Very Weak Solutions to the Stationary Stokes and Stokes Resolvent Problem in Weighted Function Spaces

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We investigate very weak solutions to the stationary Stokes and Stokes resolvent problem in function spaces with Muckenhoupt weights. The notion used here is similar but even more general than the one used in [2] or [14]. Consequently the class of solutions is enlarged. To describe boundary conditions we restrict ourselves to more regular data. We introduce a Banach space that admits a restriction operator and that contains the solutions according to such data.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-boundary. We consider the stationary Stokes resolvent problem with inhomogeneous data

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= F \quad \text{in } \Omega \\ \operatorname{div} u &= K \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

It is our aim to find a large class of solutions to (1.1) demanding as low regularity of the data as possible. In the most general case considered here the solutions possess a priori no weak derivatives. Consequently the notion of weak solutions is no longer suitable in this context. Thus one introduces the more general notion of very weak solutions.

To arrive at the definition of very weak solutions one multiplies the first equations in (1.1) with a solenoidal test function ϕ vanishing on the boundary, then formal integration by parts yields

$$\langle u, \lambda\phi\rangle - \langle u, \Delta\phi\rangle_\Omega = \langle F, \phi\rangle_\Omega - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}. \tag{1.2}$$

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Applying the same method to the second equation with a sufficiently smooth test function ψ we obtain

$$-\langle u, \nabla\psi\rangle_\Omega = \langle K, \psi\rangle_\Omega - \langle g, N\cdot\psi\rangle_{\partial\Omega}. \tag{1.3}$$

The equations (1.2) and (1.3) can be used for the definition of very weak solutions. This or similar formulations have been introduced by Amann in [2], by Amrouche and Girault in [3] and by Galdi, Simader and Sohr in [14]. In these articles as well as by Farwig, Galdi and Sohr in [6], [5], [7] and by Giga in [16] solvability with low-regularity data has been shown.

We use an even more general notion considering each right hand side of (1.2) and (1.3) as one functional in ϕ or ψ, respectively. Since these right hand sides in (1.2) and (1.3) contain a component that is supported on the boundary, it is natural that these functionals are not contained in spaces of distributions on Ω. The advantage of this approach is a simple characterization of the space of solutions, more precise a priori estimates and a shorter proof of the existence and uniqueness theorem. Moreover, it is shown in Sections 4 and 5 that the classes of strong and of very weak solutions considered in [14] are contained in the class of very weak solutions corresponding to the non-distributional data that are considered in Section 3.

In the most general context considered here every L^q-function can be considered as a very weak solution with respect to appropriate data. Thus the restriction of a general solution to the boundary is not well defined. However if one restricts oneself to more regular data similar to those considered in [14] it is again possible to prescribe boundary values. More precisely in Section 5 it is shown that very weak solutions corresponding to the restricted data are contained in a Banach space that permits restrictions to the boundary.

We investigate this problem in weighted function spaces. More precisely, we consider Lebesgue and Sobolev spaces with respect to the measure $w\,dx$, where w is a weight function contained in the Muckenhoupt class A_q. This is the class of nonnegative and locally integrable weight functions, for which the expression

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{q-1}}\,dx\right)^{q-1} \tag{1.4}$$

is finite, where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Classical tools for the treatment of partial differential equations extend to function spaces with Muckenhoupt weights. As important examples we mention the continuity of the maximal operator [15], [26], the multiplier theorems [15], [26], extension theorems of functions on a domain to functions on $\mathbb{R}^n$ shown by Chua [4], extension theorems of functions on the boundary to functions on the domain by Fröhlich [11], see also [22] and embedding theorems by Fröhlich [12] using the continuity of singular integral operators by Sawyer and Wheeden [21].

These tools were the base to treat the solvability of the Stokes and Navier-Stokes equations in weighted function spaces by Farwig and Sohr in [8] and by Fröhlich in [9], [10], [11].

As shown in [8] examples of Muckenhoupt weights are

$$\begin{array}{ll} w(x) = (1+|x|)^\alpha, & -n < \alpha < n(q-1) \ \text{ or} \\ \operatorname{dist}(x, M)^\alpha, & -(n-k) < \alpha < (n-k)(q-1), \end{array}$$

where M is a compact k-dimensional Lipschitzian manifold. Thus, if one chooses a particular weight function, the developed theory can be used for a better control of the growth of the solution, for example for $|x| \to \infty$, in the neighborhood of a point or close to the boundary.

2 Preliminaries

All over this paper let $q \in (1, \infty)$ and we consider a Muckenhoupt weight $w \in A_q$, cf. (1.4). Moreover, let $k \in \mathbb{N}_0$ and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then we define the following weighted versions of Lebesgue and Sobolev spaces.

1. $L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\, dx\right)^{\frac{1}{q}} < \infty \right\}$. Then it is an easy consequence of the corresponding result in the unweighted case that
$$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \;\text{ with }\; \frac{1}{q} + \frac{1}{q'} = 1 \;\text{ and }\; w' = w^{-\frac{1}{q-1}}. \tag{2.1}$$

2. By $C_0^\infty(\Omega)$ we denote the set of all smooth and compactly supported functions, the space $C_{0,\sigma}^\infty(\Omega)$ consists of all functions that are in addition divergence free.

3. Set $W^{k,q}_w(\Omega) = \left\{ u \in L^q_w(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha| \le k} \|D^\alpha u\|_{q,w} < \infty \right\}$.

4. Moreover we set $W^{k,q}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,q,w}}$. The dual space of it is denoted by $W^{-k,q}_w(\Omega) := (W^{k,q'}_{w',0}(\Omega))'$.

5. Using this for $k > 0$ we set $W^{-k,q}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{-k,q}_w(\mathbb{R}^n)}}$.

6. Moreover, we consider the spaces of boundary values $T^{k,q}_w(\partial\Omega) := (W^{k,q}_w(\Omega))|_{\partial\Omega}$, equipped with the norm $\|\cdot\|_{T^{k,q}_w} = \|\cdot\|_{T^{k,q}_w(\partial\Omega)}$ of the factor space.

By [9], [11] and [4] the spaces $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$, $W^{k,q}_{w,0}(\Omega)$ and $T^{k,q}_w(\partial\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$, $(C_0^\infty(\Omega)$, $C^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively) are dense.

Note that by Nečas [20], Chapitre 2, §5, in the unweighted case one has
$$T^{k,q}_1(\partial\Omega) = W^{k-\frac{1}{q},q}(\partial\Omega) \;\text{ for }\; k \in \mathbb{N} \;\text{ and }\; T^{0,q}_1(\partial\Omega) = W^{-\frac{1}{q},q}(\partial\Omega).$$
In particular, the space $T^{0,q}_w(\partial\Omega)$ does not consist of functions but of distributions on $\partial\Omega$.

By [4] the following extension theorem holds for weighted Sobolev spaces.

Theorem 2.1. *Let Ω be a bounded Lipschitz domain and $N \in \mathbb{N}$. Choose $p_i \in [1, \infty)$, $w_i \in A_{p_i}$ and $k_i \in \mathbb{N}_0$, $i = 1, ..., N$. Then there exists an extension operator*
$$E : \bigcap_{i=1}^N W^{k_i,p_i}_{w_i}(\Omega) \to \bigcap_{i=1}^N W^{k_i,p_i}_{w_i}(\mathbb{R}^n),$$
i.e., $Eu|_\Omega = u$ and $\|Eu\|_{W^{k_i,p_i}_{w_i}(\mathbb{R}^n)} \le c\|u\|_{W^{k_i,p_i}_{w_i}(\Omega)}$ for $i = 1, ..., N$ and for every $u \in \bigcap_{i=1}^N W^{k_i,p_i}_{w_i}(\Omega)$.

Proof. This is a special case of [4, Theorem 4.1]. There Chua proves extension theorems for the class of (ε,∞)-domains. By [19] this class includes Lipschitz domains. $\square$

Lemma 2.2. *Let Ω be a bounded Lipschitz domain. For $k\in\mathbb{Z}$, $q\in(1,\infty)$ and $w\in A_q$ one has*

$$W^{-k,q}_{w,0}(\Omega)=\left(W^{k,q'}_{w'}(\Omega)\right)'.$$

Proof. For $k=0$ this follows from the density of $C_0^\infty(\Omega)$ in $L^q_w(\Omega)$ and (2.1).

