# The Instationary Stokes Equations in Weighted Bessel-Potential Spaces

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We investigate the solvability of the instationary Stokes equations with fully inhomogeneous data in $L^r(0,T;H^{\beta,q}_w(\Omega))$, where $H^{\beta,q}_w(\Omega)$ is a Bessel-Potential space with a Muckenhoupt weight $w$. Depending on the order of this Bessel-Potential space we are dealing with strong solutions or with very weak solutions. Whereas in the context of lowest regularity one obtains solvability with respect to inhomogeneous data by dualization, this is more delicate in the case of higher regularity, where one has to introduce some additional time regularity. As a preparation, we introduce a generalization of the Stokes operator that is appropriate to the context of very weak solutions in weighted Bessel-Potential spaces.

*Key Words and Phrases:* Instationary Stokes equations, Muckenhoupt weights, very weak solutions, Bessel Potential spaces, nonhomgeneous data
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## 1 Introduction

We consider the instationary Stokes equations with fully inhomogeneous data on some time interval $[0,T)$, $0 < T \leq \infty$, and in a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^{1,1}$,

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= F \quad \text{in } (0,T)\times\Omega, \\ \operatorname{div} u &= K \quad \text{in } (0,T)\times\Omega, \\ u &= g \quad \text{on } (0,T)\times\partial\Omega, \\ u(0) &= u_0 \quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

It is our aim to find a class of solutions to (1.1) in $L^r(0,T;H^{\beta,q}(\Omega))$ where $H^{\beta,q}(\Omega)$ is a Bessel-potential space for $\beta \in [0,2]$. This means we develop a solution theory that includes strong solutions in the case $\beta = 2$ and weak solutions in the case $\beta = 1$. However, if $\beta = 0$, it is also possible that the solutions are only contained in $L^r(0,T;L^q(\Omega))$, i.e., they do not possess any weak derivatives. Consequently the notion of weak solutions

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is no longer suitable in this context. Thus one introduces the more general notion of very weak solutions. To arrive there one multiplies the first equation in (1.1) with a test function $\phi$, solenoidal in space and vanishing on the boundary and at time $T$, then formal integration by parts yields

$$-\langle u, \partial_t \phi\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} = \langle F, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T}. \tag{1.2}$$

Applying the same method to the second equation with a sufficiently smooth test function $\psi$ we obtain

$$-\langle u(t), \nabla\psi\rangle = \langle K(t), \psi\rangle - \langle g(t), N\cdot\psi\rangle_{\partial\Omega} \tag{1.3}$$

for almost every $t$. The equations (1.2) and (1.3) can be used for the definition of very weak solutions. A similar formulation has been introduced by Amann in [3] in the case of the Navier-Stokes equations. In this article as well as by Farwig, Galdi and Sohr in [8], [9] and by Farwig, Kozono and Sohr [10] solvability with low-regularity data has been shown.

We investigate this problem in function spaces weighted in the space variable. More precisely, we consider Lebesgue-, Sobolev- and Bessel potential spaces with respect to the measure $w\,dx$, where $w$ is a weight function contained in the Muckenhoupt class $A_q$, cf., (2.1) below.

Classical tools for the treatment of partial differential equations extend to function spaces with Muckenhoupt weights. As important examples we mention the continuity of the maximal operator and the multiplier theorems that can be found in the books of García-Cuerva and Rubio de Francia [18] and Stein [25]; extension theorems of functions on a domain to functions on $\mathbb{R}^n$ have been shown by Chua [5], extension theorems of functions on the boundary to functions on the domain by Fröhlich [15], see also [20].

These tools were the base to treat the solvability of the Stokes and Navier-Stokes equations in weighted function spaces by Farwig and Sohr in [11] and by Fröhlich in [13], [14], [15]. Using particular weight functions this theory may be used for a better dscription of the solution, e.g. close to the boundary or in the neighborhood of a point. However the mathematical significance of Muckenhoupt weights is given by the Extrapolation Theorem [18, IV Lemma 5.18]. An even more powerful extrapolation theorem by Curbera, García-Cuerva, Martell and Pérez [7] guarantees estimates in very general Banach function spaces provided that the estimates in weighted function spaces are known for all weights from the Muckenhoupt class $A_q$. Moreover, this property may be used to derive the $R$-boundedness of families of operators from their uniform boundedness in weighted function spaces. This fact was used by Fröhlich [14] to give a new proof of the maximal regularity of the Stokes operator in $L^q$; in this paper this is the crucial method in Section 4.3.

The outline of this paper is as follows. In Section 3 we introduce a generalization of the Stokes operator that is appropriate to the context of very weak solutions in weighted Bessel-Potential spaces. As in the classical case this generalized Stokes operator generates an analytic semigroup and has maximal regularity.

The crucial method in Section 4.1 is dualization. Based on the existence and uniqueness of strong solutions in [14] we obtain the solvability in the lowest regularity context treated in the paper where the solutions are merely contained in $L^r(0,T;L^q_w(\Omega))$. This dualization procedure automatically yields solutions with respect to inhomogeneous boundary values and divergences as shown in Theorem 4.3. The boundary conditions are included implicitly in the inhomogeneous force and divergence, since they may contain a part that is concentrated on the boundary.

In the Sections 4.3 and 4.4 we are looking for solutions with respect to higher regularity. In this case, the inhomogeneous boundary condition and divergence complicates the situation strongly. In particular, one needs some additional time-regularity depending on the order of the Bessel potential space we are working in and a more complex theory is required. The component of the boundary condition that is tangential to the boundary is treated by means of an operator-valued Fourier Multiplier Theorem by Weis [29]. We obtain the solution to an inhomogeneous force by interpolation between the very weak and the strong solution. The initial condition is represented by the semigroup generated by the generalized Stokes operator. The divergence and the normal part of the boundary condition can be realized by a gradient. If we put all these parts together we can prove our main result Theorem 4.17.

# 2 Preliminaries

## 2.1 Weighted Function Spaces

Let $A_q$, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all $0 \le w \in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left( \frac{1}{|Q|} \int_Q w \, dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} dx \right)^{q-1} < \infty. \tag{2.1}$$

The supremum is taken over all cubes $Q$ in $\mathbb{R}^n$. To avoid trivial cases, we exclude the case where $w$ vanishes almost everywhere.

A constant $C = C(w)$ is called $A_q$-consistent if for every $c_0 > 0$ it can be chosen uniformly for all $w \in A_q$ with $A_q(w) < c_0$. The $A_q$-consistence is of great importance since it is needed for the application of the Extrapolation Theorem [18, IV Lemma 5.18]. In particular this is used when showing the continuity of operator-valued Fourier multipliers and the maximal regularity of an operator; see e.g. [15] for details and applications, in this paper we make use of this method in Section 4.3.

Let $k \in \mathbb{N}_0$, $q \in (1, \infty)$, $w \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then we define the following weighted versions of Lebesgue and Sobolev spaces.

- $L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left( \int_\Omega |f|^q w \, dx \right)^{\frac{1}{q}} < \infty \right\}$.

  It is an easy consequence of the corresponding result in the unweighted case that

  $$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \;\text{ with }\; \frac{1}{q} + \frac{1}{q'} = 1 \;\text{ and }\; w' = w^{-\frac{1}{q-1}} \in A_{q'}. \tag{2.2}$$

- Set $W^{k,q}_w(\Omega) = \left\{ u \in L^q_w(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha| \le k} \|D^\alpha u\|_{q,w} < \infty \right\}$.

- By $C_0^\infty(\Omega)$ we denote the set of all smooth and compactly supported functions, the space $C^\infty_{0,\sigma}(\Omega)$ consists of all functions that are in addition divergence free.

- Moreover we set $W^{k,q}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,q,w}}$. The dual space of it is denoted by $W^{-k,q}_w(\Omega) := (W^{k,q'}_{w',0}(\Omega))'$. We also consider the divergence-free versions $W^{k,q}_{w,0,\sigma}(\Omega) := \{\phi \in W^{k,q}_{w,0}(\Omega) \mid \operatorname{div}\phi = 0\}$ and $L^q_{w,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{L^q_w(\Omega)}$.

- Using this for $k > 0$ we set $W^{-k,q}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{-k,q}_w(\mathbb{R}^n)}}$.

- Moreover, we consider the spaces of boundary values $T^{k,q}_w(\partial\Omega) := (W^{k,q}_w(\Omega))|_{\partial\Omega}$, equipped with the norm $\|\cdot\|_{T^{k,q}_w} = \|\cdot\|_{T^{k,q}_w(\partial\Omega)}$ of the factor space and finally $T^{0,q}_w(\partial\Omega) := (T^{1,q'}_{w'}(\partial\Omega))'$.

By [13], [15] and [5] the spaces $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$, $W^{k,q}_{w,0}(\Omega)$ and $T^{k,q}_w(\partial\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$, ($C_0^\infty(\Omega)$, $C^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively) are dense.

**Theorem 2.1. (Hörmander-Michlin Multiplier Theorem with Weights)** *Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ fulfill the property*

$$|\partial^\alpha m(\xi)| \le K|\xi|^{-|\alpha|}, \quad \text{for every } \ \xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| = 0, 1, ..., n,$$

*for some constant $K > 0$. Then $T$ defined by*

$$\widehat{Tf} = m\hat{f} \quad \text{for} \quad f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$$

*extends to a continuous operator on $L^q_w(\Omega)$ for every $q \in (1, \infty)$ and $w \in A_q$.*

*More precisely there exists an $A_q$-consistent $c$ such that $\|Tf\|_{q,w} \le c\|f\|_{q,w}$ for every $f \in L^q_w(\Omega)$.*

*Proof.* This is an immediate consequence of [18], Theorem 3.9. The same proof can be used to show the $A_q$-consistence of the continuity constant. □

## 2.2 Weighted Bessel-Potential Spaces

For $\xi \in \mathbb{R}^n$ we set $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. On the space $\mathcal{S}'(\mathbb{R}^n;\mathbb{R})$ of temperate distributions we define for all $\beta \in \mathbb{R}$ the operator

$$\Lambda^\beta f = \mathcal{F}^{-1}\langle\xi\rangle^\beta \mathcal{F}f, \quad f \in S'(\mathbb{R}^n;\mathbb{R}),$$

where $\mathcal{F}$ stands for the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n;\mathbb{R})$. Then for $1 < q < \infty$, $w \in A_q$ and $\beta \in \mathbb{R}$ the weighted Bessel potential space is given by

$$H^{\beta,q}_w(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n;\mathbb{R}) \;\middle|\; \|f\|_{H^{\beta,q}_w(\mathbb{R}^n)} := \|\Lambda^\beta f\|_{q,w,\mathbb{R}^n} < \infty \right\}.$$

**Theorem 2.2.** *If $1 < q < \infty$, $w \in A_q$, $l, k \in \mathbb{Z}$ and $l < \beta < k$ then one has for the complex interpolation spaces*

$$\left[H^{l,q}_w(\mathbb{R}^n), H^{k,q}_w(\mathbb{R}^n)\right]_\theta = H^{\beta,q}_w(\mathbb{R}^n),$$

*where $\theta = \frac{\beta - l}{k - l}$. The norms are equivalent with $A_q$-consistent equivalence constants.*

*Proof.* This can be proven analogously to [26, Proposition 13.6.2]. For the weighted version in the case $l = 0$ and $k \in \mathbb{N}$ see also [13, Satz 8.3]. The proof given there can be repeated to obtain the more general assertion of this theorem. It is based on the boundedness of the purely imaginary powers $\Lambda^{iy}$ in $L^q_w(\mathbb{R}^n)$ which is a consequence of the weighted Multiplier Theorem 2.1. Thus rereading the proof one also obtains the $A_q$-consistence of the constants. $\square$

**Corollary 2.3.** *For $q \in (1,\infty)$ and $w \in A_q$ one has $W^{1,q}_w(\mathbb{R}^n) = H^{1,q}_w(\mathbb{R}^n)$. If in addition $\beta \in [0,1]$ then*

$$\Lambda^\beta : [L^q_w(\mathbb{R}^n), W^{1,q}_w(\mathbb{R}^n)]_\beta \to L^q_w(\mathbb{R}^n)$$

*is continuous. The equivalence and continuity constants are $A_q$-consistent.*

*Proof.* $W^{1,q}_w(\mathbb{R}^n) = H^{1,q}_w(\mathbb{R}^n)$ follows from the Multiplier Theorem 2.1. In praticular the equivalence constants are $A_q$-consistent. Thus

$$\|\Lambda^\beta u\|_{L^q_w(\mathbb{R}^n)} = \|u\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{[L^q_w(\mathbb{R}^n),H^{1,q}_w(\mathbb{R}^n)]_\beta} \le c\|u\|_{[L^q_w(\mathbb{R}^n),W^{1,q}_w(\mathbb{R}^n)]_\beta},$$

where $c > 0$ is $A_q$-consistent. $\square$

We call a domain $\Omega$ an extension domain if for every $k \in \mathbb{N}$ and $q \in (1,\infty)$ there exists an extension operator

$$E : W^{j,q}_w(\Omega) \to W^{j,q}_w(\mathbb{R}^n)$$

that is continuous for $j = 0, ..., k$. By [5] in particular bounded Lipschitz domains are extension domains.

For an extension domain $\Omega$ we define the weighted Bessel potential space on $\Omega$ by

$$H^{\beta,q}_w(\Omega) = \left\{ g|_\Omega \mid g \in H^{\beta,q}_w(\mathbb{R}^n) \right\}$$

equipped with the norm $\|u\|_{H^{\beta,q}_w(\Omega)} := \inf \left\{ \|U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \mid U \in H^{\beta,q}_w(\mathbb{R}^n),\ U|_\Omega = u \right\}$. Note that if $\beta < 0$ then the restriction $g|_\Omega$ has to be understood in the sense of distributions as $g|_{C^\infty_0(\Omega)}$.

Moreover, we set $H^{\beta,q}_{w,0}(\Omega) = \overline{(C^\infty_0(\Omega))}^{H^{\beta,q}_w(\mathbb{R}^n)}$ for $\beta \in \mathbb{R}$, equipped with the norm $\|\cdot\|_{\beta,q,w,0,\Omega} := \|E_0(\cdot)\|_{\beta,q,w,\mathbb{R}^n}$, where $E_0$ denotes the extension of a function by 0 to the whole space $\mathbb{R}^n$. The space $H^{\beta,q}_{w,0}(\Omega)$ is a reflexive Banach space and it is easy to verify (see e.g. [23]) that $H^{\beta,q}_w(\Omega) = (H^{-\beta,q'}_{w',0}(\Omega))'$ for every $\beta \in \mathbb{R}$.

**Theorem 2.4.** *Let $\Omega$ be an extension domain, $1 < q < \infty$, $w \in A_q$.*

1. *For $k \in \mathbb{N}_0$ one has $H^{k,q}_w(\Omega) = W^{k,q}_w(\Omega)$ and $H^{k,q}_{w,0}(\Omega) = W^{k,q}_{w,0}(\Omega)$ with equivalent norms.*

2. *For $k \in \mathbb{N}$, $0 < \beta < k$ one has $H^{\beta,q}_w(\Omega) = [L^q_w(\Omega), W^{k,q}_w(\Omega)]_{\frac{\beta}{k}}$.*

*Proof.* [16] $\square$

For spaces of boundary values we consider the spaces

$$T_w^{\beta,q}(\Omega) := \begin{cases} H_w^{\beta,q}(\Omega) & \text{for } \beta \in [1,2] \\ [T_w^{0,q}(\Omega), T_w^{1,q}(\Omega)]_\beta & \text{for } \beta \in [0,1). \end{cases}$$

As spaces for our solutions we need spaces of functions that vanish on the boundary. Thus for an extension domain $\Omega \subset \mathbb{R}^n$, $1 < q < \infty$, $w \in A_q$ we set $Y_w^{2,q}(\Omega) := \{u \in W_w^{2,q}(\Omega) \mid u|_{\partial\Omega} = 0\}$. For $0 \le \beta \le 2$ we define the space

$$Y_w^{\beta,q}(\Omega) := \begin{cases} \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\mathbb{R}^n)}, & \text{if } 0 \le \beta \le 1 \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\mathbb{R}^n)}, \\ \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\Omega)}, & \text{if } 1 < \beta \le 2 \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\Omega)}, \end{cases}$$

where in the case $0 \le \beta \le 1$ the functions of $Y_w^{2,q}(\Omega)$ are assumed to be extended by 0 to functions defined on the whole space $\mathbb{R}^n$. This is possible, since $C_0^\infty(\Omega)$ is dense in $W_{w,0}^{1,q}(\Omega) \supset Y_w^{2,q}(\Omega)$ and $W_{w,0}^{1,q}(\Omega) \hookrightarrow W_w^{1,q}(\mathbb{R}^n) \hookrightarrow H_w^{\beta,q}(\mathbb{R}^n)$. We also consider the dual spaces $Y_w^{-\beta,q}(\Omega) := (Y_{w'}^{\beta,q'}(\Omega))'$.

