The Instationary Navier-Stokes Equations in Weighted Bessel-Potential Spaces

Katrin Schumacher*

We investigate the solvability of the instationary Navier-Stokes equations with fully inhomogeneous data in a bounded domain $\Omega \subset \mathbb{R}^n$. The class of solutions is contained in $L^r(0,T;H_w^{\beta,q}(\Omega))$, where $H_w^{\beta,q}(\Omega)$ is a Bessel-Potential space with a Muckenhoupt weight w. In this context we derive solvability for small data, where this smallness can be realized by the restriction on a short time interval. Depending on the order of this Bessel-Potential space we are dealing with strong solutions, weak solutions, or with very weak solutions.

Key Words and Phrases: Navier-Stokes equations, Muckenhoupt weights, very weak solutions, Bessel-Potential spaces, nonhomgeneous data
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1 Introduction

We consider the Navier-Stokes equations with inhomogeneous data

$$\begin{aligned} \partial_t u - \Delta u + u\nabla u + \nabla p &= f && \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= k && \text{in } (0,T)\times\Omega \\ u &= g && \text{on } (0,T)\times\partial\Omega \\ u(0) &= u_0 && \text{in } \Omega \end{aligned} \tag{1.1}$$

on a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a time interval $[0,T)$ with $T \in (0,\infty]$. For simplicity we assume that the coefficient of viscosity is equal to 1.

It is our aim to find a class of solutions to (1.1) in $L^r(0,T;H^{\beta,q}(\Omega))$ where $H^{\beta,q}(\Omega)$ is a Bessel-potential space for $\beta \in [0,2]$. This means we develop a solution theory that includes strong solutions in the case $\beta = 2$ and weak solutions in the case $\beta = 1$. However, if $\beta = 0$, it is also possible that the solutions are only contained in $L^r(0,T;L^q(\Omega))$, i.e., they do not possess any weak derivatives. Consequently the notion of weak solutions

*Department of Mathematics, Technische Universität Darmstadt, Schlossgartenstraße 7, 64289 Darmstadt, Germany, e-mail: schumacher@mathematik.tu-darmstadt.de

is no longer suitable in this context. Thus an appropriate formulation of the problem is needed, the so-called *very weak solutions* to the Navier-Stokes equations. To come to this formulation one multiplies (1.1) with a sufficiently smooth test function ϕ with $\phi(t)|_{\partial\Omega} = 0$ and $\operatorname{div}\phi(t) = 0$ for every t and $\operatorname{supp}\phi \subset [0,T) \times \overline{\Omega}$. Then one applies formal integration by parts and obtains

$$\begin{aligned} &-\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} \\ &= \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} + \langle uu, \nabla\phi\rangle_{\Omega,T} + \langle ku, \phi\rangle_{\Omega,T} - \langle u_0, \phi(0)\rangle_{\Omega} \end{aligned} \tag{1.2}$$

using the identity $u\cdot\nabla u = \operatorname{div}(uu) - (\operatorname{div} u)u$. Applying the same procedure to the second equation in (1.1) and a test function ψ, which does not necessarily vanish on the boundary, yields

$$-\langle u(t), \nabla\psi\rangle_{\Omega} = \langle k(t), \psi\rangle_{\Omega} - \langle g(t), N\psi\rangle_{\Omega} \tag{1.3}$$

for almost every t. Now, u is called a very weak solution to the Navier-Stokes equations if (1.2) and (1.3) are fulfilled for all test functions ϕ and ψ. Note that the information about the boundary values is preserved because $\nabla\phi$ and ψ do not necessarily vanish on the boundary. This or similar formulations have been introduced by Amann in [1], by Amrouche and Girault in [2] and by Galdi, Simader and Sohr in [14]. In these articles as well as by Farwig, Galdi and Sohr in [6], [5], [7] and by Giga in [16] solvability with low-regularity data has been shown. In particular, the boundary conditions under consideration are contained in spaces of distributions on the boundary.

We investigate this problem in function spaces weighted in the space variable. More precisely, we consider Lebesgue-, Sobolev- and Bessel potential spaces with respect to the measure $w\,dx$, where w is a weight function that is contained in the Muckenhoupt class A_q, cf., (2.1) below.

Classical tools for the treatment of partial differential equations extend to function spaces with Muckenhoupt weights. As important examples we mention the continuity of the maximal operator and the multiplier theorems that can be found in the books of García-Cuerva and Rubio de Francia [15] and Stein [24]; extension theorems of functions on a domain to functions on $\mathbb{R}^n$ have been shown by Chua [3], extension theorems of functions on the boundary to functions on the domain by Fröhlich [12], see also [19].

These tools were the base to treat the solvability of the Stokes and Navier-Stokes equations in weighted function spaces by Farwig and Sohr in [9] and by Fröhlich in [13], [12]. If one uses particular weight functions this theory may be used for a better description of the solution, e.g. close to the boundary or in the neighborhood of a point. A further mathematical significance of Muckenhoupt weights is given by the Extrapolation Theorem [15, IV Lemma 5.18]. An even more powerful extrapolation theorem by Curbera, García-Cuerva, Martell and Pérez [4] guarantees estimates in very general Banach function spaces provided that the estimates in weighted function spaces are known for all weights from the Muckenhoupt class A_q. Moreover, this property may be used to derive the R-boundedness of families of operators from their uniform boundedness in weighted function spaces. This fact was used by Fröhlich [13] to give a new proof of the maximal regularity of the Stokes operator in L^q.

In the Main Theorem 4.4 we prove existence, uniqueness, and a priori estimates of solutions to the instationary Navier-Stokes equations in weighted Bessel-Potential spaces under the assumption that the data is small. This smallness of the data can be guaranteed by the restriction to a short time interval.

It has been pointed out in [20] that solutions to inhomogeneous boundary values and divergences exist naturally in the case of lowest regularity, i.e. of solutions in $L^r(0,T;L^q_w(\Omega))$. However, if we want to obtain more regular solutions, one requires, in addition to the space-regularity of the data, a higher time-regularity of the boundary condition and the divergence. Naturally this problem remains in the nonlinear case. However, in the nonlinear case that is treated here, it turns out that in addition the lower regularity causes more difficulties in the sense that we have to put stronger assumptions to the weight functions, while in the case of strong solutions we can deal with the full generality. The reason is that weighted versions of the Sobolev Embedding Theorems require strong assumptions to the weight functions. In the case of higher regularity these assumptions are easier to fulfill. Finally, it turns out that our solution theory yields spaces of solutions $L^r(0,T;H^{\beta,q}_w(\Omega))$ that embed into Serrin's class $L^r(0,T;L^p(\Omega))$ with $\frac{2}{r}+\frac{n}{p}\le 1$.

2 Weighted Function Spaces

Let A_q, $1<q<\infty$, the set of Muckenhoupt weights, be given by all $0\le w\in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{q-1}}\,dx\right)^{q-1} < \infty. \tag{2.1}$$

The supremum is taken over all cubes Q in $\mathbb{R}^n$. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

In [24] there are shown the following facts about the class of Muckenhoupt weights.

- $A_q \subset A_p$ for $q<p$.
- Let $w\in A_q$ for $q>1$. Then there exists $s<q$ such that $w\in A_s$.

Let $k\in\mathbb{N}_0$, $q\in(1,\infty)$, $w\in A_q$ and let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain. Then we define the following weighted versions of Lebesgue and Sobolev spaces.

- $L^q_w(\Omega) := \left\{ f\in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\,dx\right)^{\frac{1}{q}} < \infty\right\}$.

  It is an easy consequence of the corresponding result in the unweighted case that

  $$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \ \text{ with } \ \frac{1}{q}+\frac{1}{q'} = 1 \ \text{ and } \ w' = w^{-\frac{1}{q-1}} \in A_{q'}. \tag{2.2}$$

- Set $W^{k,q}_w(\Omega) = \left\{ u\in L^q_w(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha|\le k}\|D^\alpha u\|_{q,w} < \infty\right\}$.
- By $C^\infty_0(\Omega)$ we denote the set of all smooth and compactly supported functions, the space $C^\infty_{0,\sigma}(\Omega)$ consists of all functions that are in addition divergence free.
- Moreover we set $W^{k,q}_{w,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{k,q,w}}$. The dual space of it is denoted by $W^{-k,q}_w(\Omega) := (W^{k,q'}_{w',0}(\Omega))'$. We also consider the divergence-free versions $W^{k,q}_{w,0,\sigma}(\Omega) := \{\phi\in W^{k,q}_{w,0}(\Omega) \mid \operatorname{div}\phi = 0\}$ and $L^q_{w,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{L^q_w(\Omega)}$.

