of self-adjoint elements, hence the norm of every element.

$(c)$ The 'only if' part of assertion $(c)$ follows from the Lifting theorem 5.7 $(d)$. Conversely, let $\mathbf{A} \in \mathcal{A}$ be a sequence for which all operators $W_t(\mathbf{A})$ are Fredholm and for which there is a finite subset $T_0$ of $T$ which consists of all $t$ such that $W_t(\mathbf{A})$ is not invertible. Then all operators $W_t(\mathbf{A}^*\mathbf{A})$ are Fredholm, and they are invertible if $t \notin T_0$. Let $t \in T_0$. Then $W_t(\mathbf{A}^*\mathbf{A})$ is a Fredholm operator of index 0. Hence, there is a compact operator $K_t$ such that $W_t(\mathbf{A}^*\mathbf{A}) + K_t$ is invertible. Choose a sequence $\mathbf{K}_t \in \mathcal{J}$ with $W_t(\mathbf{K}_t) = K_t$ and $W_s(\mathbf{K}_t) = 0$ for $s \neq t$ (which is possible by the separation property in Theorem 5.7), and set

$$\mathbf{K} := \sum_{t\in T_0} \mathbf{K}_t.$$

Then $\mathbf{K}$ belongs to the ideal $\mathcal{J}$, and all operators $W_t(\mathbf{A}^*\mathbf{A} + \mathbf{K})$ are invertible. By assertion $(a)$, the sequence $\mathbf{A}^*\mathbf{A} + \mathbf{K}$ is stable. Similarly, one finds a sequence $\mathbf{L} \in \mathcal{J}$ such that $\mathbf{A}\mathbf{A}^* + \mathbf{L}$ is a stable sequence. Consequently, the sequence $\mathbf{A}$ is invertible modulo $\mathcal{J}$, whence the $\mathcal{J}$-Fredholmness of that sequence.

$(d)$ Let now $\mathbf{K}$ be a sequence in $\mathcal{J}$. Since $\mathcal{J}/\mathcal{G}$ is a dual algebra, the 'only if' part follows from the Lifting theorem 5.7 $(c)$. For the 'if' part, let $\mathbf{K} \in \mathcal{A}$ be a sequence such that, for every $\varepsilon > 0$, there are only finitely many $t \in T$ with $\|W_t(\mathbf{A})\| > \varepsilon$. For $n \in \mathbb{N}$, let $T_n$ stand for the (finite) subset of $T$ which collects all $t$ with $\|W_t(\mathbf{K})\| > 1/n$. For each $t \in T_n$, choose a sequence $\mathbf{K}^t \in \mathcal{J}$ with $W_t(\mathbf{K}^t) = W_t(\mathbf{K})$ and $W_s(\mathbf{K}^t) = 0$ for $s \neq t$ (which can be done by the separation property in Theorem 5.7 again), and set $\mathbf{K}_n := \sum_{t\in T_n} \mathbf{K}^t$. Then $W_t(\mathbf{K} - \mathbf{K}_n) = 0$

for $t \in T_n$ and $W_t(\mathbf{K}_n) = 0$ for $t \notin T_n$. Hence,

$$\sup_{t\in T} \|W_t(\mathbf{K} - \mathbf{K}_n)\| \leq 1/n \quad \text{for every } n \in \mathbb{N}.$$

By Theorem 5.16 ($b$), the left-hand side coincides with $\|\mathbf{K} - \mathbf{K}_n + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}}$. Being the norm limit of a sequence in $\mathcal{J}$, the sequence $\mathbf{K}$ belongs to $\mathcal{J}$ itself. ■

An example for a complete Silbermann pair is $(\mathcal{S}(\mathsf{T}(C)), \mathcal{J})$ consisting of the algebra of the finite sections method for Toeplitz operators and its distinguished ideal (21). A sequence in the algebra $\mathcal{S}(\mathsf{T}(C))$ is Fredholm if and only its strong limit is a Fredholm operator (note that $T(a)$ and $T(\tilde{a})$ are Fredholm only simultaneously). Equivalently, the sequence $\mathbf{A} := (P_nT(a)P_n + P_nKP_n + R_nLR_n + G_n)$ with $a \in C(\mathbb{T})$, $K$, $L$ compact and $(G_n) \in \mathcal{G}$ is Fredholm if and only if $T(a)$ is a Fredholm operator. In this case,

$$\alpha(\mathbf{A}) = \dim\ker(T(a) + K) + \dim\ker(T(\tilde{a}) + L). \tag{38}$$

In particular, if $K = L = 0$, then

$$\begin{aligned} \alpha(\mathbf{A}) &= \dim\ker T(a) + \dim\ker T(\tilde{a}) \\ &= \max\{\dim\ker T(a),\, \dim\ker T(\tilde{a})\} \end{aligned}$$

where the second equality holds by a theorem of Coburn which states that one of the quantities $\dim\ker T(a)$ and $\dim\ker T(\tilde{a})$ for each non-zero Toeplitz operator.

In Figures 2 and 4 below, there are plotted the singular values of the Toeplitz matrices $P_nT(a)P_n$ and $P_nT(b)P_n$ with $n$ between 1 and 150 and with

$$a(t) = 5t^{-3} + t^{-2} + 3t^{-1} + 1 + 4t + 7t^2 + t^3$$

and

$$b(t) = 0.7t + t^5$$

respectively. The generating functions $a$ and $b$ have winding numbers 1 and 5 (Figures 1 and 3), and Figures 2 and 4 show exactly the predicted splitting of the singular values. These computations were done by Florian Meyer using standard matlab.

Thus, the singular value splitting is an effect which can be observed numerically. On the other hand, the generating functions $a$ and $b$ are polynomials which is the true reason for the excellent convergence in Figures 2 and 4. A detailed analysis of the speed of the convergence of singular values for band Toeplitz matrices can be found in [4].

The identity (38) remains valid if $A$ is a piecewise continuous function and $(T(a)$ is Fredholm. There is an example due to Tyrtyshnikov of a piecewise continuous function $c$ such that the smallest singular value of $P_nT(c)P_n$ decays to zero as $(\ln n)^{-1}$ (see also [4] for this fact and detailed references).

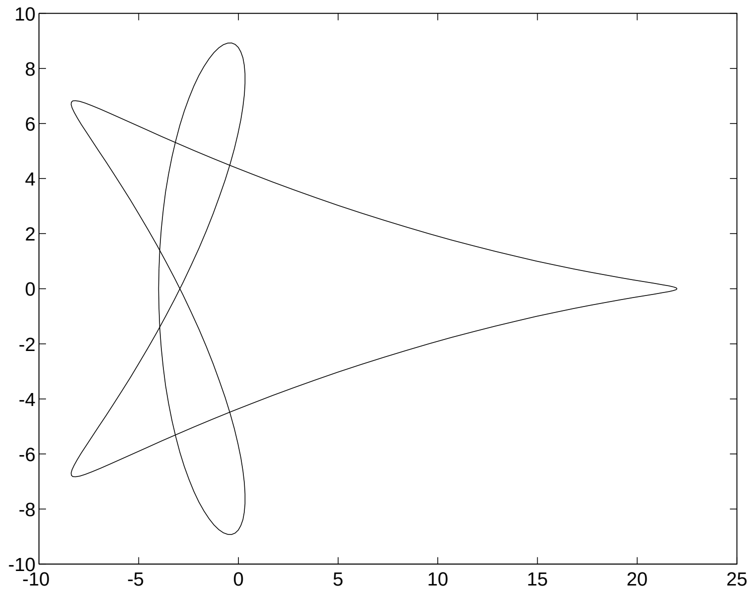



Figure 1: Image of the unit circle under the generating function $a$.

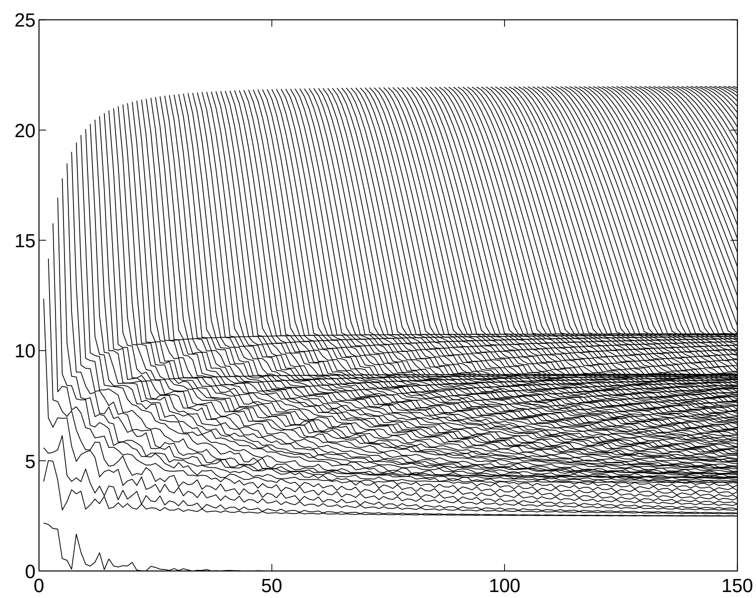



Figure 2: Singular values of $P_nT(a)P_n$ for $n$ between 1 and 150.

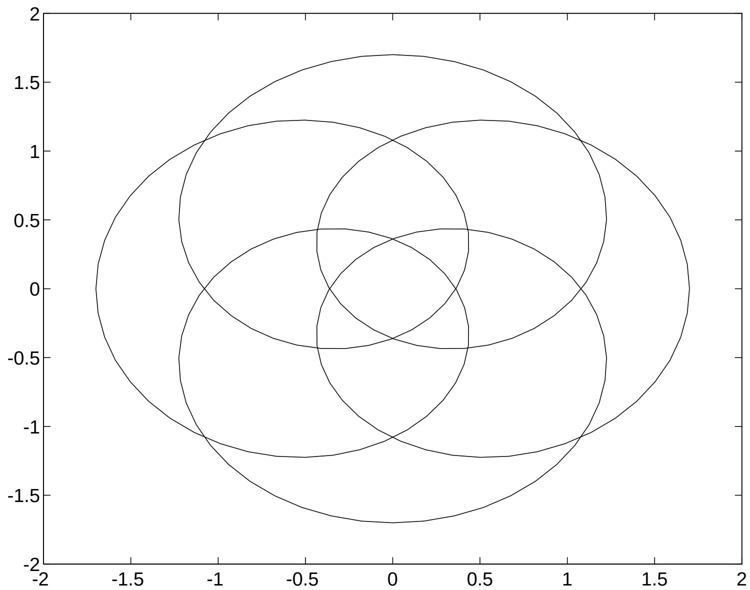



Figure 3: Image of the unit circle under the generating function $b$.

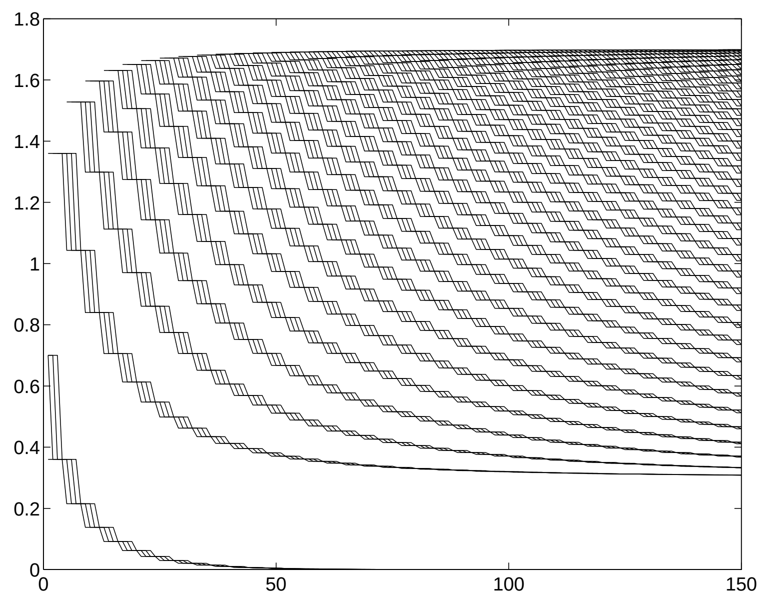



Figure 4: Singular values of $P_n T(b) P_n$ for $n$ between 1 and 150.

# 6 Discretization of band-dominated operators

## 6.1 Band-dominated operators

We provide the Hilbert space $l^2(\mathbb{Z})$ with its standard basis $(e_i)_{i\in\mathbb{Z}}$. An operator $A \in L(l^2(\mathbb{Z}))$ with matrix representation $(a_{ij})$ with respect to the standard basis is called a *band operator* if there is a number $k$ such that $a_{ij} = 0$ whenever $|i-j| > k$. The closure in $L(l^2(\mathbb{Z}))$ of the set of all band operators is a $C^*$-subalgebra of $L(l^2(\mathbb{Z}))$ which we denote by $\mathsf{BDO}(\mathbb{Z})$. The elements of $\mathsf{BDO}(\mathbb{Z})$ are called *band-dominated operators*.

The identity operator is a band operator, and every compact operator is band-dominated (indeed, every compact operator $K$ is the norm limit of the band operators $R_nKR_n$ where $R_n$ refers to the orthogonal projection onto the linear span of $e_{-n}, \ldots, e_{n-1}$). Needless to say that band-dominated operators also occur at many other instances. A typical source of band operators are discretizations and discrete versions of differential operators. For example, the tridiagonal (or *Jacobi*) operator with matrix representation

$$\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & a_{-2} & 1 & & & \\ & 1 & a_{-1} & 1 & & \\ & & 1 & a_0 & 1 & \\ & & & 1 & a_1 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \tag{39}$$

with $a_0$ standing at the 00-position is considered as a *discrete Schrödinger operator* with potential $a = (a_k) \in l^\infty(\mathbb{Z})$. In the special case when $a_k = \lambda \cos 2\pi(\alpha k + \beta)$ with real parameters $\alpha$, $\beta$ and $\lambda$ this operator is known as the *Almost Mathieu operator*.

One can think of $\mathsf{BDO}(\mathbb{Z})$ as a concrete representation of the crossed product $l^\infty(\mathbb{Z}) \times_\alpha \mathbb{Z}$ of the $C^*$-dynamical system $(l^\infty(\mathbb{Z}), \mathbb{Z}, \alpha)$ consisting of the $C^*$-algebra $l^\infty(\mathbb{Z})$, the group $\mathbb{Z}$, and the group homomorphism $\alpha : \mathbb{Z} \to \operatorname{Aut} l^\infty(\mathbb{Z})$, where $\alpha_k$ acts at $a \in l^\infty(\mathbb{Z})$ by $(\alpha_k(a))(n) = a(n-k)$, $n \in \mathbb{Z}$. In the case at hand, the group $\mathbb{Z}$ is commutative; so the crossed product $l^\infty(\mathbb{Z}) \times_\alpha \mathbb{Z}$ coincides with the reduced crossed product $l^\infty(\mathbb{Z}) \times_{\alpha r} \mathbb{Z}$. I will not go into detail here and refer to [11] and other textbooks on $C^*$-algebras for details, since we will work exclusively with the concrete representation $\mathsf{BDO}(\mathbb{Z})$ of $l^\infty(\mathbb{Z}) \times_{\alpha r} \mathbb{Z}$ in what follows.

One should also mention another spectacular arena where band-dominated operators occur, namely coarse geometry. With every coarse geometry space $X$, there is associated a certain $C^*$-algebra $\mathcal{A}(X)$ which exactly reflects the coarse properties of $X$. In case $X = \mathbb{Z}$, the algebra $\mathcal{A}(\mathbb{Z})$ is just the algebra of the band-dominated operators introduced above. For detailed accounts of coarse geometry consult the monographs [14, 25, 26].

Every band operator is constituted by two kinds of special band operators: the shift operators

$$U_k : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (U_k x)(m) := x(m-k)$$

with $k \in \mathbb{Z}$, and the multiplication operators

$$aI : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (ax)(m) := a(m)x(m)$$

with $a \in l^\infty(\mathbb{Z})$. For example, the discrete Schrödinger operator (39) is just the band operator $U_{-1} + aI + U_1$. More general, every finite sum $\sum a_k U_k$ with $a_k \in l^\infty(\mathbb{Z})$ is a band operator and, conversely, every band operator can be uniquely written in this way. The functions $a_k$ are called the coefficients of the operator.

Sometimes we will have to work on $l^2$-spaces over the non-negative integers $\mathbb{Z}^+$ and over the negative integers $\mathbb{Z}^-$. We identify $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z}^-)$ with closed subspaces of $l^2(\mathbb{Z})$ in the obvious way, and we denote the orthogonal projections from $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z}^-)$ by $P$ and $Q$, respectively. Thus, $P + Q = I$ and $PQ = QP = 0$. Given a band-dominated operator $A$ on $l^2(\mathbb{Z})$, we refer to $PAP$ and $QAQ$ as band-dominated operators on $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z}^-)$, respectively. The compression $PU_kP$ of the shift operator $U_k$ to $l^2(\mathbb{Z}^+)$ will be denoted by $V_k$.

There is a Fredholm criterion for a general band-dominated operator $A$ which expresses the Fredholm property of $A$ in terms of the limit operators of $A$. To state this result, we have to introduce a few notations. Let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}$ which tend to infinity in the sense that given $C > 0$, there is an $n_0$ such that $|h(n)| > C$ for all $n \geq n_0$. An operator $A_h \in L(l^2(\mathbb{Z}))$ is called a *limit operator* of $A \in L(l^2(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if $U_{-h(n)} A U_{h(n)}$ tends $^*$-strongly to $A_h$ as $n \to \infty$. Every operator possesses at most one limit operator with respect to a given sequence $h \in \mathcal{H}$. The set $\sigma_{op}(A)$ of all limit operators of a given operator $A$ is the *operator spectrum* of $A$.

Further, an operator $A \in L(l^2(\mathbb{Z}))$ is said to be *rich* or to possess a *rich operator spectrum* if every sequence $h \in \mathcal{H}$ possesses a subsequence $g$ such that the limit operator $A_g$ with respect to $g$ exists. Richness is a compactness property: An operator is rich if and only if the set $\{U_{-n} A U_n : n \in \mathbb{Z}\}$ of all shifts of $A$ is relatively sequentially compact in the $^*$-strong topology. We denote the set of all rich operators on $L(l^2(\mathbb{Z}))$ by $L^\$(l^2(\mathbb{Z}))$. Further we agree upon calling an operator $A$ on $l^2(\mathbb{Z}^+)$ rich if the operator $PAP$ thought of as acting on $L(l^2(\mathbb{Z}))$ is rich. The following is shown by a standard Cantor diagonal argument.

**Proposition 6.1** *$L^\$(l^2(\mathbb{Z}))$ is a $C^*$-subalgebra of $L(l^2(\mathbb{Z}))$ which contains all band-dominated operators.*

It is not hard to see that every limit operator of a compact operator is 0 and that every limit operator of a Fredholm operator is invertible. A basic result of [17] claims that the operator spectrum of a *band-dominated operator* is rich enough in order to guarantee the reverse implications. Here is a summary of the results from [17, 18] needed in what follows.

**Theorem 6.2** *Let* $A \in \mathsf{BDO}(\mathbb{Z})$. *Then*

(*a*) *A is compact if and only if* $\sigma_{op}(A) = \{0\}$.

(*b*) *A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.*

(*c*) *if A is a band operator, then A is Fredholm if and only if each of its limit operators is invertible.*

An elegant proof of Theorem 6.2 which also works for band-dominated operators on discrete groups different from $\mathbb{Z}$ is due to Roe [27].

