Very Weak, Weak and Strong Solutions to the Instationary Navier-Stokes System[1]

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1. Introduction

In this survey paper we discuss the theory of very weak solutions to the stationary and instationary (Navier-)Stokes system in a bounded domain of $\mathbb{R}^3$ and show how this new notion of solutions may be used to prove regularity locally or globally in space and time of a given weak solution.

Consider the instationary Navier-Stokes equations for a viscous incompressible fluid with density $\rho = 1$, i.e.,

$$
\begin{aligned}
u_t - \nu\Delta u + \operatorname{div}(uu) + \nabla p &= f && \text{in } \Omega \times (0,T)\\
\operatorname{div} u &= k && \text{in } \Omega \times (0,T)\\
u &= g && \text{on } \partial\Omega \times (0,T)\\
u &= u_0 && \text{at } t = 0
\end{aligned}
\tag{1.1}
$$

for the unknown velocity $u = (u_1, u_2, u_3)$ and pressure p in a domain $\Omega \subset \mathbb{R}^3$ and a time interval $(0,T)$, $0 < T \le \infty$. Here f denotes the external force (force density), $u_0 = u_0(x)$ the initial value, and $\nu > 0$ is the given viscosity of the fluid. In the physical model the divergence $k = \operatorname{div} u$ is assumed to vanish. However, for mathematical reasons it will be convenient in particualar for linear problems to consider the more general case of a prescribed divergence $k \neq 0$; see also Remark 1.9(1) below. Moreover, the boundary data $g = u|_{\partial\Omega}$ is a generalization of the classical no-slip or adhesion condition $u|_{\partial\Omega} = 0$. Obviously, for a bounded domain, k and g must satisfy the necessary compatibility condition

$$
\int_\Omega k\,dx = \int_{\partial\Omega} g \cdot N\,do\,;
\tag{1.2}
$$

here $N = N(x)$ is the external normal vector at $x \in \partial\Omega$, and do denotes the surface measure on $\partial\Omega$.

This survey is organized as follows. In this Introduction (Section 1) we discuss the notions of weak, strong, regular and very weak solutions and summarize some well-known results and important tools. Section 2 deals with the theory of very weak solutions in the stationary and instationary, linear and also nonlinear case. Finally, in Section 3 we consider applications of the theory of very weak solutions to the question under which additional assumptions a given weak solution is regular, either locally in time and globally in space or locally in time and space. The assumptions are either beyond the classical Serrin criterion of regularity or use the kinetic energy as a function of time.

For further surveys on the instationary Navier-Stokes equations we refer to [**34**], [**66**].

1.1. Weak Solutions in the Sense of Leray-Hopf. Let us test the Navier-Stokes system (with $k = 0$, $g = 0$) formally with the solution u and use integration by parts in space. Then, since $\operatorname{div} u = 0$, $\operatorname{div}(uu) = u \cdot \nabla u$ and $u = 0$ on $\partial\Omega$,

$$\int_\Omega \nabla p \cdot u \, dx = 0 \quad \text{and} \quad \int_\Omega (u \cdot \nabla u) \cdot u \, dx = \int_\Omega u \cdot \nabla\big(\frac{1}{2}|u|^2\big) \, dx = 0$$

so that (1.1) yields the identity

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \nu \|\nabla u(t)\|_2^2 = (f, u)(t);$$

here $(\cdot, \cdot)$ denotes the L^2-scalar product on Ω. A further integration in time on the interval (s, t) leads to the *energy identity*

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_s^t \|\nabla u\|_2^2 \, d\tau = \frac{1}{2} \|u(s)\|_2^2 + \int_s^t (f, u) \, d\tau \tag{1.3}$$

for $0 \le s < t \le T$. Assume that the external force f has the form

$$f = f_0 + \operatorname{div} F, \quad f_0 \in L^1\big(0, T; L^2(\Omega)\big), \quad F \in L^2\big(0, T; L^2(\Omega)\big). \tag{1.4}$$

Then Young's inequality and Gronwall's Lemma yield the integrability properties

$$u \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2_{\text{loc}}\big([0, T); H^1_0(\Omega)\big) \tag{1.5}$$

for every time interval $(0, T)$. Now (1.5) serves as starting point for the definition of a weak solution.

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a domain, let the initial value u_0 belong to the space

$$L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}, \quad C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega) : \operatorname{div} u = 0\},$$

and let f satisfy (1.4). Then a solenoidal vector field u satisfying (1.5) is called a *weak solution in the sense of Leray-Hopf* of the instationary Navier-Stokes system (1.1) with data f, u_0 (and with $k = 0$, $g = 0$) if

$$\begin{aligned} -\int_0^T (u, \varphi_t) d\tau + \nu \int_0^T (\nabla u, \nabla \varphi) d\tau + \int_0^T (u \cdot \nabla u, \varphi) d\tau \\ = \big(u_0, \varphi(0)\big) + \int_0^T \langle f, \varphi \rangle d\tau \end{aligned} \tag{1.6}$$

for all test functions $\varphi \in C^\infty_0\big([0, T); C^\infty_{0,\sigma}(\Omega)\big)$.

In (1.6) $\langle \cdot, \cdot \rangle$ denotes the duality product of $H^{-1}(\Omega) = H^1_0(\Omega)^*$ and $H^1_0(\Omega)$, and $(\cdot, \cdot)$ is used for measurable functions η, ψ on Ω in the sense $(\eta, \psi) = \int_\Omega \eta \cdot \psi \, dx$ provided $\eta \cdot \psi \in L^1(\Omega)$. Note that the same symbol, say $u \in C^\infty_0(\Omega)$, is used for a function as well as for vector fields or even matrix fields.

By the Galerkin approximation method or by the theory of analytic semigroups in the space $L^2_\sigma(\Omega)$ using Yosida approximation arguments it is shown that the Navier-Stokes system (1.6) has at least one weak solution in the sense of Leray-Hopf, see e.g. [**24**, §§2–3], [**61**, V.3]. Moreover, as a consequence of (1.6),

$$u : [0, T) \to L^2_\sigma(\Omega) \quad \text{is weakly continuous,} \tag{1.7}$$

and the initial value u_0 is attained in the sense: $\big(u(t), \varphi\big) \to (u_0, \varphi)$ as $t \to 0+$ for all $\varphi \in L^2_\sigma(\Omega)$ and even for all $\varphi \in L^2(\Omega)$.

However, due to the selection of a weakly convergent subsequence in the construction of the weak solution it cannot be guaranteed that u still satisfies the energy identity (1.3). The lower semicontinuity of norms with respect to weak convergences implies only that u satisfies the *energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_0^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f,u\rangle d\tau \tag{1.8}$$

rather than the energy identity (1.3). It is not clear whether any weak solution u according to Definition 1.1 does satisfy the energy inequality. However, each known construction method yields a weak solution satisfying (1.8).

If the domain $\Omega\subset\mathbb{R}^3$ is bounded, the compact embedding $H_0^1(\Omega)\subset L^2(\Omega)$ allows to construct a weak solution u satisfying also the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_s^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u(s)\|_2^2 + \int_s^t \langle f,u\rangle\,d\tau \tag{1.9}$$

for almost all $s\in[0,T)$ including $s=0$ and for all $t\in[s,T)$, see e.g. [**61**, Theorem V.3.6.2]. For unbounded domains the compactness argument is no longer available and more sophisticated tools based on maximal regularity, see §1.4 below, are needed to prove the existence of a weak solution satisfying the strong energy inequality; see [**40**], [**62**] for exterior domains and [**16**] for general unbounded domains with uniform C^2-regularity of the boundary.

Using (1.5) and the embedding $H_0^1(\Omega)\subset L^6(\Omega)$, we obtain for a weak solution u the space-time integrability $u\in L^s\big(0,T;L^q(\Omega)\big)$ for the pairs of exponents $s=\infty$, $q=2$ and $s=2$, $q=6$, satisfying both the condition

$$\frac{2}{s}+\frac{3}{q}=\frac{3}{2}\,. \tag{1.10}$$

More generally, using the so-called *Serrin number*

$$\mathcal{S}=\mathcal{S}(s,q)=\frac{2}{s}+\frac{3}{q}\quad\text{for}\quad s,\,q\in[1,\infty],$$

Hölder's inequality yields

$$u\in L^s\big(0,T;L^q(\Omega)\big)\quad\text{when}\quad \mathcal{S}=\frac{3}{2},\quad 2<s,\,q<\infty, \tag{1.11}$$

see [**61**, Lemma V.1.2.1]. However, it is an open problem whether a weak solution with $\mathcal{S}=\frac{3}{2}$ is unique. But the uniqueness is known if $\mathcal{S}\le 1$.

THEOREM 1.2. *Let $\Omega\subseteq\mathbb{R}^3$ be any domain, and let u,v be weak solutions of the Navier-Stokes system* (1.1) *with the same data f,u_0 (and with $k=0$, $g=0$). Assume that u satisfies the energy inequality* (1.8) *and that*

$$v\in L^s\big(0,T;L^q(\Omega)\big)\quad \textit{where}\quad \mathcal{S}(s,q)\le 1,\ 2<s<\infty,\ 3<q<\infty.$$

Then $u=v$.

For a proof we refer to [**58**]. The same result holds in the limit case $s=\infty$, $q=3$ when $\Omega\subset\mathbb{R}^3$ is a bounded or exterior domain with boundary of class C^2, see [**35**].

1.2. Regular Solutions. One of the seven Millennium Problems of Clay Mathematics Institute in 2000 is the question whether a weak solution of the Navier-Stokes equations in a three-dimensional domain is smooth, i.e., whether $u \in C^\infty\big(\Omega \times (0,T)\big)$ when $f = 0$ or, more generally, $f \in C^\infty\big(\Omega \times (0,T)\big)$. The first step in this direction is the question whether u is a strong solution.

DEFINITION 1.3. A weak solution u of the Navier-Stokes equations (with $k = 0$, $g = 0$) is called a *regular solution* if there exist exponents s, q such that

$$u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big), \;\; \mathcal{S}(s,q) \le 1, \;\; 3 < q < \infty, \; 2 < s < \infty. \tag{1.12}$$

For short, we say that u is *regular in the sense* $u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big)$. Moreover, u is called a *strong solution* if

$$u \in L^\infty_{\mathrm{loc}}\big([0,T); H^1_0(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); H^2(\Omega)\big). \tag{1.13}$$

Note that in (1.13), compared to (1.5), the regularity in space has been increased by one. Since $H^1_0(\Omega) \subset L^6(\Omega)$, we get $u \in L^\infty_{\mathrm{loc}}\big([0,T); L^6(\Omega)\big)$ with Serrin's number $\mathcal{S} = \frac{1}{2}$ so that u also satisfies (1.12).

The next two theorems state the local existence of a regular solution and the global regularity of a given weak solution under an additional assumption.

THEOREM 1.4. *Let $\Omega \subset \mathbb{R}^3$ be any domain, $u_0 \in \mathcal{D}(A_2^{1/4})$, where A_2 denotes the Stokes operator on $L^2_\sigma(\Omega)$, see §1.4, and let $f = f_0 + \operatorname{div} F$ with $f_0 \in L^{4/3}\big(0,T; L^2(\Omega)\big)$, $F \in L^4\big(0,T; L^2(\Omega)\big)$. Then there exists $T' = T'(\nu, u_0, f_0, F) \in (0,T)$ such that the Navier-Stokes equations* (1.1) *with data u_0, f (and with $k = 0$, $g = 0$) have a uniquely determined regular solution*

$$u \in L^8\big(0,T'; L^4(\Omega)\big).$$

PROOF. We refer to [**24**] for a proof of this result for a bounded domain Ω with $\partial\Omega \in C^2$ when $f = 0$ and $u_0 \in H^1_0(\Omega)$. In this case u even satisfies (1.13) in $(0,T')$. The more general result can be found in [**61**, Theorem V.4.2.2]. □

THEOREM 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^2$ and let u be a weak solution of* (1.1) *with data $f \in L^2\big(0,T; L^2(\Omega)\big)$, $u_0 \in L^2_\sigma(\Omega) \cap H^1_0(\Omega)$, $0 < T \le \infty$, (and with $k = 0$, $g = 0$) satisfying*

$$u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big), \quad \mathcal{S}(s,q) \le 1, \; 2 < s \le \infty, \; 3 \le q < \infty. \tag{1.14}$$

Then u is regular, uniquely determined by u_0, f, and a strong solution.

If $f \in C^\infty_0\big(\overline{\Omega} \times (0,T)\big)$ and $\partial\Omega \in C^\infty$, then $u \in C^\infty\big(\overline{\Omega} \times (0,T)\big)$.

PROOF. The classical implication from (1.14) when $2 < s < \infty$, $3 < q < \infty$, i.e. from (1.12), to (1.13) can be found in [**24**], see also [**61**, Theorem V.1.8.1]. The limit case $s = \infty$, $q = 3$ was proved more recently in [**11**], [**39**] [**52**], [**53**], [**54**], [**55**] starting from a result [**41**] on the finite number of singular points in time and space for a weak solution $u \in L^\infty\big(0,T; L^3(\Omega)\big)$.

Interior regularity results in the sense $u \in C^\infty\big(\overline{\Omega'} \times (0,T)\big)$ for every subdomain $\Omega' \subset\subset \Omega$ are proved in [**57**], [**58**], [**64**]. Moreover, regularity up to the boundary $\partial\Omega$ of Ω is shown [**29**], [**60**]. □

At this point, several remarks are in order, for later use in § 3 and for interest in its own. Concerning the energy identity and the energy inequality (1.8) which holds for every weak solution constructed so far in the literature, we note that every strong and every regular solution satisfies the energy identity, see the following Lemma 1.6.

LEMMA 1.6. *Let $\Omega \subseteq \mathbb{R}^3$ be any domain, and let u be a weak solution of* (1.1) *with data $u_0 \in L^2_\sigma(\Omega)$, $f = f_0 + \operatorname{div} F$, where $f_0 \in L^1\big(0,T;L^2(\Omega)\big)$, $F \in L^2\big(0,T;L^2(\Omega)\big)$ (and with $k = 0$, $g = 0$).*

(1) *Suppose additionally that*

$$u \in L^4\big(0,T;L^4(\Omega)\big)$$

or, more generally, that

$$u \in L^s\big(0,T;L^q(\Omega)\big), \quad \mathcal{S}(s,q) \le 1, \quad 2 \le s \le \infty, \quad 3 \le q \le \infty. \tag{1.15}$$

Then u satisfies the energy identity and is strongly continuous from $[0,T)$ to $L^2_\sigma(\Omega)$.

(2) *If also v satisfies the integrability condition* (1.5), *then*

$$u \cdot \nabla v \in L^s\big(0,T;L^q(\Omega)\big), \quad \mathcal{S}(s,q) = 4, \quad 1 \le s,q < 2.$$

PROOF. (1) The assumption $u \in L^4\big(0,T;L^4(\Omega)\big)$ implies that $uu \in L^2\big(0,T;L^2(\Omega)\big)$ so that $u \cdot \nabla u = \operatorname{div}(uu)$ may be written on the right-hand side of the equation as part of the external force $\operatorname{div} F$. Then u can be considered as the weak solution of a (linear) instationary Stokes system, and linear theory shows that u satisfies the energy identity.

Under the second assumption we may assume that $\frac{2}{s} + \frac{3}{q} = 1$. Since the given weak solution u also satisfies $u \in L^{s_1}\big(0,T;L^{q_1}(\Omega)\big)$ where $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$, and since $\frac{2}{4} + \frac{3}{4} = \frac{5}{4} \in (1, \frac{3}{2})$, Hölder's inequality easily implies that $u \in L^4\big(0,T;L^4(\Omega)\big)$, for details see [**61**, V.1.4]

(2) The proof is based on embedding theorems and Hölder's inequality, see [**61**, Lemma V.1.2.1]. □

REMARK 1.7. The condition (1.15) for u to satisfy the energy identity may be relaxed to the condition that $u \in L^s\big(0,T;L^q(\Omega)\big)$ and

$$\mathcal{S}(s,q) \le \min\big(1 + \frac{1}{q}, 1 + \frac{1}{s}\big), \quad 2 \le s \le \infty, \quad 3 \le q \le \infty. \tag{1.16}$$

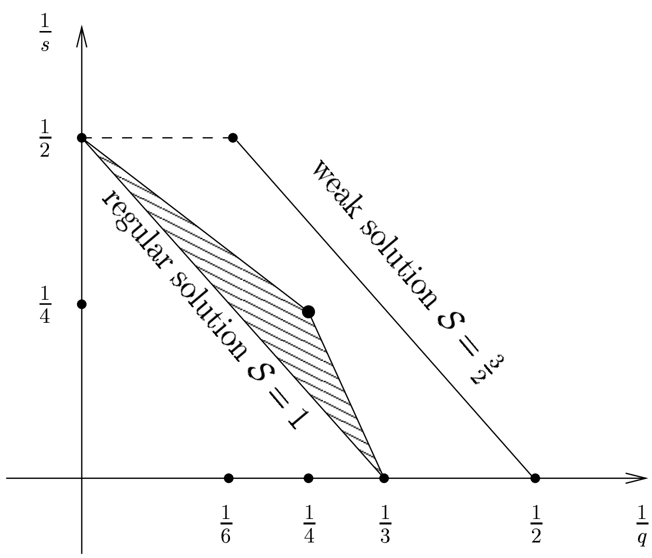

Fig 1.1. Weak and regular solutions represented by lines in the $\big(\frac{1}{q}, \frac{1}{s}\big)$-plane. The hatched region indicates the set described by (1.16) where the energy identity holds.

The proof follows the lines of [**61**, V.1.4]; note that the region in the $\big(\frac{1}{q}, \frac{1}{s}\big)$-plane described by (1.16) is the closed convex hull of the line $\mathcal{S} = 1$ and the point $\big(\frac{1}{4}, \frac{1}{4}\big)$ in the first quadrant of the $\big(\frac{1}{q}, \frac{1}{s}\big)$-plane. Hence the point $\big(\frac{1}{4}, \frac{1}{4}\big)$ can be written as a convex combination of any two points of this region and of the line $\mathcal{S}(s, q) = \frac{3}{2}$, respectively; see also Fig 1.1.

For a further discussion of the energy inequality, energy identity and regularity of a weak solution we refer to the first paragraphs of §3 as well as §3.2 and to §3.2 in general.

1.3. The Concept of Very Weak Solutions. In contrast to the definition of weak solutions, see Definition 1.1, where one integration by parts in space was used, the concept of very weak solutions allows all derivatives in space and time to be applied to the test functions. To give a precise definition we will use the spaces of test functions (vector fields)

$$C^2_{0,\sigma}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0\}$$

such that in general ∇v does not vanish on $\partial\Omega$, and

$$C^1_0\big([0,T);\, C^2_{0,\sigma}(\overline{\Omega})\big)$$

of solenoidal vector fields w satisfying $\operatorname{supp} w \subset \overline{\Omega} \times [0,T)$.