- Using this for $k > 0$ we set $W^{-k,q}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W_w^{-k,q}(\mathbb{R}^n)}}$.
- Moreover, we consider the spaces of boundary values $T^{k,q}_w(\partial\Omega) := (W^{k,q}_w(\Omega))|_{\partial\Omega}$, equipped with the norm $\|\cdot\|_{T^{k,q}_w} = \|\cdot\|_{T^{k,q}_w(\partial\Omega)}$ of the factor space and finally $T^{0,q}_w(\partial\Omega) := (T^{1,q'}_{w'}(\partial\Omega))'$.

By [12] and [3] the spaces $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$, $W^{k,q}_{w,0}(\Omega)$ and $T^{k,q}_w(\partial\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$, $(C_0^\infty(\Omega)$, $C^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively) are dense.

It has been shown in [12, Lemma 2.2.] that on a bounded domain a weighted Lebesgue space embeds into an unweighted one. More precisely, one has

$$L^q_w(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for} \quad q \geq sp \quad \text{if} \quad w \in A_s \quad \text{for} \quad s < q. \tag{2.3}$$

Next we introduce weighted Bessel-Potential spaces on $\mathbb{R}^n$ and on a bounded Lipschitz domain. For $\xi \in \mathbb{R}^n$ we set $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. On the space $\mathcal{S}'(\mathbb{R}^n;\mathbb{R})$ of temperate distributions we define for all $\beta \in \mathbb{R}$ the operator

$$\Lambda^\beta f = \mathcal{F}^{-1}\langle\xi\rangle^\beta \mathcal{F} f, \quad f \in S'(\mathbb{R}^n;\mathbb{R}),$$

where $\mathcal{F}$ stands for the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n;\mathbb{R})$. Then for $1 < q < \infty$, $w \in A_q$ and $\beta \in \mathbb{R}$ the weighted Bessel potential space is given by

$$H^{\beta,q}_w(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n;\mathbb{R}) \;\middle|\; \|f\|_{H^{\beta,q}_w(\mathbb{R}^n)} := \|\Lambda^\beta f\|_{q,w,\mathbb{R}^n} < \infty \right\}.$$

For a bounded Lipschitz domain Ω we define the weighted Bessel potential space on Ω by

$$H^{\beta,q}_w(\Omega) = \left\{ g|_\Omega \mid g \in H^{\beta,q}_w(\mathbb{R}^n) \right\}$$

equipped with the norm $\|u\|_{H^{\beta,q}_w(\Omega)} := \inf\left\{ \|U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \mid U \in H^{\beta,q}_w(\mathbb{R}^n),\ U|_\Omega = u \right\}$. Note that if $\beta < 0$ then the restriction $g|_\Omega$ has to be understood in the sense of distributions as $g|_{C_0^\infty(\Omega)}$.

For spaces of boundary values we consider the spaces

$$T^{\beta,q}_w(\Omega) := \begin{cases} H^{\beta,q}_w(\Omega) & \text{for} \quad \beta \in [1,2] \\ [T^{0,q}_w(\Omega), T^{1,q}_w(\Omega)]_\beta & \text{for} \quad \beta \in [0,1). \end{cases}$$

As spaces for our solutions we need spaces of functions that vanish on the boundary. Thus for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $1 < q < \infty$, $w \in A_q$ we set $Y^{2,q}_w(\Omega) := \{u \in W^{2,q}_w(\Omega) \mid u|_{\partial\Omega} = 0\}$. For $0 \leq \beta \leq 2$ we define the space

$$Y^{\beta,q}_w(\Omega) := \begin{cases} \overline{Y^{2,q}_w(\Omega)}^{H^{\beta,q}_w(\mathbb{R}^n)}, & \text{if} \quad 0 \leq \beta \leq 1 \quad \text{equipped with } \|\cdot\|_{H^{\beta,q}_w(\mathbb{R}^n)}, \\ \overline{Y^{2,q}_w(\Omega)}^{H^{\beta,q}_w(\Omega)}, & \text{if} \quad 1 < \beta \leq 2 \quad \text{equipped with } \|\cdot\|_{H^{\beta,q}_w(\Omega)}, \end{cases}$$

where in the case $0 \leq \beta \leq 1$ the functions of $Y^{2,q}_w(\Omega)$ are assumed to be extended by 0 to functions defined on the whole space $\mathbb{R}^n$. This is possible, since $C_0^\infty(\Omega)$ is dense in

$W^{1,q}_{w,0}(\Omega) \supset Y^{2,q}_w(\Omega)$ and $W^{1,q}_{w,0}(\Omega) \hookrightarrow W^{1,q}_w(\mathbb{R}^n) \hookrightarrow H^{\beta,q}_w(\mathbb{R}^n)$. We also consider the dual spaces $Y^{-\beta,q}_w(\Omega) := (Y^{\beta,q'}_{w'}(\Omega))'$.

Now we define the divergence free version of $Y^{\beta,q}_w(\Omega)$ by

$$Y^{\beta,q}_{w,\sigma}(\Omega) := \left\{ u \in Y^{\beta,q}_w(\Omega) \mid \langle u, \nabla\phi\rangle = 0 \text{ for every } \phi \in C^\infty(\overline{\Omega}) \right\}.$$

By Theorem 2.1 and (3.1) below one has $Y^{1,q}_{w,\sigma}(\Omega) = \{\phi \in W^{1,q}_{w,0}(\Omega) \mid \operatorname{div}\phi = 0\}$ and $Y^{0,q}_{w,\sigma}(\Omega) = L^q_{w,\sigma}(\Omega)$.

We also consider the dual spaces $Y^{-\beta,q}_{w,\sigma}(\Omega) := \left(Y^{\beta,q'}_{w',\sigma}(\Omega)\right)'$. By the Hahn-Banach theorem the space $Y^{-\beta,q}_{w,\sigma}(\Omega)$ is the restriction of all elements of $Y^{-\beta,q}_w(\Omega)$ to $Y^{\beta,q'}_{w',\sigma}(\Omega)$.

See [21] for further properties and discussions about these spaces. In particular there have been proved the following interpolation properties.

Theorem 2.1. *If Ω is a bounded $C^{1,1}$-domain then one has*

$$\left[L^q_w(\Omega), Y^{2,q}_w(\Omega)\right]_\theta = Y^{\beta,q}_w(\Omega), \quad \theta = \frac{\beta}{2}, \quad 0 \le \beta \le 2$$

and

$$\left[L^q_{w,\sigma}(\Omega), Y^{2,q}_{w,\sigma}(\Omega)\right]_\theta = Y^{\beta,q}_{w,\sigma}(\Omega), \quad \theta = \frac{\beta}{2}, \quad 0 \le \beta \le 2$$

with equivalent norms.

Finally as a space for the divergence we set

$$H^{\gamma,q}_{w,*}(\Omega) := \begin{cases} H^{\gamma,q}_w(\Omega), & \text{if } \gamma \ge 0 \\ H^{\gamma,q}_{w,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{H^{\gamma,q}_w(\mathbb{R}^n)}, & \text{if } \gamma < 0. \end{cases}$$

The following embeddings between weighted spaces have been proved in [21], where there has also been shown the existence and uniqueness of solutions to the stationary Navier-Stokes equations. In the present paper it is used to estimate the nonlinear term.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Moreover, let $1 \le s \le r \le q < \infty$, $r > 1$ and assume $0 \le \beta < n$ such that*

$$\frac{1}{q} \ge \frac{1}{r} - \frac{\beta}{ns}. \tag{2.4}$$

Then for every $w \in A_s$ the following embeddings are true:

1. $H^{\beta,r}_w(\Omega) \hookrightarrow L^q_w(\Omega)$.
2. $H^{\beta,q'}_{w_q}(\Omega) \hookrightarrow L^{r'}_{w_r}(\Omega)$, *where $w_q = w^{-\frac{1}{q-1}}$ and $w_r = w^{-\frac{1}{r-1}}$.*
3. $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_w(\Omega)$, $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_{w,0}(\Omega)$ *and for $\beta \in [0,1]$ one has $W^{-1,r}_w(\Omega) \hookrightarrow Y^{-1-\beta,q}_w(\Omega)$.*
4. *If $\beta \in [0,1]$, then one has $H^{1,r}_w(\Omega) \hookrightarrow H^{1-\beta,q}_w(\Omega)$.*

3 The Linear Stokes Equations and the Stokes Operator

Throughout this section let Ω be a bounded domain that is at least of class $C^{1,1}$.