We turn over to the Fredholm index. Let $A \in \mathsf{BDO}(\mathbb{Z})$. Then the operators $PAQ$ and $QAP$ are compact (they are of finite rank if $A$ is a band operator), and so are the operators $A-(PAP+Q)(P+QAQ)$ and $A-(P+QAQ)(PAP+Q)$. Hence, a band-dominated operator $A$ is Fredholm if and only if both $PAP+Q$ and $P+QAQ$ are Fredholm operators, and the Fredholm index of $A$ is the sum of the Fredholm indices of $PAP+Q$ and $P+QAQ$. We call $\mathrm{ind}_+(A) := \mathrm{ind}\,(PAP+Q)$ and $\mathrm{ind}_-(A) := \mathrm{ind}\,(P+QAQ)$ the *plus-index* and the *minus-index* of $A$. Evidently,

$$\mathrm{ind}\, A = \mathrm{ind}_+(A) + \mathrm{ind}_-(A) \tag{40}$$

for every Fredholm band-dominated operator $A$. Notice further that the operator spectrum of $A$ splits into $\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$ where $\sigma_+(A)$ and $\sigma_-(A)$ collect the limit operators of $A$ which correspond to sequences tending to $+\infty$ and to $-\infty$, respectively.

**Theorem 6.3** *Let* $A \in L(l^2(\mathbb{Z}))$ *be a Fredholm band-dominated operator. Then*

(*a*) *for all* $B_+ \in \sigma_+(A)$ *and* $B_- \in \sigma_-(A)$,

$$\mathrm{ind}_+(B_+) = \mathrm{ind}_+(A) \quad \textit{and} \quad \mathrm{ind}_-(B_-) = \mathrm{ind}_-(A).$$

(*b*) *all operators in* $\sigma_+(A)$ *have the same plus-index, and all operators in* $\sigma_-(A)$ *have the same minus-index.*

(*c*) *for arbitrarily chosen operators* $B_+ \in \sigma_+(A)$ *and* $B_- \in \sigma_-(A)$,

$$\mathrm{ind}\, A = \mathrm{ind}_+(B_+) + \mathrm{ind}_-(B_-). \tag{41}$$

If $A$ is Fredholm, then all operators in $\sigma_+(A)$ also have the same minus-index, which follows from (*b*), from the invertibility of all limit operators of $A$, and from (40). It is also evident that (*b*) and (*c*) are immediate consequences of (*a*).

A natural approach to the Fredholm index of an operator is via $K$-theory for $C^*$-algebras. Indeed, the original proof of Theorem 6.3 in [16] is heavily based upon calculations of the $K$-groups of the $C^*$-algebra $\mathsf{BDO}(\mathbb{Z})$ and of its related ideals. Below we will present a completely different proof of the index formula (41) which is exclusively based on ideas and results from asymptotic numerical analysis.

## 6.2 Stability of the finite sections method for BDO

As before, let $\mathcal{F}$ denote the algebra of matrix sequences and $\mathcal{G}$ the ideal of all sequences in $\mathcal{F}$ which tend to zero in the norm. We associate to each sequence $\mathbf{A} = (A_n) \in \mathcal{F}$ the block diagonal operator

$$\mathrm{Op}\,(\mathbf{A}) := \mathrm{diag}\,(A_1,\ A_2,\ A_3,\ \ldots) \tag{42}$$

considered as acting on $l^2(\mathbb{N}) = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \ldots$. The mapping Op implies a $^*$-homomorphic embedding of the algebra $\mathcal{F}$ into $L(l^2(\mathbb{N}))$, which allows one to think of $\mathcal{F}$ as a closed $^*$-subalgebra of $L(l^2(\mathbb{N}))$. Moreover, $\mathrm{Op}\,(\mathcal{F}) \cap K(l^2(\mathbb{N})) = \mathrm{Op}\,(\mathcal{G})$.

**Theorem 6.4** *A sequence $\mathbf{A} \in \mathcal{F}$ is stable if and only if the operator $\mathrm{Op}\,(\mathbf{A}) \in L(l^2(\mathbb{N}))$ is Fredholm.*

**Proof.** Let $\mathbf{A} \in \mathcal{F}$ be a sequence for which $\mathrm{Op}\,(\mathbf{A})$ is a Fredholm operator, i.e., the coset $\mathrm{Op}\,(\mathbf{A}) + K(l^2(\mathbb{N}))$ is invertible in the Calkin algebra $L(l^2(\mathbb{N}))/K(l^2(\mathbb{N}))$. Then this coset is already invertible in $(\mathrm{Op}\,(\mathcal{F}) + K(l^2(\mathbb{N})))/K(l^2(\mathbb{N}))$ due to inverse closedness of $C^*$-algebras. The canonical isomorphisms

$$\begin{aligned}&(\mathrm{Op}\,(\mathcal{F}) + K(l^2(\mathbb{N})))/K(l^2(\mathbb{N}))\\ &\qquad \cong \mathrm{Op}\,(\mathcal{F})/(\mathrm{Op}\,(\mathcal{F}) \cap K(l^2(\mathbb{N}))) \cong \mathrm{Op}\,(\mathcal{F})/\mathrm{Op}\,(\mathcal{G}) \cong \mathcal{F}/\mathcal{G}\end{aligned}$$

imply that the coset $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$. Thus, the sequence $\mathbf{A}$ is stable.

Conversely, let $\mathbf{A}$ be a stable sequence. Choose $n_0 \in \mathbb{N}$ such that the matrices $A_n$ are invertible for $n \geq n_0$ and consider the sequence $\mathbf{B} = (B_n)$ with $B_n := 0$ for $n < n_0$ and $B_n = A_n^{-1}$ for $n \geq n_0$. The operator $\mathrm{Op}\,(\mathbf{B})$ is a two-sided inverse of $\mathrm{Op}\,(\mathbf{A})$ modulo compact operators. Hence, $\mathrm{Op}\,(\mathbf{A})$ is a Fredholm operator. ∎

In general, the stability criterion stated in Theorem 6.4 seems to be of less use. But if one starts with the sequence $\mathbf{A} = (P_n A P_n)$ of the finite sections of a band-dominated operator $A$, then one ends up with the band-dominated operator $\mathrm{Op}\,(\mathbf{A})$, and Theorem 6.2 applies to study the Fredholmness of $\mathrm{Op}\,(\mathbf{A})$. Basically, one has to compute the limit operators of $\mathrm{Op}\,(\mathbf{A})$. This will be done in the following theorem in the more general context of rich operators.

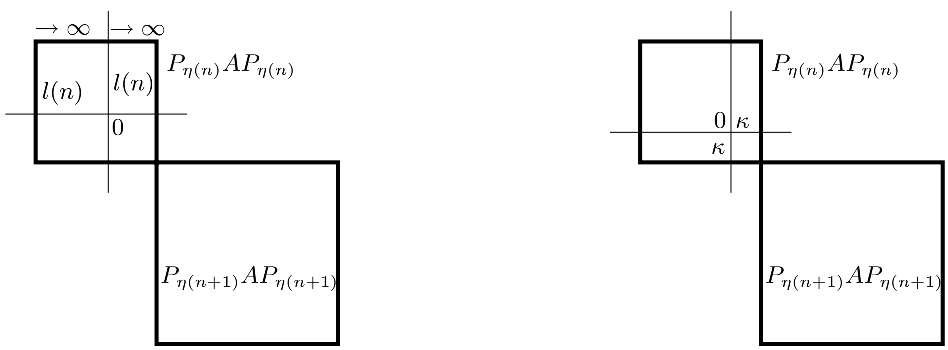
Note that simple examples show that the sequence $(P_n A P_n)$ of the finite sections of a band-dominated operator $A$ is not fractal in general. It will be therefore of vital importance to examine not only the stability of the sequence $(P_n A P_n)$ itself, but also of each of its infinite subsequences. Thus, we choose and fix a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$. Further, we write $\mathcal{H}_\eta$ for the set of all (infinite) subsequences of $\eta$ and $\sigma_{op,\eta}(A)$ for the collection of all limit operators of $A$ with respect to subsequences of $\eta$.

**Theorem 6.5** *Let $A \in L(l^2(\mathbb{Z}^+))$ be a rich operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence, and set $\mathbf{A}_\eta := (P_{\eta(n)} A P_{\eta(n)})_{n \ge 1}$. Then the operator $\text{Op}\,(\mathbf{A}_\eta)$ is rich, and its operator spectrum $\sigma_{op}(\text{Op}\,(\mathbf{A}_\eta))$ is equal to*

$$\sigma_+(A) \cup \{U_{-\kappa}(QA_hQ + PAP)U_\kappa : \kappa \in \mathbb{Z},\, A_h \in \sigma_{op,\eta}(A)\}. \tag{43}$$

Note that only limit operators of $A$ with respect to subsequences of $\eta$ appear.

**Proof.** Let $h : \mathbb{N} \to \mathbb{N}$ be a sequence which tends to infinity. We call numbers of the form $\eta(1) + \eta(2) + \ldots + \eta(n)$ $\eta$-triangular and distinguish between two cases: Either there is a subsequence $g$ of $h$ such that the distance from $g(n)$ to the set of all $\eta$-triangular numbers tends to infinity as $n \to \infty$, or there are a $\kappa \in \mathbb{Z}$ and a subsequence $g$ of $h$ such that $g(n) + \kappa$ is $\eta$-triangular for all $n$. The figures below illustrate the shifted operator $U_{-g(n)}\text{Op}\,(\mathbf{A}_\eta)U_{g(n)}$ in the neighborhood of its 00-entry (marked by 0).




Case 1 Case 2

In the first case, we let $\Delta_n$ denote the largest $\eta$-triangular number which is less than $g(n)$. Then the sequence $l$ defined by $l(n) := g(n) - \Delta_n$ still tends to infinity. Since $A$ is rich, there is a strongly monotonically increasing sequence $k : \mathbb{N} \to \mathbb{N}$ such that the limit operator $A_{l \circ k}$ of $A$ exists. Then $g \circ k$ is a subsequence of $h$ for which the limit operator $\text{Op}\,(\mathbf{A}_\eta)_{g \circ k}$ exists, and this limit operator coincides with the limit operator $A_{l \circ k}$ of $A$.

Let now $g$ be a subsequence of $h$ and $\kappa$ an integer such that each $g(n) + \kappa$ is $\eta$-triangular. Let $d(n)$ be the (uniquely determined) positive integer such that

$$g(n) + \kappa = \eta(1) + \eta(2) + \ldots + \eta(d(n)).$$

Since $g$ is strongly monotonically increasing, the sequence $d$ is strongly monotonically increasing, too. Thus, the sequence $\eta \circ d$ is a subsequence of $\eta$ and tends to infinity. Since $A$ is rich, there is a strongly monotonically increasing sequence $k : \mathbb{N} \to \mathbb{N}$ such that the limit operator $A_{\eta \circ d \circ k}$ of $A$ with respect to the subsequence $\eta \circ d \circ k$ of $\eta$ exists. It is now clear that the limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$ with respect to the sequence $(g \circ k) + \kappa$ exists and that

$$\mathrm{Op}\,(\mathbf{A}_\eta)_{(g \circ k)+\kappa} = QA_{\eta \circ d \circ k}Q + PAP.$$

This fact finally implies that the limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$ with respect to the subsequence $g \circ k$ of $h$ exists and that

$$\begin{aligned} \mathrm{Op}\,(\mathbf{A}_\eta)_{g \circ k} &= U_\kappa \mathrm{Op}\,(\mathbf{A}_\eta)_{(g \circ k)+\kappa} U_{-\kappa} \\ &= U_\kappa (QA_{\eta \circ d \circ k}Q + PAP) U_{-\kappa}. \end{aligned}$$

Thus, the operator $\mathrm{Op}\,(\mathbf{A}_\eta)$ is rich, and every limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$ is either a limit operator of $A$ or of the form

$$U_\kappa (QA_{\eta \circ d}Q + PAP) U_{-\kappa} \quad \text{with } \kappa \in \mathbb{Z} \text{ and } A_{\eta \circ d} \in \sigma_{op,\eta}(A). \tag{44}$$

Conversely, we are now going to show that each limit operator of $A$ and each operator of the form (44) appears as a limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$. Let $A_l$ be a limit operator of $A$ with respect to a sequence $l \in \mathcal{H}$. Choose a strongly monotonically increasing sequence $d : \mathbb{N} \to \mathbb{N}$ such that $\eta(d(n) + 1) - l(n) \to \infty$ and set

$$h(n) := (\eta(1) + \eta(2) + \ldots + \eta(d(n))) + l(n).$$

Then $h \in \mathcal{H}$, the limit operator $\mathrm{Op}\,(\mathbf{A}_\eta)_h$ exists, and this limit operator is equal to $A_l$. Let now $d : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence such that the limit operator $A_{\eta \circ d}$ of $A$ exists, and let $\kappa \in \mathbb{Z}$. Consider

$$h(n) := (\eta(1) + \eta(2) + \ldots + \eta(d(n))) + \kappa.$$

Again, $h \in \mathcal{H}$, the limit operator $\mathrm{Op}\,(\mathbf{A}_\eta)_h$ exists, but now this limit operator is $U_{-\kappa}(QA_{\eta \circ d}Q + PAP)U_\kappa$. Thus, the operator spectrum of $\mathrm{Op}\,(\mathbf{A}_\eta)$ coincides with (43). ∎

Now we combine Theorems 6.2, 6.4 and 6.5 to obtain a stability result for the finite sections method for band-dominated operators. In order to avoid operators acting on the negative integers, it is convenient to introduce the flip operator $J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, $(Jx)_i := x_{-i-1}$. Note that the operators $JQA_hQJ$ on $l^2(\mathbb{Z}^+)$ and $U_{-\kappa}(QA_hQ + P)U_\kappa$ on $l^2(\mathbb{Z})$ are invertible only simultaneously and that the norms of their inverses can differ at most by 1.

**Theorem 6.6** *Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)} A P_{\eta(n)})$ is stable if and only if the operator $A$ and all operators*

$$JQA_hQJ \quad \textit{with} \quad A_h \in \sigma_{op,\eta}(A)$$

*are invertible on $l^2(\mathbb{Z}^+)$ and if the norms of their inverses are uniformly bounded.*

**Corollary 6.7** *Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator, and let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence for which the limit operator $A_\eta$ exists. Then the sequence $(P_{\eta(n)} A P_{\eta(n)})_{n \geq 1}$ is stable if and only if the operators $A$ and $JQA_\eta QJ$ are invertible on $l^2(\mathbb{Z}^+)$.*

Indeed, under the conditions of the corollary, the set $\sigma_{op,\eta}(A)$ is a singleton. ■

The main result of the present section is the following. It says that the (in concrete situations quite nasty) condition of the uniform boundedness in Theorem 6.6 is redundant. This result came as a big surprise since it is still an open question whether the uniform boundedness condition in Theorem 6.2 ($c$) is redundant.

**Theorem 6.8** *Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)} A P_{\eta(n)})$ is stable if and only if the operator $A$ and all operators*

$$JQA_hQJ \quad \textit{with} \quad A_h \in \sigma_{op,\eta}(A)$$

*are invertible on $l^2(\mathbb{Z}^+)$.*

**Proof.** The necessity of invertibility of the mentioned operators follows from Theorem 6.6. Conversely, let $A$ and all operators $JQA_hQJ$ with $A_h \in \sigma_{op,\eta}(A)$ be invertible on $l^2(\mathbb{Z}^+)$. Contrary to what we want to show, assume that the sequence $\mathbf{A}_\eta = (P_{\eta(n)} A P_{\eta(n)})$ fails to be stable. Then there is a strongly monotonically increasing sequence $d : \mathbb{N} \to \mathbb{N}$ such that

$$\|(P_{\eta(d(n))} A P_{\eta(d(n))})^{-1}\| \geq n \quad \text{for all } n \in \mathbb{N}$$

where we agree upon writing $\|A_n^{-1}\| = \infty$ if the matrix $A_n$ fails to be invertible. Thus, no subsequence of the sequence $\mathbf{A}_{\eta \circ d}$ is stable.

Let $g$ be a subsequence of $\eta \circ d$ for which the limit operator $A_g$ exists. (The existence of a sequence $d$ with these properties follows from Theorem 6.2 ($a$).) Then $A_g \in \sigma_{op,\eta}(A)$, and the operators $A$ and $JQA_gQJ$ are invertible on $l^2(\mathbb{Z}^+)$ by hypothesis. Corollary 6.7 implies the stability of the subsequence $\mathbf{A}_g$ of $\mathbf{A}_{\eta \circ d}$. Contradiction. ■

The $l^2(\mathbb{Z})$-versions of the previous results can be proved in the same vein. Here we consider the orthogonal projections

$$P_n^{\mathbb{Z}} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (P_n^{\mathbb{Z}} x)(m) := \begin{cases} x(m) & \text{if } -n \leq m < n \\ 0 & \text{otherwise} \end{cases}$$

and are interested in the stability of the finite sections sequences $(P_n^{\mathbb{Z}} A P_n^{\mathbb{Z}})$ when $A$ is an operator on $l^2(\mathbb{Z})$.

**Theorem 6.9** *Let $A \in L(l^2(\mathbb{Z}))$ be a band-dominated operator. Then the finite sections sequence $(P_n^{\mathbb{Z}} A P_n^{\mathbb{Z}})_{n\geq 1}$ is stable if and only if the operator $A$, all operators*

$$QA_hQ + P \quad \textit{with} \quad A_h \in \sigma_+(A)$$

*and all operators*

$$PA_hP + Q \quad \textit{with} \quad A_h \in \sigma_-(A)$$

*are invertible on $l^2(\mathbb{Z})$.*

The results in this section remain valid for operators on $l^p$ with $1 < p < \infty$. We would further like to mention that the stability of the finite sections sequence for band-dominated operators on $l^\infty$ can be studied as well. This involves some technical subtleties (when working with adjoint sequences, for instance), but it is easier with respect to one main concern of the present section: For $p = \infty$, already the uniform boundedness condition in Theorem 6.2 $(b)$ is redundant. For much more on this topic, consult the textbook [15].

## 6.3 The $C^*$-algebra of the finite sections method

The goal of this section is to examine the smallest closed subalgebra $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ of $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $(P_nAP_n)$ with a band-dominated operator $A \in \mathsf{BDO}(\mathbb{N})$. The results hold - with obvious modifications - for the algebra $\mathcal{S}(\mathsf{BDO}(\mathbb{Z}))$ as well.

Our first goal is a stability criterion for sequences in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$. For each sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ and each strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, we write $\sigma_{op,\eta}(\mathbf{A})$ for the set of all $^*$-strong limits of subsequences of the sequence $(U_{-\eta(n)}A_{\eta(n)}U_{\eta(n)})_{n\geq 1}$. Further we let $W$ denote the homomorphism which associates with every sequence in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ its strong limit. The following theorem can be proved in the same way as its predecessor Theorem 6.8.

**Theorem 6.10** *Let $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $\mathbf{A}_\eta := (A_{\eta(n)})$ is stable if and only if the operator $W(\mathbf{A})$ and all operators in $\sigma_{op,\eta}(\mathbf{A})$ are invertible.*

We know from Proposition 2.2 that the algebra $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ splits into the direct sum of the linear space $\{(P_nAP_n) : A \in \mathsf{BDO}(\mathbb{N})\}$ and the quasicommutator ideal $\mathcal{J}(\mathsf{BDO}(\mathbb{N}))$ and that the latter is just the kernel of the (restriction of the) homomorphism $W$. We would like to mention a further characterization of sequences in $\mathcal{J}(\mathsf{BDO}(\mathbb{N}))$ which will be needed in what follows.