Given a sufficiently smooth solution u of the fully inhomogeneous Navier-Stokes system (1.1) and test functions $w \in C^1_0\big([0,T);\, C^2_{0,\sigma}(\Omega)\big)$ we are led to the identity

$$\int_0^T \big(-(u, w_t) - \nu(u, \Delta w) + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - (uu, \nabla w)\big)\, d\tau = \big(u_0, w(0)\big) + \int_0^T \langle f, w\rangle\, d\tau$$

where $\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle_{\partial\Omega}$ are pairings between corresponding spaces on Ω and $\partial\Omega$, respectively, see Definition 1.8 below. The term $\langle g, N \cdot \nabla w\rangle_{\partial\Omega}$ is due to the inhomogeneous boundary data $g = u|_{\partial\Omega}$ and the fact that in general the normal derivative $N \cdot \nabla w$ of w on $\partial\Omega$ does not vanish. Since $\operatorname{div} w = 0$ for all $t \in [0,T)$, the term $N \cdot \nabla w$ is purely tangential on $\partial\Omega$; this fact is easily checked when $\partial\Omega$ is planar. Hence, the term $\langle g, N \cdot \nabla w\rangle_{\partial\Omega}$ carries only the information of the tangential component of $g = u|_{\partial\Omega}$. Secondly we test the equation $\operatorname{div} u = k$ in $\Omega \times (0,T)$ with test functions $\psi \in C^0_0\big((0,T); C^1(\overline{\Omega})\big)$ and get the identity

$$\int_0^T (k, \psi) d\tau = \int_0^T \big(-(u, \nabla\psi) + \langle g \cdot N, \psi\rangle_{\partial\Omega}\big)\, d\tau.$$

This identity may be rewritten in the pointwise form

$$\operatorname{div} u = k \quad \text{in } \Omega \times (0,T); \quad u \cdot N = g \cdot N \quad \text{on } \partial\Omega \times (0,T)$$

giving information on $\operatorname{div} u$ and the normal component of u on $\partial\Omega$. Summarizing the previous reasoning we are led to

Definition 1.8. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$-boundary, let $f = \operatorname{div} F$ and

$$\begin{aligned} &F \in L^s\big(0,T; L^r(\Omega)\big), && k \in L^s\big(0,T; L^r(\Omega)\big) \\ &g \in L^s\big(0,T; W^{-1/q,q}(\partial\Omega)\big), && u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega) \end{aligned} \tag{1.17}$$

where $\mathcal{J}^{q,s}_\sigma(\Omega)$ is a space of initial values to be defined below, see Definition 2.10, k, g satisfy the compatibility condition (1.2) in the sense

$$\int_\Omega k(t) dx = \langle g(t), N\rangle_{\partial\Omega} \quad \text{for a.a. } t \in (0,T), \tag{1.18}$$

and q, r, s satisfy the conditions

$$\mathcal{S} = \frac{2}{s} + \frac{3}{q} = 1, \quad \frac{1}{3} + \frac{1}{q} = \frac{1}{r}, \quad 2 < s < \infty,\ 1 < r < 3 < q < \infty. \tag{1.19}$$

Then a vector field

$$u \in L^s\big(0, T; L^q(\Omega)\big)$$

is called a *very weak solution* of the instationary Navier-Stokes system (1.1) if

$$\begin{aligned} &\int_0^T \big(-(u, w_t) - \nu(u, \Delta w) + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - (uu, \nabla w)\big)\, d\tau \\ &\quad = \big(u_0, w(0)\big) - \int_0^T (F, \nabla w)\, d\tau \end{aligned} \tag{1.20}$$

for all test fields $w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$, and additionally

$$\operatorname{div} u = k \quad \text{in } \Omega \times (0,T), \qquad u \cdot N = g \cdot N \quad \text{on } \partial\Omega \times (0,T). \tag{1.21}$$

Remark 1.9. (1) Note that in [**12**], [**14**], [**19**], [**21**], [**26**] the authors considered the variational problem

$$\begin{aligned} &\int_0^T \big(-(u, w_t) - \nu(u, \Delta w) + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - (uu, \nabla w) - (ku, w)\big)\, d\tau \\ &\quad = \big(u_0, w(0)\big) - \int_0^T (F, \nabla w)\, d\tau \end{aligned} \tag{1.22}$$

instead of (1.20). The additional term (ku, w) in (1.22) or equivalently $-ku$ on the left-hand side of the first equation of (1.1) is due to the identity

$$u \cdot \nabla u = \operatorname{div}(uu) - ku, \quad \text{where} \quad k = \operatorname{div} u.$$

The difference of these variational problems originates from the derivation of the Navier-Stokes equations, see e.g. [**48**]. On the one hand, considering compressible fluids with density $\rho = \rho(x,t)$ the term $(\rho u)_t + \operatorname{div}(\rho uu)$ appears in the equation for the balance of momentum; for constant ρ and in the time-independent case we are left with the term $\operatorname{div}(uu)$ as in (1.1). On the other hand, the term $u_t + u \cdot \nabla u$ denotes the acceleration of particles and leads to the additional term $-ku$ in (1.1). We note that both models are unphysical, since the equation for the conservation of mass $\rho_t + \operatorname{div}(\rho u) = 0$ leads to $\operatorname{div} u = 0$ when the density ρ is constant. For the model (1.1) the proofs of Theorems 2.9 and 2.18 below are shorter compared to the proofs in [**12**], [**14**], [**19**], [**21**], [**26**], although the assumptions on $k = \operatorname{div} u$ and the complexity of the proofs are the same.

(2) The conditions (1.19) on q, r, s are needed to give each term in (1.20) a well-defined meaning, particularly to define the nonlinear term $(uu, \nabla w)$. The exponents q, r are chosen such that the embeddings $W^{1,r}(\Omega) \subset L^q$, $L^r(\Omega) \subset W^{-1,q}(\Omega) := W_0^{1,q'}(\Omega)^*$ (= the dual space of $W_0^{1,q'}(\Omega)^*$, $q' = \frac{q}{q-1}$) and $L^{q'}(\Omega) \subset W^{-1,r'}(\Omega)$ hold.

(3) The information on $\operatorname{div} u$ can be recovered only from (1.21), but not from (1.20).

(4) Analogous definitions of very weak solutions will be given also for the stationary Stokes and Navier-Stokes system, see §2. In these cases the conditions on q, r, s in (1.19) are more general.

Before turning to theorems on existence in §2 let us discuss the main features of this concept.

- The concept of very weak solutions was introduced in a series of papers by H. Amann [**2**], [**3**] in the setting of Besov spaces when $k = 0$.

- More recently this concept was modified by G.P. Galdi, C. Simader and the authors to a setting in classical L^q-spaces including the inhomogeneous data k, see [**12**], [**13**], [**14**], [**19**], [**21**], [**26**].
- By definition very weak solutions have no differentiability, neither in space nor in time, except for the existence of the divergence $k = \operatorname{div} u \in L^r(\Omega)$ for a.a. t.
- In general, a very weak solution does neither have a bounded kinetic energy in $L^\infty\big(0,T;L^2(\Omega)\big)$ nor a finite dissipation energy in $L^2\big(0,T;H^1(\Omega)\big)$. In particular, a very weak solution is not necessarily a weak solution.
- By definition, a very weak solution is contained in Serrin's uniqueness class $L^s\big(0,T;L^q(\Omega)\big)$ with $\mathcal{S} = 1$. Very weak solutions can be shown to be unique, see §2. However, in general, the regularity of the data is too low to guarantee any kind of regularity of the very weak solution.
- The concept of very weak solutions has been generalized by K. Schumacher to a setting in weighted Lebesgue and Bessel potential spaces using arbitrary Muckenhoupt weights, see [**51**].
- Although the data in Definition 1.8 imply no regularity for a very weak solution, the concept may be even further generalized so that neither boundary values nor initial values of a very weak solution can be defined, see [**51**] and §2.
- The concept of very weak solutions is strongly based on duality arguments concerning the theory of strong (or regular) solutions. Therefore, the boundary regularity required in this theory is the same as for strong solutions.
- The boundary is usually assumed to be of class $C^{2,1}$. Due to a new smoothing argument in the proof of an extension theorem, see [**51**], it suffices to require that $\partial\Omega \in C^{1,1}$.

1.4. Preliminaries. We summarize several auxiliary results on the Helmholtz projection and the Stokes operator introduced for later use only for bounded domains.

LEMMA 1.10. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1-boundary and let $1 < q < \infty$.*

(1) *There exists a bounded projection*

$$P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$$

from the space of all L^q-vector fields onto the subspace

$$L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$$

of all solenoidal vector fields u such that the normal component $u \cdot N$ of u vanishes on $\partial\Omega$ in the weak sense. In particular,

$$\mathcal{R}(P_q) = L^q_\sigma(\Omega), \quad \mathcal{N}(P_q) = G_q(\Omega) := \{\nabla p : \ p \in W^{1,q}(\Omega)\}.$$

Every vector field $u \in L^q(\Omega)$ has a unique decomposition

$$u = u_0 + \nabla p, \quad u_0 \in L^q_\sigma(\Omega), \quad \nabla p \in G_q(\Omega),$$

satisfying

$$\|u_0\|_q + \|\nabla p\|_q \le c\|u\|_q$$

with a constant $c = c(q,\Omega) > 0$.

(2) *The adjoint operator $(P_q)^*$ of P_q equals $P_{q'}$, where $q' = \frac{q}{q-1}$, and the dual space $L^q(\Omega)^*$ is isomorphic to $L^{q'}(\Omega)$.*

PROOF. See e.g. [**59**]. □

LEMMA 1.11. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$-boundary and let $1 < q < \infty$.*

(1) *The Stokes operator, defined by*

$$\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega), \quad A_q u = -P_q \Delta u,$$

is a closed bijective operator from $\mathcal{D}(A_q) \subset L^q_\sigma(\Omega)$ onto $L^q_\sigma(\Omega)$. If $u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_\rho)$ for $1 < \rho < \infty$, then $A_q u = A_\rho u$.

(2) *For $0 \le \alpha \le 1$ the fractional powers*

$$A_q^\alpha : \mathcal{D}(A_q^\alpha) \subset L^q_\sigma(\Omega \to L^q_\sigma(\Omega)$$

are well-defined, closed, bijective operators. In particular, the inverses $A_q^{-\alpha} := (A_q^\alpha)^{-1}$ are bounded operators on $L^q_\sigma(\Omega)$ with $\mathcal{R}(A_q^{-\alpha}) = \mathcal{D}(A_q^\alpha)$. The space $\mathcal{D}(A_q^\alpha)$ endowed with the graph norm $\|u\|_q + \|A_q^\alpha u\|_q$, equivalent to $\|A_q^\alpha u\|_q$ for bounded domains, is a Banach space. Moreover, for $1 > \alpha > \beta > 0$,

$$\mathcal{D}(A_q) \subset \mathcal{D}(A_q^\alpha) \subset \mathcal{D}(A_q^\beta) \subset L^q_\sigma(\Omega)$$

with strict dense inclusions, and $(A_q^\alpha)^* = A_{q'}^\alpha$ is the adjoint to A_q^α.

(3) *The norms $\|u\|_{W^{2,q}}$ and $\|A_q u\|_q$ are equivalent for $u \in \mathcal{D}(A_q)$. Analogously, the norms $\|\nabla u\|_q$, $\|u\|_{W^{1,q}}$ and $\|A_q^{1/2} u\|_q$ are equivalent for $u \in \mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$. More generally, the embedding estimate*

$$\|u\|_q \le c\|A_\gamma^\alpha u\|_\gamma \quad 1 < \gamma \le q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma} \tag{1.23}$$

holds for every $u \in \mathcal{D}(A_\gamma^\alpha)$; here $c = c(q, \gamma, \Omega) > 0$.

(4) *The Stokes operator A_q generates a bounded analytic semigroup e^{-tA_q}, $t \ge 0$, on $L^q_\sigma(\Omega)$. Moreover, there exists a constant $\delta_0 = \delta_0(q, \Omega) > 0$ such that*

$$\|A_q^\alpha e^{-tA_q} u\|_q \le c e^{-\delta_0 t} t^{-\alpha} \|u\|_q \quad \text{for } u \in L^q_\sigma(\Omega),\ t > 0, \tag{1.24}$$

with $c = c(q, \alpha, \Omega) > 0$.

PROOF. See [**1**], [**20**], [**27**], [**28**], [**30**], [**61**]. Usually these results are proved for bounded domains with $\partial\Omega \in C^2$ or even $C^{2,\mu}$, $0 < \mu < 1$. However, a careful inspection of the proofs shows that $C^{1,1}$-regularity is sufficient. □

We note that most of the results of Lemma 1.11 also hold for exterior domains $\Omega \subset \mathbb{R}^3$. However, some results are more restrictive, since the Poincaré inequality on $W_0^{1,q}(\Omega)$ does not hold for an exterior domain.

The next auxiliary tool concerns the instationary Stokes system

$$\begin{aligned} u_t - \nu\Delta u + \nabla p &= f, && \operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T) \\ u &= 0 && \text{on } \partial\Omega \times (0,T) \\ u(0) &= u_0 && \text{at } t = 0 \end{aligned} \tag{1.25}$$

for data $f \in L^s\big(0,T; L^q(\Omega)\big)$ and $u_0 \in L^q_\sigma(\Omega)$, $1 < s,\ q < \infty$.

Applying the Helmholtz projection P_q to (1.25) we get the abstract evolution equation

$$u_t + \nu A_q u = P_q f, \quad u(0) = u_0, \tag{1.26}$$

where we are looking for a solution u with $u(t) \in \mathcal{D}(A_q)$. The variation of constants formula yields the solution

$$u(t) = e^{-\nu t A_q} u_0 + \int_0^t e^{-\nu(t-\tau)A_q} P_q f(\tau)\, d\tau, \quad 0 \le t < T \le \infty. \tag{1.27}$$

Conversely, the solution of (1.26) yields $P_q(u_t - \nu\Delta u - f) = 0$ so that by Lemma 1.10 there exists a function p with $u_t - \nu\Delta u - f = -\nabla p$, i.e., (u,p) solves (1.25). To estimate u given by (1.27) (with $u_0 = 0$) and ∇p we introduce the notion of *maximal regularity.*

LEMMA 1.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$-boundary, let $1 < s,\ q < \infty$, $f \in L^s\big(0,T;L^q(\Omega)\big)$ and $u_0 = 0$. Then the Stokes equation* (1.26) *has a unique solution u satisfying the maximal regularity estimate*

$$\|u_t\|_{L^s(0,T;L^q(\Omega))} + \|\nu A_q u\|_{L^s(0,T;L^q(\Omega))} \le c\|f\|_{L^s(0,T;L^q(\Omega))} \tag{1.28}$$

where $c = c(q,s,\Omega) > 0$ is independent of ν and T. Moreover, there exists a function $p \in L^s\big(0,T;W^{1,q}(\Omega)\big)$ such that (u,p) satisfies (1.25) *and the estimate*

$$\|(u_t, \nabla p, \nu\nabla^2 u)\|_{L^s(0,T;L^q(\Omega))} \le c\|f\|_{L^s(0,T;L^q(\Omega))}. \tag{1.29}$$

PROOF. The first proof of this result for $s = q \in (1,\infty)$ can be found in [**63**] and is based on potential theory, the generalization to arbitrary $s \in (1,\infty)$ is a consequence of abstract theory, see [**1**], [**8**], [**30**]. Different approaches are based on the theory of pseudodifferential operators [**28**], [**31**] and on the theory of weighted estimates, see A. Fröhlich [**22**], [**23**]. □

2. Theory of Very Weak Solutions

As already outlined in §1.3 the concept of very weak solutions introduces a new class of solutions to stationary and nonstationary Stokes and Navier-Stokes equations with data of very low regularity such that solutions may have (almost) no differentiability and no finite energy, but they are unique even in the nonlinear case.

2.1. The Stationary Stokes System. First we consider the stationary Stokes problem

$$-\Delta u + \nabla p = f = \operatorname{div} F,\ \operatorname{div} u = k \text{ in } \Omega,\ u|_{\partial\Omega} = g \tag{2.1}$$

for suitable data $f = \operatorname{div} F$, k and g in a bounded domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{1,1}$ and – for simplicity – with viscosity $\nu = 1$. Let

$$C^2_{0,\sigma}(\overline{\Omega}) = \{w \in C^2(\overline{\Omega}) : \operatorname{div} w = 0,\ w|_{\partial\Omega} = 0\}$$

denote the corresponding space of test functions.

Definition 2.1. Let $1 < r \le q < \infty$ and $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$. Given data

$$F \in L^r(\Omega),\ k \in L^r(\Omega),\ g \in W^{-1/q,q}(\partial\Omega) \tag{2.2}$$

satisfying the compatibility condition

$$\int_\Omega k\, dx = \langle g, N\rangle_{\partial\Omega}\,, \tag{2.3}$$

a vector field $u \in L^q(\Omega)$ is called a *very weak solution* to (2.1) if

$$\begin{aligned} -(u, \Delta w) &= -\langle g, N\cdot\nabla w\rangle_{\partial\Omega} - (F, \nabla w) \quad \forall w \in C^2_{0,\sigma}(\overline{\Omega}) \\ \operatorname{div} u &= k \text{ in } \Omega,\ u\cdot N = g\cdot N \text{ on } \partial\Omega. \end{aligned} \tag{2.4}$$

Here $(\eta,\psi) := \int_\Omega \eta\psi\, dx$ for measurable functions η,ψ on Ω provided $\eta\cdot\psi \in L^1(\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the evaluation of the functional $g \in W^{-1/q,q}(\partial\Omega)$ at the admissible test function $N\cdot\nabla w = \frac{\partial w}{\partial N} \in W^{1-1/q',q'}(\partial\Omega)$; note that $N \in C^{0,1}(\partial\Omega) \subset W^{1-1/q',q'}(\partial\Omega)$ for every $q \in (1,\infty)$.

Since $N\cdot\nabla w$ is purely tangential on $\partial\Omega$ for $w \in C^2_{0,\sigma}(\overline{\Omega})$, the term $\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$ concerns only the tangential component of $g = u|_{\partial\Omega}$ on $\partial\Omega$. Testing the equation $\operatorname{div} u = k$ with an arbitrary scalar-valued test function $\psi \in C^1(\overline{\Omega})$, we get the second and third identity in (2.4) via the variational problem

$$-(u,\nabla\psi) = (k,\psi) - \langle g, \psi N\rangle_{\partial\Omega}. \tag{2.5}$$

Now let us define the functionals

$$\begin{aligned} \langle\mathcal{F}, w\rangle &= -(F,\nabla w) - \langle g, N\cdot\nabla w\rangle_{\partial\Omega}, \quad & w \in Y^{2,q'}_\sigma(\Omega), \\ \langle\mathcal{K}, \psi\rangle &= (k,\psi) - \langle g, \psi N\rangle_{\partial\Omega}, & \psi \in W^{1,q'}(\Omega), \end{aligned} \tag{2.6}$$

where

$$Y^{2,q'}_\sigma(\Omega) := \mathcal{D}(A_{q'}) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega).$$

Then the embeddings

$$W^{1,q'}(\Omega) \subset L^{r'}(\Omega),\ Y^{2,q'}_\sigma(\Omega) \subset W^{1,r'}(\Omega),$$

cf. Remark 1.9 (2), and the trace estimate

$$\|\psi\cdot N\|_{W^{1-1/q',q'}(\partial\Omega)} \le c\,\|\psi\|_{W^{1-1/q',q'}(\partial\Omega)} \le c\|\psi\|_{W^{1,q'}(\Omega)}$$

imply that

$$\begin{aligned}\mathcal{F} \in Y_\sigma^{-2,q}(\Omega) &:= Y_\sigma^{2,q'}(\Omega)^* \\ \mathcal{K} \in W_0^{-1,q}(\Omega) &:= W^{1,q'}(\Omega)^*.\end{aligned} \tag{2.7}$$

However, the functionals $\mathcal{F}$ and $\mathcal{K}$ are not distributions in the classical sense on their respective spaces of test functions, since in each case $C_0^\infty(\Omega)$ is *not* a dense subspace. Nevertheless, (2.6), (2.7) leads to a further useful generalization of the concept of very weak solutions, see [**51**].

Definition 2.2. Let $1 < q < \infty$ and let $\mathcal{F} \in Y_\sigma^{-2,q}(\Omega)$, $\mathcal{K} \in W_0^{-1,q}(\Omega)$ be given. Then $u \in L^q(\Omega)$ is called a *very weak solution of the Stokes problem with data* $\mathcal{F}, \mathcal{K}$ if

$$\begin{aligned}-(u, \Delta w) &= \langle \mathcal{F}, w\rangle, \quad w \in Y_\sigma^{2,q'}(\Omega), \\ -(u, \nabla\psi) &= \langle \mathcal{K}, \psi\rangle, \quad \psi \in W^{1,q'}(\Omega).\end{aligned} \tag{2.8}$$

The concept of Definition 2.2 has the drawback that *any* vector field $u \in L^q(\Omega)$ is the very weak solution of the Stokes problem for suitable data $\mathcal{F} \in Y_\sigma^{-2,q}(\Omega)$, $\mathcal{K} \in W_0^{-1,q}(\Omega)$, namely,

$$\langle \mathcal{F}, w\rangle := -(u, \Delta w), \quad \langle \mathcal{K}, \psi\rangle := -(u, \nabla\psi).$$

Hence there is no possibility to define boundary values of u in this very general setting. However, this concept immediately leads to the existence of a unique very weak solution using duality arguments.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$, let $1 < q < \infty$ and $\mathcal{F} \in Y_\sigma^{-2,q}(\Omega)$, $\mathcal{K} \in W_0^{-1,q}(\Omega)$ be given. Then the Stokes problem* (2.8) *has a unique very weak solution $u \in L^q(\Omega)$; moreover, u satisfies the estimate*

$$\|u\|_q \le c\left(\|\mathcal{F}\|_{Y_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{W_0^{-1,q}(\Omega)}\right) \tag{2.9}$$

with a constant $c = c(\Omega, q) > 0$.

