The Stationary Navier-Stokes Equations in Weighted Bessel-Potential Spaces

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We investigate the stationary Navier-Stokes equations in Bessel-potential spaces with Muckenhoupt weights. Since in this setting it is possible that the solutions do not posses any weak derivatives, we use the notation of very weak solutions introduced by Amann [1]. The basic tool is complex interpolation, thus we give a characterization of the interpolation spaces of the spaces of data and solutions. Then we establish a theory of solutions to the Stokes equations in weighted Bessel-potential spaces and use this to prove solvability of the Navier-Stokes equations for small data by means of Banach's Fixed Point Theorem.

Key Words and Phrases: Stokes and Navier-Stokes equations, Muckenhoupt weights, very weak solutions, Bessel Potential spaces, nonhomgeneous data
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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-boundary. We consider the stationary Navier-Stokes problem with inhomogeneous data

$$\begin{aligned} -\Delta u + u \cdot \nabla u + \nabla p &= F && \text{in } \Omega \\ \operatorname{div} u &= K && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

It is our aim to find a class of solutions to (1.1) in a Bessel-potential space $H^{\beta,q}(\Omega)$, $\beta \in [0, 2]$. This means we develop a solution theory that includes strong solutions in the case $\beta = 2$ and weak solutions in the case $\beta = 1$. However, if $\beta = 0$, it is also possible that the solutions are only contained in $L^q(\Omega)$, i.e., they do not possess any

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weak derivatives. Consequently the notion of weak solutions is no longer suitable in this context. Thus one introduces the more general notion of very weak solutions. To arrive there one multiplies the first equation in (1.1) with a solenoidal test function ϕ vanishing on the boundary, then formal integration by parts yields

$$-\langle u, \Delta\phi\rangle - \langle uu, \nabla\phi\rangle - \langle Ku, \phi\rangle = \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}. \tag{1.2}$$

Applying the same method to the second equation with a sufficiently smooth test function ψ we obtain

$$-\langle u, \nabla\psi\rangle = \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}. \tag{1.3}$$

The equations (1.2) and (1.3) can be used for the definition of very weak solutions. This or similar formulations have been introduced by Amann in [1], by Amrouche and Girault in [2] and by Galdi, Simader and Sohr in [14]. In these articles as well as by Farwig, Galdi and Sohr in [7], [6], [8] and by Giga in [16] solvability with low-regularity data has been shown.

We investigate this problem in weighted function spaces. More precisely, we consider Lebesgue- and Sobolev- and Bessel potential spaces with respect to the measure $w\,dx$, where w is a weight function contained in the Muckenhoupt class A_q, cf., (2.1) below.

Classical tools for the treatment of partial differential equations extend to function spaces with Muckenhoupt weights. As important examples we mention the continuity of the maximal operator and the multiplier theorems that can be found in the books of García-Cuerva and Rubio de Francia [15] and Stein [25]; extension theorems of functions on a domain to functions on $\mathbb{R}^n$ have been shown by Chua [4], extension theorems of functions on the boundary to functions on the domain by Fröhlich [12], see also [20] and embedding theorems by Fröhlich [13] using the continuity of singular integral operators by Sawyer and Wheeden [19].

These tools were the base to treat the solvability of the Stokes and Navier-Stokes equations in weighted function spaces by Farwig and Sohr in [9] and by Fröhlich in [10], [11], [12].

As shown in [9] examples of Muckenhoupt weights are

$$\begin{array}{ll} w(x) = (1+|x|)^\alpha, & -n < \alpha < n(q-1) \ \text{ or} \\ \operatorname{dist}(x, M)^\alpha, & -(n-k) < \alpha < (n-k)(q-1), \end{array}$$

where M is a compact k-dimensional Lipschitzian manifold. Thus, if one chooses a particular weight function, the developed theory can be used for a better control of the growth of the solution, for example in the neighborhood of a point or close to the boundary.

In Section 4 we prove the solvability of the linear Stokes equations in weighted Bessel potential spaces. To arrive there, we use complex interpolation between the strong and the very weak solutions. The notion of very weak solutions used in this context is slightly more general than the one mentioned above. More precisely, one considers each right hand side of (1.2) and (1.3) as one functional

$$f = [\phi \mapsto \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}] \ \text{ or } \ k = [\psi \mapsto \langle K, \psi\rangle_\Omega - \langle g, N\cdot\psi\rangle_{\partial\Omega}].$$

As a consequence it is no longer distinguished between boundary condition and force, or between boundary condition and divergence, respectively, and since the data may contain a part that is concentrated on the boundary, the functionals f and k are no longer contained in the class of distributions on Ω. In this context the regularity of the data can be chosen so low that every function $u \in L^q_w(\Omega)$ occurs as a very weak solution with respect to appropriate data. It turns out that this setting is convenient to deal with complex interpolation. As a preparation we give a characterization of the interpolation spaces of the spaces of solutions and of the spaces of the data in Sections 3.2 and 3.3. The main results in the linear case are given in the Theorems 4.3 and 4.4.

When dealing with the Navier-Stokes equations in Section 5 the nonlinearity gives us reason to demand higher regularity of data and solutions. First of all, the nonlinear term can be written as

$$u \cdot \nabla u = \operatorname{div} uu - Ku.$$

To ensure that the multiplication on the right hand side is well-defined, it is reasonable to demand that K is given by a function.

Moreover, when estimating the nonlinear term, one needs a weighted analogue to the Sobolev Embedding Theorem. A good replacement proved in [13] requires strong assumptions to the weight function. This can be compensated for the price of restrictions to the generality of the data and consequently of a smaller class of solutions. It turns out that the more general the weight function is the higher one has to choose the regularity of data and solutions. Thus it is natural to consider the problem in Bessel potential spaces, where we are able to adapt the regularity of data and solutions precisely to the quality of the weight function. Using the results from the linear case we prove existence and uniqueness results for the Navier-Stokes equations if the data is sufficiently small, cf., Theorems 5.6, 5.8 and 5.9.

2 Preliminaries

2.1 Weighted Function Spaces

Let A_q, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all $0 \le w \in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} \, dx \right)^{q-1} < \infty. \tag{2.1}$$

The supremum is taken over all cubes Q in $\mathbb{R}^n$. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

Lemma 2.1. *1. Every $w \in A_q$, $q \ge 1$ defines a locally finite Borel measure $w(F) = \int_F w \, dx$ and for $q > 1$ one has*

$$w(Q) \le \left(\frac{|Q|}{|F|} \right)^q w(F)$$

for all cubes Q and all Borel sets $F \subset Q$ with $|F| > 0$.

2. $A_q \subset A_p$ for $q < p$.

3. Let $w \in A_q$ for $q > 1$. Then there exists $s < q$ such that $w \in A_s$.

Proof. 1. [25, V.1.7], 2. [15, IV Theorem 1.14], 3. [25, IX Prop. 4.5] ◻

Let $k \in \mathbb{N}_0$, $q \in (1,\infty)$, $w \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then we define the following weighted versions of Lebesgue and Sobolev spaces.

- $L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\, dx\right)^{\frac{1}{q}} < \infty \right\}$.

  It is an easy consequence of the corresponding result in the unweighted case that
  $$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \;\text{ with }\; \frac{1}{q} + \frac{1}{q'} = 1 \;\text{ and }\; w' = w^{-\frac{1}{q-1}} \in A_{q'}. \tag{2.2}$$

- Set $W^{k,q}_w(\Omega) = \left\{ u \in L^q_w(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{q,w} < \infty \right\}$.

- By $C^\infty_0(\Omega)$ we denote the set of all smooth and compactly supported functions, the space $C^\infty_{0,\sigma}(\Omega)$ consists of all functions that are in addition divergence free.

- Moreover we set $W^{k,q}_{w,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{k,q,w}}$. The dual space of it is denoted by $W^{-k,q}_w(\Omega) := (W^{k,q'}_{w',0}(\Omega))'$. We also consider the divergence-free versions $W^{k,q}_{w,0,\sigma}(\Omega) := \{\phi \in W^{k,q}_{w,0}(\Omega) \mid \operatorname{div}\phi = 0\}$ and $L^q_{w,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{L^q_w(\Omega)}$.

- Using this for $k > 0$ we set $W^{-k,q}_{w,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{-k,q}_w(\mathbb{R}^n)}}$.

- Moreover, we consider the spaces of boundary values $T^{k,q}_w(\partial\Omega) := (W^{k,q}_w(\Omega))|_{\partial\Omega}$, equipped with the norm $\|\cdot\|_{T^{k,q}_w} = \|\cdot\|_{T^{k,q}_w(\partial\Omega)}$ of the factor space and finally $T^{0,q}_w(\partial\Omega) := (T^{1,q'}_{w'}(\partial\Omega))'$.

By [10], [12] and [4] the spaces $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$, $W^{k,q}_{w,0}(\Omega)$ and $T^{k,q}_w(\partial\Omega)$ are reflexive Banach spaces in which $C^\infty_0(\overline{\Omega})$, ($C^\infty_0(\Omega)$, $C^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively) are dense.

Note that by Nečas [18], Chapitre 2, §5, in the unweighted case one has
$$T^{k,q}_1(\partial\Omega) = W^{k-\frac{1}{q},q}(\partial\Omega) \;\text{ for }\; k \in \mathbb{N} \;\text{ and }\; T^{0,q}_1(\partial\Omega) = W^{-\frac{1}{q},q}(\partial\Omega).$$

Lemma 2.2. *Let Ω be a bounded domain. If $1 \le s$, $w \in A_s$ and $s < p < \infty$, then for every $q \ge sp$ and some $r > q$ one has*
$$L^r(\Omega) \hookrightarrow L^q_w(\Omega) \hookrightarrow L^p(\Omega).$$

Proof. The second embedding is shown in [12, Lemma 2.2] the first one follows by dualization from the second and Lemma 2.1. ◻

Theorem 2.3. *Let Ω be a bounded Lipschitz domain or $\Omega = \mathbb{R}^n_+$ and $N \in \mathbb{N}$. Choose $p_i \in [1,\infty)$, $w_i \in A_{p_i}$ and $k_i \in \mathbb{N}_0$, $i = 1,...,N$. Then there exists an extension operator*
$$E : \bigcap_{i=1}^N W^{k_i,p_i}_{w_i}(\Omega) \to \bigcap_{i=1}^N W^{k_i,p_i}_{w_i}(\mathbb{R}^n),$$
i.e., $Eu|_\Omega = u$ and $\|Eu\|_{W^{k_i,p_i}_{w_i}(\mathbb{R}^n)} \le c\|u\|_{W^{k_i,p_i}_{w_i}(\Omega)}$ for $i = 1,...,N$ and for every $u \in \bigcap_{i=1}^N W^{k_i,p_i}_{w_i}(\Omega)$.

Proof. This is a special case of [4, Theorem 1.4, Theorem 1.5]. There Chua proves extension theorems for the class of (ε, ∞)-domains. By [17] this class includes bounded Lipschitz domains and $\mathbb{R}^n_+$. □

From now on we call any domain that permits an extension operator as in Theorem 2.3 an extension domain. In particular bounded Lipschitz domains are extension domains.

Theorem 2.4. (Hörmander-Michlin Multiplier Theorem with Weights) *Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ fulfill the property*

$$|\partial^\alpha m(\xi)| \le K|\xi|^{-|\alpha|}, \quad \textit{for every} \;\; \xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| = 0, 1, ..., n,$$

for some constant $K > 0$. Then T defined by

$$\widehat{Tf} = m\hat{f} \quad \textit{for} \;\; f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$$

extends to a continuous operator on $L^q_w(\Omega)$ for every $q \in (1, \infty)$ and $w \in A_q$.

Proof. This is an immediate consequence of [15], Theorem 3.9. □

By [22] one has the following weighted version of Bogowski's Theorem.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded and locally lipschitzian domain. Assume $f \in W^{k,q}_{w,0}(\Omega)$ such that $\int f = 0$. Then there exists a function $u \in W^{k+1,q}_{w,0}(\Omega)$ such that*

$$\operatorname{div} u = f \quad \textit{and} \quad \|u\|_{k+1,q,w} \le c\|f\|_{k,q,w},$$

with $c = c(\Omega, q, w, k) > 0$. Moreover, u can be chosen such that it depends linearly on f and such that $u \in C^\infty_0(\Omega)$ if $f \in C^\infty_0(\Omega)$.

2.2 Complex Interpolation Theory

The fundamental tool in the Sections 3.3 and 4 is complex interpolation. Thus we fix some basic notation and facts in this field.

Let $\{X_1, X_2\}$ an interpolation couple and $D = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$. We define $F(X_1, X_2)$ to be the space of all bounded and holomorphic functions f from D to $X_1 + X_2$ which are extendable to continuous functions on $\overline{D}$ such that $f(j + yi)$ is continuous on $\mathbb{R}$ with values in X_{j+1}, $j = 0, 1$, and such that

$$\|f\|_{F(X_1,X_2)} = \max\left\{\sup_{y\in\mathbb{R}} \|f(iy)\|_{X_1}, \sup_{y\in\mathbb{R}} \|f(iy+1)\|_{X_2}\right\} < \infty.$$

Then for $0 < \theta < 1$ the complex interpolation space is given by $[X_1, X_2]_\theta = \{f(\theta) \mid f \in F(X_1, X_2)\}$, equipped with the norm

$$\|x\|_{[X_1,X_2]_\theta} = \inf\{\|f\|_{F(X_1,X_2)} \mid f \in F(X_1, X_2) \text{ and } f(\theta) = x\}.$$

Theorem 2.6. *Let $0 < \theta < 1$ and $X_1 \subset X_2$ with continuous and dense embedding. Then one has*

1. *X_1 is densely and continuously embedded into $[X_1, X_2]_\theta$.*

2. *(Reiteration)* $\big[[X_1,X_2]_\lambda,[X_1,X_2]_\mu\big]_\theta = [X_1,X_2]_\eta$, *where* $\lambda,\mu\in[0,1]$ *and* $\eta = (1-\theta)\lambda+\theta\mu$.

