Fredholm properties of band-dominated operators on periodic discrete structures

V. S. Rabinovich, S. Roch
Vladimir S. Rabinovich, Instituto Politécnico Nacional,
ESIME-Zacatenco, Av. IPN, edif.1, México D.F.,
07738, MÉXICO, e-mail: vladimir.rabinovich@gmail.com
Steffen Roch, Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany,
e-mail: roch@mathematik.tu-darmstadt.de

Abstract

Let $(X, \sim)$ be a combinatorial graph the vertex set X of which is a discrete metric space. We suppose that a discrete group G acts freely on $(X, \sim)$ and that the fundamental domain with respect to the action of G contains only a finite set of points. A graph with these properties is called periodic with respect to the group G. We examine the Fredholm property and the essential spectrum of band-dominated operators acting on the spaces $l^p(X)$ or $c_0(X)$, where $(X, \sim)$ is a periodic graph. Our approach is based on the thorough use of band-dominated operators. It generalizes the necessary and sufficient results obtained in [38] in the special case $X = G = \mathbb{Z}^n$ and in [41] in case $X = G$ is a general finitely generated discrete group.

1 Introduction

A first aim of this paper is to study the essential spectrum of general difference operators (or band operators, with the Schrödinger operators as prominent examples) acting on the spaces $l^p(X)$ on combinatorial graphs $(X, \sim)$. We consider the set X of the vertices of the graph as a discrete metric space which is invariant with respect to the action of a discrete countable group G. For each $p \in [1, \infty)$, we introduce a Banach algebra $\mathcal{A}_p(X)$ of so-called band-dominated operators on $l^p(X)$, which is generated by band (= difference) operators. For each operator $A \in \mathcal{A}_p(X)$, we introduce a family $\text{op}_p(A)$ of *limit operators* of A. We prove that an operator $A \in \mathcal{A}_p(X)$ is *Fredholm* on $l^p(X)$ if and only if all limit operators $A_h \in \text{op}_p(A)$ are invertible and if the norms of their inverses are uniformly bounded. In general, the limit operators of A have a simpler structure than the operator A itself, and the limit operators approach provides, thus, an effective tool to study the Fredholm property of band-dominated operators.

Nevertheless, the uniform boundedness of the inverses of the limit operators remains hard to verify in many instances, and it is of vital importance to relax this condition. This brings us to our second aim of this paper: to show that the uniform boundedness of the norms of the inverse limit operators follows already from their invertibility, *if A is a band operator*. This crucial fact implies the formula

$$\mathrm{sp}_{ess}\, A = \bigcup_{A_h \in \mathrm{op}\, A} \mathrm{sp}\, A_h \tag{1}$$

for the essential spectrum of band operators A which shows in particular that the essential spectrum $\mathrm{sp}_{ess}\, A$ of A, considered as an operator on $l^p(X)$, is independent of p.

In case G is the commutative group $\mathbb{Z}^n$, spectral properties of Schrödinger operators on combinatorial and quantum graphs have attracted a lot of attention in the last time, both due to their interesting properties and due to existing and expected applications in chemistry [30, 42] and physics [5, 19], in particular and quite recently, in the physics of nano-structures [5, 13, 43]). The spectral properties of Schrödinger operators on quantum graphs considered by P. Kuchment and collaborators in a series of papers [18, 19, 20, 21, 22, 23]. Direct and inverse spectral problems for Schrödinger operators on graphs connected with zig-zag carbon nano-tubes were considered in [16, 17].

In the special case $G = \mathbb{Z}^n$, formula (1) was obtained in [36, 37], see also [38]. In [33], this formula is applied to study electro-magnetic Schrödinger operators on the lattice $\mathbb{Z}^n$, and in [34] a generalization to $\mathbb{Z}^n$-periodic graphs is derived. A similar formula for essential spectra of perturbed pseudodifferential operators on $\mathbb{R}^n$ were obtained in [31] and applied for investigation of location of essential spectra of electro-magnetic Schrödinger operators, square-root Klein-Gordon, and Dirac operators under general assumptions with respect to the behavior of magnetic and electric potentials at infinity. Also a simple and transparent proof of the Hunziker, van Winter, Zjislin theorem on the location of essential spectra of multi-particle Hamiltonians was obtained in [31] on the basis of (1). Similar results in the spirit of (1) were derived in [25] by means of admissible geometric methods and in [9, 8, 26, 3] via C^*-algebra techniques.

Our paper is organized as follows. In Section 2 we collect some material on discrete metric spaces, discrete graphs, and group actions thereon. Section 3 introduces the basic objects of this paper: band and band-dominated operators. Under the basic assumption that X/G is a finite set, we show that the algebra of all band-dominated operators on a pseudo-homogeneous and periodic with respect to the action of a discrete countable group G metric space X can be identified with the algebra of all $n \times n$-matrices with entries in the algebra of the band-dominated operators on the group G.

Sections 4 and 5 form certainly the heart of the paper. Applying the isomorphism established in Section 3.3, we derive necessary and sufficient conditions for band-dominated operators on $l^p(X)$ to be Fredholm operators. These conditions follow more or less straightforwardly from Roe's paper [41] which treats the case when $X = G$ and which is recalled here as far as necessary. The derived Fredholm criterion involves the uniform boundedness of the norms of the inverses of the limit operators of A, which is hard to control. It is the aim of Section 5 to show that this condition is indeed redundant for band operators. In this section, we also prove formula (1) for band operators. A basic device in this section is an operator algebra $\mathcal{W}(X)$ of Wiener type.

In the concluding Section 6 we briefly discuss some first applications. We consider periodic (= shift invariant) band-dominated operators A, for which one has $\mathrm{sp}_{ess}\, A = \mathrm{sp}\, A$, and we examine operators in the Wiener algebra with slowly oscillating coefficients. The latter operators are distinguished by two remarkable properties: their limit operators are periodic operators, and all limit operators belong to the Wiener algebra again. Via formula (1), one thus obtains a complete description of the essential spectra of band operators with slowly oscillating coefficients.

We will use the following notations. Given a complex Banach space X, let $\mathcal{L}(X)$ refer to the Banach algebra of all bounded linear operators X and $\mathcal{K}(X)$ to its ideal consisting of all compact operators. An operator $A \in \mathcal{L}(X)$ is called a *Fredholm operator* if its kernel $\ker A := \{x \in X : Ax = 0\}$ and its cokernel $\mathrm{coker}\, A := X/A(X)$ are finite dimensional linear spaces. Equivalently, A is Fredholm if the coset $A + \mathcal{K}(X)$ is invertible in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. The *essential spectrum* of A is the set of all complex numbers λ for which the operator $A - \lambda I$ is not Fredholm on X. We denote the essential spectrum of A by $\mathrm{sp}_{ess}\, A$ and the usual spectrum by $\mathrm{sp}\, A$.

The authors are grateful for the support by CONACYT (Project 43432) and by the German Research Foundation (Grant 444 MEX-112/2/05).

2 Spaces, groups, and actions

2.1 Discrete spaces

By a *discrete space* we mean a countable set X provided with the discrete topology, i.e. every singleton $\{x\}$ with $x \in X$ is open. In this paper, we will have to work on the following standard Banach spaces over X. For $p \in [1, \infty)$, let $l^p(X)$ denote the Banach space of all complex-valued functions on X with norm

$$\|u\|^p_{l^p(X)} := \sum_{x \in X} |u(x)|^p,$$

and for p as above and for every positive integer N, let $l^p(X)^N$ stand for the Banach space of all vectors $u = (u_1, \ldots, u_N)$ of functions $u_i \in l^p(X)$ with norm

$$\|u\|^p_{l^p(X)^N} := \sum_{i=1}^{N} \|u_i\|^p_{l^p(X)}$$

Likewise, one can identify $l^p(X)^N$ with the Banach space $l^p(X, \mathbb{C}^N)$ of all functions $u : X \to \mathbb{C}^N$ for which

$$\|u\|^p_{l^p(X,\,\mathbb{C}^N)} := \sum_{x\in X} \sum_{i=1}^{N} |u_j(x)|^p < \infty.$$

Clearly, the Banach spaces $l^p(X)^N$ and $l^p(X, \mathbb{C}^N)$ are isometric to each other.

Further, let $l^\infty(X)$ refer to the Banach space of all bounded functions $u : X \to \mathbb{C}$ with norm

$$\|u\|_{l^\infty(X)} := \sup_{x\in X} |u(x)|.$$

The closure in $l^\infty(X)$ of the linear space of all compactly supported functions is denoted by $c_0(X)$. Since X is a discrete space, its compact subsets are the finite subsets, and X is locally compact. Thus, a function $u \in l^\infty(X)$ belongs to $c_0(X)$ if $\lim_{x\to\infty} |u(x)| = 0$ in the following sense: For every $\varepsilon > 0$, the set of all $x \in X$ with $|u(x)| > \varepsilon$ is finite. For every positive integer N, we also consider the Banach spaces $l^\infty(X)^N$ and $l^\infty(X, \mathbb{C}^N)$ with norms

$$\|u\|_{l^\infty(X)^N} := \sup_{1\le i\le N} \|u_i\|_{l^\infty(X)}$$

and

$$\|u\|_{l^\infty(X,\,\mathbb{C}^N)} := \sup_{x\in X} \sup_{1\le i\le N} |u_i(x)|$$

as well as their closed subspaces $c_0(X)^N$ and $c_0(X, \mathbb{C}^N)$. Again, the spaces $l^\infty(X)^N$ and $l^\infty(X, \mathbb{C}^N)$ are isometric to each other in a natural way. Note also that $l^\infty(X, \mathbb{C}^{N\times N})$ can be made to a C^*-algebra by providing the matrix algebra $\mathbb{C}^{N\times N}$ with a C^*-norm, and that $c_0(X, \mathbb{C}^{N\times N})$ then becomes a closed two-sided ideal of that algebra.

For convenience, we will also use the (non-standard) notations $l^0(X)$, $l^0(X)^N$ and $l^0(X, \mathbb{C}^N)$ instead of $c_0(X)$, $c_0(X)^N$ and $c_0(X, \mathbb{C}^N)$, respectively.

2.2 Periodic discrete spaces

A *periodic discrete space* is a discrete space provided with the free action of a (not necessarily commutative) group. More precisely, let X be a discrete space and G a group with unit ϵ which operates from the left on X. Thus, there is a mapping

$$\pi : G \times X \to X, \quad (\alpha, x) \to \alpha \cdot x$$

satisfying

$$\epsilon \cdot x = x \quad \text{and} \quad (\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$$

for arbitrary elements $\alpha, \beta \in G$ and $x \in X$. The triple (X, G, π) is also referred to as a discrete dynamical system. We shall call X a G*-periodic space*. Recall also that the group G acts *freely* on X if whenever the equality $x = \alpha \cdot x$ holds for some elements $x \in X$ and $\alpha \in G$ then, necessarily, $\alpha = \epsilon$.

For each element $x \in X$, consider its orbit $\{\alpha \cdot x \in X : \alpha \in G\}$ with respect to the action of G. Any two orbits are either disjoint or they coincide. Hence, there is a binary equivalence relation on X, by calling two points equivalent if they belong to the same orbit. The set of all orbits of X with respect to the action of G is denoted by X/G. A basic assumption throughout what follows is that

- **the orbit space X/G is finite.**

Thus, there is a finite subset $\mathcal{M} := \{x_1, x_2, \ldots, x_N\}$ of X such that the orbits

$$X_j := \{\alpha \cdot x_j \in X : \alpha \in G\}$$

satisfy $X_i \cap X_j = \emptyset$ if $x_i \neq x_j$ and $\cup_{i=1}^N X_i = X$. The free action of G on X guarantees that the mapping

$$U_j : G \to X_j, \, \alpha \mapsto \alpha \cdot x_j$$

is a bijection for every $j = 1, \ldots, N$. In particular, the group G must be countable.