Proof. Consider an arbitrary vector field $v \in L^{q'}(\Omega)$. Then there exists a unique strong solution $w \in Y_\sigma^{2,q'}(\Omega)$, $\psi \in W^{1,q'}(\Omega)$ of the Stokes problem

$$-\Delta w - \nabla\psi = v, \ \operatorname{div} w = 0 \text{ in } \Omega, \ w|_{\partial\Omega} = 0, \int_\Omega \psi\, dx = 0; \tag{2.10}$$

moreover, w, ψ linearly depend on v and

$$\|w\|_{W^{2,q'}(\Omega)} + \|\psi\|_{W^{1,q'}(\Omega)} \le c\,\|v\|_{q'}$$

with a constant $c = c(\Omega, q) > 0$. Now, using the duality $L^q(\Omega) = L^{q'}(\Omega)^*$, define $u \in L^q(\Omega)$ by

$$(u, v) = \langle \mathcal{F}, w\rangle + \langle \mathcal{K}, \psi\rangle$$

such that

$$\begin{aligned}|(u, v)| &\le \|\mathcal{F}\|_{Y_\sigma^{-2,q}(\Omega)}\|w\|_{W^{2,q'}(\Omega)} + \|\mathcal{K}\|_{W_0^{-1,q}(\Omega)}\|\psi\|_{W^{1,q'}(\Omega)} \\ &\le c\big(\|\mathcal{F}\|_{Y_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{W_0^{-1,q}(\Omega)}\big)\|v\|_{q'}.\end{aligned}$$

Hence u satisfies the *a priori* estimate (2.9).

To show that u is a very weak solution to the data $\mathcal{F}, \mathcal{K}$, choose arbitrary test functions $w \in Y_\sigma^{2,q'}(\Omega)$ and $\psi \in W^{1,q'}(\Omega)$ and define $v = -\Delta w - \nabla\psi \in L^{q'}(\Omega)$. Then

$$(u, -\Delta w) - (u, \nabla\psi) = (u, v) = \langle \mathcal{F}, w\rangle + \langle \mathcal{K}, \psi\rangle,$$

i.e., (2.8) is satisfied.

To prove uniqueness, let $u \in L^q(\Omega)$ satisfy (2.8) with $\mathcal{F} = 0$, $\mathcal{K} = 0$. Then for all $v \in L^{q'}(\Omega)$ and corresponding solutions $w \in Y^{2,q'}_\sigma(\Omega)$, $\psi \in W^{1,q'}(\Omega)$ of (2.10) we get

$$(u, v) = (u, -\Delta w) - (u, \nabla\psi) = \langle \mathcal{F}, w\rangle + \langle \mathcal{K}, \psi\rangle = 0.$$

Thus $u = 0$. □

We note that the proof of Theorem 2.3 was based on duality arguments related to the (strong) Stokes operator

$$A_{q'} : Y^{2,q'}_\sigma(\Omega) \to L^{q'}_\sigma(\Omega)$$

where $A_{q'} = -P_{q'}\Delta$ is considered as a bounded bijective operator from $Y^{2,q'}_\sigma(\Omega) \subset W^{2,q'}(\Omega)$, equipped with the norm of $W^{2,q'}(\Omega)$, onto $L^{q'}_\sigma(\Omega)$, and to its adjoint

$$(A_{q'})^* : L^q_\sigma(\Omega) \to Y^{-2,q}_\sigma(\Omega),$$

which defines an isomorphism as well.

To return to Definition 2.1 of very weak solutions and to interpret their boundary values let us introduce the notion of normal and tangential components of ($\mathbb{R}^3$-valued) traces on $\partial\Omega$ and of functionals on $\partial\Omega$. Given $h = (h_1, h_2, h_3) \in W^{1-1/q',q'}(\partial\Omega)$ let

$$h_N = (h \cdot N)N \quad \text{and} \quad h_\tau = h - h_N \quad \text{for a.a. } x \in \partial\Omega$$

denote its normal and tangential component, respectively. Obviously

$$h_N \in W^{1-1/q',q'}_N(\partial\Omega) := \{\varphi \in W^{1-1/q',q'}(\partial\Omega) : \varphi \| N \text{ on } \partial\Omega \text{ a.e.}\},$$
$$h_\tau \in W^{1-1/q',q'}_\tau(\partial\Omega) := \{\varphi \in W^{1-1/q',q'}(\partial\Omega) : \varphi \cdot N = 0 \text{ on } \partial\Omega \text{ a.e.}\},$$

and

$$\|h_N\|_{1-1/q',q',\partial\Omega} + \|h_\tau\|_{1-1/q',q',\partial\Omega} \leq c\|h\|_{1-1/q',q',\partial\Omega}.$$

Actually,

$$W^{1-1/q',q'}_N(\partial\Omega) \oplus W^{1-1/q',q'}_\tau(\partial\Omega) = W^{1-1/q',q'}(\partial\Omega)$$

as a topological and algebraic direct decomposition.

For $g = (g_1, g_2, g_3) \in W^{-1/q,q}(\partial\Omega)$, we define the functionals

$$g_N \in W^{-1/q,q}_N(\partial\Omega) := W^{1-1/q',q'}_N(\partial\Omega)^*$$
$$g_\tau \in W^{-1/q,q}_\tau(\partial\Omega) := W^{1-1/q',q'}_\tau(\partial\Omega)^*$$

by

$$\langle g_N, h_N\rangle_{\partial\Omega} := \langle g, h_N\rangle_{\partial\Omega},\ \ h_N \in W^{1-1/q',q'}_N(\partial\Omega),$$

and

$$\langle g_\tau, h_\tau\rangle_{\partial\Omega} := \langle g, h_\tau\rangle_{\partial\Omega},\ \ h_\tau \in W^{1-1/q',q'}_\tau(\partial\Omega),$$

respectively. Hence

$$\|g_N\|_{W^{-1/q,q}_N(\partial\Omega)} + \|g_\tau\|_{W^{-1/q,q}_\tau(\partial\Omega)} \leq c\|g\|_{-1/q,q,\partial\Omega}.$$

Since $g \in W^{-1/q,q}(\partial\Omega)$ is given, it is reasonable to extend g_N from $W^{-1/q,q}_N(\partial\Omega)$ to $W^{-1/q,q}(\partial\Omega)$ by defining $\langle g_N, h_\tau\rangle := 0$ for all tangential traces $h_\tau \in W^{1-1/q',q'}_\tau(\partial\Omega)$ and to extend g_τ from $W^{-1/q,q}_\tau(\partial\Omega)$ to $W^{-1/q,q}(\partial\Omega)$ by defining $\langle g_\tau, h_N\rangle := 0$ for all normal traces $h_N \in W^{1-1/q',q'}_N(\partial\Omega)$. That way, $W^{-1/q,q}_N(\partial\Omega)$ and $W^{-1/q,q}_\tau(\partial\Omega)$ may be considered as closed subspaces of $W^{-1/q,q}(\partial\Omega)$.

Hence

$$g = g_N + g_\tau \quad \text{on } W^{1-1/q',q'}(\partial\Omega), \tag{2.11}$$

and we get the topological and algebraic decomposition

$$W_N^{-1/q,q}(\partial\Omega) \oplus W_\tau^{-1/q,q}(\partial\Omega) = W^{-1/q,q}(\partial\Omega). \tag{2.12}$$

Finally, we define the functional $g \cdot N \in W^{-1/q,q}(\partial\Omega)$ by

$$\langle g \cdot N, \psi \rangle_{\partial\Omega} := \langle g, \psi N \rangle_{\partial\Omega}, \quad \psi \in W^{1,1/q',q'}(\partial\Omega),$$

satisfying $\|g \cdot N\|_{-1/q,q,\partial\Omega} \le c\,\|g\|_{-1/q,q,\partial\Omega}$. Obviously, $g \cdot N = g_N \cdot N$ and $g_\tau \cdot N = 0$. Moreover, $g_N = (g \cdot N)N$ formally and also in the pointwise sense when g is a vector field on $\partial\Omega$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{1,1}$, and let $1 < r \le q < \infty$ satisfy $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$.*

(1) *Given data F, k and g as in (2.2), (2.3) there exists a unique very weak solution $u \in L^q(\Omega)$ of (2.4). This solution satisfies the a priori estimate*

$$\|u\|_q \le c\Big(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\Big) \tag{2.13}$$

with a constant $c = c(q, r, \Omega) > 0$.

(2) *The very weak solution $u \in L^q(\Omega)$ in (1) has a* normal trace *$u \cdot N = g \cdot N \in W^{-1/q,q}(\partial\Omega)$ and a* tangential trace component *$u_\tau = g_\tau \in W_\tau^{-1/q,q}(\partial\Omega)$ in the following sense: The normal trace $u \cdot N = g \cdot N$ exists via the identity*

$$\langle u \cdot N, \psi \rangle_{\partial\Omega} = (k, \psi) + (u, \nabla\psi), \quad \psi \in W^{1,q'}(\Omega). \tag{2.14}$$

For the tangential component of the trace, u_τ, we use a bounded linear extension operator

$$E_\tau : W_\tau^{1-1/q',q'}(\partial\Omega) \to Y_\sigma^{2,q'}(\Omega)$$

such that

$$h = N \cdot \nabla E_\tau(h)\big|_{\partial\Omega} \quad \text{for all } h \in W_\tau^{1-1/q',q'}(\partial\Omega).$$

Then

$$\langle u_\tau, h \rangle = \big(u, \Delta E_\tau(h)\big) - \big(F, \nabla E_\tau(h)\big), \;\; h \in W_\tau^{1-1/q',q'}(\partial\Omega), \tag{2.15}$$

is uniquely defined (not depending on the extension operator E_τ with the above properties). Moreover,

$$\begin{aligned} \|u \cdot N\|_{-1/q,q,\partial\Omega} &\le c\,\|g_N\|_{W_N^{-1/q,q}(\partial\Omega)}, \\ \|u_\tau\|_{W_\tau^{-1/q,q}(\partial\Omega)} &\le c\,\|g_\tau\|_{W_\tau^{-1/q,q}(\partial\Omega)}. \end{aligned} \tag{2.16}$$

Defining the functional $u_N = (u \cdot N)N \in W_N^{-1/q,q}(\partial\Omega)$ by $\langle u_N, h_N \rangle_{\partial\Omega} := \langle u \cdot N, h_N \cdot N \rangle_{\partial\Omega}$ for $h_N \in W_N^{1-1/q',q'}(\partial\Omega)$, it holds in view of (2.11), (2.12)

$$u = u_N + u_\tau = g \in W^{-1/q,q}(\partial\Omega) \tag{2.17}$$

and

$$\|u\|_{-1/q,q,\partial\Omega} \le c\Big(\|u \cdot N\|_{-1/q,q,\partial\Omega} + \|u_\tau\|_{W_\tau^{-1/q,q}(\partial\Omega)}\Big). \tag{2.18}$$

(3) *Assume that $\mathcal{F} \in Y_\sigma^{-2,q}(\Omega)$ and $\mathcal{K} \in W_0^{-1,q}(\Omega)$ have the representations*

$$\begin{aligned} \langle \mathcal{F}, w \rangle &= -(F, \nabla w) - \langle g_\tau, N \cdot \nabla w \rangle_{\partial\Omega}, \quad w \in Y_\sigma^{2,q'}(\Omega), \\ \langle \mathcal{K}, \psi \rangle &= (k, \psi) - \langle \hat{g}, \psi \rangle_{\partial\Omega}, \qquad\qquad\quad \psi \in W^{1,q'}(\Omega), \end{aligned} \tag{2.19}$$

respectively, with

$$F, k \in L^r(\Omega) \quad \textit{and} \quad g_\tau \in W_\tau^{-1/q,q}(\partial\Omega),\ \hat{g} \in W^{-1/q,q}(\partial\Omega).$$

Then F, g_τ and $k, \hat{g}$ are uniquely determined by $\mathcal{F}$ and $\mathcal{K}$, respectively; see the proof below for details concerning the uniqueness of F.

Proof. (1) Given F, k, g as in (2.2), (2.3) define $\mathcal{F}, \mathcal{K}$ as in (2.6), and let $u \in L^q(\Omega)$ be the unique very weak solution of (2.8) due to Theorem 2.3. In view of (2.6), (2.9) u satisfies (2.13).

(2) Testing in $(2.8)_2$ with $\psi \in C_0^\infty(\Omega)$ we see from $(2.6)_2$ that $\operatorname{div} u = k \in L^r(\Omega)$ in the sense of distributions. Since $u \in L^q(\Omega) \subset L^r(\Omega)$, a classical result implies that u has a normal trace $u \cdot N \in W^{-1/r,r}(\partial\Omega)$ which by $(2.6)_2$, $(2.8)_2$ coincides with $g \cdot N \in W^{-1/q,q}(\partial\Omega)$.

Concerning the tangential trace we first construct the extension operator E_τ. Let $h \in W^{1-1/q',q'}(\partial\Omega)$. Then we find $w_h = E_1(h) \in W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega)$ such that

$$w_h\big|_{\partial\Omega} = 0 \quad \text{and} \quad N \cdot \nabla w_h = h;$$

moreover, w_h depends linearly and continuously on h. The existence of an extension operator E_1 with these porperties is well-known in the case of bounded domains with boundary of class $C^{2,1}$, see [**47**], [**65**]. However, a mollification procedure, see [**51**], allows this extension even in the case when $\partial\Omega \in C^{1,1}$ only. Next, assume that $h \in W_\tau^{1-1/q',q'}(\partial\Omega)$. Then an easy calculation shows that $\operatorname{div} w_h\big|_{\partial\Omega} = 0$ so that $\operatorname{div} w_h \in W_0^{1,q'}(\Omega)$ and $\int_\Omega \operatorname{div} w_h \, dx = 0$. Next we need properties of Bogovskii's operator concerning the divergence problem ([**6**], [**61**]): There exists a bounded linear operator

$$B : \Big\{ f \in W_0^{1,q'}(\Omega) : \int_\Omega f \, dx = 0 \Big\} \to W_0^{2,q'}(\Omega)$$

such that $\operatorname{div} Bf = f$ for these f. Now we define the extension operator $E_\tau = E_1 - B \circ E_1$. Obviously, E_τ is a bounded operator from $W_\tau^{1-1/q',q'}(\partial\Omega)$ to $W^{2,q'}(\Omega)$ such that $E_\tau(h) = 0$ on $\partial\Omega$ and $\operatorname{div} E_\tau(h) = 0$ in Ω, i.e. $E_\tau(h) \in Y_\sigma^{2,q'}(\Omega)$. Moreover, $N \cdot \nabla E_\tau(h) = N \cdot \nabla w_h = h$ on $\partial\Omega$ due to the properties of B.

Let $h \in W_\tau^{1-1/q',q'}(\partial\Omega)$. Then we use $w = E_\tau(h) \in Y_\sigma^{2,q'}(\Omega)$ as a test function in $(2.8)_1$ to see that

$$\begin{aligned} -\big(u, \Delta E_\tau(h)\big) &= \langle \mathcal{F}, E_\tau(h) \rangle \\ &= -\big(F, \nabla E_\tau(h)\big) - \langle g, N \cdot \nabla E_\tau(h) \rangle_{\partial\Omega} \\ &= -\big(F, \nabla E_\tau(h)\big) - \langle g_\tau, h \rangle_{\partial\Omega}. \end{aligned}$$

With $u_\tau := g_\tau$ the former identity coincides with (2.15) and does not depend on the particular choice of the extension operator E_τ.

(3) It suffices to consider F, g_τ or $k, \hat{g}$ such that $\mathcal{F} = 0$ or $\mathcal{K} = 0$, respectively. If $\mathcal{K} = 0$ so that $0 = (k, \psi) - \langle \hat{g}, \psi \rangle_{\partial\Omega}$ for all $\psi \in W^{1,q'}(\Omega)$, then $k = 0$ since we may consider the dense subset $C_0^\infty(\Omega)$ of $L^{r'}(\Omega)$ for the test functions ψ. Hence $0 = \langle \hat{g}, \psi \rangle_{\partial\Omega}$ for all $\psi \in W^{1,q'}(\Omega)$ and consequently $\hat{g} = 0$.

Now let $\mathcal{F} = 0$ so that, using the notation $f = \operatorname{div} F$,

$$0 = \langle f, w \rangle - \langle g_\tau, N \cdot \nabla w \rangle_{\partial\Omega} \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega). \tag{2.20}$$

Hence

$$\langle f, w \rangle = 0 \quad \text{for all } w \in C_{0,\sigma}^\infty(\Omega),$$

and a classical theorem on weak solutions of the Stokes problem proves that $f = \nabla p$ with $p \in L^r(\Omega)$. Therefore,

$$-(F, \nabla w) = \langle f, w \rangle = \langle \nabla p, w \rangle = -\int_\Omega p \operatorname{div} w \, dx = 0$$

for all $w \in Y_\sigma^{2,q'}(\Omega)$ and even for all $w \in W_{0,\sigma}^{1,r'}(\Omega) := W_0^{1,r'}(\Omega) \cap L_\sigma^{r'}(\Omega)$. In this sense $F = 0$ and $f = 0$, and (2.20) implies that

$$\langle g_\tau, N \cdot \nabla w \rangle_{\partial\Omega} = 0 \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega).$$

Using the operator E_τ we get that $\langle g_\tau, h \rangle_{\partial\Omega} = 0$ for all $h \in W_\tau^{1-1/q',q'}(\Omega)$ and hence $g_\tau = 0$. $\square$

Let us introduce a further notation for very weak solutions of the Stokes system which will be helpful in the analysis of nonstationary problems, see §§2.3 - 2.4.

Definition 2.5. For $f \in Y_\sigma^{-2,q}(\Omega)$ let $A_q^{-1} P_q f$ denote the unique vector field in $L_\sigma^q(\Omega)$ satisfying

$$(A_q^{-1} P_q f, v) = \langle f, A_{q'}^{-1} v \rangle \quad \text{for all } v \in L_\sigma^{q'}(\Omega),$$

or, equivalently, with $v = A_{q'} w$,

(2.21) $$(A_q^{-1} P_q f, A_{q'} w) = \langle f, w \rangle \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega).$$

Remark 2.6. (1) Formally, every gradient field ∇p, $p \in L^{q'}(\Omega)$, vanishes when being considered as an element of $Y_\sigma^{-2,q}(\Omega)$. In this sense we have to identify two elements $f, f' \in Y_\sigma^{-2,q}(\Omega)$ when $f - f'$ is a gradient field, or, formally, when $P_q f = P_q f'$. The notation $P_q f$ and $A_q^{-1} P_q f$ in Definition 2.5 is formal and indicates that only solenoidal test functions v are used.

(2) Since $A_q^{-1} P_q f \in L_\sigma^q(\Omega)$ for $f \in Y_\sigma^{-2,q}(\Omega)$, (2.21) also reads

$$-(A_q^{-1} P_q f, \Delta w) = \langle f, w \rangle \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega).$$

Hence $A_q^{-1} P_q f$ is the unique very weak solution of (2.8) with $\mathcal{F} = f$ and $\mathcal{K} = 0$, i.e.,

$$A_q^{-1} P_q : Y_\sigma^{-2,q}(\Omega) \to L_\sigma^q(\Omega)$$

is the corresponding bounded solution operator. In particular,

(2.22) $$\|A_q^{-1} P_q \operatorname{div} F\|_q \le c \, \|F\|_r, \quad F \in L^r(\Omega),$$

by (2.6), (2.7), (2.13) when using $\mathcal{F} = f = \operatorname{div} F$.