We define the divergence free version of $Y_w^{\beta,q}(\Omega)$ by

$$Y_{w,\sigma}^{\beta,q}(\Omega) := \left\{ u \in Y_w^{\beta,q}(\Omega) \mid \langle u, \nabla\phi \rangle = 0 \text{ for every } \phi \in C^\infty(\overline{\Omega}) \right\}.$$

By Theorem 2.5 and (3.2) below one has $Y_{w,\sigma}^{1,q}(\Omega) = W_{w,0,\sigma}^{1,q}(\Omega)$ and $Y_{w,\sigma}^{0,q}(\Omega) = L_{w,\sigma}^q(\Omega)$.

We also consider the dual spaces $Y_{w,\sigma}^{-\beta,q}(\Omega) := \left(Y_{w',\sigma}^{\beta,q'}(\Omega)\right)'$. By the Hahn-Banach theorem the space $Y_{w,\sigma}^{-\beta,q}(\Omega)$ is the restriction of all elements of $Y_w^{-\beta,q}(\Omega)$ to $Y_{w',\sigma}^{\beta,q'}(\Omega)$.

See [23] for further properties and discussions about these spaces. In particular there have been proved the following interpolation properties.

**Theorem 2.5.** *If $\Omega$ is a bounded $C^{1,1}$-domain then one has*

$$\left[L_w^q(\Omega), Y_w^{2,q}(\Omega)\right]_\theta = Y_w^{\beta,q}(\Omega), \quad \theta = \frac{\beta}{2}, \quad 0 \le \beta \le 2$$

*with equivalent norms.*

Now we prove two technical Lemmas that are needed in Section 4.3.

**Lemma 2.6.** *Let $\Omega$ be a bounded $C^{1,1}$-domain. Then the norm in $W_w^{1,q}(\Omega)$ is equivalent to the one in $[L_w^q(\Omega), W_w^{2,q}(\Omega)]_{\frac{1}{2}}$ with an equivalence constant depending $A_q$-consistently on $w$.*

*Proof.* We start defining an extension operator $E_{\mathbb{R}_+^n}$ by

$$E_{\mathbb{R}_+^n} u(x) = \begin{cases} u(x) & \text{for } x_n > 0 \\ \sum_{j=1}^3 \lambda_j u(x', -jx_n) & \text{for } x_n < 0, \end{cases}$$

where $\lambda_j$, $j = 1, ..., 3$ is chosen such that $\sum_{j=1}^3 \lambda_j (-j)^l = 1$ for $l = 0, ..., 3$. Then one shows as in the unweighed case [1] that

$$E_{\mathbb{R}_+^n} : W_w^{k,q}(\mathbb{R}_+^n) \to W_{\tilde{w}}^{k,q}(\mathbb{R}^n), \qquad k = 0, 1, 2,$$

is continuous where $\tilde{w}$ is given by

$$\tilde{w} = \begin{cases} w(x', x_n) & \text{if } x_n > 0 \\ \min_{j=1,...,3} w(x', -jx_n) & \text{if } x_n < 0. \end{cases}$$

The continuity constant of $E_{\mathbb{R}^n_+}$ and $A_q(\tilde{w})$ depend $A_q$-consistently on $w$.

Take an open covering $(U_j)_{j=1}^m$ of $\overline{\Omega}$, a collection of charts $(\alpha_j)_{j=1}^m$, $\alpha_j : V_j \to U_j$, and a partition of unity $(\phi_j)_{j=1}^m$ subordinate to the covering $(U_j)_j$. Assume that each $\alpha_j$ is extended to a $C^{1,1}$-diffeomorphism on $\mathbb{R}^n$. Moreover, let

$$E_{\mathbb{R}^n_+,j} : W^{2,q}_{w\circ\alpha_j}(\mathbb{R}^n_+) \to W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)$$

be the extension operator defined above. We define the mapping

$$\begin{aligned} P : \prod_{j=1}^m W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n) &\to W^{2,q}_w(\Omega), \\ (u_1, ..., u_m) &\mapsto \sum_{j=1}^m \psi_j R_\Omega(u_j \circ \alpha_j^{-1}), \end{aligned}$$

where $\psi_j \in C_0^\infty(U_j)$ with $\psi_j \equiv 1$ on $\operatorname{supp} \phi_j$ and $R_\Omega$ denotes the restriction of functions defined on $\mathbb{R}^n$ to $\Omega$. Note that $\widetilde{w \circ \alpha_j} \circ \alpha_j^{-1} = w$ on $U_j \cap \Omega \supset \operatorname{supp} \psi_j$. Set

$$\begin{aligned} I : W^{2,q}_w(\Omega) &\to \prod_{j=1}^m W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n) \\ u &\mapsto \left(E_{\mathbb{R}^n_+,1}((\phi_1 u) \circ \alpha_1), ..., E_{\mathbb{R}^n_+,m}((\phi_m u) \circ \alpha_m)\right). \end{aligned}$$

Since multiplication and concatenation with sufficiently smooth functions is continuous between weighted Sobolev spaces, $P$ and $I$ are continuous with $A_q$-consistent continuity constants also if they are considered as operators

$$P : \prod_{j=1}^m L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n) \to L^q_w(\Omega) \quad \text{and} \quad I : L^q_w(\Omega) \to \prod_{j=1}^m L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n).$$

Moreover, for $u \in L^q_w(\Omega)$ one has

$$PIu = \sum_{j=1}^m \psi_j R_\Omega(E_{\mathbb{R}^n_+,j}((\phi_j u) \circ \alpha_j) \circ \alpha_j^{-1}) = \sum_{j=1}^m \psi_j \phi_j u = u.$$

Thus, the retraction principle of interpolation [4] together with the assertion for $\Omega = \mathbb{R}^n$ in Crollary 2.3 yields

$$\begin{aligned} [L^q_w(\Omega), W^{2,q}_w(\Omega)]_{\frac{1}{2}} &= P\left[\prod_{j=1}^m L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n), \prod_{j=1}^m W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)\right]_{\frac{1}{2}} \\ &= P\left(\prod_{j=1}^m W^{1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)\right) = W^{1,q}_w(\Omega). \end{aligned}$$

The constants are $A_q$ consistent since so are the constants of $P$ and $I$. $\square$

**Lemma 2.7.** *Let $\Omega = \mathbb{R}^n$ or a bounded $C^{1,1}$-domain and let $\beta \in [1,2]$. Then for every $u \in H^{\beta,q}_w(\Omega)$ one has the estimate*

$$\|u\|_{H^{\beta,q}_w(\Omega)} \le c\left(\|u\|_{H^{\beta-1,q}_w(\Omega)} + \|\nabla u\|_{H^{\beta-1,q}_w(\Omega)}\right),$$

*where $c = c(\beta, q, w, \Omega)$.*

*Proof.* In $\mathbb{R}^n$ the inequality follows from the Multiplier Theorem 2.1.

Let $E_{\mathbb{R}^n_+} : W^{k,q}_w(\mathbb{R}^n_+) \to W^{k,q}_{\tilde{w}}(\mathbb{R}^n)$, $k = 0,1,2$ be the operator constructed in the proof of Lemma 2.6. Analogously, one shows for $\mathbb{R}^n$-valued functions that the extension operator

$$\tilde{E}_{\mathbb{R}^n_+} : v(x) = (v', v_n)(x', x_n) \mapsto \begin{cases} v(x', x_n) & \text{on } \mathbb{R}^n_+ \\ \begin{pmatrix} E_{\mathbb{R}^n_+}(v')(x', x_n) \\ \sum_{j=1}^3 \lambda_j(-j) v_n(x', -jx_n) \end{pmatrix} & \text{on } \mathbb{R}^n_- \end{cases}$$

is continuous as an operator $\tilde{E}_{\mathbb{R}^n_+} : W^{k,q}_w(\mathbb{R}^n_+) \to W^{k,q}_{\tilde{w}}(\mathbb{R}^n)$, $k = 0,1$. Interpolation shows that

$$\tilde{E}_{\mathbb{R}^n_+} : H^{\beta-1,q}_{\tilde{w}}(\mathbb{R}^n_+) \to H^{\beta-1,q}_{\tilde{w}}(\mathbb{R}^n),$$

and by construction one has $\nabla E_{\mathbb{R}^n_+} = \tilde{E}_{\mathbb{R}^n_+} \nabla$.

To prove the result for a bounded domain $\Omega$ let $(\alpha_j)_{j=1}^m$ be a collection of charts and $(\psi_j)_{j=1}^m$ a decomposition of unity subordinate to the corresponding covering of $\Omega$. Then we can calculate using the retraction principle of interpolation

$$\begin{aligned} \|u\|_{[W^{1,q}_w(\Omega), W^{2,q}_w(\Omega)]_{\beta-1}} &\le c\sum_{j=1}^m \|\psi_j u\|_{H^{\beta,q}_w(\Omega)} \le c\sum_{j=1}^m \|\psi_j u\|_{H^{\beta,q}_w(H_{\alpha_j})} \\ &\le c\sum_{j=1}^m \|(\psi_j u)\circ\alpha_j\|_{H^{\beta,q}_{w\circ\alpha_j}(\mathbb{R}^n_+)} \le c\sum_{j=1}^m \|E_{\mathbb{R}^n_+}((\psi_j u)\circ\alpha_j)\|_{H^{\beta,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)}, \end{aligned}$$

where by $H_{\alpha_j}$ we denote the bent half spaces with boundary $\alpha_j(\mathbb{R}^{n-1}\times\{0\})$. Using the result in the whole space case and $\widetilde{w\circ\alpha_j} = w$ on $\operatorname{supp}\psi_j \cap \Omega$ we obtain

$$\begin{aligned} \|u\|_{H^{\beta,q}_w(\Omega)} &\le c\|u\|_{[W^{1,q}_w(\Omega), W^{2,q}_w(\Omega)]_{\beta-1}} \\ &\le c\sum_{j=1}^m \left(\|E_{\mathbb{R}^n_+}((\psi_j u)\circ\alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)} + \|\nabla E_{\mathbb{R}^n_+}((\psi_j u)\circ\alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)}\right) \\ &\le c\sum_{j=1}^m \left(\|E_{\mathbb{R}^n_+}((\psi_j u)\circ\alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)} + \|\tilde{E}_{\mathbb{R}^n_+}\nabla((\psi_j u)\circ\alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)}\right) \\ &\le c(\|u\|_{H^{\beta-1,q}_w(\Omega)} + \|\nabla u\|_{H^{\beta-1,q}_w(\Omega)}). \end{aligned}$$

This is the asserted estimate. □

**Lemma 2.8.** *Let $-1 \le \beta \le 1$. Let $p \in (C_0^\infty(\Omega))'$ with $\nabla p \in H^{\beta-1,q}_w(\Omega)$. Then $p \in H^{\beta,q}_w(\Omega)$ and there exists a constant $c = c(\Omega, q, w)$ such that*

$$\|p\|_{H^{\beta,q}_w/const.} \le c\|\nabla p\|_{H^{\beta-1,q}_w}.$$

*Proof.* [23] □

## 2.3 The Stationary Stokes Equations in Bessel-Potential Spaces

**Definition 2.9.** Let $f \in Y_w^{-2,q}(\Omega)$ and $k \in W_{w,0}^{-1,q}(\Omega)$. A function $u \in L_w^q(\Omega)$ is called a very weak solution to the stationary Stokes problem with respect to the data $f$ and $k$, if

$$-\langle u, \Delta\phi\rangle = \langle f, \phi\rangle, \qquad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \text{ and} \tag{2.3}$$

$$-\langle u, \nabla\psi\rangle = \langle k, \psi\rangle, \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). \tag{2.4}$$

The existence and uniqueness of very weak solutions in $L_w^q(\Omega)$ has been shown in [24]. In general the regularity of very weak solutions is not sufficient to ensure that the restriction $u|_{\partial\Omega}$ is well defined. However, if we restrict ourselves to a certain class of data then a good definition of boundary values is again possible. More precisely the following theorem has been shown in [24] where one can also find further details and discussions.

**Theorem 2.10.** *Assume that $f \in Y_w^{-2,q}(\Omega)$ and $k \in W_{w,0}^{-1,q}(\Omega)$ allow a decomposition into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for all } \phi \in Y_{w'}^{2,q'}(\Omega), \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega} && \text{for all } \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned} \tag{2.5}$$

*with $g \in T_w^{0,q}(\partial\Omega), F \in W_{\tilde{w}}^{-1,r}(\Omega), K \in L_{\tilde{w}}^r(\Omega)$, where $1 < r < \infty$ and $\tilde{w} \in A_r$ are chosen such that $W_{w'}^{1,q'}(\Omega) \hookrightarrow L_{\tilde{w}'}^{r'}(\Omega) \hookrightarrow L_{w'}^{q'}(\Omega)$. Let $u$ be a very weak solution to the Stokes problem corresponding to the data $f$ and $k$. Then*

$$u \in \tilde{W}_{w,\tilde{w}}^{q,r} := \big\{u \in L_w^q(\Omega) \;\big|\; \exists c > 0,\ |\langle u, \Delta\phi\rangle| \le c\|\phi\|_{1,r',\tilde{w}'}\ \forall\phi \in C_{0,\sigma}^\infty(\Omega)\big\}.$$

*There exists an operator $\gamma : \tilde{W}_{w,\tilde{w}}^{q,r} \to T_w^{0,q}(\partial\Omega)$ that coincides with the tangential trace on $W_w^{1,q}(\Omega)$. The fact that $\operatorname{div} u = K \in L_{\tilde{w}}^r(\Omega)$ permits to define the normal component of the trace $N\cdot u|_{\partial\Omega}$. In this sense $u|_{\partial\Omega}$ is well-defined and $u|_{\partial\Omega} = g$.*

We now turn to the stationary Stokes equations in weighted Bessel-Potential spaces. As a space for the divergence we define

$$H_{w,*}^{\beta,q}(\Omega) = \begin{cases} H_{w,0}^{\beta,q}(\Omega) = (H_{w'}^{-\beta,q'}(\Omega))' & \text{if } \beta < 0, \\ H_w^{\beta,q}(\Omega) & \text{if } \beta \ge 0. \end{cases} \tag{2.6}$$

By [23] one has the interpolation property

$$\big[W_{w,0}^{-1,q}(\Omega), W_w^{1,q}(\Omega)\big]_{\frac{1+\beta}{2}} = H_{w,*}^{\beta,q}(\Omega) \ \text{ for } -1 \le \beta \le 1$$

and the following theorem.

**Theorem 2.11.** *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$. Assume that $f \in Y_w^{-2,q}(\Omega)$ and $k \in H_{w,0}^{-1,q}(\Omega)$ allow decompositions into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for every } \phi \in Y_{w'}^{2,q'}(\Omega) \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega} && \text{for every } \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned} \tag{2.7}$$

*with $F \in Y_w^{\beta-2,q}(\Omega)$, $K \in H_{w,*}^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$. Assume in addition that $K$ and $g$ fulfill the compatibility condition $\langle K, 1\rangle_\Omega = \langle g, N\rangle_{\partial\Omega}$.*

*Then there exists a unique very weak solution $u \in L_w^q(\Omega)$ with respect to $f$ and $k$. It is contained in $H_w^{\beta,q}(\Omega)$ and fulfills the estimate*

$$\|u\|_{\beta,q,w} \leq c\left(\|F\|_{Y_w^{\beta-2,q}(\Omega)} + \|K\|_{H_{w,0}^{\beta-1,q}(\Omega)} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right). \tag{2.8}$$

# 3 The Generalized Stokes Operator

For this section we always assume that $q \in (1,\infty)$, $w \in A_q$ and $\beta \in [0,2]$.