For each complex-valued function f on X, let $Uf : G \to \mathbb{C}^N$ be the function

$$(Uf)(\alpha) := ((U_1 f)(\alpha), \ldots, (U_N f)(\alpha)) \in \mathbb{C}^N.$$

Clearly, the mapping U is a linear isometry from $l^p(X)$ onto $l^p(G, \mathbb{C}^N)$, and the mapping $A \mapsto UAU^{-1}$ is an isometric isomorphism from $\mathcal{L}(l^p(X))$ onto $\mathcal{L}(l^p(G, \mathbb{C}^N))$ for every $p \in \{0\} \cup [1, \infty]$.

2.3 Pseudo-homogeneous discrete metric spaces

In what follows we have to assume more structure of the discrete space X. In particular, we assume that the (discrete) topology on X is generated by a metric ρ and that every closed ball

$$B_r(x_0) := \{x \in X : \rho(x, x_0) \leq r\}$$

is a finite set. Let $w(x_0, r)$ denote the number of points in $B_r(x_0)$. Following Shubin [44, 45], we call the discrete metric space X *pseudo-homogeneous* if there is a continuous and non-decreasing function $w : [0, \infty) \to [0, \infty)$ such that

$$w(r + s) \leq w(r) \, w(s) \quad \text{for all } r, s \in [0, \infty)$$

and

$$w(x_0, r) \leq w(r) \quad \text{for all } x_0 \in X. \tag{2}$$

A function w with these properties is called a *weight* for X.

We also consider discrete metric spaces which are periodic in the sense of the previous section. To relate the group action with the metric structure, we assume that the group action leaves the metric invariant,

$$\rho(\alpha \cdot x, \alpha \cdot y) = \rho(x, y) \tag{3}$$

for all elements $\alpha \in G$ and $x, y \in X$. This assumption implies that the number of the elements of a closed ball $B_r(x_0)$ does only depend on the radius r, not on the center x_0.

In the present paper we will be mainly interested in the following closely related examples of pseudo-homogeneous discrete metric spaces.

Example A: X a discrete group. Let G be a finitely generated discrete (not necessarily commutative) group with identity ϵ, and let $\mathcal{S}$ be a finite and symmetric set of generators, the latter meaning that $\epsilon \notin \mathcal{S}$ and that if α belongs to $\mathcal{S}$, then α^{-1} is also in $\mathcal{S}$. The number of generators in $\mathcal{S}$ is denoted by $|\mathcal{S}|$.

The system $\mathcal{S}$ determines a word metric $\rho = \rho_{\mathcal{S}}$ on G by defining $\rho(\alpha, \beta)$ as the word length of $\alpha^{-1}\beta$. In this case, $B_r(\alpha_0)$ contains at most $|\mathcal{S}|^r$ elements. Thus, one can choose $w(r) := |\mathcal{S}|^r$, which makes G to a pseudo-homogeneous discrete metric space.

The group G acts freely on itself by left multiplication

$$G \times G \to G, \quad (\alpha, \beta) \mapsto \alpha\beta.$$

This action leaves every word metric invariant. The corresponding set X/G of the orbits is a singleton, and one can choose $\mathcal{M} := \{\epsilon\}$, for instance. Note that there is an action of g by right multiplication as well. ∎

Example B: X a discrete graph. By a *discrete graph* we mean a countable set X together with a binary relation $\sim$ which is anti-reflexive (i.e., there is no $x \in X$ such that $x \sim x$) and symmetric and which has the property that for each $x \in X$ there are only finitely many $y \in X$ such that $x \sim y$. The points of X are called the *vertices* and the pairs (x, y) with $x \sim y$ are the *edges* of the graph. Due to anti-reflexivity, the graphs under consideration do not possess loops. We write $m(x)$ for the number of edges starting (or ending) at the vertex x of X.

For technical reasons it will be convenient to assume that the graph $(X, \sim)$ is connected, i.e., given distinct points $x, y \in X$, there are finitely many points $x_0, x_1, \ldots, x_n \in X$ such that $x_0 = x$, $x_n = y$ and $x_i \sim x_{i+1}$ for $i = 0, \ldots, n$. The smallest number n with this property defines the *graph distance* $\rho(x, y)$ of x and y. Together with $\rho(x, x) := 0$, this definition gives a metric ρ on X which makes X to a discrete metric space. (If the graph is not connected in this sense, i.e., if there are points $x, y \in X$ which are not connected by a finite chain of edges, then one should allow infinite distances and put $\rho(x, y) := \infty$.)

The discrete metric space X is pseudo-homogeneous if the degrees of the vertices of the graph are uniformly bounded. Indeed, let $m < \infty$ denote the supremum of all $m(x)$ when x runs through X. Then $w(x_0, r) \leq m^r$. Thus, the choice $w(r) := m^r$ makes X to a pseudo-homogeneous discrete metric space. ∎

Example C: The Cayley graph of a group. The previous examples are not unrelated: If G is a group as in Example A, then one defines a relation $\sim$ on G by calling two elements $\alpha, \beta \in G$ related if there is a $\sigma \in \mathcal{S}$ such that $\beta = \sigma\alpha$.

The symmetry of the generating system $\mathcal{S}$ ensures that $\sim$ is a symmetric relation. Thus, the pair $(G, \sim)$ is a discrete graph in the sense of Example B, and the associated graph distance coincides with the word metric $\rho_{\mathcal{S}}$. The graph $(G, \sim)$ is called the *Cayley graph* of the group G. Note that the *right* action of G on $(G, \sim)$ is compatible with the relation $\sim$: If $\beta = \sigma\alpha$ for some generating element σ, then $\beta\gamma = \sigma\alpha\gamma$. (Likewise, one can define $\sim$ by right action of $\mathcal{S}$, in which case one G operates from the left on its Cayley graph.) ■

As a first simple consequence of the pseudo-homogeneity we mention the following.

Lemma 1 *Let X be a periodic pseudo-homogeneous discrete metric space with respect to the action of a discrete group G. Then*

$$\lim_{G \ni \alpha \to \infty} \rho(x, \alpha \cdot y) = \infty \tag{4}$$

for all points $x, y \in X$.

Indeed, suppose that (4) is wrong. Then there are points $x, y \in X$, a positive constant M, and a sequence α of pair-wise different elements in G such that

$$\rho(x, \alpha(n) \cdot y) \leq M \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

The free action of G on X implies that $(\alpha(n) \cdot y)_{n \in \mathbb{N}}$ is a sequence of pair-wise different points in X. Hence, (5) implies that the ball with center x and radius M contains infinitely many points, a contradiction. ■

3 Band-dominated operators (BDO)

3.1 BDO on pseudo-homogeneous spaces

Let (X, ρ) be a pseudo-homogeneous discrete metric space. We consider functions $k \in l^\infty(X \times X)$ with the property that there is an $R > 0$ such that $k(x, y) = 0$ whenever $\rho(x, y) > R$. The smallest integer R with this property is also called the *band width* of k. Each function k of this form determines via

$$(Au)(x) := \sum_{y \in X} k(x, y)u(y) \quad \text{for all } x \in X \tag{6}$$

a linear operator A which is defined on the linear space of all functions $u : X \to \mathbb{C}$. Indeed, for every fixed $x \in X$, there are only finitely many $y \in X$ which belong to the ball with center x and radius R. Thus, the series in (6) is actually finite for every $x \in X$.

Every operator of this form is called a *band operator over* X, and the band width of k is also referred to as the band width of A. The function k is called the generating function of the operator A. It is clear that every band operator A determines its generating function uniquely, since

$$(A\delta_y)(x) = k(x, y) \tag{7}$$

where δ_y is the function on X which is 1 at y and 0 at all other points. We shall usually write k_A for the generating function of A. Note that the equality (7) allows one to define the generating function (or a "matrix representation") for every operator on $l^p(X)$.

Proposition 2 *Let X be a pseudo-homogeneous discrete metric space and $p \in \{0\} \cup [1, \infty]$. Then*
(a) every band operator is bounded on $l^p(X)$.
(b) the set of all band operators forms a (non-closed) subalgebra of $\mathcal{L}(l^p(X))$.
(c) the adjoint of a band operator on $l^p(X)$ with $1 < p < \infty$ is a band operator on $l^q(X)$ with $1/p + 1/q = 1$.

Proof. We will only show assertion (a) in case $p \in [1, \infty)$. Let $u \in l^p(X)$, A be given by (6), and $k_A(x, y) = 0$ for $\rho(x, y) > R$. Then

$$\sum_{x\in X} |(Au)(x)|^p = \sum_{x\in X} \left| \sum_{y\in X} k_A(x, y)u(y) \right|^p \leq \sum_{x\in X} \left(\sum_{y\in B_R(x)} |k_A(x, y)|\, |u(y)| \right)^p .$$

Since X is pseudo-homogeneous, no ball $B_R(x)$ contains more than $w(R)$ elements. Since 1-norm and p-norm are equivalent on each finite-dimensional space, there is a constant C (which depends of the band width R but is independent of x) such that

$$\begin{aligned} \sum_{x\in X} |(Au)(x)|^p &\leq C \sum_{x\in X} \sum_{y\in B_R(x)} |k_A(x, y)|^p\, |u(y)|^p \\ &\leq C\, \|k_A\|_{l^\infty(X\times X)} \sum_{x\in X} \sum_{y\in B_R(x)} |k_A(x, y)|^p\, |u(y)|^p \\ &\leq C\, \|k_A\|_{l^\infty(X\times X)}\, w(R) \sum_{x\in X} |u(x)|^p \\ &\leq C\, \|k_A\|_{l^\infty(X\times X)}\, w(R)\, \|u\|^p_{l^p(X)}, \end{aligned}$$

whence the boundedness of A. ■

For $p \in \{0\} \cup [1, \infty]$, let $\mathcal{A}_p(X)$ stand for the closure in $\mathcal{L}(l^p(X))$ of the set of all band operators. The operators in $\mathcal{A}_p(X)$ are called *band-dominated operators* (BDO for short). Note that the class $\mathcal{A}_p(X)$ depends heavily on p (whereas the class of the band operators is independent of p). The preceding proposition implies that $\mathcal{A}_p(X)$ is a Banach algebra for every $p \in \{0\} \cup [1, \infty]$ and a C^*-algebra for $p = 2$.

3.2 BDO on discrete groups

In case G is a discrete countable group, there is a way to define band operators without having recourse to any metric structure on G.

We consider functions $k \in l^\infty(G \times G)$ with the property that there is a finite subset G_0 of G such that $k(\alpha, \beta) = 0$ whenever $\alpha^{-1}\beta \notin G_0$. Then

$$(Au)(\alpha) := \sum_{\beta \in G} k(\alpha, \beta)\, u(\beta) \quad \text{for all } \alpha \in G \tag{8}$$

defines a linear operator A on the linear space of all functions $u : G \to \mathbb{C}$, since the occurring series is finite for every $\alpha \in G$. We call operators of this form again *band operators over* G and have the following analogue of Proposition 2 for these operators. The proof is the same as that of Proposition 2.

Proposition 3 *Let G be a discrete countable group and $p \in \{0\} \cup [1, \infty]$. Then*
(a) every band operator is bounded on $l^p(G)$.
(b) the set of all band operators forms a (non-closed) subalgebra of $\mathcal{L}(l^p(G))$.
(c) the adjoint of a band operator on $l^p(G)$ with $1 < p < \infty$ is a band operator on $l^q(G)$ with $1/p + 1/q = 1$.

It turns out that band operators on G are constituted by two kinds of "elementary" band operators: the operators R_α of right shift by $\alpha \in G$ and the operators bI of multiplication by a function $b \in l^\infty(G)$, where

$$R_\alpha : l^p(G) \to l^p(G), \quad (R_\alpha u)(\beta) = u(\beta\alpha)$$

and

$$bI : l^p(G) \to l^p(G), \quad (bu)(\beta) = b(\beta)\, u(\beta).$$

The operators R_α act as isometries for every p and as unitary operators if $p = 2$.