If $k<0$ it follows from the definition and the reflexivity of $W^{-k,q}_{w,0}(\Omega)$.

It remains to prove the case $k>0$. By definition, $W^{-k,q}_{w,0}(\Omega)$ is a closed subspace of $W^{-k,q}_w(\mathbb{R}^n)$. Thus, for $u\in\left(W^{-k,q}_{w,0}(\Omega)\right)'$ there exists by the Hahn-Banach theorem a functional $U\in\left(W^{-k,q}_w(\mathbb{R}^n)\right)'=W^{k,q'}_{w',0}(\mathbb{R}^n)=W^{k,q'}_{w'}(\mathbb{R}^n)$ with $\|U\|_{W^{k,q'}_{w'}(\mathbb{R}^n)}=\|u\|_{(W^{-k,q}_{w,0}(\Omega))'}$ and $U|_{C_0^\infty(\Omega)}=u|_{C_0^\infty(\Omega)}$.

This means u can be identified with the function $U|_\Omega\in W^{k,q'}_{w'}(\Omega)$ which fulfills

$$\|U|_\Omega\|_{W^{k,q'}_{w'}(\Omega)}\le c\|u\|_{(W^{-k,q}_{w,0}(\Omega))'}.$$

Vice versa let $u\in W^{k,q'}_{w'}(\Omega)$. Then by Theorem 2.1 there exists $U=Eu\in W^{k,q'}_{w'}(\mathbb{R}^n)$ with $U|_\Omega=u$ and we obtain by the continuity of E and the Hahn-Banach theorem

$$\begin{aligned}c\|u\|_{W^{k,q'}_{w'}(\Omega)}\ge&\|U\|_{W^{k,q'}_{w'}(\mathbb{R}^n)}=\sup_{\phi\in\mathcal{S},\|\phi\|_{W^{-k,q}_w(\mathbb{R}^n)}=1}|\langle U,\phi\rangle|\\ \ge&\sup_{\phi\in C_0^\infty(\Omega),\|\phi\|_{W^{-k,q}_w(\mathbb{R}^n)}=1}|\langle u,\phi\rangle|=\|u\|_{(W^{-k,q}_{w,0}(\Omega))'}.\end{aligned}$$

Thus we have shown $(W^{-k,q}_{w,0}(\Omega))'=W^{k,q'}_{w'}(\Omega)$. Now the reflexivity of the spaces proves the assertion. $\square$

By [22] one has the following extension theorem of functions defined on the boundary $\partial\Omega$ to functions defined on Ω.

Theorem 2.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{k-1,1}$-domain, $k\ge1$. Then there exists a continuous linear operator*

$$L:\prod_{j=0}^{k-1}T^{k-j,q}_w(\partial\Omega)\to W^{k,q}_w(\Omega)$$

such that $\frac{\partial^j}{\partial N^j}L(g)|_{\partial\Omega}=g_j$, $0\le j\le k-1$, where $g=(g_0,...,g_{k-1})$.

By [24] there holds the following weighted version of Bogowski's Theorem.

Theorem 2.4. *Let $\Omega\subset\mathbb{R}^n$, $n\ge2$, be a bounded and locally lipschitzian domain. Assume that $f\in W^{k,q}_{w,0}(\Omega)$ such that $\int f=0$. Then there exists a function $u\in W^{k+1,q}_{w,0}(\Omega)$ such that*

$$\operatorname{div}u=f\quad\textit{and}\quad\|u\|_{k+1,q,w}\le c\|f\|_{k,q,w},$$

where $c=c(\Omega,q,w,k)$. Moreover, u can be chosen such that it depends linearly on f and such that $u\in C_0^\infty(\Omega)$ if $f\in C_0^\infty(\Omega)$.

Theorem 2.5. *Let $1 < q < \infty$ and $w \in A_q$. Then the maximal operator M defined by*

$$(Mf)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

is continuous on $L^q_w(\mathbb{R}^n)$. More precisely, there exists a constant $c > 0$ such that

$$\|Mf\|_{q,w} \le c\|f\|_{q,w} \quad \text{for every} \;\; f \in L^q_w(\mathbb{R}^n).$$

Vice versa if μ is a nonnegative Borel measure and M is bounded on $L^q(\mathbb{R}^n, \mu)$, then μ is absolutely continuous and $d\mu = w\,dx$ for some $w \in A_q$.

Proof. See [15], Theorems 2.1 and 2.9. The reverse inclusion can be found in [26, 2.2]. □

3 Very Weak Solutions Concerning Non-Distributional Data

For a good formulation of our notion of very weak solutions, we need to define some spaces of functions and functionals. Thus for $w \in A_q$ we set

$$\begin{aligned} Y^{2,q'}_{w'}(\Omega) &:= \{u \in W^{2,q'}_{w'}(\Omega) \mid u|_{\partial\Omega} = 0\}, \\ Y^{-2,q}_{w}(\Omega) &:= (Y^{2,q'}_{w'}(\Omega))' \;\text{ and} \\ W^{-1,q}_{w,0}(\Omega) &= (W^{1,q'}_{w'}(\Omega))'. \end{aligned} \tag{3.1}$$

Moreover, we define the divergence-free versions

$$Y^{2,q'}_{w',\sigma}(\Omega) := \{\phi \in Y^{2,q'}_{w'}(\Omega) \mid \operatorname{div}\phi = 0\} \;\text{ and }\; Y^{-2,q}_{w,\sigma}(\Omega) := (Y^{2,q'}_{w',\sigma}(\Omega))'. \tag{3.2}$$

Then for suitable F, K and g one obtains for the right hand sides of (1.2) and (1.3)

$$\begin{aligned} [\phi \mapsto \langle F, \phi\rangle_\Omega - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}] &\in Y^{-2,q}_w(\Omega) \\ [\psi \mapsto \langle K, \psi\rangle_\Omega - \langle g, N\cdot\psi\rangle_{\partial\Omega}] &\in W^{-1,q}_{w,0}(\Omega). \end{aligned}$$

In the sequel we consider external forces $f \in Y^{-2,q}_w(\Omega)$ and divergences $k \in W^{-1,q}_{w,0}(\Omega)$.

Lemma 3.1. *$C^\infty(\overline{\Omega})$ is dense in $Y^{-2,q}_w(\Omega)$ and in $W^{-1,q}_{w,0}(\Omega)$.*

Proof. $Y^{2,q'}_{w'}(\Omega)$ is reflexive being a closed subspace of the reflexive space $W^{2,q'}_{w'}(\Omega)$. Let $x \in Y^{-2,q}_w(\Omega)' = Y^{2,q'}_{w'}(\Omega)$ such that $\langle\phi, x\rangle = 0$ for all $\phi \in C^\infty(\overline{\Omega})$. This yields $x = 0$ and the assertion is proved. The assertion about $W^{-1,q}_{w,0}(\Omega)$ is proved in the same way. □

Note that these spaces do not consist of distributions on Ω since $C^\infty_0(\Omega)$ is neither dense in $Y^{2,q'}_{w'}(\Omega)$ nor in $W^{1,q'}_{w'}(\Omega)$. This leads to some difficulties when talking about derivatives. However, restricting f or k to test functions $\phi \in C^\infty_0(\Omega)$ one obtains an element of $W^{-2,q}_w(\Omega)$ or $W^{-1,q}_w(\Omega)$, respectively. If we say that equations are fulfilled in the distributional sense, we consider these restrictions.