As in the classical unweighted case one defines the Stokes operator

$$\mathcal{A} = \mathcal{A}_{0,q,w} : L^q_{w,\sigma}(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L^q_{w,\sigma}(\Omega), \quad u \mapsto -P_{q,w}\Delta,$$

where $P_{q,w} : L^q_w(\Omega) \to L^q_w(\Omega)$ is the Helmholtz projection that is the projection to the space of divergence free vector fields

$$L^q_{w,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{L^q_w(\Omega)} = \{u \in L^q_w(\Omega) \mid \langle u, \nabla\phi\rangle = 0 \text{ for every } \phi \in W^{1,q'}_{w'}(\Omega)\}. \quad (3.1)$$

The kernel of $P_{q,w}$ is equal to the space of gradients $\{\nabla p \mid p \in W^{1,q}_w(\Omega)\}$.

All these facts about the Helmholtz projection in weighted spaces have been shown by Fröhlich in [11]. The domain of the Stokes operator is $\mathcal{D}(\mathcal{A}) = Y^{2,q}_{w,\sigma}(\Omega)$. In the weighted context the Stokes operator has been introduced and discussed in [12] and [13].

In the following, we consider an analogue to the Stokes operator which is adequate in the context of very weak solutions in the Bessel potential spaces $H^{\beta,q}_w(\Omega)$.

Theorem 3.1. *For every $0 \le \beta \le 2$ the Stokes operator $\mathcal{A}$ has an extension to an element of $\mathcal{L}(Y^{\beta,q}_{w,\sigma}(\Omega), Y^{\beta-2,q}_{w,\sigma}(\Omega))$ with the following properties.*

1. *It describes a closed and densely defined linear operator in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ again denoted by $\mathcal{A}$. For $u \in Y^{\beta,q}_{w,\sigma}(\Omega)$ one has*

$$\mathcal{A}u = [Y^{2-\beta,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto -\langle u, \Delta\phi\rangle_\Omega].$$

2. *The resolvent set $\rho(-\mathcal{A})$ contains a sector*

$$\Sigma_\varepsilon \cup \{0\} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| < \frac{\pi}{2} + \varepsilon\}, \quad \varepsilon \in (0, \frac{\pi}{2}),$$

and for every $0 < \delta < \varepsilon$ there exists a constant M_δ such that

$$\|\lambda(\mathcal{A} + \lambda)^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}(\Omega))} \le M_\delta \quad \text{for all} \quad \lambda \in \Sigma_\delta. \quad (3.2)$$

Proof. This has been shown in [20]. □

For $-2 \le \mu \le 0$ let $\mathcal{A}_{\mu,q,w}$ be the extension of the Stokes operator whose existence has been stated in Theorem 3.1. Then we call

$$\mathcal{A}_{\mu,q,w} : \mathcal{D}(\mathcal{A}_{\mu,q,w}) := Y^{\mu+2,q}_{w,\sigma}(\Omega) \subset Y^{\mu,q}_{w,\sigma}(\Omega) \to Y^{\mu,q}_{w,\sigma}(\Omega)$$

the generalized Stokes operator in $Y^{\mu,q}_{w,\sigma}(\Omega)$. If no confusion can occur, we omit the indices and write $\mathcal{A}$ instead of $\mathcal{A}_{\mu,q,w}$.

Corollary 3.2. *The negative generalized Stokes operator $-\mathcal{A}$ in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ is the generator of a bounded analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\in\Delta_\varepsilon}$ for every $\varepsilon \in (0, \frac{\pi}{2})$, where $\Delta_\varepsilon = \{\lambda \in \mathbb{C} \mid \lambda \neq 0, \ |\arg\lambda| < \varepsilon\}$.*

In addition one has $e^{-t\mathcal{A}_{\gamma-2,\rho,w}} = e^{-t\mathcal{A}_{\beta-2,q,w}}|_{Y^{\gamma-2,\rho}_{w,\sigma}(\Omega)}$ for $0 \le \beta \le \gamma \le 2$, $q, \rho \in (0, \infty)$ and $w \in A_q \cap A_\rho$ with $Y^{\gamma,\rho}_w(\Omega) \hookrightarrow Y^{\beta,q}_w(\Omega)$.

Proof. The first assertion follows immediately from the resolvent estimate in 3.1 the second follows from the corresponding fact on the resolvent and the representation formula of the semigroup in terms of a path integral [18]. □

For a Banach space X we denote the space of X-valued tempered distributions by $\mathcal{S}'(\mathbb{R};X) := \mathcal{L}(\mathcal{S}(\mathbb{R};\mathbb{R}),X)$. Accordingly, for an interval I we denote the set of distributions by $\mathcal{D}'(I;X) := \mathcal{L}(C_0^\infty(I),X)$.

For the treatment of solutions to the instationary Stokes and Navier-Stokes problem in Bessel-Potential spaces with inhomogeneous divergence and boundary conditions we need a higher time regularity of this part of the data. To measure this time regularity we work in Banach space-valued Bessel-Potential spaces.

For $\beta \in \mathbb{R}$ we set $\Lambda_t^\beta := \mathcal{F}^{-1}\langle\tau\rangle^\beta\mathcal{F}$, where $\langle\tau\rangle^\beta = (1+|\tau|^2)^{\frac{\beta}{2}}$, $\tau \in \mathbb{R}$. Using this, for $r > 1$ we define the X-valued Bessel-potential space by

$$H^{\beta,r}(\mathbb{R};X) := \left\{u \in \mathcal{S}'(\mathbb{R};X) \mid \Lambda_t^\beta u \in L^r(\mathbb{R};X)\right\},$$

equipped with the norm $\|u\|_{H^{\beta,r}(\mathbb{R};X)} := \|\Lambda_t^\beta u\|_{L^r(\mathbb{R};X)}$. Moreover, we set for $\beta \geq 0$

$$H_0^{\beta,r}((0,T];X) := \left\{U|_{C_0^\infty(0,T;\mathbb{R})} \mid U \in H^{\beta,r}(\mathbb{R};X),\ \operatorname{supp} U \subset [0,\infty)\right\}$$

equipped with

$$\begin{aligned}\|u\|_{H_0^{\beta,r}((0,T];X)} := \inf\big\{\|U\|_{H^{\beta,r}(\mathbb{R};X)} \mid U \in H^{\beta,r}(\mathbb{R};X),\\ \operatorname{supp} U \subset [0,\infty),\ U|_{C_0^\infty(0,T;\mathbb{R})} = u\big\}.\end{aligned}$$

As a space of initial values we consider the space

$$\mathcal{I}_w^{\beta,q,r} = \mathcal{I}_w^{\beta,q,r}(\Omega) := \left\{u_0 \in Y_{w,\sigma}^{\beta-2,q}(\Omega) \;\middle|\; \int_0^\infty \|e^{-t\mathcal{A}}u_0\|_{\beta,q,w}^r dt < \infty\right\},$$

where $e^{-t\mathcal{A}}$ is the semigroup that is generated by the generalized Stokes operator $\mathcal{A}$ in $Y_{w,\sigma}^{\beta-2,q}(\Omega)$ with

$$e^{-t\mathcal{A}} : Y_{w,\sigma}^{\beta-2,q}(\Omega) \to \mathcal{D}(\mathcal{A}) = Y_{w,\sigma}^{\beta,q}(\Omega) \subset H_{w,\sigma}^{\beta,q}(\Omega).$$

It is equipped with the norm $\|u_0\|_{\mathcal{I}_w^{\beta,q,r}} := \|u_0\|_{Y_{w,\sigma}^{\beta-2,q}} + \|e^{-t\mathcal{A}}u_0\|_{L^r(H_w^{\beta,q})}$.

The following theorem has been shown in [20]

Theorem 3.3. *Let $1 < q < \infty$, $\beta \in [0,2]$ and let $w \in A_q$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,1}$-domain if $\beta > 1$ and a bounded $C^{1,1}$-domain if $\beta \leq 1$. Moreover, we take*

$$\begin{aligned} f &\in L^r(0,T;Y_w^{\beta-2,q}(\Omega)),\\ k &\in H_0^{\frac{\beta}{2},r}((0,T];W_{w,0}^{-1,q}(\Omega)) \cap L^r(0,T;H_{w,*}^{\beta-1,q}(\Omega)),\\ g &\in H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q}(\partial\Omega)) \cap L^r(0,T;T_w^{\beta,q}(\partial\Omega)),\\ u_0 &\in \mathcal{I}_w^{\beta,q,r}(\Omega),\end{aligned}$$

fulfilling the compatibility condition $\langle k(t),1\rangle_\Omega = \langle g(t),N\rangle_{\partial\Omega}$, for almost all $t \in (0,T)$.