(3) Let us discuss the relation of Definition 2.5 to the weak Stokes problem. Given $F \in L^\rho(\Omega)$, $1 < \rho < \infty$, there exists a unique weak solution $u \in W_{0,\sigma}^{1,\rho}(\Omega) = \mathcal{D}(A_\rho^{1/2})$ such that

$$(\nabla u, \nabla v) = \langle \operatorname{div} F, v \rangle = -(F, \nabla v) \quad \text{for all } W_{0,\sigma}^{1,\rho'}(\Omega)$$

(2.23) $$\|\nabla u\|_\rho \le c \, \|F\|_\rho$$

where $c = c(\rho, \Omega) > 0$. Using as a test function $v \in Y_\sigma^{2,\rho'}(\Omega)$ we get that

$$\langle \operatorname{div} F, v \rangle = -(u, \Delta v) = (u, A_{\rho'} v).$$

Hence u coincides with the unique very weak solution $A_\rho^{-1} P_\rho \operatorname{div} F \in L_\sigma^\rho(\Omega)$, and we conclude that $A_\rho^{-1} P_q \operatorname{div} F \in \mathcal{D}(A_\rho^{1/2})$, and, from (2.23), that

(2.24) $$\|A_\rho^{1/2} A_\rho^{-1} P_\rho \operatorname{div} F\|_\rho \le c \, \|F\|_\rho$$

where $c = c(\rho, \Omega) > 0$. For short, we will write $A_\rho^{-1/2} P_\rho \operatorname{div} F = A_\rho^{1/2} A_\rho^{-1} P_\rho \operatorname{div} F$ so that (2.24) reads

$$\|A_\rho^{-1/2} P_\rho \operatorname{div} F\|_\rho \le c\,\|F\|_\rho$$

2.2. The Stationary Navier-Stokes System.

DEFINITION 2.7. Let $1 < r,\ q < \infty$ satisfy $\frac{2}{q} \le \frac{1}{r} \le \frac{1}{3} + \frac{1}{q}$ and let the data F, k, g be given as in (2.2), (2.3). Then $u \in L^q(\Omega)$ is called a *very weak solution* of the stationary Navier-Stokes system

$$-\nu\Delta u + \operatorname{div}(uu) + \nabla p = f = \operatorname{div} F, \quad \operatorname{div} u = k \text{ in } \Omega,\ u|_{\partial\Omega} = g \tag{2.25}$$

if for all $w \in C^2_{0,\sigma}(\overline{\Omega})$

$$-\nu(u, \Delta w) - (uu, \nabla w) = -(F, \nabla w) - \nu\langle g,\ N\cdot\nabla w\rangle_{\partial\Omega} \tag{2.26}$$

and

$$\operatorname{div} u = k \text{ in } \Omega,\ u\cdot N|_{\partial\Omega} = g\cdot N. \tag{2.27}$$

REMARK 2.8. As already noted in Remark 1.9, the variational problem (2.26) is missing the term (ku, w) compared to the approach in [**12**], [**14**], [**19**], [**21**], [**26**] where the authors considered the equation

$$-\nu(u, \Delta w) - (uu, \nabla w) - (ku, w) = -(F, \nabla w) - \nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega},$$

$w \in C^2_{0,\sigma}(\overline{\Omega})$. The only reason for this change is to keep the proofs shorter than for the model including the term ku.

THEOREM 2.9. *There exists a constant $\varepsilon_* = \varepsilon_*(q, r, \Omega)$ independent of the data F, k, g and the viscosity $\nu > 0$ with the following property:*

(1) *If*

$$\|F\|_r + \nu\|k\|_r + \nu\|g\|_{-1/q,q,\partial\Omega} \le \varepsilon_*\nu^2, \tag{2.28}$$

then there exists a very weak solution $u \in L^q(\Omega)$ to the stationary Navier-Stokes system (2.25)*. This solution satisfies the a priori estimate*

$$\nu\|u\|_q \le c\big(\|F\|_r + \nu\|k\|_r + \nu\|g\|_{-1/q,q,\partial\Omega}\big) \tag{2.29}$$

where $c = c(q, r, \Omega) > 0$.

(2) *A very weak solution u to data F, k, g is unique in $L^q(\Omega)$ under the smallness condition $\|u\|_q \le \varepsilon_*\nu$.*

We note that in Definition 2.7 and Theorem 2.9 we need the restrictions $2r \le q$ and $q \ge 3$ in contrast to the linear case. The proof of existence (and hence of local uniqueness) is based on Banach's Fixed Point Theorem, whereas the proof of uniqueness in all of $L^q(\Omega)$ requires a bootstrapping argument; the case $q = 3$ needs a further approximation step and will be omitted.

PROOF. (1) Since $2r \le q$, every vector field $u \in L^q(\Omega)$ satisfies the estimate

$$\|uu\|_r \le c\,\|u\|^2_{2r} \le c\,\|u\|^2_q. \tag{2.30}$$

Now, for arbitrary data F, k, g as in (2.2), (2.3), let $u = S(F, k, g) \in L^q(\Omega)$ denote the very weak solution of the Stokes problem (2.1) with $\nu = 1$. Then, in view of (2.30) a very weak solution $u \in L^q(\Omega)$ of the Navier-Stokes system (2.25) is a fixed point of the nonlinear map

$$\mathcal{N}(u) = S\big(\frac{1}{\nu}(F - uu), k, g\big) = S(\frac{1}{\nu}F, k, g) - \frac{1}{\nu}S(uu, 0, 0).$$

To apply Banach's fixed Point Theorem we estimate $\mathcal{N}(u)$ by using (2.30) and the *a priori* estimate (2.13) for the operator S as follows:

$$\begin{aligned} \|\mathcal{N}(u)\|_q &\le c\big(\tfrac{1}{\nu}(\|F\|_r + \|u\|_q^2) + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big) \\ &= a\|u\|_q^2 + b \end{aligned} \tag{2.31}$$

where $a = \frac{c}{\nu}$ and $b = c\big(\frac{1}{\nu}\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big)$. Moreover, for $u, u' \in L^q(\Omega)$ we get the estimate

$$\begin{aligned} \|\mathcal{N}(u) - \mathcal{N}(u')\|_q &= \|\tfrac{1}{\nu}S(uu - u'u', 0, 0)\|_q \\ &\le \tfrac{c}{\nu}\|u - u'\|_q(\|u\|_q + \|u'\|_q) \end{aligned} \tag{2.32}$$

with the same constant $c > 0$ as above. Now consider the closed ball $\mathcal{B}_\rho \subset L^q(\Omega)$ of radius $\rho > 0$ and center 0 where ρ is the smallest positive root of the quadratic equation $y = ay^2 + b$; for the existence of $\rho > 0$ we need the smallness condition

$$4\,ab < 1$$

which is equivalent to (2.28) with a suitable constant $\varepsilon_* = \varepsilon_*(q, r, \Omega) > 0$. Furthermore note that $\rho < \frac{1}{2a}$ so that by (2.32)

$$\|\mathcal{N}(u) - \mathcal{N}(u')\|_q \le \kappa\|u - u'\|_q, \;\; u, u' \in \mathcal{B}_\rho,$$

with $\kappa = 2a\rho < 1$. Since $\mathcal{N}$ maps $\mathcal{B}_\rho$ into $\mathcal{B}_\rho$ by (2.31) and is a strict contraction on $\mathcal{B}_\rho$, Banach's Fixed Point Theorem yields a unique fixed point $u \in \mathcal{B}_\rho$ of $\mathcal{N}$. Finally the trivial bound $\rho \le 2b$ yields the *a priori* estimate (2.29).

(2) To prove uniqueness of a very weak solution u in $L^q(\Omega)$ we start with the case when $q > 3$. Let $u, v \in L^q(\Omega)$ be fixed points of $\mathcal{N}$. Then $w = u - v$ is the unique very weak solution of the linear Stokes system

$$-\nu\Delta w + \nabla p = -\text{div}\,(wu + vw), \;\; \text{div}\, w = 0 \quad \text{in } \Omega, \;\; w_{|_{\partial\Omega}} = 0 \tag{2.33}$$

with "known" right-hand side $-\text{div}\,(wu + vw)$. Since $u, v \in L^q(\Omega)$ and consequently $w \in L^{q_1}_\sigma(\Omega)$ where $q_1 = q$, we get that

$$wu + vw \in L^{\rho_1}(\Omega), \;\; \frac{1}{\rho_1} = \frac{1}{q} + \frac{1}{q_1}.$$

Hence w coincides with the unique *weak* solution of the Stokes problem (2.33) and satisfies

$$w \in \mathcal{D}(A_{\rho_1}^{1/2}) = W^{1,\rho_1}_{0,\sigma}(\Omega) \subset L^{q_1}_\sigma(\Omega), \;\; \frac{1}{q_1} = \frac{1}{\rho_1} - \frac{1}{3} = \frac{1}{q} + (\frac{1}{q} - \frac{1}{3}).$$

If $\rho_1 < 2$, i.e., $q < 4$, we repeat this argument finitely many times to get in the m-th step, $m = 1, 2, 3, \ldots$, that

$$w \in L^{q_m}_\sigma(\Omega), \;\; \frac{1}{q_m} = \frac{1}{q} + m(\frac{1}{q} - \frac{1}{3}).$$

Since $q > 3$, we will arrive at the property

$$wu + vw \in L^{\rho_m}(\Omega), \;\; \frac{1}{\rho_m} = \frac{1}{q} + \frac{1}{q_m} = \frac{2}{q} + m(\frac{1}{q} - \frac{1}{3}) \le \frac{1}{2}$$

for sufficiently large $m \in \mathbb{N}$. Now we see that $wu + vw \in L^2(\Omega)$, consequently $w \in \mathcal{D}(A_2^{1/2}) = W^{1,2}_{0,\sigma}(\Omega)$, and that we may test in (2.33) with w. By these means we get that

$$\begin{aligned} \nu \|\nabla w\|_2^2 &= \int_\Omega u(w\cdot\nabla w)\,dx + \int_\Omega w(v\cdot\nabla w)\,dx = \int_\Omega u(w\cdot\nabla w)\,dx \\ &\le \|u\|_3\,\|w\|_6\,\|\nabla w\|_2 \\ &\le c\,\|u\|_q\,\|\nabla w\|_2^2. \end{aligned}$$

Hence, under the smallness condition $\|u\|_q \le \varepsilon_* \nu$ we may conclude that $\nabla w = 0$ and $u = v$.

The limit case $q = 3$, in which the above iteration is stationary ($q_m = q$ for all $m \in \mathbb{N}$), requires a complicated approximation and smoothing argument. For details we refer to [**21**]. $\square$

2.3. The Instationary Stokes System. Looking at very weak solutions $u \in L^s\big(0,T;L^q(\Omega)\big)$, $1 < s,q < \infty$, of the initial-boundary value problem of the Stokes system we carefully introduce the set of admissible initial values, $\mathcal{J}^{q,s}_\sigma(\Omega)$, as a subset of $Y^{-2,q}_\sigma(\Omega)$. In this subsection we set $\nu = 1$ for simplicity.

Definition 2.10. Given $1 < s,q < \infty$ let

$$\mathcal{J}^{q,s}_\sigma(\Omega) = \Big\{u_0 \in Y^{-2,q}_\sigma(\Omega) : \int_0^\infty \|A_q e^{-\tau A_q}(A_q^{-1}P_q u_0)\|_q^s\,d\tau < \infty\Big\},$$

equipped with the norm

$$\|u_0\|_{\mathcal{J}^{q,s}_\sigma} := \Big(\int_0^\infty \|A_q e^{-\tau A_q}(A_q^{-1}P_q u_0)\|_q^s\,d\tau\Big)^{1/s}.$$

Remark 2.11. (1) The term $\|\cdot\|_{\mathcal{J}^{q,s}_\sigma}$ defines a norm on $\mathcal{J}^{q,s}_\sigma(\Omega)$: If $\|u_0\|_{\mathcal{J}^{q,s}_\sigma} = 0$, then $A_q e^{-tA_q}(A_q^{-1}P_q u_0) = 0$ and consequently $e^{-tA_q}A_q^{-1}P_q u_0 = 0$ for a.a. $t > 0$; as $t \to 0+$, we conclude that $A_q^{-1}P_q u_0 = 0$, i.e., $u_0 = 0$ as an element of $Y^{-2,q}_\sigma(\Omega)$. Note that $\|u_0\|_{\mathcal{J}^{q,s}_\sigma(\Omega)}$ equals the $L^s\big(0,T;L^q(\Omega)\big)$-norm of $Au(t)$ where $u(t)$ denotes the strong solution of the homogeneous instationary Stokes problem with initial value $A_q^{-1}P_q u_0 \in L^q_\sigma(\Omega)$.

(2) The spaces $\mathcal{J}^{q,s}_\sigma(\Omega)$ can be considered as real interpolation spaces and identified with solenoidal subspaces of Besov spaces. Actually,

$$u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega) \Leftrightarrow A_q^{-1}P_q u_0 \in \big(\mathcal{D}(A_q), L^q_\sigma(\Omega)\big)_{1/s,s}$$

and

$$\|u_0\|_{\mathcal{J}^{q,s}_\sigma} + \|A_q^{-1}P_q u_0\|_q \sim \|A_q^{-1}P_q u_0\|_{(\mathcal{D}(A_q),L^q_\sigma(\Omega))_{1/s,s}}$$

in the sense of norm equivalence, see [**30**, (2.5)], [**65**]. Moreover, consider the solenoidal Besov spaces $\mathbb{B}^{2-2/s}_{q,s}(\Omega)$ introduced in [**3**, (0.6)], with the property

$$\mathbb{B}^{2-2/s}_{q,s}(\Omega) = \begin{cases} \{u \in B^{2-2/s}_{q,s}(\Omega) : \operatorname{div} u = 0,\ u|_{\partial\Omega} = 0\}, & \frac{1}{q} < 2 - \frac{2}{s}, \\ \{u \in B^{2-2/s}_{q,s}(\Omega) : \operatorname{div} u = 0,\ u\cdot N|_{\partial\Omega} = 0\}, & \frac{1}{q} > 2 - \frac{2}{s}, \end{cases}$$

cf. [**65**], where $B^{2-2/s}_{q,s}(\Omega)$ are the usual Besov spaces. By [**3**, Proposition 3.4]

$$u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega) \Leftrightarrow A_q^{-1}P_q u_0 \in \big(\mathcal{D}(A_q), L^q_\sigma(\Omega)\big)_{1/s,s} = \mathbb{B}^{2-2/s}_{q,s}(\Omega).$$

(3) Consider $u_0 \in Y^{-2,q}_\sigma(\Omega)$ such that

$$|\langle u_0, w\rangle| \le c\,\|A_{q'}^{-1/s+\varepsilon} w\|_{q'}, \quad w \in Y^{2,q'}_\sigma(\Omega),$$

where $0<\varepsilon<\frac{1}{s}$. Then by (1.24) $u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega)$.

Definition 2.12. Let $1<s,q<\infty$, $1<r\le q$, $\frac{1}{3}+\frac{1}{q}\ge\frac{1}{r}$, $0<T\le\infty$, let the data F,k,g satisfy

$$F\in L^s\big(0,T;L^r(\Omega)\big),\ k\in L^s\big(0,T;L^r(\Omega)\big),\ g\in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big) \tag{2.34}$$

$$\int_\Omega k(t)\,dx=\langle g(t),N\rangle_{\partial\Omega}\quad\text{for a.a. } t\in(0,T), \tag{2.35}$$

and let $u_0\in\mathcal{J}^{q,s}_\sigma(\Omega)$. Then $u\in L^s\big(0,T;L^q(\Omega)\big)$ is called a *very weak solution* of the instationary Stokes system

$$\begin{aligned} u_t-\Delta u+\nabla p &= \operatorname{div}F, & \operatorname{div}u &= k \text{ in } \Omega\times(0,T)\\ u(0) &= u_0 \text{ at } t=0, & u &= g \text{ on } \partial\Omega\times(0,T)\end{aligned} \tag{2.36}$$

if

$$\begin{aligned} -(u,w_t)_{\Omega,T}-(u,\Delta w)_{\Omega,T} &= \langle u_0,w(0)\rangle-(F,\nabla w)_{\Omega,T}-\langle g,N\cdot\nabla w\rangle_{\partial\Omega,T}\\ \operatorname{div}u=k \text{ in } \Omega\times(0,T),&\quad u\cdot N=g\cdot N \text{ on } \partial\Omega\times(0,T)\end{aligned} \tag{2.37}$$

for all test functions $w\in C^1_0\big([0,T);C^2_{0,\sigma}(\overline{\Omega})\big)$.

Remark 2.13. (1) As shown in Theorem 2.14 below the very weak solution $u\in L^s\big(0,T;L^q(\Omega)\big)$ of (2.36), (2.37) has the property $A_q^{-1}P_qu(\cdot)\in C^0\big([0,T);L^q(\Omega)\big)$ or equivalently, $u\in C^0\big([0,T);Y^{-2,q}_\sigma(\Omega)\big)$. Hence the initial value $u(0)=u_0$ in $(2.36)_2$ is attained in $Y^{-2,q}_\sigma(\Omega)$, i.e.

$$\langle u(0),w\rangle=\langle u_0,w\rangle \text{ for all } w\in Y^{2,q'}_\sigma(\Omega),$$

or equivalently $(A_q^{-1}P_qu)(0)=A_q^{-1}P_qu_0$.

(2) Definition 2.12 may be extended, correspondingly to Definition 2.2, to the problem

$$\begin{aligned} (u,w_t)_{\Omega,T}-(u,\Delta w)_{\Omega,T} &= \langle\mathcal{F},w\rangle\\ -(u,\nabla\psi)_{\Omega,T} &= \langle\mathcal{K},\psi\rangle\end{aligned} \tag{2.38}$$

with data $\mathcal{F}\in L^s\big(0,T;Y^{-2,q}_\sigma(\Omega)\big)$ and $\mathcal{K}\in L^s\big(0,T;W^{-1,q}_0(\Omega)\big)$ and for suitable test function w and ψ, cf. [**51**]. Then existence and uniqueness of a very weak solution $u\in L^s\big(0,T;L^q(\Omega)\big)$ to (2.38) is a direct consequence of duality arguments and results on the strong instationary Stokes system in $L^{s'}\big(0,T;L^{q'}(\Omega)\big)$. As in §2.1, in this very general setting neither initial values nor boundary values of u are well-defined. Actually, every $u\in L^s\big(0,T;L^q(\Omega)\big)$ is the very weak solution of (2.38) for certain data $\mathcal{F}$ and $\mathcal{K}$. However, in contrast to our approach in §2.1, we will follow a different idea to solve (2.37).