Proposition 4 *Let G be a discrete countable group. An operator A is a band operator if and only if it can be written as a finite sum $\sum b_i R_{\alpha_i}$ where $b_i \in l^\infty(G)$ and $\alpha_i \in G$.*

Proof. Let A be an operator of the form (8) and let $G_0 := \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a finite subset of G such that $k(\alpha, \beta) = 0$ if $\alpha^{-1}\beta \notin G_0$ or, equivalently, if β is not of the form $\alpha\alpha_i$ for some i. Thus,

$$(Au)(\alpha) = \sum_{i=1}^{r} k(\alpha, \alpha\alpha_i)\, u(\alpha\alpha_i) \quad \text{for all } \alpha \in G.$$

Set $b_i(\alpha) := k(\alpha, \alpha\alpha_i)$. The functions b_i are in $l^\infty(G)$, and one has

$$A = \sum_{i=1}^{r} b_i R_{\alpha_i}. \tag{9}$$

Conversely, one easily checks that the operators R_α and bI are band operators. Since the band operators form an algebra by Proposition 3, each finite sum $\sum b_i R_{\alpha_i}$ is a band operator. ∎

It is easy to see that the representation of a band operator on G in the form (9) is unique. We call the functions b_i also the *diagonals* of the operator A.

Assume now there is also a pseudo-homogeneous and shift-invariant metric ρ on G, which happens, for instance, when G is finitely generated and ρ is a word metric. Then we have two notions of band operators on G, to which we refer for a moment as *metric* band operators (if defined as in the previous section) and *combinatorial* band operators (if defined as in this section), respectively. The following proposition shows that these notions coincide. [1]

Proposition 5 *Let G be a discrete countable group, provided with a pseudo-homogeneous and shift-invariant metric ρ. Then every combinatorial band operator on G is a metric band operator, and conversely.*

Proof. Let $k \in l^\infty(G \times G)$, and let $G_0 \subset G$ be a finite set such that $k(\alpha, \beta) = 0$ for $\alpha^{-1}\beta \notin G_0$. Set

$$R := \max_{\gamma \in G_0} \rho(\epsilon, \gamma).$$

Then, if $\rho(\alpha, \beta) = \rho(\epsilon, \alpha^{-1}\beta) > R$, one has $\alpha^{-1}\beta \notin G_0$, whence $k(\alpha, \beta) = 0$ for these pairs α, β.

Conversely, let R be a constant such that $k(\alpha, \beta) = 0$ for $\rho(\alpha, \beta) > R$. Set

$$G_0 := \{\gamma \in G : \rho(\epsilon, \gamma) \leq R\}.$$

This set is finite due to pseudo-homogeneity, and if $\alpha^{-1}\beta \notin G_0$ for some elements $\alpha, \beta \in G$, then the inequality $R < \rho(\epsilon, \alpha^{-1}\beta) = \rho(\alpha, \beta)$ implies that $k(\alpha, \beta) = 0$. ■

This result justifies to omit the attributes "metric" and "combinatorial" of a band operator and to denote the closure in $\mathcal{L}(l^p(G))$ of all (combinatorial) band operators by $\mathcal{A}_p(G)$ again. As before, $\mathcal{A}_p(G)$ is a Banach algebra for every $p \in \{0\} \cup [1, \infty]$, and we call the elements of $\mathcal{A}_p(G)$ band-dominated operators again.

In the following section, we will also have to deal with (finite) matrices the entries of which are band-dominated operators on G. For, we agree upon the following notation. Let $N \geq 1$. An operator A on $l^p(G)^N = l^p(G, \mathbb{C}^N)$ is called a band (band-dominated) operator if each of the entries of the canonical $N \times N$-matrix representation of A is a band (band-dominated) operator on $l^p(G)$. We denote the class of all band-dominated operators on $l^p(G)^N$ by $\mathcal{A}_p(G, \mathbb{C}^N)$. This class is a closed subalgebra of $\mathcal{L}(l^p(G)^N)$ for every p and a C^*-algebra for $p = 2$.

3.3 BDO on G-periodic spaces

The following result reduces the study of BDO on periodic pseudo-homogeneous spaces to that of "block" or "matrix" BDO on discrete groups.

[1] Coarse geometry provides a frame in which both types of band operators can be understood as special realizations of coarse structures on G, see Roe's textbook [40].

Proposition 6 *Let X be a pseudo-homogeneous discrete metric space which is periodic under the action of a discrete countable group G, and let $p \in \{0\} \cup [1, \infty]$. Then the mapping $A \mapsto UAU^{-1}$ is an isomorphism between the Banach algebras $\mathcal{A}_p(X)$ and $\mathcal{A}_p(G, \mathbb{C}^N)$.*

Proof. The operator UAU^{-1} has the matrix representation

$$(UAU^{-1}f)_i(\alpha) = \sum_{j=1}^{N} \sum_{\beta \in G} r_A^{ij}(\alpha, \beta) f_j(\beta) \tag{10}$$

where $\alpha \in G$, $i = 1, \ldots, N$, N the number of elements of X/G, and

$$r_A^{ij}(\alpha, \beta) := k_A(\alpha \cdot x_i, \beta \cdot x_j). \tag{11}$$

From Lemma 1 we conclude that

$$\rho(\alpha \cdot x_i, \beta \cdot x_j) = \rho(x_i, (\alpha^{-1}\beta) \cdot x_j) \to \infty$$

as $\alpha^{-1}\beta \to \infty$. Thus, there is a finite subset G_0 of G such that $r_A^{ij}(\alpha, \beta) = 0$ if $\alpha^{-1}\beta \notin G_0$. In other words, every r_A^{ij} is the generating function of a (combinatorial) band operator on $l^p(G)$, implying that UAU^{-1} is a band operator on $l^p(G, \mathbb{C}^N)$. Now the assertion of the proposition follows since the mapping $A \mapsto UAU^{-1}$ is a continuous isomorphism between the Banach algebras $\mathcal{L}(l^p(X))$ and $\mathcal{L}(l^p(G, \mathbb{C}^N))$. ■

Example: Operators of right shift. Some interesting examples of band operators occur if the group G is allowed to operate also from the right on X (as in the previous example for $X = G$). That is, besides the left action, there is a mapping

$$G \times X \to X, \quad (\alpha, x) \to x \cdot \alpha$$

satisfying

$$x \cdot \epsilon = x \quad \text{and} \quad x \cdot (\alpha\beta) = (x \cdot \alpha) \cdot \beta$$

for arbitrary elements $\alpha, \beta \in G$ and $x \in X$. Moreover, we assume that left and right action are related by

$$(\alpha \cdot x) \cdot \beta = \alpha \cdot (x \cdot \beta)$$

for all $\alpha, \beta \in G$ and $x \in X$. Every element $\alpha \in G$ generates an operator R_α of right shift via $(R_\alpha u)(x) := u(x \cdot \alpha)$ for $x \in X$.

One easily checks that $L_\alpha^{-1} R_\beta L_\alpha = R_\beta$ for arbitrary group elements α, β and that the generating function of R_β is

$$k_{R_\beta}(x, y) = \begin{cases} 1 & \text{if } y = x \cdot \beta \\ 0 & \text{else.} \end{cases}$$

Since the metric ρ is invariant with respect to left shifts and X/G is finite, one has

$$\begin{aligned}\sup_{x\in X} \rho(x,\, x\cdot\beta) &= \sup_{\alpha\in G,\, x_j\in\mathcal{M}} \rho(\alpha\cdot x_j,\, (\alpha\cdot x_j)\cdot\beta)\\ &= \sup_{x_j\in\mathcal{M}} \rho(x_j,\, x_j\cdot\beta) < \infty.\end{aligned}$$

Hence, the R_β are periodic band operators. In particular, every finite sum

$$\sum a_i R_{\beta_i} \tag{12}$$

with group elements β_i and functions $a_i \in l^\infty(X)$ is a band operator. But not every band operator if of the form (12). To see an example, let $\mathcal{M}$ have at least two elements x_1 and x_2 and consider the band operator A with generating function k which is 1 at the points $(x_1,\, x_2)$ and $(x_2,\, x_1)$ and 0 at all other points of $X \times X$. Since x_1 and x_2 belong to different orbits, A is not of the form (12). ■

4 Limit operators and Fredholmness of BDO

Let X be a pseudo-homogeneous discrete metric space which is periodic with respect to a discrete countable group G. The goal of this section is a criterion for the Fredholmness of band-dominated operators on $l^2(X)$. The main step will be a Fredholm criterion for operators on $l^2(G)$ from which the general case will easily follow. In the next section we will generalize this criterion to a class of BDO acting on $l^p(X)$ with $p \in [1,\, \infty)$ or on $c_0(X)$.

Fortunately, a Fredholm criterion for operators on $l^2(G)$ is already known: for $G = \mathbb{Z}^N$ it was derived in joint work with Silbermann in [36, 37], see also the monograph [38]. Then John Roe [41] came up with a very smooth and elegant treatment of the general case.

It is certainly not our intention to recall Roe's proof in detail. But for later reference we have to mention its basic steps. Roe's proof makes thoroughly use of the Stone-Čech compactification of G and of the related ultra-filters, which is one reason for the elegance of his approach: it avoids the nasty passage from sequences to subsequences to subsubsequences... Nevertheless we prefer the language of sequences in this section. It allows us to work on a more elementary level which is certainly useful for some applications in physics and engineering.

Another point should be mentioned in advance. Roe proves his general Fredholm criterion for finitely generated discrete groups which are exact. By definition, a group G is *exact*, if its reduced translation algebra is an exact C^*-algebra. The latter is defined as the reduced crossed product of $l^\infty(G)$ by G. We will also not use the language of crossed products here. It is sufficient to have in mind that the reduced crossed product of $l^\infty(G)$ by G *is* exactly our algebra $\mathcal{A}_2(G)$ of band-dominated operators, and that the class of exact groups is extremely rich: it contains, for example, all amenable (hence, all solvable groups like the discrete Heisenberg group and in particular all commutative groups) and all hyperbolic groups (see [40], Chapter 3).

Our Fredholm criterion for band-dominated operators makes use of the concept of limit operators which we introduce in the context of general G-periodic discrete metric spaces. For this part, the exactness of G is not needed.

Every element $g \in G$ generates an operator L_g of left shift via $(L_g u)(x) := u(g \cdot x)$ for $x \in X$. Clearly, L_g is an isometry on each of the spaces $l^p(X)$ and $c_0(X)$, and $L_g^{-1} = L_{g^{-1}}$.

Let $p \in \{0\} \cup [1, \infty)$ and $h : \mathbb{N} \to G$ a sequence tending to infinity. We say that A_h is a limit operator of $A \in \mathcal{L}(l^p(X))$ defined by the sequence h if

$$L_{h(m)}^{-1} A L_{h(m)} \to A_h \quad \text{and} \quad L_{h(m)}^{-1} A^* L_{h(m)} \to A_h^*$$

strongly on $l^p(X)$ and $l^p(X)^*$, respectively. We denote the set of all limit operators of $A \in \mathcal{L}(l^p(X))$ by $\mathrm{op}_p\,(A)$ and call this set the *operator spectrum* of A. Recall that $l^p(X)^* = l^q(G)$ with $1/p + 1/q = 1$ if $p \geq 1$ and that $c_0(x)^* = l^1(X)$.

Note that the generating function of the shifted operator $L_\alpha^{-1} A L_\alpha$ is related with the generating function of A by

$$k_{L_\alpha^{-1} A L_\alpha}(x,\, y) = k_A(\alpha^{-1} \cdot x,\, \alpha^{-1} \cdot y) \tag{13}$$

and that the generating functions of $L_{h(m)}^{-1} A L_{h(m)}$ converge point-wise on $X \times X$ to the generating function of the limit operator A_h if the latter exists.

It is an important property of band-dominated operators that their operator spectrum is not empty. More general, one has the following result which can be proved by a standard Cantor diagonal argument (see [36, 37, 38]).