3. *(Duality) Let* X_1 *and* X_2 *be reflexive. Then* $[X_1,X_2]'_\theta = [X'_1,X'_2]_\theta$.

4. *Let* $\{Y_1,Y_2\}$ *be another interpolation couple with* $Y_1\subset Y_2$. *Moreover let* $T: X_i\to Y_i$ *be a continuous linear operator for* $i=1,2$. *Then* $T:[X_1,X_2]_\theta\to[Y_1,Y_2]_\theta$ *is continuous with operator norm bounded by* $\|T\|^{1-\theta}_{\mathcal{L}(X_1,Y_1)}\|T\|^{\theta}_{\mathcal{L}(X_2,Y_2)}$.

5. *Let* $\{X_1,X_2\}$ *and* $\{Y_1,Y_2\}$ *be interpolation couples such that* $\{X_1,X_2\}$ *is a retract of* $\{Y_1,Y_2\}$, *i.e., there exist continuous linear operators*

$$I: X_1+X_2\to Y_1+Y_2 \quad \textit{and} \quad P: Y_1+Y_2\to X_1+X_2,$$

such that $PI = \mathrm{id}_{X_1+X_2}$ *and* $I: X_i\to Y_i$ *and* $P: Y_i\to X_i$, $i=1,2$ *are continuous. Then* $[X_1,X_2]_\theta = P[Y_1,Y_2]_\theta$ *for* $\theta\in[0,1]$. *The norms* $\|u\|_{[X_1,X_2]_\theta}$ *and* $\inf\{\|U\|_{[Y_1,Y_2]_\theta}\mid PU=u\}$ *are equivalent.*

Proof. All assertions can be found in [28] or [3]. □

2.3 Very Weak Solutions to the Stokes Equations

The existence and uniqueness of very weak solutions in weighted L^q-spaces have been shown in [23]. We quote the basic definitions and facts that are needed in this paper.

Definition 2.7. Let $f\in Y_w^{-2,q}(\Omega)$ and $k\in W_{w,0}^{-1,q}(\Omega)$. A function $u\in L^q_w(\Omega)$ is called a very weak solution to the Stokes problem with respect to the data f and k, if

$$-\langle u,\Delta\phi\rangle = \langle f,\phi\rangle, \qquad \text{for all } \phi\in Y^{2,q'}_{w',\sigma}(\Omega) \text{ and} \tag{2.3}$$

$$-\langle u,\nabla\psi\rangle = \langle k,\psi\rangle, \qquad \text{for all } \psi\in W^{1,q'}_{w'}(\Omega). \tag{2.4}$$

Setting $\psi=1$ in (2.4) it follows that a necessary condition for the existence of a very weak solution u is $\langle k,1\rangle = 0$. This condition is the analogue to the compatibility condition $\langle k,1\rangle = \langle g,N\rangle_{\partial\Omega}$ between divergence and boundary values in the case of weak solutions.

Theorem 2.8. *Let* $f\in Y_w^{-2,q}(\Omega)$ *and* $k\in W_{w,0}^{-1,q}(\Omega)$ *with* $\langle k,1\rangle=0$. *Then there exists a unique very weak solution* $u\in L^q_w(\Omega)$ *to the Stokes problem in the sense of Definition 2.7.2. It fulfills the a priori estimate*

$$\|u\|_{q,w}\le c\left(\|f\|_{Y_w^{-2,q}(\Omega)}+\|k\|_{W_{w,0}^{-1,q}}\right) \tag{2.5}$$

with $c=c(\Omega,q,w)>0$.

Moreover, there exists a pressure functional $p\in W_{w,0}^{-1,q}(\Omega)$ *(unique modulo constants) such that* (u,p) *solves*

$$-\langle u,\Delta\phi\rangle - \langle p,\operatorname{div}\phi\rangle = \langle f,\phi\rangle \quad \textit{for all} \;\; \phi\in Y^{2,q'}_{w'}(\Omega).$$

In particular $-\Delta u + \nabla p|_{C_0^\infty(\Omega)} = f|_{C_0^\infty(\Omega)}$ *in the sense of distributions. The functionals* (u,p) *fulfill the inequality*

$$\|u\|_{q,w} + \|p\|_{W^{-1,q}_{w,0}} \le c\left(\|f\|_{Y_w^{-2,q}} + \|k\|_{W^{-1,q}_{w,0}}\right), \tag{2.6}$$

where $c = c(\Omega, q, w) > 0$.

Theorem 2.9. *Assume that* $f \in Y_w^{-2,q}(\Omega)$ *and* $k \in W^{-1,q}_{w,0}(\Omega)$ *allow a decomposition into*

$$\begin{aligned} \langle f,\phi\rangle &= \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for all } \phi \in Y^{2,q'}_{w'}(\Omega),\\ \langle k,\psi\rangle &= \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega} && \text{for all } \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned} \tag{2.7}$$

with $g \in T^{0,q}_w(\partial\Omega), F \in W^{-1,r}_{\tilde w}(\Omega), K \in L^r_{\tilde w}(\Omega)$, *where* $1 < r < \infty$ *and* $\tilde w \in A_r$ *are chosen such that* $W^{1,q'}_{w'}(\Omega) \hookrightarrow L^{r'}_{\tilde w'}(\Omega) \hookrightarrow L^{q'}_{w'}(\Omega)$. *Then one has:*

1. *Such a decomposition is uniquely defined by* f *and* k.
2. *Every strong solution* $u \in W^{2,q}_w(\Omega)$ *to the Stokes problem corresponding to the data* $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ *and* $K \in W^{1,q}_w(\Omega)$ *is a very weak solution corresponding to the data* f *and* k *with the notation of (2.7).*
3. *If* $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ *and* $K \in W^{1,q}_w(\Omega)$ *with* $\int_\Omega K = \int_{\partial\Omega} N\cdot g$, *then the very weak solution* u *to the Stokes problem with respect to* f *and* k *is a strong solution with respect to* F, K *and* g. *In particular* $u \in W^{2,q}_w(\Omega)$, *there exists a pressure function* $p \in W^{1,q}_w(\Omega)$, *unique modulo constants, such that the Stokes equations are fulfilled in the sense of distributions and one has*

$$\|u\|_{2,q,w} + \|p\|_{1,q,w} \le c(\|F\|_{q,w} + \|K\|_{1,q,w} + \|g\|_{T^{2,q}_w}). \tag{2.8}$$

4. *Let* u *be a very weak solution to the Stokes problem corresponding to the data* f *and* k *as in (2.7). Then*

$$u \in \tilde W^{q,r}_{w,\tilde w} := \left\{u \in L^q_w(\Omega) \;\middle|\; \exists c>0,\ |\langle u,\Delta\phi\rangle| \le c\|\phi\|_{1,r',\tilde w'}\ \forall \phi \in C^\infty_{0,\sigma}(\Omega)\right\}.$$

   There exists an operator $\gamma : \tilde W^{q,r}_{w,\tilde w} \to T^{0,q}_w(\partial\Omega)$ *that coincides with the tangential trace on* $W^{1,q}_w(\Omega)$. *The fact that* $\operatorname{div} u = K \in L^r_{\tilde w}(\Omega)$ *permits to define the normal component of the trace* $N\cdot u|_{\partial\Omega}$. *In this sense* $u|_{\partial\Omega}$ *is well-defined and* $u|_{\partial\Omega} = g$.

3 Weighted Bessel Potential Spaces

3.1 Definition and Simple Properties

For $\xi \in \mathbb{R}^n$ we set $\langle\xi\rangle := (1+|\xi|^2)^{\frac12}$. On the space $\mathcal{S}'(\mathbb{R}^n;\mathbb{R})$ of temperate distributions we define for all $\beta \in \mathbb{R}$ the operator

$$\Lambda^\beta f = \mathcal{F}^{-1}\langle\xi\rangle^\beta \mathcal{F} f, \quad f \in S'(\mathbb{R}^n;\mathbb{R}),$$

where $\mathcal{F}$ stands for the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n;\mathbb{R})$. Then for $1 < q < \infty$, $w \in A_q$ and $\beta \in \mathbb{R}$ the weighted Bessel potential space is given by

$$H^{\beta,q}_w(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n;\mathbb{R}) \;\middle|\; \|f\|_{H^{\beta,q}_w(\mathbb{R}^n)} := \|\Lambda^\beta f\|_{q,w,\mathbb{R}^n} < \infty \right\}.$$

Theorem 3.1. *If $1 < q < \infty$, $w \in A_q$, $l, k \in \mathbb{Z}$ and $l < \beta < k$ then*

$$\left[H_w^{l,q}(\mathbb{R}^n), H_w^{k,q}(\mathbb{R}^n)\right]_\theta = H_w^{\beta,q}(\mathbb{R}^n),$$

where $\theta = \frac{\beta - l}{k - l}$.

Proof. This can be proven analogously to [26, Proposition 13.6.2]. For the weighted version in the case $l = 0$ and $k \in \mathbb{N}$ see also [10, Satz 8.3]. The proof given there can be repeated to obtain the more general assertion of this theorem. $\square$

For an extension domain Ω we define the weighted Bessel potential space on Ω by

$$H_w^{\beta,q}(\Omega) = \left\{ g|_\Omega \mid g \in H_w^{\beta,q}(\mathbb{R}^n) \right\}$$

equipped with the norm

$$\|u\|_{H_w^{\beta,q}(\Omega)} := \inf \left\{ \|U\|_{H_w^{\beta,q}(\mathbb{R}^n)} \mid U \in H_w^{\beta,q}(\mathbb{R}^n),\ U|_\Omega = u \right\}.$$

Note that if $\beta < 0$ then the restriction $g|_\Omega$ has to be understood in the sense of distributions as $g|_{C_0^\infty(\Omega)}$.

Moreover, we set

$$H_{w,0}^{\beta,q}(\Omega) = \overline{(C_0^\infty(\Omega))}^{H_w^{\beta,q}(\mathbb{R}^n)}, \quad \beta \in \mathbb{R},$$

equipped with the norm $\|\cdot\|_{\beta,q,w,0,\Omega} := \|E_0(\cdot)\|_{\beta,q,w,\mathbb{R}^n}$, where E_0 denotes the extension of a function by 0 to the whole space $\mathbb{R}^n$. The space $H_{w,0}^{\beta,q}(\Omega)$ is a reflexive Banach space being a closed subspace of $H_w^{\beta,q}(\mathbb{R}^n)$, which is reflexive since it is isomorphic to $L_w^q(\Omega)$.

Note that by (3.6) below this norm is in general not equivalent to $\|\cdot\|_{\beta,q,w,\Omega}$. Moreover, if $\beta < 0$ the space $H_{w,0}^{\beta,q}(\Omega)$ does in general not consist of distributions on Ω but of distributions on $\mathbb{R}^n$ supported by $\overline{\Omega}$.

We choose this definition because in this way one obtains a good behavior of the dual spaces and interpolation properties, see Lemma 3.3 below.

Theorem 3.2. *Let Ω be an extension domain, $1 < q < \infty$, $w \in A_q$.*

1. *For $k \in \mathbb{N}_0$ one has $H_w^{k,q}(\Omega) = W_w^{k,q}(\Omega)$ and $H_{w,0}^{k,q}(\Omega) = W_{w,0}^{k,q}(\Omega)$ with equivalent norms.*

2. *For $k \in \mathbb{N}$, $0 < \beta < k$ one has $H_w^{\beta,q}(\Omega) = [L_w^q(\Omega), W_w^{k,q}(\Omega)]_{\frac{\beta}{k}}$.*

3. *The spaces $H_w^{\beta,q}(\Omega)$, $\beta > 0$, are independent of the values of the weight function $w \in A_q$ outside Ω, i.e., if $w_1, w_2 \in A_q$, $w_1|_\Omega = w_2|_\Omega$ then $H_{w_1}^{\beta,q}(\Omega) = H_{w_2}^{\beta,q}(\Omega)$ with equivalent norms.*

Proof. The assertions of 1. and 2. can be found in [13] except for the assertion on $H_{w,0}^{k,q}(\Omega)$ in 1. Since one has $H_w^{k,q}(\mathbb{R}^n) = W_w^{k,q}(\mathbb{R}^n)$ with equivalent norms, the equation $H_{w,0}^{k,q}(\Omega) = W_{w,0}^{k,q}(\Omega)$ follows from the definition of $H_{w,0}^{k,q}(\Omega)$. 3. follows from 2. $\square$

3.2 Bessel Potential Spaces of Negative Order

Throughout this section let $1 < q < \infty$ and $w \in A_q$. It follows in a straight-forward way from the definition of the spaces $H^{\beta,q}_w(\mathbb{R}^n)$ that for every $\beta > 0$ one has

$$H^{-\beta,q}_w(\mathbb{R}^n) = \left(H^{\beta,q'}_{w'}(\mathbb{R}^n)\right)' \quad \text{isometrically.} \tag{3.1}$$

Lemma 3.3. *For $\beta \in \mathbb{R}$ one has $H^{-\beta,q}_w(\Omega) = \left(H^{\beta,q'}_{w',0}(\Omega)\right)'$ with equivalent norms. In particular, for $k \in \mathbb{N}$ one has $H^{-k,q}_w(\Omega) = W^{-k,q}_w(\Omega)$.*

Proof. Let $u \in H^{-\beta,q}_w(\Omega)$. Then by definition there exists $U \in H^{-\beta,q}_w(\mathbb{R}^n)$ such that $U|_{C_0^\infty(\Omega)} = u$ with

$$\begin{aligned} 2\|u\|_{-\beta,q,w,\Omega} &\geq \|U\|_{-\beta,q,w,\mathbb{R}^n} = \sup_{\phi\in\mathcal{S}(\mathbb{R}^n),\|\phi\|_{\beta,q',w',\mathbb{R}^n}\leq 1} \langle U,\phi\rangle \\ &\geq \sup_{\phi\in C_0^\infty(\Omega),\|\phi\|_{\beta,q',w',\mathbb{R}^n}\leq 1} \langle u,\phi\rangle = \|u\|_{(H^{\beta,q'}_{w',0}(\Omega))'} \end{aligned}$$

using (3.1). Thus $u \in (H^{\beta,q'}_{w',0}(\Omega))'$.