Then there exists a unique very weak solution $u \in L^r(0,T;H_w^{\beta,q}(\Omega))$ *to the instationary Stokes system, i.e.,*

$$\begin{aligned} -\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} &= -\langle u_0, \phi(0)\rangle_\Omega + \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} \\ -\langle u(t), \nabla\psi\rangle_\Omega &= \langle k(t), \psi\rangle_\Omega - \langle g(t), N\psi\rangle_{\partial\Omega} \quad \text{for a.e. } t\in[0,T] \end{aligned}$$

for all $\phi \in L^{r'}(0,T;Y^{2,q'}_{w',\sigma}(\Omega)) \cap W^{1,r'}(0,T;L^{q'}_{w'}(\Omega))$ *with* $\operatorname{supp}\phi$ *compact in* $\overline{\Omega}\times[0,T)$ *and* $\psi \in W^{1,q'}_{w'}(\Omega)$*. This solution* u *fulfills the estimate*

$$\begin{aligned} &\|u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^r(0,T;Y^{\beta-2,q}_{w,\sigma}(\Omega))} + \|u\|_{L^r(H^{\beta,q}_w)} \\ &\qquad \le c\Bigg(\|f\|_{L^r(H^{\beta-2,q}_w)} + \|k\|_{H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0})\cap L^r(H^{\beta-1,q}_{w,*})} \\ &\qquad\quad + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w)\cap L^r(T^{\beta,q}_w)} + \|u_0\|_{\mathcal{I}^{\beta,q,r}_w}\Bigg) \end{aligned} \tag{3.3}$$

with $c = c(\Omega, r, \beta, q, w) > 0$.

4 Instationary Navier-Stokes Equations

The nonlinear term of the Navier-Stokes equations in the variational formulation can be rewritten by the functional

$$\phi \mapsto -\langle uu, \nabla\phi\rangle - \langle u\operatorname{div} u, \phi\rangle.$$

To make the multiplication $u\operatorname{div} u$ well-defined, we assume that the divergence is given by a function. More precisely for $\beta \le 1$ we choose $\mu > 1$ such that

$$L^\mu_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega) \tag{4.1}$$

and assume that $u(t) \in L^\mu_w(\Omega)$ for almost every t. See Lemma 2.2 for sufficient conditions for (4.1).

Definition 4.1. Let $\beta \in [0,2]$, $r, q \in (1,\infty)$, $w \in A_q$. Moreover, in the case $\beta \le 1$ choose $\mu > 1$ such that it fulfills (4.1). Take

$$\begin{aligned} &f \in L^r(0,T;Y^{\beta-2,q}_w(\Omega)), \\ &k \in L^r(0,T;L^\mu_w(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0}(\Omega)) \quad \text{if } \beta < 1 \text{ and} \\ &k \in L^r(0,T;H^{\beta-1,q}_w(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0}(\Omega)) \quad \text{if } \beta \ge 1, \\ &g \in L^r(0,T;T^{\beta,q}_w(\partial\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w(\partial\Omega)), \\ &u_0 \in \mathcal{I}^{\beta,q,r}_w(\Omega). \end{aligned}$$

Then $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ is called a very weak solution to the Navier-Stokes problem if

$$\begin{aligned} &-\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} \\ &\qquad = \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} + \langle uu, \nabla\phi\rangle_{\Omega,T} + \langle ku, \phi\rangle_{\Omega,T} - \langle u_0, \phi(0)\rangle_\Omega \end{aligned}$$

for every $\phi \in W^{1,r'}(0,T;Y^{2,q'}_{w',\sigma}(\Omega))$ with $\operatorname{supp}\phi \subset [0,T)\times\overline{\Omega}$, $\operatorname{div} u = k$ is fulfilled in the sense of distributions and $u\cdot N|_{\partial\Omega} = g\cdot N$.

Since a very weak solutions $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ is in general not regular enough to guarantee that its restriction to the boundary is well-defined, some discussions on the boundary values are in order. The component of the boundary condition that is normal to the boundary is simple, since u fulfills $\operatorname{div} u(t) = k(t) \in L^\mu_w(\Omega)$. Thus we find for almost every t

$$\langle u(t), \psi N\rangle_{\partial\Omega} = \langle \operatorname{div} u(t), \psi\rangle_\Omega + \langle u(t), \nabla\psi\rangle_\Omega = \langle k(t), \psi\rangle_\Omega + \langle u(t), \nabla\psi\rangle_\Omega = \langle g(t), \psi N\rangle.$$

In this sense one has $u \cdot N|_{\partial\Omega} = g \cdot N$. The tangential part of the boundary condition causes more difficulties. By [22] a sufficient condition that guarantees the well-definedness of the boundary condition is $|\langle u(t), \Delta\phi\rangle| \le C(t)\|\phi\|_{1,\eta,v}$ for every $\phi \in C^\infty_{0,\sigma}(\Omega)$ and almost every t. It is obvious that this requires a further restriction of the exterior force and, in addition, some appropriate time regularity of the solution. However, it is not difficult to show that every sufficiently regular solution fulfills the boundary condition $u(t)|_{\partial\Omega} = g(t)$ almost everywhere.

The evaluation of every very weak solution to the Navier-Stokes equation with respect to the above data at time 0 is well-defined and satisfies $u(0)|_{Y^{2,q'}_{w',\sigma}} = u_0$. This holds because by the a priori estimate (4.5) below one has $u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_{w,\sigma}(\Omega))$. See [20] for more details.

For $\alpha > 0$ the fractional powers A^α of the generator of an analytic semigroup are well-defined by [18]. In the unweighted case the boundedness of imaginary powers of the Stokes operator is used to prove an exact characterization of the domains of fractional powers of the Stokes operator, see Giga [17]. However, in weighted function spaces this is not established. We use the following Theorem by Franzke [10] as a replacement.

Theorem 4.2. *Let X be a Banach space and A a densely defined positive operator in X, i.e.,*

$$\|(\lambda + A)^{-1}\| \le \frac{K}{1+\lambda} \quad \text{for every} \quad \lambda \ge 0.$$

Then for $m \in \mathbb{N}$, $0 < \theta < 1$ and $0 < \alpha_- < \theta m < \alpha_+$ one has

$$\mathcal{D}(A^{\alpha_+}) \hookrightarrow [X, \mathcal{D}(A^m)]_\theta \hookrightarrow \mathcal{D}(A^{\alpha_-}).$$

In particular if $A = \mathcal{A}$ is the Stokes operator in $L^q_{w,\sigma}(\Omega)$, $m = 1$, $0 < \beta < 2$ and $\varepsilon > 0$, then we find by Theorem 2.1

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}\beta+\varepsilon}) \hookrightarrow Y^{\beta,q}_{w,\sigma}(\Omega) \hookrightarrow \mathcal{D}(\mathcal{A}^{\frac{1}{2}\beta-\varepsilon}). \tag{4.2}$$

Thus one has the estimate

$$c_1\|A^{\frac{1}{2}\beta-\varepsilon}u\|_{q,w} \le \|u\|_{Y^{\beta,q}_{w,\sigma}} \le c_2\|A^{\frac{1}{2}\beta+\varepsilon}u\|_{q,w}.$$

Moreover, if we consider the generalized Stokes operator in $Y^{-1,\rho}_{w,\sigma}(\Omega)$, one obtains by Theorems 2.1 and 3.1

$$\begin{aligned} \|u\|_{\rho,w} &\le c\|\mathcal{A}u\|_{Y^{-2,\rho}_{w,\sigma}} = c\|\mathcal{A}^{\frac{1}{2}-\varepsilon}\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{Y^{-2,\rho}_{w,\sigma}} \\ &\le c\|\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{[Y^{-2,\rho}_{w,\sigma},L^q_{w,\sigma}]_{\frac{1}{2}}} \le c\|\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{Y^{-1,\rho}_{w,\sigma}} \\ &= c\|\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{H^{-1,\rho}_{w,\sigma}}. \end{aligned} \tag{4.3}$$

The proof of existence and uniqueness of very weak solutions to the instationary Navier-Stokes equations requires the Variation of Constants Formula established in the following lemma.