Proposition 7 *Let $p \in \{0\} \cup [1, \infty)$ and $A \in \mathcal{A}_p(X)$. Then every sequence $h : \mathbb{N} \to G$ which tends to infinity possesses a subsequence g such that the limit operator A_g of A with respect to g exists.*

Let A be a band-dominated operator and $h : \mathbb{N} \to G$ a sequence which tends to infinity for which the limit operator A_h of A exists. Let B be another band-dominated operator. By Proposition 7 we can choose a subsequence g of h such that the limit operator B_g exists. Then the limit operators of A, $A + B$ and AB with respect to g exist, and

$$A_g = A_h, \qquad (A + B)_g = A_g + B_g, \qquad (AB)_g = A_g B_g.$$

Thus, the mapping $A \mapsto A_h$ acts, at least partially, as an algebra homomorphism.

An operator $T \in \mathcal{A}_p(X)$ is called a *ghost* if its generating function k_T is in $c_0(X \times X)$, i.e., if for every $\varepsilon > 0$, there is a finite subset F of $X \times X$ such that $|k_T(x,\, y)| < \varepsilon$ whenever $(x,\, y) \notin F$.

Proposition 8 *Let $p \in \{0\} \cup [1, \infty)$. Further, let $T \in \mathcal{A}_p(X)$ be a ghost and let $h : \mathbb{N} \to G$ be a sequence which tends to infinity. Then the limit operator T_h of T with respect to h exists, and $T_h = 0$.*

Proof. We prove the assertion for $p \geq 1$. Given $\varepsilon > 0$, write T as a sum $T_1 + T_2 + T_3$ where T_1 is a band operator the generating function of which has a finite support, where $|k_{T_2}(x, y)| \leq \varepsilon$ for all pairs $(x, y) \in X \times X$ and where $\|T_3\|_{\mathcal{L}(l^p(X))} \leq \varepsilon$. Let $y \in X$. Then

$$L_{h(m)}^{-1} T_1 L_{h(m)} \delta_y = 0 \tag{14}$$

for all sufficiently large m. Let now R be such that $k_{T_2}(x, y) = 0$ whenever $\rho(x, y) > R$. By (13), one has for every $x \in X$,

$$\begin{aligned} (L_{h(m)}^{-1} T_2 L_{h(m)} \delta_y)(x) &= \sum_{z \in X} k_{T_2}(h(m)^{-1} \cdot x, h(m)^{-1} \cdot z)\, \delta_y(z) \\ &= k_{T_2}(h(m)^{-1} \cdot x, h(m)^{-1} \cdot y). \end{aligned}$$

Since X is pseudo-homogeneous, $k_{T_2}(h(m)^{-1} \cdot x, h(m)^{-1} \cdot y)$ is zero whenever

$$\rho(h(m)^{-1} \cdot x, h(m)^{-1} \cdot y) = \rho(x, y) > R.$$

Thus, there are at most $w(R)$ points $x \in X$ for which $k_{T_2}(h(m)^{-1} \cdot x, h(m)^{-1} \cdot y)$ is not zero, and for these x one has

$$|k_{T_2}(h(m)^{-1} \cdot x, h(m)^{-1} \cdot y)| \leq \varepsilon.$$

Hence,

$$\|L_{h(m)}^{-1} T_2 L_{h(m)} \delta_y\|_{l^p(X)}^p = \sum_{x \in X} |k_{T_2}(h(m)^{-1} \cdot x, h(m)^{-1} \cdot y)|^p \leq w(R)\, \varepsilon^p. \tag{15}$$

Finally,

$$\|L_{h(m)}^{-1} T_3 L_{h(m)} \delta_y\|_{l^p(X)} \leq \|T_3\|_{\mathcal{L}(l^p(X))} \leq \varepsilon. \tag{16}$$

From (14) – (16) we conclude that

$$L_{h(m)}^{-1} T L_{h(m)} \delta_y \to 0 \quad \text{as} \quad m \to \infty.$$

Since the functions δ_y span a dense subset of $l^p(X)$ and since the shifted operators $L_{h(m)}^{-1} T L_{h(m)}$ are uniformly bounded with respect to m, the assertion follows. ■

As announced, we let now $p = 2$ and $X = G$. Let $A \in \mathcal{A}_2(X)$, and consider the smallest closed subalgebra $\mathcal{A}_A$ of $\mathcal{L}(l^2(G))$ which contains the operators A, A^*, the identity I, and all ghosts. Clearly, $\mathcal{A}_A$ is a C^*-algebra. Let $\mathcal{H}_A$ stand for the set of all sequences $h : \mathbb{N} \to G$ which converge to infinity and for which the limit operator A_h exists. Using the previous proposition one easily checks that then the limit operator B_h exists for every operator $B \in \mathcal{A}_A$ and for every sequence $h \in \mathcal{H}_A$. Thus, one obtains a *-homomorphism

$$\text{smb} : \mathcal{A}_A \to \mathcal{F}, \quad B \mapsto (h \mapsto B_h) \tag{17}$$

from $\mathcal{A}_A$ into the C^*-algebra $\mathcal{F}$ of all bounded functions on $\mathcal{H}_A$ with values in $\mathcal{L}(l^2(X))$.

Theorem 9 *The kernel of the mapping* smb *defined by* (17) *consists exactly of the ghosts.*

Proof. From Proposition 8 we infer that all ghosts belong to the kernel of smb. Conversely, let $T \in \mathcal{A}_A$ be an operator in the kernel of smb. Contrary to what we want to show, suppose T is not a ghost. Then there is a constant $C > 0$ and a sequence of points $(x_m, y_m) \in G \times G$ which tends to infinity and for which $|k_T(x_m, y_m)| \geq C$. Since T is band-dominated, we can single out a *finite* number of elements $\alpha_1, \ldots, \alpha_r$ of G such that, for every m, there is an i with $y_m = x_m \alpha_i$. Then, of course, one finds at least one $\alpha \in \{\alpha_1, \ldots, \alpha_r\}$ such that $y_m = x_m \alpha$ for infinitely many m. Consequently, there is a sequence $h : \mathbb{N} \to G$ which tends to infinity and for which one has

$$|k_T(h(n)^{-1}, h(n)^{-1}\alpha)| \geq C \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

Without loss one can assume that the sequence h belongs to $\mathcal{H}_A$ (otherwise we pass to a subsequence of h). Thus, the limit operator T_h exists, and

$$k_T(h(n)^{-1}, h(n)^{-1}\alpha) \to k_{T_h}(\epsilon, \alpha).$$

The right hand side $k_{T_h}(\epsilon, \alpha)$ is zero since T is in the kernel of the homomorphism smb, which contradicts (18). ■

The relation to Fredholm theory is given by the following *compact ghost theorem.* This is the place where exactness of G is needed.

Theorem 10 *Every compact operator on $l^2(G)$ is a ghost. If G is exact then, conversely, every ghost is compact.*

A proof is in [40], see also [41, 46]. The following is a main result of this section. It follows immediately from Theorems 9 and 10 and from the inverse closedness of C^*-algebras. The latter is used to get that an operator $B \in \mathcal{A}_A$ is Fredholm if and only if its coset $B + \mathcal{K}(l^2(G))$ is invertible in $\mathcal{A}_A/\mathcal{K}(l^2(G))$, and this happens if and only if $\operatorname{smb} B$ is invertible in $\mathcal{F}$ by the cited theorems.

Theorem 11 *Let G be a finitely generated discrete and exact group, and let $A \in \mathcal{A}_2(G)$. Then the operator A is Fredholm on $l^2(G)$ if and only if all limit operators of A are invertible and if*

$$\sup_{A_h \in \mathrm{op}_2(A)} \|A_h^{-1}\| < \infty. \tag{19}$$

Let now $N \geq 1$ and consider band-dominated operators in $\mathcal{A}_2(G, \mathbb{C}^N)$ which we identify with the matrix algebra $\mathcal{A}_2(G)^{N \times N}$. We say that an operator $A \in \mathcal{A}_2(G)^{N \times N}$ possesses a limit operator with respect to a sequence $h : \mathbb{N} \to G$ which tends to infinity if every entry $A_{ij} \in \mathcal{A}_2(G)$ of the matrix representation $A = (A_{ij})_{i,j=1}^N$ has a limit operator with respect to h, and we define $A_h := ((A_{ij})_h)_{i,j=1}^N$ in this case. The operator spectrum of A is again the set of all limit operators of A. Then we have the following matrix version of Theorem 11.

Corollary 12 *Let G be a finitely generated discrete and exact group, $N \geq 1$, and $A \in \mathcal{A}_2(G, \mathbb{C}^N)$. The operator A is Fredholm on $l^2(G, \mathbb{C}^N)$ if and only if all limit operators of A are invertible and if*

$$\sup_{A_h \in \mathrm{op}_2(A)} \|A_h^{-1}\| < \infty.$$

This corollary follows easily from the previous theorem and its proof if one takes into account that every *-homomorphism $W : \mathcal{A} \to \mathcal{B}$ between C^*-algebras extends component-wise to a *-homomorphism $W_N : \mathcal{A}^{N\times N} \to \mathcal{B}^{N\times N}$ and that

$$\ker W_N = (\ker W)^{N\times N}.$$

Moreover,

$$\mathcal{K}(X^N) = (\mathcal{K}(X))^{N\times N} \tag{20}$$

for every Banach space X. ■

Let now X be a pseudo-homogeneous discrete metric space which is periodic with respect to a finitely generated discrete group G. The following proposition is immediate from this definition and from the definition of a limit operator. It holds for general $l^p(X)$.

Proposition 13 *Let $p \in \{0\} \cup [1, \infty)$. The operator A_h is a limit operator of $A \in \mathcal{L}(l^p(X))$ if and only if the operator UA_hU^{-1} is a limit operator of $UAU^{-1} \in \mathcal{L}(l^p(G, \mathbb{C}^N))$.*

Since also the operator $A \in \mathcal{L}(l^p(X))$ is Fredholm if and only if UA_hU^{-1} is Fredholm (which follows easily from (20), we arrive at the following consequence of the preceding corollary.

Theorem 14 *Let X be a pseudo-homogeneous discrete metric space which is periodic with respect to a finitely generated discrete and exact group G, and let $A \in \mathcal{A}_2(X)$. The operator A is Fredholm on $l^2(X)$ if and only if all limit operators of A are invertible and if*

$$\sup_{A_h \in \mathrm{op}_2(A)} \|A_h^{-1}\| < \infty. \tag{21}$$

5 A Wiener algebra of BDO

The goal of this section is to show that the uniform boundedness condition (21) is redundant in case A is a band operator and if the group G satisfies two additional conditions which will be presented later in more detail:

- the growth of the group G is moderate, and
- G contains non-cyclic elements.

The first of these conditions will be needed to verify a "partial inverse closedness" property of the Wiener algebra defined below, whereas the second one is needed to produce a band operator the Fredholm properties of which are independent on the space $l^p(X)$. This operator serves as a "gauge" to compare the Fredholm properties of band operators on different l^p-spaces.

For our goal we introduce a Wiener algebra of operators over X. Again we start with the case when X is a group.

Let G be a finitely generated discrete group. An operator of the form

$$(Au)(\alpha) = \sum_{\beta \in G} k_A(\alpha, \beta) u(\beta) \quad \text{for all } \alpha \in G \tag{22}$$

(thought of as acting on functions on G with finite support) is said to belong to the class $\mathcal{W}(G)$ if there is a (real and non-negative) function $h_A \in l^1(G)$

$$|k_A(\alpha, \beta)| \le h_A(\beta^{-1}\alpha) \tag{23}$$

for all pairs $\alpha, \beta \in G$.