Definition 3.2. Let $f \in Y_w^{-2,q}(\Omega)$ and $k \in W_{w,0}^{-1,q}(\Omega)$. A function $u \in L_w^q(\Omega)$ is called

1. a very weak solution to the Stokes problem with respect to the data f and k if

$$\begin{aligned} -\langle u, \Delta\phi\rangle = &\quad \langle f, \phi\rangle, \qquad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \text{ and} & (3.3)\\ -\langle u, \nabla\psi\rangle = &\quad \langle k, \psi\rangle, \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). & (3.4)\end{aligned}$$

2. a very weak solution to the Stokes resolvent problem with respect to the data f and k and $\lambda \in \mathbb{C}$, if

$$\begin{aligned} \langle \lambda u, \phi\rangle - \langle u, \Delta\phi\rangle = &\quad \langle f, \phi\rangle, \qquad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \text{ and} & (3.5)\\ -\langle u, \nabla\psi\rangle = &\quad \langle k, \psi\rangle, \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). & (3.6)\end{aligned}$$

Setting $\psi = 1$ in (3.4) and (3.6) it follows that a necessary condition for the existence of such a very weak solution u is $\langle k, 1\rangle = 0$. This condition is the analogue to the compatibility condition $\langle k, 1\rangle = \langle g, N\rangle_{\partial\Omega}$ between divergence and boundary values in the case of weak solutions.

Remark 3.3. Some comments about the missing boundary values:

1. For every $u \in L_w^q(\Omega)$ one has

$$[\phi \mapsto \langle u, \Delta\phi\rangle] \in Y_w^{-2,q}(\Omega) \quad \text{and} \quad [\psi \mapsto \langle u, \nabla\psi\rangle] \in W_{w,0}^{-1,q}(\Omega).$$

   Thus any $u \in L_w^q(\Omega)$ appears as a very weak solution to the Stokes problem with respect to appropriate data. However, since $C_0^\infty(\Omega)$ is dense in $L_w^q(\Omega)$, it is impossible to define boundary values for arbitrary L_w^q-functions in the sense of a continuous linear operator from $L_w^q(\Omega)$ into some boundary space which coincides with the usual trace on smooth functions.

2. Dealing with very weak solutions one can define boundary values adding the term $\langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}$ on the right hand side of (3.3) and $\langle g, N \cdot \psi\rangle_{\partial\Omega}$ on the right hand side of (3.4). This is done in e.g. in [2], [6] and [14] in the case of more regular data. However, one easily sees that if $g \in T_w^{0,q}(\partial\Omega)$ then

$$G = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \in Y_w^{-2,q}(\Omega) \;\text{ and }\; K = [\psi \mapsto \langle g, N \cdot \psi\rangle_{\partial\Omega}] \in W_{w,0}^{-1,q}(\Omega),$$

   the spaces of external forces and divergences, respectively. This means

$$\begin{aligned} -\langle u, \Delta\phi\rangle &= \langle f, \phi\rangle + \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} = \langle f + G, \phi\rangle \text{ and}\\ -\langle u, \nabla\psi\rangle &= \langle k, \psi\rangle + \langle g, N \cdot \psi\rangle_{\partial\Omega} = \langle k + K, \psi\rangle.\end{aligned}$$

   Hence, since the data is so irregular, it is impossible to distinguish between force or divergence and boundary value.

3. In Section 5 we will consider the case of more regular (distributional) forces and divergences. It will be described how to regain the possibility of prescribing boundary data. Moreover, we will discuss why the existence and uniqueness of very weak solutions in the sense of Definition 3.2 does not contradict the theory of strong solutions to the Stokes equations in weighted spaces established in [9], [10].

In the following theorem it is stated that the constant in the a priori estimate is A_q-consistent. A constant $C = C(w)$ is called A_q-consistent if for every $c_0 > 0$ it can be chosen uniformly for all $w \in A_q$ with $A_q(w) < c_0$.

The A_q-consistence is of great importance since it is needed for the application of the Extrapolation Theorem [15, IV Lemma 5.18]. In particular this is used when showing the continuity of operator-valued Fourier multipliers and the maximal regularity of an operator; see e.g. [11] for details and applications.

Theorem 3.4. *Let $f \in Y_w^{-2,q}(\Omega)$, $k \in W_{w,0}^{-1,q}(\Omega)$ with $\langle k, 1\rangle = 0$ and let $\lambda \in \Sigma_\varepsilon \cup \{0\} = \left\{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \varepsilon + \frac{\pi}{2}\right\} \cup \{0\}$ with $0 < \varepsilon < \frac{\pi}{2}$. Then there exists a unique very weak solution $u \in L_w^q(\Omega)$ to the Stokes resolvent problem in the sense of Definition 3.2.2. It fulfills the a priori estimate*

$$\lambda \|u|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{Y^{-2,q}_{w,\sigma}} + \|u\|_{q,w} \le c\left(\|f\|_{Y_w^{-2,q}(\Omega)} + \|k\|_{W^{-1,q}_{w,0}}\right) \tag{3.7}$$

with $c = c(\Omega, q, w, \varepsilon) > 0$ depending A_q-consistently on w.

Proof. Step 1. Let $v \in L^{q'}_{w'}(\Omega)$. By the existence of strong solutions to the Stokes resolvent problem ([10, Theorem 3.3] in the case of weighted and [13], [25] in the case of unweighted spaces) there are unique functions $\phi \in W^{2,q'}_{w'}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$ which depend linearly on v and such that

$$\lambda\phi - \Delta\phi + \nabla\psi = v \quad \text{and} \quad \operatorname{div}\phi = 0 \quad \text{in } \Omega, \quad \phi|_{\partial\Omega} = 0 \quad \text{and} \quad \int \psi = 0. \tag{3.8}$$

This solution satisfies

$$\lambda\|\phi\|_{q',w'} + \|\phi\|_{2,q',w'} + \|\psi\|_{1,q',w'} \le c\|v\|_{q',w'}$$

with an A_q-consistent constant c that is independent of $\lambda \in \Sigma_\varepsilon \cup \{0\}$.
Step 2. (Existence and a priori estimates) Setting for $v \in L^{q'}_{w'}(\Omega)$

$$\langle u, v\rangle := \langle f, \phi\rangle - \langle k, \psi\rangle, \quad \text{with } (\phi, \psi) \text{ as in } (3.8), \tag{3.9}$$

we obtain

$$|\langle u, v\rangle| \le |\langle f, \phi\rangle| + |\langle k, \psi\rangle| \le c\left(\|f\|_{Y_w^{-2,q}} + \|k\|_{W^{-1,q}_{w,0}}\right)\|v\|_{q',w'}.$$

Thus $u \in (L^{q'}_{w'}(\Omega))' = L^q_w(\Omega)$ and fulfills $\|u\|_{q,w} \le c(\|f\|_{Y_w^{-2,q}} + \|k\|_{W^{-1,q}_{w,0}})$ with c independent of λ and depending A_q-consistently on w.

We now show that u is a very weak solution to the Stokes problem with respect to f and k. Choose test functions $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$. Then setting $v = \lambda\phi - \Delta\phi + \nabla\psi$ we obtain from the uniqueness of strong solutions

$$\langle u, \lambda\phi - \Delta\phi + \nabla\psi\rangle = \langle u, v\rangle = \langle f, \phi\rangle - \langle k, \psi\rangle.$$

Since ϕ and ψ were chosen arbitrarily, (3.5) and (3.6) are fulfilled.

Moreover, let $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$. Then we obtain

$$\begin{aligned} |\langle \lambda u, \phi\rangle| &\le |\langle u, \Delta\phi\rangle| + |\langle f, \phi\rangle| \le (\|u\|_{q,w} + \|f\|_{Y_w^{-2,q}})\|\phi\|_{2,q',w'} \\ &\le c(\|f\|_{Y_w^{-2,q}} + \|k\|_{W^{-1,q}_{w,0}(\Omega)})\|\phi\|_{2,q',w'}. \end{aligned}$$

Combining this with the previous estimate we get (3.7),

Step 3. (Uniqueness) Assume $U \in L^q_w(\Omega)$ is a very weak solution to the Stokes resolvent problem with respect to f and k. As above for every $v \in L^{q'}_{w'}(\Omega)$ we find $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$ such that $\lambda u - \Delta\phi + \nabla\psi = v$. If we add the equations (3.5) and (3.6) we obtain

$$\langle U, v\rangle = \langle U, \lambda\phi - \Delta\phi + \nabla\psi\rangle = \langle f, \phi\rangle - \langle k, \psi\rangle = \langle u, v\rangle.$$

Since $v \in L^{q'}_{w'}(\Omega)$ was arbitrary, we obtain $u = U$. □

Theorem 3.5. *Let f and k be chosen as in Theorem 3.4 and let $u \in L^q_w(\Omega)$ be the associated very weak solution to the Stokes problem. Then there exists a unique pressure functional $p \in W^{-1,q}_{w,0}(\Omega)$ (unique modulo constants) such that (u,p) solves*

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \langle f, \phi\rangle \quad \textit{for all} \ \ \phi \in Y^{2,q'}_{w'}(\Omega).$$

In particular

$$-\Delta u + \nabla p|_{C_0^\infty(\Omega)} = f|_{C_0^\infty(\Omega)}$$

in the sense of distributions. The functionals (u,p) fulfill the inequality

$$\|u\|_{q,w} + \|p\|_{W^{-1,q}_{w,0}} \le c\left(\|f\|_{Y_w^{-2,q}} + \|k\|_{W^{-1,q}_{w,0}}\right), \tag{3.10}$$

where $c = c(\Omega, q, w) > 0$.