Lemma 4.3. *Let $1 < q, r < \infty$, $0 \le \beta \le 2$. Moreover, take $f \in L^r(0,T; Y^{\beta-2,q}_{w,\sigma}(\Omega))$ and let $u \in L^r(0,T; Y^{\beta,q}_{w,\sigma}(\Omega))$ be the solution to*

$$u_t + \mathcal{A}u = f \quad \text{in} \;\; \mathcal{D}'(0,T; Y^{\beta-2,q}_{w,\sigma}(\Omega)) \quad \text{and} \quad u(0) = 0,$$

where $\mathcal{A} = \mathcal{A}_{\beta-2,q,w}$ is the generalized Stokes operator in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$. Then

$$u(t) = \int_0^t e^{-\mathcal{A}(t-\tau)} f(\tau)\, d\tau \quad \text{for almost every } t \in (0,T).$$

Proof. From the embeddings $H^{\beta,q}_{w,\sigma}(\Omega) \hookrightarrow L^q_{w,\sigma}(\Omega)$ and $Y^{\beta-2,q}_{w,\sigma}(\Omega) \hookrightarrow Y^{-2,q}_{w,\sigma}(\Omega)$ we know that

$$u \in L^r(0,T; L^q_{w,\sigma}(\Omega)) \quad \text{and} \quad f \in L^r(0,T; Y^{-2,q}_{w,\sigma}(\Omega)).$$

Thus we obtain $\mathcal{A}^{-1}_{0,q,w} u \in L^r(0,T; Y^{2,q}_{w,\sigma}(\Omega))$ and $\mathcal{A}^{-1}_{-2,q,w} f \in L^r(0,T; L^q_{w,\sigma}(\Omega))$.

Since the generalized Stokes operator is defined in Theorem 3.1 as an extension of the classical one, we obtain that $\mathcal{A}^{-1}_{-2,q,w} u = \mathcal{A}^{-1}_{0,q,w} u$ is the strong solution to the instationary Stokes problem

$$(\mathcal{A}^{-1}_{0,q,w} u)_t + \mathcal{A}_{0,q,w}(\mathcal{A}^{-1}_{0,q,w} u) = \mathcal{A}^{-1}_{-2,q,w} f.$$

By the maximal regularity of the classical Stokes operator [13] and the uniqueness of strong solutions the Variation of Constants Formula holds in the case of strong solutions. Thus one obtains

$$\mathcal{A}^{-1}_{0,q,w} u(t) = \int_0^t e^{-\mathcal{A}_{0,q,w}(t-\tau)} \mathcal{A}^{-1}_{-2,q,w} f(\tau)\, d\tau. \tag{4.4}$$

Moreover, by Theorem 3.2 one has

$$\begin{aligned}\mathcal{A}_{0,q,w} e^{-(t-\tau)\mathcal{A}_{0,q,w}} \mathcal{A}^{-1}_{-2,q,w} f &= \mathcal{A}_{-2,q,w} e^{-(t-\tau)\mathcal{A}_{-2,q,w}} \mathcal{A}^{-1}_{-2,q,w} f = e^{-(t-\tau)\mathcal{A}_{-2,q,w}} f \\ &= e^{-(t-\tau)\mathcal{A}_{\beta-2,q,w}} f.\end{aligned}$$

Thus if one applies $\mathcal{A}_{0,q,w}$ to both sides of (4.4) the proof of the lemma is finished. □

Theorem 4.4. *Let $\beta \in [0,2]$ with $\beta > \frac{ns}{q} - 1$, where $q \in (1,\infty)$ and $w \in A_s$ for some $s < q$. Moreover, let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be a bounded $C^{1,1}$-domain, if $\beta \le 1$, and a bounded $C^{2,1}$-domain, if $\beta > 1$.*

Choose $r \in (1,\infty)$ such that

$$\begin{aligned}
&\frac{1}{r} < \min\left\{ -\frac{ns}{2q} + \frac{\beta}{2} + \frac{1}{2}, \frac{1-\beta}{2} \right\} && \text{if } 0 \le \beta < 1, \\
&\frac{1}{r} < \min\left\{ -\frac{ns}{2q} + \frac{\beta}{2} + \frac{1}{2}, \frac{2-\beta}{2} \right\} && \text{if } 1 \le \beta < 2 \text{ and} \\
&\frac{1}{r} < \min\left\{ -\frac{ns}{2q} + \frac{3}{2}, \frac{1}{2} \right\} \quad \text{and} \quad \frac{2}{q} - \frac{2}{ns} < \frac{1}{s} && \text{if } \beta = 2.
\end{aligned}$$

In the case $n = 2$ and $\beta \in [1,2)$ we assume in addition that $\beta > \frac{2s}{q} - \frac{1}{2}$.

Take f, k, g and u_0 as in Definition 4.1 with μ chosen such that

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{\mu} = 0 \quad \text{in the case} \quad \beta \le 1.$$

Then, if $\beta \le 1$ there exists a constant $\eta = \eta(\Omega, \beta, q, w, r) > 0$ with the following property: If $0 < T' \le T$ with

$$\left(\int_0^{T'} \|e^{-\tau\mathcal{A}} u_0\|_{\beta,q,w}^r d\tau \right)^{\frac{1}{r}} + \|f\|_{L^r(0,T';Y_w^{\beta-2,q})}$$
$$+ \|k\|_{L^r(0,T';L_w^\mu)\cap H_0^{\frac{\beta}{2},r}((0,T'];W_{w,0}^{-1,q})} + \|g\|_{L^r(0,T';T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}((0,T'];T_w^{0,q})} \le \eta,$$

then there exists a unique very weak solution $u \in L^r(0,T';H_w^{\beta,q}(\Omega))$ to the Navier-Stokes equations. For every $T'' \in (0,T']$, $T'' < \infty$ this solution u satisfies the estimate

$$\begin{aligned} \|u\|_{L^r(0,T';H_w^{\beta,q})} + &\|u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^{\frac{r}{2}}(0,T'';Y_{w,\sigma}^{\beta-2,q}(\Omega))} \\ \le c\Bigg(&\left(\int_0^{T'} \|e^{-\tau\mathcal{A}} u_0\|_{\beta,q,w}^r d\tau \right)^{\frac{1}{r}} + \|f\|_{L^r(0,T';Y_w^{\beta-2,q})} \\ &+ \|k\|_{L^r(0,T';L_w^\mu)\cap H_0^{\frac{\beta}{2},r}((0,T'];W_{w,0}^{-1,q})} + \|g\|_{L^r(0,T';T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}((0,T'];T_w^{0,q})}\Bigg); \end{aligned} \tag{4.5}$$

here c increases with increasing T'' but can be chosen independently of T and T'.

If $\beta > 1$ then the same assertion holds if $L^r(0,T';L_w^\mu(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T'];W_{w,0}^{-1,q}(\Omega))$ is replaced by $L^r(0,T';H_w^{\beta-1,q}(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T'];W_{w,0}^{-1,q}(\Omega))$.

Proof. Let $E \in L^r(0,T;H_w^{\beta,q}(\Omega))$ be the very weak solution to the instationary Stokes problem with respect to the data f, k, g and u_0 in the sense of Theorem 3.3.

Assume that $u \in L^r(0,T;H_w^{\beta,q}(\Omega))$ is the very weak solution to the Navier-Stokes equations we are looking for. Then $\tilde{u} := u - E$ solves

$$\partial_t \tilde{u} - \Delta\tilde{u} + \nabla\tilde{p} = -W(u), \quad \operatorname{div}\tilde{u} = 0, \quad \tilde{u}|_{\partial\Omega} = 0 \ \text{ and } \ \tilde{u}(0) = 0$$

in the very weak sense with

$$W(u)(t) := [Y^{2,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto -\langle u(t)u(t), \nabla\phi\rangle_\Omega - \langle k(t)u(t), \phi\rangle_\Omega]$$

for almost every t. This means

$$-\langle \tilde{u}, \phi_t\rangle_{\Omega,T} - \langle \tilde{u}, \Delta\phi\rangle_{\Omega,T} = \langle W(u), \phi\rangle_\Omega$$

with ϕ as in Definition 4.1. Then the Variation of Constants Formula proved in Lemma 4.3 yields

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)\mathcal{A}} W(u) d\tau =: \mathcal{G}(\tilde{u})(t).$$

As a first step we assume $\beta < 1$. By the definition of μ and the assumptions on β one has $s \le \mu < q$ and by Lemma 2.2 one obtains $L_w^\mu(\Omega) \hookrightarrow H_w^{\beta-1,q}(\Omega)$. Put

$$\alpha = \frac{1}{r} - 1 < 0 \quad \text{and} \quad \varepsilon := \min\left\{ \frac{1}{5}\left(-\alpha - \frac{\beta}{2} - \frac{1}{2}\right), \frac{1}{5}\left(-\frac{ns}{q} + \beta + 1 - \frac{2}{r}\right) \right\} > 0,$$

where ε is positive by the assumption on r. Moreover, if $\beta < \frac{ns}{q}$ we set $\rho := \frac{nsq}{2ns-2q\beta}$. Then one obtains by the assumptions on β:

- ρ is well-defined and $\rho > \frac{ns}{2} \geq s$.
- $\frac{1}{q} \geq \frac{1}{\rho} - \frac{-2\alpha-5\varepsilon-\beta-1}{ns}$ with $0 < -2\alpha - 5\varepsilon - \beta - 1 \leq 1 < n$.
- $\frac{1}{2\rho} = \frac{1}{q} - \frac{\beta}{ns}$.