Proposition 15 *The set $\mathcal{W}(G)$ is a Banach algebra with respect to the norm*

$$\|A\|_{\mathcal{W}(G)} := \inf \|h_A\|_{l^1(G)} \tag{24}$$

where the infimum is taken over all functions h_A for which (23) holds. If k_A is the generating function of an operator $A \in \mathcal{W}(G)$, then $k_{A^*}(\alpha, \beta) := \overline{k_A(\beta, \alpha)}$ is the generating function of an operator $A^* \in \mathcal{W}(G)$, and the mapping $A \mapsto A^*$ defines an involution on $\mathcal{W}$.

Notice that if $A \in \mathcal{W}(G)$ is considered as acting on the Hilbert space $l^2(G)$, then its Hilbert adjoint coincides with A^* which justifies the notation.

Proof. It is clear that $\mathcal{W}(G)$ is a linear space and that (24) defines a norm on $\mathcal{W}(G)$. Let $A, B \in \mathcal{W}(G)$. The generating function of the operator AB is given by

$$k_{AB}(\alpha, \gamma) := \sum_{\beta \in G} k_A(\alpha, \beta)\, k_B(\beta, \gamma)$$

from which one concludes that

$$\begin{aligned} |k_{AB}(\alpha, \gamma)| &\le \sum_{\beta \in G} |k_A(\alpha, \beta)|\, |k_B(\beta, \gamma)| \\ &\le \sum_{\beta \in G} h_A(\beta^{-1}\alpha)\, h_B(\gamma^{-1}\beta) \\ &\le \sum_{\delta \in G} h_A(\delta^{-1}\gamma^{-1}\alpha)\, h_B(\delta^{-1}) \\ &= (h_A * h_B)(\gamma^{-1}\alpha) \end{aligned}$$

with $l^1(G)$-functions h_A and h_B. Since the convolution of h_A and h_B is in $l^1(G)$ again, the operator AB is in $\mathcal{W}(G)$, and the convolution $h_{AB} := h_A * h_B$ is one of the functions h such that (23) holds for that operator. The convolution theorem implies further that

$$\|AB\|_{\mathcal{W}(G)} \leq \|h_A * h_B\|_{l^1(G)} \leq \|h_A\|_{l^1(G)} \|h_B\|_{l^1(G)}$$

for all possible choices of functions h_A and h_B. Passing to the infimum yields $\|AB\|_{\mathcal{W}(G)} \leq \|A\|_{\mathcal{W}(G)} \|B\|_{\mathcal{W}(G)}$, thus $\mathcal{W}(G)$ is a normed algebra.

For the next part of the proof, it is convenient to have another description of this algebra available. For a moment, let $\mathcal{W}'$ stand for the set of all operators A of the form (22) for which

$$\|A\|_{\mathcal{W}'} := \sum_{\beta \in G} \sup_{\alpha \in G} |k_A(\alpha, \alpha\beta^{-1})| < \infty.$$

We claim that $\mathcal{W}(G) = \mathcal{W}'$ and $\|A\|_{\mathcal{W}(G)} = \|A\|_{\mathcal{W}'}$ for every operator $A \in \mathcal{W}(G)$. Let $A \in \mathcal{W}(G)$, and let h_A be a function such that (23) holds. Then,

$$\sum_{\beta \in G} \sup_{\alpha \in G} |k_A(\alpha, \alpha\beta^{-1})| \leq \sum_{\beta \in G} \sup_{\alpha \in G} h_A(\beta) = \|h_A\|_{l^1(G)}.$$

Hence, $A \in \mathcal{W}'$, and $\|A\|_{\mathcal{W}'} \leq \|h_A\|_{l^1(G)}$. Passing to the infimum yields $\|A\|_{\mathcal{W}'} \leq \|A\|_{\mathcal{W}(G)}$. Let now, conversely, $A \in \mathcal{W}'$. Then the function h_A defined by

$$h_A(\beta) := \sup_{\alpha \in G} |k_A(\alpha, \alpha\beta^{-1})|$$

belongs to $l^1(G)$ and $\|A\|_{\mathcal{W}'} = \|h_A\|_{l^1(G)}$. From $|k_A(\alpha, \alpha\beta^{-1})| \leq h_A(\beta)$ we conclude that

$$|k_A(\alpha, \gamma)| \leq h_A(\gamma^{-1}\alpha) \quad \text{for all } \alpha, \gamma \in G,$$

which implies that $A \in \mathcal{W}(G)$ and $\|A\|_{\mathcal{W}(G)} \leq \|h_A\|_{l^1(G)} = \|A\|_{\mathcal{W}'}$. This proves the claim. Now the completeness of $\mathcal{W}(G)$ follows in a standard way from

$$\| \sum b_i R_{\alpha_i} \|_{\mathcal{W}(G)} = \| \sum b_i R_{\alpha_i} \|_{\mathcal{W}'} = \sum \|b_i\|_{l^\infty(G)}$$

by using the completeness of both $l^1(G)$ and $l^\infty(G)$. Also the isometry of the mapping $A \mapsto A^*$ follows immediately from the above claim. The remaining properties of an involution can be checked straightforwardly. ■

For $N > 1$, we define a norm and an involution on $\mathcal{W}(G)^{N \times N}$ by

$$\|(A_{ij})\|_{\mathcal{W}(G)^{N \times N}} := \sum_{i,\, j=1}^{N} \|A_{ij}\|_{\mathcal{W}(G)} \quad \text{and} \quad (A_{ij})^* := (A_{ji}^*),$$

respectively, which makes $\mathcal{W}(G)^{N \times N}$ to an involutive Banach algebra. Finally, if X is a G-periodic discrete metric space, then we define $\mathcal{W}(X)$ as the set of all operators A of the form (6) for which the operator UAU^{-1} belongs to $\mathcal{W}(G)^{N \times N}$. Clearly, $\mathcal{W}(X)$ is an involutive Banach algebra with respect to the involution and the norm inherited from the involution and the norm on $\mathcal{W}(G)^{N \times N}$.

Proposition 16 *Let X be a G-periodic discrete metric space. Then every operator in $\mathcal{W}(X)$ is bounded on each of the spaces $l^p(X)$ with $p \in \{0\} \cup [1, \infty]$.*

Proof. It is evidently sufficient to prove this for $X = G$. Let $A \in \mathcal{W}(G)$ and let $h_A \in l^1(G)$ be a function such that (23) holds. First let $f \in l^\infty(G)$. Then

$$\begin{aligned}\|Af\|_{l^\infty(G)} &= \sup_{\alpha \in G} \left| \sum_{\beta \in G} k_A(\alpha, \beta) f(\beta) \right| \\ &\leq \sup_{\alpha \in G} \sum_{\beta \in G} h_A(\beta^{-1}\alpha) \|f\|_{l^\infty(G)} \\ &\leq \|h_A\|_{l^1(G)} \|f\|_{l^\infty(G)}.\end{aligned}$$

Thus, A is bounded on $l^\infty(X)$ and, since A maps $c_0(X)$ to $c_0(X)$, also on $c_0(X)$. If $f \in l^1(G)$ then the convolution theorem implies

$$\begin{aligned}\|Af\|_{l^1(G)} &= \sum_{\alpha \in G} \left| \sum_{\beta \in G} k_A(\alpha, \beta) f(\beta) \right| \\ &\leq \sum_{\alpha \in G} \sum_{\beta \in G} h_A(\beta^{-1}\alpha) |f(\beta)| \\ &\leq \|h_A\|_{l^1(G)} \|f\|_{l^1(G)}.\end{aligned}$$

Thus, by the interpolation theorem, $\|A\|_{\mathcal{L}(l^p(G))} \leq \|h_A\|_{l^1(G)}$ for every $p \in [1, \infty]$. ■

Proposition 17 *Every band operator belongs to $\mathcal{W}(X)$.*

Proof. Since the mapping $A \mapsto UAU^{-1}$ sends band operators to band operators, it is again sufficient to prove the assertion for $X = G$. Let A be a band operator, and let R such that $k_A(\alpha, \beta) = 0$ if $\rho(\alpha, \beta) = \rho(\beta^{-1}\alpha, \epsilon) > R$. Then the function

$$h_A(\gamma) := \begin{cases} \|k_A\|_{l^\infty(G \times G)} & \text{if } \rho(\gamma, \epsilon) \leq R \\ 0 & \text{if } \rho(\gamma, \epsilon) > R \end{cases}$$

satisfies the inequality

$$|k_A(\alpha, \beta)| \leq h_A(\beta^{-1}\alpha)$$

for all pairs $\alpha, \beta \in G$. Moreover, the function h_A is finitely supported since

$$\{\gamma \in G : \rho(\gamma, \epsilon) \leq R\}$$

is a finite set. Hence, $h_A \in l^1(G)$. ■

In [36, 38] we made essentially use of the fact that the Wiener algebra $\mathcal{W}(\mathbb{Z}^N)$ is inverse closed in each of the algebras $\mathcal{L}(l^p(\mathbb{Z}^N))$, that is, whenever an operator $A \in \mathcal{W}(\mathbb{Z}^N)$ is invertible on one of the spaces $l^p(\mathbb{Z}^N)$ with $p \in [1, \infty]$ then its inverse A^{-1} belongs to $\mathcal{W}(\mathbb{Z}^N)$. The proof given in [38] seems to work for every countable discrete and *commutative* group, but we were not able to prove this fact for a general countable discrete group G in place of $\mathbb{Z}^N$. Rather we have to assume some restrictions for the operator (only band operators are allowed) and for the group (only a moderate growth is allowed).

A finitely generated discrete group is said to have *sub-exponential growth* if the weight function w can be chosen such that

$$\lim_{k\to\infty} \frac{w(k)}{e^{\delta k}} = 0 \quad \text{for all } \delta > 0. \tag{25}$$

This condition is satisfied, for example, for the abelian groups $\mathbb{Z}^n$, the discrete Heisenberg group and, more general, for nilpotent groups. In all these examples one observes polynomial growth. Grigorchuk [11] succeeded in proving the existence of finitely generated groups with a growth between polynomial and exponential, which are also subject to condition (25). On the other hand, every free group of $n > 1$ generators has exponential growth. For more details see [12].

Theorem 18 *Let G be a finitely generated group with sub-exponential growth which acts freely on a pseudo-homogeneous discrete metric space X. The following assertions are equivalent for every band operator A :*

(a) *the operator A is invertible on one of the spaces $l^p(X)$ with $p \in \{0\} \cup [1, \infty]$;*

(b) *the operator A is invertible in $\mathcal{W}(X)$;*

(c) *the operator A is invertible on each of the spaces $l^p(X)$ with $p \in \{0\} \cup [1, \infty]$.*

The essential implication $(a) \Rightarrow (b)$ will follow almost straightforwardly from a result by Shubin (Theorem 2 in [44]; see also [45] for more details) which is cited here in a special version which is sufficient for our purposes.

Theorem 19 *Let X be a pseudo-homogeneous discrete metric space, and let A be an operator of the form* (6) *the generating function of which admits an estimate*

$$|k_A(x, y)| \le Ce^{-\rho(x, y)} w^{-1-\delta}(\rho(x, y)) \tag{26}$$

with some positive constants C, δ. If the operator K possesses a bounded inverse L on one of the spaces $l^p(X)$ with $p \in [1, \infty]$ then the generating function of L satisfies the estimate

$$|k_L(x, y)| \le C_1 e^{-\varepsilon\rho(x, y)} \tag{27}$$

with some positive constants C_1 and ε.