**Proposition 3.1.** *If $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-domain, then $\left[L_{w,\sigma}^q(\Omega), Y_{w,\sigma}^{2,q}(\Omega)\right]_\theta = Y_{w,\sigma}^{\beta,q}(\Omega)$ and $\left[L_{w,\sigma}^q(\Omega), Y_{w,\sigma}^{-2,q}(\Omega)\right]_\theta = Y_{w,\sigma}^{-\beta,q}(\Omega)$, where $\theta = \frac{\beta}{2}$ with equivalent norms.*

*Proof.* From Theorem 2.11 we obtain that the operator $\mathcal{S} : Y_w^{\beta-2,q}(\Omega) \to Y_{w,\sigma}^{\beta,q}(\Omega)$, defined by

$$\begin{aligned} \langle f, \varphi\rangle &= -\langle \mathcal{S}f, \Delta\varphi\rangle \qquad \text{for all } \varphi \in Y_{w',\sigma}^{2,q'}(\Omega) \;\text{ and} \\ 0 &= -\langle \mathcal{S}f, \nabla\psi\rangle \qquad \text{for all } \psi \in W_{w'}^{1,q}(\Omega), \end{aligned} \tag{3.1}$$

is continuous. In addition, the operator

$$A : Y_w^{\beta,q}(\Omega) \to Y_w^{\beta-2,q}(\Omega), \quad u \mapsto [\phi \mapsto \langle u, \Delta\phi\rangle] \in Y_w^{\beta-2,q}(\Omega)$$

is continuous. For $\beta = 0$ and $\beta = 2$ this is obvious, for $\beta \in (0,2)$ it follows by interpolation from Theorem 2.5.

Moreover $x = \mathcal{S}A|_{Y_{w,\sigma}^{\beta,q}} x$ for every $x \in Y_{w,\sigma}^{\beta,q}(\Omega)$ and it follows from the retraction principle for interpolation spaces [4, Theorem 6.4.2] that

$$\left[L_{w,\sigma}^q(\Omega), Y_{w,\sigma}^{2,q}(\Omega)\right]_\theta = \mathcal{S}\left([Y_w^{-2,q}(\Omega), L_w^q(\Omega)]_\theta\right) = \mathcal{S}\left(Y_w^{\beta-2,q}(\Omega)\right) = Y_{w,\sigma}^{\beta,q}(\Omega).$$

The second assertion follows when considering the dual spaces in the first. ☐

As in the classical unweighted case one defines the Stokes operator

$$\mathcal{A} = \mathcal{A}_{0,q,w} : L_{w,\sigma}^q(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L_{w,\sigma}^q(\Omega), \quad u \mapsto -P_{q,w}\Delta,$$

where $P_{q,w} : L_w^q(\Omega) \to L_w^q(\Omega)$ is the Helmholtz projection that is the projection to the space of divergence free vector fields

$$L_{w,\sigma}^q(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{L_w^q(\Omega)} = \{u \in L_w^q(\Omega) \mid \langle u, \nabla\phi\rangle = 0 \text{ for every } \phi \in W_{w'}^{1,q'}(\Omega)\}. \tag{3.2}$$

The kernel of $P_{q,w}$ is equal to the space of gradients $\{\nabla p \mid p \in W_w^{1,q}(\Omega)\}$. Moreover $(1 - P_{q,w})f = \nabla p$, where $p$ solves the weak Neumann problem

$$\langle \nabla p, \nabla\phi\rangle_\Omega = \langle f, \nabla\phi\rangle_\Omega \;\text{ for every }\; \phi \in W_{w'}^{1,q'}(\Omega). \tag{3.3}$$

All these facts about the Helmholtz projection in weighted spaces have been shown by Fröhlich in [12]. The domain of the Stokes operator is $\mathcal{D}(\mathcal{A}) = Y_{w,\sigma}^{2,q}(\Omega)$. In the weighted context it has been introduced and discussed in [15] and [14].

In the following, we find an analogue to the Stokes operator which is adequate in the context of very weak solutions in the Bessel potential spaces $H_w^{\beta,q}(\Omega)$.

**Theorem 3.2.** *For every $0 \le \beta \le 2$ the Stokes operator $\mathcal{A}$ has an extension to an element of $\mathcal{L}(Y^{\beta,q}_{w,\sigma}(\Omega), Y^{\beta-2,q}_{w,\sigma}(\Omega))$ with the following properties.*

1. *It describes a closed and densely defined linear operator in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ again denoted by $\mathcal{A}$. For $u \in Y^{\beta,q}_{w,\sigma}(\Omega)$ one has*

$$\mathcal{A}u = [Y^{2-\beta,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto -\langle u, \Delta\phi\rangle_\Omega].$$

2. *The resolvent set $\rho(-\mathcal{A})$ contains a sector $\Sigma_\varepsilon \cup \{0\} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| < \frac{\pi}{2} + \varepsilon\}$, $\varepsilon \in (0, \frac{\pi}{2})$, and for $\lambda \in \Sigma_\varepsilon \cup \{0\}$ the operator $\lambda + \mathcal{A}$ is an isomorphism in $\mathcal{L}(Y^{\beta,q}_{w,\sigma}(\Omega), Y^{\beta-2,q}_{w,\sigma}(\Omega))$. The norm of the inverse $\|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}, Y^{\beta,q}_{w,\sigma})}$ is independent of $\lambda \in \Sigma_\delta$ for every $0 < \delta < \varepsilon$.*

3. *For every $0 < \delta < \varepsilon$ there exists a constant $M_\delta$ such that*

$$\|\lambda(\mathcal{A}+\lambda)^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}(\Omega))} \le M_\delta \quad \textit{for all} \quad \lambda \in \Sigma_\delta. \tag{3.4}$$

For $-2 \le \mu \le 0$ let $\mathcal{A}_{\mu,q,w}$ be the extension of the Stokes operator whose existence has been stated in Theorem 3.2. Then we call

$$\mathcal{A}_{\mu,q,w} : \mathcal{D}(\mathcal{A}_{\mu,q,w}) := Y^{\mu+2,q}_{w,\sigma}(\Omega) \subset Y^{\mu,q}_{w,\sigma}(\Omega) \to Y^{\mu,q}_{w,\sigma}(\Omega)$$

the generalized Stokes operator in $Y^{\mu,q}_{w,\sigma}(\Omega)$. If no confusion can occur, we omit the indices and write $\mathcal{A}$ instead of $\mathcal{A}_{\mu,q,w}$.

*Proof.* For $\beta = 2$ one has $Y^{\beta,q}_{w,\sigma}(\Omega) = Y^{2,q}_{w,\sigma}(\Omega) = \mathcal{D}(\mathcal{A})$, the domain of the classical Stokes operator in $L^q_{w,\sigma}(\Omega)$. Hence, in this case the assertion of this theorem is shown in [14], where the Stokes operator in $L^q_{w,\sigma}(\Omega)$ is introduced.

Our aim is to show the assertion for $\beta = 0$ and to apply complex interpolation to obtain the results for arbitrary $0 \le \beta \le 2$.
*Step 1:* We consider $\lambda + \mathcal{A}_{0,q',w'}$, where $\mathcal{A}_{0,q',w'}$ is the Stokes operator in $L^{q'}_{w',\sigma}(\Omega)$, as a continuous linear operator

$$\lambda + \mathcal{A}_{0,q',w'} : Y^{2,q'}_{w',\sigma}(\Omega) \to L^{q'}_{w',\sigma}(\Omega).$$

Let $\mathcal{A}_{-2,q,w} := \mathcal{A}^*_{0,q',w'} : L^q_{w,\sigma}(\Omega) \to Y^{-2,q}_{w,\sigma}(\Omega)$ be the associated dual operator. Then one has for $u \in Y^{2,q}_{w,\sigma}(\Omega)$ and $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$

$$\begin{aligned}\langle(\lambda + \mathcal{A}_{-2,q,w})u, \phi\rangle &= \langle u, (\lambda + \mathcal{A}_{0,q',w'})\phi\rangle = \langle u, \lambda\phi - \Delta\phi\rangle \\ &= \langle(\lambda - P_{q,w}\Delta)u, \phi\rangle = \langle(\lambda + \mathcal{A}_{0,q',w'})u, \phi\rangle.\end{aligned}$$

Thus we obtain using the properties of the dual operator, see e.g. [6]

- $(\lambda + \mathcal{A}_{-2,q,w})|_{Y^{2,q}_{w,\sigma}} = (\lambda + \mathcal{A}_{0,q,w})|_{Y^{2,q}_{w,\sigma}}$.
- For $\lambda \in \Sigma_\varepsilon \cup \{0\}$ one has $\lambda + \mathcal{A}_{-2,q,w} = (\lambda + \mathcal{A}_{0,q',w'})^*$, which implies $\|\lambda + \mathcal{A}_{-2,q,w}\|_{\mathcal{L}(L^q_{w,\sigma}, Y^{-2,q}_{w,\sigma})} = \|\lambda + \mathcal{A}_{0,q',w'}\|_{\mathcal{L}(Y^{2,q'}_{w',\sigma}, L^{q'}_{w',\sigma})}$.

- $\Sigma_\varepsilon \cup \{0\}$ is contained in the resolvent set of $\mathcal{A}_{-2,q,w}$ and there exists $M_\delta > 0$ such that for all $\lambda \in \Sigma_\delta$, $0 < \delta < \varepsilon$,

$$\|(\lambda + \mathcal{A}_{-2,q,w})^{-1}\|_{\mathcal{L}(Y^{-2,q}_{w,\sigma}, L^q_{w,\sigma})} = \|(\lambda + \mathcal{A}_{0,q',w'})^{-1}\|_{\mathcal{L}(L^{q'}_{w',\sigma}, Y^{2,q'}_{w',\sigma})} \le M_\delta.$$

This implies by the definition of the resolvent

$$\|\lambda(\lambda + \mathcal{A}_{-2,q,w})^{-1} f\|_{Y^{-2,q}_{w,\sigma}} + \|(\lambda + \mathcal{A}_{-2,q,w})^{-1} f\|_{q,w} \le M_\delta \|f\|_{Y^{-2,q}_{w,\sigma}}.$$

Since the resolvent set is nonempty, we know that the operator $\mathcal{A}_{-2,q,w}$ is closed in $Y^{-2,q}_{w,\sigma}(\Omega)$. Using the Hahn-Banach theorem one shows that $L^q_{w,\sigma}(\Omega)$, which is equal to the domain of $\mathcal{A}_{-2,q,w}$ in $Y^{-2,q}_{w,\sigma}(\Omega)$, is dense in $Y^{-2,q}_{w,\sigma}(\Omega)$.
*Step 2:* Combining Proposition 3.1 and the assertions for $\beta = 0$ and $\beta = 2$ we obtain by complex interpolation that

$$\mathcal{A} : Y^{\beta,q}_{w,\sigma}(\Omega) \to Y^{\beta-2,q}_{w,\sigma}(\Omega) \quad \text{and} \quad (\lambda - \mathcal{A})^{-1} : Y^{\beta-2,q}_{w,\sigma}(\Omega) \to Y^{\beta,q}_{w,\sigma}(\Omega), \quad \lambda \in \Sigma_\delta \cup \{0\}$$

are continuous operators. Moreover, by the same arguments we obtain from (3.4) for $\beta = 0$ and $\beta = 2$ that $\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}(\Omega))} \le M_\delta |\lambda|^{-1}$ for every $\lambda \in \Sigma_\delta$ and $M_\delta$ independent of $\lambda$. This completes the proof. □

For $\varepsilon \in (0, \frac{\pi}{2})$ one defines

$$\Delta_\varepsilon := \{\lambda \in \mathbb{C} \mid \lambda \neq 0,\ |\arg \lambda| < \varepsilon\}.$$

**Corollary 3.3.** *The negative generalized Stokes operator $-\mathcal{A}$ in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ is the generator of a bounded analytic semigroup $\{e^{-t\mathcal{A}}\}_{t \in \Delta_\varepsilon}$ for every $\varepsilon \in (0, \frac{\pi}{2})$.*

*Proof.* This follows immediately when combining Theorem 3.2 with [19, Theorem 2.5.2]. □

# 4 Instationary Stokes Equations

## 4.1 Very Weak Solutions

We define some function spaces that are appropriate to the instationary and very weak context. First, for $T < \infty$ and $1 < r, q < \infty$ we set

$$X^{r',q'}_{w'}(0,T) = \left\{\phi \in L^{r'}(0,T; Y^{2,q'}_{w'}(\Omega)) \cap W^{1,r'}(0,T; L^{q'}_{w'}(\Omega)) \mid \phi(T) = 0\right\}$$

and for $T = \infty$

$$X^{r',q'}_{w'}(0,\infty) = \Big\{\phi \in L^{r'}(0,\infty; Y^{2,q'}_{w'}(\Omega)) \cap W^{1,r'}(0,\infty; L^{q'}_{w'}(\Omega)) \ \Big| \ \operatorname{supp}\phi \text{ compact in } \overline{\Omega} \times [0,\infty)\Big\}.$$

Both spaces are equipped with the norm $\|\phi\|_{X^{r',q'}_{w'}} := \|\phi\|_{L^{r'}(W^{2,q'}_{w'})} + \|\phi_t\|_{L^{r'}(L^{q'}_{w'})}$. If there is no danger of confusion, we omit the $(0,T)$ and write $X^{r',q'}_{w'}$. We choose the data

$$f \in \left(X^{r',q'}_{w'}(0,T)\right)' \quad \text{and} \quad k \in L^r(0,T; W^{-1,q}_{w,0}(\Omega)). \tag{4.1}$$

As a space of test functions we choose

$$X^{r',q'}_{w',\sigma}(0,T) = \left\{ \phi \in X^{r',q'}_{w'}(0,T) \mid \operatorname{div}\phi = 0 \right\}.$$

**Definition 4.1.** If $f$ and $k$ are given as in (4.1), then a function $u \in L^r(0,T;L^q_w(\Omega))$ is called a very weak solution to the instationary Stokes equations if

$$\begin{aligned} -\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} &= \langle f, \phi\rangle_{\Omega,T}, && \text{for every } \phi \in X^{r',q'}_{w',\sigma} \text{ and} \\ -\langle u(t), \nabla\psi\rangle_\Omega &= \langle k(t), \psi\rangle_\Omega, && \text{for every } \psi \in W^{1,q'}_{w'}(\Omega) \\ & && \text{and almost every } t \in (0,T). \end{aligned}$$

Note that there does not occur any explicit initial condition $u(0)$. It is hidden implicitly in the definition, since the test functions do not vanish at time $t = 0$. Moreover such explicit initial conditions would not be reasonable, as shown in the following considerations. Let $u \in L^r(0,T;L^q_w(\Omega))$. Then

$$\begin{aligned} f &:= [\phi \mapsto \langle u, -\phi_t - \Delta\phi\rangle] \\ &\in \left\{ \phi \in W^{1,r'}(0,T;L^{q'}_{w'}(\Omega)) \mid \phi(T) = 0 \right\}' + (L^{r'}(0,T;Y^{2,q'}_{w'}(\Omega)))' = (X^{r',q'}_{w'})', \\ k(t) &:= [\psi \mapsto \langle u(t), \nabla\psi\rangle] \in W^{-1,q}_{w,0}(\Omega) \text{ for almost every } t \in (0,T), \end{aligned}$$

and since $\|k(t)\|_{-1,q,w,0} \le \|u(t)\|_{q,w}$ for almost every $t$, one has $k \in L^r(0,T;W^{-1,q}_{w,0}(\Omega))$. Thus according to Definition 4.1 every $u \in L^r(0,T;L^q_w(\Omega))$ is a very weak solution to the instationary Stokes problem with respect to appropriate data.

To obtain the solvability of the instationary Stokes equations in the very weak sense in Theorem 4.3 below, we dualize the strong solutions that have been treated in [14]. More precisely one has:

**Theorem 4.2.** *Let $1 < q < \infty$, $w \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain. Moreover, let $0 < T \le \infty$. Then for every $f \in L^r(0,T;L^q_w(\Omega))$ there exists a unique solution $u \in L^r(0,T;\mathcal{D}(\mathcal{A}_{q,w})) = L^r(0,T;Y^{2,q}_{w,\sigma}(\Omega))$ with $u_t \in L^r(0,T;L^q_{w,\sigma}(\Omega))$ to the Stokes equations*

$$u_t + \mathcal{A}u = P_{q,w}f \quad \text{a.e. in } (0,T),\ u(0) = 0,$$

*where $\mathcal{A}$ is the classical Stokes operator in $L^q_{w,\sigma}(\Omega)$. This solution fulfills the estimate*

$$\|u_t\|_{L^r(L^q_{w,\sigma})} + \|\mathcal{A}u\|_{L^r(L^q_{w,\sigma})} \le c\|P_{q,w}f\|_{L^r(L^q_{w,\sigma})},$$

*where $c$ is independent of $f$ and $T$.*

Let $\phi \in L^r(0,T;Y^{2,q}_{w,\sigma}(\Omega)) \cap W^{1,r}(0,T;L^q_w(\Omega))$ be a strong solution to the instationary Stokes problem in the sense of Theorem 4.2 with respect to the exterior force $v \in L^r(0,T;L^q_w(\Omega))$. Then, by de Rham's Theorem [27] there exists a distribution $\psi(t) \in C_0^\infty(\Omega)'$ such that

$$-\Delta\phi(t) + \nabla\psi(t) = v(t) - \phi_t(t)$$

for almost every $t$. Then from this equation and from Lemma 2.8 we obtain, if we assume in addition that $\int_\Omega \psi(t) = 0$ for every $t \in (0,T)$ that $\psi \in L^r(0,T;W^{1,q}_w(\Omega))$ and that $\|\psi\|_{L^r(W^{1,q}_w)} \le c\|\nabla\psi\|_{L^r(L^q_w)} \le c\|v\|_{L^r(L^q_w)}$.