Theorem 2.14. *Suppose that the data F,k,g satisfy the conditions* (2.34), (2.35), *and that $u_0\in\mathcal{J}^{q,s}_\sigma(\Omega)$ where $1<s$, $q<\infty$, $1<r\le q$, $\frac{1}{q}+\frac{1}{3}\ge\frac{1}{r}$. Then there exists a unique very weak solution $u\in L^s\big(0,T;L^q(\Omega)\big)$ of* (2.36), *satisfying*

$$u_t\in L^s\big(0,T;Y^{-2,q}_\sigma(\Omega)\big),\ u\in C^0\big([0,T);Y^{-2,q}_\sigma(\Omega)\big).$$

Moreover, there exists a constant $c=c(q,r,s,\Omega)>0$ independent of $T>0$ such that

$$\begin{aligned}&\|u\|_{L^s(L^q)}+\|u_t\|_{L^s(Y^{-2,q}_\sigma)}\\ &\quad\le c\big(\|F\|_{L^s(L^r)}+\|k\|_{L^s(L^r)}+\|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}+\|u_0\|_{\mathcal{J}^{q,s}_\sigma}\big).\end{aligned} \tag{2.39}$$

PROOF. For almost all $t \in (0,T)$ let $H(t)$ denote the solution of the weak Neumann problem

$$\Delta H = k \text{ in } \Omega, \;\; N \cdot (\nabla H - g) = 0 \text{ on } \partial\Omega.$$

Since $k(t) \in L^r(\Omega) \subset W_0^{-1,q}(\Omega)$, we find a unique solution $\nabla H(t) \in L^q(\Omega)$ satisfying

$$\nabla H(t) \in L^s\big(0,T; L^q(\Omega)\big), \;\; \|\nabla H\|_{L^s(L^q)} \le c\,\big(\|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}\big). \tag{2.40}$$

Moreover, for almost all $t \in (0,T)$ let $\gamma(t) = \gamma_{F(t),k(t),g(t)} \in L^q(\Omega)$ denote the very weak solution of the inhomogeneous Stokes problem

$$-\Delta\gamma + \nabla p = \operatorname{div} F, \;\; \operatorname{div}\gamma = k \text{ in } \Omega, \;\; \gamma\big|_{\partial\Omega} = g, \tag{2.41}$$

satisfying the estimate

$$\|\gamma\|_{L^s(L^q)} \le c\big(\|F\|_{L^s(L^r)} + \|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}\big). \tag{2.42}$$

Assume that $u \in L^s\big(0,T; L^q(\Omega)\big)$ is a very weak solution of (2.36). Obviously

$$P_q u = u - \nabla H \quad \text{and} \quad P_q\gamma = \gamma - \nabla H \quad \text{for a.a. } t \in (0,T),$$

where P_q denotes the usual Helmholtz projection on $L^q(\Omega)$. Thus

$$\hat u := P_q u = u - \nabla H = u - \gamma + P_q\gamma \in L^s\big(0,T; L^q_\sigma(\Omega)\big).$$

Next let us prove that $U = A_q^{-1}\hat u \in L^s\big(0,T; \mathcal{D}(A_q)\big)$ is a strong solution of the Stokes system

$$U_t + A_q U = P_q\gamma \text{ on } (0,T), \;\; U(0) = A_q^{-1}P_q u_0. \tag{2.43}$$

For this reason consider any test function $v \in C_0^1\big([0,T); L^{q'}_\sigma(\Omega)\big)$ and also $w = A_{q'}^{-1}v \in C_0^1\big([0,T); Y^{2,q'}_\sigma(\Omega)\big)$. Then

$$\begin{aligned} &-(U,v_t)_{\Omega,T} + (A_q U, v)_{\Omega,T} - (P_q\gamma, v)_{\Omega,T} \\ &\quad= -(\hat u, w_t)_{\Omega,T} + (\hat u, A_{q'}w)_{\Omega,T} - (P_q\gamma, A_{q'}w)_{\Omega,T} \\ &\quad= -(u, w_t)_{\Omega,T} - (u-\gamma, \Delta w)_{\Omega,T}, \end{aligned}$$

since $(\nabla H, w_t)_{\Omega,T} = 0$ and $\operatorname{div}(u-\gamma) = 0$. Due to (2.41) we know that

$$-(\gamma, \Delta w)_{\Omega,T} = -(F, \nabla w)_{\Omega,T} - \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T},$$

so that we may proceed as follows:

$$\begin{aligned} &-(U,v_t)_{\Omega,T} + (A_q U, v)_{\Omega,T} - (P_q\gamma, v)_{\Omega,T} \\ &\quad= -(u, w_t)_{\Omega,T} - (u, \Delta w)_{\Omega,T} + (F, \nabla w)_{\Omega,T} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} \\ &\quad= \langle u_0, w(0)\rangle \\ &\quad= \big(A_q^{-1}P_q u_0, v(0)\big). \end{aligned}$$

This identity, valid for all $v \in C^1\big([0,T); L^{q'}_\sigma(\Omega)\big)$, proves that U satisfies (2.43) and that $U(0) = A_q^{-1}P_q u_0$. Moreover, by Lemma 1.12 on maximal regularity, the estimates (1.28), (2.42) and the variation of constants formula (1.27) we know that $U_t \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$, in particular, $U \in C^0\big([0,T); L^q_\sigma(\Omega)\big)$,

$$U(t) = e^{-A_q t}(A_q^{-1}P_q u_0) + \int_0^t e^{-A_q(t-\tau)}P_q\gamma(\tau)\,d\tau \tag{2.44}$$

and

$$
\begin{aligned}
&\|U_t\|_{L^s(L^q)} + \|A_q U\|_{L^s(L^q)} \\
(2.45) \qquad &\leq \; c\Big(\int_0^T \|A_q\, e^{-A_q t}(A_q^{-1} P_q u_0)\|_q^s\, dt\Big)^{1/s} + \|P_q\gamma\|_{L^s(L^q)} \\
&\leq \; c\big(\|u_0\|_{\mathcal{J}_\sigma^{q,s}} + \|F\|_{L^s(L^r)} + \|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}\big).
\end{aligned}
$$

Since $u = \hat{u} + \nabla H = A_q U + \nabla H$, we proved so far that u necessarily has the representation

$$
(2.46) \qquad u = \nabla H + A_q\, e^{-A_q t}(A_q^{-1} P_q u_0) + \int_0^t A_q e^{-A_q(t-\tau)} P_q\gamma(\tau)\, d\tau.
$$

Hence u is uniquely defined by the data F, k, g and u_0 and satisfies (2.36) in the very weak sense, since we may pass through the previous computations in reverse order. Finally, (2.45) and (2.46) imply (2.39). □

Remark 2.15. The very weak solution $u \in L^s\big(0,T; L^q(\Omega)\big)$ constructed in Theorem 2.14 has a trace $u_{|_{\partial\Omega}} \in L^s\big(0,T; W^{-1/q,q}(\partial\Omega)\big)$. Actually, since $k = \operatorname{div} u \in L^s\big(0,T; L^r(\Omega)\big)$, we get that $u\cdot N_{|_{\partial\Omega}} \in L^s\big(0,T; W^{-1/r,r}(\partial\Omega)\big)$ and even

$$
u\cdot N_{|_{\partial\Omega}} = g\cdot N \in L^s\big(0,T; W^{-1/q,q}(\partial\Omega)\big).
$$

Concerning the tangential component of u on $\partial\Omega$ we consider $h \in C_0^1\big((0,T); W_\tau^{1-1/q',q'}(\partial\Omega)\big)$ and $w = E_\tau(h) \in C_0^1\big((0,T); Y_\sigma^{2,q'}(\Omega)\big)$ satisfying $h = N\cdot\nabla w_{|_{\partial\Omega}}$, cf. Theorem 2.4. Inserting w in (2.37) we obtain the formula

$$
\langle g, h\rangle_{\partial\Omega,T} = (u, w_t)_{\Omega,T} + (u, \Delta w)_{\Omega,T} - (F, \nabla w)_{\Omega,T}.
$$

This formula yields a well-defined expression for the tangential component $g_\tau = g - (g\cdot N)N$ of the boundary values. Obviously, if u is sufficiently smooth, integration by parts shows that $u_{\tau}{}_{|_{\partial\Omega}} = g_\tau$.

2.4. The Instationary Navier-Stokes System. Let us consider the instationary Navier-Stokes system

$$
(2.47) \qquad
\begin{aligned}
u_t - \nu\Delta u + \operatorname{div}(uu) + \nabla p = f, &\quad \operatorname{div} u = k \text{ in } \Omega\times(0,T) \\
u(0) = u_0 \text{ at } t = 0, &\quad u = g \text{ on } \partial\Omega\times(0,T).
\end{aligned}
$$

Definition 2.16. Let the data F, k, g satisfy (2.34), (2.35) and let $u_0 \in \mathcal{J}_\sigma^{q,s}(\Omega)$ where

$$
(2.48) \qquad 2 < s < \infty,\; 3 < q < \infty,\; \frac{2}{s} + \frac{3}{q} = 1 \quad \text{and} \quad \frac{1}{3} + \frac{1}{q} \geq \frac{1}{r} \geq \frac{2}{q}.
$$

Then $u \in L^s\big(0,T; L^q(\Omega)\big)$ is called a very weak solution of (2.47) if for all test functions $w \in C_0^1\big([0,T); C_{0,\sigma}^2(\overline{\Omega})\big)$

$$
(2.49) \qquad
\begin{aligned}
&-(u, w_t)_{\Omega,T} - \nu(u, \Delta w)_{\Omega,T} - (uu, \nabla w)_{\Omega,T} \\
&\qquad = -(F, \nabla w)_{\Omega,T} - \nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} + \langle u_0, w(0)\rangle, \\
&\operatorname{div} u = k \text{ in } \Omega\times(0,T), \quad u\cdot N_{|_{\partial\Omega}} = g\cdot N \text{ on } \partial\Omega\times(0,T).
\end{aligned}
$$

REMARK 2.17. (1) In (2.48) we added the condition $\mathcal{S}(s,q) = \frac{2}{s} + \frac{3}{q} = 1$ in order to allow an estimate of the nonlinear term $(uu, \nabla w)_{\Omega,T}$. Compared to (1.19) in Definition 1.8 the assumptions on q, r, s are a little bit weaker in (2.48).

(2) Looking at [**12**], [**19**] we omitted the term $(-k, uw)_{\Omega,T}$ on the left-hand side of $(2.49)_1$ leading to some simplifications in the proof, cf. Remarks 1.9 and 2.8.

THEOREM 2.18. *Given data F, k, g, u_0 as in Definition 2.16 there exists some $T' = T'(\nu, F, k, g, u_0) \in (0,T]$ and a unique very weak solution $u \in L^s\big(0,T'; L^q(\Omega)\big)$ of the Navier-Stokes system* (2.47). *Moreover, u satisfies*

$$u_t \in L^{s/2}_{\mathrm{loc}}\big([0,T'); Y_\sigma^{-2,q}(\Omega)\big),$$

and the interval of existence, $[0,T')$, is determined by the condition

$$\Big(\int_0^{T'} \|\nu A_q e^{-\nu t A_q}(A_q^{-1}P_q u_0)\|_q^s\,dt\Big)^{1/s} + \|F\|_{L^s(0,T';L^r)} + \|\nu k\|_{L^s(0,T';L^r)} + \|\nu g\|_{L^s(0,T';W^{-1/q,q}(\partial\Omega))} \le \varepsilon_* \nu^{2-1/s}. \tag{2.50}$$

We note that the first term in (2.50) coincides with $\|u_0\|_{\mathcal{J}_\sigma^{s,q}}$ except for the interval of integration $(0,T')$ and the viscosity $\nu > 0$. If $T = \infty$, the case $T' = \infty$ is possible provided the data F, k, g, u_0 are sufficiently small. Formally, (2.50) contains the smallness condition (2.28) in the case $s = \infty$ which, however, is excluded by (2.48).

PROOF OF THEOREM 2.18. Let $\gamma(t) = \gamma_{F(t),k(t),g(t),u_0}$ denote the unique very weak solution in $L^s\big(0,T;L^q(\Omega)\big)$ of the linear system

$$\begin{aligned} \tfrac{\partial\gamma}{\partial t} - \nu\Delta\gamma + \nabla p = \operatorname{div} F, \quad \operatorname{div}\gamma &= k \text{ in } \Omega\times(0,T),\\ \gamma(0) = u_0, \qquad \gamma &= g \text{ on } \partial\Omega\times(0,T), \end{aligned}$$

as constructed in §2.3 when $\nu = 1$. Obviously Theorem 2.14 extends to the case of a general viscosity $\nu > 0$, and the *a priori* estimate (2.39) reads as follows:

$$\begin{aligned} \|\nu\gamma\|_{L^s(0,T',L^q)} \le c\bigg(\Big(\int_0^{T'} \|\nu A_q e^{-\nu\tau A_q}(A_q^{-1}P_q u_0)\|_q^s\,d\tau\Big)^{1/s} \\ + \|F\|_{L^s(0,T';L^r)} + \|\nu k\|_{L^s(0,T';L^r)} + \|\nu g\|_{L^s(0,T';W^{-1/q,q}(\partial\Omega))}\bigg) \end{aligned} \tag{2.51}$$

for every $T' \in (0,T]$ with a constant $c = c(q,r,s,\Omega) > 0$ independent of $\nu > 0$ and T'.

Assume that $u \in L^s\big(0,T';L^q(\Omega)\big)$ is a very weak solution of (2.47). Then $\tilde u = u - \gamma$ is a very weak solution of the system

$$\begin{aligned} \tilde u_t - \nu\Delta\tilde u + \nabla p &= -\operatorname{div}(uu), \quad \operatorname{div}\tilde u = 0 \text{ in } \Omega\times(0,T')\\ \tilde u &= 0 \text{ at } t = 0, \qquad \tilde u = 0 \text{ on } \partial\Omega\times(0,T') \end{aligned} \tag{2.52}$$

with the right-hand side $-\operatorname{div}(uu) = -\operatorname{div}\big((\tilde u+\gamma)(\tilde u+\gamma)\big)$. Since $2r \le q$, we get $\|(\tilde u+\gamma)(\tilde u+\gamma)(t)\|_r \le c\,\|\tilde u+\gamma\|_q^2$ for a.a $t \in (0,T')$ and consequently $(\tilde u+\gamma)(\tilde u+\gamma) \in L^{s/2}\big(0,T';L^r(\Omega)\big)$, cf. (2.30). Hence by Theorem 2.14, $\tilde u$ in (2.52) is the unique very weak solution in $L^{s/2}\big(0,T';L^q(\Omega)\big)$ and

$$\tilde u(t) = \mathcal{N}(\tilde u)(t) := -\int_0^t A_q e^{-\nu A_q(t-\tau)}A_q^{-1}P_q \operatorname{div}(\tilde u+\gamma)(\tilde u+\gamma)(\tau)\,d\tau \tag{2.53}$$

for a.a. $t \in (0,T')$, cf. (2.46).

To find $\tilde{u}$ as the fixed point of the nonlinear map $\mathcal{N}$ in $L^s\big(0,T';L^q(\Omega)\big)$ we estimate $\mathcal{N}(\tilde{u})$. Let $\alpha = \frac{1}{2} - \frac{1}{s}$ so that $2\alpha + \frac{3}{q} = \frac{3}{q/2}$ since $\frac{2}{s} + \frac{3}{q} = 1$. Then by Lemma 1.11 (4), (3) and (2.24)

$$\begin{aligned}
\|\mathcal{N}(\tilde{u})(t)\|_q &\le c\int_0^t \frac{1}{\big(\nu(t-\tau)\big)^{1/2+\alpha}}\|A_q^{1/2-\alpha}A_q^{-1}P_q\,\mathrm{div}\,(uu)(\tau)\|_q\,d\tau\\
&\le c\int_0^t \frac{1}{\big(\nu(t-\tau)\big)^{1-1/s}}\|A_{q/2}^{1/2}A_{q/2}^{-1}P_{q/2}\,\mathrm{div}\,(uu)(\tau)\|_{q/2}\,d\tau\\
&\le c\int_0^t \frac{1}{\big(\nu(t-\tau)\big)^{1-1/s}}\|u(\tau)\|_q^2\,d\tau.
\end{aligned}$$

Next we use the Hardy-Littlewood inequality, see [**61**, p. 103],

$$\left(\int_0^T\Big|\int_0^t \frac{1}{(t-\tau)^{1-1/s}}h(\tau)\,d\tau\Big|^s dt\right)^{1/s} \le c\,\|h\|_{L^{s/2}(0,T)}$$

where $c = c(s) > 0$ is independent of T. Hence there exists a constant $c = c(q,r,s,\Omega) > 0$ independent of T' such that

$$\begin{aligned}
\|\mathcal{N}(\tilde{u})\|_{L^s(0,T';L^q)} &\le \frac{c}{\nu^{1-1/s}}\|u\|^2_{L^s(0,T';L^q)}\\
&\le \frac{c}{\nu^{1-1/s}}\left(\|\tilde{u}\|^2_{L^s(0,T';L^q)} + \|\gamma\|^2_{L^s(0,T';L^q)}\right).
\end{aligned}$$

By analogy, we prove for $u' \in L^s\big(0,T';L^q(\Omega)\big)$ and $\tilde{u}' = u' - \gamma$ that

$$\begin{aligned}
(2.54)\qquad &\|\mathcal{N}(\tilde{u}) - \mathcal{N}(\tilde{u}')\|_{L^s(0,T';L^q)}\\
&\le \frac{c}{\nu^{1-1/s}}\|\tilde{u} - \tilde{u}'\|_{L^s(0,T';L^q)}\big(\|u\|_{L^s(0,T';L^q)} + \|u'\|_{L^s(0,T';L^q)}\big).
\end{aligned}$$

Now we may proceed as in the proof of Theorem 2.9. Let $a = \frac{c}{\nu^{1-1/s}}$ and $b = \frac{c}{\nu^{1-1/s}}\|\gamma\|^2_{L^2(0,T';L^q)}$. The smallness condition $4ab < 1$ is equivalent to the estimate $\|\nu\gamma\|_{L^s(0,T':L^q)} \le \varepsilon_*\nu^{2-1/s}$, so that in view of (2.51) the condition (2.50) is sufficient to guarantee that $4ab < 1$. Since (2.51) holds for $T' \in (0,T)$ sufficiently small (or even for $T' = T = \infty$), Banach's Fixed Point Theorem proves the existence of a unique solution to the equation $\tilde{u} = \mathcal{N}(\tilde{u})$ in a sufficiently small closed ball of $L^s\big(0,T';L^q(\Omega)\big)$.

Let us write (2.53) in the form

$$A_q^{-1}\tilde{u}(t) = -\int_0^t e^{-\nu(t-\tau)A_q}A_q^{-1}P_q\,\mathrm{div}\,(uu)(\tau)\,d\tau,\ \ 0 \le t \le T'.$$

Then by the maximal regularity estimate (1.28) and (2.22)

$$\begin{aligned}
\|\big(A_q^{-1}\tilde{u}(\cdot)\big)_t\|_{L^{s/2}(0,T',L^q)} &\le c\,\|A_q^{-1}P_q\,\mathrm{div}\,(uu)\|_{L^{s/2}(0,T';L^q)}\\
&\le c\,\|uu\|_{L^{s/2}(0,T';L^r)}\\
&\le c\left(\|\tilde{u}\|^2_{L^{s/2}(0,T';L^q)} + \|\gamma\|^2_{L^{s/2}(0,T';L^q)}\right)
\end{aligned}$$

so that $\tilde{u}_t \in L^{s/2}\big(0,T';Y_\sigma^{-2,q}(\Omega)\big)$. Since by Theorem 2.14 $\gamma_t \in L^s\big(0,T;Y_\sigma^{-2,q}(\Omega)\big)$, we conclude that $u_t \in L^{s/2}\big(0,T';Y_\sigma^{-2,q}(\Omega)\big)$. Moreover, it is easily seen that $u = \tilde{u} + \gamma$ is a very weak solution of the Navier-Stokes system (2.47).

Finally we prove that u is the unique very weak solution of (2.47) in all of $L^s\big(0,T';L^q(\Omega)\big)$. Assume that $v \in L^s\big(0,T';L^q(\Omega)\big)$ is also a very weak solution to (2.47). Then $U = u - v \in L^s\big(0,T';L^q(\Omega)\big)$ is a very weak solution to the system

$$
\begin{aligned}
U_t - \nu\Delta U + \nabla P &= -\mathrm{div}\,(Uu + vU), \quad \mathrm{div}\, U = 0 \text{ in } \Omega \times (0,T') \\
U &= 0 \text{ at } t = 0, \qquad\qquad U = 0 \text{ on } \partial\Omega \times (0,T').
\end{aligned}
$$

Using similar estimates as in the derivation of (2.54) we get that for all $T'' \in (0,T')$

$$
\|U\|_{L^s(0,T'';L^q)} \le \frac{c}{\nu^{1-1/s}} \|U\|_{L^s(0,T'';L^q)} \big(\|u\|_{L^s(0,T'';L^q)} + \|v\|_{L^s(0,T'';L^q)}\big) \tag{2.55}
$$

with a constant $c > 0$ independent of T''. Hence there exists some $T'' \in (0,T')$ depending on u, v such that (2.55) is reduced to the inequality $\|U\|_{L^s(0,T'';L^q)} \le \frac{1}{2}\|U\|_{L^s(0,T'';L^q)}$ and that consequently $U = 0$, $u = v$ holds on $[0,T'']$. This argument may be repeated finitely many times with the same T'' on the intervals $(T'',2T'')$, $(2T'',3T'')$ etc. and finally leads to $u = v$ on $[0,T')$. Now the proof of Theorem 2.18 is complete. □

3. Regularity of Weak Solutions

Let u be a weak solution of the instationary Navier-Stokes system

$$\begin{array}{rclrclll} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p & = & f, & \operatorname{div} u & = & 0 & \text{in } \Omega\times(0,T) \\ u_{|\partial\Omega} & = & 0, & u(0) & = & u_0 & \text{at } t=0, \end{array} \tag{3.1}$$

in the bounded domain $\Omega\subset\mathbb{R}^3$. Besides the classical Serrin condition

$$u\in L^s\big(0,T;L^q(\Omega)\big),\quad \mathcal{S}(s,q)\le 1,\ 2<s\le\infty,\ 3\le q<\infty, \tag{3.2}$$

cf. (1.14) in Theorem 1.5, there are numerous other assumptions of *conditional regularity* imposed on specific components of u, ∇u or $\omega = \operatorname{rot} u$ to imply regularity of u. Most of these conditions are related to (3.2) with a different upper bound for $\mathcal{S}$, cf. [**9**], [**42**], [**43**], [**49**], [**50**]; other conditions have a more geometric character, see [**4**], [**10**], [**44**], [**45**], [**46**], or are related to the pressure [**5**], [**56**], [**68**]. In the following we describe new results of Serrin's type, i.e., we assume

$$u\in L^r\big(0,T;L^q(\Omega)\big)$$

where $\frac{2}{r}+\frac{3}{q}$ is allowed to be larger than 1 such that u is regular locally or globally in time or locally in space and time. The proofs are based on a local or global identification of the weak solution u with a very weak solution v having the same initial value at $t_0\ge 0$ and the same boundary value as u.