Proof. By Lemma 3.1 there exist sequences $(f_n)_n, (k_n)_n \subset C^\infty(\overline{\Omega})$ such that

$$f_n \xrightarrow{Y_w^{-2,q}(\Omega)} f \quad \text{and} \quad k_n \xrightarrow{W^{-1,q}_{w,0}(\Omega)} k.$$

Then by [10, Theorem 3.3] there exist unique solutions $(u_n, p_n) \in W^{2,q}_w(\Omega) \times W^{1,q}_w(\Omega)$ such that

$$-\Delta u_n + \nabla p_n = f_n, \quad \operatorname{div} u_n = k_n, \quad u_n|_{\partial\Omega} = 0, \quad \int p_n = 0.$$

Integration by parts implies that u_n is a very weak solution with respect to f_n, k_n. Now the a priori estimate (3.2) shows $u_n \xrightarrow{L^q_w(\Omega)} u$. For $\phi \in W^{1,q'}_{w'}(\Omega)$ with $\int\phi = 0$ let $\zeta \in Y^{2,q'}_{w'}(\Omega)$ be the solution to $-\Delta\zeta + \nabla\pi = 0$ and $\operatorname{div}\zeta = \phi$. Then $\|\zeta\|_{2,q',w'} \le c\|\phi\|_{1,q',w'}$. Thus we obtain

$$\begin{aligned} |\langle p_n - p_m, \phi\rangle| &= |\langle p_n - p_m, \operatorname{div}\zeta\rangle| = |\langle \nabla(p_n - p_m), \zeta\rangle| \\ &\le |\langle \Delta(u_n - u_m), \zeta\rangle| + |\langle f_n - f_m, \zeta\rangle| \\ &\le c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{Y_w^{-2,q}})\|\phi\|_{1,q',w'}. \end{aligned}$$

Thus $\|p_n - p_m\|_{-1,q,w,0} \le c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{Y_w^{-2,q}}) \xrightarrow{n,m\to\infty} 0$ and $(p_n)_n$ is a Cauchy sequence converging to some $p \in W^{-1,q}_{w,0}(\Omega)$. For this p

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \lim_{n\to\infty}(-\langle u_n, \Delta\phi\rangle - \langle p_n, \operatorname{div}\phi\rangle) = \lim_{n\to\infty}\langle f_n, \phi\rangle = \langle f, \phi\rangle$$

holds for every $\phi \in Y^{2,q'}_{w'}(\Omega)$. The estimate (3.10) follows from the estimates for p_n and u_n. □

4 Regularity

The following Theorem 4.2 describes how strong solutions fit into the context of very weak solutions considered in the previous section. Moreover, it prepares further considerations about boundary values in the case of low regularity data.

Lemma 4.1. *Let $1 < q, r < \infty$, $w \in A_q$ and $\tilde{w} \in A_r$ such that*

$$W^{1,q'}_{w'}(\Omega) \hookrightarrow L^{r'}_{\tilde{w}'}(\Omega) \hookrightarrow L^{q'}_{w'}(\Omega). \tag{4.1}$$

Then

$$L^r_{\tilde{w}}(\Omega) \hookrightarrow W^{-1,q}_{w,0}(\Omega) \quad \text{and} \quad W^{-1,r}_{\tilde{w}}(\Omega) \hookrightarrow Y^{-2,q}_w(\Omega)$$

Proof. Both assertions follow from (4.1) by duality. □

The reason why we require these embeddings is that Sobolev-like inequalities in weighted spaces need strong assumptions on the weight-functions. In [12] sufficient conditions for such embeddings are proved using the continuity of singular integral operators shown in [21].

Theorem 4.2. *Assume that $f \in Y^{-2,q}_w(\Omega)$ and $k \in W^{-1,q}_{w,0}(\Omega)$ allow a decomposition into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for all } \phi \in Y^{2,q'}_{w'}(\Omega), \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega} && \text{for all } \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned} \tag{4.2}$$

with $g \in T^{0,q}_w(\partial\Omega), F \in W^{-1,r}_{\tilde{w}}(\Omega), K \in L^r_{\tilde{w}}(\Omega)$, where $1 < r < \infty$ and $\tilde{w} \in A_r$ are chosen according to (4.1). Then one has:

1. *Such a decomposition is uniquely defined by f and k.*

2. *For $\lambda \in \Sigma_\varepsilon \cup \{0\}$ every strong solution u to the Stokes resolvent problem corresponding to the data $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ and $K \in W^{1,q}_w(\Omega)$ is a very weak solution corresponding to the data f and k with the notation of (4.2).*

3. *If $\lambda \in \Sigma_\varepsilon \cup \{0\}$, $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ and $K \in W^{1,q}_w(\Omega)$ with $\int_\Omega K = \int_{\partial\Omega} N\cdot g$, then the very weak solution u to the Stokes resolvent problem with respect to f and k is a strong solution with respect to F, K and g. In particular $u \in W^{2,q}_w(\Omega)$ and*

$$|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \le c(\|F\|_{q,w} + \|K\|_{1,q,w} + \|\lambda K\|_{W^{-1,q}_{w,0}} + \|g\|_{T^{2,q}_w} + \|\lambda g\|_{T^{0,q}_w}). \tag{4.3}$$

Proof. 1. Let $\langle f, \phi\rangle = \langle F_i, \phi\rangle - \langle g_i, N\cdot\nabla\phi\rangle_{\partial\Omega}$ for $i = 1, 2$ with F_i, g_i as in the assumption. Then

$$\langle F_1 - F_2, \phi\rangle = \langle g_1 - g_2, N\cdot\nabla\phi\rangle_{\partial\Omega} \quad \text{for } \phi \in Y^{2,q'}_{w'}(\Omega).$$

The latter functional vanishes on $C_0^\infty(\Omega)$ and since $F_1 - F_2$ is a distribution on Ω, it follows that $F_1 - F_2 = 0$ and hence $\langle g_1 - g_2, N\cdot\nabla\phi\rangle = 0$ for every $\phi \in Y^{2,q'}_{w'}(\Omega)$. By Theorem 2.3 the mapping

$$\phi \mapsto N\cdot\nabla\phi : \ Y^{2,q'}_{w'}(\Omega) \to T^{1,q'}_{w'}(\partial\Omega)$$

is surjective, hence $g_1 = g_2$. Analogously one shows that the decomposition of the divergence k is unique.
2. This follows immediately by integration by parts.
3. By Theorem 2.3 there exists $v_1 \in W^{2,q}_w(\Omega)$ with $v_1|_{\partial\Omega} = g$ and $\|v_1\|_{2,q,w} \le c\|g\|_{T^{2,q}_w}$ and one has

$$\langle K - \operatorname{div} v_1, 1\rangle = \langle K, 1\rangle - \langle g, N\rangle_{\partial\Omega} = 0.$$

Hence, by [10, Theorem 3.3] there exists a strong solution $v_2 \in Y^{2,q}_w(\Omega)$ with respect to the exterior force $F - \lambda v_1 + \Delta v_1$ and divergence $K - \operatorname{div} v_1$. It fulfills the estimate

$$\begin{aligned}
&|\lambda|\|v_2\|_{q,w} + \|v_2\|_{2,q,w} \\
&\le c\left(\|F\|_{q,w} + \|\Delta v_1\|_{q,w} + |\lambda|\|v_1\|_{q,w} + \|K - \operatorname{div} v_1\|_{1,q,w} + |\lambda|\|K - \operatorname{div} v_1\|_{W^{-1,q}_{w,0}}\right) \\
&\le c\left(\|F\|_{q,w} + |\lambda|\|v_1\|_{q,w} + \|K\|_{1,q,w} + |\lambda|\|K - \operatorname{div} v_1\|_{W^{-1,q}_{w,0}} + \|g\|_{T^{2,q}_w}\right).
\end{aligned} \tag{4.4}$$

Then $u = v_1 + v_2$ is a strong solution to the Stokes resolvent problem with respect to the given data. Moreover, in the case $\lambda = 0$, also the estimate is proved.