**Theorem 6.11** *The following conditions are equivalent for a sequence* $(K_n) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))$:

*(a)* $(K_n) \in \mathcal{J}(\mathsf{BDO}(\mathbb{N}))$*;*

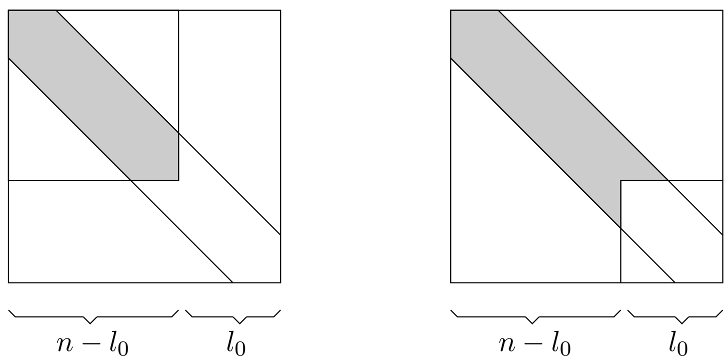
*(b)* $(K_n)$ *tends strongly to zero;*

*(c)* $(K_n)$ *is localized at the right-hand end of the interval* $\{0, 1, \ldots, n-1\}$ *in the sense that, given* $\varepsilon > 0$*, there are non-negative integers* $n_0$ *and* $l_0$ *such that*

$$\sup_{n \geq n_0} \|P_{n-l_0} K_n P_{n-l_0}\| < \varepsilon. \tag{45}$$

*(d)* $(K_n)$ *is localized at the right-hand end of the interval* $\{0, 1, \ldots, n-1\}$ *in the sense that, given* $\varepsilon > 0$*, there are non-negative integers* $n_0$ *and* $l_0$ *such that*

$$\sup_{n \geq n_0} \|K_n - (P_n - P_{n-l_0}) K_n (P_n - P_{n-l_0})\| < \varepsilon. \tag{46}$$




$P_{n-l_0} K_n P_{n-l_0}$ $\qquad$ $K_n - (P_n - P_{n-l_0}) K_n (P_n - P_{n-l_0})$

The proof can be found in [22]. Let me only mention here that the localization effect is well known for sequences in the quasicommutator ideal of the algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections method for Toeplitz operators (note that the latter can be viewed as band-dominated operators with constant coefficients). Indeed, in this case a sequence is in $\mathcal{J}(\mathsf{T}(C))$ is and only if it is of the form $(R_n L R_n) + (G_n)$ with $L$ compact and $(G_n) \in \mathcal{G}$. Approximate $L$ by an operator of the form $P_{n_0} L P_{n_0}$ as closely as desired. Then $(R_n L R_n) + (G_n)$ can be written as the sum of the "small" sequence $(R_n(L - P_{n_0} L P_{n_0}) R_n) + (G_n)$ and the sequence $(R_n(P_{n_0} L P_{n_0}) R_n)$ each entry of which has its non-vanishing entries in the right lower corner.

## 6.4 Compact and Fredholm sequences in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$

Here we are going to look at compact sequences and Fredholm sequences in the algebras $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$. Again, all results will hold with evident modifications for the algebra $\mathcal{S}(\mathsf{BDO}(\mathbb{Z}))$ as well. Recall that $\mathcal{K}$ refers to the ideal of all compact sequences in the algebra $\mathcal{F}$.

**Theorem 6.12** *The intersection* $\mathcal{S}(\mathsf{BDO}(\mathbb{N})) \cap \mathcal{K}$ *consists of all sequences* $(A_n) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ *which converge* $^*$*-strongly to a compact operator. Equivalently, a sequence in* $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ *is compact if and only if it is of the form* $(P_nKP_n)+(K_n)$ *with a compact operator* $K$ *and a sequence* $(K_n) \in \mathcal{J}(\mathsf{BDO}(\mathbb{N}))$.

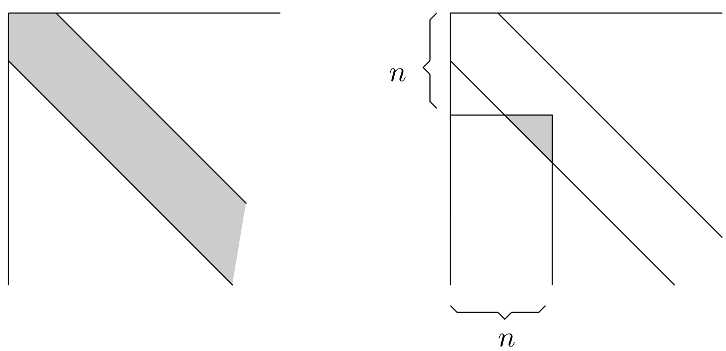
**Proof.** It is a general fact that if a sequence $(K_n) \in \mathcal{K}$ converges $^*$-strongly, then its strong limit is compact (cp. Theorem 4.8). Conversely, if $(A_n) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ is a sequence with compact strong limit $K$ then, by the direct sum decomposition,

$$(A_n) = (P_nKP_n) + (K_n) \quad \text{with} \quad (K_n) \in \mathcal{J}(\mathsf{BDO}(\mathbb{N})).$$

Evidently, the sequence $(P_nKP_n)$ is compact. It remains to verify the compactness of the sequences in the quasicommutator ideal $\mathcal{J}(\mathsf{BDO}(\mathbb{N}))$. Let $A$ and $B$ be band operators, and let $r \geq 0$ be the band width of $B$, i.e., the entries $b_{ij}$ in the matrix representation of $B$ with respect to the standard basis vanish for $|i-j| > r$. We have to verify the compactness of

$$(P_nABP_n) - (P_nAP_n)(P_nBP_n) = (P_nAQ_nBP_n).$$

Looking at the matrix representation of $Q_nBP_n$, one easily realizes that this operator has rank at most $r$ for each $n \in \mathbb{N}$. Thus, the sequence $(P_nAQ_nBP_n)$ is of essential rank not greater than $r$, whence its compactness. ■




$B$ $\qquad$ $Q_n\,B\,P_n$

The following result determines the essential rank of a compact sequence in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ in terms of the ranks of its limit operators. For each sequence $\mathbf{A} := (A_n)$ in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ and each strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, we define the *operator spectrum* $\sigma_{op,\,\eta}(\mathbf{A})$ as the set of all $^*$-strong limits of subsequences of the sequence $(U_{-\eta(n)}A_{\eta(n)}U_{\eta(n)})_{n\geq 1}$. In case $\mathbf{A}$ is a compact sequence in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$, all operators in $\sigma_{op,\,\eta}(\mathbf{A})$ are compact.

**Theorem 6.13** *Let $K \in l^2(\mathbb{N})$ be a compact operator and let* $\mathbf{K} := (K_n) \in \mathcal{J}(\mathsf{BDO}(\mathbb{N}))$. *Then the sequence* $(P_{\eta(n)} K P_{\eta(n)} + K_{\eta(n)}) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))_\eta$ *is compact, and*

$$\begin{aligned} &\operatorname{ess\,rank}(P_{\eta(n)} K P_{\eta(n)} + K_{\eta(n)}) \\ &\quad = \operatorname{rank} K + \max\{\operatorname{rank} \mathbf{K}_h : \mathbf{K}_h \in \sigma_{op,\eta}(\mathbf{K})\}. \end{aligned} \tag{47}$$

We split the proof into several lemmas which we formulate and prove only for the identity sequence $\eta(n) = n$.

**Lemma 6.14** *Let $K \in L(l^2(\mathbb{N}))$ be compact and $(K_n) \in \mathcal{J}(\mathsf{BDO}(\mathbb{N}))$. Then*

$$\operatorname{ess\,rank}(P_n K P_n + K_n) = \operatorname{ess\,rank}(P_n K P_n) + \operatorname{ess\,rank}(K_n). \tag{48}$$

The idea of the proof is as follows. We infer from Theorem 6.11 $(d)$ that, up to a small sequence, the matrices $K_n$ are localized at the right-hand end of the interval $\{0, 1, \ldots, n-1\}$ whereas the $P_n K P_n$ are localized at the left-hand end of that interval. Thus, the non-vanishing entries of these matrices sit in different corners, which implies that the rank of their sum is the sum of their ranks. For details, see [22]. ■

**Lemma 6.15** *If $K \in L(l^2(\mathbb{N}))$ is compact, then $(P_n K P_n)$ is a compact sequence, and* $\operatorname{ess\,rank}(P_n K P_n) = \operatorname{rank} K$.

Indeed, this is easy to see. The next lemma completes the proof of Theorem 6.13.

**Lemma 6.16** *Let* $\mathbf{K} := (K_n) \in \mathcal{J}(\mathsf{BDO}(\mathbb{N}))$. *Then*

$$\operatorname{ess\,rank} \mathbf{K} = \max\{\operatorname{rank} \mathbf{K}_h : \mathbf{K}_h \in \sigma_{op,1}(\mathbf{K})\}. \tag{49}$$

**Proof.** Let $\operatorname{ess\,rank} \mathbf{K} =: r$. One can show that the strong limit of operators of rank $r$ has rank at most $r$. This settles the estimate $\operatorname{rank} \mathbf{K}_h \le r$ for each limit operator $\mathbf{K}_h \in \sigma_{op,1}(\mathbf{K})$. It remains to establish the existence of a limit operator of $\mathbf{K}$ for which $\operatorname{rank} \mathbf{K}_h \ge r$.

By Theorem 6.11 $(d)$ and Corollary 4.5 (lower semi-continuity of the essential rank), there is an $n_0$ such that the sequence $\mathbf{K}^{(1)} = (K_n^{(1)})$ with

$$K_n^{(1)} := (P_n - P_{n-n_0}) K_n (P_n - P_{n-n_0})$$

has essential rank $l \ge r$. In fact $\mathbf{K}^{(1)}$ has essential rank $r$ since it is a product of the essential rank $r$ sequence $\mathbf{K}$ by other sequences. Moreover, we can choose $n_0 > r$. Since $\mathbf{K}^{(1)}$ has essential rank $r$, there is a monotonically increasing sequence $h : \mathbb{N} \to \mathbb{N}$ such that

$$\Sigma_r(K_{h(n)}^{(1)}) > C > 0 \quad \text{for all } n \in \mathbb{N} \tag{50}$$

whereas

$$\Sigma_{r+1}(K_n^{(1)}) \to 0 \quad \text{as } n \to \infty. \tag{51}$$

Let $K_n^{(1)} = E_n^* \operatorname{diag}(\Sigma_1(K_n^{(1)}), \ldots, \Sigma_n(K_n^{(1)}))F_n$ be the singular value decomposition of $K_n^{(1)}$ which can be chosen in such a way that $P_{n-n_0}E_n^*P_rE_n = 0$ and $P_{n-n_0}F_n^*P_rF_n = 0$. Write $K_n^{(1)}$ as $K_n^{(2)} + K_n^{(3)}$ with

$$K_n^{(2)} := E_n^*P_rE_nK_n^{(1)}F_n^*P_rF_n.$$

Then $P_{n-n_0}K_n^{(2)} = K_n^{(2)}P_{n-n_0} = 0$ for all $n$ and $\lim_{n\to\infty} \|K_n^{(3)}\| = 0$ by (51).

Consider the sequence $(U_{-h(n)}K_{h(n)}U_{h(n)})_{n\in\mathbb{N}}$. One can assume that this sequence tends $^*$-strongly to a limit operator $\mathbf{K}_h$ of the sequence $\mathbf{K}$ (otherwise pass to a suitable subsequence of $h$). Then, since $P_{n_0}$ is compact, the sequence $(P_{n_0}JU_{-h(n)}K_{h(n)}U_{h(n)}JP_{n_0})$ tends in the operator norm to $P_{n_0}J\mathbf{K}_hJP_{n_0}$. The entries of that sequence can be written as

$$\begin{aligned}
&P_{n_0}JU_{-h(n)}(P_{h(n)} - P_{h(n)-n_0})K_{h(n)}(P_{h(n)} - P_{h(n)-n_0})U_{h(n)}JP_{n_0}\\
&\quad = P_{n_0}JU_{-h(n)}K_{h(n)}^{(1)}U_{h(n)}JP_{n_0} = P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0} + G_n
\end{aligned}$$

with a sequence $(G_n)$ tending to zero in the norm. Consequently,

$$P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0} \to P_{n_0}J\mathbf{K}_hJP_{n_0} \quad \text{in the norm as } n \to \infty.$$

Write the operators on the left hand side as

$$\begin{aligned}
&P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0}\\
&\quad = P_{n_0}JU_{-h(n)}E_{h(n)}^*P_rE_{h(n)}K_{h(n)}^{(2)}F_{h(n)}^*P_rF_{h(n)}U_{h(n)}JP_{n_0}\\
&\quad = P_{n_0}JU_{-h(n)}E_{h(n)}^*P_rE_{h(n)}U_{h(n)}JP_{n_0}\\
&\qquad \times P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0}\\
&\qquad \times P_{n_0}JU_{-h(n)}F_{h(n)}^*P_rF_{h(n)}U_{h(n)}JP_{n_0}
\end{aligned}$$

(recall that the matrices $E_n^*P_rE_n$ and $F_n^*P_rF_n$ both commute with $P_n - P_{n-n_0}$), and abbreviate the operators

$$P_{n_0}JU_{-h(n)}E_{h(n)}^*P_rE_{h(n)}U_{h(n)}JP_{n_0} \quad \text{and} \quad P_{n_0}JU_{-h(n)}F_{h(n)}^*P_rF_{h(n)}U_{h(n)}JP_{n_0}$$

to $P_n^{E,r}$ and $P_n^{F,r}$, respectively. Each of the operators $P_n^{E,r}$ is an orthogonal projection which acts of the range of $P_{n_0}$, i.e., on a finite dimensional space with a dimension independent of $n$. Thus, one can choose a subsequence of $(P_n^{E,r})$ which converges in the norm to an operator $P^{E,r}$. For simplicity we will assume that the sequence $(P_n^{E,r})$ itself enjoys this property. Then $P^{E,r}$ is an orthogonal projection, and being the norm limit of orthogonal projections of rank $r$, the rank of $P^{E,r}$ is also $r$. Similarly, one can assume that the sequence $(P_n^{F,r})$ tends in the norm to an orthogonal projection $P^{F,r}$ of rank $r$. Thus,

$$P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0} \to P^{E,r}P_{n_0}J\mathbf{K}_hJP_{n_0}P^{F,r} \quad \text{in the norm as } n \to \infty.$$

Moreover, by the definition of $K_{h(n)}^{(2)}$, each operator

$$P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0} = P_n^{E,r}P_{n_0}JU_{-h(n)}K_{h(n)}^{(2)}U_{h(n)}JP_{n_0}P_n^{F,r}, \tag{52}$$

thought of as acting from $\operatorname{im} P_n^{F,r}$ to $\operatorname{im} P_n^{E,r}$ is invertible, and the norm of its inverse is $\Sigma_n(K_{h(n)}^{(1)})^{-1}$, which is less than $1/C$ due to (50). Thus, the inverses of the operators (52) are uniformly bounded. This implies the norm convergence of these inverses from which one concludes that the norm limit of the operators (52), i.e. the operator $P^{E,r}P_{n_0}J\mathbf{K}_hJP_{n_0}P^{F,r}$, is invertible when considered as acting from $\operatorname{im} P^{F,r}$ to $\operatorname{im} P^{E,r}$. Since both ranges have dimension $r$, this implies that the rank of $\mathbf{K}_h$ is at least $r$. ∎

In order to get an identity for the essential rank of a compact sequence in $\mathcal{S}(\mathbb{Z})$ in terms of limit operators, one has to guarantee that both ends of the interval $\{-n, -n+1, \ldots, n-1\}$ are treated simultaneously. This can be done by identifying a sequence $(K_n) \in \mathcal{S}(\mathbb{Z})$ with the $2 \times 2$-matrix of sequences

$$\begin{pmatrix} (PK_nP) & (PK_nQJ) \\ (JQL_nP) & (JQK_nQJ) \end{pmatrix} \tag{53}$$

which belongs to $\mathcal{S}(\mathbb{N})_{2\times 2}$, and which is in $\mathcal{K}_{2\times 2}$ if $(K_n) \in \mathcal{K}(\mathbb{Z})$. The essential rank of the sequence (53) can be determined by the (evident analogue of) identity (47), i.e., as a sum of two terms corresponding to the points 0 and 1 again.

Next we turn to the Fredholmness of sequences in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$. The invertibility of the band-dominated operator $A$ is a necessary condition for the stability of the sequence $(P_nAP_n)$, but it is by no means sufficient. Indeed, Theorem 6.8 involves a bulk of extra conditions (the invertibility of all limit operators) which, together with the invertibility of $A$, imply the stability. In contrast to this observation, the Fredholmness of $A$ implies the Fredholmness of the sequence $(P_nAP_n)$ without any additional ingredients.

**Theorem 6.17** (*a*) *Let $A \in \mathsf{BDO}(\mathbb{N})$. The sequence $(P_nAP_n)$ of the finite sections is Fredholm if and only if the operator $A$ is Fredholm.*
(*b*) *A sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ is Fredholm if and only if its strong limit $W(\mathbf{A})$ is a Fredholm operator.*
(*c*) *The mapping $\mathsf{BDO}(\mathbb{N})/K(l^2(\mathbb{N})) \to \mathcal{S}(\mathsf{BDO}(\mathbb{N}))/(\mathcal{S}(\mathsf{BDO}(\mathbb{N})) \cap \mathcal{K})$,*

$$A + K(l^2(\mathbb{N})) \mapsto (P_nAP_n) + \mathcal{S}(\mathbb{N}) \cap \mathcal{K}, \tag{54}$$

*is a $^*$-isomorphism.*

**Proof.** We only prove assertion $(a)$. Let $A$ be a Fredholm operator, and let $R$ be a regularizer of $A$, i.e., the operators $K_r := I - RA$ and $K_l := I - AR$ are compact. Then

$$P_nRP_nAP_n = P_nRAP_n - P_nRQ_nAP_n = P_n - P_nK_rP_n - P_nRQ_nAP_n$$

with $Q_n := I - P_n$. The sequence $(P_nK_rP_n)$ belongs to the ideal $\mathcal{K}$ since the compact operator $K_r$ can be approximated as closely as desired by finite rank operators. We claim that the sequence $(P_nRQ_nAP_n)$ is compact, too. Since $A$ can be approximated as closely as desired by band operators, it is sufficient to check this claim for a band operator $A$. This has already been done in the proof of Theorem 6.12. So we get $(P_nRP_n)(P_nAP_n) - (P_n) \in \mathcal{K}$ and, analogously, $(P_nAP_n)(P_nRP_n) - (P_n) \in \mathcal{K}$. Thus, $(P_nAP_n)$ is a Fredholm sequence.

Conversely, if $(P_nAP_n)$ is a Fredholm sequence, then the coset $(P_nAP_n) + \mathcal{K}$ is invertible in $\mathcal{F}/\mathcal{K}$. By inverse closedness of $C^*$-algebras, this coset is also invertible in $(\mathcal{F}^C + \mathcal{K})/\mathcal{K}$ where $\mathcal{F}^C$ refers to the $C^*$-subalgebra of $\mathcal{F}$ consisting of all $^*$-strongly convergent sequences. Further, the $^*$-isomorphy

$$(\mathcal{F}^C + \mathcal{K})/\mathcal{K} \cong \mathcal{F}^C/(\mathcal{F}^C \cap \mathcal{K})$$

implies that the coset $(P_nAP_n) + (\mathcal{F}^C \cap \mathcal{K})$ is invertible in $\mathcal{F}^C/(\mathcal{F}^C \cap \mathcal{K})$. Thus, there are a sequence $(B_n) \in \mathcal{F}^C$ as well as sequences $(K_n)$, $(L_n) \in \mathcal{F}^C \cap \mathcal{K}$ with strong limits $B$, $K$ and $L$, respectively, such that

$$(B_n)(P_nAP_n) = (P_n) + (K_n) \quad \text{and} \quad (P_nAP_n)(B_n) = (P_n) + (L_n).$$

Passing to the strong limit as $n \to \infty$ yields $BA = I + K$ and $AB = I + L$ with compact operators $K$ and $L$. Hence, $A$ is compact. ∎

Our next goal are the $\alpha$-numbers of the finite sections sequence $(P_nAP_n)$ for Fredholm band-dominated operators $A$ on $l^2(\mathbb{N})$. The main result of this section reads as follows. We formulate this result also for subsequences of $(P_nAP_n)$.