3.1. Local in Time Regularity. In addition to the definition of the global regularity in $(0,T)$, see (1.12), we say that u *is regular at* $t\in(0,T)$ if there exists $0<\delta'<\min(t,T-t)$, such that

$$u\in L^{s_*}\big(t-\delta',t+\delta';L^{q_*}(\Omega)\big),\quad \mathcal{S}(s_*,q_*)=1,\ 2<s_*<\infty,\ 3<q_*<\infty. \tag{3.3}$$

By analogy, u *is regular in* $(a,b)\subset(0,T)$, if u is regular at every $t\in(a,b)$. Note that in §§3.1 – 3.3 we will use the notation s_*, q_* for exponents satisfying $\mathcal{S}(s_*,q_*)=1$, but s,q if $\mathcal{S}(s,q)\ge 1$ is allowed.

Now our first result, see also [**17**], [**18**], reads as follows:

THEOREM 3.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega\in C^{1,1}$, and let*

$$2<s_*<\infty,\quad 3<q_*<\infty,\quad \mathcal{S}(s_*,q_*)=1,\quad \frac{1}{3}+\frac{1}{q_*}=\frac{1}{\rho},\quad 1\le s\le s_*\,. \tag{3.4}$$

Given data

$$f=\operatorname{div}F,\ F\in L^2\big(0,T;L^2(\Omega)\big)\cap L^{s_*}\big(0,T;L^\rho(\Omega)\big)\quad\text{and}\quad u_0\in L^2_\sigma(\Omega), \tag{3.5}$$

let u be a weak solution of the Navier-Stokes system (3.1) *satisfying the strong energy inequality* (1.9) *on $[0,T)$, where $0<T\le\infty$.*

(1) Left-side $L^{s_*}(L^{q_*})$-condition: *If for $t\in(0,T)$*

$$u\in L^{s_*}\big(t-\delta,t;L^{q_*}(\Omega)\big)\quad \textit{for some } 0<\delta=\delta(t)<t, \tag{3.6}$$

then u is regular at t.

(2) Left-side $L^s(L^{q_*})$-condition: *If at $t\in(0,T)$*

$$\liminf_{\delta\to 0+}\frac{1}{\delta}\int_{t-\delta}^{t}\|u(\tau)\|_{q_*}^s\,d\tau<\infty, \tag{3.7}$$

then u is regular at t. Assumption (3.7) *may be replaced by the essentially weaker condition*

$$\liminf_{\delta\to 0+} \frac{1}{\delta^{1-s/s_*}} \int_{t-\delta}^{t} \|u(\tau)\|_{q_*}^{s}\, d\tau = 0, \tag{3.8}$$

which includes (3.6) *when $s = s_*$. Moreover,* (3.8) *is even a necessary condition for regularity of u at t.*

(3) Global $L^s(L^{q_*})$-condition. *There exists a constant $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$ independent of u, u_0, f and ν with the following property: If $u_0 \in L^{q_*}_\sigma(\Omega)$, $u \in L^s\big(0,T; L^{q_*}(\Omega)\big)$,*

$$\int_0^T \|F(\tau)\|_\rho^{s_*}\, d\tau \le \varepsilon_* \nu^{2s_*-1} \quad \textit{and} \quad \int_0^T \|u(\tau)\|_{q_*}^{s}\, d\tau < \varepsilon_* \frac{\nu^{s_*-1}}{\|u_0\|_{q_*}^{s_*-s}}, \tag{3.9}$$

then u is regular in the sense $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$.

The proof of Theorem 3.1 is based on a key lemma, see Lemma 3.2, combining the notions of weak and very weak solutions, and on a technical lemma, see Lemma 3.4, from which the results of Theorem 3.1 and also of §3.2 will follow easily.

Lemma 3.2. *In addition to the assumptions of Theorem 3.1 assume $u_0 \in L^{q_*}_\sigma(\Omega)$. Then there exists a constant $\varepsilon_* = \varepsilon_*(q_*, \Omega) > 0$ independent of u_0, f and ν with the following property: If*

$$\int_0^T \|F\|_\rho^{s_*}\, d\tau \le \varepsilon_* \nu^{2s_*-1} \quad \textit{and} \quad \int_0^T \|e^{-\nu\tau A_{q_*}} u_0\|_{q_*}^{s_*}\, d\tau \le \varepsilon_* \nu^{s_*-1}, \tag{3.10}$$

then the Navier-Stokes system (3.1) *has a unique weak solution u in the sense of Leray and Hopf satisfying Serrin's condition $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ and moreover the energy inequality* (1.8).

We note that the weak solution $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ constructed in Lemma 3.2 even satisfies the energy identity (1.3), see Lemma 1.6 (1).

Proof of Lemma 3.2. Given the smallness condition (3.10) Theorem 2.18 yields a unique very weak solution $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ of (3.1). Moreover,

$$u(t) = \gamma(t) + \tilde{u}(t)$$

where γ solves the instationary Stokes system with data u_0, f in $\Omega \times (0,T)$, i.e.

$$\gamma(t) = e^{-\nu t A_{q_*}} u_0 + \int_0^t A_{q_*} e^{-\nu(t-\tau)A_{q_*}}\, A_{q_*}^{-1} P_{q_*} \operatorname{div} F(\tau)\, d\tau\ , \tag{3.11}$$

and where $\tilde{u}$ solves the nonlinear equation

$$\tilde{u}(t) = -\int_0^t A^{1/2}_{q_*/2}\, e^{-\nu(t-\tau)A_{q_*/2}} A^{-1/2}_{q_*/2} P_{q_*/2} \operatorname{div}(uu)\, d\tau. \tag{3.12}$$

Since $F \in L^2\big(0,T; L^2(\Omega)\big)$ and $u_0 \in L^2_\sigma(\Omega)$, we see that γ is the weak solution of the instationary Stokes system; in particular,

$$\gamma \in L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega)\big).$$

The major part of the proof concerns the property

$$\tilde{u} \in L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega)\big) \tag{3.13}$$

so that $u = \gamma + \tilde{u} \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ is a weak solution in the sense of Leray and Hopf. Hence u satisfies the energy (in-)equality, and Serrin's Uniqueness Theorem 1.2 shows that u is the unique weak solution with these properties.

To prove (3.13) we recall from (2.24) that

$$\|A_{q_*/2}^{-1/2} P_{q_*} \mathrm{div}\,(uu)\|_{q_*/2} \le c\|uu\|_{q_*/2} \le c\|u\|_{q_*}^2 \quad \text{for a.a. } t \in (0,T). \tag{3.14}$$

Consequently, (3.12) implies the identity

$$A_{q_*/2}^{1/2} \tilde{u}(t) = -A_{q_*/2}\Big(\int_0^t e^{-\nu(t-\tau)A_{q_*/2}} A_{q_*/2}^{-1/2} P_{q_*/2}\, \mathrm{div}\,(uu)\, d\tau \Big). \tag{3.15}$$

Now the maximal regularity estimate (1.28), Lemma 1.11 (3) and (3.14) yield the estimate

$$\begin{aligned} \nu\|\nabla\tilde{u}\|_{L^{s_*/2}(L^{q_*/2})} &\le c\nu\|A_{q_*/2}^{1/2}\tilde{u}\|_{L^{s_*/2}(L^{q_*/2})} \\ &\le c\|uu\|_{L^{s_*/2}(L^{q_*/2})} \le c\|u\|_{L^{s_*}(L^{q_*})}^2 \end{aligned} \tag{3.16}$$

and particularly the result

$$\nabla\tilde{u} \in L^{s_*/2}\big(0,T; L^{q_*/2}(\Omega)\big). \tag{3.17}$$

We will consider four cases concerning the exponent s_*, starting with the case $2 < s_* < 4$ (and $q_* > 6$). Let $s_1 = s_*$, $q_1 = q_*$. Then (3.12) and (1.24) (with $\alpha = \frac{1}{2}$) imply that

$$\|\tilde{u}(t)\|_{q_1/2} \le \frac{c}{\sqrt{\nu}} \int_0^t \frac{1}{(t-\tau)^{1/2}}\, \|uu\|_{q_1/2}\, d\tau,$$

where $\|uu(\tau)\|_{q_1/2} \in L^{s_1/2}(0,T)$. Hence the Hardy-Littlewood inequality proves with

$$\frac{1}{s_2} = \frac{1}{s_1/2} - \frac{1}{2}, \quad q_2 = \frac{q_1}{2}$$

that

$$\tilde{u} \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big).$$

Here $\frac{2}{s_2} + \frac{3}{q_2} = 1$ since $\frac{2}{s_1} + \frac{3}{q_1} = 1$, and $s_2 > s_1$, $q_2 < q_1$. To get the same result for γ, note that

$$\gamma_1(t) := e^{-\nu t A_{q_*}} u_0 \in L^\infty\big(0,T; L^{q_*}(\Omega)\big) \subset L^{s_2}\big(0,T; L^{q_2}(\Omega)\big).$$

Concerning $\gamma_2(t) = \gamma(t) - \gamma_1(t)$, the second term on the right-hand side of (3.11), we use (1.23) with $\alpha = \frac{1}{s_1}$ and conclude, since $A_\rho^{-1/2} P_\rho\, \mathrm{div}\, F \in L^\rho(\Omega)$, see (2.24), that

$$v := A_\rho^{-1/s_1} A_\rho^{-1/2} P_\rho\, \mathrm{div}\, F \in L^{s_1}\big(0,T; L^{q_2}(\Omega)\big).$$

Hence $\gamma_2(t)$ satisfies the estimate

$$\|\gamma_2(t)\|_{q_2} \le c_\nu \int_0^t \frac{1}{(t-\tau)^{1/2+1/s_1}}\, \|v(\tau)\|_{q_2}\, d\tau,$$

from which we deduce by the Hardy-Littlewood inequality that $\gamma_2 \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big)$; here we used that $\frac{1}{2} + \frac{1}{s_1} = 1 - \big(\frac{1}{s_1} - \frac{1}{s_2}\big)$.

Summarizing the results for γ_1 and γ_2 we get that $\gamma \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big)$ so that also $u \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big)$ and

$$\nabla\tilde{u} \in L^{s_2/2}\big(0,T; L^{q_2/2}(\Omega)\big),$$

cf. (3.17). Repeating this step finitely many times, we finally arrive at exponents $s_k \in [4,\infty)$, $q_k \in (3,6]$. The problem of exponents $s \ge 4$, $q \le 6$ will be considered in the following three cases.

Now let $s_* = 4$, $q_* = 6$. In this special case (3.16) yields $\nabla\tilde{u} \in L^2\big(0,T;L^2(\Omega)\big)$. Since by (3.14)

$$A_{q_*/2}^{-1/2} P_{q_*/2} \operatorname{div}(uu) \in L^{s_*/2}\big(0,T;L^{q_*/2}(\Omega)\big) \subset L^2\big(0,T;L^2(\Omega)\big),$$

we may consider $A_{q_*/2}^{-1/2}\tilde{u}$ as the strong solution of the instationary Stokes system with an external force in $L^2\big(0,T;L^2(\Omega)\big)$ and vanishing initial value. Hence

$$\tilde{u} = A_{q_*/2}^{1/2} A_{q_*/2}^{-1/2}\tilde{u} \in L^\infty\big(0,T;L^2(\Omega)\big)$$

and $\nabla\tilde{u} \in L^2\big(0,T;L^2(\Omega)\big)$ so that $u = \gamma + \tilde{u}$ satisfies

$$u \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big).$$

Moreover, since $u \in L^{s_*}\big(0,T;L^{q_*}(\Omega)\big)$, we see that $uu \in L^2\big(0,T;L^2(\Omega)\big)$. An elementary calculation shows that u is not only a very weak solution, but also a weak one in the sense of Leray and Hopf. Hence u is even a regular solution by Theorem 1.5 and satisfies the energy (in-)equality. Furthermore, the uniqueness assertion follows from Theorem 1.2.

Next let $4 < s_* \le 8$ (and $4 \le q_* < 6$) so that (3.17) immediately yields $\nabla\tilde{u} \in L^2\big(0,T;L^2(\Omega)\big)$ and $\tilde{u} \in L^2\big(0,T;H_0^1(\Omega)\big)$. Applying (1.24) and (3.14) to (3.12), Hölder's inequality implies the estimate

$$\begin{aligned}
\|\tilde{u}(t)\|_2 &\le \frac{c}{\sqrt{\nu}} \int_0^t \frac{1}{(t-\tau)^{1/2}}\, e^{-\nu\delta(t-\tau)}\, \|uu\|_2\, d\tau \\
&\le \frac{c}{\sqrt{\nu}} \int_0^t \frac{1}{(t-\tau)^{1/2}} e^{-\nu\delta(t-\tau)}\, \|uu\|_{q_*/2}\, d\tau \\
&\le c\nu^{-1+2/s_*}\, \|uu\|_{L^{s_*/2}(0,T;L^{q_*/2}(\Omega))} \\
&\le c\nu^{-1+2/s_*}\, \|u\|^2_{L^{s_*}(0,T;L^{q_*}(\Omega))}.
\end{aligned}$$

Consequently, $\tilde{u}$ and even u belong to $L^\infty\big(0,T;L^2(\Omega)\big)$. Now we complete the proof as in the previous case.

Finally assume that $8 < s_* < \infty$ (and $3 < q_* < 4$). Now we need finitely many steps to reduce this case to the former one. Let $s_1 = s_*$ and $q_1 = q_*$. Then $\nabla\tilde{u} \in L^{s_1/2}\big(0,T;L^{q_1/2}(\Omega)\big)$ by (3.17). Defining $s_2 < s_1$, $q_2 > q_1$ by

$$s_2 = \frac{s_1}{2}, \quad \frac{1}{3} + \frac{1}{q_2} = \frac{2}{q_1}$$

we get by Sobolev's embedding theorem that $\tilde{u} \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big)$. By Lemma 1.11 we conclude that also $\gamma \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big)$ so that

$$u \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big)$$

where again $\frac{2}{s_2} + \frac{3}{q_2} = 1$. Repeating this step finitely many times, if necessary, we arrive at exponents $s_k \in (4,8]$, $q_k \in [4,6)$, i.e. in the previous case.

Now Lemma 3.2 is completely proved. $\square$

Corollary 3.3. *In the situation of Lemma 3.2 assume that* $T = \infty$. *Then there exists a constant* $\varepsilon_* = \varepsilon_*(q_*, \Omega) > 0$ *with the following property: If*

$$\int_0^\infty \|F\|_\rho^{s_*}\, d\tau \le \varepsilon_* \nu^{2s_*-1} \quad \text{and} \quad \|u_0\|_{q_*} \le \varepsilon_* \nu,$$

then the Navier-Stokes system (3.1) *has a unique weak solution* u *in* $\Omega \times (0,\infty)$ *satisfying* $u \in L^{s_*}\big(0,\infty; L^{q_*}(\Omega)\big)$ *and the energy inequality.*

Proof. From (1.24) with $\alpha = 0$ we obtain that

$$\int_0^\infty \|e^{-\nu t\, A_{q_*}} u_0\|_{q_*}^{s_*}\, dt \le c\|u_0\|_{q_*}^{s_*} \int_0^\infty e^{-\nu s_* \delta_0 t}\, dt \le \frac{c}{\nu}\, \|u_0\|_{q_*}^{s_*}.$$

Now the result follows from Lemma 3.2 when using a different constant $\varepsilon_* = \varepsilon_*(q_*, \Omega) > 0$. $\square$

The next lemma has a technical character, but will immediately imply the assertions of Theorem 3.1. We will use the notation

$$⨍_a^b h(\tau)\, d\tau = \frac{1}{b-a} \int_a^b h(\tau)\, d\tau$$

for the mean value of an integral.

Lemma 3.4. *Under the assumptions of Theorem 3.1 there exists a constant* $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$ *with the following property:*

If $0 < t_0 < t \le t_1 < T$, $0 \le \beta \le \frac{s}{s_*}$ *and if*

$$\int_{t_0}^{t_1} \|F\|_\rho^{s_*}\, d\tau \le \varepsilon_* \nu^{2s_*-1} \quad \text{and} \quad ⨍_{t_0}^{t} (t_1 - \tau)^\beta\, \|u\|_{q_*}^{s}\, d\tau \le \varepsilon_* \nu^{s-\beta}, \tag{3.18}$$

then u *is regular in the interval* $(t-\delta, t_1)$ *for some* $\delta > 0$ *in the sense that* $u \in L^{s_*}\big(t-\delta, t_1; L^{q_*}(\Omega)\big)$. *In particular, if* $t_1 > t$, *then* t *is a regular point of* u. *If* $\beta = 0$, *then* $t_1 = T \le \infty$ *is allowed.*

Proof. From the second condition in (3.18) and the fact that u satisfies the strong energy inequality we find a null set $N \subset (t_0, t)$ such that for $\tau_0 \in (t_0, t)\backslash N$

$$\frac{1}{2}\|u(\tau_1)\|_2^2 + \nu \int_{\tau_0}^{\tau_1} \|\nabla u\|_2^2\, d\tau \le \frac{1}{2}\|u(\tau_0)\|_2^2 + \int_{\tau_0}^{\tau_1} \langle f, u\rangle d\tau, \quad \tau_0 < \tau_1 < T, \tag{3.19}$$

and $u(\tau_0) \in L^{q_*}_\sigma(\Omega)$. Now, if we find $\tau_0 \in (t_0, t)\backslash N$ such that

$$\int_0^{t_1-\tau_0} \|e^{-\nu\tau A_{q_*}} u(\tau_0)\|_{q_*}^{s_*}\, d\tau \le \varepsilon_* \nu^{s_*-1}, \tag{3.20}$$

Lemma 3.2 will yield a unique weak solution $v \in L^{s_*}\big([\tau_0, t_1); L^{q_*}_\sigma(\Omega)\big)$ to the Navier-Stokes system (3.1) with initial value $v(\tau_0) = u(\tau_0)$ at τ_0. Then (3.19) and Serrin's Uniqueness Theorem 1.2 show that

$$u = v \in L^{s_*}\big(\tau_0, t_1; L^{q_*}_\sigma(\Omega)\big)$$

and complete the proof.

To prove (3.20) note that the second condition in (3.18) yields the existence of $\tau_0 \in (t_0, t)\backslash N$ such that

$$(t_1 - \tau_0)^\beta\, \|u(\tau_0)\|_{q_*}^{s} \le ⨍_{t_0}^{t} (t_1 - \tau)^\beta\, \|u(\tau)\|_{q_*}^{s}\, d\tau \le \varepsilon_* \nu^{s-\beta}; \tag{3.21}$$

otherwise $(t_1-\tau)^\beta \|u(\tau)\|_{q_*}^s$ is strictly larger than $\fint_{t_0}^t (t_1-\tau)^\beta \|u\|_{q_*}^s \, d\tau$ for every $\tau \in (t_0,T)\backslash N$, and we are led to a contradiction. Now, by Lemma 1.11, Hölder's inequality and (3.21),

$$\begin{aligned}\int_0^{t_1-\tau_0} \|e^{-\nu\tau A_{q_*}} u(\tau_0)\|_{q_*}^{s_*} \, d\tau &\leq \int_0^{t_1-\tau_0} e^{-\delta_0 \nu s_* \tau} \, d\tau \, \|u(\tau_0)\|_{q_*}^{s_*} \\ &\leq c(t_1-\tau_0)^{\beta s_*/s} \, \nu^{-1+\beta s_*/s} \|u(\tau_0)\|_{q_*}^{s_*} \\ &\leq c\, \varepsilon_*^{s_*/s} \, \nu^{s_*-1}.\end{aligned}$$

Hence, with a new constant $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$, (3.20) is proved. If $\beta = 0$, then $t_1 = T \leq \infty$ is admitted. $\square$

Proof of Theorem 3.1. (1) Assuming (3.6) we choose $s = s_*$, $\beta = \frac{s}{s_*} = 1$. Furthermore, let $t_0 = t - \delta$, $t_1 = t + \delta$ where $\delta > 0$ is chosen so small that

$$\fint_{t-\delta}^t (t_1-\tau) \|u\|_{q_*}^s \, d\tau \leq 2 \int_{t-\delta}^t \|u\|_{q_*}^s \, d\tau \leq \varepsilon_* \nu^{s-\beta}$$

and

$$\int_{t-\delta}^t \|F\|_\rho^{s_*} \, d\tau \leq \varepsilon_* \nu^{2s_*-1}.$$

Then Lemma 3.4 implies that u is regular at t.