Vice versa, by Hahn-Banach's theorem every $u \in \left(H^{\beta,q'}_{w',0}(\Omega)\right)'$ can be extended to an element

$$U \in \left(H^{\beta,q'}_{w'}(\mathbb{R}^n)\right)' = H^{-\beta,q}_w(\mathbb{R}^n) \quad \text{with} \quad \|U\|_{-\beta,q,w,\mathbb{R}^n} = \|u\|_{(H^{\beta,q'}_{w',0}(\Omega))'}.$$

Then a similar calculation as above yields $u \in H^{-\beta,q}_w(\Omega)$ with $\|u\|_{-\beta,q,w,\Omega} \leq \|u\|_{(H^{\beta,q'}_{w',0}(\Omega))'}$.

To obtain the result for $k \in \mathbb{N}$ one combines the first assertion with Theorem 3.2.1. $\square$

Lemma 3.3 also yields the completeness of $H^{-\beta,q}_w(\Omega)$ in the case $\beta > 0$.

Lemma 3.4. *Let Ω be a bounded $C^{1,1}$-domain or the half space. There exists a continuous linear extension operator*

$$E : H^{-1,q}_w(\Omega) \to H^{-1,q}_w(\mathbb{R}^n)$$

such that $Eu|_{C_0^\infty(\Omega)} = u$ for all $u \in H^{-1,q}_w(\Omega)$ and which is also continuous as a mapping $E : H^{1,q}_w(\Omega) \to H^{1,q}_w(\mathbb{R}^n)$.

Proof. We begin with showing the assertion for the half space $\Omega = \mathbb{R}^n_+$.

By [12] for every $f \in W^{-1,q}_w(\mathbb{R}^n_+)$ there exists a unique $u \in W^{1,q}_{w,0}(\mathbb{R}^n_+)$ solving the equation $(1-\Delta)u = f$. This solution u depends linearly on f and fulfills the estimate $\|u\|_{1,q,w} \leq c\|f\|_{-1,q,w}$. We write $u = (1-\Delta_D)^{-1}f$. As shown in [21] one can prove as in the unweighted case [5] the regularity of solutions to the Laplace equation. In particular $f \in W^{1,q}_w(\mathbb{R}^n_+)$ yields $u \in W^{3,q}_w(\mathbb{R}^n_+)$ with $\|u\|_{3,q,w} \leq c\|f\|_{1,q,w}$.

To construct E we remind that by Theorem 2.3 there exists a linear continuous extension operator

$$\tilde{E} : W^{1,q}_w(\mathbb{R}^n_+) \to W^{1,q}_w(\mathbb{R}^n) \quad \text{and} \quad \tilde{E} : W^{3,q}_w(\mathbb{R}^n_+) \to W^{3,q}_w(\mathbb{R}^n) \text{ with } \tilde{E}u|_{\mathbb{R}^n_+} = u.$$

Now we set $Eu = (1-\Delta)\tilde{E}(1-\Delta_D)^{-1}u$ for every $u \in H_w^{-1,q}(\mathbb{R}^n_+)$. Then E has the asserted properties on the half space $\mathbb{R}^n_+$.

For a bounded $C^{1,1}$-domain Ω we take a collection of charts $(\alpha_j)_{j=1}^m$ and a decomposition of unity $(\psi_j)_{j=1}^m$ subordinate to the corresponding covering $(U_j)_j$ of $\overline{\Omega}$. Then for $u \in W_w^{1,q}(\Omega)$ we set

$$E_\Omega u = \sum_{j=1}^m \phi_j \cdot E_{\mathbb{R}^n_+}((u\psi_j)\circ\alpha_j)\circ\alpha_j^{-1},$$

where $E_{\mathbb{R}^n_+} : W^{1,q}_{w\circ\alpha_j}(\mathbb{R}^n_+) \to W^{1,q}_{w\circ\alpha_j}(\mathbb{R}^n)$ is the operator just constructed and $\phi_j \in C_0^\infty(U_j)$ with $\phi_j\psi_j = \psi_j$. Obviously $E_\Omega : W_w^{1,q}(\Omega) \to W_w^{1,q}(\mathbb{R}^n)$ is continuous. Moreover, change of variables yields that $u \mapsto u\circ\alpha_j$ is a continuous operation from $W_w^{-1,q}(\Omega) \to W^{-1,q}_{w\circ\alpha_j}(\alpha_j^{-1}(\Omega))$. This shows the continuity of $E_\Omega : W_w^{-1,q}(\Omega) \to W_w^{-1,q}(\mathbb{R}^n)$, and combined with Lemma 3.3 the proof is complete. $\square$

Theorem 3.5. *Let $1 < q < \infty$, $w \in A_q$, $-1 \le \beta \le 1$ and $\Omega = \mathbb{R}^n_+$ or a bounded $C^{1,1}$-domain. Then*

1. $[H_w^{-1,q}(\Omega), H_w^{1,q}(\Omega)]_\theta = H_w^{\beta,q}(\Omega)$, *where* $\theta = \frac{1+\beta}{2}$.
2. *For* $\theta = \frac{1+\beta}{2}$ *one has*

$$\left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_\theta = \begin{cases} H^{\beta,q}_{w,0}(\Omega), & \text{if } \beta < 0 \\ H^{\beta,q}_w(\Omega), & \text{if } \beta \ge 0. \end{cases}$$

Proof. 1. $\{H_w^{-1,q}(\Omega), H_w^{1,q}(\Omega)\}$ is a retract of $\{H_w^{-1,q}(\mathbb{R}^n), H_w^{1,q}(\mathbb{R}^n)\}$ where the retraction is the restriction operator

$$R_\Omega : H_w^{\pm1,q}(\mathbb{R}^n) \to H_w^{\pm1,q}(\Omega), \quad u \mapsto u|_{C_0^\infty(\Omega)},$$

and the coretraction is the extension operator E constructed in Lemma 3.4. Thus the assertion in 1. follows from Theorem 2.6 and the corresponding interpolation property on $\mathbb{R}^n$ stated in Theorem 3.1
2. An application of the Duality Theorem 2.6 to 1. together with Lemma 3.3 yields

$$\left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_\theta = H_{w,0}^{\beta,q}(\Omega). \tag{3.2}$$

Since $F(H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)) \subset F(H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega))$, F as in (2.2), and the same is true when replacing q by q' and w by w', we have by (3.2)

$$L^q_w(\Omega) = \left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_{\frac{1}{2}} \hookrightarrow \left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}} \tag{3.3}$$

and

$$L^{q'}_{w'}(\Omega) \hookrightarrow \left[H_{w',0}^{-1,q'}(\Omega), H_{w'}^{1,q'}(\Omega)\right]_{\frac{1}{2}} = \left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]'_{\frac{1}{2}}. \tag{3.4}$$

By the density of the embedding $H^{1,q'}_{w'}(\Omega) \hookrightarrow \left[H_{w',0}^{-1,q'}(\Omega), H_{w'}^{1,q'}(\Omega)\right]_{\frac{1}{2}}$ we obtain that the embedding (3.4) is dense. Thus we dualize (3.4) and combine it with (3.3) to obtain

$$\left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}} = L^q_w(\Omega) = \left[H_w^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}} = \left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_{\frac{1}{2}}.$$

Now the assertion follows by the reiteration property in Theorem 2.6. $\square$

3.3 Bessel Potential Spaces with Zero Boundary Values

For an extension domain $\Omega \subset \mathbb{R}^n$, $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$ we define the space

$$Y_w^{\beta,q}(\Omega) := \begin{cases} \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\mathbb{R}^n)}, & \text{if } 0 \le \beta \le 1 \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\mathbb{R}^n)}, \\ \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\Omega)}, & \text{if } 1 < \beta \le 2 \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\Omega)}, \end{cases}$$

where in the case $0 \le \beta \le 1$ the functions of $Y_w^{2,q}(\Omega)$ are assumed to be extended by 0 to functions defined on the whole space $\mathbb{R}^n$. This is possible, since $C_0^\infty(\Omega)$ is dense in $W_{w,0}^{1,q}(\Omega) \supset Y_w^{2,q}(\Omega)$ and $W_{w,0}^{1,q}(\Omega) \hookrightarrow W_w^{1,q}(\mathbb{R}^n) \hookrightarrow H_w^{\beta,q}(\mathbb{R}^n)$.

In particular, this implies that in the case $0 \le \beta \le 1$ one has

$$Y_w^{\beta,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{H_w^{\beta,q}(\mathbb{R}^n)} = H_{w,0}^{\beta,q}(\Omega). \tag{3.5}$$

Moreover, for such β it follows immediately from the definition of $Y_w^{\beta,q}(\Omega)$ that the extension $E_0 u$ of functions $u \in Y_w^{\beta,q}(\Omega)$ by 0 to functions on $\mathbb{R}^n$ is a continuous linear map to $H_w^{\beta,q}(\mathbb{R}^n)$.

Finally, since $H_w^{1,q}(\Omega) = W_w^{1,q}(\Omega)$ and the norm in $W_w^{1,q}(\Omega)$ is local, for $\beta = 1$ the two definitions are equivalent, i.e.,

$$Y_w^{1,q}(\Omega) = W_{w,0}^{1,q}(\Omega) = \overline{Y_w^{2,q}(\Omega)}^{H_w^{1,q}(\Omega)},$$

where the latter space is equipped with $\|\cdot\|_{H_w^{1,q}(\Omega)}$.

For symmetry reasons the question arises whether $Y_w^{\beta,q}(\Omega) = \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\Omega)}$ for all $0 \le \beta \le 2$. However this is not the case, not even in the unweighted case. Indeed, by Triebel [29, I.5.23] one has

$$\overline{Y_1^{2,q}(\Omega)}^{H^{\frac{1}{q},q}(\Omega)} = \overline{C_0^\infty(\Omega)}^{H^{\frac{1}{q},q}(\Omega)} \ne \{u \in H^{\frac{1}{q},q}(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\} = Y_1^{\frac{1}{q},q}(\Omega). \tag{3.6}$$

We choose the spaces $Y_w^{\beta,q}(\Omega)$ because of their good properties with respect to interpolation.

Theorem 3.6. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$. Then*

$$\left[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)\right]_\theta = Y_w^{\beta,q}(\mathbb{R}_+^n), \quad \theta = \frac{\beta}{2}$$

with equivalent norms.

Proof. As a preparation we note that the norm in $Y_w^{\beta,q}(\mathbb{R}_+^n)$ is equivalent to the one in $Y_{\tilde{w}}^{\beta,q}(\mathbb{R}_+^n)$ if $\tilde{w} \in A_q$ with $\tilde{w}|_{\mathbb{R}_+^n} = w|_{\mathbb{R}_+^n}$. In the case $\beta \ge 1$ this is true by Theorem 3.2.

If $\beta < 1$ one has by Theorem 3.5 and (3.5)

$$Y_w^{\beta,q}(\mathbb{R}_+^n) = H_{w,0}^{\beta,q}(\mathbb{R}_+^n) = \left(H_{w'}^{-\beta,q'}(\mathbb{R}_+^n)\right)' = [H_{w'}^{1,q'}(\mathbb{R}_+^n), H_{w'}^{-1,q'}(\mathbb{R}_+^n)]'_{\frac{\beta+1}{2}}.$$

The latter interpolation space is independent of the weight function outside $\mathbb{R}^n_+$, because $H^{1,q'}_{w'}(\mathbb{R}^n_+)$ and $H^{-1,q'}_{w'}(\mathbb{R}^n_+)$ are.

As shown in [12] if $w \in A_q$ then

$$w^*(x) := \begin{cases} w(x) & \text{on } \mathbb{R}^n_+ \\ w(x', -x_n) & \text{on } \mathbb{R}^n_- \end{cases}$$

is also contained in A_q. Thus we may assume from now on that $w = w^*$ is even.

Step 1: We show that

$$\left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta \hookrightarrow Y^{\beta,q}_w(\mathbb{R}^n_+).$$

To see this let $u \in \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta$.

We begin with the case $1 \le \beta \le 2$. Then there is a function $U \in F(L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+))$ such that $U(\theta) = u$ and $\|U\|_{F(L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+))} \le 2\|u\|_{[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)]_\theta}$.