Proof of Theorem 18. $(a) \Rightarrow (b)$: It is evident that the generating function of a band operator satisfies (26) (simply because $k_A(x, y) = 0$ if $\rho(x, y)$ is sufficiently large). It remains to show that every operator L the generating function of which is subject to the estimate (27) belongs to the Wiener algebra $\mathcal{W}(X)$. It is further sufficient to show this fact for $X = G$, in which case (27) implies

$$|k_L(\alpha, \beta)| \le C_1 e^{-\varepsilon\rho(\beta^{-1}\alpha, \epsilon)}$$

for all elements $\alpha,\,\beta \in G$. Consider the function $h_L : G \to \mathbb{C}$,

$$h_L(\gamma) := e^{-\varepsilon\rho(\gamma,\,\epsilon)}.$$

There at most $w(k) - w(k-1)$ words of length k (with $w(-1) := 0$). Thus,

$$\begin{aligned}\sum_{\gamma\in G} h_L(\gamma) &= \sum_{\gamma\in G} e^{-\varepsilon\rho(\gamma,\,\epsilon)} \\ &\leq \sum_{k\in\mathbb{N}} (w(k) - w(k-1))e^{-\varepsilon\, k} \\ &= \sum_{k\in\mathbb{N}} \frac{w(k) - w(k-1)}{e^{\varepsilon/2\, k}} e^{-\varepsilon/2\, k}.\end{aligned}$$

The sub-exponentiality of G ensures that the quotient in this sum remains bounded. Hence, h_L is in $l^1(G)$. This settles the implication $(a) \Rightarrow (b)$ for band operators A which are invertible on $l^p(X)$ for $p \geq 1$. If a band operator A is invertible on $c_0(X)$, then A^* is a band operator which is invertible on $l^1(X)$. By what has already been proved, A^* is invertible in the Wiener algebra. Since $\mathcal{W}(X)$ is an involutive algebra with respect to $A \mapsto A^*$, the operator A itself is invertible in $\mathcal{W}(X)$.

The implication $(b) \Rightarrow (c)$ follows from Proposition 16, and the implication $(c) \Rightarrow (a)$ is evident. ■

A first consequence of this "partial inverse closedness" is the following Theorem 20. Note that this theorem would hold for every operator in the Wiener algebra (not only for band operators) if one were able to prove the inverse closedness of the full algebra $\mathcal{W}(X)$. The latter happens, for example, if $X = G$ is the abelian group $\mathbb{Z}^N$ (see, for example, [38], Theorem 2.5.2).

Theorem 20 *Let G be a finitely generated group with sub-exponential growth which acts freely on a pseudo-homogeneous discrete metric space X. In addition assume that not every element of G is cyclic of finite order. The following assertions are equivalent for every band operator on X:*

(a) *the operator A is Fredholm on one of the spaces $l^p(X)$ with $p \in \{0\} \cup (1,\,\infty)$;*

(b) *the operator A is Fredholm on each of the spaces $l^p(X)$ with $p \in \{0\} \cup [1,\,\infty]$.*

If one of these conditions is satisfied, then the Fredholm index of A does not depend on the underlying space.

In connection with the hypotheses of the previous theorem, recall that in 1968 Novikov and Adian [29] gave a positive solution to Burnside's problem of 1902 asking whether there exists an infinite but finitely generated group each element of which is cyclic.

Proof. We prepare the proof by showing that there is a band operator V with the following properties (a similar construction is in [39]):

- V is Fredholm on each space $l^p(X)$ with $p \in \{0\} \cup [1, \infty]$,
- the Fredholm index of V is 1, independently of the space,
- there is a Fredholm inverse of V which is a band operator again and which can also be chosen independently of the space.

Let α be a non-cyclic element of G and set $\alpha^0 := \epsilon$. Then the semi-group G_0 of all powers α^k with $k \geq 0$ is infinite, and no two of its elements coincide. Let P denote the operator of multiplication by the characteristic function of G_0 and consider the operators

$$V := PR_\alpha P + (I - P) \quad \text{and} \quad V_{-1} := PR_{\alpha^{-1}} P + (I - P)$$

on G. One easily checks that $V_{-1}V = I$ and that $I - VV_{-1}$ is the operator of multiplication by the characteristic function of the singleton $\{\epsilon\}$, hence compact. Thus, V is a band operator with the desired properties in case $X = G$. In the general case we make use of the fact that $l^p(X) = l^p(G, \mathbb{C}^N)$. Then the $N \times N$ diagonal matrix operator

$$U^{-1} \text{diag}\,(V,\, I,\, \ldots,\, I) U \tag{28}$$

does the job. We denote the operator (28) by V again.

For brevity, let $\mathcal{X}$ be one of the spaces $l^p(X)$ with $p \in \{0\} \cup (1, \infty)$, and let A be a Fredholm band operator on $\mathcal{X}$ with Fredholm index k. Then the operator AV^k is again a Fredholm band operator on $\mathcal{X}$, but now with Fredholm index 0. It is well known (see, e.g. [10], Chapter 4, Theorem 6.2 and the remark after its proof) that then there is a compact operator K such that $AV^k + K$ is invertible on $\mathcal{X}$. Let P_R denote the operator of multiplication by the characteristic function of the ball with center 0 and radius R in X. Since balls are finite, the operators P_R are compact. Moreover, the P_R as well as their adjoints tend strongly to the identity operator on $\mathcal{X}$ as $R \to \infty$. Then the operators $P_R K P_R$ tend to K in the operator norm. Thus, for sufficiently large R, the operators $AV^k + P_R K P_R$ are also invertible on $\mathcal{X}$. We fix an R_0 with this property and write for simplicity again K instead of $P_{R_0} K P_{R_0}$. So we arrived at an invertible *band* operator $AV^k + K$ on $\mathcal{X}$.

From Theorem 18 we conclude that the operator $AV^k + K$ is invertible on each of the spaces $l^p(X)$ with $p \in \{0\} \cup [1, \infty]$. Since K is compact on each of these spaces, the operator AV^k is a Fredholm operator of index zero on each of these spaces. Since, finally, V^k is a Fredholm operator of index $-k$ on each of these spaces, the assertion follows. ∎

Now we can formulate and prove the main result of this section. Again, this result holds for all operators in the Wiener algebra if the inverse closedness of this algebra is known.

Theorem 21 *Let G be a finitely generated and exact group with sub-exponential growth which possesses at least one non-cyclic element and which acts freely on* *a pseudo-homogeneous discrete metric space X. Then the following assertions are equivalent for every band operator on X :*

(a) *the operator A is Fredholm on $l^q(X)$ for some $q \in \{0\} \cup (1, \infty)$;*

(b) *all limit operators of A are invertible on $l^q(X)$ for some $q \in \{0\} \cup (1, \infty)$;*

(c) *all limit operators of A are invertible on $c_0(X)$;*

(d) *all limit operators of A are invertible on $c_0(X)$, and the norms of their inverses are uniformly bounded;*

(e) *all limit operators of A are invertible on $l^p(X)$ for every $p \in \{0\} \cup [1, \infty)$, and the norms of their inverses are uniformly bounded;*

(f) *all limit operators of A are invertible on $l^2(X)$, and the norms of their inverses are uniformly bounded;*

(g) *the operator A is Fredholm on $l^2(X)$;*

(h) *the operator A is Fredholm on $l^p(X)$ for every $p \in \{0\} \cup [1, \infty]$.*

If one of these conditions is satisfied, then the essential spectrum $\mathrm{sp}_{ess} A$ does not depend on the underlying space $l^p(X)$ with $p \in \{0\} \cup [1, \infty)$, and

$$\mathrm{sp}_{ess} A = \bigcup_{A_h \in \mathrm{op}(A)} \mathrm{sp}\, A_h. \tag{29}$$

Proof. $(a) \Rightarrow (b)$: For definiteness, let A be Fredholm on $l^q(X)$ with $q \in (1, \infty)$. (The proof for $c_0(X)$ runs analogously.) We have seen in the proof of Theorem 20 that every Fredholm band operator possesses a Fredholm inverse which belongs to the Wiener algebra. Thus, there are a operator $B \in \mathcal{W}(X)$ and compact operators K_1, K_2 on $l^q(X)$ such that $BA = I + K_1$ and $AB = I + K_2$. Let $h : \mathbb{N} \to G$ be a sequence which tends to infinity and for which the limit operator A_h exists. By Proposition 7, there is a subsequence g of h such that the limit operators B_g, $(K_1)_g$ and $(K_2)_g$ exist. The latter two operators vanish since the K_i are compact. Hence, $B_h A_h = A_h B_h = I$, whence the invertibility of the limit operators.

$(b) \Rightarrow (c)$: Since limit operators of band operators are band operators again, this implication follows from Theorem 18.

$(c) \Rightarrow (d)$: Fix $x_0 \in X$ and, for each positive integer k, let P_k stand for the operator of multiplication with the characteristic function of the ball $\{x \in X : \rho(x, x_0) \le k\}$. We claim that there are constants $C > 0$ and $k \in \mathbb{N}$ such that

$$\|u\|_{c_0(X)} \le C\,(\|Au\|_{c_0(X)} + \|P_k u\|_{c_0(X)}) \tag{30}$$

for all $u \in c_0(X)$. This claim is clearly equivalent to the existence of constants $C > 0$ and $k \in \mathbb{N}$ such that

$$1/C \le \|Au\|_{c_0(X)} + \|P_k u\|_{c_0(X)}$$

for all unit vectors $u \in c_0(X)$. Contrary to what we want to show, assume that such constants do not exist. Then, for each $C > 0$ and each $k \in \mathbb{N}$, there is a unit vector $u_{k,C} \in c_0(X)$ such that

$$1/C > \|Au_{k,C}\|_{c_0(X)} + \|P_k u_{k,C}\|_{c_0(X)}.$$

Choose $C := k$. Thus, for each k, there is a unit vector $u_k \in c_0(X)$ such that

$$1/k > \|Au_k\|_{c_0(X)} + \|P_k u_k\|_{c_0(X)}. \tag{31}$$

From $\|u_k\|_{c_0(X)} = 1$ and $\|P_k u_k\|_{c_0(X)} < 1/k$ we conclude that there is a sequence (x_k) of points in X which tend to infinity and which satisfy

$$|u_k(x_k)| \geq 1/2 \quad \text{for every } k \in \mathbb{N}.$$

Since the number of the orbits of X/G is finite by assumption, we can assume without loss of generality that all x_k belong to the same orbit, thus they are of the form $x_k = \alpha_k x_0$ with some fixed point $x_0 \in X$. Set $v_k := L_{\alpha_k^{-1}} u_k$. Since the sequence (v_k) is bounded in $c_0(X)$, it possesses a subsequence which converges point-wise on X (thus, uniform on compact = finite subsets of X) to a certain sequence $v_\infty \in c_0(X)$. From $|v_k(x_0)| = |u_k(x_k)| \geq 1/2$ we get that $|v_\infty(x_0)| \geq 1/2$; in particular, v_∞ is *not* the zero sequence. For simplicity we will assume that already the sequence (v_n) itself converges to v_∞. Otherwise we pass to a suitable subsequence.

Define a sequence $h : \mathbb{N} \to G$ by $h(k) := \alpha_k$. Since A is a band operator, there is a subsequence g of h such that the limit operator A_g exists. Again we assume for the sake of simplicity the already the limit operator of A with respect to the sequence h exists. From (31) we then infer that $\|AL_{h(k)} v_k\|_{c_0(X)} = \|Au_k\|_{c_0(X)} < 1/k$, whence

$$\|L_{h(k)}^{-1} A L_{h(k)} v_k\|_{c_0(X)} < 1/k \tag{32}$$

for all $k \in \mathbb{N}$.

For every finite (= compact) subset F of X, we denote by χ_F the operator of multiplication by the characteristic function of F. Let F be an arbitrary finite subset of X. Since all shifted operators $L_{h(k)}^{-1} A L_{h(k)}$ are band operators of the same band width as A, there is another finite subset E of X (which depends only on the band width, not on k) such that

$$\chi_F L_{h(k)}^{-1} A L_{h(k)} = \chi_F L_{h(k)}^{-1} A L_{h(k)} \chi_E.$$

Thus, (32) implies that

$$\|\chi_F L_{h(k)}^{-1} A L_{h(k)} v_k\|_{c_0(X)} = \|\chi_F L_{h(k)}^{-1} A L_{h(k)} \chi_E v_k\|_{c_0(X)} < 1/k.$$

Passing to the limit as $k \to \infty$ we obtain

$$\|\chi_F A_h \chi_E v_\infty\|_{c_0(X)} = 0,$$

whence $\|\chi_F A_h v_\infty\|_{c_0(X)} = 0$ for every finite subset F of X. Hence, $A_h v_\infty = 0$. Since v_∞ is not the zero sequence, this shows that the limit operator A_h of A

has a non-trivial kernel and is, thus, not invertible. This contradiction proves the claim (30).