This proves

$$H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{2\rho}(\Omega) \quad \text{and} \quad H_w^{-2\alpha-5\varepsilon-\beta-1,\rho}(\Omega) \hookrightarrow L_w^q(\Omega), \tag{4.6}$$

using Lemma 2.2 if $\rho < q$. For $q \leq \rho$ the latter embedding is obvious.

If $1 > \beta \geq \frac{ns}{q}$ then $H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{2\rho}(\Omega)$ for every $\rho \in (1,\infty)$. In this case we choose ρ with $\frac{1}{q} < \frac{1}{\rho} < \frac{1}{q} + \frac{-2\alpha-5\varepsilon-\beta-1}{ns}$. Then the embeddings (4.6) hold for ρ.

By (4.2), (4.3) and well-known estimates about analytic semigroups [18] one obtains the estimate

$$\begin{aligned}
\|\mathcal{G}(\tilde{u})(t)\|_{H_w^{\beta,q}} \leq & \|\mathcal{G}(\tilde{u})(t)\|_{Y_w^{\beta,q}} \\
\leq & \int_0^t \left\|e^{-(t-\tau)\mathcal{A}} W(u)\right\|_{Y_w^{\beta,q}} d\tau \\
\leq & \int_0^t \left\|\mathcal{A}^{-\alpha} e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{\alpha+\frac{\beta}{2}+\varepsilon} W(u)\right\|_{q,w} d\tau \\
\leq & c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left\|\mathcal{A}^{(\alpha+2\varepsilon+\frac{\beta}{2}+\frac{1}{2})-\frac{1}{2}-\varepsilon} W(u)\right\|_{q,w} d\tau \\
\leq & c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left\|\mathcal{A}^{(\alpha+2\varepsilon+\frac{\beta}{2}+\frac{1}{2})-\frac{1}{2}-\varepsilon} W(u)\right\|_{-2\alpha-5\varepsilon-\beta-1,\rho,w} d\tau \\
\leq & c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left\|\mathcal{A}^{-\frac{1}{2}-\varepsilon} W(u)\right\|_{\rho,w} d\tau \\
\leq & c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \|W(u)\|_{-1,\rho,w} d\tau,
\end{aligned} \tag{4.7}$$

where we have used $\mathcal{A}_{-2,q,w}|_{Y_{w,\sigma}^{-1,\rho}} = \mathcal{A}_{-1,\rho,w}$, where $\mathcal{A}_{-2,q,w}$ is the generalized Stokes operator in $Y_{w,\sigma}^{-2,q}(\Omega)$ and $\mathcal{A}_{-1,\rho,w}$ is the generalized Stokes operator in $Y_{w,\sigma}^{-1,\rho}(\Omega) = (W_{0,w',\sigma}^{1,\rho'}(\Omega))'$.

We have to estimate $\|W(u)\|_{-1,\rho,w}$. For $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$ one has

$$\begin{aligned}
|\langle W(u), \phi\rangle_\Omega| \leq & |\langle uu, \nabla\phi\rangle_\Omega| + |\langle ku, \phi\rangle_\Omega| \\
\leq & \|uu\|_{\rho,w}\|\nabla\phi\|_{\rho',w_\rho} + \|ku\|_{\tilde{\rho},w}\|\phi\|_{\tilde{\rho}',w_{\tilde{\rho}}} \\
\leq & \left(\|u\|_{2\rho,w}^2 + \|k\|_{\mu,w}\|u\|_{2\rho,w}\right)\|\phi\|_{1,\rho',w_\rho} \\
\leq & c\left(\|u\|_{\beta,q,w}^2 + \|k\|_{\mu,w}\|u\|_{\beta,q,w}\right)\|\phi\|_{1,\rho',w_\rho}
\end{aligned} \tag{4.8}$$

since for $\tilde{\rho}$ given by $\frac{1}{\tilde{\rho}} = \frac{1}{\mu} + \frac{1}{2\rho}$ an elementary computation shows $\frac{1}{\rho} = \frac{1}{\tilde{\rho}} - \frac{1}{ns}$ which implies $H_{w_\rho}^{1,\rho'}(\Omega) \hookrightarrow L_{w_{\tilde{\rho}}}^{\tilde{\rho}'}(\Omega)$.

Thus we continue combining (4.7) and (4.8), the Hardy-Littlewood inequality [24, VIII 4.2] and the equality $\frac{1}{r} + \alpha + 1 = \frac{1}{\frac{r}{2}}$. Then we may estimate

$$
\begin{aligned}
&\|\mathcal{G}(\tilde{u})\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} \\
\leq & c \left\| \int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left(\|u(\tau)\|^2_{\beta,q,w} + \|k(\tau)\|_{\mu,w} \|u(\tau)\|_{\beta,q,w} \right) d\tau \right\|_{L^r(0,T)} \\
\leq & c \left\| \|u(\tau)\|^2_{\beta,q,w} + \|k(\tau)\|_{\mu,w} \|u(\tau)\|_{\beta,q,w} \right\|_{L^{\frac{r}{2}}(0,T)} \\
\leq & c \left(\|u\|^2_{L^r(0,T;H^{\beta,q}_w(\Omega))} + \|k\|_{L^r(0,T;L^\mu_w(\Omega))} \|u\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} \right) \\
\leq & c \Big(\left(\|\tilde{u}\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} + \|E\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} \right)^2 \\
& \quad + \|k\|_{L^r(0,T;L^\mu_w(\Omega))} \left(\|\tilde{u}\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} + \|E\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} \right) \Big).
\end{aligned} \tag{4.9}
$$

Now assume

$$
\begin{aligned}
&\left(\int_0^{T'} \|e^{-\tau \mathcal{A}} u_0\|^r_{\beta,q,w} d\tau \right)^{\frac{1}{r}} + \|f\|_{L^r(0,T';Y^{\beta-2,q}_w)} \\
&+ \|k\|_{L^r(0,T';L^\mu_w) \cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})} + \|g\|_{L^r(0,T';T^{\beta,q}_w) \cap H_0^{\frac{\beta}{2},r}((0,T'];T^{0,q}_w)} \leq \eta
\end{aligned}
$$

and $\|\tilde{u}\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} < \delta$, where η and δ are positive but will be chosen sufficiently small later on. Then one has $\|E\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} \leq K\eta$, where K is the constant from the a priori estimate for the solution to the instationary Stokes equations, Theorem 3.3. Thus we obtain from (4.9)

$$
\|\mathcal{G}(\tilde{u})\|_{L^r(0,T;H^{\beta,q}_w(\Omega))} \leq c((\delta + K\eta)^2 + \eta(\delta + K\eta)) < \delta,
$$

if η and δ are sufficiently small. This shows $\mathcal{G}(B_\delta(0)) \subset \overline{B_\delta(0)}$, where $\overline{B_\delta(0)}$ is the closed ball with radius δ in $L^r(0,T;H^{\beta,q}_w(\Omega))$.

We show that $\mathcal{G}$ is a contraction on $B_\delta(0)$. As above we find the pointwise estimate

$$
\begin{aligned}
&|\langle W(E+\tilde{u}) - W(E+\tilde{v}), \phi \rangle_\Omega| \\
\leq & |\langle (E+\tilde{u})^2 - (E+\tilde{v})^2, \nabla\phi \rangle_\Omega| + |\langle k(E+\tilde{u}) - k(E+\tilde{v}), \phi \rangle| \\
\leq & |\langle 2E(\tilde{u}-\tilde{v}), \nabla\phi \rangle_\Omega| + |\langle \tilde{u}\tilde{u} - \tilde{v}\tilde{v}, \nabla\phi \rangle_\Omega| + |\langle k(\tilde{u}-\tilde{v}), \phi \rangle_\Omega| \\
\leq & c \big(\|E\|_{\beta,q,w} \|\tilde{u}-\tilde{v}\|_{\beta,q,w} + \|\tilde{u}\|_{\beta,q,w} \|\tilde{u}-\tilde{v}\|_{\beta,q,w} \\
& \quad + \|\tilde{v}\|_{\beta,q,w} \|\tilde{u}-\tilde{v}\|_{\beta,q,w} + \|k\|_{\mu,w} \|\tilde{u}-\tilde{v}\|_{\beta,q,w} \big) \|\phi\|_{1,\rho',w_\rho}
\end{aligned}
$$

almost everywhere in t. Thus

$$
\|W(E+\tilde{u}) - W(E+\tilde{v})\|_{-1,\rho,w} \leq c \big(\|E\|_{\beta,q,w} + \|\tilde{u}\|_{\beta,q,w} + \|\tilde{v}\|_{\beta,q,w} + \|k\|_{\mu,w} \big) \|\tilde{u}-\tilde{v}\|_{\beta,q,w}.
$$