Since we use test functions that vanish at time $T$ instead of 0, we set $\tilde{\phi}(t) := \phi(T-t)$ and $\tilde{\psi}(t) := -\psi(T-t)$. Then we obtain $-\tilde{\phi}_t - \Delta\tilde{\phi} - \nabla\tilde{\psi} = v(T-\cdot)$ with $\tilde{\phi}(T) = 0$, and $\tilde{\phi}$ and $\tilde{\psi}$ fulfill the estimate

$$\|\tilde{\phi}\|_{X_w^{r,q}} + \|\tilde{\psi}\|_{L^r(W_w^{1,q})} \leq c\|v\|_{L^r(L_w^q)}. \tag{4.2}$$

**Theorem 4.3.** *Let $\Omega$ be a bounded $C^{1,1}$-domain and $0 < T \leq \infty$. Let $f$ and $k$ be given as in (4.1) with $\langle k(t), 1\rangle = 0$ for almost every $t \in (0,T)$.*

*Then there exists a unique very weak solution $u \in L^r(0,T; L_w^q(\Omega))$ to the instationary Stokes problem. This function $u$ satisfies the estimate*

$$\|u\|_{L^r(L_w^q)} \leq c\left(\|f\|_{(X_{w'}^{r',q'})'} + \|k\|_{L^r(W_{w,0}^{-1,q})}\right) \tag{4.3}$$

*with a constant $c = c(r,q,w,\Omega) > 0$.*

*Proof.* First assume that $T < \infty$.

As explained above for every $v \in L^{r'}(0,T; L_{w'}^{q'}(\Omega))$ there exists a unique tuple $(\phi,\psi) \in X_{w',\sigma}^{r',q'} \times L^{r'}(0,T; W_{w'}^{1,q'}(\Omega))$, with

$$-\phi_t - \Delta\phi - \nabla\psi = v, \quad \int_\Omega \psi(t)\,dx = 0 \ \text{ for almost every } t.$$

We define a functional $u$ by

$$\langle u, v\rangle_{\Omega,T} := \langle f,\phi\rangle_{\Omega,T} + \langle k,\psi\rangle_{\Omega,T} \ \text{ for all } v \in L^{r'}(0,T; L_{w'}^{q'}(\Omega)).$$

Then the a priori estimate for the strong solution (4.2) implies

$$\begin{aligned} |\langle u, v\rangle_{\Omega,T}| \leq& \|f\|_{(X_{w'}^{r',q'})'}\|\phi\|_{X_{w'}^{r'q'}} + \|k\|_{L^r(W_{w,0}^{-1,q})}\|\psi\|_{L^{r'}(W_{w'}^{1,q'})} \\ \leq& c\left(\|f\|_{(X_{w'}^{r',q'})'} + \|k\|_{L^r(W_{w,0}^{-1,q})}\right)\|v\|_{L^{r'}(L_{w'}^{q'})}. \end{aligned} \tag{4.4}$$

Thus we obtain $u \in \left(L^{r'}(0,T; L_{w'}^{q'}(\Omega))\right)' = L^r(0,T; L_w^q(\Omega))$ with

$$\|u\|_{L^r(L_w^q)} \leq c\left(\|f\|_{(X_{w'}^{r',q'})'} + \|k\|_{L^r(W_{w,0}^{-1,q})}\right).$$

Moreover, for every $(\phi,\psi) \in X_{w',\sigma}^{r',q'} \times L^{r'}(0,T; W_{w'}^{1,q'}(\Omega))$ we have

$$-\langle u,\phi_t\rangle_{\Omega,T} - \langle u,\Delta\phi\rangle_{\Omega,T} - \langle u,\nabla\psi\rangle_{\Omega,T} = \langle f,\phi\rangle_{\Omega,T} + \langle k,\psi\rangle_{\Omega,T},$$

where we used that the mapping $v = -\phi_t - \Delta\phi - \nabla\psi \mapsto (\phi,\psi)$ is well-defined. This shows that $u$ is a very weak solution to the instationary Stokes problem according to Definition 4.1 and finishes the proof of existence and of the a priori estimate.

To show the uniqueness let $U \in L^r(0,T; L_w^q(\Omega))$ be another very weak solution with respect to the data $f$ and $k$. Moreover, let $v \in L^{r'}(0,T; L_{w'}^{q'}(\Omega))$ and let $\phi \in X_{w',\sigma}^{r',q'}$ and $\psi \in L^{r'}(0,T; W_{w'}^{1,q'}(\Omega))$ solve $v = -\phi_t - \Delta\phi - \nabla\psi$ as above. Then one has

$$\langle U, v\rangle = -\langle U,\phi_t\rangle - \langle U,\Delta\phi\rangle - \langle U,\nabla\psi\rangle = \langle f,\phi\rangle + \langle k,\psi\rangle = \langle u, v\rangle.$$

Since $v$ was arbitrary, this implies $U = u$ and the proof for $T < \infty$ is complete.

For $T = \infty$ we take $v \in L^{r'}(\mathbb{R}_+; L^{q'}_{w'}(\Omega))$, with $\operatorname{supp} v \subset (0, N) \times \overline{\Omega}$ for some $N \in \mathbb{N}$ and let

$$(\phi, \psi) \in X^{r',q'}_{w',\sigma}(0, N) \times L^{r'}(0, N; W^{1,q'}_{w'}(\Omega))$$

with $\int_\Omega \psi(t) = 0$ for almost every $t$ be the unique solution of $-\phi_t - \Delta\phi - \nabla\psi = v$ with $\phi(N) = 0$. Extending the functions $\phi$ and $\psi$ by 0 on $[N, \infty) \times \Omega$ one obtains $\phi \in X^{r',q'}_{w',\sigma}(0, \infty)$ and $\psi \in L^{r'}(0, \infty; W^{1,q'}_{w'}(\Omega))$. Thus the mapping

$$u := \left[ \bigcup_{N=1}^{\infty} L^{r'}(0, N, L^{q'}_{w'}(\Omega)) \ni v \mapsto \langle f, \phi \rangle_{\Omega,\infty} + \langle k, \psi \rangle_{\Omega,\infty} \right]$$

is well-defined, where every $v \in L^{r'}(0, N, L^{q'}_{w'}(\Omega))$ is assumed to be extended by zero to $\mathbb{R}_+$.

We obtain that $u|_{(0,N)} \in L^r(0, N, L^q_w(\Omega))$ for every $N \in \mathbb{N}$. Moreover, since the set of functions with compact support in time is dense in $L^{r'}(0, \infty, L^{q'}_{w'}(\Omega))$ and the estimates in (4.4) are independent of $T$, this yields $u \in L^r(0, \infty; L^q_w(\Omega))$ and the asserted estimate.

The uniqueness in the case $T = \infty$ follows from the uniqueness in the case $T < \infty$. $\square$

Using a slightly more restricted space for the data one obtains the following estimate for the time derivative. In particular the corollary below shows that the generalized Stokes operator in $Y^{-2,q}_{w,\sigma}(\Omega)$ has maximal regularity.

**Corollary 4.4.** *Assume $f \in L^r(0, T; Y^{-2,q}_w(\Omega))$ and $k \in L^r(0, T; W^{-1,q}_{w,0}(\Omega))$. One has $L^r(0, T; Y^{-2,q}_w(\Omega)) \subset \left( X^{r',q'}_{w'}(0, T) \right)'$ and the associated very weak solution which exists according to Theorem 4.3 satisfies the stronger estimate*

$$\left\| u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \right\|_{L^r(Y^{-2,q}_{w,\sigma})} + \|u\|_{L^r(L^q_w)} \le c \left( \|f\|_{L^r(Y^{-2,q}_w)} + \|k\|_{L^r(W^{-1,q}_{w,0})} \right) \tag{4.5}$$

*with $c = c(r, q, w, \Omega) > 0$. If in addition $k = 0$ then $u$ solves the equation*

$$u'|_{Y^{2,q'}_{w',\sigma}(\Omega)} + \mathcal{A}u = f|_{Y^{2,q'}_{w',\sigma}(\Omega)},$$

*where $\mathcal{A}$ is the generalized Stokes operator in $Y^{-2,q}_{w,\sigma}(\Omega)$.*

*Proof.* Let $\phi \in C^\infty_0(0, T; Y^{2,q'}_{w',\sigma}(\Omega))$. Then we can estimate using (4.3)

$$\begin{aligned} |\langle u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}, \phi \rangle_{\Omega,T}| &\le |\langle u, \Delta\phi \rangle_{\Omega,T}| + |\langle f, \phi \rangle_{\Omega,T}| \\ &\le c \left( \|f\|_{L^r(Y^{-2,q}_w)} + \|k\|_{L^r(W^{-1,q}_{w,0})} \right) \|\phi\|_{L^{r'}(Y^{2,q'}_{w'})}. \end{aligned}$$

Together with (4.3), the a priori estimate in Theorem 4.3, this proves the assertion.

The last assertion follows from the characterization of the Stokes operator in Theorem 3.2 and the formulation of very weak solutions. $\square$

According to Definition 4.1 every $u \in L^r(0,T;L^q_w(\Omega))$ is a very weak solution with respect to appropriate data. This means that such solutions in general do not possess enough time-regularity to ensure that the initial condition $u(0) = u_0$ is well-defined.

However, if the data is chosen as in Corollary 4.4, we obtain $u \in L^r(0,T;L^q_w(\Omega))$ and $u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^r(0,T;Y^{-2,q}_{w,\sigma}(\Omega))$. By [2] this implies that $u|_{Y^{2,q'}_{w',\sigma}(\Omega)}$ is uniformly continuous and hence this regularity suffices to define $u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in Y^{-2,q}_{w,\sigma}(\Omega)$, and one has

$$\langle u(0), \phi(0)\rangle_\Omega = \langle u, \phi_t\rangle_{\Omega,T} + \langle u_t, \phi\rangle_{\Omega,T}$$

for every $\phi \in C^1_0([0,T], Y^{2,q'}_{w',\sigma}(\Omega))$ with $\phi(T) = 0$. Analogously to the case of strong solutions the gradient part of the initial condition cannot be prescribed and is not needed for the uniqueness of the solution.

**Lemma 4.5.** *If $u \in L^r(0,T;L^q_w(\Omega))$ is a very weak solution according to Definition 4.1 with respect to $f \in L^r(0,T;Y^{-2,q}_w(\Omega))$ and $k \in L^r(0,T;W^{-1,q}_{w,0}(\Omega))$ then $u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} = 0$.*

*Proof.* For $\phi \in C^1_0((0,T);Y^{2,q'}_{w',\sigma}(\Omega))$ one has

$$\langle u_t, \phi\rangle_{\Omega,T} = -\langle u, \phi_t\rangle_{\Omega,T} = \langle u, \Delta\phi\rangle_{\Omega,T} + \langle f, \phi\rangle_{\Omega,T} \tag{4.6}$$

which implies $u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^r(0,T;Y^{-2,q}_{w,\sigma}(\Omega))$ and (4.6) holds for all $\phi \in X^{r',q'}_{w'}$ because one can approximate $\phi \in X^{r',q'}_{w'}$ by a sequence in $C^1_0((0,T);Y^{2,q'}_{w'}(\Omega))$ that converges in $L^{r'}(0,T;Y^{2,q'}_{w'}(\Omega))$. Thus $\langle u(0), \phi(0)\rangle_\Omega = \langle u_t, \phi\rangle_{\Omega,T} + \langle u, \phi_t\rangle_{\Omega,T} = 0$ for every $\phi \in C^1([0,T];Y^{2,q'}_{w',\sigma}(\Omega))$ with $\phi(T) = 0$. In particular, for a fixed $\zeta \in Y^{2,q'}_{w',\sigma}(\Omega)$ and $\eta \in C^\infty_0([0,T))$ with $\eta(0) = 1$ one has $\langle u(0), \zeta\rangle_\Omega = \langle u(0), \zeta\eta(0)\rangle_\Omega = 0$. We have proved $u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} = 0$. □

## 4.2 The Spaces $H^{\beta,r}(X)$

By $\mathcal{S}(\mathbb{R};\mathbb{R})$ we denote the space of rapidly decreasing smooth functions. For a Banach space $X$ we denote the space of $X$-valued tempered distributions by $S'(\mathbb{R};X) := \mathcal{L}(\mathcal{S}(\mathbb{R};\mathbb{R}), X)$. Accordingly, for an interval $I$ we denote the set of distributions by $\mathcal{D}'(I;X) := \mathcal{L}(C^\infty_0(I), X)$.

For the treatment of solutions to the instationary Stokes Problem in Bessel potential spaces with inhomogeneous divergence and boundary conditions we need a higher time regularity of this part of the data. To measure this time regularity we work in Banach space-valued Bessel potential spaces.

For $\beta \in \mathbb{R}$ we set $\Lambda^\beta_t := \mathcal{F}^{-1}\langle\tau\rangle^\beta\mathcal{F}$, where $\langle\tau\rangle^\beta = (1+|\tau|^2)^{\frac{\beta}{2}}$, $\tau \in \mathbb{R}^n$. Using this, for $r > 1$ we define the $X$-valued Bessel-potential space by

$$H^{\beta,r}(\mathbb{R};X) := \left\{u \in \mathcal{S}'(\mathbb{R};X) \mid \Lambda^\beta_t u \in L^r(\mathbb{R};X)\right\},$$

equipped with the norm $\|u\|_{H^{\beta,r}(\mathbb{R};X)} := \|\Lambda^\beta_t u\|_{L^r(\mathbb{R};X)}$. Moreover, we define

$$H^{\beta,r}(0,T;X) := \left\{u|_{C^\infty_0(0,T;\mathbb{R})} \mid u \in H^{\beta,r}(\mathbb{R};X)\right\}$$

with the norm $\|u\|_{H^{\beta,r}(0,T;X)} := \inf\{\|U\|_{H^{\beta,r}(\mathbb{R};X)} \mid U \in H^{\beta,r}(\mathbb{R};X), U|_{C_0^\infty(0,T;\mathbb{R})} = u\}$. Finally, we set for $\beta \geq 0$

$$H_0^{\beta,r}((0,T];X) := \{U|_{C_0^\infty(0,T;\mathbb{R})} \mid U \in H^{\beta,r}(\mathbb{R};X),\ \operatorname{supp} U \subset [0,\infty)\}$$

equipped with

$$\|u\|_{H_0^{\beta,r}((0,T];X)} := \inf\{\|U\|_{H^{\beta,r}(\mathbb{R};X)} \mid U \in H^{\beta,r}(\mathbb{R};X), \\ \operatorname{supp} U \subset [0,\infty),\ U|_{C_0^\infty(0,T;\mathbb{R})} = u\}$$

and $H_0^{\beta,r}(0,T;X) := \overline{C_0^\infty(0,T;X)}^{H^{\beta,r}(\mathbb{R};X)}$ with $\|\cdot\|_{H_0^{\beta,r}(0,T;X)} = \|\cdot\|_{H^{\beta,r}(\mathbb{R};X)}$.