Since $F(L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)) \subset F(L^q_w(\mathbb{R}^n_+), H^{2,q}_w(\mathbb{R}^n_+))$, we obtain

$$u = U(\theta) \in [L^q_w(\mathbb{R}^n_+), H^{2,q}_w(\mathbb{R}^n_+)]_\theta = H^{\beta,q}_w(\mathbb{R}^n_+)$$

and

$$\begin{aligned} \|u\|_{H^{\beta,q}_w(\mathbb{R}^n_+)} &\le c \inf\left\{ \|V\|_{F(L^q_w(\mathbb{R}^n_+), H^{2,q}_w(\mathbb{R}^n_+))} \mid V \in F(L^q_w(\mathbb{R}^n_+), H^{2,q}_w(\mathbb{R}^n_+)),\ V(\theta) = u \right\} \\ &\le c\|U\|_{F(L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+))} \le 2c\|u\|_{[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)]_\theta}. \end{aligned}$$

Moreover, by Theorem 2.6 we know that $Y^{2,q}_w(\mathbb{R}^n_+)$ is dense in $[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)]_\theta$ which yields the assertion of Step 1 in the case $\beta \ge 1$.

In the case $0 \le \beta \le 1$ we assume that we already know $\left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_{\frac{1}{2}} = Y^{1,q}_w(\mathbb{R}^n_+)$. This follows from the case $1 \le \beta \le 2$ which will be shown independently. Then, since

$$Y^{1,q}_w(\mathbb{R}^n_+) = \overline{C^\infty_0(\mathbb{R}^n_+)}^{W^{1,q}_w(\mathbb{R}^n)} = W^{1,q}_{w,0}(\mathbb{R}^n_+),$$

and the extension

$$E_0 u(x) = \begin{cases} u(x) & \text{for } x \in \mathbb{R}^n_+ \\ 0 & \text{for } x \in \mathbb{R}^n_- \end{cases}$$

of functions defined on the half space is continuous from $W^{1,q}_{w,0}(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$ and from $L^q_w(\mathbb{R}^n_+)$ to $L^q_w(\mathbb{R}^n)$, we find by interpolation and the reiteration property that

$$E_0 : \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta = \left[L^q_w(\mathbb{R}^n_+), W^{1,q}_{w,0}(\mathbb{R}^n_+)\right]_{2\theta} \to H^{\beta,q}_w(\mathbb{R}^n)$$

is continuous for $0 \le \theta \le \frac{1}{2}$. Thus for every $u \in C^\infty_0(\mathbb{R}^n_+)$ we obtain

$$\|u\|_{Y^{\beta,q}_w(\mathbb{R}^n_+)} = \|E_0 u\|_{\beta,q,w,\mathbb{R}^n} \le c\|u\|_{[L^q_w(\mathbb{R}^n_+), W^{1,q}_{w,0}(\mathbb{R}^n_+)]_{2\theta}}.$$

Then the density of the embedding $C^\infty_0(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), W^{1,q}_{w,0}(\mathbb{R}^n_+)\right]_{2\theta}$ finishes the proof of Step 1.

Step 2: Claim: If the odd extension, $E_{odd} : Y^{\beta,q}_w(\mathbb{R}^n_+) \to H^{\beta,q}_w(\mathbb{R}^n)$, is continuous, where

$$E_{odd} u(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^n_+ \\ -u(x', -x_n) & \text{if } x \in \mathbb{R}^n_- \end{cases}$$

for $x = (x', x_n)$, then the assertion $Y_w^{\beta,q}(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), Y_w^{2,q}(\mathbb{R}^n_+)\right]_\theta$ is true for β.
Proof of the Claim. Let $u \in Y_w^{\beta,q}(\mathbb{R}^n_+)$ and set

$$U(z) = e^{z^2}\Lambda^{(\theta-z)2}E_{odd}u.$$

Then one has $U \in F(L^q_w(\mathbb{R}^n), W^{2,q}_w(\mathbb{R}^n))$ with $U(\theta) = e^{\theta^2}E_{odd}u$. Moreover, since for every $\mu \in \mathbb{C}$ the operator Λ^μ maps odd functions to odd functions, one has $U(iy+1)|_{\mathbb{R}^{n-1}} = 0$ which implies $U(iy+1)|_{\mathbb{R}^n_+} \in Y_w^{2,q}(\mathbb{R}^n_+)$ for every y. Thus $U|_{\mathbb{R}^n_+} \in F(L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+))$ and we obtain $u \in \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta$ with

$$\begin{aligned}\|u\|_{[L^q_w(\mathbb{R}^n_+),Y^{2,q}_w(\mathbb{R}^n_+)]_\theta} &\le \sup_y \|U(iy+1)\|_{Y^{2,q}_w(\mathbb{R}^n_+)} + \sup_y \|U(iy)\|_{L^q_w(\mathbb{R}^n_+)}\\ &\le \sup_y \|U(iy+1)\|_{Y^{2,q}_w(\mathbb{R}^n)} + \sup_y \|U(iy)\|_{L^q_w(\mathbb{R}^n)}\\ &\le c\|E_{odd}u\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{Y^{\beta,q}_w(\mathbb{R}^n_+)}.\end{aligned}$$

Step 3: The embedding $Y_w^{\beta,q}(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), Y_w^{2,q}(\mathbb{R}^n_+)\right]_\theta$ is true for $\beta < 1$.
By the definition of $Y_w^{\beta,q}(\mathbb{R}^n_+)$ for $\beta < 1$ we know that the extension E_0u of u by 0 on $\mathbb{R}^n$ is continuous from $Y_w^{\beta,q}(\mathbb{R}^n_+)$ to $H_w^{\beta,q}(\mathbb{R}^n)$ with norm 1. Thus the odd extension of u, which is equal to

$$E_{odd}u(x) = E_0u(x) - E_0u(x', -x_n),$$

is also continuous. Step 2 completes the argument.
Step 4: The embedding $Y_w^{\beta,q}(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), Y_w^{2,q}(\mathbb{R}^n_+)\right]_\theta$ is true for $1 \le \beta \le 2$.
For $g \in T^{2,q}_w(\mathbb{R}^{n-1})$ there exists an extension $S(g)$ with the following properties:

- $S(g)|_{\mathbb{R}^{n-1}} = g$.
- S is a continuous linear mapping $S : T^{2,q}_w(\mathbb{R}^{n-1}) \to W^{2,q}_w(\mathbb{R}^n)$ and $S : T^{1,q}_w(\mathbb{R}^{n-1}) \to W^{1,q}_w(\mathbb{R}^n)$.

To see this we define $S(g)|_{\mathbb{R}^n_+}$ to be the solution of

$$(1-\Delta)S(g) = 0 \quad \text{on } \mathbb{R}^n_+ \quad \text{and} \quad S(g) = g \quad \text{on } \mathbb{R}^{n-1}.$$

Then by [12, Theorem 4.5] we know that $S(g)|_{\mathbb{R}^n_+}$ is well-defined and has the two properties on $\mathbb{R}^n_+$. By Theorem 2.3 there exists an extension operator, continuous from $W^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ as well as from $W^{1,q}_w(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$. Thus the existence of such an S is proved.
Now we consider the operator

$$B : H^{2,q}_w(\mathbb{R}^n_+) \to H^{2,q}_w(\mathbb{R}^n), \quad u \mapsto S(u|_{\mathbb{R}^{n-1}}) + E_{odd}(u - S(u|_{\mathbb{R}^{n-1}})).$$

Since $w = \tilde{w}$ and $Y^{2,q}_w(\mathbb{R}^n_+)|_{\mathbb{R}^{n-1}} = \{0\}$, it is easy to check that the operator E_{odd} is continuous from $Y^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ and from $W^{1,q}_{w,0}(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$. Thus, we have constructed an operator B which is continuous from $W^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ as well as from $W^{1,q}_w(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$ and which coincides with E_{odd} on $Y^{\beta,q}_w(\mathbb{R}^n_+)$, $\beta = 1, 2$. By interpolation we find that

$$B : H^{\beta,q}_w(\mathbb{R}^n_+) \to H^{\beta,q}_w(\mathbb{R}^n)$$

is continuous for every $1 \le \beta \le 2$. Thus for every $u \in Y_w^{\beta,q}(\mathbb{R}^n_+) \subset Y_w^{1,q}(\mathbb{R}^n_+)$ one has

$$\|E_{odd}u\|_{H_w^{\beta,q}(\mathbb{R}^n)} = \|Bu\|_{H_w^{\beta,q}(\mathbb{R}^n)} \le c\|u\|_{H_w^{\beta,q}(\mathbb{R}^n_+)} = c\|u\|_{Y_w^{\beta,q}(\mathbb{R}^n_+)}.$$

Thus Step 2 finishes the proof. □

Theorem 3.7. *The assertion of Theorem 3.6 holds true, when replacing $\mathbb{R}^n_+$ by a bounded $C^{1,1}$-domain Ω, i.e.,*

$$\left[L_w^q(\Omega), Y_w^{2,q}(\Omega)\right]_\theta = Y_w^{\beta,q}(\Omega), \quad \theta = \frac{\beta}{2}, \quad 0 \le \beta \le 2$$

with equivalent norms.

Proof. Let α_j, $j = 1, ..., m$, be a collection of $C^{1,1}$-charts and ψ_j a decomposition of unity subordinate to the corresponding covering of $\overline{\Omega}$. We assume that every ψ_j is extended to an element of $C_0^\infty(\mathbb{R}^n)$ and that every α_i is extended to an element of $C^{1,1}(\mathbb{R}^n)$ such that it has an inverse $\alpha_j^{-1} \in C^{1,1}(\mathbb{R}^n)$.

Then we fix j, write $\psi = \psi_j$ and $\alpha = \alpha_j$ and define the mapping

$$B : Y_{w\circ\alpha}^{\beta,q}(\mathbb{R}^n_+) \to Y_w^{\beta,q}(\Omega), \quad u \mapsto (u \cdot (\psi \circ \alpha)) \circ \alpha^{-1}. \tag{3.7}$$

Using appropriate extensions of functions in $Y_w^{\beta,q}(\mathbb{R}^n_+)$ to $\mathbb{R}^n$ and the continuity of the concatenation and multiplication with sufficiently smooth functions one shows that B is a continuous mapping into the asserted image space.

Now setting $B_j u = (u(\psi_j \circ \alpha_j)) \circ \alpha_j^{-1}$ we define the operator

$$B_\Omega : \prod_{i=1}^m Y_{w\circ\alpha_i}^{\beta,q}(\mathbb{R}^n_+) \to Y_w^{\beta,q}(\Omega), \quad (u_1, ..., u_m) \mapsto \sum_{i=1}^m B_i u_i,$$

which is continuous and surjective for every $\beta \in [0, 2]$. (Surjectivity follows if one considers the operators

$$A_j : H_w^{\beta,q}(\Omega) \ni u \mapsto (u\phi_j) \circ \alpha_j \in H_{w\circ\alpha_j}^{\beta,q}(\mathbb{R}^n_+), \quad j = 1, ..., m,$$

where ϕ_j is an appropriate cut-off function, with $\phi_j \equiv 1$ on $\operatorname{supp}\psi_j$.) By interpolation and Theorem 3.6 it follows that

$$B_\Omega : \prod_{i=1}^m Y_{w_i}^{\beta,q}(\mathbb{R}^n_+) \to \left[L_w^q(\Omega), Y_w^{2,q}(\Omega)\right]_{\frac{\beta}{2}}$$

is continuous, where $w_i := w \circ \alpha_i$.

For every $u \in Y_w^{\beta,q}(\Omega)$ there exists $(u_1, ..., u_m) \in \prod_{i=1}^m Y_{w_i}^{\beta,q}(\mathbb{R}^n_+)$ with $B_\Omega(u_1, ..., u_m) = u$ and $\|u_i\|_{Y_{w_i}^{\beta,q}(\mathbb{R}^n_+)} \le c\|u\|_{Y_w^{\beta,q}(\Omega)}$ for every $i = 1, ..., m$. Then one can estimate

$$\begin{aligned}
\|u\|_{[L_w^q(\Omega), Y_w^{2,q}(\Omega)]_\theta} &= \|B_\Omega(u_1, ..., u_m)\|_{[L_w^q(\Omega), Y_w^{2,q}(\Omega)]_\theta} \\
&\le c\sum_{i=1}^m \|u_i\|_{[L_{w_i}^q(\mathbb{R}^n_+), Y_{w_i}^{2,q}(\mathbb{R}^n_+)]_\theta} \le c\sum_{i=1}^m \|u_i\|_{Y_{w_i}^{\beta,q}(\mathbb{R}^n_+)} \\
&\le c\|u\|_{Y_w^{\beta,q}(\Omega)}.
\end{aligned}$$

Thus we obtain $[L_w^q(\Omega), Y_w^{2,q}(\Omega)]_{\frac{\beta}{2}} \supset Y_w^{\beta,q}(\Omega)$.

The inclusion "$\subset$" is proved in the same way as in the proof of Theorem 3.6, Step 1. □

4 Stokes Equations in Weighted Bessel Potential Spaces

Throughout this section let Ω be a bounded $C^{1,1}$-domain. Moreover let $\beta \in [0, 2]$, $q \in (1, \infty)$ and $w \in A_q$. As a space for exterior forces we define

$$Y_w^{-\beta,q}(\Omega) := \left(Y_{w'}^{\beta,q'}(\Omega)\right)'.$$

Note that if $0 \le \beta \le 1$ then by (3.5) one has the embedding

$$Y_w^{-\beta,q}(\Omega) = H_w^{-\beta,q}(\Omega) \hookrightarrow W_w^{-1,q}(\Omega)$$

and thus $Y_w^{-\beta,q}(\Omega)$ consists of distributions on Ω.