**Theorem 6.18** *Let $A \in l^2(\mathbb{Z}^+)$ be a Fredholm band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then $(P_{\eta(n)}AP_{\eta(n)})$ is a Fredholm sequence, and*

$$\begin{aligned} &\alpha(P_{\eta(n)}AP_{\eta(n)}) \\ &\quad = \dim\ker A + \max\{\dim\ker(QA_hQ + P) : A_h \in \sigma_{op,\eta}(A)\}. \end{aligned} \tag{55}$$

The proof is based on the formula (47) for the essential rank of compact sequences in $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$. We omit the details.

A situation of particular interest arises if the sequence $\eta$ in Theorem 6.18 is chosen such that the limit operator $A_\eta$ exists. In this case, the operator spectrum $\sigma_{op,\eta}(A)$ consists of exactly one operator, namely $A_\eta$.

**Corollary 6.19** *Let $A \in L(l^2(\mathbb{Z}^+))$ be a Fredholm band-dominated operator, and let $h : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence for which the limit operator $A_h$ exists. Then $(P_{h(n)}AP_{h(n)})$ is a Fredholm sequence, and*

$$\alpha(P_{h(n)}AP_{h(n)}) = \dim\ker A + \dim\ker(QA_hQ + P). \tag{56}$$

## 6.5 The index formula for band-dominated operators

Now we are in a position to settle the announced finite sections proof of the index formula (41) for band-dominated operators. Its main ingredients are Theorem 6.18 and its Corollary 6.19.

Let $A \in L(l^2(\mathbb{Z}^+))$ be a Fredholm band-dominated operator. Then its adjoint $A^*$ has the same property. Moreover, if the limit operator of $A$ with respect to the sequence $h$ exists, then the limit operator $(A^*)_h$ exists as well, and $(A^*)_h = (A_h)^*$. Applying Corollary 6.19 to the sequence $(P_{h(n)}A^*P_{h(n)})$ we find

$$\begin{aligned} \alpha(P_{h(n)}A^*P_{h(n)}) &= \dim\ker A^* + \dim\ker(Q(A^*)_hQ + P) \\ &= \dim\ker A^* + \dim\ker(Q(A_h)^*Q + P). \end{aligned} \tag{57}$$

We have already mentioned that

$$\alpha(P_{h(n)}AP_{h(n)}) = \alpha(P_{h(n)}A^*P_{h(n)}).$$

Thus, subtracting (57) from (56) and taking into account that $\operatorname{ind} A = \dim\ker A - \dim\ker A^*$ for every Fredholm operator $A$ one gets

$$\operatorname{ind} A = -\operatorname{ind}(QA_hQ + P) \tag{58}$$

for every Fredholm band-dominated operator $A$ on $l^2(\mathbb{Z}^+)$ and every strongly monotonically sequence $h$ for which the limit operator $A_h$ exists. Notice that the "ind" on the left-hand side of (58) refers to the Fredholm index of an operator on $l^2(\mathbb{Z}^+)$, whereas the "ind" on the right-hand side stands for the index of an operator acting on $l^2(\mathbb{Z})$. Further, since $A_h$ is invertible and by (40), one has

$$0 = \operatorname{ind} A_h = \operatorname{ind}(PA_hP + Q) + \operatorname{ind}(QA_hQ + P).$$

Thus, (58) can be written in the more symmetric form

$$\operatorname{ind}(PAP + Q) = \operatorname{ind}(PA_hP + Q). \tag{59}$$

Similar arguments apply to Fredholm band-dominated operator on $l^2(\mathbb{Z}^-)$ and give

$$\operatorname{ind}(QAQ + P) = \operatorname{ind}(QA_hQ + P). \tag{60}$$

Finally, let $A$ be a Fredholm band-dominated operator on $l^2(\mathbb{Z})$. Then $PAP$ and $QAQ$ are Fredholm band-dominated operators on $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z}^-)$, respectively, and from (59) and (60) one gets

$$\begin{aligned} \operatorname{ind} A &= \operatorname{ind}(PAP+Q) + \operatorname{ind}(QAQ+P) \\ &= \operatorname{ind}(PA_hP+Q) + \operatorname{ind}(QA_gQ+P), \end{aligned}$$

which is the assertion (41). ■

## 6.6 Fractality of subalgebras of $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$

It is evident that the algebra $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ is not fractal. A simple example is provided by the sequence of the finite sections of the unitary band operator

$$A := \operatorname{diag}\left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots \right).$$

Then $(P_{2n}AP_{2n})$ is a stable sequence (consisting of unitary matrices) whereas no entry of the sequence $(P_{2n+1}AP_{2n+1})$ is invertible. Also no restriction of $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ is fractal (note that this algebra is not separable; so Theorem 3.8 does not apply). It is, thus, of vital importance to single out restrictions of *subalgebras* of $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ which are fractal. The following constructions yields, for every band-dominated operator $A$, a restricted subalgebra of $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ which is fractal and contains the restricted sequence of the finite sections of $A$.

Let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence and let $\mathsf{BDO}^{(\eta)}(\mathbb{N})$ denote the set of all operators $A \in \mathsf{BDO}(\mathbb{N})$ for which the strong limits

$$\operatorname{s-lim} U_{-\eta(n)} A U_{\eta(n)} \quad \text{and} \quad \operatorname{s-lim} U_{-\eta(n)} A^* U_{\eta(n)}$$

exist. Then $\mathsf{BDO}^{(\eta)}(\mathbb{N})$ is a closed $C^*$-subalgebra of $\mathsf{BDO}(\mathbb{N})$, and the mapping $A \mapsto \operatorname{s-lim} U_{-\eta(n)} A U_{\eta(n)}$ is a $^*$-homomorphism on $\mathsf{BDO}^{(\eta)}(\mathbb{N})$. Let $\mathcal{S}_\eta(\mathsf{BDO}^{(\eta)}(\mathbb{N}))$ stand for the smallest closed $^*$-subalgebra of $\mathcal{F}_\eta^{\mathcal{P}}$ which contains all sequences $(P_{\eta(n)} A P_{\eta(n)})$ with $A \in \mathsf{BDO}^{(\eta)}(\mathbb{N})$.

**Theorem 6.20** *The algebra* $\mathcal{S}_\eta(\mathsf{BDO}^{(\eta)}(\mathbb{N}))$ *is fractal.*

**Proof.** Let $\mathbf{A} := (A_{\eta(n)}) \in \mathcal{S}_\eta(\mathsf{BDO}^{(\eta)}(\mathbb{N}))$. Then the strong limit

$$W_\eta(\mathbf{A}) := \operatorname{s-lim} U_{-\eta(n)} A_{\eta(n)} U_{\eta(n)}$$

exists, and the mapping $W_\eta$ is a $^*$-homomorphism on $\mathcal{S}_\eta(\mathsf{BDO}^{(\eta)}(\mathbb{N}))$. From Theorem 6.10 we infer that the sequence $\mathbf{A}$ is stable if and only if the operators $W(\mathbf{A})$ and $W_\eta(\mathbf{A})$ are invertible. Since both homomorphisms $W$ and $W_\eta$ are fractal, the algebra $\mathcal{S}_\eta(\mathsf{BDO}^{(\eta)}(\mathbb{N}))$ is fractal. ■

## 6.7 Irrational rotation algebras

Let $\theta \in (0, 1)$ be irrational. The *irrational rotation algebra* $\mathcal{A}_\theta$ is the universal $C^*$-algebra generated by two unitary elements $u$, $v$ with the relation

$$uv = e^{2\pi i\theta} vu. \tag{61}$$

Since each irrational rotation algebra is simple, every $C^*$-algebra which is generated by two unitary elements which satisfy (61) is naturally $^*$-isomorphic to $\mathcal{A}^\theta$. Nice introductions to irrational rotation algebras are [2, 11]. For a concrete model, consider the unitary operators

$$U : (x_n) \mapsto (x_{n-1}) \quad \text{and} \quad V : (x_n) \mapsto (e^{-2\pi in\theta} x_n)$$

on $l^2(\mathbb{Z})$. One easily checks that $UV = e^{2\pi i\theta} VU$, hence, $\mathcal{A}_\theta$ is naturally $^*$-isomorphic to the smallest $C^*$-subalgebra of $L(l^2(\mathbb{Z}))$ which contains $U$ and $V$. We denote this algebra by $\mathsf{A}_\theta$. Evidently, $\mathsf{A}_\theta$ is a subalgebra of $\mathsf{BDO}(\mathbb{Z})$.

We consider the smallest closed subalgebra $\mathcal{S}(\mathsf{A}_\theta)$ of $\mathcal{F}^{\mathcal{P}^\mathbb{Z}}$ which contains all sequences $(P_n^\mathbb{Z} A P_n^\mathbb{Z})$ of finite sections where $A \in \mathsf{A}_\theta$ and where the projections $P_n^\mathbb{Z}$ are defined as before. Since the algebra $\mathsf{A}_\theta$ is separable, Theorem 3.8 guarantees the existence of a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that the restricted algebra $\mathcal{S}_\eta(\mathsf{A}_\theta)$ is fractal. Theorem 6.20 offers a way to construct a sequence $\eta$ with this property. The $\mathbb{Z}$-version of this theorem states that we have to choose $\eta$ such that the strong limits

$$\text{s-lim}\, U_{-\eta(n)} A U_{\eta(n)} \quad \text{and} \quad \text{s-lim}\, U_{\eta(n)} A U_{-\eta(n)} \tag{62}$$

exist for each operator $A \in \mathsf{A}_\theta$. Taking into account that each operator $A \in \mathsf{A}_\theta$ is band-dominated one easily checks that $A$ can be approximated as closely as desired by finite sums $\sum a_r U_r$ with coefficients $a_r$ in the $C^*$-subalgebra of $l^\infty(\mathbb{Z})$ generated by the functions[1] $U_{-k} V U_k$ with $k \in \mathbb{Z}$. Since $U_{-n} U U_n = U$ for every $n$, we have thus to choose $\eta$ such that the limits (62) exist for the operators $U_{-k} V U_k$ in place of $A$. For this goal, we compute

$$U_{-\eta(n)} U_{-k} V U_k U_{\eta(n)} = e^{-2\pi i\eta(n)\theta} U_{-k} V U_k$$

for each $k \in \mathbb{Z}$ from which we infer that

$$U_{-\eta(n)} A U_{\eta(n)} = e^{-2\pi i\eta(n)\theta} A \qquad \text{for each } A \in \mathsf{A}_\theta. \tag{63}$$

We will see now that if $\theta$ is irrational, then there is strongly monotonically increasing sequence $\eta$ such that $e^{2\pi i\eta(n)\theta} \to 1$ as $n \to \infty$. Indeed, develop $\theta$ into a

[1]Note that each operator $U_{-k} V U_k$ acts on $l^2(\mathbb{Z})$ as the operator of multiplication by a bounded function.

continued fraction

$$\theta = \lim_{n\to\infty} \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots \, b_{n-1} + \cfrac{1}{b_n}}}}$$

with uniquely determined positive integers $b_i$. Write this continued fraction as $p_n/q_n$ with positive and relatively prime integers $p_n$, $q_n$. These integers satisfy the recursions

$$p_n = b_n p_{n-1} + p_{n-2}, \qquad q_n = b_n q_{n-1} + q_{n-2} \tag{64}$$

with $p_0 = 0$, $p_1 = 1$, $q_0 = 1$ and $q_1 = b_1$, and one has

$$\left| \theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2} \tag{65}$$

for every $n \geq 1$. (These facts can be found in every textbook on continued fractions.) From (65) we conclude that

$$|\theta q_n - p_n| \leq q_n \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n} \to 0,$$

whence $e^{2\pi i\theta q_n} = e^{2\pi i(\theta q_n - p_n)} \to 1$. Since moreover $q_1 < q_2 < \ldots$ due to the recursion (64), this shows that the sequence $\eta(n) := q_n$ has the desired properties. Thus, if $\eta$ is specified in this way, then we can not only conclude via (63) that the strong limits (62) exist for every $A \in \mathsf{A}_\theta$; moreover, they exist even with respect to norm convergence, and the operators $U_{\mp\eta(n)} A U_{\pm\eta(n)}$ converge in the norm to $A$.

Having these facts at our disposal, we can prove the following theorem of the structure of the algebra $\mathcal{S}_\eta(\mathsf{A}_\theta)$ in full analogy to the proof of Theorem 2.4. Details can be found in [19].

**Theorem 6.21** *Let $\theta \in (0, 1)$ be irrational and specify $\eta$ as above. Then the algebra $\mathcal{S}_\eta(\mathsf{A}_\theta)$ consists exactly of all sequences of the form*

$$(P^{\mathbb{Z}}_{\eta(n)} A P^{\mathbb{Z}}_{\eta(n)} + P^{\mathbb{Z}}_{\eta(n)} U_{-n} K U_n P^{\mathbb{Z}}_{\eta(n)} + P^{\mathbb{Z}}_{\eta(n)} U_n JLJ U_{-n} P^{\mathbb{Z}}_{\eta(n)} + G_{\eta(n)}) \tag{66}$$

*with $A \in \mathsf{A}_\theta$, $K$, $L \in L(l^2(\mathbb{Z}))$ are compact and $\|G_{\eta(n)}\| \to 0$ as $n \to \infty$, and each sequence in $\mathcal{S}_\eta(\mathsf{A}_\theta)$ can be written in the form (66) in a unique way. Moreover, the quasicommutator ideal of $\mathcal{S}_\eta(\mathsf{A}_\theta)$ consists of all sequences*

$$(P^{\mathbb{Z}}_{\eta(n)} U_{-n} K U_n P^{\mathbb{Z}}_{\eta(n)} + P^{\mathbb{Z}}_{\eta(n)} U_n JLJ U_{-n} P^{\mathbb{Z}}_{\eta(n)} + G_{\eta(n)})$$

*with $K$, $L$ and $G_{\eta(n)}$ as before.*

This theorem gives us full control over discretized irrational rotation algebras. Here are some more or less immediate consequences.

**Corollary 6.22** *Let* $\mathbf{A} = (A_{\eta(n)}) \in \mathcal{S}_\eta(\mathsf{A}_\theta)$. *Then the strong limits*

$$W_{\pm 1}(\mathbf{A}) := \text{s-lim}_{n\to\infty} U_{\mp\eta(n)} A_{\eta(n)} U_{\pm\eta(n)}$$

*exist, and* $\mathbf{A}$ *is a stable sequence if and only if the operators* $W_1(\mathbf{A})$ *and* $W_{-1}(\mathbf{A})$ *are invertible on* $l^2(\mathbb{Z}^-)$ *and* $l^2(\mathbb{Z}^+)$*, respectively.*

In particular, if the sequence $\mathbf{A}$ is written as in (66), then $W_1(\mathbf{A}) = QAQ + QLQ$ and $W_{-1}(\mathbf{A}) = PAP + PKP$.

**Corollary 6.23** *The algebra* $\mathcal{S}_\eta(\mathsf{A}_\theta)$ *is fractal.*

**Corollary 6.24** *The compact sequences in* $\mathcal{S}_\eta(\mathsf{A}_\theta)$ *are exactly the sequences in the quasicommutator ideal.*

**Corollary 6.25** *A sequence* $\mathbf{A}$ *in* $\mathcal{S}_\eta(\mathsf{A}_\theta)$ *is Fredholm if and only if* $W_1(\mathbf{A})$ *and* $W_{-1}(\mathbf{A})$ *are Fredholm operators on* $l^2(\mathbb{Z}^-)$ *and* $l^2(\mathbb{Z}^+)$*, respectively. In this case,*

$$\alpha(\mathbf{A}) = \dim\ker W_1(\mathbf{A}) + \dim\ker W_{-1}(\mathbf{A}).$$

In case $\mathbf{A}$ is written as in (66), this sequence is Fredholm if and only if $PAP + PKP$ and $QAQ + QLQ$ are Fredholm operators. Clearly, this happens if and only if $PAP$ and $QAQ$ are Fredholm operators. Employing the special properties of $A$ (more precisely: employing that the coefficients of $A$ are almost periodic functions) one can show that $PAP$ is Fredholm if and only if $QAQ$ is Fredholm if and only if $A$ is invertible.

**Corollary 6.26** *A sequence* $\mathbf{A}$ *in* $\mathcal{S}_\eta(\mathsf{A}_\theta)$ *is Fredholm if and only if its strong limit* $W(\mathbf{A})$ *is invertible on* $l^2(\mathbb{Z})$.

# 7 Spatial discretization of Cuntz algebras

Our running example, the Toeplitz algebra $\mathsf{T}(C)$, is (isomorphic to) the universal algebra generated by one isometry. Here we go one step further and consider the spatial discretization of algebras which are generated by a finite number of non-commuting non-unitary isometries, namely the Cuntz algebras.

Let $N \geq 2$. The *Cuntz algebra* $\mathcal{O}_N$ is the universal $C^*$-algebra generated by $N$ isometries $s_0, \ldots, s_{N-1}$ with the property that

$$s_0 s_0^* + \ldots + s_{N-1} s_{N-1}^* = I. \tag{67}$$

Cuntz algebras cannot be obtained as inductive limits of type I $C^*$-algebras. In particular, they cannot be approximated by finite dimensional algebras in the sense of $AF$-algebras. (For these and other facts, consult Cuntz' pioneering paper [10]. A nice introduction is also in [11].) The importance of Cuntz algebras in theory and applications cannot be overestimated. Let me only mention Kirchberg's deep result that a separable $C^*$-algebra is exact if and only if it embeds in the Cuntz algebra $\mathcal{O}_2$, and the role that representations of Cuntz algebras play in wavelet theory and signal processing (see [1, 6] and the references therein).

To discretize the Cuntz algebra $\mathcal{O}_N$ by the finite sections method, we represent this algebra as an algebra of operators on $l^2(\mathbb{Z}^+)$. Since Cuntz algebras are simple, every $C^*$-algebra which is generated by $N$ isometries $S_0, \ldots, S_{N-1}$ which fulfill (67) in place of the $s_i$ is $^*$-isomorphic to $\mathcal{O}_N$. Thus, $\mathcal{O}_N$ is $^*$-isomorphically to the smallest $C^*$-subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains the operators

$$S_i : (x_k)_{k\geq 0} \mapsto (y_k)_{k\geq 0} \quad \text{with} \quad y_k := \begin{cases} x_r & \text{if } k = rN + i \\ 0 & \text{else} \end{cases} \tag{68}$$

for $i = 0, \ldots, N-1$. We denote the (concrete) Cuntz algebra generated by the operators $S_i$ in (68) by $\mathsf{O}_N$. It is this concrete Cuntz algebra for which we will examine the sequence algebra $\mathcal{S}(\mathsf{O}_N)$ (modulo zero sequences) in what follows.

One should mention that the abstract Cuntz algebra $\mathcal{O}_N$ has an uncountable set of equivalence classes of irreducible representations. Representations of $\mathcal{O}_N$ different from (68) will certainly lead to sequence algebras different from $\mathcal{S}(\mathsf{O}_N)$. The relation between these algebras is not yet understood. The chosen representation of $\mathcal{O}_N$ is distinguished by the fact that it is both irreducible and *permutative* in the sense that every isometry $S_i$ maps elements of the standard basis to elements of the standard basis.