**Lemma 4.6.** *Let $X$ be a reflexive Banach space and $\beta \geq 0$. Then one has*

$$H^{-\beta,r}(\mathbb{R};X) \cong (H^{\beta,r'}(\mathbb{R};X'))' \quad \textit{and} \quad H^{-\beta,r}(0,T;X) \cong (H_0^{\beta,r'}(0,T;X'))'$$

*with equivalent norms. Every $u \in H^{-\beta,r}(\mathbb{R};X)$ is identified with the element of the space $(H^{\beta,r'}(\mathbb{R};X'))'$ fulfilling*

$$\phi x^* \mapsto \langle u, \phi x^*\rangle_{X,X',\mathbb{R}} := \big\langle \langle u(t), \phi(t)\rangle_{\mathbb{R}}, x^*\big\rangle_{X,X'}, \tag{4.7}$$

*where $\phi \in \mathcal{S}(\mathbb{R};\mathbb{R})$ and $x^* \in X'$. With this identification one has*

$$\langle u, \psi\rangle_{X,X',\mathbb{R}} = \int_{\mathbb{R}} \Big\langle \Lambda_t^{-\beta} u(s), \Lambda_t^{\beta}\psi(s)\Big\rangle_{X,X'}\, ds \tag{4.8}$$

*for every $u \in H^{-\beta,r}(\mathbb{R};X)$ and $\psi \in H^{\beta,r'}(\mathbb{R};X')$.*

*Proof.* Let $u \in H^{-\beta,r}(\mathbb{R};X)$. The linear hull of $\{\phi x^* \mid \phi \in \mathcal{S}(\mathbb{R};\mathbb{R}),\quad x^* \in X'\}$ is dense in $H^{\beta,r'}(\mathbb{R};X')$. Moreover for $u \in H^{-\beta,r}(\mathbb{R};X)$ and $\phi \in \mathcal{S}(\mathbb{R},\mathbb{R})$, $x^* \in X'$ one has

$$\langle u, \phi x^*\rangle_{X,X',\mathbb{R}} = \int_{\mathbb{R}} \langle \Lambda_t^{-\beta} u(s), \Lambda_t^{\beta}\phi(s)x^*\rangle_{X,X'} ds,$$

thus $|\langle u, \phi x^*\rangle_{X,X',\mathbb{R}}| \leq \|u\|_{H^{-\beta,r}(\mathbb{R};X)}\|\phi x^*\|_{H^{\beta,r'}(\mathbb{R};X')}$, and we obtain that $\langle u, \cdot\rangle_{X,X',\mathbb{R}}$ extends in a unique way to a continuous functional on $H^{\beta,r'}(\mathbb{R};X')$. This extension fulfills (4.8).

Vice versa let $u \in \big(H^{\beta,r'}(\mathbb{R};X')\big)'$. Then, since $X$ is reflexive, $u$ defines a distribution $u \in \mathcal{S}'(\mathbb{R};X)$ by

$$\mathcal{S}(\mathbb{R};\mathbb{R}) \ni \phi \mapsto [X' \ni x^* \mapsto \langle u, \phi x^*\rangle] \in X'' = X.$$

For $\phi \in \mathcal{S}(\mathbb{R},\mathbb{R})$, $x^* \in X'$ one has

$$\begin{aligned} \big|\big\langle \langle \Lambda_t^{-\beta} u, \phi\rangle_{\mathbb{R}}, x^*\big\rangle_{X,X'}\big| &\leq \|u\|_{(H^{-\beta,r'}(\mathbb{R};X'))'}\|\Lambda_t^{-\beta}\phi x^*\|_{H^{\beta,r'}(\mathbb{R};X')} \\ &= \|u\|_{(H^{-\beta,r'}(\mathbb{R};X'))'}\|\phi x^*\|_{L^{r'}(\mathbb{R};X')}. \end{aligned}$$

Thus the functional $\Lambda_t^{-\beta} u$ can be identified with an element of $L^r(\mathbb{R};X)$, or $u$ with an element of $H^{-\beta,r}(\mathbb{R},X)$.

The assertion $H^{-\beta,r}(0,T;X) \cong (H_0^{\beta,r'}(0,T;X'))'$ follows from the assertion on $\mathbb{R}$ as follows. For $u \in H^{-\beta,r}(0,T;X)$ there exists $U \in H^{-\beta,r}(\mathbb{R};X) = \big(H^{\beta,r'}(\mathbb{R};X')\big)'$ with $U|_{C_0^\infty(0,T)} = u$. Thus it follows for $\phi \in C_0^\infty(0,T)$ and $x^* \in X'$

$$\big\langle \langle u, \phi\rangle_T, x^*\big\rangle_{X,X'} = \langle U, \phi x^*\rangle_{X,X',T}.$$

This extends by density and continuity to a functional in $(H_0^{\beta,r'}(0,T;X'))'$.

Vice versa, for $u \in (H_0^{\beta,r'}(0,T;X'))'$ there exists by the Hahn-Banach theorem a functional $U \in (H^{\beta,r'}(\mathbb{R};X'))' \cong H^{-\beta,r}(\mathbb{R};X)$ such that $U|_{H_0^{\beta,r'}(0,T;X')} = u$. Since $X$ is reflexive, one has

$$\big[\mathcal{S}(\mathbb{R};\mathbb{R}) \ni \phi \mapsto [X' \ni x^* \mapsto \langle U, \phi x^*\rangle]\big] \in H^{-\beta,r}(\mathbb{R};X'') = H^{-\beta,r}(\mathbb{R};X)$$

and $U|_{C_0^\infty(0,T)} \in H^{-\beta,r}(0,T;X)$. □

A Banach space $X$ is called a UMD-space if the Hilbert transform,

$$Hf(x) = PV - \int_{\mathbb{R}} \frac{1}{t-s} f(s)ds, \quad f \in \mathcal{S}(\mathbb{R};X),$$

extends to a bounded linear operator on $L^p(\mathbb{R};X)$ for every $1 < p < \infty$.

**Lemma 4.7.** *Let $X$ be a UMD-space and $\beta \in \mathbb{R}$.*

1. *The derivative $\partial_t$ is continuous*

$$\begin{array}{rll} \partial_t : & H^{\beta,r}(\mathbb{R};X) & \to H^{\beta-1,r}(\mathbb{R};X), \\ \partial_t : & H^{\beta,r}(0,T;X) & \to H^{\beta-1,r}(0,T;X), \\ \partial_t : & H_0^{\beta,r}((0,T];X) & \to H_0^{\beta-1,r}((0,T];X). \end{array}$$

2. *For $k \in \mathbb{Z}$ one has $H^{k,r}(\mathbb{R};X) \cong W^{k,r}(\mathbb{R};X)$ and $H^{k,r}(0,T;X) \cong W^{k,r}(0,T;X)$ with equivalent norms. The isomorphism is given by the identification in (4.7).*

3. *Let $\beta \in [0,1]$ and let $X_1, X_2$ be UMD-spaces with $X_1 \hookrightarrow X_2$. Then there exists a continuous linear extension operator*

$$E : H_0^{\beta,r}((0,T];X_2) \cap L^r(0,T;X_1) \to H^{\beta,r}(\mathbb{R};X_2) \cap L^r(\mathbb{R};X_1)$$

*with $Eu(t) = 0$ for every $t < 0$.*

*Proof.* The assertions of 1. and 2. for the case $H^{k,r}(\mathbb{R};X)$ follows from a the continuity of scalar-valued Fourier multipliers between UMD-spaces proved by Zimmermann [30] and duality.

For $u \in W^{k,r}(0,T;X)$, $k > 0$, we construct an extension

$$Eu(x) = \begin{cases} \phi(-x)\sum_{j=1}^{k+1} \lambda_j u(-jx) & \text{if } \ -\frac{T}{k+1} < x < 0, \\ u(x) & \text{if } \ x \in [0,T], \\ \phi(x-T)\sum_{j=1}^{k+1} \lambda_j u(T - j\cdot(x-T)) & \text{if } \ T < x < T + \frac{T}{k+1}, \\ 0 & \text{else,} \end{cases} \tag{4.9}$$

with $\sum_j \lambda_j(-j)^l = 1$ for $l = 0, ..., k$, where $\phi$ is a smooth cut-off function with $\phi = 0$ in a neighborhood of $\frac{T}{k+1}$.

Thus for $u \in W^{k,r}(0,T;X)$ one has $Eu \in W^{k,r}(\mathbb{R};X) = H^{k,r}(\mathbb{R};X)$ which shows that $u \in H^{k,r}(0,T;X)$ with

$$\|u\|_{H^{k,r}(0,T;X)} \leq \|Eu\|_{H^{k,r}(\mathbb{R};X)} \leq c\|Eu\|_{W^{k,r}(\mathbb{R};X)} \leq c\|u\|_{W^{k,r}(0,T;X)}.$$

Vice versa for $u \in H^{k,r}(0,T;X)$ an appropriate extension exists by definition. Hence an analogous argument completes the proof for $k \geq 0$.

For $k < 0$ the assertion follows by the duality stated in Lemma 4.6.

3. We begin to consider the extension by 0 to the negative half axis

$$E_0 : H_0^{\beta,r}((0,T];X_2) \cap L^r(0,T;X_1) \to H^{\beta,r}(-\infty,T;X_2) \cap L^r(-\infty,T;X_1),$$

which is continuous by the definition of $H_0^{\beta,r}((0,T];X_2)$. Moreover, by $E$ we denote the extension to $t > T$ defined in the same way as in (4.9) with $k = 1$. By construction

$$\begin{array}{lll} E: & L^r(-\infty,T;X_i) & \to L^r(\mathbb{R};X_i), \quad i = 1,2 \quad \text{and} \\ E: & H^{1,r}(-\infty,T;X_2) & \to H^{1,r}(\mathbb{R};X_2) \end{array}$$

is continuous. Since $X_2$ is a UMD-space one, has

$$\left[L^r(-\infty;X_2), H^{1,r}(-\infty,T;X_2)\right]_\beta = H^{\beta,r}(-\infty,T;X_2).$$

This is proved in the same way as in the scalar-valued case, cf. [26] 13, Prop. 6.2, replacing the scalar-valued multiplier theorem by the Banach space-valued version in [30]. Thus the assertion follows by interpolation. □

## 4.3 Inhomogeneous Tangential Boundary Conditions

Our next aim is to develop a solution theory of the instationary Stokes equations in weighted Bessel potential spaces. In the context of lowest regularity, in which the class of solutions is contained in $L^r(0,T;L^q_w(\Omega))$ the data could be chosen fully inhomogeneous. Now, turning to higher regularity, we do not want to loose this possibility. However, this requires a more complex theory and a higher regularity of the data than before.

We start with purely tangential boundary conditions. If $g(t) \in T_w^{\beta,q}(\Omega)$ for almost every $t$, this means

$$\begin{array}{lll} g(t,x)\cdot N & = 0 & \text{for almost every } \ x \in \partial\Omega \ \text{ if } \ \beta \in [1,2] \ \text{ and} \\ \langle g(t), Nh\rangle_{\partial\Omega} & = 0 & \text{for every scalar-valued } \ h \in C^\infty(\overline{\Omega})|_{\partial\Omega} \ \text{ if } \ \beta \in [0,1]. \end{array}$$

The reason why we deal with tangential boundary data is that such data can be represented by

$$f := \left[Y^{2,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto \langle g(t), N\cdot\nabla\phi\rangle_{\partial\Omega}\right] \in Y^{-2,q}_{w,\sigma}(\Omega). \tag{4.10}$$

In the latter space we have defined the generalized Stokes operator $\mathcal{A}$, see Section 3. In general very weak solutions are not regular enough to ensure that their restriction to the boundary is well-defined. However, since $f|_{C_0^\infty(\Omega)} = 0$ we can give a sense to $\mathcal{A}^{-1}f|_{\partial\Omega}$ and it follows that $\mathcal{A}^{-1}f|_{\partial\Omega} = g$. This has been shown and discussed in [17] and [24].

**Lemma 4.8.** *$H^{\beta,q}_w(\Omega)$ is a UMD-space for every $\beta \geq 0$ and $T^{\beta,q}_w(\partial\Omega)$ is a UMD-space for $\beta \in [0,2]$.*

*Proof.* By [2, Theorem 4.5.2] spaces isomorphic to $L^q_w(\Omega)$, their dual spaces, factor spaces and complex interpolation spaces are UMD-spaces. This proves the assertion. ☐

**Definition 4.9.** Let $X, Y$ be Banach spaces. A subset $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $R$-bounded if there is a constant $C > 0$ such that for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$ and $n \in \mathbb{N}$ one has

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j(x_j) \right\|_Y du \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\|_X du,$$

where $(r_j)$ is a sequence of independent, symmetric $\{1,-1\}$-valued random variables on $[0,1]$, e.g. the Rademacher functions.

The following theorem has been shown by Weis in [29, Theorem 3.4].

**Theorem 4.10.** *Let $X$ and $Y$ be UMD-spaces. Let*

$$\mathbb{R} \setminus \{0\} \ni t \mapsto M(t) \in \mathcal{L}(X,Y)$$

*be a differentiable function such that the sets*

$$\{M(t) \mid t \in \mathbb{R} \setminus \{0\}\} \quad \text{and} \quad \{tM'(t) \mid t \in \mathbb{R} \setminus \{0\}\}$$

*are $R$-bounded. Then $\mathcal{K}f = [M(\cdot)\hat{f}(\cdot)]^\vee$, $f \in C_0^\infty(\mathbb{R}, X)$, extends to a bounded linear operator*

$$\mathcal{K} : L^r(\mathbb{R}; X) \to L^r(\mathbb{R}; Y) \quad \text{for} \quad 1 < r < \infty.$$

By Theorem 2.4 one has $H^{\beta,q}_w(\Omega) = [L^q_w(\Omega), W^{k,q}_w(\Omega)]_{\frac{\beta}{k}}$ and by Theorem 2.5 one has $[L^q_w(\Omega), Y^{2,q}_w(\Omega)]_\theta = Y^{\beta,q}_w(\Omega)$. However, we do not know whether the equivalence constants depend $A_q$-consistently on the weight function. To fix notation and to ensure that interpolation preserves the $A_q$-consistence of the constants we assume for the rest of this section that the norm on $H^{\beta,q}_w(\Omega)$ is given by the norm in the interpolation space, i.e.

$$\| \cdot \|_{H^{\beta,q}_w(\Omega)} = \| \cdot \|_{[W^{k,q}_w(\Omega), W^{k+1,q}_w(\Omega)]_\theta}, \text{ where } \beta \in [k, k+1] \text{ and } \theta = \beta - k.$$

In particular $H^{k,q}_w(\Omega)$ is equipped with the norm in $W^{k,q}_w(\Omega)$ for every $k \in \mathbb{N}_0$. Accordingly we assume $\| \cdot \|_{Y^{\beta,q}_w(\Omega)} = \| \cdot \|_{H^{\beta,q}_w(\Omega)}$ for $\beta \in [1,2]$ and

$$\| \cdot \|_{Y^{\beta,q}_w(\Omega)} = \| \cdot \|_{[L^q_w(\Omega), W^{1,q}_{w,0}(\Omega)]_\beta} \quad \text{for} \quad \beta \in [0,1).$$

**Theorem 4.11.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$ and let $I$ be an interval.*

1. *For $2 \geq \beta \geq 0$ let $B(t) : L^q_w(\Omega) \to H^{\beta,q}_w(\Omega)$, $t \in I$, be uniformly bounded for every $w \in A_q$ with an $A_q$-consistent bound of the continuity constant. Then $B(t)$, $t \in I$, is $R$-bounded.*

2. *The assertion of 1. holds true if one replaces $H^{\beta,q}_w(\Omega)$ by $Y^{\beta,q}_w(\Omega)$.*

*Proof.* 1. We begin with the case $0 \le \beta < 1$. Let $(\psi_j)_{j=1}^N$, $\psi_j : \mathbb{R}^n_+ \supset U_j \to V_j \subset \overline{\Omega}$ be a collection of $C^{1,1}$-charts and assume that each $\psi_j$ is extended to a $C^{1,1}$-diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $(\phi_j)_j$ be a decomposition of unity subordinate to the covering $(V_j)_j$ of $\overline{\Omega}$.

For $v \in A_q$ we set $w_j := v \circ \psi_j^{-1}$ and by $E_e : H^{\beta,q}_v(\mathbb{R}^n_+) \to H^{\beta,q}_{v^*}(\mathbb{R}^n)$ we denote the even extension

$$E_e u(x) = \begin{cases} u(x) & \text{for } \ x_n \ge 0 \\ u(x', -x_n) & \text{for } \ x_n \le 0 \end{cases} \quad \text{for } \ u \in H^{\beta,q}_v(\mathbb{R}^n_+)$$

and by $E_0 : L^q_w(\Omega) \to L^q_w(\mathbb{R}^n)$ we denote the extension by 0. We consider the mapping $M_j(t) : L^q_v(\mathbb{R}^n) \to L^q_v(\mathbb{R}^n)$, which is defined by the composition

$$\begin{aligned} M_j(t) : L^q_v(\mathbb{R}^n) &\xrightarrow{C_{\psi_j^{-1}} : h \mapsto h\circ\psi_j^{-1}} L^q_{w_j}(\mathbb{R}^n) \xrightarrow{R_\Omega} L^q_{w_j}(\Omega) \xrightarrow{B(t)} H^{\beta,q}_{w_j}(\Omega) \\ &\xrightarrow{M_{\phi_j} : h \mapsto \phi_j h} H^{\beta,q}_{w_j}(H_{\psi_j}) \xrightarrow{C_{\psi_j} : h \mapsto h\circ\psi_j} H^{\beta,q}_v(\mathbb{R}^n_+) \xrightarrow{E_e} H^{\beta,q}_{v^*}(\mathbb{R}^n) \\ &\xrightarrow{\Lambda^\beta} L^q_{v^*}(\mathbb{R}^n) \xrightarrow{R_{\mathbb{R}^n_+}} L^q_v(\mathbb{R}^n_+) \xrightarrow{E_0} L^q_v(\mathbb{R}^n), \end{aligned}$$

where $H_{\psi_j}$ is the bent half space with boundary $\psi_j(\mathbb{R}^{n-1}\times\{0\})$ and $v^*(x', x_n) = v(x', x_n)$ for $x_n \ge 0$ and $v^*(x', x_n) = v(x', -x_n)$ for $x_n < 0$. This operator $M_j(t)$ is the composition of $B(t)$ with operators constant in $t$ and with norms depending $A_q$-consistently on the weight functions $v$ and $w$. The $A_q$-consistence of the norms of $C_{\psi_j}$, $M_{\phi_j}$, $E_0$ and $E_e$ is easy to check in the cases $\beta = 0$ and $\beta = 1$ and it is preserved by interpolation. For $\Lambda^\beta$ we refer to Corollary 2.3.