If $\beta > 1$ then this is in general not the case. In particular, if β is large enough, then a functional $f \in Y_w^{-\beta,q}(\Omega)$ might include a part that is supported on the boundary and which can be considered as a boundary condition.

As a space for divergences we choose

$$H_{w,*}^{\gamma,q}(\Omega) := \begin{cases} H_w^{\gamma,q}(\Omega), & \text{if } \gamma \ge 0 \\ H_{w,0}^{\gamma,q}(\Omega), & \text{if } \gamma < 0, \end{cases}$$

for every $\gamma \in [-1, 1]$. This space is equipped with the norm $\|\cdot\|_{\gamma,q,w,*,\Omega} := \|\cdot\|_{H_{w,*}^{\gamma,q}(\Omega)}$.

We use the notion of very weak solutions introduced in Definition 2.7, however if $\beta \ge 1$, i.e., the solution is contained in $W_w^{1,q}(\Omega)$, then we also speak of weak solutions.

Theorem 4.1. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$. Moreover, let $f \in Y_w^{\beta-2,q}(\Omega)$ and $k \in H_{w,*}^{\beta-1,q}(\Omega)$ with $\langle k, 1\rangle = 0$. Then there exists a unique very weak solution $u \in Y_w^{\beta,q}(\Omega)$ to the Stokes problem with respect to the data f, k in the sense of Definition 2.7. This function u fulfills the estimate*

$$\|u\|_{Y_w^{\beta,q}(\Omega)} \le c\left(\|f\|_{Y_w^{\beta-2,q}(\Omega)} + \|k\|_{\beta-1,q,w,*,\Omega}\right). \tag{4.1}$$

Moreover, there exists a pressure functional $p \in H_w^{\beta-1,q}(\Omega)$, unique modulo constants, such that

$$-\Delta u + \nabla p = f|_{C_0^\infty(\Omega)} \quad \textit{in} \quad C_0^\infty(\Omega)'.$$

Proof. From the results in Sections 3.2 and 3.3 it follows that

$$\left[Y_w^{-2,q}(\Omega) \times H_{w,0}^{-1,q}(\Omega), L_w^q(\Omega) \times H_w^{1,q}(\Omega)\right]_\theta = Y_w^{\beta-2,q}(\Omega) \times H_{w,*}^{\beta-1,q}(\Omega),$$

where $\theta = \frac{\beta}{2}$. It is immediate that

$$k \mapsto K := k - \langle k, 1\rangle \in \mathcal{L}(H_{w,0}^{-1,q}(\Omega)) \cap \mathcal{L}(H_w^{1,q}(\Omega)).$$

By Theorem 2.8 the mapping

$$\mathcal{S} : Y_w^{-2,q}(\Omega) \times H_{w,0}^{-1,q}(\Omega) \ni (f, k) \mapsto u \in L_w^q(\Omega),$$

is continuous, where $u \in L_w^q(\Omega)$ is the very weak solution to the Stokes problem with respect to the data f and $K = k - \langle k, 1\rangle$.

If u is a solution in the sense of Definition 2.7 with sufficiently regular data f and k, then by Theorem 2.9 we find that u is a strong solution with zero boundary values. In particular, $\mathcal{S}$ is also continuous from $L^q_w(\Omega) \times H^{1,q}_w(\Omega)$ to $Y^{2,q}_w(\Omega)$. Now we obtain from the interpolation properties in Theorems 3.5 and 3.7 together with the duality Theorem 2.6 that

$$\mathcal{S} : Y^{\beta-2,q}_w(\Omega) \times H^{\beta-1,q}_{w,*}(\Omega) \to Y^{\beta,q}_w(\Omega)$$

is continuous, which finishes the proof of existence and estimates of u. Uniqueness follows from the uniqueness of very weak solutions in $L^q_w(\Omega)$ (Theorem 2.8).

It remains to show the existence of p. By the theory of strong solutions in [12] there exists a pressure function $p \in H^{1,q}_w(\Omega)$. Moreover, by Theorem 2.8 there exists a pressure functional $p \in H^{-1,q}_{w,0}(\Omega)$ that belongs to a very weak solution. In both cases p is unique if $\langle p, 1\rangle = 0$. Thus by the interpolation Theorem 3.5.2 we obtain a functional $\tilde{p} \in H^{\beta-1,q}_{w,*}(\Omega)$ such that

$$-\langle u, \Delta\phi\rangle - \langle \tilde{p}, \operatorname{div}\phi\rangle = \langle F, \phi\rangle \quad \text{for all} \quad \phi \in Y^{2,q'}_{w'}(\Omega).$$

The restriction $p := \tilde{p}|_{C_0^\infty(\Omega)}$ solves the problem. $\square$

By the definition of $Y^{\beta,q}_w(\Omega)$ it follows, that whenever a trace operator

$$\operatorname{tr} : H^{\beta,q}_w(\Omega) \to T(D)$$

for a boundary portion $D \subset \partial\Omega$ is well-defined (as a continuous linear operator into some boundary space $T(D)$, which coincides with the usual trace $u|_D$ on $W^{1,q}_w(\Omega)$), then for the solution $u \in Y^{\beta,q}_w(\Omega)$ one has $\operatorname{tr} u = 0$.

In the case, where data and solutions are regular enough (including the case $\beta = 1$ of weak solutions), we want to deal with inhomogeneous boundary values.

If $\beta \geq 1$, then $H^{\beta,q}_w(\Omega) \hookrightarrow W^{1,q}_w(\Omega)$ which implies the existence of a continuous trace operator

$$\operatorname{tr} : H^{\beta,q}_w(\Omega) \to T^{1,q}_w(\partial\Omega), \quad \operatorname{tr} u = u|_{\partial\Omega} \text{ if } \quad u \in C^\infty(\overline{\Omega}).$$

As in the case of weighted Sobolev spaces we define the associated boundary space by

$$T^{\beta,q}_w(\partial\Omega) = \operatorname{tr}\left(H^{\beta,q}_w(\Omega)\right)$$

equipped with the norm $\|g\|_{T^{\beta,q}_w(\partial\Omega)} = \inf\left\{\|u\|_{\beta,q,w,\Omega} \mid u \in H^{\beta,q}_w(\Omega),\ \operatorname{tr} u = g\right\}$.

Lemma 4.2. *For every $\beta \in [1,2]$ one has $[T^{1,q}_w(\partial\Omega), T^{2,q}_w(\partial\Omega)]_{\beta-1} = T^{\beta,q}_w(\partial\Omega)$ and there exists a continuous linear extension operator* $\operatorname{ext} : T^{\beta,q}_w(\partial\Omega) \to H^{\beta,q}_w(\Omega)$*, independent of β.*

Proof. As shown in [21] one can prove as in the unweighted case that there exists a unique solution to the Dirichlet problem $(1-\Delta)u = 0$, $u|_{\partial\mathbb{R}^n_+} = g$. This solution u is regular according to the data, i.e., $\|u\|_{k,q,w} \leq c\|g\|_{T^{k,q}_w(\mathbb{R}^{n-1})}$ for $k \in \mathbb{N}$. Using this a straight-forward localization procedure yields that there exists a continuous linear extension operator

$$\operatorname{ext} : T^{1,q}_w(\partial\Omega) \to W^{1,q}_w(\Omega) \ \text{ and } \ \operatorname{ext} : T^{2,q}_w(\partial\Omega) \to W^{2,q}_w(\Omega) \tag{4.2}$$

with $\operatorname{ext} g|_{\partial\Omega} = g$. Moreover, by definition the trace operator $\operatorname{tr} : W^{1,q}_w(\Omega) \to T^{1,q}_w(\partial\Omega)$ and $\operatorname{tr} : W^{2,q}_w(\Omega) \to T^{2,q}_w(\partial\Omega)$ is continuous.

Obviously one has $\operatorname{tr} \circ \operatorname{ext} = \operatorname{id}_{T^{1,q}_w(\partial\Omega)}$ and thus Theorem 2.6.5 shows

$$[T^{1,q}_w(\partial\Omega), T^{2,q}_w(\partial\Omega)]_{\beta-1} = \operatorname{tr}\, [W^{1,q}_w(\Omega), W^{2,q}_w(\Omega)]_{\beta-1} = \operatorname{tr} H^{\beta,q}_w(\Omega) = T^{\beta,q}_w(\partial\Omega).$$

Thus the first assertion is proved. The second assertion follows from the first combined with (4.2). □

Theorem 4.3. *Let $1 < q < \infty$, $w \in A_q$ and $1 \le \beta \le 2$. Moreover, let $F \in H^{\beta-2,q}_w(\Omega)$, $K \in H^{\beta-1,q}_w(\Omega)$ and $g \in T^{\beta,q}_w(\partial\Omega)$ with $\int_\Omega K = \int_{\partial\Omega} g \cdot N$. Then there exists a unique weak solution $u \in H^{\beta,q}_w(\Omega)$, i.e.,*

$$(\nabla u, \nabla\phi) = \langle F, \phi\rangle, \quad \textit{for all} \;\; \phi \in W^{1,q}_{w,0,\sigma}(\Omega)$$

fulfilling $u|_{\partial\Omega} = g$ and $\operatorname{div} u = K$ in the sense of distributions. This solution fulfills the estimate

$$\|u\|_{\beta,q,w} \le c \left(\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T^{\beta,q}_w(\partial\Omega)} \right).$$

Moreover, there exists a pressure function $p \in H^{\beta-1,q}_w(\Omega)$, unique modulo constants, such that the Stokes equations are fulfilled in the sense of distributions.

Proof. First of all recall that if $\beta \in [1,2]$, then $\beta - 2 \in [-1,0]$, which implies $F \in H^{\beta-2,q}_w(\Omega) = Y^{\beta-2,q}_w(\Omega)$.
Existence: For $g \in T^{\beta,q}_w(\partial\Omega)$ there exists $v \in H^{\beta,q}_w(\Omega)$ such that $\operatorname{tr} v = g$ and $\|v\|_{\beta,q,w,\Omega} \le 2\|g\|_{T^{\beta,q}_w(\partial\Omega)}$. Since there exists an extension V of v to the whole space $\mathbb{R}^n$ that fulfills the estimate $\|V\|_{\beta,q,w,\mathbb{R}^n} \le c\|v\|_{\beta,q,w,\Omega}$, one has

$$\Delta v = (\Delta V)|_{C^\infty_0(\Omega)} \in H^{\beta-2,q}_w(\Omega) = Y^{\beta-2,q}_w(\Omega).$$

Hence by Theorem 4.1 there exists $U \in H^{\beta,q}_w(\Omega)$ solving

$$\begin{aligned} \langle F + \Delta v, \phi\rangle \;&= -\langle U, \Delta\phi\rangle \qquad \text{for all } \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \;\text{ and} \\ \langle K - \operatorname{div} v, \psi\rangle \;&= -\langle U, \nabla\psi\rangle \qquad \text{for all } \psi \in W^{1,q'}_{w'}(\Omega). \end{aligned}$$

Since $U \in Y^{\beta,q}_w(\Omega) \subset W^{1,q}_{w,0}(\Omega)$, we obtain by integration by parts for $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$, which is dense in $W^{1,q'}_{w',0}(\Omega)$, that

$$(\nabla(U + v), \nabla\phi) = -(U, \Delta\phi) - \langle \Delta v, \phi\rangle = \langle F, \phi\rangle,$$

where by the density of $C^\infty_0(\Omega)$ in $W^{1,q'}_{w',0}(\Omega)$ one can apply the definition of the derivatives div to ∇v in the sense of distributions. Setting $u := U + v$ we obtain $\operatorname{div} u = K$ in the sense of distributions and $\operatorname{tr} u = \operatorname{tr} v + \operatorname{tr} U = \operatorname{tr} v = g$. Moreover by the a priori estimate (4.1)

$$\begin{aligned} \|u\|_{\beta,q,w,\Omega} \le& c\big(\|g\|_{T^{\beta,q}_w(\partial\Omega)} + \|F\|_{\beta-2,q,w,\Omega} + \|\Delta v\|_{\beta-2,q,w,\Omega} \\ &\quad + \|K\|_{\beta-1,q,w,\Omega} + \|\operatorname{div} v\|_{\beta-1,q,w,\Omega}\big) \\ \le& c \left(\|g\|_{T^{\beta,q}_w(\partial\Omega)} + \|F\|_{\beta-2,q,w,\Omega} + \|K\|_{\beta-1,q,w,\Omega} \right). \end{aligned}$$

Uniqueness: Note that $[\phi \mapsto -\langle F, \phi\rangle + \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \in Y_w^{-2,q}(\Omega)$. Thus the uniqueness of u follows from the one of very weak solutions shown in Theorem 2.8.
Pressure: To show the existence of p we use that by de Rham's Theorem [27, Ch.1 Proposition 1.1] there exists $p \in (C_0^\infty(\Omega))'$ such that the Stokes equations are fulfilled in the sense of distributions. From the equation we obtain $\nabla p \in H_w^{\beta-2,q}(\Omega)$. It remains to show $p \in H_w^{\beta-1,q}(\Omega)$. However, this follows by Lemma 4.7 below and the proof is complete □

Now we turn to the case $0 \le \beta \le 1$. In this case the functions in $H_w^{\beta,q}(\Omega)$ in general do not possess enough regularity to guarantee the well-definedness of a trace operator. Here we define boundary spaces by

$$T_w^{\beta,q}(\partial\Omega) = \left[T_w^{0,q}(\partial\Omega), T_w^{1,q}(\partial\Omega)\right]_\beta, \tag{4.3}$$

equipped with the norm of the interpolation space.