Now we will see how to get the uniform boundedness of the inverse limit operators from (30). The a priori estimate (30) implies that, for each choice of $u \in c_0(X)$, $\alpha \in G$ and $k \in \mathbb{N}$,

$$\|L_\alpha P_k u\|_{c_0(X)} \leq C \left(\|A L_\alpha P_k u\|_{c_0(X)} + \|P_k L_\alpha P_k u\|_{c_0(X)} \right).$$

Let $h : \mathbb{N} \to G$ be a sequence for which the limit operator A_h of A exists. Since every L_α acts as an isometry, we get

$$\|P_k u\|_{c_0(X)} \leq C \left(\|L_{h(n)}^{-1} A L_{h(n)} P_k u\|_{c_0(X)} + \|L_{h(n)}^{-1} P_k L_{h(n)} P_k u\|_{c_0(X)} \right).$$

Passing to the limit $n \to \infty$ yields

$$\|P_k u\|_{c_0(X)} \leq C \, \|A_h P_k u\|_{c_0(X)},$$

and passage to the limit $k \to \infty$ implies

$$\|u\|_{c_0(X)} \leq C \, \|A_h u\|_{c_0(X)}$$

for all $u \in c_0(X)$, whence $\|A_h^{-1}\|_{\mathcal{L}(c_0(X))} \leq 1/C$ for every sequence h for which a limit operator of A exists.

$(d) \Rightarrow (e)$: The proof of this implication is based on the existence of an involution on the Wiener algebra. Let $A \in \mathcal{W}(X)$ be an operator with

$$C_\infty(A) := \sup \{ \|A_h^{-1}\|_{\mathcal{L}(c_0(X))} : A_h \in \mathrm{op}\,(A) \} < \infty. \tag{33}$$

The limit operators of A^* are just the Wiener adjoints of the limit operators of A. Thus, the invertibility of all limit operators of A implies the invertibility of all limit operators of A^*. This shows that

$$C_0(A^*) := \sup \{ \|(A_h^*)^{-1}\|_{\mathcal{L}(c_0(X))} : A_h \in \mathrm{op}\,(A) \} < \infty. \tag{34}$$

The operator A, thought of as acting on $l^1(X)$, can be identified with the usual Banach dual operator of $A^* \in \mathcal{L}(c_0(X))$. Hence,

$$C_1(A) := \sup \{ \|A_h^{-1}\|_{\mathcal{L}(l^1(X))} : A_h \in \mathrm{op}\,(A) \} = C_0(A^*) < \infty.$$

By the Riess-Thorin interpolation theorem (which is usually stated for interpolation between $l^1(X)$ and $l^\infty(X)$ but which holds for $c_0(X)$ in place of $l^\infty(X)$ as well, see Theorem 5.6.1 in [4]), we have for every $p \in [1, \infty)$ and for every operator $A_h \in \mathrm{op}\,(A)$,

$$\|A_h^{-1}\|_{\mathcal{L}(l^p(X))}^p \leq \|A_h^{-1}\|_{\mathcal{L}(c_0(X))}^{p-1} \|A_h^{-1}\|_{\mathcal{L}(l^1(X))} \leq C_\infty(A)^{p-1} C_1(A),$$

which verifies the uniform boundedness of the norms of the inverses of the limit operators of A on $l^p(X)$.

Finally, the implication $(e) \Rightarrow (f)$ is evident, the implication $(f) \Rightarrow (g)$ is a consequence of Theorem 14, the implication $(g) \Rightarrow (h)$ follows from Theorem 20, and the implication $(h) \Rightarrow (a)$ is evident again. ■

The following simple proposition gives a sufficient condition for the absence of the discrete spectrum of band-dominated operators. The criterion holds for general groups, since only the trivial implication of Theorem 11 is used in the proof.

Proposition 22 *Let $A \in \mathcal{A}_p(X)$ and let $h : \mathbb{N} \to G$ be a sequence tending to infinity for which the limit operator A_h exists with respect to norm convergence, i.e.,*

$$\lim_{m\to\infty} \|L_{h(m)}^{-1} A L_{h(m)} - A_h\| = 0. \tag{35}$$

Then $\mathrm{sp}_{ess}\, A = \mathrm{sp}\, A$.

Proof. Let $\lambda \notin \mathrm{sp}_{ess}\, A$. Then, by Theorem 11, $\lambda \notin \mathrm{sp}\, A_h$. It follows from (35) that $\lambda \notin \mathrm{sp}\, A$. Hence, $\mathrm{sp}\, A \subseteq \mathrm{sp}_{ess}\, A$, whence the assertion. ■

6 Some applications

6.1 Periodic operators on periodic graphs

Let X be a discrete graph. We call $(X, \sim)$ a *periodic discrete graph* if it is a connected discrete graph and if there is a finitely generated discrete group G which freely operates from the left on X and for which

$$x \sim y \qquad \text{if and only if} \qquad g \cdot x \sim g \cdot y$$

for arbitrary vertices $x, y \in X$ and group elements $g \in G$. Clearly, every group with these properties leaves the graph distance invariant, that is, X becomes a periodic discrete metric space under the action of G.

Let $(X, \sim)$ be a periodic discrete graph with respect to a group G operating on X. An operator $A \in \mathcal{L}(l^p(X))$ is said to be *periodic* if A is invariant with respect to left shifts by elements of G, that is

$$L_g A = A L_g \qquad \text{for every } g \in G.$$

Example: Laplace operators. Every G-periodic discrete graph $\Gamma := (X, \sim)$ induces a canonical band operator Δ_Γ on $l^p(X)$, called the (discrete) *Laplace operator* or *Laplacian* of Γ, via

$$(\Delta_\Gamma u)(x) := \frac{1}{m(x)} \sum_{y\sim x} u(y), \quad x \in X. \tag{36}$$

Evidently, Δ_Γ is a G-periodic band operator.

For example, if G is the commutative group $\mathbb{Z}^2$ with standard generating system

$$\mathcal{S} = \{(0,\, 1),\, (0,\, -1),\, (1,\, 0),\, (-1,\, 0)\}$$

then its Cayley graph $\Gamma = (G,\, \sim)$ is the usual lattice, and

$$(\Delta_\Gamma u)(x_1,\, x_2) = \frac{1}{4}(u(x_1+1,\, x_2) + u(x_1-1,\, x_2) + u(x_1,\, x_2+1) + u(x_1,\, x_2-1)$$

for $(x_1,\, x_2) \in \mathbb{Z} \times \mathbb{Z}$. ■

The following is a straightforward consequence of Proposition 22.

Proposition 23 *Let $A \in \mathcal{A}_p(X)$ be a periodic operator. Then*

$$\mathrm{sp}_{ess}\, A = \mathrm{sp}\, A.$$

One easily checks that $A \in \mathcal{W}(X)$ is a periodic operator on the periodic discrete graph X if and only if the generating function k_A of A satisfies the following periodicity condition: For all group elements $\gamma \in G$ and all vertices $x,\, y \in X$,

$$k_A(\gamma \cdot x,\, \gamma \cdot y) = k_A(x,\, y).$$

This equality implies that the functions $r_A^{ij}(\alpha,\, \beta) = k_A(\alpha \cdot x_i,\, \beta \cdot x_j)$ satisfy

$$r_A^{ij}(\alpha,\, \beta) = k_A((\gamma^{-1}\alpha) \cdot x_i,\, (\gamma^{-1}\beta) \cdot x_j)$$

for all $\gamma \in G$, whence $r_A^{ij}(\alpha,\, \beta) = r_A^{ij}(\beta^{-1}\alpha,\, \epsilon)$. Hence, for $i = 1,\, \ldots,\, N$,

$$\begin{aligned}
(UAU^{-1}f)_i(\alpha) &= \sum_{j=1}^{N} \sum_{\beta \in G} r_A^{ij}(\alpha,\, \beta)\, (U_j f)(\beta) \\
&= \sum_{j=1}^{N} \sum_{\beta \in G} r_A^{ij}(\beta^{-1}\alpha,\, \epsilon)\, (U_j f)(\beta) \\
&= \sum_{j=1}^{N} \sum_{\beta \in G} r_A^{ij}(\beta,\, \epsilon)\, (R_{\beta^{-1}} U_j f)(\alpha)
\end{aligned}$$

where

$$|r_A^{ij}(\beta,\, \epsilon)| \le h(\beta)$$

for a some non-negative function $h \in l^1(\mathbb{Z}^n)$. Thus, we arrived at the following proposition.

Proposition 24 *Every periodic operator $A \in \mathcal{W}(X)$ is isometrically equivalent to the shift invariant matrix operator $UAU^{-1} \in W(G,\, \mathbb{C}^N)$.*

In case $G = \mathbb{Z}^n$, the explicit description of the spectrum (= the essential spectrum) of a periodic operator can be given by means of the Fourier transform. Under the conditions of the previous proposition, one associates with A a function $\sigma_A : \mathbb{T}^n \to \mathbb{C}^{N\times N}$, referred to as the *symbol* of A, by

$$\sigma_A(t) := \sum_{\beta\in\mathbb{Z}^n} r_A(\beta)\, t^\beta$$

where $\mathbb{T}$ is the torus $\{z \in \mathbb{C} : |z| = 1\}$, $r_A(\beta)$ is the matrix $(r_A^{ij}(\beta, 0))_{i,\,j=1}^N$, and $t^\beta := t_1^{\beta_1} \ldots t_n^{\beta_n}$ for $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. Let $\lambda_A^j(t)$ with $j = 1, \ldots, N$ denote the eigenvalues of the matrix $\sigma_A(t)$. The enumeration of the eigenvalues can be chosen in such a way that $\lambda_A^j(t)$ depends continuously on t for every j. Thus, the sets

$$\mathcal{C}_j(A) := \{\lambda \in \mathbb{C} : \lambda = \lambda_A^j(t),\, t \in \mathbb{T}^n\}, \quad j = 1, \ldots, N \tag{37}$$

are compact and connected curves in the complex plane, called the *spectral* or *dispersion curves* of A.

Proposition 25 *Let $G = sZ^n$ and $A \in \mathcal{W}(X)$ a periodic operator. Then*

$$\operatorname{sp} A = \operatorname{sp}_{ess} A = \bigcup_{j=1}^N \mathcal{C}_j(A). \tag{38}$$

If, moreover, $A \in \mathcal{W}(X)$ is a *self-adjoint* periodic operator on $l^2(X)$, then σ_A is a Hermitian matrix-valued function. Hence, the λ_A^j are continuous real-valued functions, and

$$\mathcal{C}_j(A) = [\alpha_j(A),\, \beta_j(A)] \quad \text{for } j = 1, \ldots, N$$

where $\alpha_j(A) := \min_{t\in\mathbb{T}^n} \lambda_A^j(t)$ and $\beta_j(A) := \max_{t\in\mathbb{T}^n} \lambda_A^j(t)$. Thus, the spectrum of a self-adjoint periodic operator on a periodic graph is the union of not more than N compact intervals (with N the number of orbits of X under the action of G).

A detailed analysis of band operators (in particular, perturbed Schrödinger operators) on $\mathbb{Z}^n$-periodic graphs (like the honeycomb graph) can be found in [34]; see also the references therein.

6.2 Operators with slowly oscillating coefficients on periodic graphs

Let X be a periodic discrete graph on which a discrete group G operates. A function $a \in l^\infty(X)$ is called *slowly oscillating* if, for every pair of vertices $x, y \in X$,

$$\lim_{G\ni\alpha\to\infty} (a(\alpha\cdot x) - a(\alpha\cdot y)) = 0. \tag{39}$$

The set of all slowly oscillating functions on X forms a C^*-algebra which we denote by $SO(X)$ (but note that $SO(X)$ depends on the group G and its action, too).