Coburn's already mentioned result suggests to consider the Toeplitz algebra $\mathsf{T}(C)$ as the Cuntz algebra $\mathsf{O}_1$. But one should have in mind that the main properties of $\mathsf{O}_1$ and of $\mathsf{O}_N$ for $N > 1$ are quite different from each other. For example, the compact operators $K(l^2(\mathbb{Z}^+))$ form a closed ideal of $\mathsf{O}_1$, and the quotient $\mathsf{O}_1/K(l^2(\mathbb{Z}^+))$ is isomorphic to $C(\mathbb{T})$, whereas $\mathsf{O}_N$ is simple if $N \geq 2$. These differences continue to the corresponding sequence algebras $\mathcal{S}(\mathsf{O}_1)$ and $\mathcal{S}(\mathsf{O}_N)$ for $N > 1$. A main point is that $\mathcal{S}(\mathsf{O}_1)/\mathcal{G}$ contains an ideal which is constituted of two exemplars of the ideal $K(l^2(\mathbb{Z}^+))$, and the irreducible representations, $W_1$ and $W_2$ say, of $\mathcal{S}(\mathsf{O}_1)$ which come from this ideal are *sufficient* in the sense that a sequence $\mathbf{A} = (A_n)$ in $\mathcal{S}(\mathsf{O}_1)$ is stable if and only if $W_1(\mathbf{A})$ and $W_2(\mathbf{A})$ are invertible. We have seen that this fact implies an effective criterion to check the stability of a sequence in $\mathcal{S}(\mathsf{O}_1)$. In contrast to this, if $N > 1$, then $\mathcal{S}(\mathsf{O}_N)$ has only one non-trivial ideal. We will construct an injective representation of this ideal, and will then observe that this representation extends to a representation, $W$ say, of $\mathcal{S}(\mathsf{O}_N)$ which is injective on all of $\mathcal{S}(\mathsf{O}_N)$. Thus, roughly speaking, our stability result will say that a sequence $\mathbf{A}$ in $\mathcal{S}(\mathsf{O}_N)$ is stable if and only if the operator $W(\mathbf{A})$ is invertible. At the first glance, this result might seem to be useless since the stability of $\mathbf{A}$ is not easier to check than the invertibility of $W(\mathbf{A})$. So why this effort, if many canonical homomorphisms on $\mathcal{S}(\mathsf{O}_N)$ own the same property as $W$: the identical mapping and the faithful representation via the GNS-construction, for example. What is important is the concrete form of the mapping $W$ constructed below: it is defined by means of strong limits of operator sequences, and this special form implies immediately the fractality of (some restriction of) the algebra $\mathcal{S}(\mathsf{O}_N)$.

## 7.1 The full algebra $\mathcal{S}(\mathsf{O}_N)$

In accordance with earlier notations, let $\mathcal{F}$ be the $C^*$-algebra of all bounded sequences $\mathbf{A} = (A_n)$ of matrices $A_n \in \mathbb{C}^{n\times n}$, and let $\mathcal{S}(\mathsf{O}_N)$ denote the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences $(P_nAP_n)$ with $A$ in the concrete Cuntz algebra $\mathsf{O}_N$. Since $(P_nAP_n)^* = (P_nA^*P_n)$, $\mathcal{S}(\mathsf{O}_N)$ is a $C^*$-algebra. The isometries $S_i$ are defined as in (68). We further abbreviate $\Omega := \{0, 1, \ldots, N-1\}$.

**Lemma 7.1** *$\mathcal{S}(\mathsf{O}_N)$ is the smallest $C^*$-subalgebra of $\mathcal{F}$ which contains all sequences $(P_nS_jP_n)$ with $j \in \Omega$.*

**Proof.** Let $\mathcal{S}'$ denote the smallest $C^*$-subalgebra of $\mathcal{F}$ which contains all sequences $(P_nS_jP_n)$ with $j \in \Omega$. Evidently, $\mathcal{S}' \subseteq \mathcal{S}(\mathsf{O}_N)$. For the reverse inclusion, note that

$$S_i^*S_j = 0 \quad \text{whenever } i \neq j. \tag{69}$$

Indeed, this follows by straightforward calculation, but it also a consequence of the Cuntz axiom (67): Multiply (67) from the left by $S_i^*$ and from the right by $S_i$ and take into account that a sum of positive elements in a $C^*$-algebra is zero if and only if each of the elements is zero. From (69) we conclude that every finite word with letters in the alphabet $\{S_1, \ldots, S_N, S_1^*, \ldots, S_N^*\}$ is of the form

$$S_{i_1}S_{i_2}\ldots S_{i_k}S_{j_1}^*S_{j_2}^*\ldots S_{j_l}^* \quad \text{with} \quad i_s,\, j_t \in \Omega \tag{70}$$

(Lemma 1.3 in [10]). Further one easily checks that

$$P_nS_j = P_nS_jP_n \quad \text{and} \quad S_j^*P_n = P_nS_j^*P_n \tag{71}$$

for every $j \in \Omega$ and every $n \in \mathbb{N}$. Thus, if $A$ is any word of the form (70), then

$$P_nAP_n = P_nS_{i_1}P_n \cdot P_nS_{i_2}P_n \ldots P_nS_{i_k}P_n \cdot P_nS_{j_1}^*P_n \cdot P_nS_{j_2}^*P_n \ldots P_nS_{j_l}^*P_n \in \mathcal{S}'.$$

Since the set of all linear combinations of the words (70) is dense in $\mathsf{O}_N$, it follows that $\mathcal{S}(\mathsf{O}_N) \subseteq \mathcal{S}'$. ■

Recall that an element $S$ of a $C^*$-algebra is called a partial isometry if $SS^*S = S$. If $S$ is a partial isometry, then $SS^*$ and $S^*S$ are projections (i.e., self-adjoint idempotents), called the *range projection* and the *initial projection* of $S$, respectively. Conversely, if $S^*S$ (or $SS^*$) is a projection for an element $S$, then $S$ is a partial isometry. Recall also that projections $P$ and $Q$ are called orthogonal if $PQ = 0$.

**Lemma 7.2** *Every sequence* $(P_nS_iP_n)$, $i \in \Omega$, *is a partial isometry in* $\mathcal{F}$, *and the corresponding range projections are orthogonal if* $i \neq j$. *Moreover,*

$$P_nS_i^*P_nS_jP_n = 0 \quad \textit{if } i \neq j, \tag{72}$$

*and*

$$P_nS_0P_nS_0^*P_n + \ldots + P_nS_{N-1}P_nS_{N-1}^*P_n = P_n. \tag{73}$$

**Proof.** The identities (71) imply that $P_nS_iS_i^*P_n = P_nS_iP_nS_i^*P_n$ for every $i \in \Omega$ and every $n \in \mathbb{Z}^+$. The operators $S_iS_i^*$ are projections, and their matrices with respect to the standard basis of $l^2(\mathbb{Z}^+)$ are of diagonal form. Hence, the projections $S_iS_i^*$ and $P_n$ commute, which implies that $P_nS_iS_i^*P_n$ is a projection. Hence, $(P_nS_iP_n)$ is a partial isometry in $\mathcal{F}$, and $(P_nS_iS_i^*P_n)$ is the associated range projection.

Let $i \neq j$ be in $\Omega$. The fact that $P_n$ and $S_iS_i^*$ commute further implies together with (69) that

$$(P_nS_iS_i^*P_n)(P_nS_jS_j^*P_n) = P_nS_iS_i^*S_jS_j^*P_n = 0.$$

Multiplying $P_nS_iS_i^*P_nS_jS_j^*P_n = 0$ from the left by $P_nS_i^*P_n$ and from the right by $P_nS_jP_n$ yields (72). Finally, (73) follows by summing up the equalities (72) over $i \in \Omega$ and from axiom (67). ∎

Thus, the generating sequences $(P_nS_iP_N)$, $i \in \Omega$, are still subject of the Cuntz axiom (67), but note they are partial isometries only and no longer isometries. The result of the preceding lemma holds more general. For $i = (i_1, i_2, \ldots, i_k) \in \Omega^k$, abbreviate $S_i := S_{i_1}S_{i_2}\ldots S_{i_k}$. Further, for every real number $x$, let $\{x\}$ denote the smallest integer which is greater than or equal to $x$. The first assertion of the following proposition follows as in Lemma 7.2, the second one by straightforward calculation.

**Proposition 7.3** *Let* $i = (i_1, i_2, \ldots, i_k) \in \Omega^k$. *Each sequence*

$$(P_nS_{i_1}P_nS_{i_2}P_n\ldots P_nS_{i_k}P_n)$$

*is a partial isometry in* $\mathcal{F}$. *The corresponding range projection is given by*

$$P_nS_i^*P_nS_iP_n = P_{\{(n-v_{i,k})/N^k\}}, \tag{74}$$

*where* $v_{i,k} := i_1 + i_2N + \ldots + i_kN^{k-1}$.

We specialize the result of Proposition 7.3 to the case $k = 1$ and consider two examples. If $n = jN$ is a multiple of $N$, then $\{(n-i)/N\} = \{(jN-i)/N\} = \{j - i/N\} = j$, whence

$$P_{jN}S_i^*P_{jN}S_iP_{jN} = P_j \quad \text{for all } i \in \Omega. \tag{75}$$

On the other hand, one has

$$P_n S_0^* P_n S_0 P_n - P_n S_1^* P_n S_1 P_n = \begin{cases} P_{j+1} - P_j & \text{if } n = jN+1, \\ 0 & \text{else.} \end{cases} \tag{76}$$

Thus, the sequence

$$(P_n S_0^* P_n S_0 P_n - P_n S_1^* P_n S_1 P_n)_{n \geq 1} \tag{77}$$

possesses both a subsequence consisting of zeros only (take $\eta(n) := nN$) and a subsequence consisting of non-zero projections (if $\eta(n) := nN + 1$). This shows that the algebra $\mathcal{S}(\mathsf{O}_N)$ cannot be fractal. Moreover, a similar argument shows that also the restricted algebra $\mathcal{S}_\eta(\mathsf{O}_N)$, with $\eta(n) := nN$, cannot be fractal. We will see later on (and, in some sense, this is our main goal in this section) that

$$\eta(n) := N^n \tag{78}$$

is the correct choice for the restriction $\eta$, since it will indeed guarantee the fractality of the restricted algebra $\mathcal{S}_\eta(\mathsf{O}_N)$.

A first hint that the choice (78) is a natural one is given by the following lemma which states that, up to sequences in the ideal $\mathcal{G}_\eta$, the initial projections of the partial isometries $(P_{\eta(n)} S_i P_{\eta(n)})$ with $i \in \Omega^k$ only depend on the length $k$ of the multi-index, not on the multi-index $i$ itself. The proof is again a straightforward calculation.

**Lemma 7.4** *Let $i \in \Omega^k$ and $n = N^j$ with $j \in \mathbb{Z}^+$. Then*

$$P_n S_i^* P_n S_i P_n = \begin{cases} 0 & \text{if } j < k \text{ and } N^j \leq v_{i,k}, \\ P_1 & \text{if } j < k \text{ and } N^j > v_{i,k}, \\ P_{N^{j-k}} & \text{if } j \geq k. \end{cases}$$

## 7.2 The finite sections algebra $\mathcal{S}_N$ and its ideal $\mathcal{J}_N$

In what follows we will exclusively deal with the restricted algebra $\mathcal{S}_\eta(\mathsf{O}_N)$ where $\eta(n) = N^n$.

**Proposition 7.5** *The algebra $\mathcal{S}_\eta(\mathsf{O}_N)$ contains the ideal $\mathcal{G}_\eta$.*

We skip the proof since it is not needed in what follows. But note that we cannot refer to Proposition 2.5 since $\mathsf{O}_N$ does not contain non-zero compact operators.

By Proposition 7.5, the quotient algebra $\mathcal{S}_N := \mathcal{S}_\eta(\mathcal{O}_N)/\mathcal{G}_\eta$ is well defined. Recall that $\mathcal{S}_N$ is generated by the partial isometries

$$s_i := (P_{N^n} S_i P_{N^n})_{n \geq 0} + \mathcal{G}_\eta, \quad i \in \Omega$$

and contains the identity element $e$ of $\mathcal{F}_\eta/\mathcal{G}_\eta$. More general, for each multi-index $i \in \Omega^k$ we set $s_i := s_{i_1} s_{i_2} \ldots s_{i_k}$. These elements are partial isometries by Proposition 7.3 and the initial projection of $s_i$ does only depend on the length of $i$ by Lemma 7.4. We denote the length of the multi-index $i$ by $|i|$ and write $p_k$ for the joint initial projection of all partial isometries $s_i$ with length $k$. Further we write $\Omega_\infty$ for the set of all multi-indices (of arbitrary length). The following axioms collect the basic properties of these elements.

(A1) for every $i \in \Omega_\infty$, the coset $s_i$ is a partial isometry, the initial projection of which depends on $|i|$ only: $s_i^* s_i = p_{|i|}$.

(A2) $s_0 s_0^* + s_1 s_1^* + \ldots + s_{N-1} s_{N-1}^* = e$.

All results in this section will follow from these axioms. For later reference we list some relations between partial isometries $s_i$ and the projections $p_k$.

**Lemma 7.6** *Let $k$, $l$ positive integers and $i \in \Omega^k$. Then*
*(a) $s_i = s_i p_k$ and $s_i^* = p_k s_i^*$.*
*(b) $s_i^* p_l s_i = p_{k+l}$.*
*(c) $p_k p_l = p_k$ if $k \geq l$.*
*(d) $p_l s_i = s_i p_{k+l}$.*
*(e) The generalized Cuntz condition $\sum_{i \in \Omega^k} s_i s_i^* = e$ holds for every $k \geq 1$.*

**Proof.** The first three assertions follow immediately from the definitions. Assertions $(b)$ and $(c)$ imply that

$$\begin{aligned}(p_l s_i - s_i p_{k+l})^*(p_l s_i - s_i p_{k+l}) &= (s_i^* p_l - p_{k+l} s_i^*)(p_l s_i - s_i p_{k+l}) \\ &= s_i^* p_l s_i - s_i^* p_l s_i p_{k+l} - p_{k+l} s_i^* p_l s_i + p_{k+l} s_i^* s_i p_{k+l} \\ &= p_{k+l} - p_{k+l} - p_{k+l} + p_{k+l} p_k p_{k+l} = 0,\end{aligned}$$

whence $(d)$ via the $C^*$-axiom. Finally, assertion $(e)$ follows easily by induction. ■

Let $k$ a positive integer. By axiom (A1), every partial isometry $s_i$ of length $k$ is an isometry modulo the smallest closed ideal $\mathcal{J}^{(k)}$ of $\mathcal{S}_N$ which contains the projection $e - p_k$. Note that, by Lemma 7.4,

$$e - p_1 = (0,\, P_N - P_1,\, P_{N^2} - P_N,\, P_{N^3} - P_{N^2},\, \ldots) + \mathcal{G}_\eta.$$

We claim that $\mathcal{J}^{(k)} = \mathcal{J}^{(1)}$ for every $k$. Indeed, by Lemma 7.6 $(c)$,

$$(e - p_k)(e - p_1) = e - p_k - p_1 + p_k p_1 = e - p_1.$$

Hence, $e - p_1 \in \mathcal{J}^{(k)}$, whence $\mathcal{J}^{(1)} \subseteq \mathcal{J}^{(k)}$. For the reverse inclusion recall from Lemma 7.6 $(b)$ that $(s_0^*)^l (e - p_1) s_0^l = p_l - p_{l+1}$ for every $l \in \mathbb{Z}^+$. Adding these identities for $l$ between 0 and $k-1$ gives $e - p_k$ on the right-hand side, whereas the element of the left-hand side belongs to $\mathcal{J}^{(1)}$. Thus, $e - p_k \in \mathcal{J}^{(1)}$, whence

$\mathcal{J}^{(k)} \subseteq \mathcal{J}^{(1)}$. In what follows we write $\mathcal{J}_N$ for the ideal $\mathcal{J}^{(1)}$ of $\mathcal{S}_N$. Note that *every* partial isometry $s_i$ with $i \in \Omega_\infty$ is an isometry modulo $\mathcal{J}_N$.

The following property of the $p_n$ will be needed later.

**Lemma 7.7** *For each $j \in \mathcal{J}_N$, one has* $\lim_{n\to\infty} \|p_n j\| = 0$.

**Proof.** If $j$ is of the form

$$s_i s_k^*(e - p_1) s_l s_m^* \quad \text{with} \quad |i| \leq |k| \tag{79}$$

then the assertion holds since

$$p_n s_i s_k^*(e - p_1) = s_i s_k^* p_{n+|i|-|k|}(e - p_1) = 0$$

for $n > |k| - |i|$. Hence, the assertion also holds if $j$ is a linear combination of elements of the form (79). Since these linear combinations form a dense subset of $\mathcal{J}_N$ and $\|p_n\| = 1$ for each $n$, the assertion holds for every $j \in \mathcal{J}_N$. ■

Our further analysis of the algebra $\mathcal{S}_N$ is based on the following elementary fact. It settles a condition which guarantees than every element which is invertible modulo an ideal can be lifted to an invertible element.

**Proposition 7.8** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{I}$ a closed ideal of $\mathcal{A}$. Further suppose there is a unital $^*$-homomorphism $\pi$ from $\mathcal{A}$ into a unital $C^*$-algebra $\mathcal{B}$ such that the restriction of $\pi$ onto $\mathcal{I}$ is injective. Then the following assertions are equivalent for every element $a \in \mathcal{A}$:*
*(a) $a$ is invertible in $\mathcal{A}$.*
*(b) The coset $a+\mathcal{I}$ is invertible in the quotient algebra $\mathcal{A}/\mathcal{I}$, and $\pi(a)$ is invertible in $\mathcal{B}$.*

We shall apply this result with $\mathcal{A} := \mathcal{S}_N$ and $\mathcal{I} := \mathcal{J}_N$. By Proposition 7.8, the problem to derive a criterion for the invertibility of elements of $\mathcal{S}_N$ (and thus, for the stability of sequences in $\mathcal{S}_\eta(\mathsf{O}_N)$) splits into two separate tasks:

- to describe the quotient algebra $\mathcal{S}_N/\mathcal{J}_N$, and
- to construct an injective $^*$-homomorphism on $\mathcal{J}_N$.

The solution of the first task is evident: The quotient algebra $\mathcal{S}_N/\mathcal{J}_N$ is generated by the cosets $s_i + \mathcal{J}_N$ with $i \in \Omega$. These cosets are isometries and they satisfy the Cuntz axiom

$$(s_0 + \mathcal{J}_N)(s_0 + \mathcal{J}_N)^* + \ldots + (s_{N-1} + \mathcal{J}_N)(s_{N-1} + \mathcal{J}_N)^* = e + \mathcal{J}_N.$$

By the universal property of Cuntz algebras, $\mathcal{S}_N/\mathcal{J}_N$ is $^*$-isomorphic to the (abstract) Cuntz algebra $\mathcal{O}_N$. It is also not hard to construct an explicit isomorphism from $\mathcal{S}_N/\mathcal{J}_N$ onto the (concrete) Cuntz algebra $\mathsf{O}_N$. Let $W_\eta$ denote the mapping which associates with each sequence in $\mathcal{S}_\eta(\mathsf{O}_n)$ its strong limit. Since the ideal $\mathcal{G}_\eta$ lies in the kernel of $W_\eta$, there is a correctly defined quotient homomorphism

$$W_\eta^{\mathcal{G}} : \mathcal{F}_\eta/\mathcal{G}_\eta \to L(l^2(\mathbb{Z}^+)), \quad \mathbf{A} + \mathcal{G}_\eta \mapsto W_\eta(\mathbf{A}). \tag{80}$$

Applying this homomorphism to both sides of the equality $s_0^* s_0 = p_1$ we get $S_0^* S_0 = W_\eta^{\mathcal{G}}(p_1)$, whence $W_\eta^{\mathcal{G}}(p_1) = I$. Hence, the ideal $\mathcal{J}_N$ lies in the kernel of $W_\eta^{\mathcal{G}}$, which implies that the quotient homomorphism

$$(\mathcal{S}_\eta(\mathcal{O}_N)/\mathcal{G}_\eta)/\mathcal{J}_N \to L(l^2(\mathbb{Z}^+)), \quad (\mathbf{A} + \mathcal{G}_\eta) + \mathcal{J}_N \mapsto W_\eta^{\mathcal{G}}(\mathbf{A} + \mathcal{G}) \tag{81}$$

is correctly defined, too; we denote it by $W^{\mathcal{J}}$. The $^*$-homomorphism $W^{\mathcal{J}}$ maps the generating cosets $s_i + \mathcal{J}_N$, $i \in \Omega$, to the generating operators $S_i$ of $\mathsf{O}_N$, respectively. Since both sets of generators consist of isometries which satisfy the (same) Cuntz axiom, the following is a consequence of the universal property of Cuntz algebras.