Theorem 4.4. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 1$. Assume that $f \in Y_w^{-2,q}(\Omega)$ and $k \in H_{w,0}^{-1,q}(\Omega)$ allow decompositions into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} \qquad \textit{for every} \;\; \phi \in Y_{w'}^{2,q'}(\Omega) \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega} \qquad \textit{for every} \;\; \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned} \tag{4.4}$$

with $F \in Y_w^{\beta-2,q}(\Omega)$, $K \in H_{w,0}^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$. Assume in addition that K and g fulfill the compatibility condition $\langle K, 1\rangle_\Omega = \langle g, N\rangle_{\partial\Omega}$.

Then the very weak solution $u \in L_w^q(\Omega)$ with respect to f and k, which exists according to Theorem 2.8 is contained in $H_w^{\beta,q}(\Omega)$ and fulfills the estimate

$$\|u\|_{\beta,q,w} \le c\left(\|F\|_{Y_w^{\beta-2,q}(\Omega)} + \|K\|_{H_{w,0}^{\beta-1,q}(\Omega)} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right). \tag{4.5}$$

Remark 4.5. The regularity of the data in Theorem 4.4 is in general not sufficient to guarantee that the restriction of the corresponding solution u to the boundary is well-defined. Accordingly, without the additional regularity the decomposition of the data (4.4) is in general not unique.

If we assume in addition that $F \in W_{\tilde{w}}^{-1,r}(\Omega)$ and $K \in L_{\tilde{w}}^r(\Omega)$, where r and $\tilde{w} \in A_r$ are chosen such that

$$W_{\tilde{w}}^{-1,r}(\Omega) \hookrightarrow Y_w^{\beta-2,q}(\Omega) \;\text{ and }\; L_{\tilde{w}}^r(\Omega) \hookrightarrow H_{w,0}^{\beta-1,q}(\Omega) \tag{4.6}$$

one obtains by Theorem 2.9.3 that the trace $u|_{\partial\Omega}$ is well-defined and that one has $u|_{\partial\Omega} = g$, where g is the given boundary condition.

Proof of Theorem 4.4. Step 1: We consider the operator

$$B : T_w^{0,q}(\partial\Omega) \to L_w^q(\Omega), \quad g \mapsto u,$$

where u is the very weak solution to the Stokes problem with data

$$f = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \;\text{ and }\; k = [\psi \mapsto \langle g, N\psi\rangle_{\partial\Omega}].$$

Obviously, B is linear and continuous, also considered as an operator $B : T_w^{1,q}(\partial\Omega) \to W_w^{1,q}(\Omega)$. This follows from Theorem 4.3 in the case $\beta = 1$ since the very weak solution with respect to f and k coincides with the weak solution with 0 force and divergence and boundary condition g. Thus interpolation yields that $B : T_w^{\beta,q}(\partial\Omega) \to H_w^{\beta,q}(\Omega)$ is continuous.

Step 2: Let $U = Bg \in H_w^{\beta,q}(\Omega)$ be given by Step 1. Moreover, let $v \in Y_w^{\beta,q}(\Omega)$ be the very weak solution to the Stokes problem with respect to the data F, K, which exists according to Theorem 4.1. Then $u := U + v$ is a very weak solution with respect to f and k and fulfills the estimate (4.5).

The uniqueness of the solution follows from Theorem 2.8. $\square$

Corollary 4.6. *Let Ω be a bounded $C^{1,1}$-domain. Moreover, let $1 < q, r < \infty$, $w \in A_q$, $v \in A_r$ and $0 \le \beta \le 2$ be given such that $H_w^{\beta,q}(\Omega) \hookrightarrow L_v^r(\Omega)$. Then*

$$T_w^{\beta,q}(\partial\Omega) \hookrightarrow T_v^{0,r}(\partial\Omega).$$

Proof. Let $g \in T_w^{\beta,q}(\partial\Omega)$. Then the very weak solution $u \in H_w^{\beta,q}(\Omega)$ to

$$\begin{aligned} -\langle u, \Delta\phi\rangle &= \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \quad \text{for all} \quad \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \\ -\langle u, \nabla\psi\rangle &= \langle g, N\psi\rangle_{\partial\Omega} \qquad \text{for all} \quad \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned}$$

fulfills $\|u\|_{\beta,q,w} \le c\|g\|_{T_w^{\beta,q}(\partial\Omega)}$. Moreover, one has $u \in \tilde W_{v,v}^{r,r}$ (defined in Theorem 2.9.4) with $\|u\|_{\tilde W_{v,v}^{r,r}} = \|u\|_{r,v}$ and $\operatorname{div} u = 0$. Thus the tangential and the normal trace of u are well-defined in the sense of Theorem 2.9.4. Since $u|_{\partial\Omega} = g$, we obtain

$$\|g\|_{T_v^{0,r}(\partial\Omega)} \le c\|u\|_{r,v} \le c\|u\|_{\beta,q,w} \le c\|g\|_{T_w^{\beta,q}(\partial\Omega)}. \qquad \square$$

The results of this section can be used for the proof of the following Lemma which is needed to estimate the pressure in Theorem 4.3. Since the pressure is well-defined only modulo constants, we consider the space $H_w^{\beta,q}(\Omega)/const.$ If $\beta \ge 0$ this space can be identified with the space of all $u \in H_w^{\beta,q}(\Omega)$ such that $\langle u, 1\rangle_\Omega = 0$. If $\beta < 0$ one has

$$H_w^{\beta,q}(\Omega)/const. \cong \left\{\phi \in H_{w',0}^{-\beta,q'}(\Omega) \;\Big|\; \int_\Omega \phi = 0\right\}'$$

via the isomorphism $u + \mathbb{R} \mapsto u|_{\left\{\phi\in H_{w',0}^{-\beta,q'}(\Omega) \;|\; \int_\Omega \phi=0\right\}}$.

Lemma 4.7. *Let $-1 \le \beta \le 1$. Let $p \in (C_0^\infty(\Omega))'$ with $\nabla p \in H_w^{\beta-1,q}(\Omega)$. Then $p \in H_w^{\beta,q}(\Omega)$ and there exists a constant $c = c(\Omega, q, w)$ such that*

$$\|p\|_{H_w^{\beta,q}/const.} \le c\|\nabla p\|_{H_w^{\beta-1,q}}.$$

Proof. Case 1: Let $\beta \le 0$. By Theorem 2.5 for every $\phi \in W_{w',0}^{1,q'}(\Omega)$ with $\int_\Omega \phi = 0$ there exists $\zeta \in W_{w',0}^{2,q'}(\Omega)$ such that $\operatorname{div}\zeta = \phi$ and $\|\zeta\|_{2,q',w'} \le c\|\phi\|_{1,q',w'}$. The function ζ can be chosen such that the mapping $\phi \mapsto \zeta$ is linear and fulfills the additional estimate $\|\zeta\|_{1,q',w'} \le c\|\phi\|_{q',w'}$.

For a moment we consider the mapping $\phi \mapsto \zeta$ as a mapping from $L_{w'}^{q'}(\Omega)$ to $H_{w'}^{1,q'}(\mathbb{R}^n)$ and from $H_{w',0}^{1,q'}(\Omega)$ to $H_{w'}^{2,q'}(\mathbb{R}^n)$ assuming that ζ is extended by 0 to a function defined on $\mathbb{R}^n$. Thus by interpolation we obtain for $\gamma \in [0,1]$ that $\|\zeta\|_{H^{\gamma+1,q'}_{w'}(\mathbb{R}^n)} \le c\|\phi\|_{H^{\gamma,q'}_{w',0}(\Omega)}$. Since for $\phi \in C_0^\infty(\Omega)$ one has $\operatorname{supp}\zeta \subset \Omega$, we have shown $\|\zeta\|_{H^{\gamma+1,q'}_{w',0}(\Omega)} \le c\|\phi\|_{H^{\gamma,q'}_{w',0}(\Omega)}$. This implies the estimate

$$|\langle p,\phi\rangle_\Omega| = |\langle p, \operatorname{div}\zeta\rangle_\Omega| \le \|\nabla p\|_{H^{\beta-1,q}_w}\|\zeta\|_{H^{1-\beta,q'}_{w',0}} \le c\|\nabla p\|_{H^{\beta-1,q}_w}\|\phi\|_{H^{-\beta,q'}_{w',0}}$$

for every $\phi \in C_0^\infty(\Omega)$. This is the assertion for $\beta \le 0$.
Case 2: Let $\beta > 0$. We consider the solution operator $\mathcal{S} : f \mapsto p$ where (u,p) solves

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \langle f,\phi\rangle, \quad \text{for all} \quad \phi \in Y^{2,q'}_{w'}(\Omega)$$

and $\langle u, \nabla\psi\rangle = 0$ for $\psi \in W^{1,q'}_{w'}(\Omega)$, $\langle p, 1\rangle = 0$. By the Theorems 2.8 and 2.9

$$\mathcal{S} : Y^{-2,q}_w(\Omega) \to W^{-1,q}_{w,0}(\Omega) \quad \text{and} \quad \mathcal{S} : L^q_w(\Omega) \to W^{1,q}_w(\Omega)$$

is continuous. By the interpolation theorems proved in the Sections 3.2 and 3.3 and the fact that $\beta \in (0,1]$ and $\mathcal{S}\nabla p = p - \langle p,1\rangle$ we obtain the estimate

$$\begin{aligned}\|p - \langle p,1\rangle\|_{H^{\beta,q}_w(\Omega)} &\le c\|\mathcal{S}\nabla p\|_{[W^{-1,q}_{w,0}(\Omega), W^{1,q}_w(\Omega)]_{\frac{\beta+1}{2}}} \le c\|\nabla p\|_{[Y^{-2,q}_w(\Omega), L^q_w(\Omega)]_{\frac{\beta+1}{2}}} \\ &\le c\|\nabla p\|_{Y^{\beta-1,q}_w(\Omega)} = c\|\nabla p\|_{H^{\beta-1,q}_w(\Omega)}\end{aligned}$$

□

5 The Stationary Navier-Stokes Equations

5.1 Estimates of the Nonlinear Term

We prepare some embedding theorems. These theorems are proved by the use of weakly singular integral operators. Thus for $0 < \beta < n$ we define

$$I_\beta g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\beta}}dy = c\mathcal{F}^{-1}|\xi|^{-\beta}\hat{g}(x), \tag{5.1}$$

where the second equality holds by [24, V. Lemma 2] for an appropriate constant $c \in \mathbb{R}$.

Theorem 5.1. *Let $0 < \beta < n$ and $1 < p < q < \infty$, $v \in A_p$ and $w \in A_q$. Moreover, assume that v and w fulfill the condition*

$$|Q|^{\frac{\beta}{n}-1}\left(\int_Q w\right)^{\frac{1}{q}}\left(\int_Q v^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}} < c \quad \textit{for every cube} \quad Q \subset \mathbb{R}^n$$

with a constant $c > 0$ independent of Q. Then

$$\|I_\beta f\|_{q,w} \le c\|f\|_{p,v} \quad \textit{for every} \quad f \in L^p_v(\mathbb{R}^n).$$

Proof. This is a special case of [19, Theorem 1 (B)]. □

Lemma 5.2. *Let $w \in A_q$, $v \in A_p$ with*

$$|Q|^{\frac{\beta}{n}-1}\left(\int_Q w\right)^{\frac{1}{q}}\left(\int_Q v^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}} < c \quad \text{for every cube} \quad Q \subset \mathbb{R}^n$$

with a constant $c > 0$ independent of Q. Then one has

$$H_v^{\gamma,p}(\mathbb{R}^n) \hookrightarrow L_w^q(\mathbb{R}^n) \quad \text{for every} \quad \gamma \geq \beta.$$

Proof. By [13, Lemma 3.2] the embedding

$$\mathcal{M} := \left\{f \in \mathcal{S}(\mathbb{R}^n) \mid \hat{f} \equiv 0 \text{ in a neighborhood of } 0\right\} \hookrightarrow H_v^{\beta,p}(\mathbb{R}^n)$$

is dense. Moreover, we define $J_\beta f := c\mathcal{F}^{-1}|\xi|^\beta(1+|\xi|^2)^{-\frac{\beta}{2}}\mathcal{F}f$, where c is the constant from (5.1). Then by the Multiplier Theorem 2.4 the operator $J_\beta : L_v^p(\Omega) \to L_v^p(\Omega)$ is continuous. Moreover, for $f \in \mathcal{M}$ one has $f = I_\beta J_\beta \Lambda_\beta f$. Thus one obtains using Theorem 5.1 for every $f \in \mathcal{M}$

$$\|f\|_{L_w^q(\mathbb{R}^n)} = \|I_\beta J_\beta \Lambda_\beta f\|_{L_w^q(\mathbb{R}^n)} \leq c\|J_\beta \Lambda_\beta f\|_{L_v^p(\mathbb{R}^n)} \leq c\|\Lambda_\beta f\|_{L_v^p(\mathbb{R}^n)} = c\|f\|_{H_v^{\beta,p}(\mathbb{R}^n)}.$$

Thus by the density of $\mathcal{M}$ in $H_v^{\beta,p}(\mathbb{R}^n)$ the inequality holds for every $f \in H_v^{\beta,p}(\mathbb{R}^n)$ and one obtains $H_v^{\gamma,p}(\mathbb{R}^n) \hookrightarrow H_v^{\beta,p}(\mathbb{R}^n) \hookrightarrow L_w^q(\mathbb{R}^n)$. $\square$