Let $a \in SO(X)$ and h a sequence in G tending to infinity. The Bolzano-Weierstrass Theorem and a Cantor diagonal argument imply that there is a subsequence g of h such that the functions $x \mapsto a(g(m) \cdot x)$ converge point-wise to a function $a_g \in l^\infty(X)$ as $m \to \infty$. The condition (39) ensures that the limit function a_g is periodic on X. Indeed, for every $\alpha \in G$,

$$\begin{aligned} a_g(\alpha \cdot x) &= \lim_{m\to\infty} a(g(m) \cdot (\alpha \cdot x)) \\ &= \lim_{m\to\infty} a((g(m)\alpha) \cdot x)) \\ &= \lim_{m\to\infty} a(g(m) \cdot x)) = a_g(x). \end{aligned}$$

Let $N_\infty := \infty$ if $G = \mathbb{Z}^n$ and let N_∞ be a finite number in case the group G is different from $\mathbb{Z}^n$. We consider operators of the form

$$A = \sum_{k,\, l=1}^{N_\infty} b_k\, A_{kl}\, c_l I \tag{40}$$

where the A_{kl} are periodic operators in $\mathcal{W}(X)$ if $G = \mathbb{Z}^n$ and periodic band operators for other groups G, and where the b_k and c_l are slowly oscillating functions satisfying

$$\sum_{k,\, l=1}^{N_\infty} \|b_k\|_{l^\infty(X)}\, \|A_{kl}\|_{\mathcal{W}(X)}\, \|c_k\|_{l^\infty(X)} < \infty.$$

Let $h : \mathbb{N} \to G$ be a sequence tending to infinity. Then

$$L_{h(m)}^{-1} A L_{h(m)} = \sum_{k,\, l=1}^{N_\infty} (L_{h(m)}^{-1} b_k)\, A_{kl}\, (L_{h(m)}^{-1} c_l) I.$$

One can assume without loss that the point-wise limits

$$\lim_{m\to\infty} (L_{h(m)}^{-1} b_k)(x) =: b_k^h, \qquad \lim_{m\to\infty} (L_{h(m)}^{-1} c_l)(x) =: c_l^h$$

exist (otherwise we pass to a suitable subsequence of h). Above we have mentioned that the limit functions b_k^h and c_l^h are periodic on X. Consequently, the limit operators A_h of A are periodic operators of the form

$$A_h = \sum_{k,\, l=1}^{N_\infty} b_k^h\, A_{kl}\, c_l^h I.$$

Thus, in any case, the limit operators of A are periodic with respect to G. In case $G = \mathbb{Z}^n$, the spectra of these operators can be explicitly determined via Theorem 21, which leads to the following.

Theorem 26 *Let $G = \mathbb{Z}^n$, and let A be an operator with slowly oscillating coefficients of the form* (40). *Then A is a Fredholm operator on $l^p(X)$ if and only if, for every operator $A_h \in \mathrm{op}\, A$,*

$$\det \sigma_{A_h}(t) \neq 0 \quad \text{for } t \in \mathbb{T}^n.$$

Moreover,

$$\mathrm{sp}_{ess}\, A = \bigcup_{A_h \in \mathrm{op}\,(A)} \mathrm{sp}\, A_h = \bigcup_{A_h \in \mathrm{op}\,(A)} \bigcup_{j=1}^{N} \mathcal{C}_j(A_h).$$

References

[1] S. Albeverio, S. N. Lakaev, Z. I. Muminov, *The structure of the essential spectrum of the three-particle Schrödinger operators on a lattice.* arXiv:math-ph/0312050, 2003.

[2] S. Albeverio, S. N. Lakaev, J. I. Abdullaev, *On the finiteness of the discrete spectrum of four-particle lattice Schrodinger operators.* Reports Math. Phys. **51**(2003), 1, 43 - 70.

[3] W. Amrein, M. Măntoiu, R. Purice, *Propagation properties for Schrödinger operators affiliated with certain C^*-algebras.* Ann. H. Poincaré In-t **3**(2002), 6, 1215 – 1232.

[4] J. Bergh, J. Löfström, *Interpolation spaces. An Introduction.* Springer-Verlag, Berlin, Heidelberg, New York 1976.

[5] G. Berkolaiko, R. Carlson, S. Fulling, P. Kuchment (Editors), *Quantum Graphs and Their Applications.* Contemp. Math. **415**, Amer. Math. Soc., Providence, R.I., 2006.

[6] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry.* Springer-Verlag, Berlin, Heidelberg, New York 1987.

[7] P. Deift, *Orthogonal Polynomials and Random Matrices: A Rieman-Hilbert Approach.* Courant Lectures Notes Math. **3**, Amer. Math. Soc., Providence, R.I., 2000.

[8] V. Georgescu, A. Iftimovici, *Crossed products of C^*-algebras and spectral analysis of quantum Hamiltonians.* Comm. Math. Phys. **228**(2002), 519 – 560.

[9] V. Georgescu, A. Iftimovici, *Localization at infinity and essential spectrum of quantum Hamiltonians.* arXiv:math-ph/0506051v1, 20 June 2005.

[10] I. Gohberg, N. Krupnik, *One-dimensional Linear Singular Integral Operators I. Introduction.* Birkhäuser Verlag, Basel, Boston, Stuttgart 1992.

[11] R. I. Grigorchuk, On Milnor's problem of group growth. Dokl. Akad. Nauk SSSR **271**(1983), 1, 30 – 33 (Russian, Engl. transl. Soviet Math. Dokl. **28**(1983), 1, 23 – 26).

[12] P. de la Harpe, Topics in Geometric Group Theory. Univ. of Chigaco Press, Chigaco 2000.

[13] P. Harris, *Carbon Nano-tubes and Related Structure*. Cambridge Univ. Press, Cambridge 2002.

[14] A. Jirari, *Second Order Sturm-Liouville Difference Equations and Orthogonal Polynoms*. Memoirs Amer. Math. Soc. **542**, Amer. Math. Soc., Providence, R.I., 1995.

[15] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin, Heidelberg, New York 1966.

[16] E. Korotyaev, I. Lobanov, *Schrödinger operators on zigzag graphs*. arXiv: math.SP/06040006.

[17] E. Korotyaev, I. Lobanov, *Zigzag periodic nanotube in magnetic field*, arXiv: math.SP/06040007.

[18] P. Kuchment, *On some spectral problems of mathematical physics*. In: Partial Differential Equations and Inverse Problems, C. Conca, R. Manasevich, G. Uhlmann, M. S. Vogelius (Editors), Contemp. Math. **362**, Amer. Math. Soc., Providence, R.I., 2004.

[19] P. Kuchment (Editor), *Quantum graphs and their applications*. Special issue of Waves in Random Media **14**(2004), no. 1.

[20] P. Kuchment, *Quantum graphs I. Some basic structure*. Waves in Random Media **14**(2004), 107 – 128.

[21] P. Kuchment, *Quantum graphs II. Some spectral properties of quantum and combinatorial graphs*. J. Phys. A **38**(2005), 22, 4887 – 4900.

[22] P. Kuchment, O. Post, *On the spectra of carbon nano-structure*. arXiv: math-ph/0612021v4, 19 Jan 2007, 23 p.

[23] P. Kuchment, B. Vainberg, *On the structure of eigenfunctions corresponding to embedded eigenvalues of locally perturbed periodic graph operators*. Comm. Math. Phys. **268**(2006), 673 – 686.

[24] S. N. Lakaev, Z. I. Muminov, *The asymptotics of the number of eigenvalues of a three-particle lattice Schr ödinger operator*. Func. Anal. Appl. **37**(2003), 3, 228 – 231.

[25] Y. Last, B. Simon, *The essential spectrum of Schrödinger, Jacobi, and CMV operators*. Preprint **304** at http://www.math.caltech.edu/people/biblio.html.

[26] M. Măntoiu, *C^*-algebras, dynamical systems at infinity and the essential spectrum of generalized Schrödinger operators*. J. Reine Angew. Math. **550**(2002), 211 – 229.

[27] D. C. Mattis, *The few-body problem on a lattice*. Rev. Modern Phys. **58**(1986), 361 – 379.

[28] A. Mogilner, *Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: Problems and results*. Adv. Soviet Math. **5**, Amer. Math. Soc., Providence, R.I.

[29] P. S. Novikov, S. I. Adian, *On infinite periodic groups I, II, III*. Izv. Akad. Nauk SSSR, Ser. Mat. **32**(1968), 212 – 244, 251 – 524, 709 – 731 (Russian).

[30] L. Pauling, *The diamagnetic anisotropy of aromatic moleculas*, J. Chem. Phys. **4**(1936), 673 – 677.

[31] V. S. Rabinovich, *Essential spectrum of perturbed pseudodifferential operators. Applications to the Schrödinger, Klein-Gordon, and Dirac operators*. Russian J. Math. Phys. **12**(2005), 1, 62 – 80.

[32] V. S. Rabinovich, S. Roch, *Pseudodifference operators on weighted spaces and applications to discrete Schrödinger operators*. Acta Appl. Math. **84**(2004), 55 – 96.

[33] V. S. Rabinovich, S. Roch, *The essential spectrum of Schrödinger operators on lattices*. J. Phys. A: Math. Gen. **39**(2006), 8377 – 8394.

[34] V. S. Rabinovich, S. Roch, *Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs*. Preprint 2507 TU Darmstadt, April 2007, 26 pages (submitted to J. Phys. A: Math. Gen.)

[35] V. S. Rabinovich, S. Roch, J. Roe, *Fredholm indices of band-dominated operators*. Integral Eq. Oper. Theory **49**(2004), 2, 221 – 238.

[36] V. S. Rabinovich, S. Roch, B. Silbermann, *Fredholm theory and finite section method for band-dominated operators*. Integral Equations Oper. Theory **30**(1998), 452 – 495.

[37] V. S. Rabinovich, S. Roch, B. Silbermann, *Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite sections*. Integral Eq. Oper. Theory **40**(2001), 3, 342 – 381.

[38] V. S. Rabinovich, S. Roch, B. Silbermann, *Limit Operators and their Applications in Operator Theory*, Oper. Theory: Adv. Appl. **150**, Birkhäuser, Basel 2004.

[39] S. Roch, *Band-dominated operators on l^p-spaces: Fredholm indices and finite sections*. Acta Sci. Math. (Szeged) **70**(2004), 783 – 797.

[40] J. Roe, *Lectures on Coarse Geometry*. Univ. Lecture Ser. **31**, Amer. Math. Soc., Providence, R. I., 2003.

[41] J. Roe, *Band-dominated Fredholm operators on discrete groups*. Integral Eq. Oper. Theory **51**(2005), 3, 411 – 416.

[42] K. Ruedenberg, C. W. Scherr, *Free-electron network model for conjugated systems I. Theory*. J. Chem. Phys. **21**(1953), 9, 1565 – 1581.

[43] R. Saito, G. Dresselhaus, M. S. Dresselhaus, *Physical Properties of Carbon Nanotubes*. Imperial College Press, London 1998.

[44] M. A. Shubin, *Pseudodifference operators and their inversion*. Dokl. Mat. Nauk SSSR **276**(1984), 3, 567 – 570 (Russian, Engl. transl. Soviet Math. Dokl. **29**(1984), 3, 574 – 577).

[45] M. A. Shubin, *Pseudodifference operators and their Green functions*. Izv. Akad. Nauk SSSR, Ser. Mat. **49**(1985), 3, 625 – 671 (Russian, Engl. transl. Math USSR Izvestiya **26**(1986), 3, 605 – 622).

[46] G. Yu, *The coarse Connes-Baum conjecture for spaces which admit a uniform embedding into Hilbert space*. Inventiones Mathematicae **139**(2000), 201 – 240.