(2) Given (3.8) let $t_0 = t-\delta$, $t_1 = t+\delta$ such that with $\beta = \frac{s}{s_*}$

$$\fint_{t-\delta}^t (t_1-\tau)^\beta \|u\|_{q_*}^s \, d\tau \leq 2^\beta \frac{1}{\delta^{1-\beta}} \int_{t-\delta}^t \|u\|_{q_*}^s \, d\tau.$$

By (3.8) we find $\delta > 0$ such that the second condition of (3.18) is satisfied. Obviously, the condition on F in (3.18) can be fulfilled as well. Then Lemma 3.4 proves the sufficiency of (3.8) to imply regularity of u at t. The necessity of (3.8) is a simple consequence of Hölder's inequality.

(3) Given the initial value $u_0 \in L^{q_*}_\sigma(\Omega)$, Lemma 3.2 yields a unique weak solution $v \in L^{s_*}\big(0,\delta_1; L^{q_*}_\sigma(\Omega)\big)$ for some $\delta_1 > 0$ which coincides with u on $[0,\delta_1)$ by Theorem 1.2. Moreover, the elementary estimate

$$\int_0^{\delta_1} \|e^{-\nu\tau A_{q_*}} u_0\|_{q_*}^{s_*} \, d\tau \leq c\, \delta_1 \|u_0\|_{q_*}^{s_*}$$

and (3.10) imply that we may choose

$$\delta_1 = \frac{\varepsilon_* \nu^{s_*-1}}{c\|u_0\|_{q_*}^{s_*}}.$$

In Lemma 3.4 let $\beta = \frac{s}{s_*}$, $t_0 = t - \frac{\delta_1}{2}$ and $t_1 = t + \frac{\delta_1}{2}$ where $t \geq \delta_1$ is arbitrary. Then

$$\fint_{t_0}^t (t_1-\tau)^\beta \|u\|_{q_*}^s \, d\tau \leq \frac{2}{\delta_1^{1-\beta}} \int_0^T \|u\|_{q_*}^s \, d\tau$$

which by (3.9) is smaller than

$$2 \left(\frac{\varepsilon_* \nu^{s_*-1}}{c\|u_0\|_{q_*}^{s_*}} \right)^{\frac{s}{s_*}-1} \cdot \varepsilon_* \frac{\nu^{s_*-1}}{\|u_0\|_{q_*}^{s_*-s}} \;=\; c\, \varepsilon_*^{s/s_*} \nu^{s-s/s_*}.$$

Redefining ε_*, we see that (3.18) is fulfilled. Hence u is regular at every $t \in [\delta_1, T)$ by Lemma 3.4; more precisely, u is regular in $\big(t - \delta(t), t + \frac{\delta_1}{2}\big)$. This argument completes the proof when $T < \infty$.

If $T = \infty$, applying the previous result for each finite interval we obtain that $u \in L^{s_*}_{\mathrm{loc}}\big([0,\infty); L^{q_*}_\sigma(\Omega)\big)$. Due to (3.9) we find a sufficiently large τ_0 satisfying $\|u(\tau_0)\|_{q*} \le \varepsilon_* \nu$ and the energy inequality (3.19). Then Corollary 3.3 yields the existence of a unique weak solution $v \in L^{s_*}\big(\tau_0, \infty; L^{q_*}_\sigma(\Omega)\big)$ with $v(\tau_0) = u(\tau_0)$ which must coincide with u on $[\tau_0, \infty)$. This argument proves (3). □

Corollary 3.5. *Under the assumptions of Theorem 3.1 we have the following results:*

(1) *There exists $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$ such that u is regular for all $t \ge T_1$ where*

$$T_1 > \frac{1}{\varepsilon_* \nu^s} \|u\|^s_{L^s(0,\infty;L^{q_*}(\Omega))} \tag{3.22}$$

provided that $u \in L^s\big(0,\infty; L^{q_*}_\sigma(\Omega)\big)$ and $\int_0^\infty \|F\|^{s_*}_\rho \, d\tau \le \varepsilon_* \nu^{2s_*-1}$.

(2) *Assume that $t \in (0,T)$ is a singular point of the weak solution u in the sense that $u \notin L^{s_*}\big(t-\delta, t+\delta; L^{q_*}(\Omega)\big)$ for any $\delta > 0$. Then*

$$\liminf_{\delta\to 0+} \frac{1}{\delta^{1-\beta}} \int_{t-\delta}^t \|u\|^s_{q_*} d\tau > 0 \quad \textit{for all } \beta \in \Big[0, \frac{s}{s_*}\Big] \tag{3.23}$$

and even

$$\lim_{\delta\to 0+} \fint_{t-\delta}^t \|u\|^s_{q_*} \, d\tau = \infty. \tag{3.24}$$

The set of singular points of u is either empty or at least a set of Lebesgue measure zero, if $u \in L^s\big(0,T; L^{q^*}(\Omega)\big)$.

Proof. (1) Let $\beta = 0$ in Lemma 3.4. Then by assumption

$$\lim_{t_0\to 0+} \fint_{t_0}^{T_1} \|u\|^s_{q_*} d\tau < \varepsilon_* \nu^s,$$

and Lemma 3.4 yields the regularity of u for $t \ge T_1$.

(2) Let $t \in (0,T)$ be a singular point of u and assume that the left hand side of (3.23) is zero. Then, setting $t_0 = t - \delta$, $t_1 = t + \delta$ we conclude that there exists some sufficiently small $\delta > 0$ such that (3.18) is satisfied. Hence we get the contradiction that u is regular at t. If (3.24) does not hold, then $\liminf_{\delta\to 0+} \fint_{t-\delta}^t \|u\|^s_{q_*} d\tau < \infty$ and consequently $\liminf_{\delta\to 0+} \frac{1}{\delta^{1-\beta}} \int_{t-\delta}^t \|u\|^s_{q_*} d\tau = 0$ for $\beta \in \big(0, \frac{s}{s_*}\big]$ which is a contradiction to (3.23).

It is a simple consequence of Leray's Structure Theorem, see [**24**], that the Lebesgue measure of the set of singular points in time vanishes. Here we may also argue as follows if $u \in L^s\big(0,T; L^{q_*}_\sigma(\Omega)\big)$. By Lebesgue's Differentiation Theorem

$$\lim_{\delta\to 0+} \fint_{t-\delta}^t \|u\|^s_{q_*} \, d\tau = \|u(t)\|^s_{q_*} \quad \text{for almost all} \quad t \in (0,T).$$

Hence (3.24) can hold only on a Lebesgue null set. □

3.2. Energy-Based Criteria for Regularity. Let u be a weak solution in the sense of Leray and Hopf satisfying the energy inequality. Assume that $f = 0$ and $0 \neq u_0 \in H_0^1(\Omega) \cap L_\sigma^2(\Omega)$ so that there exists an interval $[0, T)$ on which u is a strong solution and satisfies even the energy identity (1.3). Then the *kinetic energy*

$$E(t) = \frac{1}{2} \|u(t)\|_2^2$$

is a strictly decreasing continuous function of $t \in [0, T)$. However, at $t = T$ the energy identity could loose its validity; either the kinetic energy has a jump discontinuity downward at $t = T$ or $E(t)$ will be strictly less than the continuously decreasing function

$$-\nu \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau + E(0)$$

for certain $t > T$ close to T. In the first case the jump must be downward since $\|u(t)\|_2$ is lower semicontinuous by (1.7). Assuming that $\|u(t)\|_2$ is continuous and decreasing in an open interval to the right of T, there are three possibilities: $E(T+) := \lim_{t \to T+} E(t)$ equals either $E(T)$ or

$$E(T) < E(T+) < E(T-),$$

where $E(T-) := \lim_{t \to T-} E(t)$, or $E(T+) = E(T-)$. The fourth possibility $E(T+) > E(T-)$ is excluded since u satisfies the energy inequality for $t \geq T$ as well; if we want to exclude this possibility at a further jump discontinuity $\tilde{T} > T$, we have to use the strong energy inequality. If u satisfies the strong energy inequality and T is an initial point in time where the energy inequality holds ($T = s$ in (1.9)), then necessarily $E(T+) = E(T)$; otherwise the other two possibilities cannot be ruled out.

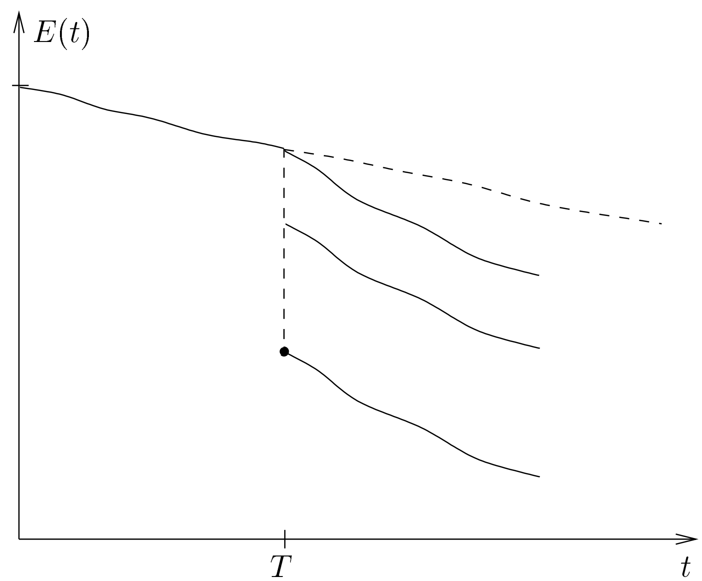

Fig 3.1 The kinetic energy $E(t)$ in the neighborhood of a jump discontinuity T

In the following assume that $E(\cdot)$ is continuous in time, so that (1.7) implies $u \in C^0\big([0, T); L_\sigma^2(\Omega)\big)$ rather than only $u \in L^\infty\big(0, T; L_\sigma^2(\Omega)\big)$. Nevertheless we are not allowed to conclude that u is a regular solution. Actually, this conclusion is related to the modulus of continuity of the function $E(t)$ (or to that of the function $t \mapsto \|u(t)\|_2$ since $u \in L^\infty\big(0, T; L_\sigma^2(\Omega)\big)\big)$.

THEOREM 3.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$ and let u be a weak solution of the instationary Navier-Stokes system (3.1) satisfying the strong energy inequality on $(0,T)$. The data u_0, f satisfy $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^{s_*/s}(0,T; L^2(\Omega))$, $f = \operatorname{div} F$, $F \in L^2(0,T; L^2(\Omega)) \cap L^{s_*}((0,T; L^\rho(\Omega))$ where ρ, s, s_* will be given in (3.29) below.*

(1) *Let $\alpha \in (\frac{1}{2}, 1)$ and let u satisfy at $t \in (0,T)$ the condition*

$$\sup_{t' \neq t} \frac{|E(t) - E(t')|}{|t - t'|^\alpha} < \infty$$

or only

$$\liminf_{\delta \to 0+} \frac{1}{\delta^\alpha} |E(t) - E(t-\delta)| < \infty, \tag{3.25}$$

where $E(\cdot)$ denotes the kinetic energy. Then u is regular at t.

(2) *(The case $\alpha = \frac{1}{2}$) There exists a constant $\varepsilon_* = \varepsilon_*(\Omega) > 0$ such that if*

$$\sup_{t' \neq t} \frac{|E(t) - E(t')|}{|t - t'|^{1/2}} \leq \varepsilon_* \, \nu^{5/2}$$

or only

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{1/2}} |E(t) - E(t-\delta)| \leq \varepsilon_* \, \nu^{5/2}, \tag{3.26}$$

then u is regular at $t \in (0,T)$.

REMARK 3.7. (1) By Theorem 3.6 (1), Hölder continuity of the kinetic energy $E(\tau)$ from the left at t implies regularity at t if the Hölder exponent α is larger than $\frac{1}{2}$. In the case $\alpha = \frac{1}{2}$ the corresponding Hölder seminorm (from the left) is assumed to be sufficiently small. In both cases the function $E(\tau)$ may be replaced by the function $\|u(\tau)\|_2$.

(2) The proof of Theorem 3.6, see (3.30), (3.31) below, will yield the following regularity criterion using $\|\nabla u\|_2$ instead of $\|u\|_2$. If

$$\alpha \in (\frac{1}{2}, 1) \quad \text{and} \quad \liminf_{\delta \to 0+} \frac{1}{\delta^\alpha} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 \, d\tau < \infty \tag{3.27}$$

or

$$\alpha = \frac{1}{2} \quad \text{and} \quad \liminf_{\delta \to 0+} \frac{1}{\delta^{1/2}} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 \, d\tau \leq \varepsilon_* \, \nu^{5/2}, \tag{3.28}$$

then u is regular at t.

(3) In the case $\alpha = \frac{1}{2}$ a smallness condition as in (3.26) or (3.28) is necessary. Indeed, if $f = 0$ and $(0,t)$, $0 < t < \infty$, is a *maximal* regularity interval of u, then

$$\|\nabla u(\tau)\|_2 \geq \frac{c_0}{(t-\tau)^{1/4}} \quad , 0 < \tau < t,$$

where $c_0 = c_0(\Omega) > 0$, see [**24**]. Hence

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{1/2}} \int_{t-\delta}^t \|\nabla u\|_2^2 \, d\tau \geq 2\, c_0^2 > 0,$$

and due to the strong energy inequality,

$$\liminf_{\delta\to 0+} \frac{E(t-\delta)-E(t)}{\delta^{1/2}} \geq 2\,\nu\, c_0^2 > 0.$$

PROOF OF THEOREM 3.6. (see also [**15**] for the proof of (1)). The proof is based on Lemma 3.4 with $t_0 = t-\delta$, $t_1 = t+\delta$ and the exponents

$$\begin{cases} \text{if } \alpha > \frac{1}{2}: \; s = 4\alpha - \varepsilon > 2, \; \frac{2}{s} + \frac{3}{q_*} = \frac{3}{2}, \; \frac{2}{s_*} + \frac{3}{q_*} = 1, \; \beta = \frac{s}{s_*}, \\ \text{if } \alpha = \frac{1}{2}: \; s = 2, \; \varepsilon = 0, \; q_* = 6, \; s_* = 4, \; \beta = \frac{1}{2}. \end{cases} \tag{3.29}$$

In both cases the weak solution u satisfies $u \in L^s\big(0,T; L^{q_*}(\Omega)\big)$, cf. (1.11), and $1 - \frac{s}{s_*} = \frac{s}{4}$. To control the second term in (3.18) we will use the interpolation inequality

$$\|u\|_{q_*} \leq c\,\|u\|_2^{1-2/s}\, \|\nabla u\|_2^{2/s}, \;\; c = c\,(q_*, \Omega) > 0,$$

and get that

$$\begin{aligned} I(\delta) := \int_{t_0}^{t} (t_1-\tau)^\beta \|u\|_{q_*}^s \, d\tau \;&\leq\; 2^\beta\, \delta^{\beta-1} \int_{t-\delta}^{t} \|u\|_{q_*}^s \, d\tau \\ &\leq\; c\,\delta^{-s/4} \int_{t-\delta}^{t} \|\nabla u\|_2^2\, \|u\|_2^{s-2}\, d\tau \\ &\leq\; c\,\|u\|_{L^\infty(L^2)}^{s-2}\; \delta^{-s/4} \int_{t-\delta}^{t} \|\nabla u\|_2^2\, d\tau\,. \end{aligned} \tag{3.30}$$

Since u is supposed to satisfy the strong energy inequality, we may proceed for almost all $\delta > 0$ as follows:

$$\begin{aligned} I(\delta) \;&\leq\; \frac{c}{\nu}\, \delta^{-s/4} \Big(|E(t-\delta) - E(t)| + \Big| \int_{t-\delta}^{\delta} (f,u)\, d\tau \Big| \Big) \\ &=\; \frac{c}{\nu}\, \delta^{\varepsilon/4} \left(\frac{|E(t-\delta)-E(t)|}{\delta^\alpha} + \Big| \frac{1}{\delta^\alpha} \int_{t-\delta}^{t} (f,u)\, d\tau \Big| \right), \end{aligned} \tag{3.31}$$

where the constant c depends on $\|u_0\|_2$ when $\alpha > \frac{1}{2}$.

First consider the case $\alpha > \frac{1}{2}$ in which $\varepsilon > 0$. Then

$$\Big| \frac{1}{\delta^{s/4}} \int_{t-\delta}^{t} (f,u)\, d\tau \Big| \leq \frac{c}{\delta^{s/4}} \int_{t-\delta}^{t} \|f\|_2\, d\tau \leq c \left(\int_{t-\delta}^{t} \|f\|_2^{4/(4-s)}\, d\tau \right)^{(4-s)/4}.$$

Hence, if $f \in L^{4/(4-s)}\big(0,T; L^2(\Omega)\big)$, the left-hand term in the previous inequality converges to 0 as $\delta \to 0+$. Moreover, due to the assumption (3.25), the term

$$\frac{c}{\nu}\, \delta^{\varepsilon/4} \cdot \frac{|E(t-\delta)-E(t)|}{\delta^\alpha} \tag{3.32}$$

in (3.31) converges to 0 as $\delta \to 0+$. Hence the right-hand side in (3.31) converges to 0 as $\delta \to 0+$, and the continuity of $I(\delta)$ for $\delta > 0$ implies that the condition $(3.18)_2$ can be fulfilled for some $\delta' > 0$. Finally, the assumption $F \in L^{s_*}\big(0,T; L^\rho(\Omega)\big)$ shows that also $(3.18)_1$ can be satisfied.

Secondly, in the case $\alpha = \frac{1}{2}$ (and $\varepsilon = 0$), the assumption $f \in L^2\big(0,T; L^2(\Omega)\big)$ implies as above that

$$\frac{1}{\delta^\alpha} \int_{t-\delta}^{t} (f,u)\, d\tau \to 0 \text{ as } \delta \to 0+\,.$$

Moreover, the term (3.32) is bounded by $2\,c\varepsilon_*\nu^{3/2}$ for a sequence (δ_j), $0<\delta_j\to 0$ as $j\to\infty$, due to the assumption (3.26). Hence the continuity of $I(\delta)$, $\delta>0$, proves that $(3.18)_2$ can be satisfied. Concerning $(3.18)_1$ we proceed as before.

Now Theorem 3.6 is completely proved. □

3.3. Local in Space-Time Regularity. Consider a weak solution u of the Navier-Stokes system (3.1) in a general domain $\Omega\subset\mathbb{R}^3$. In this subsection we are looking for conditions on u locally in space and time to guarantee that u is regular locally in space and time. The fundamental result in this direction is due to L. Caffarelli, R. Kohn and L. Nirenberg [**7**] and requires the definition of a suitable weak solution.

Definition 3.8. A weak solution u to (3.1) is called a *suitable weak solution* if the associated pressure term satisfies

$$\nabla p\in L^q_{\rm loc}\big(0,\infty;L^q_{\rm loc}(\overline{\Omega})\big)\quad\text{with } q=\frac{5}{4} \tag{3.33}$$

and the *localized energy inequality*

$$\begin{aligned}\frac12\|\varphi u(t)\|_2^2+\nu\int_{t_0}^t\|\varphi\nabla u\|_2^2\,d\tau \;&\le\; \frac12\|\varphi u(t_0)\|_2^2+\int_{t_0}^t(\varphi f,\varphi u)\,d\tau\\ &-\frac12\int_{t_0}^t(\nabla|u|^2,\nabla\varphi^2)\,d\tau+\int_{t_0}^t\Big(\frac12|u|^2+p,\;u\cdot\nabla\varphi^2\Big)\,d\tau\end{aligned}\tag{3.34}$$

holds for almost all $t_0\ge 0$, all $t\ge t_0$ and all $\varphi\in C_0^\infty(\mathbb{R}^3)$.

Using a standard mollification procedure we obtain from (3.34) the inequality

$$\begin{aligned}\int_{\Omega\times(0,T)}|\nabla u|^2\,\phi\,dx\,dt\;&\le\;\int_{\Omega\times(0,T)}u\cdot f\phi\,dx\,d\tau\\ &+\frac12\int_{\Omega\times(0,T)}|u|^2(\phi_t+\Delta\phi)\,dx\,dt+\int_{\Omega\times(0,T)}\Big(\frac12|u|^2+p,\;u\cdot\nabla\phi\Big)\,dx\,dt\end{aligned}\tag{3.35}$$

for all non-negative test functions $\phi\in C_0^\infty\big(\Omega\times(0,T)\big)$. This version of the localized energy inequality was used in [**7**]. However, note that (3.34) is a stronger condition than (3.35) in the sense that the test functions in (3.34) are not assumed to vanish in a neighborhood of $\partial\Omega$. The existence of a suitable weak solution satisfying (3.35) has been proved, under certain smoothness assumptions on the boundary $\partial\Omega$, for a bounded domain in [**7**], for an exterior domain in [**25**], and for a general uniform C^2-domain in [**16**], with (3.34) instead of (3.35).