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Moreover, let $1 \leq s \leq r \leq q < \infty$, $r > 1$ and assume $0 \leq \beta < n$ such that*

$$\frac{1}{q} \geq \frac{1}{r} - \frac{\beta}{ns}. \tag{5.2}$$

Then for every $w \in A_s$ the following embeddings are true:

1. *$H_w^{\beta,r}(\Omega) \hookrightarrow L_w^q(\Omega)$.*
2. *$H_{w_q}^{\beta,q'}(\Omega) \hookrightarrow L_{w_r}^{r'}(\Omega)$, where $w_q = w^{-\frac{1}{q-1}}$ and $w_r = w^{-\frac{1}{r-1}}$.*
3. *$L_w^r(\Omega) \hookrightarrow H_w^{-\beta,q}(\Omega)$, $L_w^r(\Omega) \hookrightarrow H_{w,0}^{-\beta,q}(\Omega)$ and for $\beta \in [0,1]$ one has $W_w^{-1,r}(\Omega) \hookrightarrow Y_w^{-1-\beta,q}(\Omega)$.*
4. *If $\beta \in [0,1]$, then one has $H_w^{1,r}(\Omega) \hookrightarrow H_w^{1-\beta,q}(\Omega)$.*

Proof. We begin with showing that without loss of generality we may assume that $1 \leq s < r$. Let $s = r$. Since $r > 1$ and $w \in A_r$ by Lemma 2.1.3 there exists $t \in [1,r)$ such that $w \in A_t$. If (5.2) holds for s, it holds for s replaced by t in any case. Thus we may replace s by $t < r$.
1. By [13, Corollary 3.2] the asserted embedding holds if there exists a constant $C > 0$ such that $|Q|^{\frac{\beta}{n}}w(Q)^{\frac{1}{q}-\frac{1}{r}} < C$ for all $Q \subset U$ for some open set $U \supset \overline{\Omega}$. By Lemma 2.1.1 we know that for every $Q \subset U$ and $w \in A_s$ it holds $|Q|^s \leq \frac{|U|^s}{w(U)}w(Q) = cw(Q)$. Thus

$$|Q|^{\frac{\beta}{n}}w(Q)^{\frac{1}{q}-\frac{1}{r}} \leq cw(Q)^{\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}} \leq cw(U)^{\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}} =: C$$

since $\frac{\beta}{sn} + \frac{1}{q} - \frac{1}{r} \geq 0$ by assumption.
2. As above Lemma 2.1.1 states that $w \in A_s$ implies $w(Q) \geq c(U)|Q|^s$ for every $Q \subset U$, where U is some bounded domain with $\overline{\Omega} \subset U$. It has been shown in [10, Lemma A.2] that in this case there exists a weight function $W \in A_q$ such that $W = w$ on Ω and $W(Q) \geq c(U)|Q|^s$ for every cube $Q \subset \mathbb{R}^n$.

Now by Theorem 3.2 we know that

$$H^{\gamma,q'}_{w_q}(\Omega) = H^{\gamma,q'}_{W_q}(\Omega)$$

with equivalent norms. By Lemma 5.2 the condition

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q W_r\right)^{\frac{1}{r'}}\left(\int_Q (W_q)^{-\frac{1}{q'-1}}\right)^{\frac{1}{q}} < c \;\text{ for every cube }\; Q \subset \mathbb{R}^n \tag{5.3}$$

implies $H^{\gamma,q'}_{W_q}(\mathbb{R}^n) \hookrightarrow L^{r'}_{W_r}(\mathbb{R}^n)$ for every $\gamma \geq \alpha$. Thus we have to show (5.3). Since $W_r^{-\frac{1}{r'-1}} = W^{\frac{1}{r'-1}\frac{1}{r-1}} = W = (W_q)^{-\frac{1}{q'-1}}$, we calculate using the definition of Muckenhoupt weights, $W \in A_r$ and $\frac{1}{q} - \frac{1}{r} \leq 0$

$$\begin{aligned} |Q|^{\frac{\alpha}{n}-1}\left(\int_Q W_r\right)^{\frac{1}{r'}}\left(\int_Q (W_q)^{-\frac{1}{q'-1}}\right)^{\frac{1}{q}} &= |Q|^{\frac{\alpha}{n}-1} W_r(Q)^{\frac{1}{r'}} W(Q)^{\frac{1}{q}} \\ &\leq c|Q|^{\frac{\alpha}{n}} W(Q)^{(\frac{1}{q}-\frac{1}{r})} \leq c|Q|^{\frac{\alpha}{n}+s(\frac{1}{q}-\frac{1}{r})}. \end{aligned}$$

The last term is bounded if $\frac{\alpha}{n} + s(\frac{1}{q} - \frac{1}{r}) = 0$. There exists $0 \leq \alpha \leq \beta$ so that this is true, because $s\left(\frac{1}{q} - \frac{1}{r}\right) \leq 0$ and for $\alpha = \beta$ one has $\frac{\beta}{n} + s(\frac{1}{q} - \frac{1}{r}) \geq \frac{\beta}{n} - s\frac{\beta}{sn} = 0$.

Now for $f \in H^{\gamma,q'}_{w_q}(\Omega)$ there exists an extension $F \in H^{\gamma,q'}_{W_q}(\mathbb{R}^n)$ with $\|F\|_{H^{\gamma,q'}_{W_q}(\mathbb{R}^n)} \leq 2\|f\|_{H^{\gamma,q'}_{W_q}(\Omega)} \leq c\|f\|_{H^{\gamma,q'}_{w_q}(\Omega)}$. One obtains

$$\|f\|_{L^{r'}_{w_r}(\Omega)} \leq \|F\|_{L^{r'}_{W_r}(\mathbb{R}^n)} \leq c\|F\|_{H^{\gamma,q'}_{W_q}(\mathbb{R}^n)} \leq c\|f\|_{H^{\gamma,q'}_{W_q}(\mathbb{R}^n)},$$

and the asserted embedding is proved.
3. Considering the dual spaces in 2. we obtain $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_{w,0}(\Omega)$. Moreover, since $H^{\beta,q'}_{w',0}(\Omega) \hookrightarrow H^{\beta,q'}_{w'}(\Omega) \hookrightarrow L^{r'}_{w_r}(\Omega)$, one also has $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_w(\Omega)$.

Finally, for $u \in W^{-1,r}_w(\Omega)$ and $\phi \in Y^{2,q'}_{w'}(\Omega)$ one has by the Poincaré inequality

$$|\langle u, \phi\rangle| \leq c\|u\|_{-1,r,w}\|\nabla\phi\|_{r',w'} \leq c\|u\|_{-1,r,w}\|\nabla\phi\|_{\beta,q',w'} \leq c\|u\|_{-1,r,w}\|\phi\|_{\beta+1,q',w'}.$$

This proves the last embedding.
4. For $u \in H^{1,r}_w(\Omega)$ one has by Lemma 4.7 and 3.

$$\left\|u - \int_\Omega u\,dx\right\|_{1-\beta,q,w} \leq c\|\nabla u\|_{-\beta,q,w} \leq c\|\nabla u\|_{r,w} \leq c\|u\|_{1,r,w}.$$

Thus $\|u\|_{1-\beta,q,w} \leq c\|u\|_{1,r,w} + \int_\Omega |u|\,dx \leq c\|u\|_{1,r,w} + c\|u\|_{r,w} \leq c\|u\|_{1,r,w}$. □

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain. Assume $w \in A_s$ for some $1 \le s < q$ and $\beta > \frac{ns}{q} - 1$ in the case $n \ge 3$ and $\beta > \frac{2s}{q} - \frac{1}{2}$ in the case $n = 2$.*

1. *In addition, let $0 \le \beta \le 1$ and $1 < t < \infty$ with*

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{t} = 0. \tag{5.4}$$

Then $w \in A_t$, $L^t_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega)$ and

a) for every $u, v \in H^{\beta,q}_w(\Omega)$ and $\psi \in H^{1-\beta,q'}_{w'}(\Omega)$ one has

$$\left| \int uv\psi \, dx \right| \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}\|\psi\|_{t',w'},$$

b) for every $k \in L^t_w(\Omega)$, $u \in H^{\beta,q}_w(\Omega)$ and $\phi \in H^{2-\beta,q'}_{w'}(\Omega)$ one has

$$\left| \int ku\phi \, dx \right| \le c\|k\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w'}.$$

2. *If $1 \le \beta \le 2$ then $\|u \cdot \nabla v\|_{\beta-2,q,w} \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}$ for every $u, v \in H^{\beta,q}_w(\Omega)$.*

Proof. One has

$$t = \frac{nsq}{q(1-\beta) + ns} > \frac{nsq}{q(2 - \frac{ns}{q}) + ns} = \frac{ns}{2} \ge s.$$

Thus, by Lemma 5.3 one has $L^t_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega)$ and $H^{1-\beta,q'}_{w_q}(\Omega) \hookrightarrow L^{t'}_{w_t}(\Omega)$.
1. a) Let $r := 2t$. Then one has

- $\frac{1}{r} - \frac{1}{q} + \frac{\beta}{ns} \ge 0$ and hence $H^{\beta,q}_w(\Omega) \hookrightarrow L^r_w(\Omega)$. If $q \le r$ this follows from Lemma 5.3 and if $q > r$ then one obtains from the definition of the spaces $H^{\beta,q}_w(\Omega) \hookrightarrow L^q_w(\Omega) \hookrightarrow L^r_w(\Omega)$.
- $\frac{1}{r} + \frac{1}{r} + \frac{1}{t'} = 1$.
- $-\frac{1}{(t-1)t'} + \frac{1}{r} + \frac{1}{r} = 0$.

$$\begin{aligned} \left| \int uv\phi \, dx \right| &= \left| \int uw^{\frac{1}{r}} vw^{\frac{1}{r}} \psi w_t^{\frac{1}{t'}} \, dx \right| \\ &\le \|u\|_{r,w}\|v\|_{r,w}\|\psi\|_{t',w_t} \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}\|\psi\|_{t',w_t}. \end{aligned}$$

1. b) First we assume that $\beta < \frac{ns}{q}$. We set $r = \frac{nsq}{-q\beta+ns}$ and $\eta = \left(1 - \frac{1}{r} - \frac{1}{t}\right)^{-1} = \frac{rt}{rt-t-r}$. Then

- $\eta' = \frac{rt}{r+t} = \frac{nsq}{q+2ns-2q\beta} > \frac{nqs}{3q} \ge s$ if $n \ge 3$. If $n = 2$ one needs the stronger assumption on β to ensure $\eta' \ge s$.
- $-\frac{1}{\eta'} + \frac{1}{t} + \frac{1}{ns} = -\frac{1}{r} + \frac{1}{ns} = \frac{1+\beta-\frac{ns}{q}}{ns} > 0$. Hence $H^{1,t'}_{w_t}(\Omega) \hookrightarrow L^{\eta}_{w_{\eta'}}(\Omega)$.

- $\frac{1}{t}+\frac{1}{r}+\frac{1}{\eta}=1$ and $-\frac{1}{(\eta'-1)\eta}+\frac{1}{t}+\frac{1}{r}=0$.

Thus we can estimate as above

$$\left|\int ku\phi\,dx\right| \leq \|k\|_{t,w}\|u\|_{r,w}\|\phi\|_{\eta,w_{\eta'}} \leq c\|k\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w_t}.$$

If $\beta \geq \frac{ns}{q}$ then $H^{\beta,q}_w(\Omega)\hookrightarrow L^r_w(\Omega)$ for every $r\in(1,\infty)$. Moreover, we find some $\eta > t'$ such that $H^{1,t'}_{w_t}(\Omega)\hookrightarrow L^\eta_{w_{\eta'}}(\Omega)$. Choosing r such that $\frac{1}{r}+\frac{1}{\eta}+\frac{1}{t}=1$ we can repeat the above estimate.
2. As above we begin with the case $\beta < \frac{ns}{q}$. Let $\eta := \frac{nsq}{ns-q\beta}$, $\mu := \frac{nsq}{ns-q\beta+q}$ and $r := \frac{nsq}{2ns-2\beta q+q}$. Then one has

- $\frac{1}{r}=\frac{1}{\eta}+\frac{1}{\mu}$.
- $r > \frac{ns}{3}\geq s$ if $n\geq 3$. If $n=2$ we need the stronger assumption on β to ensure $r>s$. Moreover, $\frac{1}{q}>\frac{1}{r}-\frac{2-\beta}{ns}$, thus $L^r_w(\Omega)\hookrightarrow H^{\beta-2,q}_w(\Omega)$.
- $\frac{1}{\eta}=\frac{1}{q}-\frac{\beta}{ns}$ which implies $H^{\beta,q}_w(\Omega)\hookrightarrow L^\eta_w(\Omega)$.
- $\frac{1}{q}-\frac{\beta-1}{ns}=\frac{1}{\mu}$ which shows $H^{\beta-1,q}_w(\Omega)\hookrightarrow L^\mu_w(\Omega)$.

Thus it follows from Hölder's inequality

$$\|u\nabla v\|_{\beta-2,q,w}\leq c\|u\nabla v\|_{r,w}\leq c\|u\|_{\eta,w}\|\nabla v\|_{\mu,w}\leq c\|u\|_{\beta,q,w}\|\nabla v\|_{\beta-1,q,w}.$$

If $2\geq\beta\geq\frac{ns}{q}$ then $H^{\beta,q}_w(\Omega)\hookrightarrow L^\eta_w(\Omega)$ for every $\eta\in(1,\infty)$. Thus if $\beta\neq 2$ we repeat the above estimate with r as above, $\mu=q$ and η such that $\frac{1}{\eta}+\frac{1}{\mu}=\frac{1}{r}$.