By the assumptions on $B(t)$ we obtain that $M_j(t)$ is uniformly bounded in $t$ with an $A_q$-consistent bound. Thus by [14, Theorem 4.3] we obtain that $M_j(t)$ is $R$-bounded.

Next we show that

$$B(t) = \sum_{j=1}^n M_{\tilde\phi_j} \circ C_{\psi_j^{-1}} \circ R_{\mathbb{R}^n_+} \circ \Lambda^{-\beta} \circ E_e \circ R_{\mathbb{R}^n_+} \circ M_j(t) \circ C_{\psi_j} \circ E_0, \tag{4.11}$$

where $M_{\tilde\phi_j} : H^{\beta,q}_{(w\circ\psi_j)^*\circ\psi_j^{-1}}(H_{\psi_j}) \to H^{\beta,q}_w(\Omega)$ is the multiplication with some cut-off function $\tilde\phi_j \in C_0^\infty(V_j)$ with $\tilde\phi_j \equiv 1$ on $\operatorname{supp}\phi_j$. One has the equations

$$R_\Omega \circ C_{\psi_j^{-1}} \circ C_{\psi_j} \circ E_0 = \mathrm{id}_{L^q_w(\Omega)},$$

$$\sum_{j=1}^N M_{\tilde\phi_j} \circ C_{\psi_j^{-1}} \circ \underbrace{R_{\mathbb{R}^n_+} \circ \Lambda^{-\beta} \circ E_e \circ R_{\mathbb{R}^n_+} \circ E_0 \circ R_{\mathbb{R}^n_+} \circ \Lambda^\beta \circ E_e}_{=\mathrm{id}_{H^{\beta,q}_w(\mathbb{R}^n_+)},\ \text{since } \Lambda^\beta\circ E_e \text{ is even}} \circ C_{\psi_j} \circ M_{\phi_j} = \mathrm{id}_{H^{\beta,q}_w(\Omega)}.$$

We have used that the Fourier transform and the inverse Fourier transform as well as the multiplication with the even function $\langle\xi\rangle^\beta$ maps even functions to even functions. This shows that the image space of $\Lambda^\beta \circ E_e$ consists of even functions. Thus (4.11) holds.

We find that $B(t)$ is $R$-bounded as a sum and composition of the $R$-bounded operators $M_j(t)$ with bounded operators which are constant in $t$.

We turn to the case $1 < \beta \le 2$. If $B(t) : L^q_w(\Omega) \to H^{\beta,q}_w(\Omega)$, $t \in I$, fulfills the assumptions of this theorem, then $\partial_i B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega)$ is uniformly bounded for $i = 1, ..., n$ as well, by a constant depending $A_q$-consistently on $w$. Moreover, by the embedding $H^{\beta,q}_w(\Omega) \hookrightarrow H^{\beta-1,q}_w(\Omega)$ the same is true for $B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega)$.

Since $0 < \beta - 1 \le 1$, we are in the case just treated and we find that

$$\partial_i B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega), \quad i = 1, ..., n, \quad \text{and} \quad B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega)$$

are $R$-bounded. Thus using the notation of Definition 4.9 we find by Lemma 2.7

$$\begin{aligned}
&\int_0^1 \left\| \sum_{k=1}^m r_k(u) B(t_k) h_k \right\|_{H^{\beta,q}_w(\Omega)} du \\
&\le c \left( \sum_{j=1}^n \int_0^1 \left\| \partial_j \sum_{k=1}^m r_k(u) B(t_k) h_k \right\|_{H^{\beta-1,q}_w(\Omega)} du + \int_0^1 \left\| \sum_{k=1}^m r_k(u) B(t_k) h_k \right\|_{H^{\beta-1,q}_w(\Omega)} du \right) \\
&\le c \int_0^1 \left\| \sum_{k=1}^m r_k(u) h_k \right\|_{L^q_w(\Omega)} du.
\end{aligned}$$

Hence $B(t)$ is $R$-bounded.

2. For $1 \le \beta \le 2$ one has $\| \cdot \|_{Y^{\beta,q}_w} = \| \cdot \|_{\beta,q,w}$. Thus, if $B(t) : L^q_w(\Omega) \to Y^{\beta,q}_w(\Omega) \subset H^{\beta,q}_w(\Omega)$ fulfills the assumptions of the theorem, then $B(t) : L^q_w(\Omega) \to H^{\beta,q}_w(\Omega)$ is $R$-bounded. Since $B(t)$ takes values in $Y^{\beta,q}_w(\Omega)$, we obtain the asserted $R$-boundedness of $B(t) : L^q_w(\Omega) \to Y^{\beta,q}_w(\Omega)$.

Now we assume $0 \le \beta < 1$. We choose some ball $B_r$ such that $\overline{\Omega} \subset B_r$. Then the operator

$$E_{0,B_r} : Y^{\beta,q}_w(\Omega) \to H^{\beta,q}_w(B_r), \quad E_{0,B_r}(u)(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in B_r \setminus \Omega \end{cases} \tag{4.12}$$

is continuous with continuity constant 1. This is clear for $\beta = 0$ and $\beta = 1$, for $\beta \in (0, 1)$ it follows by interpolation.

We set

$$D(t) : L^q_w(B_r) \to H^{\beta,q}_w(B_r), \quad D(t)u = E_{0,B_r} \circ B(t) \circ R_\Omega,$$

where $R_\Omega$ is the restriction to $\Omega$. Then $D(t)$ is uniformly bounded by a constant depending $A_q$-consistently on $w$. Hence it is $R$-bounded by 1.

Let $u \in H^{\beta,q}_w(B_r)$ with $u|_{B_r \setminus \Omega} = 0$. Then by the Theorems 2.5 and 2.4 the norm in the interpolation space is equivalent to the one defined by restrictions. The constants are maybe no longer $A_q$-consistent, but in this step of the proof this is no longer needed.

Thus we may estimate, denoting by $E_{0,\mathbb{R}^n}$ the extension by 0 to the whole space $\mathbb{R}^n$,

$$\begin{aligned}
\|R_\Omega u\|_{Y^{\beta,q}_w(\Omega)} \le& c\|E_{0,\mathbb{R}^n} R_\Omega u\|_{H^{\beta,q}_w(\mathbb{R}^n)} \\
=& c\|\psi U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{H^{\beta,q}_w(B_r)},
\end{aligned}$$

where $\psi$ is some cut-off function with $\operatorname{supp} \psi \subset B_r$ and $\psi = 1$ in $\Omega$ and $U \in H^{\beta,q}_w(\mathbb{R}^n)$ is some extension of $(E_{0,\mathbb{R}^n} R_\Omega u)|_{B_r} = u$ with $\|U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{H^{\beta,q}_w(B_r)}$.

Now the $R$-boundedness of $B(t)$ follows from the $R$-boundedness of $D(t)$ as before. $\square$

**Lemma 4.12.** *Let $0 < \delta < \varepsilon$, $\varepsilon \in (0, \frac{\pi}{2})$ and $w \in A_q$. Then the operator*

$$\langle\lambda\rangle^{1-\frac{\beta}{2}}(\lambda + \mathcal{A})^{-1} : L^q_{w,\sigma}(\Omega) \to Y^{\beta,q}_w(\Omega)$$

*is bounded uniformly with respect to $\lambda \in \Sigma_\delta \cup \{0\}$. This uniform bound depends $A_q$-consistently on $w$.*

*Proof.* For the cases $\beta = 0$ and $\beta = 2$ we observe that by [14] the strong solution $u$ of $(\lambda + \mathcal{A})u = f$ fulfills the estimate

$$|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \leq c\|f\|_{q,w}$$

with $c$ depending $A_q$-consistently on $w$. This yields $\|u\|_{2,q,w} \leq c\|f\|_{q,w}$, which is the assertion for $\beta = 2$ and $\langle\lambda\rangle\|u\|_{q,w} \leq c(|\lambda| + 1)\|u\|_{q,w} \leq c\|f\|_{q,w}$, which is the assertion for $\beta = 0$. Thus we have shown

$$\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma}, H^{\beta,q}_w)} \leq c\langle\lambda\rangle^{\frac{\beta}{2}-1} \text{ for } \beta = 0, 2.$$

Next we consider the case $\beta = 1$. By interpolation we obtain

$$\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma}, [L^q_w, H^{2,q}_w]_{\frac{1}{2}})} \leq c^{1-\frac{1}{2}}\langle\lambda\rangle^{-(1-\frac{1}{2})}c^{\frac{1}{2}} = c\langle\lambda\rangle^{-\frac{1}{2}},$$

where $c$ is independent of $\lambda$ and depends $A_q$-consistently on $w$. Now Lemma 2.6 yields

$$\|(\lambda + \mathcal{A})^{-1}f\|_{Y^{1,q}_w} \leq M\|(\lambda + \mathcal{A})^{-1}f\|_{[L^q_w, H^{2,q}_w]_{\frac{1}{2}}} \leq cM\langle\lambda\rangle^{-\frac{1}{2}}\|f\|_{q,w}.$$

This is the assertion for $\beta = 1$. For $\beta \in (0, 1)$ and $\beta \in (1, 2)$ we use reiteration. ☐

We obtain the following regularity result in the case of purely tangential boundary conditions.

**Lemma 4.13.** *Let $0 \leq \beta \leq 2$ and*

$$g \in L^r(0, T; T^{\beta,q}_w(\partial\Omega)) \cap H_0^{\frac{\beta}{2},r}((0, T]; T^{0,q}_w(\partial\Omega))$$

*be purely tangential. Let $u \in L^r(0, T; L^q_w(\Omega))$ be the unique very weak solution to the instationary Stokes problem with zero initial values, force and divergence and boundary condition $g$, i.e.,*

$$\begin{aligned} -\langle u, \partial_t\phi\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} &= -\langle g, N \cdot \nabla\phi\rangle_{\partial\Omega,T} && \textit{for all } \phi \in X^{r',q'}_{w',\sigma} \\ \langle u(t), \psi\rangle_\Omega &= 0 && \textit{for all } \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned} \tag{4.13}$$

*and almost every $t$. Then $u \in L^r(0, T; H^{\beta,q}_w(\Omega))$ and it fulfills the estimate*

$$\|u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} + \|u\|_{L^r(H^{\beta,q}_w)} \leq c\left(\|g\|_{L^r(T^{\beta,q}_w)} + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w)}\right),$$

*with $c = c(r, \Omega, q, A_q(w)) > 0$.*

*Proof.* By the Lemmas 4.7 and 4.8 we may assume that $g$ is extended to an element of $L^r(\mathbb{R}; T_w^{\beta,q}(\partial\Omega)) \cap H^{\frac{\beta}{2},r}(\mathbb{R}; T_w^{0,q}(\partial\Omega))$ with $g(t) = 0$ for $t < 0$. This is possible without increasing the magnitude of the norm of $g$. The extension is again denoted by $g$. Let

$$\begin{aligned} B : \{g \in T_w^{\beta,q}(\partial\Omega) \mid g \text{ purely tangential}\} \to & Y_w^{-2,q}(\Omega), \\ g \mapsto & [\phi \mapsto -\langle g, N \cdot \nabla\phi\rangle]. \end{aligned}$$

Let $u \in L^r(\mathbb{R}; L_w^q(\Omega))$ with $u(t) = 0$ for $t < 0$ and such that, for $t \geq 0$, it is the very weak solution to the instationary Stokes problem with exterior force $Bg$, for the extended function $g$. This solution exists by Theorem 4.3, is uniquely defined by $g$ and solves the Stokes equations in the sense of (4.13) with $T$ replaced by $\infty$. Moreover, by the uniqueness of very weak solutions, this function $u$ coincides on $[0, T]$ with the very weak solution with respect to the original $g$, given in the assumption of this theorem.

We have to show that it satisfies $u \in L^r(\mathbb{R}; H_w^{\beta,q}(\Omega))$ and fulfills the estimate. Set $u_1(t) := \mathcal{A}^{-1}Bg(t)$, where $\mathcal{A}$ is the generalized Stokes operator on $Y_{w,\sigma}^{-2,q}(\Omega)$. Then $u_1(t)|_{\partial\Omega} = g(t)$ in the sense of Theorem 2.10 for almost every $t$ since $g$ is purely tangential.

Since $\mathcal{A}^{-1}B : T_w^{0,q}(\partial\Omega) \to L_w^q(\Omega)$ is continuous, one obtains

$$\|u_1\|_{H^{\frac{\beta}{2},r}(\mathbb{R};L_w^q)} = \|\Lambda_t^{\frac{\beta}{2}}\mathcal{A}^{-1}Bg\|_{L^r(\mathbb{R};L_w^q)} = \|\mathcal{A}^{-1}B\Lambda_t^{\frac{\beta}{2}}g\|_{L^r(\mathbb{R};L_w^q)} \leq c\|g\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q})}.$$

Moreover, from the pointwise estimate in Theorem 2.11 we obtain $u_1 \in L^r(\mathbb{R}; H_w^{\beta,q}(\Omega))$ and the estimate $\|u_1\|_{L^r(\mathbb{R};H_w^{\beta,q})} \leq c\|g\|_{L^r(\mathbb{R};T_w^{\beta,q})}$. Now $u_2 := u - u_1$ solves

$$\partial_t u_2 + \mathcal{A}u_2 = -\partial_t u_1 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}, Y_{w,\sigma}^{-2,q}(\Omega)).$$

An application of the Fourier transformation with respect to the time variable $t$ yields $\hat{u}_2 = -it(it + \mathcal{A})^{-1}\hat{u}_1$.

As a next step we show that

$$M(t) := \langle t\rangle^{-\frac{\beta}{2}} t(it + \mathcal{A})^{-1}P_{q,w} \in \mathcal{L}(L_w^q(\Omega), Y_w^{\beta,q}(\Omega))$$

is a Fourier multiplier. Since

$$\|M(t)\|_{\mathcal{L}(L_w^q(\Omega),Y_w^{\beta,q}(\Omega))} \leq \|\langle t\rangle^{-\frac{\beta}{2}+1}(it + \mathcal{A})^{-1}P_{q,w}\|_{\mathcal{L}(L_w^q(\Omega),Y_w^{\beta,q}(\Omega))}$$

for every $t$, we find by Lemma 4.12 that $M(t)$ is uniformly bounded by a constant that depends $A_q$-consistently on $w$. By Theorem 4.11 this implies that $M(t)$ is $R$-bounded.

Moreover,

$$tM'(t) = (t\langle t\rangle^{-\frac{\beta}{2}} - \frac{\beta}{2}t^3\langle t\rangle^{-\frac{\beta}{2}-2})(it + \mathcal{A})^{-1}P_{q,w} - it^2\langle t\rangle^{-\frac{\beta}{2}}(it + \mathcal{A})^{-2}P_{q,w}.$$

Since $t(it+\mathcal{A})^{-1} : L_w^q(\Omega) \to L_w^q(\Omega)$ is uniformly bounded with an $A_q$-consistent constant and $t\langle t\rangle^{-\frac{\beta}{2}} - \frac{\beta}{2}t^3\langle t\rangle^{-\frac{\beta}{2}-2} \leq (1 + \frac{\beta}{2})\langle t\rangle^{1-\frac{\beta}{2}}$, this is $R$-bounded as before.