Now we repeat the above arguments with v_1 replaced by the solution to the Stokes problem

$$-\Delta v_1 + \nabla p = 0, \quad \operatorname{div} v_1 = 0 \ \text{ and } \ v_1|_{\partial\Omega} = g.$$

Then v_1 fulfills the estimate $\|v_1\|_{2,q,w} \le c\|g\|_{T^{2,q}_w(\partial\Omega)}$. In addition, by 2. we know that v_1 is also a very weak solution with respect to the data

$$\tilde{f} = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle] \ \text{ and } \ \tilde{k} = [\psi \mapsto \langle g, N \cdot \psi\rangle].$$

Thus we obtain the estimate

$$\|v_1\|_{q,w} \le c\left(\|\tilde{f}\|_{Y^{-2,q}_w} + \|\tilde{k}\|_{W^{-1,q}_{w,0}}\right) \le c\|g\|_{T^{0,q}_w}.$$

Inserting this in (4.4) we obtain

$$\begin{aligned}
|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \le& |\lambda|\|v_1\|_{q,w} + \|v_1\|_{2,q,w} + |\lambda|\|v_2\|_{q,w} + \|v_2\|_{2,q,w} \\
\le& c\left(\|F\|_{q,w} + \|K\|_{1,q,w} + |\lambda|\|K\|_{W^{-1,q}_{w,0}} + \|g\|_{T^{2,q}_w} + |\lambda|\|g\|_{T^{0,q}_w}\right).
\end{aligned}$$

Thus there exists a strong solution to the Stokes resolvent problem with respect to the given data which fulfills the estimate.

The uniqueness of very weak solutions proved in Theorem 3.4 together with 2. yields that u coincides with the very weak solution. In particular the very weak solution is regular according to the data. □

Remark 4.3. If there exist decompositions for the data f and k as in (4.2) even with smooth functions F, K, g this does not mean that f and k are smooth. The reason is that if $g \neq 0$, then $\phi \mapsto \langle g, N \cdot \nabla\phi\rangle$ can never be a function since it is a functional supported by the boundary and depending on derivatives.

Vice versa, if f and k are regular, e.g. $f \in W^{-1,q}_w(\Omega)$ and $k \in L^q_w(\Omega)$ then they also allow a decomposition according to (4.2) and we automatically obtain $g = 0$, which means that the very weak solution with respect to f and k has zero boundary values.

5 Boundary Values in the Case of More Regular Data

Our next aim is to define boundary values for very weak solutions to the Stokes problem presumed the data is sufficiently regular. To this aim we find a Banach space containing all solutions corresponding to such data and a continuous linear operator on this space that coincides with the usual trace on $C^\infty(\overline{\Omega})$.

From now on let $1 < r < \infty$, $\tilde{w} \in A_r$ such that (4.1) is fulfilled and take $F \in W_{\tilde{w}}^{-1,r}(\Omega)$ and $K \in L^r_{\tilde{w}}(\Omega)$ and $g \in T_w^{0,q}(\partial\Omega)$. Then

$$[\phi \mapsto \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \in Y_w^{-2,q}(\Omega) \text{ and}$$
$$[\psi \mapsto \langle K, \psi\rangle - \langle g, N \cdot \psi\rangle_{\partial\Omega}] \in W_{w,0}^{-1,q}(\Omega).$$

Thus by Theorem 3.4 there exists a unique function $u \in L^q_w(\Omega)$ such that

$$-\langle u, \Delta\phi\rangle = \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} \quad \forall \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \text{ and}$$
$$-\langle u, \nabla\psi\rangle = \langle K, \psi\rangle - \langle g, N \cdot \psi\rangle_{\partial\Omega} \quad \forall \psi \in W_w^{1,q'}(\Omega).$$

However, the question arises in which sense this solution u fulfills $u|_{\partial\Omega} = g$.

As a large space of functions in which the definition of tangential boundary conditions is possible we define

$$\begin{aligned}\tilde{W}^{q,r}_{w,\tilde{w}}(\Omega) :=& \big\{u \in L^q_w(\Omega) \;\big|\; (\Delta u)|_{C^\infty_{0,\sigma}(\Omega)} \text{ extends to an element of } (W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))'\big\} \\ =& \big\{u \in L^q_w(\Omega) \;\big|\; \exists c > 0,\ |\langle u, \Delta\phi\rangle| \le c\|\phi\|_{1,r',\tilde{w}'}\ \forall \phi \in C^\infty_{0,\sigma}(\Omega)\big\},\end{aligned} \tag{5.1}$$

where we denote $W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega) := \Big\{u \in W^{k,r'}_{\tilde{w}',0}(\Omega) \mid \operatorname{div} u = 0\Big\}$. We will omit the symbol Ω and write $\tilde{W}^{q,r}_{w,\tilde{w}}$ if no confusion can occur.

To guarantee that the extension in (5.1) is uniquely defined by the values of $\langle u, \Delta\phi\rangle$ for $\phi \in C^\infty_{0,\sigma}(\Omega)$ we use the following Lemma.

Lemma 5.1. *Let $r' > 1$, $\tilde{w} \in A_{r'}$ and $k \in \mathbb{N}$. Then one has*

$$\overline{C^\infty_{0,\sigma}(\Omega)}^{W^{k,r'}_{\tilde{w}'}(\Omega)} = W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega).$$

Proof. We have to prove the density $C^\infty_{0,\sigma}(\Omega) \hookrightarrow W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega)$. To do this let

$$v \in (W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega))', \quad \langle v, \phi\rangle = 0 \text{ for all } \phi \in C^\infty_{0,\sigma}(\Omega).$$

By the Hahn-Banach theorem v extends to an element $V \in W_{\tilde{w}}^{-k,r}(\Omega)$. Since $\langle V, \phi\rangle = 0$ for every $\phi \in C^\infty_{0,\sigma}(\Omega)$, it follows by de Rham's theorem [27] that $V = \nabla U$ for some $U \in C_0^\infty(\Omega)'$. By Theorem 2.4 there exists for every $\phi \in C_0^\infty(\Omega)$ with $\int_\Omega \phi = 0$ some $\zeta \in C_0^\infty(\Omega)$ with $\operatorname{div}\zeta = \phi$ and $\|\zeta\|_{k,r',\tilde{w}'} \le c\|\phi\|_{k-1,r',\tilde{w}'}$. Thus we can estimate

$$|\langle U, \phi\rangle| = |\langle U, \operatorname{div}\zeta\rangle| = |\langle \nabla U, \zeta\rangle| \le c\|V\|_{-k,r,\tilde{w}}\|\phi\|_{k-1,r',\tilde{w}'}$$

for every ϕ with $\int_\Omega \phi = 0$. This proves $U \in W_{\tilde{w}}^{1-k,r}(\Omega)$ and we obtain for every $\psi \in W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega)$ using the definition of the distributional derivative and the fact that we can approximate ψ by $C_0^\infty(\Omega)$-functions in the norm of $W^{k,r'}_{\tilde{w}',0}(\Omega)$

$$\langle v, \psi\rangle = \langle V, \psi\rangle = \langle \nabla U, \psi\rangle = -\langle U, \operatorname{div}\psi\rangle = 0.$$