If $\beta=2$ let $r=q$ and we may choose $\mu>q$ such that such that $H^{\beta-1,q}_w(\Omega)\hookrightarrow L^\mu_w(\Omega)$ and η such that $\frac{1}{\eta}+\frac{1}{\mu}=\frac{1}{r}$. $\square$

5.2 Stationary Navier-Stokes Equations in Bessel Potential Spaces

In this section we always assume

- $\Omega\subset\mathbb{R}^n$ is a bounded $C^{1,1}$-domain,
- $1<q<\infty$ and $w\in A_s$ for some $1\leq s<q$,
- $\beta\in[0,2]$ with $\frac{ns}{q}-1<\beta$.

If $n\leq 3$ one can always choose such a β since by Lemma 2.1 for every $w\in A_q$ there exists s as above with $s<q$ and $w\in A_s$. Thus $\frac{ns}{q}-1<n-1\leq 2$.

Definition 5.5. Let $0\leq\beta\leq 2$, $1<q<\infty$ and $w\in A_q$. Moreover, let $g\in T^{\beta,q}_w(\partial\Omega)$, $F\in Y^{\beta-2,q}_w(\Omega)$ and $K\in L^t_w(\Omega)$. Then $u\in H^{\beta,q}_w(\Omega)$ is called a very weak solution to the stationary Navier-Stokes equations, if

$$\begin{aligned} -\langle u,\Delta\phi\rangle-\langle uu,\nabla\phi\rangle-\langle Ku,\phi\rangle &= \langle F,\phi\rangle-\langle g,N\cdot\nabla\phi\rangle_{\partial\Omega} &&\text{for every } \phi\in Y^{2,q'}_{w',\sigma}(\Omega),\\ -\langle u,\nabla\psi\rangle &= \langle K,\psi\rangle-\langle g,N\psi\rangle_{\partial\Omega} &&\text{for every } \psi\in W^{1,q'}_{w'}(\Omega). \end{aligned}$$

Theorem 5.6. *Let $q > 1$, $w \in A_s$ for some $1 \le s < q$, $0 \le \beta < 1$ and $\beta > \frac{ns}{q} - 1$ if $n \ge 3$ and $\beta > -\frac{1}{2} + \frac{2s}{q}$ if $n = 2$. Moreover, let $F \in Y_w^{\beta-2,q}(\Omega)$, $K \in L_w^t(\Omega)$ with*

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{t} = 0 \tag{5.5}$$

and $g \in T_w^{\beta,q}(\partial\Omega)$ with $\langle K, 1\rangle_\Omega = \langle g, N\rangle_{\partial\Omega}$. Then there exists a constant $\rho > 0$ independent of the data such that, if

$$\|F\|_{Y_w^{\beta-2,q}} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} \le \rho,$$

then there exists a very weak solution $u \in H_w^{\beta,q}(\Omega)$ to the stationary Navier-Stokes equations. This solution satisfies the estimate

$$\|u\|_{\beta,q,w} \le c\left(\|F\|_{-1,t,w} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right) \tag{5.6}$$

with $c = c(\beta, q, w, \Omega) > 0$. Furthermore, if we assume in addition that $F \in W_w^{-1,t}(\Omega)$, then u fulfills $u|_{\partial\Omega} = g$ in the sense of Theorem 2.9.4.

Proof. By the Lemmas 5.3 and 5.4 one has

$$L_w^t(\Omega) \hookrightarrow H_{w,0}^{\beta-1,q}(\Omega) \quad \text{and} \quad W_w^{-1,t}(\Omega) \hookrightarrow Y_w^{\beta-2,q}(\Omega).$$

For $u \in H_w^{\beta,q}(\Omega)$ let $W(u) \in (C_0^\infty(\Omega))'$ be given by

$$\langle W(u), \phi\rangle = \langle uu, \nabla\phi\rangle + \langle Ku, \phi\rangle \quad \text{for all} \quad \phi \in C_0^\infty(\Omega).$$

By Lemma 5.4.1 one has for $\phi \in C_0^\infty(\Omega)$

$$|\langle W(u), \phi\rangle| \le c(\|u\|_{\beta,q,w}^2 + \|K\|_{t,w}\|u\|_{\beta,q,w})\|\phi\|_{1,t',w'}$$

and hence $W(u) \in W_w^{-1,t}(\Omega) \hookrightarrow Y_w^{\beta-2,q}(\Omega)$ with

$$\|W(u)\|_{Y_w^{\beta-2,q}} \le c_1\|W(u)\|_{-1,t,w} \le c(\|u\|_{\beta,q,w}^2 + \|K\|_{t,w}\|u\|_{\beta,q,w}). \tag{5.7}$$

We define the mapping $S : H_w^{\beta,q}(\Omega) \to H_w^{\beta,q}(\Omega)$ by

$$\begin{aligned} -\langle Su, \Delta\phi\rangle \quad &= \langle F, \phi\rangle + \langle W(u), \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \qquad &&\text{for every} \quad \phi \in Y_{w',\sigma}^{2,q'}(\Omega), \\ -\langle Su, \nabla\psi\rangle \quad &= \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega} &&\text{for every} \quad \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned}$$

The operator S is well-defined by Theorem 4.4.

We want to use Banach's Fixed Point Theorem to show that S has a fixed point under the assumption that the data is small enough.

By the a priori estimate in Theorem 4.4 we know that

$$\|v\|_{\beta,q,w} \le D(\|F\|_{Y_w^{\beta-2,q}} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}), \tag{5.8}$$

if v is a very weak solution to the Stokes problem with respect to the data $F \in Y_w^{\beta-2,q}(\Omega)$, $K \in L_w^t(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$.

We assume that the data F, K and g are chosen small enough such that the right hand side of (5.8) is strictly smaller then $\rho := \frac{1}{6cD}$, where c is the constant in the estimate (5.7) and D is the constant in the a priori estimate (5.8). Without loss of generality we assume that $D \geq 1$, which implies that additionally $\|K\|_{t,w} < \rho$.

Furthermore, it follows from (5.7) and (5.8) that for such data and $\delta = \frac{2}{6cD}$ the closed ball $\overline{B_\delta(0)}$ in $H_w^{\beta,q}(\Omega)$ is mapped by S into itself.

The next step is to show that S is a contraction on $B_\delta(0)$. Take $u, v \in B_\delta(0)$. Then $Su - Sv$ is a solution of

$$\begin{aligned} -\langle Su - Sv, \Delta\phi\rangle &= \langle W(u) - W(v), \phi\rangle \qquad \text{for every } \ \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \\ -\langle Su - Sv, \nabla\psi\rangle &= 0 \qquad\qquad\qquad\qquad\quad \text{for every } \ \psi \in W^{1,q'}_{w'}(\Omega). \end{aligned}$$

Moreover, from Lemma 5.4.1 we obtain

$$\begin{aligned} |\langle W(u) - W(v), \phi\rangle| &\leq |\langle (u-v)u, \nabla\phi\rangle| + |\langle v(u-v), \nabla\phi\rangle| + |\langle K(u-v), \phi\rangle| \\ &\leq c(\|u\|_{\beta,q,w} + \|v\|_{\beta,q,w} + \|K\|_{t,w})\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t} \\ &= \frac{5}{6D}\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t}. \end{aligned}$$

Thus we obtain from the a priori estimate (5.8) that

$$\|Su - Sv\|_{\beta,q,w} \leq D\|W(u) - W(v)\|_{-1,t,w} \leq \frac{5}{6}\|u-v\|_{\beta,q,w}.$$

Now Banach's fixed point theorem gives us the existence of a unique fixed point of S within the ball $B_\delta(0)$ and hence of a solution $u \in H_w^{\beta,q}(\Omega)$ to the stationary Navier-Stokes system.

The a priori estimate (5.6) follows from

$$\begin{aligned} &\|u\|_{\beta,q,w} = \|S(u)\|_{\beta,q,w} \\ &\qquad \leq D\left(\|F\|_{Y_w^{\beta-2,q}} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} + c(\|u\|^2_{\beta,q,w} + \|K\|_{t,w}\|u\|_{\beta,q,w})\right) \end{aligned}$$

since $Dc(\|u\|_{\beta,q,w} + \|K\|_{t,w}) \leq \frac{3}{6}$ and we may subtract $\frac{3}{6}\|u\|_{\beta,q,w}$ from both sides of the above equation.

Now assume that $F \in W_w^{-1,q}(\Omega)$. It remains to show that in this case the solution u fulfills the boundary condition $u|_{\partial\Omega} = g$. To see this one uses the fact that u is a very weak solution to the Stokes equations with respect to the data

$$\begin{aligned} f &= [\phi \ \mapsto \langle F, \phi\rangle + \langle W(u), \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \\ k &= [\psi \ \mapsto \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega}], \end{aligned}$$

where $f|_{C_0^\infty(\Omega)} = [\phi \mapsto \langle F, \phi\rangle + \langle W(u), \phi\rangle] \in W_w^{-1,t}(\Omega)$. Then the assertion about the boundary values follows from Theorem 2.9.4. $\square$

Definition 5.7. Let $1 \leq \beta \leq 2$. Moreover, let $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$. Then $u \in H_w^{\beta,q}(\Omega)$ is called a weak solution to the stationary Navier-Stokes equations, if

$$(\nabla u, \nabla\phi) + (u \cdot \nabla u, \phi) = \langle F, \phi\rangle \ \text{ for every } \ \phi \in C^\infty_{0,\sigma}(\Omega),$$

$\operatorname{div} u = K$ and $u|_{\partial\Omega} = g$.

Theorem 5.8. *Let $1 \le \beta \le 2$ and $\beta > \frac{ns}{q} - 1$ if $n \ge 3$ and $\beta > \frac{2s}{q} - \frac{1}{2}$ if $n = 2$. Moreover, let $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$ with $\int K\,dx = \int_{\partial\Omega} gN\,dS$. Then there exists a constant $\rho > 0$ such that, if*

$$\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} \le \rho,$$

then there exists a weak solution $u \in H_w^{\beta,q}(\Omega)$ to the stationary Navier-Stokes equations. This solution satisfies the estimate

$$\|u\|_{\beta,q,w} \le c\left(\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right)$$

with $c = c(\beta, q, w, \Omega) > 0$.

Proof. This can be proved in the same way as Theorem 5.6 using Lemma 5.4.2. instead of Lemma 5.4.1. and Theorem 4.3 instead of Theorem 4.4. □

The very weak solution is unique even without the assumption of the smallness of the exterior force f and the boundary condition g. In the case $n \ge 3$ this follows from the uniqueness of very weak solutions to the stationary Navier-Stokes equations in the unweighted case which has been proved in [6]. This is shown in the following theorem.

Theorem 5.9. *Let the data F, K and g be given as in Theorem 5.6 or Theorem 5.8, respectively, and let u be a very weak solution to the stationary Navier-Stokes system with respect to the data F, K and g.*

Then there exists a constant $\rho > 0$ such that under the condition that

$$\|u\|_{\beta,q,w} + \|K\|_{t,w} \le \rho$$

there exists at most one very weak solution to the stationary Navier-Stokes equations according to Definition 5.5.

Proof. By Lemma 5.3 and Lemma 2.2 one has for $\beta < \frac{ns}{q}$

$$u \in H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{\frac{nsq}{-q\beta+ns}}(\Omega) \hookrightarrow L^{\frac{nq}{-q\beta+ns}}(\Omega) = L^{\eta}(\Omega),$$

where, by the assumptions on β, one has $\eta := \frac{nq}{-q\beta+ns} > n$.

For $\beta \ge \frac{ns}{q}$ the embedding $H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{\mu}(\Omega)$ holds for every $\mu > 1$. If we choose $\mu = \eta s$ with $\eta > n$, then we obtain that also in this case

$$H_w^{\beta,q}(\Omega) \hookrightarrow L^{\eta}(\Omega) \tag{5.9}$$

We want to show that $\eta > n$ in (5.9) can be chosen such that $K \in L^{\frac{\eta n}{\eta+n}}(\Omega)$ and $F \in W^{-1,\frac{n\eta}{\eta+n}}(\Omega)$ is fulfilled additionally. If $\beta \le 1$ then one has by assumption $K \in L_w^t(\Omega)$ and $F \in W_w^{-1,t}(\Omega)$ and by the proof of Lemma 5.4 one has $t > \frac{ns}{2} = \frac{n^2 s}{n+n}$. Thus we find η with the asserted properties, since again by Lemma 2.2 one has the embeddings

$$L_w^t(\Omega) \hookrightarrow L^{\frac{t}{s}}(\Omega) \text{ and } W_w^{-1,t}(\Omega) \hookrightarrow W^{-1,\frac{t}{s}}(\Omega).$$

Now let $\beta > 1$. Then the embedding $H_w^{\beta-1,q}(\Omega) \hookrightarrow L_w^t(\Omega)$ follows directly from Lemma 5.3 and $Y_w^{\beta-2,q}(\Omega) \hookrightarrow W_w^{-1,t}(\Omega)$ follows when taking the dual spaces in the embedding $W_{w',0}^{1,t'}(\Omega) \hookrightarrow Y_{w'}^{2-\beta,q'}(\Omega)$, that is shown in Lemma 5.3.

Moreover, from Corollary 4.6 we obtain that $g \in W^{-\frac{1}{\eta},\eta}(\partial\Omega) := T_1^{0,\eta}(\partial\Omega)$. Hence data and solution are contained in the same spaces as in [6, Theorem 1.5]. Thus exactly the same proof as given there can be used to show that two solutions that correspond to the same data coincide. $\square$

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