Combining the above with Theorem 4.10 and Lemma 4.8 shows that $M(t)$ is a multiplier. Thus

$$\begin{aligned} \|u_2\|_{L^r(H_w^{\beta,q})} \leq \|u_2\|_{L^r(Y_w^{\beta,q})} &= \|\mathcal{F}^{-1}iM(t)\langle t\rangle^{\frac{\beta}{2}}\hat{u}_1\|_{L^r(H_w^{\beta,q})} \\ &\leq c\|\mathcal{F}^{-1}\langle t\rangle^{\frac{\beta}{2}}\hat{u}_1\|_{L^r(L_w^q)} = c\|u_1\|_{H^{\frac{\beta}{2},r}(L_w^q)} \leq c\|g\|_{H^{\frac{\beta}{2},r}(T_w^{0,q})}. \end{aligned}$$

Using this we are in the position to estimate the time derivative of $u$ because

$$\|\partial_t u\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} = \|\mathcal{A}u_2\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} \le c\|u_2\|_{L^r(Y^{\beta,q}_{w,\sigma})} \le c\|g\|_{H^{\frac{\beta}{2},r}(T^{0,q}_w)}.$$

Combining this with the estimate for $u_1$ implies

$$\begin{aligned}\|u_t\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} + \|u\|_{L^r(H^{\beta,q}_w)} &\le c\left(\|g\|_{L^r(T^{\beta,q}_w)} + \|g\|_{H^{\frac{\beta}{2},r}(T^{0,q}_w)}\right)\\ &= c\|g\|_{L^r(T^{\beta,q}_w)\cap H^{\frac{\beta}{2},r}(T^{0,q}_w)}.\end{aligned}$$

□

## 4.4 Solutions to Fully Inhomogeneous Data

In the following we consider external forces

$$f \in L^r(0,T;Y^{\beta-2,q}_w(\Omega)) = \left[L^r(0,T;Y^{-2,q}_w(\Omega)), L^r(0,T;L^q_w(\Omega))\right]_{\frac{\beta}{2}} \quad \text{for} \quad 0 \le \beta \le 2,$$

where the equality of the spaces follows from [28, 1.18.4] combined with Theorem 2.5. For such forces one obtains very weak solutions to the instationary Stokes problem by interpolation.

**Lemma 4.14.** *For every $f \in L^r(0,T;Y^{\beta-2,q}_w(\Omega))$ there exists a unique solution $u \in L^r(0,T;Y^{\beta,q}_{w,\sigma}(\Omega))$ to the Stokes equation*

$$u_t + \mathcal{A}u = f|_{Y^{2,q'}_{w',\sigma}(\Omega)} \quad \textit{in} \quad \mathcal{D}'(0,T;Y^{\beta-2,q}_{w,\sigma}(\Omega)) \quad \textit{with} \quad u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} = 0.$$

*It fulfills the estimate*

$$\|u\|_{L^r(Y^{\beta,q}_{w,\sigma})} \le c\|f|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^r(Y^{\beta-2,q}_{w,\sigma})}.$$

*Proof.* By Corollary 4.4 and Lemma 4.5 this is true for $\beta = 0$. Since for $f(t) \in L^q_{w,\sigma}(\Omega)$ one has $f|_{Y^{2,q'}_{w',\sigma}(\Omega)} = P_{q,w}f|_{Y^{2,q'}_{w',\sigma}(\Omega)}$ the solution operator

$$L : L^r(0,T;L^q_w(\Omega)) \ni f \mapsto u \in L^r(0,T;Y^{2,q}_{w,\sigma}(\Omega)),$$

where $u$ is the strong solution to the instationary Stokes equations with force $f$, is well-defined, continuous by Theorem 4.2 and it coincides with the very weak solution with respect to $\phi \mapsto \langle f, \phi\rangle$ by the uniqueness of the very weak solution in Theorem 4.3.

Thus we may apply interpolation to the solution operator $L : f \mapsto u$

$$\begin{aligned}&L : L^r(0,T;Y^{-2,q}_w(\Omega)) \to L^r(0,T;L^q_{w,\sigma}(\Omega)) \quad \text{and}\\ &L : L^r(0,T;L^q_w(\Omega)) \to L^r(0,T;Y^{2,q}_{w,\sigma}(\Omega))\end{aligned}$$

and we obtain the assertion. □

Our space of initial values is

$$\mathcal{I}_w^{\beta,q,r} = \mathcal{I}_w^{\beta,q,r}(\Omega) := \left\{ u_0 \in Y_{w,\sigma}^{\beta-2,q}(\Omega) \;\Big|\; \int_0^\infty \|e^{-t\mathcal{A}}u_0\|_{\beta,q,w}^r dt < \infty \right\},$$

where $e^{-t\mathcal{A}}$ is the semigroup that is generated by the generalized Stokes operator $\mathcal{A}$ in $Y_{w,\sigma}^{\beta-2,q}(\Omega)$ with

$$e^{-t\mathcal{A}} : Y_{w,\sigma}^{\beta-2,q}(\Omega) \to \mathcal{D}(\mathcal{A}) = Y_{w,\sigma}^{\beta,q}(\Omega) \subset H_{w,\sigma}^{\beta,q}(\Omega).$$

It is equipped with the norm $\|u_0\|_{\mathcal{I}_w^{\beta,q,r}} := \|u_0\|_{Y_{w,\sigma}^{\beta-2,q}} + \|e^{-t\mathcal{A}}u_0\|_{L^r(H_w^{\beta,q})}$.

**Lemma 4.15.** *$\mathcal{I}_w^{2,q,r}$ is dense in $\mathcal{I}_w^{\beta,q,r}$ for every $\beta \in [0,2]$.*

*Proof.* If $\beta = 2$ nothing is to show. Thus we assume $\beta \in [0,2)$.

For $u_0 \in \mathcal{I}_w^{\beta,q,r}$ and $\lambda > 0$ we set $u_\lambda := \lambda(\lambda + \mathcal{A})^{-1}u_0$. Recall the inequalities

$$\|(\lambda + \mathcal{A})^{-1}x\|_{L_{w,\sigma}^q} \le c\|x\|_{Y_{w,\sigma}^{-2,q}} \quad \text{and} \quad \|(\lambda + \mathcal{A})^{-1}x\|_{Y_{w,\sigma}^{2,q}} \le c\|x\|_{L_{w,\sigma}^q},$$

which are true with $c$ independent of $\lambda$ by Theorem 3.2. Using this and the definition of the norm in $\mathcal{I}_w^{\beta,q,r}$ one shows that $\|u_\lambda\|_{\mathcal{I}_w^{2,q,r}} \le c(\lambda)\|u_0\|_{\mathcal{I}_w^{0,q,r}} \le c(\lambda)\|u_0\|_{\mathcal{I}_w^{\beta,q,r}}$. This yields $u_\lambda \in \mathcal{I}_w^{2,q,r}$. Moreover, since $x(t) := e^{-t\mathcal{A}}u_0 \in Y_{w,\sigma}^{2,q}(\Omega)$ we find by Lemma 4.12

$$\|\lambda(\lambda + \mathcal{A})^{-1}x(t) - x(t)\|_{Y_{w,\sigma}^{\beta,q}} \le \frac{1}{\langle\lambda\rangle^{1-\frac{\beta}{2}}}\|\mathcal{A}x(t)\|_{q,w} \xrightarrow{\lambda\to\infty} 0. \tag{4.14}$$

Since $\|\mathcal{A}(\lambda+\mathcal{A})^{-1}x(t)\|_{Y_{w,\sigma}^{\beta,q}} \le c\|\mathcal{A}x(t)\|_{Y_{w,\sigma}^{\beta-2,q}} \in L^r(\mathbb{R}_+)$ with $c$ independent of $\lambda$ we have by Lebesgue's Theorem

$$\|e^{-t\mathcal{A}}u_\lambda - e^{-t\mathcal{A}}u_0\|_{Y_{w,\sigma}^{\beta,q}} = \|\lambda(\lambda + \mathcal{A})^{-1}x(t) - x(t)\|_{Y_{w,\sigma}^{\beta,q}} \to 0 \;\text{ in }\; L^r(\mathbb{R}_+)$$

as $\lambda \to \infty$. In addition Lemma [19, Lemma I.3.2] implies that $u_\lambda \to u_0$ in $Y_{w,\sigma}^{\beta-2,q}(\Omega)$ as $\lambda \to \infty$ and we obtain convergence in $\mathcal{I}_w^{\beta,q,r}$. □

**Lemma 4.16.** *Let $1 < q < \infty$, $\beta \in [0,2]$ and let $w \in A_q$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,1}$-domain if $\beta > 1$ and a bounded $C^{1,1}$-domain if $\beta \le 1$.*

*Then the Helmholtz projection $P_{q,w} : H_w^{\beta,q}(\Omega) \to H_w^{\beta,q}(\Omega)$ is continuous.*

*Proof.* This follows by interpolation from the corresponding assertions for $\beta = 0, 1, 2$. The assertion for $\beta = 0$ follows from [12] and the one for $\beta = 1$ and $\beta = 2$ follows from the regularity of solutions to the weak Neumann problem in weighted spaces that has been proved in [21, A2]. □

**Theorem 4.17.** *Let $1 < q < \infty$, $\beta \in [0,2]$ and let $w \in A_q$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,1}$-domain if $\beta > 1$ and a bounded $C^{1,1}$-domain if $\beta \le 1$. Moreover, we take*

$$\begin{aligned}
f &\in L^r(0,T; Y_w^{\beta-2,q}(\Omega)),\\
k &\in H_0^{\frac{\beta}{2},r}((0,T]; W_{w,0}^{-1,q}(\Omega)) \cap L^r(0,T; H_{w,*}^{\beta-1,q}(\Omega)),\\
g &\in H_0^{\frac{\beta}{2},r}((0,T]; T_w^{0,q}(\partial\Omega)) \cap L^r(0,T; T_w^{\beta,q}(\partial\Omega)),\\
u_0 &\in \mathcal{I}_w^{\beta,q,r}(\Omega),
\end{aligned}$$

*fulfilling the compatibility condition* $\langle k(t), 1\rangle_\Omega = \langle g(t), N\rangle_{\partial\Omega}$, *for almost all* $t \in (0,T)$.

*Then there exists a unique very weak solution* $u \in L^r(0,T; H_w^{\beta,q}(\Omega))$ *to the instationary Stokes system, i.e.,*

$$\begin{aligned} -\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} &= -\langle u_0, \phi(0)\rangle_\Omega + \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} \\ -\langle u(t), \nabla\psi\rangle_\Omega &= \langle k(t), \psi\rangle_\Omega - \langle g(t), N\psi\rangle_{\partial\Omega} \quad \textit{for a.e.} \quad t \in [0,T] \end{aligned}$$

*for all* $\phi \in X_{w',\sigma}^{r',q'}$ *and* $\psi \in W_{w'}^{1,q'}(\Omega)$.

*Moreover, there exists a pressure functional* $p \in H^{-1,r}(0,T; H_w^{\beta-1,q}(\Omega))$ *that is unique modulo constants, such that*

$$\partial_t u - \Delta u + \nabla p = f|_{C_0^\infty(\Omega)}$$

*is fulfilled in the sense of distributions on* $(0,T)\times\Omega$. *This solution* $(u,p)$ *fulfills the estimate*

$$\begin{aligned} &\|u_t|_{Y_{w',\sigma}^{2,q'}(\Omega)}\|_{L^r(0,T;Y_{w,\sigma}^{\beta-2,q}(\Omega))} + \|u\|_{L^r(H_w^{\beta,q})} + \|p\|_{H^{-1,r}(H_w^{\beta-1,q})} \\ &\qquad \le c\left(\|f\|_{L^r(H_w^{\beta-2,q})} + \|k\|_{H_0^{\frac{\beta}{2},r}((0,T];W_{w,0}^{-1,q})\cap L^r(H_{w,*}^{\beta-1,q})}\right. \\ &\qquad\quad \left. + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q})\cap L^r(T_w^{\beta,q})} + \|u_0\|_{\mathcal{I}_w^{\beta,q,r}}\right) \end{aligned} \tag{4.15}$$

*with* $c = c(\Omega, r, \beta, q, w) > 0$.

**Remark 4.18.** The right hand side in the above theorem is

$$[\phi \mapsto -\langle u_0, \phi(0)\rangle_\Omega + \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T}] \in (X_{w'}^{r',q'})'.$$

This means the case of non-zero initial conditions requires no generalization of the definition of the very weak solution given in Definition 4.1.

*Proof. Step 1.* We start with the divergence and the normal part of the boundary condition.

Let $\tilde{u}_1(t) \in H_w^{\beta,q}(\Omega)$ be the very weak solution to the stationary Stokes system with external force 0, boundary condition $g(t)$ and divergence $k(t)$. Moreover, set $u_1(t) := \tilde{u}_1(t) - P_{q,w}\tilde{u}_1(t)$. Then one has by Lemma 4.16

$$u_1(t) \in H_w^{\beta,q}(\Omega), \quad u_1(t) = \nabla\pi(t)$$

and for almost every $t \in [0,T]$ and every $\psi \in W_{w'}^{1,q'}(\Omega)$ one has by (3.3)

$$\langle\nabla\pi, \nabla\psi\rangle_\Omega = \langle u_1(t), \nabla\psi\rangle_\Omega = \langle\tilde{u}_1(t), \nabla\psi\rangle_\Omega = -\langle k(t), \psi\rangle_\Omega + \langle g(t), N\psi\rangle_{\partial\Omega}.$$

This function $\pi$ can be chosen such that $\int_\Omega \pi = 0$.

The a priori estimate of the solution to the stationary problem combined with the continuity of $P_{q,w}$ on $H_w^{\beta,q}(\Omega)$ shown in Lemma 4.16 implies $u_1 \in L^r(0,T; H_w^{\beta,q}(\Omega))$. Thus by Lemma 4.7 one has $\partial_t u_1 \in H^{-1,r}(0,T; H_w^{\beta,q}(\Omega))$ and it cannot be expected to be a function in time. However, since $u_1$ is a gradient, for $\phi \in C_0^\infty(0,T;Y^{2,q'}_{w',\sigma}(\Omega))$ one has

$$\langle \partial_t u_1, \phi\rangle_{\Omega,T} = -\langle u_1, \partial_t P_{q',w'}\phi\rangle_{\Omega,T} = -\langle P_{q,w}u_1, \partial_t \phi\rangle_{\Omega,T} = 0.$$

Thus the estimate for $\partial_t u_1|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^r(0,T;Y^{\beta-2,q}_{w,\sigma}(\Omega))$ is obvious.

Next we have to show that the tangential component of the boundary value $\gamma(u_1)$ of $u_1$ is well-defined in the sense of Theorem 2.10 and fulfills the estimate

$$\begin{aligned}\|\gamma(u_1)\|_{L^r(T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}(T^{0,q}_w)} &\le c\|u_1\|_{L^r(H^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}(L^q_w)} \\ &\le c\left(\|k\|_{L^r(H^{\beta-1,q}_{w,*})\cap H_0^{\frac{\beta}{2},r}(W^{-1,q}_{w,0})} + \|g\|_{L^r(T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}(T^{0,q}_w)}\right).\end{aligned} \tag{4.16}$$

We begin proving the following pointwise inequality

$$\|\gamma(u_1(t))\|_{T^{\beta,q}_w} \le c\|u_1(t)\|_{H^{\beta,q}_w} \le c(\|k(t)\|_{H^{\beta-1,q}_{w,*}} + \|g(t)\|_{T^{\beta,q}_w}). \tag{4.17}$$

The second inequality follows from the a priori estimate of the stationary Stokes equation in Theorem 2.11 combined with the continuity of $P_{q,w}$. Hence it remains to prove the first.