To describe the local regularity result from [**7**] we introduce the space-time cylinder

$$Q_r=Q_r(x_0,t_0)=B_r(x_0)\times(t_0-r^2,t_0),\quad B_r(x_0)=\{x\in\mathbb{R}^3:\,|x-x_0|<r\},$$

for $(x_0,t_0)\in\Omega\times(0,T)$ such that $Q_r\subset\Omega\times(0,T)$. The following result is a simplified version of the local results in [**7**], [**36**], [**37**].

Theorem 3.9. *Let u be a suitable weak solution of* (3.1) *and let $Q_r=Q_r(x_0,t_0)\subset\Omega\times(0,T)$, $r>0$. There exists an absolute constant $\varepsilon_*>0$ with the following property:*

(1) *If*

$$\|u\|^3_{L^3(Q_r)}+\|p\|^{3/2}_{L^{3/2}(Q_r)}\le\varepsilon_* r^2, \tag{3.36}$$

then $u\in L^\infty(Q_{r/2})$.

(2) *If*

$$\limsup_{\rho\to 0} \frac{1}{\rho} \|\nabla u\|^2_{L^2(Q_\rho)} \le \varepsilon_* \tag{3.37}$$

then there exists $r_0 > 0$ with $Q_{r_0} \subset \Omega \times (0,T)$ such that $u \in L^\infty(Q_{r_0})$.

REMARK 3.10. (1) The condition (3.36) requires the existence of a suitable radius $r > 0$ and information on u as well as on the pressure p. However, (3.37) needs information for ∇u only, but on all parabolic cylinders Q_r, $r > 0$ sufficiently small.

(2) The main condition on u in (3.36), i.e. $\|u\|^3_{L^3(Q_r)} \le \varepsilon_* r^2$, may be rewritten in the integral mean form

$$\fint_{t_0-r^2}^{t_0} \fint_{B_r(x_0)} |ru|^3 \, dx \, d\tau \le \varepsilon_* \, .$$

Obviously this condition is satisfied when $|u(x,t)| \le \frac{\varepsilon_*}{r}$ in Q_r. By analogy, the other terms in (3.36) and (3.37) may be treated. Conversely, if u is not regular at (x_0, t_0), then we are heuristically led to the blow-up rate

$$|u(x,t)| \ge \frac{c_0}{\big(|x-x_0|^2 + |t-t_0|\big)^{1/2}},$$

$c_0 > 0$, in a neighborhood of (x_0, t_0), see [**7**].

(3) The conclusion $u \in L^\infty(Q_{r/2})$ in Theorem 3.9 does not imply that $u \in C^\infty(Q_{r/2})$ even if $f \in C^\infty$ or $f = 0$. However, u is of class C^∞ in x, but not necessarily in t, see [**57**], [**64**]. In [**36**] it is proved that a suitable weak solution satisfying (3.37) is Hölder continuous in space and time locally.

(4) In (3.37) the term ∇u may be replaced by its symmetric part $\frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$ or its skew-symmetric part $\frac{1}{2}\big(\nabla u - (\nabla u)^T\big)$, i.e. by the vorticity $\omega = \operatorname{curl} u$, see [**38**], [**67**].

(5) More general results concerning regularity criteria for suitable weak solutions using local smallness conditions on $u, \nabla u, \operatorname{curl} u$ or $\nabla^2 u$ without any condition on the pressure can be found in [**33**]. If e.g. $1 \le \frac{2}{s} + \frac{3}{q} \le 2$ and

$$\limsup_{r\to 0} r^{-(\frac{2}{s}+\frac{3}{q}-1)} \|u\|_{L^s(t_0-r^2,t_0;\, L^q(B_r(x_0)))} \le \varepsilon_* \tag{3.38}$$

for some smallness constant $\varepsilon_* > 0$, then u is regular at (x_0, t_0) in the sense that u is essentially bounded in a space time cylinder $Q_{r'}(x_0, t_0) \subset \Omega \times (0,T)$, $0 < r' < r$. For similar results near the boundary of Ω see [**32**].

To describe our main result on local space-time regularity of suitable weak (or only weak) solutions we use the short notation

$$\|u\|_{L^sL^q(Q_r)} = \|u\|_{L^s(t_0-r^2,t_0;L^q(B_r(x_0))}$$

when $Q_r = Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$. Note that the condition (3.39) in Theorem 3.11 below does not use the $\limsup_{r\to 0}$, but requires the existence of a single sufficiently small radius $r > 0$, and only norms of u, but not of ∇u or the pressure.

THEOREM 3.11. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary domain and let u be a suitable weak solution of the Navier-Stokes system in $\Omega \times (0,T)$ where for simplicity $f = 0$. Let $2 < s < \infty$, $3 < q < \infty$ satisfy the conditions*

$$\frac{2}{s} + \frac{3}{q} \le 1 + \frac{1}{q} \quad \textit{and} \quad q \ge 4.$$

Then there exists an absolute constant $\varepsilon_* = \varepsilon_*(s,q) > 0$ *independent of* $\nu > 0$, $x_0 \in \Omega$, $t_0 \in (0,T)$ *and* $r > 0$ *with* $Q_r(x_0,t_0) \subset \Omega \times (0,T)$ *and of* u *with the following property: If*

$$\|u\|_{L^sL^q(Q_r)} \le \varepsilon_* \min\left(\nu, \nu^{1-\frac{1}{s}}\right) r^{\frac{2}{s}+\frac{3}{q}-1}, \tag{3.39}$$

then u *is regular in* $Q_{r/2}$ *in the sense*

$$u \in L^{s_*}\left(t_0 - (r/2)^2, t_0; L^{q_*}\left(B_{r/2}(x_0)\right)\right), \quad \frac{2}{s_*} + \frac{3}{q_*} = 1.$$

Here, $s_* = 4$, $q_* = 6$ *if* $s \ge 4$*; in this case, it suffices to assume that* u *is a weak solution only. If* $2 < s < 4$*, then* s_*, q_* *are defined by* $\frac{2}{s_*} + \frac{3}{q} = 1 + \frac{1}{q}$ *and* $\frac{2}{s_*} + \frac{3}{q_*} = 1$.

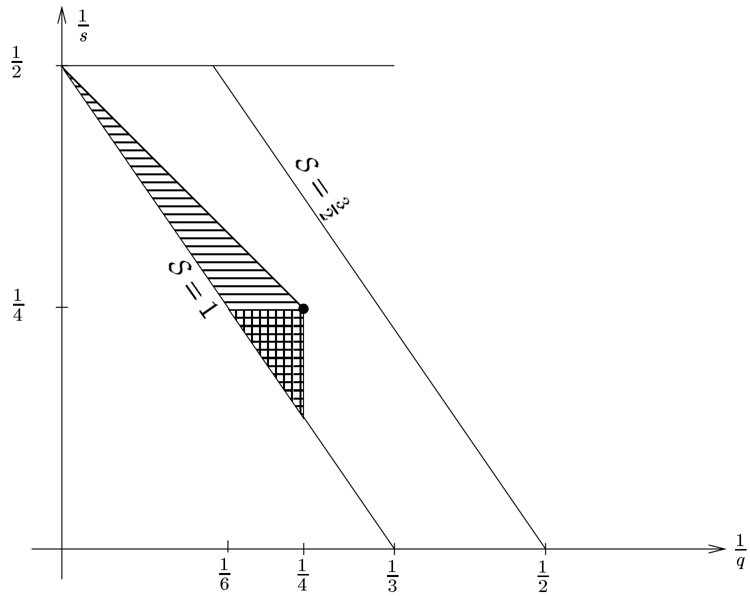

Fig 3.2 In the hatched region ($s < 4$) the localized energy inequality is needed to prove local regularity, in the doubly hatched region ($s \ge 4, q \ge 4$) no local version of an energy inequality is needed.

Proof. Rewriting (3.39) in the integral mean form

$$\left(\fint_{t_0-r^2}^{t_0} \left(\fint_{B_r(x_0)} |ru|^q \, dx\right)^{s/q} ds\right)^{1/s} \le \varepsilon_* \min\left(\nu, \nu^{1-\frac{1}{s}}\right)$$

where ε_* from (3.39) must be replaced by $\frac{\varepsilon_*}{|B_1(0)|^{1/q}}$, Hölder's inequality shows that we may replace s,q in (3.39) by any smaller s and smaller q, respectively. In particular, when $s \ge 4$ and $q \ge 4$, we may assume that $s = s_* = 4, q = 4$. When $2 < s < 4$, then let $s = s_*$ satisfy $\frac{2}{s_*} + \frac{3}{q} = 1 + \frac{1}{q}$. In both cases we get

$$s = s_* \le q, \quad \frac{2}{s_*} + \frac{3}{q} = 1 + \frac{1}{q}, \quad \frac{2}{s_*} + \frac{3}{q_*} = 1, \tag{3.40}$$

since $q \ge 4$. As a second step we may assume after a shift of coordinates in space and time that $x_0 = 0$ and $t_0 = 0$. Next we use a scaling argument and consider

$$u_r(y,\tau) = ru(ry, r^2\tau), \quad p_r(y,\tau) = r^2 p(ry, r^2\tau) \tag{3.41}$$

on $Q_1 = B_1(0) \times (-1,0)$ instead of (u,p) on Q_r. Note that u_r, p_r solve the Navier-Stokes system with the same viscosity ν and that u_r satisfies (3.39) in the form

$$\|u_r\|_{L^sL^q(Q_1)} \le \varepsilon_* \min\big(\nu, \nu^{1-1/s}\big). \tag{3.42}$$

Hence, without loss of generality, we assume that u satisfies (3.39) on Q_1 with $r = 1$ and $s = s_*$.

The idea of the proof is to construct with the help of Theorem 2.18 a very weak solution v in $Q' = B_{r'} \times (t', 0)$ for suitable $r' \in (\frac{1}{2}, 1)$ and $t' \in (-1, -\frac{1}{2})$ with data

$$v(t') = u(t'), \quad v|_{\partial B_{r'}} = u|_{\partial B_{r'}}$$

and to identify v with u on Q'; hence

$$v = u \in L^{s_*}L^{q_*}(Q') \quad \text{and} \quad v = u \text{ in } L^{s_*}L^{q_*}\big(B_{1/2} \times (-\frac{1}{2}, 0)\big).$$

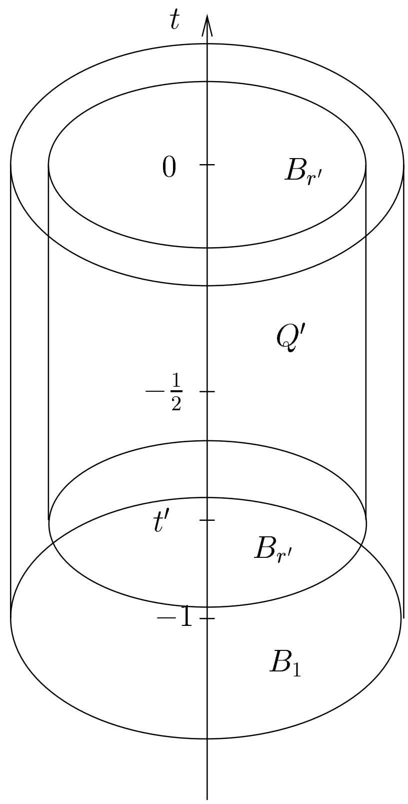

Fig 3.3 The space-time cylinders Q_1 and Q'.

For this purpose we have to find $r' \in (\frac{1}{2}, 1)$ and $t' \in (-1, -\frac{1}{2})$ such that the smallness conditions

$$\int_0^{-t'} \|A_{q_*} e^{-\nu\tau A_{q_*}} A_{q_*}^{-1} P_{q_*} u(t')\|_{q_*}^{s_*} \, d\tau \quad \le \quad \varepsilon_*^{s_*} \nu^{s_*-1} \tag{3.43}$$

$$\int_{t'}^0 \|u|_{\partial B_{r'}}\|_{W^{-1/q_*, q_*}(\partial B_{r'})}^{s_*} \, d\tau \quad \le \quad \varepsilon_*^{s_*} \nu^{s_*-1}, \tag{3.44}$$

cf. (2.50), are fulfilled; here A_{q_*} and P_{q_*} denote the Stokes operator and the Helmholtz projection, respectively, on $B_{r'}$.

Concerning (3.43) we find $t' \in (-1, -\frac{1}{2})$ satisfying

$$\|u(t')\|^s_{L^q(B_1)} \le \fint_{-1}^{-1/2} \|u\|^s_{L^q(B_1)}\,d\tau \le 2\,\|u\|^s_{L^sL^q(Q_1)} \le 2\varepsilon_*^s \nu^s.$$

Then Lemma 1.11 (3), (4) with $\alpha = \frac{1}{2q}$, $\frac{1}{q} + \frac{3}{q_*} = \frac{3}{q}$, and the property $\frac{s_*}{2q} = \frac{1}{q-2} < 1$ imply that

$$\begin{aligned}
&\int_0^{-t'} \|A_{q_*} e^{-\nu\tau A_{q_*}} A_{q_*}^{-1} P_{q_*} u(t')\|^{s_*}_{q_*}\,d\tau \\
&\quad = \int_0^{-t'} \|A_q^{1/2q} e^{-\nu\tau A_q} A_q^{-1/2q} P_q u(t')\|^{s_*}_{q_*}\,d\tau \\
&\quad \le c \int_0^{-t'} \frac{e^{-\nu\delta_0 s_*\tau}}{(\nu\tau)^{s_*/2q}}\,\|u(t')\|^{s_*}_q\,d\tau \\
&\quad \le \frac{c}{\nu}\|u(t')\|^{s_*}_q \le c\,\varepsilon_*^{s_*}\nu^{s-1}.
\end{aligned}$$

Hence (3.43) is satisfied for a sufficiently small constant ε_* in (3.42).

Now consider the problem of finding $r' \in (\frac{1}{2}, 1)$ such that (3.44) is satisfied. By the mean value argument as before, there exists $r' \in (\frac{1}{2}, 1)$ such that

$$\begin{aligned}
\|u\|^{s_*}_{L^{s_*}(-1,0;L^q(\partial B_{r'}))} &= \int_{-1}^0 \|u\|^{s_*}_{L^q(\partial B_{r'})}\,d\tau \\
&\le \fint_{1/2}^1 \Big(\int_{-1}^0 \|u\|^{s_*}_{L^q(\partial B_r)}\Big)\,d\tau\,dr \\
&= 2\int_{-1}^0 \Big(\int_{1/2}^1 \|u\|^{s_*}_{L^q(\partial B_r)}\,dr\Big)\,d\tau.
\end{aligned}$$

Since $s_* \le q$, see (3.40), we apply Hölder's inequality to the inner integral and get from (3.42) that

$$\begin{aligned}
\|u\|^{s_*}_{L^{s_*}(-1,0;L^q(\partial B_{r'}))} &\le 2\int_{-1}^0 \Big(\int_{1/2}^1 \|u\|^q_{L^q(\partial B_r)}dr\Big)^{s_*/q}\,d\tau \\
&\le 2\int_{-1}^0 \|u\|^{s_*}_{L^q(B_1)}d\tau \\
&\le 2\varepsilon_*^{s_*}\,\nu^{s_*-1}.
\end{aligned}$$

Finally, using the embedding $L^q(\partial B_{r'}) \subset W^{-1/q_*,q_*}(\partial B_{r'})$ with an embedding constant uniformly bounded in $r' \in \left(\frac{1}{2}, 1\right)$, we get that (3.44) is satisfied for a slightly different constant $\varepsilon_* > 0$.

Now Theorem 2.18 yields a unique very weak solution v in $L^{s_*}L^{q_*}(Q')$ with data $v(t') = u(t')$ and $v = u$ on $\partial B_{r'} \times (t', 0)$. For this argument it is important to note that the smallnes constant ε_* in the application of Theorem 2.18 in the space-time domain Q' may be chosen independently of $r' \in (\frac{1}{2}, 1)$ and $t' \in (-1, -\frac{1}{2})$; for its proof we have to refer to the scaling argument (3.41).

As the final step of the proof it suffices to show that $v = u$ on Q'. First consider the case $s \ge 4$ in which $s_* = 4$, $q_* = 6$, $v \in L^4L^6(Q')$ and $u \in L^4L^4(Q')$. Let γ denote the very weak solution of the Stokes system

$$\gamma_t - \nu\Delta\gamma + \nabla p = 0, \qquad \operatorname{div}\gamma = 0 \text{ in } Q', \\ \gamma(t') = u(t'), \quad \gamma\big|_{\partial B_{r'}} = u\big|_{\partial B_{r'}}.$$

By Theorem 2.14 $\gamma \in L^4L^6(Q') \subset L^4L^4(Q')$ so that $v-\gamma$ and $u-\gamma$ solve the instationary Stokes system

$$U_t - \nu\Delta U + \nabla p = -\operatorname{div}(vv) \qquad \text{and} \qquad = -\operatorname{div}(uu) \text{ in } Q', \\ \operatorname{div} U = 0 \text{ in } Q', \quad U(t') = 0, \quad U\big|_{\partial B_{r'}} = 0,$$

respectively. Since $vv \in L^2L^2(Q')$ and $uu \in L^2L^2(Q')$, in both cases the very weak solution U is even a *weak* solution satisfying the energy identity. Hence $u - v = u - \gamma - (v-\gamma)$ is a weak solution of the Stokes system

$$\begin{aligned} U_t - \nu\Delta U + \nabla p &= -\operatorname{div}(Uu + vU), \quad \operatorname{div} U = 0 \text{ in } Q', \\ U(t') &= 0, \quad u\big|_{\partial B_{r'}} = 0, \end{aligned} \tag{3.45}$$

where $Uu,\ vU \in L^2L^2(Q')$. Let $\|\cdot\|_{[\tau,t]}$, $\tau < t$, denote the norm

$$\|w\|_{[\tau,t]} = \left(\|w\|^2_{L^\infty(\tau,t;L^2(B_{r'}))} + \nu\|\nabla w\|^2_{L^2(\tau,t;L^2(B_{r'}))}\right)^{1/2}.$$

Testing (3.45) in $B_{r'} \times [t', t'+\varepsilon]$, $\varepsilon > 0$, with U we get the estimate

$$\|U\|^2_{[t',t'+\varepsilon]} \le c\|U\|^2_{[t',t'+\varepsilon]}\,\|v\|_{L^4(t',t'+\varepsilon;L^6(B_{r'}))} \tag{3.46}$$

with a constant $c > 0$ independent of t' and $\varepsilon > 0$ as well as of U, u and v; here we used that $\int_{B_{r'}} Uu\cdot\nabla U\,dx = 0$ and that

$$\left|\int_{B_{r'}} vU\cdot\nabla U\,dx\right| \le c\|\nabla U\|_2\,\|U\|_3\,\|v\|_6 \le c\|\nabla U\|_2^{3/2}\,\|U\|_2^{1/2}\,\|v\|_6.$$

Since $v \in L^4L^6(Q')$, we may choose $\varepsilon > 0$ sufficiently small so that (3.46) yields $U \equiv 0$ on $[t', t'+\varepsilon]$. Repeating this argument a finite number of times with the same $\varepsilon > 0$ we conclude that $U \equiv 0$ on $[t', 0]$, i.e., $u = v \in L^4L^6(Q')$. This proves Theorem 3.11 in the case $s \ge 4$. Note that u was not assumed to be a suitable weak solution in this case.

Secondly, let $2 < s = s_* < 4$ and consequently $q > 4$. In this case an approximation procedure is used to apply the localized energy inequality in a similar way as in Serrin's uniqueness criterion concerning the usual energy inequality. Moreover, regularity results for v allow to conclude that $U = u - v$ satisfies the inequality

$$\frac{1}{2}\,\|U(t)\|_2^2 + \nu\int_{t'}^t \|\nabla U\|_2^2\,d\tau \le \int_{t'}^t (U\cdot\nabla U, v)\,d\tau\,;$$

we omit further details of these technical arguments. Since $v \in L^{s_*}L^{q_*}(Q_{r'})$, the absorption principle may be used to get in a finite number of steps on consecutive intervals $t' = t_1 < t_2 < \ldots < t_m = 0$ that $u = v$ in Q', cf. (3.46). □

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