Now the Hahn-Banach Theorem proves the assertion. $\square$

Lemma 5.2. *$\tilde{W}^{q,r}_{w,\tilde{w}}$ is a Banach space equipped with the norm*

$$\|u\|_{\tilde{W}^{q,r}_{w,\tilde{w}}} = \|u\|_{q,w} + \|\Delta u|_{C^\infty_{0,\sigma}(\Omega)}\|_{(W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))'}.$$

Proof. Let $(u_n)_n$ be a Cauchy sequence in $W^{q,r}_{w,\tilde{w}}$. Then there exists $u \in L^q_w(\Omega)$ and $v \in (W^{1,r'}_{\tilde{w},0,\sigma}(\Omega))'$ with $u_n \xrightarrow{L^q_w(\Omega)} u$ and $\Delta u_n \xrightarrow{(W^{1,r'}_{\tilde{w},0,\sigma}(\Omega))'} v$. From the continuity of $\Delta : L^q_w(\Omega) \to W^{-2,q}_w(\Omega)$ it follows that $\langle v, \phi\rangle = \langle \Delta u, \phi\rangle$ for all $\phi \in C^\infty_{0,\sigma}(\Omega)$. □

The following Lemma is crucial when proving the well-definedness of the tangential component of the trace on $\tilde{W}^{q,r}_{w,\tilde{w}}$.

Lemma 5.3. *$C^\infty(\overline{\Omega})$ is dense in $\tilde{W}^{q,r}_{w,\tilde{w}}$.*

As a preparation for the proof of Lemma 5.3 we need two auxiliary results. The first one concerns the solvability of the Laplace equation in the very weak context.

Theorem 5.4. *For every $f \in Y^{-2,q}_w(\Omega)$ there exists a unique very weak solution $u \in L^q_w(\Omega)$ to the Laplace equation $\Delta u = f$, i.e.,*

$$\langle u, \Delta\phi\rangle = \langle F, \phi\rangle \tag{5.2}$$

holds for every $\phi \in Y^{2,q'}_{w'}(\Omega)$. This solution fulfills

$$\|u\|_{q,w} \le c\|f\|_{Y^{-2,q}_w(\Omega)}$$

with $c = c(q, w, \Omega) > 0$.

Proof. As shown in [23, A1] one has that the existence of solutions to the Laplace equation extends to weighted function spaces. More precisely, the operator

$$\Delta : Y^{2,q'}_{w'}(\Omega) \to L^{q'}_{w'}(\Omega)$$

is invertible, we denote its inverse by Δ_D^{-1}. Thus we can define a functional u by $\langle u, v\rangle := \langle f, \Delta_D^{-1} v\rangle$ for every $v \in L^{q'}_{w'}(\Omega)$. Then

$$|\langle u, v\rangle| = |\langle f, \Delta_D^{-1} v\rangle| \le \|f\|_{Y^{-2,q}_w(\Omega)}\|\Delta_D^{-1} v\|_{2,q',w'} \le c\|f\|_{Y^{-2,q}_w(\Omega)}\|v\|_{q',w'}.$$

Thus $u \in (L^{q'}_{w'}(\Omega))' = L^q_w(\Omega)$ and $\|u\|_{q,w} \le c\|f\|_{Y^{-2,q}_w(\Omega)}$.

To show that u is a very weak solution to the Laplace equation we see that for any $\phi \in Y^{2,q'}_{w'}(\Omega)$

$$\langle u, \Delta\phi\rangle = \langle f, \Delta_D^{-1}\Delta\phi\rangle = \langle f, \phi\rangle.$$

Vice versa every very weak solution to the Laplace equation fulfills

$$\langle u, \phi\rangle = \langle u, \Delta\Delta_D^{-1}\phi\rangle = \langle f, \Delta_D^{-1}\phi\rangle.$$

This proves the uniqueness. □

One problem in the weighted context is that if $u \in L^q_w(\mathbb{R}^n)$, then the dilated function $u_\lambda(x) = u(\lambda x)$ is in general not contained in $L^q_w(\mathbb{R}^n)$. However, if u is harmonic, the situation is better. More precisely one has the following Lemma.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^n$ be strictly star-shaped, i.e., Ω is star-shaped with respect to every point of a ball $B_r(0)$, $r > 0$, with $\overline{B_r(0)} \subset \Omega$. Moreover, let $u \in L^q_w(\Omega)$ with $\Delta u = 0$. For $\lambda < 1$ we set $u_\lambda(x) := u(\lambda x)$. Then*

$$u_\lambda \xrightarrow{\lambda \to 1, \lambda < 1} u \quad \textit{in } L^q_w(\Omega).$$

Proof. Let $d = \sup_{x \in \Omega} |x|$ and choose $K < \frac{r}{d}$. Then for every λ with $\frac{1}{2} < \lambda < 1$ one has

$$B_{K(1-\lambda)|x|}(\lambda x) \subset \Omega \ \text{ for every } x \in \Omega. \tag{5.3}$$

To show this let $y \in \mathbb{R}^n$, $|y - \lambda x| < K(1-\lambda)|x|$. For $z = \frac{y - \lambda x}{1 - \lambda}$ we have

$$|z| \le \frac{(1-\lambda)\frac{r}{d}|x|}{1-\lambda} \le r.$$

Since Ω is star-shaped with respect to $z \in B_r(0)$, we have $y = \lambda x + (1-\lambda)z \in \Omega$. This proves (5.3). Now we denote by $\tilde{u}$ the extension of u by 0 outside Ω. Take $x \in \Omega$ and $\lambda < 1$ fixed. Since u is harmonic, we can estimate using the mean value property [18, I. Theorem 2.1] and (5.3)

$$\begin{aligned} |u_\lambda(x)| &= |u(\lambda x)| = \frac{1}{|B_{K(1-\lambda)|x|}(\lambda x)|}\left| \int_{B_{K(1-\lambda)|x|}(\lambda x)} u(t)dt \right| \\ &\le \frac{(K+1)^3}{K^3} \frac{1}{|B_{(K+1)(1-\lambda)|x|}(x)|} \int_{B_{|x|(1-\lambda)+K(1-\lambda))}(x)} |\tilde{u}(t)|dt \le cM\tilde{u}(x). \end{aligned}$$

Since M, the maximal operator in $L^q_w(\Omega)$ is bounded by Theorem 2.5, one has $M\tilde{u} \in L^q_w(\mathbb{R}^n)$. Thus, we have found a majorant. Moreover, since the harmonic function $u \in C^\infty(\Omega)$, the convergence $u_\lambda \to u$ is pointwise. By Lebesgue's Theorem we find $u_\lambda \to u$ in $L^q_w(\Omega)$. $\square$

Proof of Lemma 5.3. Let $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$. Then by definition and Lemma 4.1 we have

$$\Delta u|_{C^\infty_{0,\sigma}} \in (W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))' \hookrightarrow Y^{-2,q}_{w,\sigma}(\Omega).$$

The Hahn-Banach theorem yields the existence of some

$$f \in (W^{1,r'}_{\tilde{w}',0}(\Omega))' = W^{-1,r}_{\tilde{w}}(\Omega) \subset Y^{-2,q}_w(\Omega)$$

such that $\langle f, \phi \rangle = \langle \Delta u, \phi \rangle$ for all $\phi \in C^\infty_{0,\sigma}(\Omega)$.

Using Hahn Banach's Theorem combined with Theorem 2.4 one shows that there exists an extension $F \in Y^{-2,q}_w(\Omega)$ of $(\langle u, \Delta \cdot \rangle - f)|_{Y^{2,q'}_{w',\sigma}(\Omega)}$ vanishing on $C^\infty_0(\Omega)$. By Theorem 5.4 there exists a $v \in L^q_w(\Omega)$ such that

$$\langle v, \Delta\phi \rangle = \langle F, \phi \rangle \ \text{ for all } \ \phi \in Y^{2,q'}_{w'}(\Omega).$$

This v is harmonic on Ω because $\langle F, \phi \rangle = 0$ for all $\phi \in C^\infty_0(\Omega)$.