If $\beta \ge 1$ this follows from the continuity of the restriction $v \mapsto v|_{\partial\Omega} : H^{\beta,q}_w(\Omega) \to T^{\beta,q}_w(\partial\Omega)$. Thus we assume $0 \le \beta < 1$. Since $\Delta u_1(t) = \nabla\Delta\pi(t)$ one has $\Delta u_1(t)|_{C^\infty_{0,\sigma}(\Omega)} = 0$. This means $\gamma(u_1(t)) \in T^{0,q}_w(\partial\Omega)$ is well-defined by Theorem 2.10. Moreover, if $\beta = 0$, this means that the mapping

$$W^{1,q}_w(\Omega) \ni \pi \mapsto \gamma(\nabla\pi) \in T^{0,q}_w(\partial\Omega)$$

is continuous and, by the definition of $T^{1,q}_w(\Omega)$, it is also bounded as an operator

$$\gamma \circ \nabla : W^{2,q}_w(\Omega) \to T^{1,q}_w(\partial\Omega).$$

Hence by interpolation we obtain the continuity of $\gamma\circ\nabla : H^{\beta+1,q}_w(\Omega) \to T^{\beta,q}_w(\partial\Omega)$ and this implies the pointwise estimate (4.17) for almost every $t$, where one uses the Lemmas 2.8 and 2.7 to verify

$$\|\pi\|_{H^{\beta+1,q}_w} \le c\left(\|\nabla\pi\|_{H^{\beta,q}_w} + \|\pi\|_{H^{\beta,q}_w}\right) \le c\left(\|\nabla\pi\|_{H^{\beta,q}_w} + \|\nabla\pi\|_{H^{\beta-1,q}_w}\right) \le c\|u_1\|_{H^{\beta,q}_w},$$

since $\pi$ has mean value 0. Thus we obtain

$$\|\gamma(u_1)\|_{L^r(T^{\beta,q}_w)} \le c\|u_1\|_{L^r(H^{\beta,q}_w)} \le c(\|k\|_{L^r(H^{\beta-1,q}_{w,*})} + \|g\|_{L^r(T^{\beta,q}_w)}). \tag{4.18}$$

In particular (4.17) holds for $\beta$ replaced by 0. Assume for a moment that $k$, $g$ and $u_1$ are defined on $\mathbb{R}\times\Omega$ with $\operatorname{supp} k, \operatorname{supp} g \subset [0,\infty)$ in time. Obviously the operator $\Lambda_t$ acting in time commutes with the continuous operator $(g(t),k(t)) \mapsto u_1(t)$ acting in space. Combining this with (4.17) implies

$$\begin{aligned}\|\gamma(u_1)\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T^{0,q}_w)} \le& c\|u_1\|_{H^{\frac{\beta}{2},r}(\mathbb{R};L^q_w)} \\ \le& c\left(\|k\|_{H^{\frac{\beta}{2},r}(\mathbb{R};H^{-1,q}_{w,0})} + \|g\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T^{0,q}_w)}\right).\end{aligned} \tag{4.19}$$

For $g$ and $k$ given as in the assumption of this theorem by Lemma 4.7 there exist extensions $Eg \in H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q}(\partial\Omega))$ and $Ek \in H^{\frac{\beta}{2},r}(\mathbb{R};H_{w,0}^{-1,q}(\Omega))$. The resulting $u_1^E$ fulfills $\operatorname{supp} u_1^E \subset \operatorname{supp} Eg \cup \operatorname{supp} Ek$ in time. Thus we obtain

$$\|\gamma(u_1)\|_{H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q})} \leq c\left(\|k\|_{H_0^{\frac{\beta}{2},r}((0,T];H_{w,0}^{-1,q})} + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q})}\right) \tag{4.20}$$

from (4.19). Combining (4.18) and (4.20) implies that the tangential component of the boundary value of $u_1$ fulfills $\gamma(u_1) \in H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q}(\partial\Omega)) \cap L^r(0,T;T_w^{\beta,q}(\partial\Omega))$ and the estimate (4.16).

*Step 2.* We consider the tangential component of the boundary condition.

Let $u_2 \in L^r(0,T;H_w^{\beta,q}(\Omega))$ be the solution to the instationary Stokes system with vanishing initial condition, exterior force, divergence and the purely tangential boundary condition

$$g_{tan} - \gamma(u_1) \in H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q}(\partial\Omega)) \cap L^r(0,T;T_w^{\beta,q}(\partial\Omega)),$$

where $g_{tan}$ is the tangential component of $g$. Such a function $u_2$ exists by Lemma 4.13 and fulfills the estimate

$$\begin{aligned}
\|(\partial_t u_2)|_{Y_{w',\sigma}^{2,q'}}\|_{L^r(Y_{w,\sigma}^{\beta-2,q})} &+ \|u_2\|_{L^r(H_w^{\beta,q})} \\
&\leq c\left(\|g_{tan}\|_{H_0^{\frac{\beta}{2},r}(T_w^{0,q})\cap L^r(T_w^{\beta,q})} + \|\gamma(u_1)\|_{H_0^{\frac{\beta}{2},r}(T_w^{0,q})\cap L^r(T_w^{\beta,q})}\right) \\
&\leq c\left(\|k\|_{H_0^{\frac{\beta}{2},r}(H_{w,0}^{-1,q})} + \|g\|_{H_0^{\frac{\beta}{2},r}(T_w^{0,q})} + \|k\|_{L^r(H_{w,*}^{\beta-1,q})} + \|g\|_{L^r(T_w^{\beta,q})}\right),
\end{aligned}$$

where in the last inequality we have used (4.16).

*Step 3.* The next step is to consider the initial values.

We set $u_3(t) = e^{-t\mathcal{A}}u_0$, where $e^{-t\mathcal{A}}$ is the semigroup generated by the generalized Stokes operator in $Y_{w,\sigma}^{\beta-2,q}(\Omega)$. Then $u_3$ is a solution to

$$\partial_t u_3 + \mathcal{A}u_3 = 0, \qquad u_3|_{Y_{w',\sigma}^{2,q'}(\Omega)}(0) = u_0.$$

By the definition of the space of initial values $\mathcal{I}_w^{\beta,q,r}$ it fulfills the estimate

$$\left\|\partial_t u_3|_{Y_{w',\sigma}^{2,q'}(\Omega)}\right\|_{L^r(Y_{w,\sigma}^{\beta-2,q})} + \|u_3\|_{L^r(H_w^{\beta,q})} \leq \|u_0\|_{\mathcal{I}_w^{\beta,q,r}}.$$

*Step 4.* It remains to treat the external force.

By Lemma 4.14 there exists a unique very weak solution $u_4 \in L^r(0,T;Y_{w,\sigma}^{\beta,q}(\Omega))$ solving

$$\partial_t u_4 + \mathcal{A}u_4 = f|_{Y_{w',\sigma}^{2,q'}(\Omega)}, \quad u_4|_{Y_{w',\sigma}^{2,q'}(\Omega)}(0) = 0.$$

It fulfills the estimate $\left\|\partial_t u_4|_{Y_{w',\sigma}^{2,q'}(\Omega)}\right\|_{L^r(Y_{w,\sigma}^{\beta-2,q})} + \|u_4\|_{L^r(Y_w^{\beta,q}(\Omega))} \leq c\|f\|_{L^r(Y_w^{\beta-2,q})}$.

*Step 5.* Summarizing the above shows that $u := u_1 + u_2 + u_3 + u_4 \in L^r(0,T;H_w^{\beta,q}(\Omega))$

is a very weak solution as required. The function $u$ fulfills the estimate

$$\left\|\partial_t u|_{Y^{2,q'}_{w',\sigma}(\Omega)}\right\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} + \|u\|_{L^r(H^{\beta,q}_w)} \leq c\bigg(\|f\|_{L^r(H^{\beta-2,q}_w)} + \|k\|_{L^r(H^{\beta-1,q}_{w,*})\cap H^{\frac{\beta}{2},r}_0(W^{-1,q}_{w,0})}$$
$$+ \|g\|_{L^r(T^{\beta,q}_w)\cap H^{\frac{\beta}{2},r}_0(T^{0,q}_w)} + \|u_0\|_{\mathcal{I}^{\beta,q,r}_w}\bigg),$$

since this is true for $u_1$, $u_2$, $u_3$ and $u_4$.
*Step 6.* The uniqueness of $u$ follows from the uniqueness of the very weak solution proved in Theorem 4.3.
*Step 7.* It remains to show existence and estimates for the pressure functional.

We approximate $f, k, g, u_0$ by functions

$$f_n \in L^r(0,T;L^q_w(\Omega)), \qquad k_n \in H^{1,r}_0(0,T;H^{1,q}_w(\Omega)),$$
$$g_n \in H^{1,r}_0(0,T;T^{2,q}_w(\partial\Omega)), \qquad u_{0,n} \in \mathcal{I}^{2,q,r}_w$$

in the norms of the corresponding spaces for the data as in the assumptions of this theorem. Then one obtains as above a strong solution

$$u_n \in L^r(0,T;H^{2,q}_w(\Omega)) \quad \text{with} \quad \partial_t u_n \in L^r(0,T;L^q_w(\Omega))$$

to the Stokes problem with respect to the data $u_{0,n}, f_n, g_n, k_n$. By the uniqueness proved in Step 5 the functions $u_n$ fulfill the a priori estimate (4.15). This implies $u_n \to u$ in $L^r(0,T;H^{\beta,q}_w(\Omega))$.

By de Rham's Theorem [27] there exists $p_n(t) \in (C^\infty_0(\Omega))'$ such that

$$\partial_t u_n(t) - \Delta u_n(t) + \nabla p_n(t) = f_n(t) \;\text{ almost everywhere on }\; (0,T)\times\Omega.$$

Since $\nabla p_n \in L^r(0,T;L^q_w(\Omega))$ one has by Lemma 2.8 that $p(t) \in W^{1,q}_w(\Omega)$ for almost every $t$. We choose $p_n(t)$ such that $\int p_n(t)\,dx = 0$ for every $t$.

Every $\nabla p_n$ fulfills the estimate

$$\|\nabla p_n\|_{H^{-1,r}(H^{\beta-2,q}_w)} \leq c\big(\|k_n\|_{L^r(H^{\beta-1,q}_{w,*})\cap H^{\frac{\beta}{2},r}_0(H^{-1,q}_{w,0})} + \|g_n\|_{L^r(T^{\beta,q}_w)\cap H^{\frac{\beta}{2},r}_0(T^{0,q}_w)}$$
$$+ \|f_n\|_{L^r(Y^{\beta-2,q}_w)} + \|u_{0,n}\|_{\mathcal{I}^{\beta,q,r}_w}\big),$$

where we have used $Y^{\beta-2,q}_w(\Omega)|_{H^{2-\beta,q'}_{w',0}} \hookrightarrow H^{\beta-2,q}_w(\Omega)$ and Lemma 4.7 to show

$$\|\partial_t u_n\|_{H^{-1,r}(H^{\beta-2,q}_w)} \leq c\|u_n\|_{L^r(H^{\beta-2,q}_w)} \leq c\|u_n\|_{L^r(H^{\beta,q}_w)}.$$

Moreover, by Lemma 4.7 one has $H^{-1,r}(0,T;H^{\beta-2,q}_w(\Omega)) = W^{-1,r}(0,T;H^{\beta-2,q}_w(\Omega))$ and for every $\phi \in W^{1,r}_0(0,T;H^{1-\beta,q'}_{w',0}(\Omega))$ with $\langle\phi(t),1\rangle = 0$ we find $\zeta \in W^{1,r}_0(0,T;H^{2-\beta,q'}_{w',0}(\Omega))$ with

$$-\langle\zeta(t),\nabla\psi\rangle_\Omega = \langle\phi(t),\psi\rangle_\Omega \quad \text{for all} \;\; \psi \in W^{1,q'}_{w'}(\Omega)$$

and $\|\zeta\|_{W^{1,r}_0(H^{2-\beta,q'}_{w',0})} \leq c\|\phi\|_{W^{1,r}_0(H^{1-\beta,q'}_{w',0})}$. For $\beta \in [0,1]$ we may choose $\zeta(t)$ to be equal to the very weak solution to the stationary Stokes equation with 0 external force and divergence $\phi(t)$ that exists by Theorem 2.11. For $\beta \in (1,2]$ we may apply complex interpolation to the Bogowski operator on $L^q_w(\Omega)$ and on $W^{1,q}_{w,0}(\Omega)$. The continuity of this operator between weighted spaces has been shown in [22].

For $\phi \in C_0^\infty((0,T)\times\Omega)$ with mean value 0 one has the estimate

$$|\langle p_n, \phi\rangle_{\Omega,T}| = |\langle \nabla p_n, \zeta\rangle_{\Omega,T}| \le c\|\nabla p_n\|_{H^{-1,r}(H_w^{\beta-2,q})}\|\phi\|_{H_0^{1,r}(H_{w',0}^{1-\beta,q'})}.$$

Combining the above yields the estimate

$$\begin{aligned}\|p_n\|_{H^{-1,r}(H_w^{\beta-1,q})} \le & c\big(\|k_n\|_{L^r(H_{w,*}^{\beta-1,q})\cap H^{\frac{\beta}{2},r}(H_{w,0}^{-1,q})} + \|g_n\|_{L^r(T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}(T_w^{0,q})} \\ & + \|f_n\|_{L^r(H_w^{\beta-2,q})} + \|u_{0,n}\|_{\mathcal{I}_w^{\beta,q,r}}\big).\end{aligned}$$

Replacing $p_n$ by $p_n - p_m$ in the above estimates shows that $(p_n)$ is a Cauchy sequence in $H^{-1,r}(0,T;H_w^{\beta-1,q}(\Omega))$ converging to some $p \in H^{-1,r}(0,T;H_w^{\beta-1,q}(\Omega))$.

The couple $(u,p)$ solves the Stokes equations in the distributional sense and fulfills the a priori estimate. □

Note that the solution constructed in Theorem 4.17 does in general not fulfill $\partial_t u \in L^r(0,T;Y_w^{\beta-2,q}(\Omega))$. Accordingly, the pressure $p$ is contained in $H^{-1,r}(0,T;H_w^{\beta-1,q}(\Omega))$. This result could be improved to

$$p \in H^{-1,r}(0,T;H_w^{\beta,q}(\Omega)) + L^r(0,T;H_w^{\beta-1,q}(\Omega)),$$

but $p$ is in general not integrable in time.

Another problem concerns the boundary values. In the above theorem boundary conditions are included even though for $0 \le \beta < 1$ the equation $u|_{\partial\Omega} = g$ in general makes no sense. The reason is that $u$ is in general not smooth enough to make its restriction to the boundary well-defined.

However, if data and solution are regular enough, this can be established a posteriori. More precisely, let $\mu \in (1,\infty)$ and $\tilde{w} \in A_\mu$ such that $L^\mu_{\tilde{w}}(\Omega) \hookrightarrow W^{\beta-1,q}_{w,0}(\Omega)$ and assume $k \in L^r(0,T;L^\mu_{\tilde{w}}(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];H^{-1,q}_{w,0}(\Omega))$. Then the normal component of the boundary condition can be defined as in the stationary case and one obtains

$$\langle u(t), N\psi\rangle_{\partial\Omega} = \langle u(t), \nabla\psi\rangle_\Omega + \langle \operatorname{div} u(t), \psi\rangle_\Omega = \langle g(t), N\psi\rangle_{\partial\Omega}$$

for almost every $t$ and every $\psi \in W^{1,q'}_{w'}(\Omega)$. Thus the normal component of $u$ is equal to the one of $g$.

The tangential component causes more difficulties than in the stationary case. The reason is that $f \in L^r(0,T;W^{-1,\mu}_{\tilde{w}}(\Omega))$ does in general not imply $\partial_t u(t) \in W^{-1,\mu}_{\tilde{w}}(\Omega)$ for almost every $t$. And this is necessary to ensure $u(t) \in \tilde{W}^{q,\mu}_{w,\tilde{w}}$, the space in which the tangential component of the boundary values is well-defined.

Hence, to ensure that the tangential boundary condition is well-defined we assume

$$f \in L^r(0,T;W^{-1,\mu}_{\tilde{w}}(\Omega)) \quad \text{and} \quad u \in L^r(0,T;H^{\beta,q}_w(\Omega)), \quad u_t(t) \in W^{-1,\mu}_{\tilde{w}}(\Omega) \tag{4.21}$$

for almost every $t$. Then, using test functions of the form $\eta\cdot\phi$ with $\eta \in C_0^\infty(0,T)$ and $\phi \in C_0^\infty(\Omega)$ one shows that for every $\phi \in C^\infty_{0,\sigma}(\Omega)$ and almost every $t$ one has

$$\langle \Delta u(t), \phi\rangle_\Omega = \langle u(t), \Delta\phi\rangle_\Omega = \langle \partial_t u(t), \phi\rangle_\Omega - \langle f(t), \phi\rangle_\Omega$$

which implies $u(t) \in \tilde{W}^{q,\mu}_{w,\tilde{w}}$ for almost every $t$ by the assumptions on $f$ and $u_t$. Moreover,

$$\langle u, N \cdot \nabla\phi\rangle_{\partial\Omega,T} = \langle u, \Delta\phi\rangle_{\Omega,T} - \langle \Delta u, \phi\rangle_{\Omega,T} = \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega,T}$$

for every $\phi \in W^{1,r}(0,T; Y^{2,q'}_{w',\sigma}(\Omega))$ with $\phi(0) = \phi(T) = 0$. This means that $u$ fulfills the tangential boundary condition almost everywhere.

In particular, (4.21) is fulfilled in the case of weak solutions. Thus one has the following proposition.

**Proposition 4.19.** *Let $\beta \in [1,2]$ and the data $f, k, g$ and $u_0$ be chosen according to Theorem 4.17 and let $u \in L^r(0,T; H^{\beta,q}_w(\Omega))$ be a very weak solution with respect to this data.*

*Then $(u,p)$ fulfills the Stokes system (1.1) in the sense of distributions. In addition $u(t)|_{\partial\Omega} = g(t)$ for almost every $t$.*

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