**Theorem 7.9** *$W^{\mathcal{J}}$ is a $^*$-isomorphism from $\mathcal{S}_N/\mathcal{J}_N$ onto $\mathsf{O}_N$.*

**Corollary 7.10** *The kernel of the restriction of the homomorphism $W_\eta^{\mathcal{G}}$ defined by* (80) *to the algebra $\mathcal{S}_N$ coincides with $\mathcal{J}_N$.*

Indeed, this follows since $\mathsf{O}_N$ is a simple algebra. The following fact sheds a first light on our second task.

**Theorem 7.11** *Every proper closed ideal of $\mathcal{S}_N$ lies in $\mathcal{J}_N$.*

**Proof.** Let $\widetilde{\mathcal{J}}$ be a proper closed ideal of $\mathcal{S}_N$. Then $\mathcal{J}_N + \widetilde{\mathcal{J}}$ is a closed ideal of $\mathcal{S}_N$ with $\mathcal{J}_N \subseteq \mathcal{J}_N + \widetilde{\mathcal{J}} \subseteq \mathcal{S}_N$. Since the quotient $\mathcal{S}_N/\mathcal{J}_N$ is $^*$-isomorphic to $\mathcal{O}_N$ and, hence, a simple algebra, one has either

- Case A: $\mathcal{J}_N + \widetilde{\mathcal{J}} = \mathcal{S}_N$, or
- Case B: $\mathcal{J}_N + \widetilde{\mathcal{J}} = \mathcal{J}_N$, i.e. $\widetilde{\mathcal{J}} \subseteq \mathcal{J}_N$.

We wish to exclude case A. Suppose we are in the situation of case A. Consider the ideals $\mathcal{I}_1 := \mathcal{J}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})$ and $\mathcal{I}_2 := \widetilde{\mathcal{J}}/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})$ of $\mathcal{B} := \mathcal{S}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})$. These ideals have a trivial intersection, their sum is $\mathcal{B}$, and the algebra

$$\mathcal{B}/\mathcal{I}_1 = \Big(\mathcal{S}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})\Big) \,/\, \Big(\mathcal{J}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})\Big) \cong \mathcal{S}_N/\mathcal{J}_N$$

is still simple. Let $W$ stand for the canonical homomorphism from $\mathcal{B}$ onto $\mathcal{B}/\mathcal{I}_2$ and write $\hat{a}$ for the coset of $a \in \mathcal{S}_N$ modulo $\mathcal{J}_N \cap \widetilde{\mathcal{J}}$. Since $W(\mathcal{B}) = W(\mathcal{I}_1)$, there is an element $\hat{\pi} \in \mathcal{I}_1$ such that $W(\hat{\pi}) = W(\hat{e})$. From

$$W(\hat{\pi}^2 - \hat{\pi}) = W(\hat{e}^2 - \hat{e}) = 0 \quad \text{and} \quad W(\hat{\pi}^* - \hat{\pi}) = W(\hat{e}^* - \hat{e}) = 0$$

we conclude that both $\hat{\pi}^2 - \hat{\pi}$ and $\hat{\pi}^* - \hat{\pi}$ belong to $\mathcal{I}_1 \cap \mathcal{I}_2$. Since this intersection is trivial, the element $\hat{\pi}$ is a (self-adjoint) projection. Moreover, since

$$W(\hat{a}\hat{\pi} - \hat{\pi}\hat{a}) = W(\hat{a})W(\hat{e}) - W(\hat{e})W(\hat{a}) = 0$$

for every element $\hat{a} \in \mathcal{B}$ we conclude as above that $\hat{\pi}$ lies in the commutant of $\mathcal{B}$. A similar reasoning shows finally that $\hat{\pi}$ is the identity element for $\mathcal{I}_1$. Similarly, $\hat{e} - \hat{\pi}$ belongs to $\mathcal{I}_2$ and is the identity element for $\mathcal{I}_2$.

Let $\pi \in \mathcal{J}_N$ be a representative of the coset $\hat{\pi}$. From Lemma 7.7 we infer that $\|(e - p_n)\pi - \pi\| \to 0$ whence $\|(\hat{e} - \widehat{p_n})\hat{\pi} - \hat{\pi}\| \to 0$ as $n \to \infty$. Since $\hat{\pi}$ is the identity element of $\mathcal{I}_1$ and $\hat{e} - \widehat{p_n} \in \mathcal{I}_1$, this implies $\|\hat{e} - \widehat{p_n} - \hat{\pi}\| \to 0$. The elements $\hat{e} - \widehat{p_n}$ and $\hat{\pi}$ are commuting projections. Thus, $\hat{e} - \widehat{p_n} = \hat{\pi}$ for all sufficiently large $n$, say $n \ge k$. Consequently, for $n \ge k$, one has $\widehat{p_n} = \hat{e} - \hat{\pi} \in \mathcal{I}_2$, whence $p_n \in \widetilde{\mathcal{J}}$. Since $s_i = s_i p_k$ for all $i \in \Omega^k$ by Lemma 7.6, this implies $s_i \in \widetilde{\mathcal{J}}$ and, thus, the smallest closed ideal of $\mathcal{S}_N$ which contains all partial isometries $s_i$ with $|i| = k$ lies in $\widetilde{\mathcal{J}}$. By Lemma 7.6 ($e$), this finally implies $e \in \widetilde{\mathcal{J}}$. Thus, $\widetilde{\mathcal{J}}$ is not a proper ideal of $\widetilde{\mathcal{J}}$. This contradiction excludes case A. ■

**Corollary 7.12** *Every $^*$-homomorphism on $\mathcal{S}_N$ which is injective on $\mathcal{J}_N$ is injective on all of $\mathcal{S}_N$.*

Indeed, if $W$ is a $^*$-homomorphism on $\mathcal{S}_N$ which is injective on $\mathcal{J}_N$, then its kernel is a proper ideal of $\mathcal{S}_N$. By Theorem 7.11, $\ker W \subset \mathcal{J}_N$. But $\mathcal{J}_N \cap \ker W = \{0\}$ by assumption. Hence, the kernel of $W$ is trivial, and $W$ is injective on $\mathcal{S}_N$. ■

In Section 7.5 we are going to construct an injective homomorphism on $\mathcal{J}_N$. We prepare this construction by a closer look at the Cuntz algebra and a related Toeplitz algebra in Sections 7.3 and 7.4.

## 7.3 Block Toeplitz operators

We will have to work with block Toeplitz operators. Let $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ denote the Hilbert space of all sequences $x = (x_n)_{n\ge 0}$ with values in $l^2(\mathbb{Z}^+)$ such that

$$\|x\|^2 := \sum_{n\ge 0} \|x_n\|^2 < \infty.$$

We think of a bounded linear operator $A$ on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ as an infinite matrix $(A_{ij})_{i,j\ge 0}$ where $A_{ij} \in L(l^2(\mathbb{Z}^+))$ is the operator which maps the $i$th component of $x$ to the $j$th component of $Ax$. It will be convenient to identify the algebra of all bounded linear operators on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ with the minimal tensor product $L(l^2(\mathbb{Z}^+)) \otimes L(l^2(\mathbb{Z}^+))$. The closure in $L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ of the set of all matrices with only finitely non-vanishing entries forms a closed ideal of this algebra which we identify with $K(l^2(\mathbb{Z}^+)) \otimes L(l^2(\mathbb{Z}^+))$ in a natural way.

Let $a$ be a function in $C(\mathbb{T}, L(l^2(\mathbb{Z}^+)))$ with $k$th Fourier coefficient

$$a_k := \int_{\mathbb{T}} a(\lambda)\lambda^{-k}\, d\lambda \in L(l^2(\mathbb{Z}^+)), \quad k \in \mathbb{Z}.$$

Then the (infinite) Toeplitz matrix $T(a) := (a_{i-j})_{i,j\ge 0}$ defines a bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ which is called a block Toeplitz operator and denoted by $T(a)$ again. Moreover, $\|T(a)\| = \|a\|_\infty$. The Toeplitz operator $T(a)$ is invertible modulo $K(l^2(\mathbb{Z}^+)) \otimes L(l^2(\mathbb{Z}^+))$ if and only if the function $a$ is invertible in $C(\mathbb{T}, L(l^2(\mathbb{Z}^+)))$.

For $i \in \Omega$, consider the infinite matrix

$$\Sigma_i := \begin{pmatrix} 0 & S_i & & & \\ & 0 & S_i & & \\ & & 0 & S_i & \\ & & & \ddots & \ddots \end{pmatrix} \tag{82}$$

with all entries left empty being zero, and for each multi-index $i \in \Omega^k$, let $\Sigma_i := \Sigma_{i_1} \dots \Sigma_{i_k}$. Clearly, every $\Sigma_i$ is a Toeplitz matrix with continuous generating function. Let $\mathcal{T}_N$ refer to the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ which contains all operators $\Sigma_i$ and $\Sigma_i^*$ with $i \in \Omega$. One easily checks that the operators $\Sigma_i$ are partial isometries which satisfy the Cuntz axiom

$$\sum_{i\in\Omega} \Sigma_i\Sigma_i^* = \sum_{i\in\Omega} \operatorname{diag}(S_iS_i^*,\, S_iS_i^*, \dots) = \operatorname{diag}(I,\, I,\, \dots).$$

Moreover, for each multi-index $i$, the initial projection $\Sigma_i^*\Sigma_i$ is equal to $I - \Pi_{|i|}$ where

$$\Pi_k := \operatorname{diag}(\underbrace{I, \dots, I}_{k}, 0, 0, \dots) \in \mathcal{T}_N \tag{83}$$

for $k \ge 1$. Thus, the algebra $\mathcal{T}_N$ contains the identity operator $I$ and all projections $\Pi_k$, and the partial isometries $\Sigma_i$ satisfy the axioms (A1) and (A2) in Section 7.2 in place of the $s_i$. Thus, all results of this section will remain valid for the algebra $\mathcal{T}_N$ in place of $\mathcal{S}_N$ and for its ideal $\mathcal{C}_N$, which is the smallest closed ideal of $\mathcal{T}_N$ which contains the projection $\Pi_1$, in place of $\mathcal{J}_N$. In particular, $\Pi_n \in \mathcal{C}_N$ for each $n \ge 1$, and $\mathcal{T}_N/\mathcal{C}_N \cong \mathcal{O}_N$.

Our next goal is a description of the ideal $\mathcal{C}_N$ of $\mathcal{T}_N$. Let $\mathsf{O}_N^{par}$ refer to the smallest closed subalgebra of $\mathsf{O}_N$ which contains all products $S_iS_j^*$ with multi-indices $i$ and $j$ of the same length $|i| = |j|$. Here we allow multi-indices of length 0 and set $S_\emptyset := I$. Thus, $\mathsf{O}_N^{par}$ is a unital algebra. One easily checks that $\mathsf{O}_N$ is the closed span of the set of all products $S_iS_j^*$, whereas $\mathsf{O}_N^{par}$ is the closed span of all products $S_iS_j^*$ with $|i| = |j|$. The algebra $\mathsf{O}_N^{par}$ is known to be isomorphic to the UHF-algebra of type $N^\infty$, which is the inductive limit

$$\mathbb{C} \to \mathbb{C}^{N\times N} \to \mathbb{C}^{N^2\times N^2} \to \mathbb{C}^{N^3\times N^3} \to \dots$$

with connecting maps $a \mapsto \mathrm{diag}\,(a,\, a,\, \ldots, a)$. Being an inductive limit of simple algebras, the algebra $\mathsf{O}_N^{par}$ is simple. For details see [10, 11].

We shall further make use of the following elementary observation. Let $\mathcal{A}$ be a $C^*$-subalgebra of a unital $C^*$-algebra $\mathcal{B}$ and let $\Sigma$ be an isometry in $\mathcal{B}$. Then the mapping

$$\mathcal{A} \to \mathcal{B}, \quad A \mapsto \Sigma A \Sigma^* \tag{84}$$

is an injective $^*$-homomorphism. We apply this observation to the algebras $\mathcal{A} := \mathcal{C}_N$ and $\mathcal{B} := L(l^2(\mathbb{Z}^+,\, l^2(\mathbb{Z}^+)))$ and with $\Sigma := \mathrm{diag}\,(I,\, S_0,\, S_0^2,\, S_0^3,\, \ldots)$. Thus, $\mathcal{C}_N$ is $^*$-isomorphic to $\Sigma \mathcal{C}_N \Sigma^*$. The latter algebra can be described as follows where we let $\Pi := \Sigma\Sigma^*$ be the image of the identity operator under the mapping (84).

**Theorem 7.13** $\Sigma \mathcal{C}_N \Sigma^* = \Pi \,(K(l^2(\mathbb{Z}^+)) \otimes \mathsf{O}_N^{par})\, \Pi$.

**Proof.** Since the algebra $\mathcal{T}_N$ is generated by the partial isometries $\Sigma_i$, the algebra $\Sigma \mathcal{T}_N \Sigma^*$ is generated by the operators

$$\Sigma \Sigma_i \Sigma^* = \begin{pmatrix} 0 & S_i S_0^* & & & \\ & 0 & S_0 S_i (S_0^*)^2 & & \\ & & 0 & S_0^2 S_i (S_0^*)^3 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

All entries of this matrix belong to $\mathsf{O}_N^{par}$. Hence,

$$\Sigma \mathcal{T}_N \Sigma^* \subseteq \Pi \left( L(l^2(\mathbb{Z}^+)) \otimes \mathsf{O}_N^{par} \right) \Pi.$$

Further, the mapping (84) sends the generator $\Pi_1$ of the ideal $\mathcal{C}_N$ to itself. Since $\Pi_1 \Pi = \Pi \Pi_1$, one has

$$\Sigma \Pi_1 \Sigma^* = \Pi_1 \in \Pi \left( K(l^2(\mathbb{Z}^+)) \otimes \mathsf{O}_N^{par} \right) \Pi,$$

whence the inclusion $\Sigma \mathcal{C}_N \Sigma^* \subseteq \Pi \,(K(l^2(\mathbb{Z}^+)) \otimes \mathsf{O}_N^{par})\, \Pi$.

The reverse inclusion will follow once we have shown that for each $n \times n$-matrix $A := (A_{ij})$ with entries in $\mathsf{O}_N^{par}$, which we identify with an operator on the range of $\Pi_n$ in the obvious way, the operator $\Pi A \Pi$ belongs to $\Sigma \mathcal{C}_N \Sigma^*$. Due to linearity we can assume that only one of the entries of $A$, say $A_{ij}$, is different from 0. Finally, since $\mathsf{O}_N^{par}$ is spanned by products $S_l S_r^*$ with multi-indices of the same length, we can assume that only the $ij$th entry of $A$ is different from zero and that this entry is $S_l S_r^*$ with $|r| = |l|$. Then $\Pi A \Pi$ is a matrix the only non-vanishing entry of which stands at the $ij$th position, and this entry is

$$S_0^i (S_0^*)^i \, S_l S_r^* \, S_0^j (S_0^*)^j. \tag{85}$$

Let $B := (\Pi_{i+1} - \Pi_i)(\Sigma_0^*)^i \Sigma_l \Sigma_r^* \Sigma_0^j (\Pi_{j+1} - \Pi_j)$. This operator is in $\mathcal{C}_N$, all entries in the matrix representation of $\Sigma B \Sigma^*$ with exception of the $ij$th entry vanish, and the $ij$th entry coincides with (85). Thus, $\Sigma B \Sigma^* = \Pi A \Pi$, which finishes the proof. ■

**Theorem 7.14** *The ideal $\mathcal{C}_N$ of $\mathcal{T}_N$ is simple.*

**Proof.** Let $\mathcal{R}$ be a closed ideal of $\mathcal{C}_N$. Then $\Pi_1 \mathcal{R} \Pi_1$ is a closed ideal of $\Pi_1 \mathcal{C}_N \Pi_1$. From Theorem 7.13 we infer that $\Pi_1 \mathcal{C}_N \Pi_1$ is $^*$-isomorphic to the algebra $\mathsf{O}_N^{par}$, which is simple as already mentioned. Hence, either $\Pi_1 \mathcal{R} \Pi_1 = \Pi_1 \mathcal{C}_N \Pi_1$ or $\Pi_1 \mathcal{R} \Pi_1 = \{0\}$. In the first case, $\Pi_1 \in \Pi_1 \mathcal{R} \Pi_1 \subseteq \mathcal{R}$. Since $\Pi_1$ generates $\mathcal{C}_N$ as an ideal, we conclude $\mathcal{R} = \mathcal{C}_N$.

Assume now that $\Pi_1 \mathcal{R} \Pi_1 = \{0\}$. Let $R = (R_{ij})_{ij \ge 0} \in \mathcal{R}$. Then, for arbitrary subscripts $i_0$, $j_0 \ge 0$, the matrix $(\Pi_{i_0+1} - \Pi_{i_0}) R (\Pi_{j_0+1} - \Pi_{j_0})$ has the entry $R_{i_0 j_0}$ at the $i_0 j_0$th position whereas all other entries are zero. Let $k$ and $l$ be multi-indices with $|k| = i_0$ and $|l| = j_0$. Then the matrix

$$\Sigma_k (\Pi_{i_0+1} - \Pi_{i_0}) R (\Pi_{j_0+1} - \Pi_{j_0}) \Sigma_l^*$$

has the entry $S_k R_{i_0 j_0} S_l^*$ at the 00th position whereas all other entries are zero. Thus, this matrix belongs to $\Pi_1 \mathcal{R} \Pi_1$, whence $S_k R_{i_0 j_0} S_l^* = 0$ by assumption. Since the $S_i$ are isometries, this implies $R_{i_0 j_0} = 0$, and since $i_0$ and $j_0$ were arbitrarily chosen, $R$ is the zero matrix. Thus, $\mathcal{R}$ is the zero ideal. ■

**Corollary 7.15** *$\mathcal{C}_N$ is the only non-trivial closed ideal of $\mathcal{T}_N$.*

## 7.4 Expectations on $\mathsf{O}_N$ and Toeplitz operators

There are at least two ways to associate with every element of the Cuntz algebra $\mathsf{O}_N$ a Toeplitz operator in $\mathcal{T}_N$ . For facts cited without proof see [10, 11].

The first way is via a special expectation. Recall that the operators $S_i$ and $S_j^*$ with $i, j \in \Omega$ generate a dense subalgebra of $\mathsf{O}_N$. Each operator $A$ in this algebra can be uniquely written as a finite sum

$$A = \sum_{k<0} (S_0^*)^{-k} A_k + A_0 + \sum_{k>0} A_k S_0^k \tag{86}$$

with coefficients $A_k \in \mathsf{O}_N^{par}$. For $k \in \mathbb{Z}$ and $A$ as in (86), define $\Phi_k(A) := A_k$. Then $\|\Phi_k(A)\| \le \|A\|$, thus the $\Phi_k$ extend by continuity to bounded mappings from $\mathsf{O}_N$ onto $\mathsf{O}_N^{par}$. These mappings own the following properties:

- $\Phi_0 : \mathsf{O}_N \to \mathsf{O}_N^{par}$ is an expectation, i.e. $\Phi_0^2 = \Phi_0$,
- $\Phi_{k+1}(A) = \Phi_k(A S_0^*)$ if $k \ge 0$, and
- $\Phi_{k-1}(A) = \Phi_k(S_0 A)$ if $k < 0$.

We associate with each operator $A \in \mathsf{O}_N$ a matrix of operators on $l^2(\mathbb{Z}^+)$ by

$$\Psi(A) := \big( (S_0^*)^i \Phi_0 (S_0^i A (S_0^*)^j) S_0^j \big)_{i,j \ge 0} . \tag{87}$$

We will see in a moment that the formal matrix $\Psi(A)$ defines a bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ and that this operator is a Toeplitz operator in $\mathcal{T}_N$.