Now an analogous estimate as for $\|\mathcal{G}(\tilde{u})\|_{L^r(0,T;H_w^{\beta,q}(\Omega))}$ shows

$$\begin{aligned}
&\|\mathcal{G}(\tilde{u})-\mathcal{G}(\tilde{v})\|_{L^r(0,T;H_w^{\beta,q}(\Omega))}\\
&\quad\le c\Big\|\int_0^t \frac{1}{(t-\tau)^{-\alpha}}\big(\|E\|_{\beta,q,w}+\|\tilde{u}\|_{\beta,q,w}+\|\tilde{v}\|_{\beta,q,w}+\|k\|_{\mu,w}\big)\|\tilde{u}-\tilde{v}\|_{\beta,q,w}d\tau\Big\|_{L^r(0,T')}\\
&\quad\le c\big(\|E\|_{L^r(0,T';H_w^{\beta,q})}+\|\tilde{u}\|_{L^r(0,T';H_w^{\beta,q})}+\|\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})}+\|k\|_{L^r(0,T';L_w^{\mu})}\big)\\
&\qquad\cdot\|\tilde{u}-\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})}\\
&\quad\le c(K\eta+\eta+2\delta)\|\tilde{u}-\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})}.
\end{aligned}$$

This means, if η and δ are sufficiently small, then $\mathcal{G}$ is a contraction. Hence by Banach's fixed point theorem there exists a unique $\tilde{u}\in B_\delta(0)$ with $\mathcal{G}(\tilde{u})=\tilde{u}$. Then $u:=E+\tilde{u}$ is the solution we have been looking for.

We turn to the case $\beta\ge 1$. In this case the proof of existence follows the same lines of the case $\beta<1$ but using different embeddings. Moreover, the fact that $k\in L^r(0,T;H_w^{\beta-1,q}(\Omega))$ gives us reason to repeat the arguments.

Let $\alpha=\frac{1}{r}-1$. Then as in (4.7) we obtain with an appropriate choice of $\varepsilon>0$

$$\begin{aligned}
\|\mathcal{G}(\tilde{u})(t)\|_{\beta,q,w}=&\|\mathcal{G}(\tilde{u})(t)\|_{Y_w^{\beta,q}}\\
\le& c\int_0^t\Big\|\mathcal{A}^{-\alpha}e^{-(t-\tau)\mathcal{A}}\mathcal{A}^{\alpha+\frac{\beta}{2}+\frac{\varepsilon}{4}}W(u)\Big\|_{q,w}d\tau\\
\le& c\int_0^t\frac{1}{(t-\tau)^{-\alpha}}\Big\|\mathcal{A}^{\alpha+\frac{\beta}{2}+\frac{\varepsilon}{4}}W(u)\Big\|_{q,w}d\tau\\
\le& c\int_0^t\frac{1}{(t-\tau)^{-\alpha}}\|W(u)\|_{\beta+2\alpha+\varepsilon,q,w}d\tau\\
\le&\int_0^t\frac{1}{(t-\tau)^{-\alpha}}\|W(u)\|_{\rho,w}d\tau,
\end{aligned}$$

where ρ is chosen according to β as follows.

If $\beta<\frac{ns}{q}$ then we choose η_1, η_2, ρ such that

$$\frac{1}{\eta_1}=\frac{1}{q}-\frac{\beta}{ns},\quad \frac{1}{\eta_2}=\frac{1}{q}-\frac{\beta-1}{ns},\quad \frac{1}{\rho}=\frac{1}{\eta_1}+\frac{1}{\eta_2}.$$

Then one has by the restrictions on $\frac{1}{r}$

- $\rho>s$. If $n=2$ one uses the additional assumption to show this.
- $\frac{1}{\rho}+\frac{\beta+2\alpha}{ns}=\frac{2}{q}-\frac{2\beta-1}{ns}+\frac{\beta+2\alpha}{ns}<\frac{1}{q}$.
- $\beta+2\alpha<0$.

This implies with an appropriate choice of ε

$$L_w^\rho(\Omega)\hookrightarrow H_w^{\beta+2\alpha+\varepsilon,q}(\Omega),\quad H_w^{\beta,q}(\Omega)\hookrightarrow L_w^{\eta_1}(\Omega),\quad H_w^{\beta-1,q}(\Omega)\hookrightarrow L_w^{\eta_2}(\Omega). \tag{4.10}$$

If $\frac{ns}{q} \leq \beta < 2$, then $H^{\beta,q}_w(\Omega) \hookrightarrow L^{\eta_1}_w(\Omega)$ for every $\eta_1 \in (1,\infty)$. Then we choose ρ with $\frac{1}{q} < \frac{1}{\rho} < \frac{1}{q} - \frac{\beta+2\alpha}{ns}$, $\eta_2 > \rho$ with $\frac{1}{\eta_2} \geq \frac{1}{q} - \frac{\beta-1}{ns}$ and η_1 such that $\frac{1}{\eta_1} + \frac{1}{\eta_2} = \frac{1}{\rho}$. This implies the embeddings (4.10). Thus in any case we may estimate

$$\|v \cdot \nabla u\|_{\rho,w} \leq \|v\|_{\eta_1,w}\|\nabla u\|_{\eta_2,w} \leq c\|v\|_{\beta,q,w}\|u\|_{\beta,q,w} \tag{4.11}$$

for every $u, v \in H^{\beta,q}_w(\Omega)$. Hence we obtain as in (4.9)

$$\begin{aligned} \|\mathcal{G}(\tilde{u})\|_{L^r(0,T';H^{\beta,q}_w)} &\leq c \left\| \int_0^t \frac{1}{(t-\tau)^{-\alpha}} (\|\tilde{u}(\tau)\|_{\beta,q,w} + \|E(\tau)\|_{\beta,q,w})^2 d\tau \right\|_{L^r(0,T')} \\ &\leq c \left(\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w)} + \|E\|_{L^r(0,T';H^{\beta,q}_w)} \right)^2 \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\|\mathcal{G}(\tilde{u}) - \mathcal{G}(\tilde{v})\|_{L^r(0,T';H^{\beta,q}_w)} \\ &\quad \leq c \left(\|E\|_{L^r(0,T';H^{\beta,q}_w)} + \|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w)} + \|\tilde{v}\|_{L^r(0,T';H^{\beta,q}_w)} \right) \|u - v\|_{L^r(0,T';H^{\beta,q}_w)}. \end{aligned}$$

Then the same iteration procedure as in the case $\beta < 1$ shows the existence of a unique fixed point $\tilde{u} = \mathcal{G}(\tilde{u})$ within a ball in $L^r(0,T';H^{\beta,q}_w(\Omega))$ with radius δ.

We turn to the case $\beta = 2$, i.e., the case of strong solutions. One uses the estimate

$$\begin{aligned} \|\mathcal{G}(u)\|_{2,q,w} \leq& c \int_0^t \frac{1}{(t-\tau)^{-\alpha}} \|W(u)\|_{2+2\alpha+\varepsilon,q,w} d\tau \\ \leq& c \int_0^t \frac{1}{(t-\tau)^{-\alpha}} \|W(u)\|_{1,\rho,w} d\tau. \end{aligned}$$

Such a ρ can be chosen because $2 + 2\alpha = \frac{2}{r} < 1$. As above we choose ρ and η such that such that

- $\frac{1}{q} > \frac{1}{\rho} - \frac{1-\frac{2}{r}}{ns}$ to guarantee the embedding $H^{1,\rho}_w(\Omega) \hookrightarrow H^{\frac{2}{r}+\varepsilon,q}_w(\Omega)$,
- $\frac{1}{2\rho} \geq \frac{1}{q} - \frac{1}{ns}$ to obtain $H^{1,q}_w(\Omega) \hookrightarrow L^{2\rho}_w(\Omega)$.
- $\frac{1}{\eta} \geq \frac{1}{q} - \frac{2}{ns}$ which yields $H^{2,q}_w(\Omega) \hookrightarrow L^{\eta}_w(\Omega)$.