Now we assume temporarily that Ω is star-shaped with respect to some ball $B_r(0)$ with center 0 and radius r. So we may set $v_\lambda(x) := v(\lambda x)$, where $\lambda \in (0,1)$ and

$v_n(x) := v_{\lambda_n}(x)$, where $(\lambda_n) \subset (0,1)$ is a sequence converging to 1. Then by Theorem 2.4 we have $v_n \xrightarrow{n\to\infty} v$ in $L^q_w(\Omega)$. Moreover, since every v_n is harmonic, we have $\Delta v_n - \Delta v = 0$ for all n which yields the convergence in $\tilde{W}^{q,r}_{w,\tilde{w}}$.

For an arbitrary bounded $C^{1,1}$-domain Ω one uses a decomposition $\Omega = \bigcup_{i=1}^N \Omega_i$ with strictly star-shaped domains Ω_i and a partition of unity $(\alpha_i)_i$ subordinate to this covering. For $i = 1, ..., N$ let $(v_n^{(i)})_n$ be the sequences of harmonic functions constructed above converging to $v^{(i)} := v|_{\Omega_i}$ in $\tilde{W}^{q,r}_{w,\tilde{w}}(\Omega_i)$. Then using the embeddings $L^q_w(\Omega) \hookrightarrow W^{-1,r}_{\tilde{w}}(\Omega)$ and $W^{-1,q}_w(\Omega) \hookrightarrow W^{-1,r}_{\tilde{w}}(\Omega)$ ons shows that

$$v_n := \sum_{j=1}^N \alpha_j v_n^{(j)} \xrightarrow{n\to\infty} v \text{ in } \tilde{W}^{q,r}_{w,\tilde{w}}(\Omega).$$

Moreover, we have

$$\begin{aligned} \langle u - v, \Delta\phi\rangle &= \langle f, \phi\rangle + \langle F, \phi\rangle - \langle F, \phi\rangle = \langle f, \phi\rangle \quad && \text{for } \phi \in Y^{-2,q'}_{w',\sigma}(\Omega) \\ \langle u - v, \nabla\psi\rangle &=: \langle k, \psi\rangle && \text{for } \psi \in W^{1,q'}_{w'}(\Omega). \end{aligned}$$

Let $(f_n)_n, (k_n)_n \subset C^\infty(\overline{\Omega})$ such that $f_n \xrightarrow{n\to\infty} f$ in $W^{-1,r}_{\tilde{w}}(\Omega)$ and $k_n \xrightarrow{n\to\infty} k$ in $W^{-1,q}_{w,0}(\Omega)$. The embedding $W^{-1,r}_{\tilde{w}}(\Omega) \hookrightarrow Y^{-2,q}_w(\Omega)$ and the a priori estimate for very weak solutions to the Stokes equations (3.7) yields that the sequence of very weak solutions $(u_n)_n$ to the Stokes problem with respect to f_n and k_n converges to $u - v$ in $L^q_w(\Omega)$. By the regularity of the data and of the boundary (Theorem 4.2) one has $u_n \in W^{2,q}_w(\Omega)$.

Since $\langle u_n, \Delta\phi\rangle = \langle f_n, \phi\rangle$ for all $\phi \in C^\infty_{0,\sigma}(\Omega)$, it follows that the sequence $(u_n + v_n)_n \subset W^{2,q}_w(\Omega)$ approximates u in the norm of $\tilde{W}^{q,r}_{w,\tilde{w}}$. Since $C^\infty(\overline{\Omega})$ is dense in $W^{2,q}_w(\Omega)$, the assertion is proved. □

It is not difficult to see that if $\phi \in W^{2,q}_w(\Omega)$ with $\phi|_{\partial\Omega} = 0$ and $\operatorname{div}\phi = 0$, then $N \cdot \nabla\phi$ is purely tangential. The next Lemma shows that vice versa every purely tangential function on the boundary is a normal derivative of such a function.

Lemma 5.6. *Let Ω be a bounded $C^{1,1}$-domain, $1 < q < \infty$ and $w \in A_q$. For every $h \in T^{1,q}_w(\partial\Omega)$ with $N \cdot h = 0$ there exists a function $\phi_h \in W^{2,q}_w(\Omega)$ such that*

$$\phi_h|_{\partial\Omega} = 0,\ N \cdot \nabla\phi_h = h \text{ and } \operatorname{div}\phi_h = 0.$$

Moreover ϕ_h can be chosen depending linearly on h and fulfilling the estimate

$$\|\phi_h\|_{2,q,w} \le c\|h\|_{T^{1,q}_w(\partial\Omega)}$$

with a constant $c = c(\Omega, q, w) > 0$.

Proof. For $h \in T^{1,q}_w(\partial\Omega)$ there exists by Theorem 2.3 a function $\psi_h \in W^{2,q}_w(\Omega)$ depending linearly on h such that

$$\psi_h|_{\partial\Omega} = 0,\ N \cdot \nabla\psi_h = h \text{ and } \|\psi_h\|_{2,q,w} \le c\|h\|_{T^{1,q}_w(\partial\Omega)}.$$

Since in addition $h = N \cdot \nabla\psi_h$ is purely tangential, one can show (see [14]) that $\operatorname{div}\psi_h \in W^{1,q}_{w,0}(\Omega)$. Thus by Theorem 2.4 there exists a function $\zeta \in W^{2,q}_{w,0}(\Omega)$ with $\operatorname{div}\zeta = \operatorname{div}\psi_h$, depending linearly on ψ_h and satisfying the estimate $\|\zeta\|_{2,q,w} \le c\|\operatorname{div}\psi_h\|_{1,q,w} \le c\|\psi_h\|_{2,q,w}$.

Now $\phi_h := \psi_h - \zeta$ solves the problem. □

Using this lemma we define the tangential component of $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ on the boundary as follows. If $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ and $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$ we use the notation

$$\langle \Delta_\sigma u, \phi \rangle := \lim_{n\to\infty} \langle u, \Delta \phi_n \rangle \tag{5.4}$$

where $(\phi_n)_n \in C^\infty_{0,\sigma}(\Omega)$ converges to ϕ in $W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega)$. This is possible by Lemma 5.1, and by the definition of $\tilde{W}^{q,r}_{w,\tilde{w}}$ the functional $\Delta_\sigma u$ is independent of the approximation (ϕ_n).

Theorem 5.7. *There exists a continuous linear operator*

$$\begin{array}{lll} \gamma : \tilde{W}^{q,r}_{w,\tilde{w}} & \to T^{0,q}_w(\partial\Omega), & \textit{such that} \\ \langle \gamma(u), h\rangle_{\partial\Omega} & = \langle u, \Delta\phi_h\rangle - \langle \Delta_\sigma u, \phi_h\rangle & \textit{if } N\cdot h = 0, \\ \langle \gamma(u), h\rangle_{\partial\Omega} & = 0 & \textit{if } h = \tilde{h}N \end{array} \tag{5.5}$$

for $h \in T^{1,q'}_{w'}(\partial\Omega)$, for some scalar-valued $\tilde{h} \in T^{1,q'}_{w'}(\partial\Omega)$, and where ϕ_h is given by Lemma 5.6. Moreover, this tangential trace is independent of the choice of the extension ϕ_h and coincides with the tangential component of the usual restriction if $u \in C^\infty(\overline{\Omega})$.

Proof. Assume that γ is defined by (5.5). Let $m \in T^{1,q'}_w(\partial\Omega)$. The function m can be decomposed into its normal and tangential components, i.e.,

$$m = (N\cdot m)N + h \quad \text{with} \quad N\cdot h = 0$$

with $\|h\|_{T^{1,q'}_{w'}(\partial\Omega)} \le c\|m\|_{T^{1,q'}_{w'}(\partial\Omega)}$. Then one obtains

$$\begin{array}{rcl} |\langle\gamma(u), m\rangle_{\partial\Omega}| & = & |\langle\gamma(u), h\rangle_{\partial\Omega}| \\ & = & |\langle u, \Delta\phi_h\rangle - \langle\Delta_\sigma u, \phi_h\rangle| \\ & \le & \|u\|_{q,w}\|\phi_h\|_{2,q',w'} + \|\Delta_\sigma u\|_{(W^{1,r'}_{\tilde{w}',0,\sigma})'}\|\phi_h\|_{1,r',\tilde{w}'} \\ & \le & c\|u\|_{\tilde{W}^{q,r}_{w,\tilde{w}}}\|m\|_{T^{1,q'}_{w'}(\partial\Omega)}. \end{array}$$

Thus γ is continuous.