A second way to associate with every operator in $\mathsf{O}_N$ a Toeplitz operator in $\mathcal{T}_N$ is via continuous functions. Let $\lambda \in \mathbb{T}$. Then the mapping $\rho_\lambda : S_i \mapsto \bar{\lambda} S_i$ extends to an automorphism of $\mathsf{O}_N$. Here, as usual, $\bar{\lambda}$ stands for the complex conjugate of $\lambda$; note that the mapping $\rho_\lambda$ is defined in [11] without the bar. For each operator $A \in \mathsf{O}_N$, consider the function

$$f_A : \mathbb{T} \to \mathsf{O}_N, \quad \lambda \mapsto \rho_\lambda(A). \tag{88}$$

**Lemma 7.16** *The function $f_A$ is continuous for each $A \in \mathsf{O}_N$, and $\|f_A\|_\infty = \|A\|$.*

**Proof.** For each $\lambda \in \mathbb{T}$, one has $\|f_A(\lambda)\| = \|\rho_\lambda(A)\| \le \|A\|$, whence $\|f_A\|_\infty \le \|A\|$. Since $f_A(1) = A$, equality holds in this estimate. Since $A \mapsto f_A$ is a linear mapping, this implies that $\|f_A - f_B\|_\infty = \|A - B\|$ for all $A, B \in \mathsf{O}_N$. Choose operators $B_n$ in the dense subalgebra of $\mathsf{O}_N$ generated by the isometries $S_i$ such that $\|A - B_n\| \to 0$ as $n \to \infty$. Then $\|f_A - f_{B_n}\| \to 0$. Being a uniform limit of the continuous functions $f_{B_n}$, the function $f_A$ is continuous. ∎

As in Section 6.1, we associate with the continuous function $f_A$ the sequence of its Fourier coefficients and consider the associated Toeplitz operator $T(f_A)$ on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$. From $\|T(a)\| = \|a\|_\infty$ and Lemma 7.16 we conclude that

$$\|T(f_A)\| = \|f_A\|_\infty = \|A\| \quad \text{for every } A \in \mathsf{O}_N. \tag{89}$$

Since the products $S_l S_m^*$ span a dense subalgebra of $\mathsf{O}_N$ and since the mapping $A \mapsto T(f_A)$ is an isometry, we conclude that $T(f_A)$ is a Toeplitz operator *in* $\mathcal{T}_N$ for every $A \in \mathsf{O}_N$.

We will see now that the two ways lead to the same goal.

**Theorem 7.17** *The mapping $\Psi$ is a linear contraction from $\mathsf{O}_N$ into $\mathcal{T}_N$. It coincides with the mapping $A \mapsto T(f_A)$.*

**Proof.** It is not hard to check that $T(f_A) = \Psi(A)$ for $A = S_l S_m^*$. Since these products span a dense subalgebra of $\mathsf{O}_N$ and $A \mapsto T(f_A)$ and $\Psi$ are linear mappings, this implies that $T(f_A) = \Psi(A) \in \mathcal{T}_N$ for all $A$ in a dense subalgebra of $\mathsf{O}_N$. Since the mapping $A \mapsto T(f_A)$ is an isometry, we further conclude that $\|\Psi(A)\| = \|T(f_A)\| = \|A\|$ for all $A$ in a dense subalgebra of $\mathsf{O}_N$. Thus, the mapping $\Psi$ can be continued to a linear contraction from $\mathsf{O}_N$ into $\mathcal{T}_N$ (which, of course, coincides with the mapping $A \mapsto T(f_A)$). Since each entry of the matrix (87) depends continuously on $A$, this contractive continuation coincides with the formal matrix in (87). ∎

The classical Toeplitz algebra decomposes into the direct sum of the linear space $\{T(f) : f \in C(\mathbb{T})\}$ and the ideal of the compact operators. A similar decomposition holds for the algebra $\mathcal{T}_N$.

**Theorem 7.18** $\mathcal{T}_N = \{T(f_A) : A \in \mathsf{O}_N\} \oplus \mathcal{C}_N = \{\Psi(A) : A \in \mathsf{O}_N\} \oplus \mathcal{C}_N$.

**Proof.** We reify Proposition 2.1 with the following algebras and mappings: $\mathsf{A}$ is the smallest closed subalgebra of $L(l^2(\mathbb{Z}, l^2(\mathbb{Z}^+)))$ which contains all operators of Laurent type represented by the two-sided infinite matrix

$$\begin{pmatrix} \ddots & \ddots & & & \\ & 0 & S_i & & \\ & & 0 & S_i & \\ & & & \ddots & \ddots \end{pmatrix}$$

with the zeros standing on the main diagonal, $\mathcal{B}$ is the algebra $\mathcal{T}_N$, $D$ is the mapping $\mathsf{A} \to \mathcal{B}$, $A \mapsto PAP$ where $P$ is the orthogonal projection from $l^2(\mathbb{Z}, l^2(\mathbb{Z}^+))$ onto $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$, and $W$ is the mapping $\mathcal{B} \to \mathsf{A}$, $B \mapsto \text{s-lim}_{n\to+\infty} V_n^* B V_n$ where $V$ is the operator of forward shift on $l^2(\mathbb{Z}, l^2(\mathbb{Z}^+))$ and $V_n := V^n$ for $n \geq 0$, and where s-lim refers to the limit in the strong operator topology.

Then Proposition 2.1 implies that $\mathcal{T}_N = \{T(f_A) : A \in \mathsf{O}_N\} \oplus \ker W$, and it remains to verify that

$$\mathcal{C}_N \ (:= \text{closid}_{\mathcal{T}_N} \{\Pi_1\}) = \ker W. \tag{90}$$

Evidently, $\Pi_1 \in \ker W$, whence the inclusion $\mathcal{C}_N \subseteq \ker W$. To get the reverse implication, we show that the quasicommutator ideal of $\mathcal{T}_N$ lies in $\mathcal{C}_N$. Since the products $S_l S_m^*$ with multi-indices $l$, $m$ span a dense subalgebra of $\mathsf{O}_N$, this fact will follow once we have shown that

$$\Psi(S_l S_m^*)\Psi(S_n S_r^*) - \Psi(S_l S_m^* S_n S_r^*) \in \mathcal{C}_N \tag{91}$$

for each choice of multi-indices $l$, $m$, $n$ and $r$. A straightforward calculation gives

$$\begin{aligned} &\Psi(S_l S_m^*)\, \Psi(S_n S_r^*) - \Psi(S_l S_m^* S_n S_r^*) \\ &\quad = \begin{cases} -\Pi_{|m|-|l|}\, \Psi(S_l S_m^* S_n S_r^*) & \text{if} \quad |m| - |l| > 0 \\ 0 & \text{if} \quad |m| - |l| \leq 0. \end{cases} \end{aligned}$$

Since $\Pi_n \in \mathcal{C}_N$ for every $n \geq 1$, the inclusion (91) follows. ■

## 7.5 The algebra $(e - p_1)\mathcal{S}_N(e - p_1)$

In order to define the desired lifing homomorphism we will need some facts on the algebra $(e - p_1)\mathcal{S}_N(e - p_1)$. To derive them, it will be convenient to compare and to operate with multi-indices. Given multi-indices $i = (i_1, \ldots, i_k) \in \Omega^k$ and $j = (j_1, \ldots, j_l) \in \Omega^l$ we define their sum as the multi-index

$$i + j := (i_1, \ldots, i_k, j_1, \ldots, j_l) \in \Omega^{k+l}.$$

Differences of multi-indices can be defined if one of the multi-indices is a part of the other. Since addition of multi-indices is not commutative, we consider differences from the left and from the right. More precisely, we write $i \prec k$ if there is a multi-index $j$ such that $i + j = k$ and $k \succ j$ if there is a multi-index $i$ with $i + j = k$. The multi-indices $j$ and $i$ are uniquely determined, and we denote them by $j := (-i) + k$ and $i := k - j$, respectively. Note that it follows from (72) that the product $s_i^* s_j$ is not zero only if $i \prec j$ or $j \prec i$. The following lemmas show how to simplify certain products in the $s_i$ and $p_k$. The simple proofs are omitted.

**Lemma 7.19** *Let $i$, $j$, $k$, $l$ be multi-indices (not necessarily of the same length). Then the product $(e - p_1) s_i s_j^* s_k s_l^*$ can be written as $(e - p_1) s_r s_t^*$ with multi-indices $r$ and $t$ such that $|r| \geq |t|$, or this product is zero.*

Repeated application of this lemma gives the following.

**Corollary 7.20** *Let $i$, $j$, $k$, $l$, $\ldots$, $m$, $n$ be multi-indices (not necessarily of the same length). Then the product $(e - p_1) s_i s_j^* s_k s_l^* \ldots s_m s_n^* (e - p_1)$ can be written as $(e - p_1) s_r s_t^* (e - p_1)$ with multi-indices $r$ and $t$ of the same length, or this product is zero.*

**Corollary 7.21** *Let $a \in \mathcal{S}_N$. Then $(e - p_1) a (e - p_1)$ can be approximated as closely as desired by linear combinations of elements of the form $(e - p_1) s_r s_t^* (e - p_1)$ with multi-indices $r$ and $t$ of the same length.*

Let $\mathcal{S}_N^{par}$ stand for the smallest closed subalgebra of $\mathcal{S}_N$ which contains all products $s_i s_j^*$ with multi-indices $i$, $j$ of the same length. Again we allow multi-indices of length zero, for which we set $s_\emptyset := e$.

**Lemma 7.22** $\mathcal{S}_N^{par} = \operatorname{clos\,span} \{ s_i s_j^* : |i| = |j| \}$.

**Proof.** Let $i$, $j$, $k$, $l$ be multi-indices with $|i| = |j|$ and $|k| = |l|$. We have to show that the product $(s_i s_j^*)(s_k s_l^*)$ can be written as $s_r s_t^*$ with multi-indices $r$, $t$ of the same length. This product is zero if not $j \prec k$ or $k \prec j$. For instance, if $j \prec k$, then $s_i s_j^* s_k s_l^* = s_{i + ((-j) + k)} s_l^*$ with $|i + ((-j) + k)| = |i| + |k| - |j| = |l|$. ■

Thus, another way to state the assertion of Corollary 7.21 is the following:

$$\text{If } a \in \mathcal{S}_N \quad \text{then} \quad (e - p_1) a (e - p_1) \in (e - p_1)\, \mathcal{S}_N^{par}\, (e - p_1). \tag{92}$$

The mapping

$$\mathcal{S}_N \to \mathcal{S}_N^{par}, \quad a \mapsto (e - p_1) a (e - p_1) \tag{93}$$

is an expectation which is related with the expectation $\Phi_0 : \mathsf{O}_N \to \mathsf{O}_N^{par}$.

**Proposition 7.23** *Let $A \in \mathsf{O}_N$, $a := (P_{N^n} A P_{N^n})_{n\geq 0} + \mathcal{G}_\eta$, and $i, j \in \mathbb{Z}^+$. Then*

$$\begin{aligned}&(e-p_1)s_0^i a (s_0^*)^j (e-p_1)\\ &\quad = (e-p_1)\,\big((P_{N^n}\Phi_0(S_0^i A (S_0^*)^j)P_{N^n})_{n\geq 0} + \mathcal{G}_\eta\big)\,(e-p_1). \end{aligned} \tag{94}$$

**Proof.** Because of

$$\begin{aligned} s_0^i a (s_0^*)^j &= (P_{N^n} S_0^i P_{N^n})\,(P_{N^n} A P_{N^n})\,(P_{N^n}(S_0^*)^j P_{N^n}) + \mathcal{G}_\eta \\ &= (P_{N^n} S_0^i P_{N^n} A P_{N^n} (S_0^*)^j P_{N^n}) + \mathcal{G}_\eta \\ &= (P_{N^n} S_0^i A (S_0^*)^j P_{N^n}) + \mathcal{G}_\eta, \end{aligned}$$

it is sufficient to prove the assertion for $i = j = 0$. Thus, we have to show that

$$(e-p_1)a(e-p_1) = (e-p_1)\,((P_{N^n}\Phi_0(A)P_{N^n})_{n\geq 0} + \mathcal{G}_\eta)\,(e-p_1). \tag{95}$$

Both sides of this equality depend linearly and continuously on $A$. It is thus sufficient to verify (95) for $A = S_k S_l^*$ with multi-indices $k, l$ of arbitrary length. If $k$ and $l$ are of the same length, then $A \in \mathsf{O}_N^{par}$, whence $\Phi_0(A) = A$, whereas otherwise $\Phi_0(A) = 0$. Thus, the right-hand side of (95) is equal to $(e-p_1)a(e-p_1)$ if $|k| = |l|$ and zero otherwise. Since

$$P_{N^n} S_k S_l^* P_{N^n} = P_{N^n} S_k P_{N^n} \cdot P_{N^n} S_l^* P_{N^n},$$

the left-hand side of (95) is also zero whenever $|k| \neq |l|$. ∎

The desired lifting homomorphism will be defined explicitly by means of strong limits which involve the following reflection operators. For $n \geq 1$, let

$$R_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k\geq 0} \mapsto (x_{n-1}, x_{n-2}, \ldots, x_0, 0, 0, \ldots).$$

**Proposition 7.24** *Let $a \in \mathcal{S}_N$ and write $(e-p_1)a(e-p_1)$ as $(A_{N^n}) + \mathcal{G}_\eta$. Then the strong limit*

$$\text{s-lim}_{n\to\infty} R_{N^n} A_{N^n} R_{N^n} \tag{96}$$

*exists, and the limit is independent of the choice of the representative of the coset $(e-p_1)a(e-p_1)$.*

**Sketch of the proof.** Evidently, the limit (96) is independent of the choice of the representative. By Corollary 7.21, it is sufficient to prove the existence of the strong limit (96) for sequences $(A_{N^n})$ which belong to the coset $e - p_1$ or to the coset $s_i s_j^*$ with multi-indices $i$, $j$ of the same length. Choose the sequence $(P_{N^n}) - (P_{N^n} S_0^* P_{N^n})\,(P_{N^n} S_0 P_{N^n})$ as representative of the coset $e - p_1$. It is not hard to check that

$$R_{N^n}(P_{N^n} - P_{N^n} S_0^* P_{N^n} S_0 P_{N^n}) R_{N^n} \to I \quad \text{strongly} \tag{97}$$

Next we choose $(P_{N^n}S_iS_j^*P_{N^n})$ as representative of the coset $s_is_j^*$. To state the result, let $V$ denote the shift operator

$$V : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k\geq 0} \mapsto (0,\, x_0,\, x_1,\, \ldots),$$

set $V_n := V^n$ and $V_{-n} := (V^*)^n$ for every positive integer $n$, and define $V_0 := I$. Further, let

$$\Pi_k := \operatorname{diag}(1,\, \underbrace{0,\, \ldots,\, 0,}_{N^k-1}\, 1,\, \underbrace{0,\, \ldots,\, 0,}_{N^k-1} \ldots).$$

and recall that $v_{i,k} := i_1 + i_2N + \ldots + i_kN^{k-1}$. Then a somewhat tedious but straightforward calculation shows

$$\begin{aligned} R_{N^n}P_{N^n}S_iS_j^*P_{N^n}R_{N^n} &\to V_{v_{j,k}-v_{i,k}}V_{N^k-v_{j,k}-1}\Pi_kV^*_{N^k-v_{j,k}-1} \\ &= V_{N^k-v_{i,k}-1}\Pi_kV^*_{N^k-v_{j,k}-1} \end{aligned} \tag{98}$$

where the latter equality holds since $N^k - v_{j,k} - 1 \geq 0$. ■

It will be useful to write the operator (98) in a different form. Note that

$$S_iS_j^* = V_{v_{i,k}}V_{-v_{j,k}}S_jS_j^* = V_{v_{i,k}}V_{-v_{j,k}}V_{v_{j,k}}\Pi_kV^*_{v_{j,k}} = V_{v_{i,k}}\Pi_kV^*_{v_{j,k}}.$$

A comparison with (98) suggests to introduce the *dual index* $\hat{\imath}$ of a multi-index $i = (i_1,\, i_2,\, \ldots,\, i_k)$ by $\hat{\imath} := (N-1-i_1,\, N-1-i_2,\, \ldots,\, N-1-i_k)$. Evidently, $|\hat{\imath}| = |i| = k$, and one easily checks that

$$v_{\hat{\imath},k} = (N-1-i_1) + (N-1-i_2)N + \ldots + (N-1-i_k)N^{k-1} = N^k - 1 - v_{i,k}.$$

Hence,

$$V_{N^k-v_{i,k}-1}\Pi_kV^*_{N^k-v_{j,k}-1} = V_{v_{\hat{\imath},k}}\Pi_kV^*_{v_{\hat{\jmath},k}} = S_{\hat{\imath}}S_{\hat{\jmath}}^*.$$

For each $i \in \Omega$ we set $S_i^\sharp := S_{\hat{\imath}} = S_{N-1-i}$. Due to the universal property of $\mathcal{O}_N$, the mapping $^\sharp : S_i \mapsto S_{N-1-i}$ can be continued to an automorphism of $\mathsf{O}_N$.

**Corollary 7.25** *If $i$ and $j$ are multi-indices of the same length then*

$$R_{N^n}P_{N^n}S_iS_j^*P_{N^n}R_{N^n} \to (S_iS_j^*)^\sharp \quad \textit{strongly as } n \to \infty. \tag{99}$$

## 7.6 The lifting homomorphism

We denote the strong limit (96) by $W_{00}(a)$ and consider $W_{00}$ as a mapping from $\mathcal{S}_N$ into $L(l^2(\mathbb{Z}^+))$. More general, for $i,\, j \in \mathbb{Z}^+$, let $W_{ij} : \mathcal{S}_N \to L(l^2(\mathbb{Z}^+))$ refer to the operator

$$a \mapsto (S^*_{N-1})^i\, W_{00}((e-p_1)s_0^ia(s_0^*)^j(e-p_1))S^j_{N-1}$$

which is consistent with the previous definition. With every element $a \in \mathcal{S}_N$, we associate the infinite matrix

$$\widetilde{W}(a) := (W_{ij}(a))_{i,\, j \geq 0} \tag{100}$$

the entries of which are operators on $L(l^2(\mathbb{Z}^+))$. For example,

$$\widetilde{W}(s_r) = \Sigma_{N-1-r}, \quad \widetilde{W}(s_r^*) = \Sigma_{N-1-r}^*, \quad \text{and} \quad \widetilde{W}(p_1) = I - \Pi_1. \tag{101}$$

We are going to show that the matrix (100) defines a linear bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ and that the mapping $\widetilde{W}$ is the desired injective lifting homomorphism. The following is the main result of this section.

**Theorem 7.26** *The mapping $\widetilde{W}$ defined by* (100) *is a $^*$-isomorphism from $\mathcal{S}_N$ onto $\mathcal{T}_N$ which maps the ideal $\mathcal{J}_N$ onto $\mathcal{C}_N$.*

**Corollary 7.27** *$\mathcal{J}_N$ is the only non-trivial closed ideal of $\mathcal{S}_N$.*

Indeed, this follows immediately from Theorem 7.26 and Corollary 7.15. ■

**Sketch of the proof of Theorem 7.26.** It is not too hard to verify that the mapping $\widetilde{W}$ acts as a $^*$-homomorphism on a *dense subalgebra* of $\mathcal{S}_N$. That it acts as a $^*$-homomorphism on the whole algebra would easily follow from this fact if one would know that $\widetilde{W}$ is bounded. Conversely, the boundedness of $\widetilde{W}$ comes as a simple consequence of the fact that $\widetilde{W}$ acts as a $^*$-homomorphism on $\mathcal{S}_N$. Unfortunately, neither the boundedness of $\widetilde{W}$ nor the homomorphy of $\widetilde{W}$ on all of $\mathcal{S}_N$ could be shown directly. Rather we have to prove both properties simultaneously by climbing step by step from small substructures of $\mathcal{S}_N$ to the whole algebra. I will not present each detail of this quite cumbersome (but mainly straightforward) way and only mark the essential steps.