If $ns - q > 0$ the above holds if $\rho = \frac{nsq}{2ns-2q}$ and $\frac{1}{\eta} = \frac{1}{\rho} - \frac{1}{q}$. If $ns - q \leq 0$ then $H^{1,q}_w(\Omega) \hookrightarrow L^{2\rho}_w(\Omega)$ for every ρ and in addition $H^{2,q}_w(\Omega) \hookrightarrow L^{\eta}_w(\Omega)$ for every η. Then we choose any ρ with $\frac{1}{q} < \frac{1}{\rho} < \frac{1}{q} + \frac{1-\frac{1}{r}}{ns}$ and $\frac{1}{\eta} = \frac{1}{\rho} - \frac{1}{q}$ to guarantee the above.

We use this to prove $\|W(u)\|_{1,\rho,w} \leq c\|u\|^2_{2,q,w}$. To this aim we calculate

$$\begin{aligned} \|\partial_k W(u)\|_{\rho,w} \leq& \|\partial_k u \cdot \nabla u\|_{\rho,w} + \|u \cdot \partial_k \nabla u\|_{\rho,w} \\ \leq& \|\nabla u\|^2_{2\rho,w} + \|u\|_{\eta,w}\|\nabla^2 u\|_{q,w} \leq c\|u\|^2_{2,q,w}. \end{aligned} \tag{4.13}$$

From now on we derive all following estimates as in the case $0 \leq \beta < 1$. This finishes the proof of existence for small data for every $\beta \in [0,2]$.

The next step is to prove the a priori estimate. Let $\tilde{u} \in B_\delta(0)$ be the fixed point of $\mathcal{G}$. Then one has by (4.9) and (4.12)

$$\begin{aligned}\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} =&\|\mathcal{G}(\tilde{u})\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ \leq& c\Big((\delta+K\eta)(\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))})\\ &+\eta(\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))})\Big).\end{aligned}$$

Choosing δ and η such that $c(\delta+2\eta)<1$ this proves

$$\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq \frac{c(\delta+2\eta)}{1-c(\delta+2\eta)}\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}.$$

Finally, we obtain for $\beta<1$

$$\begin{aligned}\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq&\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ \leq& c\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ \leq& c\Big(\Big(\int_0^{T'}\|e^{-\tau\mathcal{A}}u_0\|^r_{\beta,q,w}d\tau\Big)^{\frac{1}{r}}+\|f\|_{L^r(0,T',Y^{\beta-2}_w(\Omega))}\\ &+\|k\|_{L^r(0,T';L^u_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})}+\|g\|_{L^r(0,T';T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];T^{0,q}_w)}\Big)\end{aligned}$$

by the a priori estimate in the linear case in Theorem 3.3. If $\beta\geq 1$ one obtains the estimate

$$\begin{aligned}\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq& c\Big(\Big(\int_0^{T'}\|e^{-\tau\mathcal{A}}u_0\|^r_{\beta,q,w}d\tau\Big)^{\frac{1}{r}}+\|f\|_{L^r(0,T',Y^{\beta-2}_w(\Omega))}\\ &+\|k\|_{L^r(0,T';H^{\beta-1,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})}+\|g\|_{L^r(0,T';T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];T^{0,q}_w)}\Big)\end{aligned}$$

analogously.

Since u is a very weak solution to the instationary Stokes problem

$$\partial_t u-\Delta u+\nabla p=f-W(u)-G-U_0, \tag{4.14}$$

where $G=[\phi\mapsto\langle g,N\cdot\nabla\phi\rangle_{\partial\Omega,T}]$ and $U_0=[\phi\mapsto\langle u_0,\phi(0)\rangle_\Omega]$, we get the estimate (4.5) from the linear case. More precisely let $T''\in(0,T']$ with $T''<\infty$ and choose ρ as in the estimates (4.8), (4.11), (4.13). Then we obtain by the a priori estimate of solutions to the instationary Stokes equation in $L^{\frac{r}{2}}(0,T;H^{\beta,q}_w(\Omega))$ and Hölder's inequality in the case $\beta<1$

$$\begin{aligned}&\|\partial_t u|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_{w,\sigma}(\Omega))}\\ &\leq c\Big(\|f\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_w(\Omega))}+\|W(u)\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_{w,\sigma}(\Omega))}\\ &\quad+\|k\|_{H_0^{\frac{\beta}{2},\frac{r}{2}}((0,T];W^{-1,q}_{w,0})\cap L^{\frac{r}{2}}(H^{\beta-1,q}_{w,*})}+\|g\|_{H_0^{\frac{\beta}{2},\frac{r}{2}}((0,T];T^{0,q}_w)\cap L^{\frac{r}{2}}(T^{\beta,q}_w)}+\|u_0\|_{\mathcal{I}^{\beta,q,\frac{r}{2}}_w}\Big)\\ &\leq c(T'')\Big(\|f\|_{L^r(0,T'';Y^{\beta-2,q}_w(\Omega))}+c(\eta+\delta)\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ &\quad+\|k\|_{H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0})\cap L^r(H^{\beta-1,q}_{w,*})}+\|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w)\cap L^r(T^{\beta,q}_w)}+\|u_0\|_{\mathcal{I}^{\beta,q,r}_w}\Big).\end{aligned}$$

If $\beta \in [1,2]$ one estimates analogously. Then the estimate for u proves (4.5).

Note that the equation (4.14) is only tested with functions in $Y^{2,q'}_{w',\sigma}(\Omega)$ and only holds in this sense. Thus the distributional derivative $\partial_t u$ may contain a gradient part which is not a function in time.

Uniqueness can be proved in the same way as in [6]: Let $v \in L^r(0,T';H^{\beta,q}_w(\Omega))$ be a very weak solution corresponding to the same data f,k,g and u_0. Then $U := u - v$ solves

$$\begin{aligned} &\partial_t U - \Delta U + \nabla P = -\mathrm{div}\,(Uu) - \mathrm{div}\,(vU) + kU,\\ &\mathrm{div}\,U = 0, \quad U|_{\partial\Omega} = 0, \quad U(0) = 0 \end{aligned}$$

in the very weak sense. Then for $\beta < 1$ one obtains as above

$$\begin{aligned} \|U\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq & c\Big(\|u\|_{L^r(0,T';H^{\beta,q}_w)} + \|v\|_{L^r(0,T';H^{\beta,q}_w)} \\ & + \|k\|_{L^r(0,T';L^{\mu}_w(\Omega))}\Big)\,\|U\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \end{aligned}$$

with a constant c that is independent of T'. A corresponding inequality holds in the case $\beta \geq 1$. In particular it holds for T' replaced by any $T''' \in (0,T']$. If T''' is sufficiently small such that

$$\|u\|_{L^r(0,T''';H^{\beta,q}_w(\Omega))} + \|v\|_{L^r(0,T''';H^{\beta,q}_w(\Omega))} + \|k\|_{L^r(0,T''';L^{\mu}_w(\Omega))} < \frac{1}{2c},$$

we obtain $\|U\|_{L^r(0,T''';H^{\beta,q}_w(\Omega))} \leq 0$ or $U = 0$ on $[0,T''')$. If $T''' < T'$ we assume that T''' is maximal with the property $u = v$ on $[0,T''')$. However, then we may repeat this procedure and obtain $u = v$ on a bigger interval. This is a contradiction. Thus u is unique in $L^r(0,T';H^{\beta,q}_w(\Omega))$ and the proof is complete. $\square$

Remark 4.5. Choose β, r, q according to Theorem 4.4.

We now prove that in this case the solution $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ fulfills Serrin's condition [23] in the sense that $u \in L^r(0,T',L^{\eta}(\Omega))$, where $\frac{1}{r} + \frac{n}{\eta} < 1$.

If $ns - q\beta > 0$ then for the number ρ that fulfills $\frac{1}{2\rho} = \frac{1}{q} - \frac{\beta}{ns}$ one has by Lemma 2.2 and (2.3)

$$H^{\beta,q}_w(\Omega) \hookrightarrow L^{2\rho}_w(\Omega) \hookrightarrow L^{\frac{2\rho}{s}}(\Omega)$$

and

$$\frac{2}{r} + \frac{n}{\frac{2\rho}{s}} < -\frac{ns}{q} + \beta + 1 + \frac{ns - q\beta}{q} = 1.$$

If $ns - q\beta \leq 0$ then $H^{\beta,q}_w(\Omega) \hookrightarrow L^{\eta}(\Omega)$ for every $\eta \in (1,\infty)$. Since $r > 2$, this η can be chosen such that $\frac{2}{r} + \frac{n}{\eta} < 1$.

The reason why there appears "$<$" instead of "$\leq$" as in the unweighted case [23], [8], is that the boundedness of imaginary powers is not proved for the Stokes operator in spaces weighted with arbitrary Muckenhoupt weights. Thus we have to work without an exact characterization of the domains of fractional powers of the Stokes operator $\mathcal{D}(\mathcal{A}^{\alpha})$ and use the embedding (4.2) instead.

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