By Gauss' Theorem one obtains that for $u \in C^\infty(\overline{\Omega})$ and a purely tangential $h \in T^{1,q'}_{w'}(\partial\Omega)$ one has $\langle\gamma(u), h\rangle_{\partial\Omega} = \langle u|_{\partial\Omega}, h\rangle_{\partial\Omega}$. Thus the tangential component of $\gamma(u)$ is equal to the tangential component of $u|_{\partial\Omega}$ which is in particular independent of the extension of h. Since by Lemma 5.3 the space $C^\infty(\overline{\Omega})$ is dense in $\tilde{W}^{q,r}_{w,\tilde{w}}$ the same is true for every $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$. $\square$

The definition of normal traces is easier. If

$$u \in E^{q,r}_{w,\tilde{w}} := \{v \in L^q_w(\Omega) \mid \operatorname{div} v \in L^r_{\tilde{w}}(\Omega)\}$$

then we can define the normal trace $u \mapsto N\cdot u|_{\partial\Omega}$ using Green's formula by

$$\langle N\cdot u|_{\partial\Omega}, v\rangle_{\partial\Omega} := \langle u|_{\partial\Omega}, Nv\rangle_{\partial\Omega} := \langle \operatorname{div} u, v\rangle + \langle u, \nabla v\rangle \quad \text{for all} \quad v \in W^{1,q'}_{w'}(\Omega). \tag{5.6}$$

This defines a functional in $T^{0,q}_w(\partial\Omega)$ since by Theorem 2.3 for every $\zeta \in T^{1,q'}_{w'}(\partial\Omega)$ there exists $v \in W^{1,q}_w(\Omega)$ with

$$v|_{\partial\Omega} = \zeta \quad \text{and} \quad \|v\|_{1,q',w'} \le c\|\zeta\|_{T^{1,q'}_{w'}}. \tag{5.7}$$

Moreover, it is known that there exists $r \in (1,\infty)$ such that $W^{1,q}_w(\Omega) \hookrightarrow W^{1,r}(\Omega)$. Thus we obtain from the corresponding result in the unweighted case [25] that the right hand side in (5.6) is independent of the extension v.

Then it follows from (5.7) that the mapping

$$u \mapsto N \cdot u|_{\partial\Omega} \quad : E^{q,r}_{w,\tilde{w}} \to T^{0,q}_w(\partial\Omega)$$

is continuous. Using the above theorem for $u \in \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}}$ we write $u|_{\partial\Omega} = g$ if

$$\langle \gamma(u), h\rangle_{\partial\Omega} = \langle g, h\rangle_{\partial\Omega} \text{ for all } h \in T^{1,q'}_{w'}(\partial\Omega) \text{ with } h\cdot N = 0 \text{ and } u\cdot N|_{\partial\Omega} = g\cdot N. \quad (5.8)$$

With this notation we also define the operator

$$\mathrm{tr} : \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}} \to T^{0,q}_w(\Omega), \quad u \mapsto g.$$

Proposition 5.8. *Let u be a very weak solution to the Stokes problem corresponding to the data $\langle f,\phi\rangle = \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$ and $\langle k,\psi\rangle = \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}$ with $F \in W^{-1,r}_{\tilde{w}}(\Omega)$, $K \in L^r_{\tilde{w}}(\Omega)$, $g \in T^{0,q}_w(\partial\Omega)$.*

Then $u \in \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}}$ and $u|_{\partial\Omega} = g$.

Proof. By definition, u is the solution to the variational problem

$$\begin{aligned} -\langle u, \Delta\phi\rangle &= \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}, \quad \text{for all } \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \text{ and} \\ -\langle u, \nabla\psi\rangle &= \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}, \qquad \text{for all } \psi \in W^{1,q'}_{w'}(\Omega). \end{aligned}$$

Inserting $\phi \in C^\infty_{0,\sigma}(\Omega)$ into the first equation we obtain that $[\phi \mapsto \langle \Delta u, \phi\rangle = -\langle F,\phi\rangle]$ is extendable to an element of $(W^{1,r'}_{\tilde{w},0,\sigma}(\Omega))'$. Thus $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ and by the definition of the tangential trace we have

$$\langle \gamma(u), N\cdot\nabla\phi\rangle_{\partial\Omega} = \langle u, \Delta\phi\rangle - \langle \Delta_\sigma u, \phi\rangle = \langle u,\Delta\phi\rangle + \langle F,\phi\rangle = \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$$

for all $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$. Using the second equation one shows that $N\cdot u|_{\partial\Omega} = N\cdot g$. □

Remark 5.9. 1. It is not difficult to see that the space $\tilde{W}^{q,r}_{w,\tilde{w}}$ is equal to the space of very weak solutions to the Stokes problem with respect to data

$$f = [\phi \mapsto \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}]$$

with $F \in W^{-1,r}_{\tilde{w}}(\Omega)$ and $g \in T^{0,q}_w(\partial\Omega)$ and $k \in W^{-1,q}_{w,0}(\Omega)$. Indeed, let $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ and let $F \in W^{-1,r}_{\tilde{w}}(\Omega)$ be an extension of $-\Delta u|_{C^\infty_{0,\sigma}(\Omega)}$. Then setting $g := \gamma u \in T^{0,q}_w(\Omega)$ we obtain by the definition of γ

$$-\langle u, \Delta\phi\rangle = \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \text{ for every } \phi \in Y^{2,q'}_{w',\sigma}(\Omega).$$

2. In [14] the unweighted case is treated. There the space in which the traces are well-defined is defined in a different way. We repeat this definition and show that the out-coming space is the same in the case $w = \tilde{w} = 1$.

For $u \in W^{1,q}(\Omega)$ one sets

$$\|\mathcal{A}_r^{-\frac{1}{2}} P_r \Delta u\|_{L^r_\sigma(\Omega)} = \sup_{0 \neq v \in L^{r'}_\sigma} \left(\frac{\langle \nabla u, \nabla \mathcal{A}_{r'}^{-\frac{1}{2}} v \rangle}{\|v\|_{L^{r'}_\sigma(\Omega)}} \right),$$

where $\mathcal{A}_r$ stands for the Stokes operator and P_r for the Helmholtz projection in $L^r(\Omega)$ and $\frac{1}{r} \leq \frac{1}{n} + \frac{1}{q}$. Note that r is chosen such that by the Sobolev embedding theorems [1] one has $W^{1,r}(\Omega) \hookrightarrow L^q(\Omega)$. Then following [14] one defines

$$\widehat{W}^{1,q}(\Omega) := \overline{W^{1,q}(\Omega)}^{\|\cdot\|_{\widehat{W}^{1,q}(\Omega)}} \quad \text{where} \quad \|u\|_{\widehat{W}^{1,q}(\Omega)} := \|u\|_q + \|\mathcal{A}_r^{-\frac{1}{2}} P_r \Delta u\|_r.$$

For $u \in C^\infty(\overline{\Omega})$ one has

$$\begin{aligned} \|\Delta u|_{C^\infty_{0,\sigma}}\|_{(W^{1,r'}_{0,\sigma})'} &= \sup_{\phi \in C^\infty_{0,\sigma}, \|\phi\|_{1,r'}=1} |\langle \Delta u, \phi \rangle| \\ &\sim \sup_{\psi \in C^\infty_{0,\sigma}, \|\psi\|_{r'}=1} |\langle P_r \Delta u, \mathcal{A}_{r'}^{-\frac{1}{2}} \psi \rangle| = \|\mathcal{A}_r^{-\frac{1}{2}} P_r \Delta u\|_r, \end{aligned}$$

where we have used that by [17] one has $\|\mathcal{A}_{r'}^{\frac{1}{2}} \cdot \|_{r'} \sim \|\nabla \cdot \|_{r'}$.

Thus in the unweighted case these norms are equivalent and by the density shown in Lemma 5.3 the spaces are equal.

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