We will make use of the fact that the algebra $\mathcal{S}_N$ splits into the direct sum

$$\{(P_{N^n} A P_{N^n})_{n \geq 0} + \mathcal{G}_\eta : A \in \mathsf{O}_N\} \oplus \mathcal{J}_N. \tag{102}$$

**Proposition 7.28** *The mapping $\widetilde{W}$ acts as a linear contraction on the first summand of* (102)*, and it maps this summand into $\mathcal{T}_N$.*

For a proof note that the mapping

$$a = (P_{N^n} A P_{N^n}) + \mathcal{G}_\eta \;\mapsto\; A = \text{s-lim} P_{N^n} A P_{N^n}$$

is a linear contraction by the Banach-Steinhaus theorem. Further, as already mentioned, the mapping $A \mapsto A^\sharp$ is a linear contraction (and even an isometry) on $\mathsf{O}_N$. Finally, from Theorem 7.17 we infer that the mapping $A^\sharp \mapsto \Psi(A^\sharp)$ is a linear contraction with range in $\mathcal{T}_N$. It remains to show that $\Psi(A^\sharp) = \widetilde{W}(a)$ which follows by direct computation. ■

Now we consider the second summand in (102). Abbreviate $e-p_n$ to $\pi_n$ and recall the definition (83) of $\Pi_n$. Our starting point is the action of $\widetilde{W}$ on $\pi_n \mathcal{J}_N \pi_n$. The calculations are quite lengthy but, since we are dealing with $n\times n$-block matrices, purely algebraic.

**Proposition 7.29** *The mapping $\widetilde{W}$ is a $^*$-homomorphism from $\pi_n \mathcal{J}_N \pi_n$ into $\Pi_n \mathcal{C}_N \Pi_n$ for every $n \geq 1$.*

Consequently, $\widetilde{W}$ is a contraction on $\pi_n \mathcal{J}_N \pi_n$ for every $n$. Next extend this property to all of $\mathcal{J}_N$.

**Corollary 7.30** *The mapping $\widetilde{W} : \mathcal{J}_N \to \mathcal{C}_N$ is a linear contraction.*

To see this, let $j \in \mathcal{J}_N$. From Lemma 7.7 we infer that $\lim_{n\to\infty} \|j-\pi_n j \pi_n\| = 0$ for each $j \in \mathcal{J}_N$. Hence, $(\pi_n j \pi_n)_{n\geq 1}$ is a Cauchy sequence. Since $\widetilde{W}$ is a contraction on $\pi_n \mathcal{J}_N \pi_n$ for each $n$ and $\pi_n j \pi_n \in \pi_m \mathcal{J}_N \pi_m$ for $m \geq n$, we conclude that

$$\|\widetilde{W}(\pi_n j \pi_n) - \widetilde{W}(\pi_m j \pi_m)\| \leq \|\pi_n j \pi_n - \pi_m j \pi_m\|$$

whenever $m \geq n$. Hence, $(\widetilde{W}(\pi_n j \pi_n))_{n\geq 1}$ is a Cauchy sequence. Let $J$ denote its limit. Further, since the entries of the matrix mapping $\widetilde{W}$ are continuous, we conclude from $\|\widetilde{W}(\pi_n j \pi_n) - J\| \to 0$ that $J = \widetilde{W}(j)$. Now it is clear that $\widetilde{W}(j) \in \mathcal{C}_N$ and $\|\widetilde{W}(j)\| \leq \|j\|$ for every $j \in \mathcal{J}_N$. ■

**Corollary 7.31** *The mapping $\widetilde{W} : \mathcal{J}_N \to \mathcal{C}_N$ is a $^*$-homomorphism.*

Indeed, having now the boundedness of $\widetilde{W}$ at our disposal, we can argue as follows to get that $\widetilde{W}$ is a multiplicative on $\mathcal{J}_N$. Let $j_1$, $j_2 \in \mathcal{J}_N$. By Lemma 7.7, $j_1 j_2 = \lim \pi_n j_1 \pi_n j_2 \pi_n$. Since $\widetilde{W}$ is continuous on $\mathcal{J}_N$,

$$\widetilde{W}(j_1 j_2) = \lim \widetilde{W}(\pi_n j_1 \pi_n j_2 \pi_n).$$

Now the multiplicativity of $\widetilde{W}$ on $\pi_n \mathcal{J}_N \pi_n$ entails

$$\widetilde{W}(j_1 j_2) = \lim \widetilde{W}(\pi_n j_1 \pi_n) \widetilde{W}(\pi_n j_2 \pi_n) = \widetilde{W}(j_1)\widetilde{W}(j_2),$$

whence the assertion. ■

**Corollary 7.32** *The mapping $\widetilde{W}$ is bounded on all of $\mathcal{S}_N$.*

This can be seen as follows. Let $(A_n) + \mathcal{G}_\eta \in \mathcal{S}_N$. In accordance with (102), we write this coset as

$$(A_n) + \mathcal{G}_\eta = ((P_{N^n} A P_{N^n}) + \mathcal{G}_\eta) + ((J_n) + \mathcal{G}_\eta) =: a + j \qquad (103)$$

with $A :=$ s-lim$A_n P_{N^n}$ and $j = (J_n) + \mathcal{G}_\eta \in \mathcal{J}_N$. Then

$$\|a\| = \|(P_{N^n} A P_{N^n}) + \mathcal{G}_\eta\| \leq \|A\| \leq \|(A_n) + \mathcal{G}_\eta\|,$$

i.e., the first summand in (103) depends continuously on $(A_n) + \mathcal{G}_\eta$. From $\|a\| \leq \|a+j\|$ we obtain $\|j\| \leq \|a+j\| + \|a\| \leq 2\,\|a+j\|$, whence

$$\|\widetilde{W}(a+j)\| \leq \|\widetilde{W}(a)\| + \|\widetilde{W}(j)\| \leq \|a\| + \|j\| \leq 3\,\|a+j\|$$

due to Proposition 7.28 and Corollary 7.30. ■

**Proposition 7.33** *The mapping $\widetilde{W}$ is a $^*$-homomorphism from $\mathcal{S}_N$ into $\mathcal{T}_N$.*

It is clearly sufficient to verify that $\widetilde{W}$ is multiplicative on $\mathcal{S}_N$. The proof starts with a partial multiplicativity result,

$$\widetilde{W}(a\pi_k) = \widetilde{W}(a)\Pi_k \quad \text{for each } a \in \mathcal{S}_N, \tag{104}$$

which is shown be simple calculation. For the proof of the general assertion, let $a, b \in \mathcal{S}_N$. Since $\pi_k \in \mathcal{J}_N$ for every $k$ and $\widetilde{W}$ is multiplicative on $\mathcal{J}_N$, we get $\widetilde{W}(a\pi_m b\pi_k) = \widetilde{W}(a\pi_m)\widetilde{W}(b\pi_k)$ for all $k, m \geq 0$. By (104), $\widetilde{W}(a\pi_m b\pi_k) = \widetilde{W}(a)\Pi_m\widetilde{W}(b)\Pi_k$. For $m \to \infty$ we have $a\pi_m b\pi_k \to ab\pi_k$ by Lemma 7.7 (recall that $\pi_k \in \mathcal{J}_N$) and $\Pi_m \to I$ strongly. Thus, due to the continuity of $\widetilde{W}$,

$$\widetilde{W}(ab\pi_k) = \widetilde{W}(a)\widetilde{W}(b)\Pi_k.$$

Invoking (104) again and letting $k$ tend to infinity, we arrive at the assertion. ■

**Proposition 7.34** *The mapping $\widetilde{W}$ is injective on $\mathcal{J}_N$.*

The assertion will follow once we have shown that

$$\widetilde{W} \text{ is injective on } \pi_n\mathcal{J}_N\pi_n \text{ for every } n \geq 1. \tag{105}$$

Indeed, let (105) be satisfied, and let $j \in \mathcal{J}_N$ be an element with $\widetilde{W}(j) = 0$. Then $\Pi_n\widetilde{W}(j)\Pi_n = 0$ for every $n$. By (104), this implies $\widetilde{W}(\pi_n j\pi_n) = 0$ for every $n$. From (105) we infer that $\pi_n j\pi_n = 0$ for every $n$. Passage to the limit $n \to \infty$ yields $j = 0$, which implies the desired injectivity.

Further, (105) will follow once we have shown that

$$\widetilde{W} \text{ is injective on } \pi_1\mathcal{J}_N\pi_1 = (e-p_1)\mathcal{J}_N(e-p_1). \tag{106}$$

The reason is that, roughly speaking, $\pi_n\mathcal{J}_N\pi_n$ is constituted by a finite number of shifted copies of $\pi_1\mathcal{J}_N\pi_1$.

So it remains to prove (106). Let $j \in \mathcal{J}_N$ and $\widetilde{W}((e-p_1)j(e-p_1)) = 0$. One can show easily that $(e-p_1)j(e-p_1)$ can be written as

$$(e-p_1)\left((P_{N^n}CP_{N^n})_{n\geq 0} + \mathcal{G}_\eta\right)(e-p_1) \quad \text{with } C \in \mathsf{O}_N^{par}.$$

Consequently, $0 = W_{00}((e-p_1)j(e-p_1)) = \Phi_0(C^\sharp) = C^\sharp$ since $\Phi$ is an expectation from $\mathsf{O}_N$ onto $\mathsf{O}_N^{par}$ and $C$ belongs to the latter subalgebra. Thus, $C = 0$, which implies that $(e - p_1)j(e - p_1)$. The injectivity of $\widetilde{W}$ on $\pi_1 \mathcal{J}_N \pi_1$ follows. ■

Now we can finish the proof of Theorem 7.26 as follows. The mapping $\widetilde{W}$ is an injective $^*$-homomorphism on $\mathcal{J}_N$ as we have just seen. Hence, by Corollary 7.12, $\widetilde{W}$ is an injective $^*$-homomorphism on $\mathcal{S}_N$. The range of this homomorphism contains the generating operators $\Sigma_k$ of $\mathcal{T}_N$; thus, $\widetilde{W}$ maps $\mathcal{S}_N$ onto $\mathcal{T}_N$. Since $\widetilde{W}$ maps the generating element $e - p_1$ of the ideal $\mathcal{J}_N$ to the generating element $\Pi_1$ of $\mathcal{C}_N$, it is further clear that $\widetilde{W}$ maps $\mathcal{J}_N$ onto $\mathcal{C}_N$. ■

## 7.7 Some consequences

**Stability and fractality.** The assertion of Theorem 7.26 is equivalent to the following stability criterion.

**Theorem 7.35** *A sequence* $\mathbf{A} = (A_n)$ *in* $\mathcal{S}_\eta(\mathsf{O}_N)$ *is stable if and only if the operator* $\widetilde{W}(\mathbf{A} + \mathcal{G}_\eta)$ *is invertible.*

Specifying this result to finite sections sequences for operators in the Cuntz algebra yields

**Corollary 7.36** *Let* $A \in \mathsf{O}_N$. *Then the sequence* $(P_{N^n} A P_{N^n})_{n \geq 0}$ *is stable if and only if the block Toeplitz operator* $\Psi(A^\sharp) = T(f_{A^\sharp}) \in L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ *is invertible.*

The following is certainly the most important consequence of Theorem 7.26. It can also serve as a perfect illustration to Theorem 3.8. The proof will follow directly from the special form of the homomorphism $\widetilde{W}$.

**Corollary 7.37** *The algebra* $\mathcal{S}_\eta(\mathsf{O}_N)$ *is fractal.*

**Proof.** Recall that the entries of the matrix operator $\widetilde{W}((A_n)+\mathcal{G}_\eta)$ are defined by strong limits. Consequently, if only an (infinite) subsequence of $(A_n)$ is known, one can nevertheless determine the operator $\widetilde{W}((A_n) + \mathcal{G}_\eta) \in \mathcal{T}_N$. Since $\widetilde{W} : \mathcal{S}_N \to \mathcal{T}_N$ is an isomorphism one can, thus, reconstruct the coset of $(A_n)$ modulo $\mathcal{G}_\eta$ from each subsequence of $(A_n)$. ■

**Spectral approximation.** As already mentioned, sequences in fractal algebras are distinguished by their excellent convergence properties. To mention only a few of them, let $\sigma(a)$ denote the spectrum of an element $a$ of a $C^*$-algebra with identity element $e$, write $\sigma_2(a)$ for the set of the singular values of $a$, i.e., $\sigma_2(a)$ is the set of all non-negative square roots of elements in the spectrum of $a^*a$ and finally, for $\varepsilon > 0$, let $\sigma^{(\varepsilon)}(a)$ refer to the $\varepsilon$-pseudospectrum of $a$, i.e. to the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is not invertible or $\|(a - \lambda e)^{-1}\| \geq 1/\varepsilon$.

**Theorem 7.38** *Let $(A_n)$ be a sequence in $\mathcal{S}_\eta(\mathsf{O}_N)$ and set $a := (A_n) + \mathcal{G}_\eta$. Then the following set-sequences converge with respect to the Hausdorff metric as $n \to \infty$:*

*(a)* $\sigma(A_n) \to \sigma(\widetilde{W}(a))$ *if $a$ is self-adjoint;*

*(b)* $\sigma_2(A_n) \to \sigma_2(\widetilde{W}(a))$*;*

*(c)* $\sigma^{(\varepsilon)}(A_n) \to \sigma^{(\varepsilon)}(\widetilde{W}(a))$.

The proof follows immediately from the stability criterion in Theorem 7.26 above and from Theorems 3.20, 3.23 and 3.33 in [13]. Let us emphasize that in general one cannot remove the assumption $a = a^*$ in assertion $(a)$, whereas $(c)$ holds without any assumption.

**Compactness and Fredholm properties.** Recall the definition of the algebra $\mathcal{F}$ of all bounded sequences of matrices and of its ideal $\mathcal{K}$ ideal of the compact sequences from Section 4.3 and let $\mathcal{F}_\eta$ and $\mathcal{K}_\eta$ denote the corresponding restricted algebras.

**Proposition 7.39** *The only compact sequences in $\mathcal{S}_\eta(\mathsf{O}_N)$ are the sequences in $\mathcal{G}_\eta$.*

**Proof.** By Corollary 7.27, $\mathcal{J}_N$ is the only non-trivial closed ideal of $\mathcal{S}_N$. Thus, the intersection $\mathcal{S}_N \cap (\mathcal{K}_\eta/\mathcal{G}_\eta)$ is either $\mathcal{S}_N$, $\mathcal{J}_N$, or $\{0\}$. Since already $\mathcal{J}_N$ contains cosets of non-compact sequences (e.g., the coset $e - p_1$), the assertion follows. ■

**Corollary 7.40** *Every Fredholm sequence in $\mathcal{S}_\eta(\mathsf{O}_N)$ is stable.*

# References


[1] B. Blackadar, Operator Algebras. Theory of $C^*$-Algebras and von Neumann Algebras. – Springer, Berlin, Heidelberg 2006.

[2] F. P. Boca, Rotation $C^*$-Algebras and Almost Mathieu Operators. – Theta Series in Advanced Mathematics **1**, The Theta Foundation, Bucharest 2001.

[3] A. Böttcher, Infinite matrices and projection methods. – *In:* P. Lancaster (Ed.), Lectures on Operator Theory and its Applications, Fields Institute Monographs Vol. 3, Amer. Math. Soc., Providence, Rhode Island 1995, 1 – 72.

[4] A. Böttcher, S. M. Grudsky, Spectral Properties of Banded Toeplitz Matrices. – siam, Philadelphia 2005.

[5] A. Böttcher, B. Silbermann, Introduction to Large Truncated Toeplitz Matrices. – Springer-Verlag, Berlin, Heidelberg 1999.

[6] O. Bratteli, P. E. T. Jorgensen, Wavelets through a Looking Glass: The World of the Spectrum. – Birkhäuser, Boston 2002.

[7] N. Brown, AF embeddings and the numerical computation of spectra in irrational rotation algebras. – Numer. Funct. Anal. Optimization **27**(2006), 5, 517 – 528.

[8] N. P. Brown, N. Ozawa, $C^*$-Algebras and Finite-Dimensional Approximations. – Graduate Studies Math. **88**, Amer. Math. Soc., Providence, R. I., 2008.

[9] L. A. Coburn, The $C^*$-algebra generated by an isometry. – Bull. Am. Math. Soc. **73**(1967), 722 – 726.

[10] J. Cuntz, Simple $C^*$-algebras generated by isometries. – Comm. Math. Phys. **57**(1977), 173 – 185.

[11] K. R. Davidson, $C^*$-Algebras by Example. – Fields Institute Monographs Vol. 6, Providence, R. I., 1996.

[12] R. Hagen, S. Roch, B. Silbermann, Spectral Theory of Approximation Methods for Convolution Equations. – Birkhäuser Verlag, Basel, Boston, Berlin 1995.

[13] R. Hagen, S. Roch, B. Silbermann, $C^*$-Algebras and Numerical Analysis. – Marcel Dekker, Inc., New York, Basel 2001.

[14] N. Higson, J. Roe, Analytic K-Homology. – Oxford University Press, Oxford 2000.

[15] M. Lindner, Infinite Matrices and their Finite Sections. An Introduction to the Limit Operator Method. – Birkhäuser Verlag, Basel, Boston, Berlin 2006.

[16] V. S. Rabinovich, S. Roch, J. Roe, Fredholm indices of band-dominated operators. – Integral Equations Oper. Theory **49**(2004), 2, 221 – 238.

[17] V. S. Rabinovich, S. Roch, B. Silbermann, Fredholm theory and finite section method for band-dominated operators. – Integral Equations Oper. Theory **30**(1998), 4, 452 – 495.

[18] V. S. Rabinovich, S. Roch, B. Silbermann, Limit Operators and their Applications in Operator Theory. – Oper. Theory: Adv. Appl. **150**, Birkhäuser, Basel 2004.

[19] V. S. Rabinovich, S. Roch, B. Silbermann, Finite sections of band-dominated operators with almost periodic coefficients. – *In*: Modern Operator Theory and Applications, Oper. Theory: Adv. Appl. **170**, Birkhäuser, Basel 2006, 205 – 228.

[20] S. Roch, Algebras of approximation sequences: Fractality. – *In*: Problems and Methods in Mathematical Physics, Oper. Theory: Adv. Appl. **121**, Birkhäuser, Basel 2001, 471 – 497.

[21] S. Roch, Algebras of approximation sequences: Fredholm theory in fractal algebras. – Studia Math. **150**(2002), 1, 53 – 77.

[22] S. Roch, Finite sections of band-dominated operators. – Memoirs AMS Vol. **191**, 895, Providence, R.I., 2008.

[23] S. Roch, Spatial discretization of Cuntz algebras. – Houston Math. J., to appear.

[24] S. Roch, B. Silbermann, $C^*$-algebra techniques in numerical analysis. – J. Oper. Theory **35**(1996), 2, 241 – 280.

[25] J. Roe, Index Theory, Coarse Geometry, and Topology of Manifolds. – CBMS **90**, Amer. Math. Soc., Providence, R. I., 1996.

[26] J. Roe, Lectures on Coarse Geometry. – Univ. Lecture Series **31**, Amer. Math. Soc., Providence, R. I., 2003.

[27] J. Roe, Band-dominated Fredholm operators on discrete groups. – Integral Equations Oper. Theory **51**(2005), 3, 411 – 416.

[28] B. Silbermann, $C^*$-algebras and asymptotic spectral theory. – *In*: Operator Algebras, Operator Theory and Applications, Oper. Theory: Adv. Appl. **181**, Birkhäuser, Basel 2008, 33 – 66.