Solutions to the Equation $\operatorname{div} u = f$ in Weighted Sobolev Spaces

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Abstract

We consider the problem $\operatorname{div} u = f$ in a bounded Lipschitz domain Ω, where f with $\int_\Omega f = 0$ is given. It is shown that the solution u, that is constructed as in Bogowski's approach in [1] fulfills estimates in the weighted Sobolev spaces $W^{k,q}_w(\Omega)$, where the weight function w is contained in the class of Muckenhoupt weights A_q.

1 Introduction and Main Results

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. We consider a given function f with $\int_\Omega f = 0$ and we are looking for solutions u to the divergence equation

$$\operatorname{div} u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u|_{\partial\Omega} = 0. \tag{1.1}$$

It is an immediate consequence of Green's formula that the condition that f has mean value 0 is necessary for the existence of a solution u to (1.1).

This problem has been studied by Bogowski [1], v. Wahl [12], Galdi [8] and Sohr [10], they prove existence and estimates of a solution u to (1.1) in the framework of classical L^p- and Sobolev spaces.

We investigate this problem in weighted function spaces. More precisely, we consider weighted Lebesgue spaces $L^q_w(\Omega)$ and Sobolev spaces $W^{k,q}_w(\Omega)$ which means that we integrate with respect to the measure $w\,dx$ for an appropriate weight function w.

All weight functions, that we use are contained in the Muckenhoupt class A_q. This the class of nonnegative and locally integrable weight functions, for which the expression

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}}\,dx \right)^{q-1}$$

is finite, where the supremum is taken over all cubes in $\mathbb{R}^n$.

As shown in [4] examples of Muckenhoupt weights are $w(x) = (1+|x|)^\alpha$, with $-n < \alpha < n(q-1)$ or $\operatorname{dist}(x, M)^\alpha$, $-(n-k) < \alpha < (n-k)(q-1)$, where M is a compact k-dimensional Lipschitzian manifold. Thus such weight functions can be used for a better description of the solution close to the boundary, in a neighborhood of a point or for $|x| \to \infty$.

One reason why the class of Muckenhoupt weights is appropriate for analysis is that the maximal operator is continuous in weighted L^q-spaces, if and only if the weight function is a Muckenhoupt weight. Thus the powerful tools of harmonic analysis may be applied, cf. García-Cuerva and Rubio de Francia [9] and Stein [11].

In this paper we follow Bogowski's approach in [1] and [8] using an explicit construction in star shaped domains and a decomposition of Lipschitz domains into starshaped domains. The solution is represented by a non translation-invariant singular integral operator. Before showing its continuity by the help of Theorem 3.1 below, it is necessary to modify the kernel of this operator in a way which does effect the solution depending on the compactly supported right hand side f.

Denoting by $W^{k,q}_{w,0}(\Omega)$ the closure of all smooth and compactly supported functions in the norm of the Sobolev space $W^{k,q}_w(\Omega)$ our main result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and locally lipschitzian domain. Assume $f \in W^{k,q}_{w,0}(\Omega)$ such that $\int f = 0$. Then there exists a function $u \in W^{k+1,q}_{w,0}(\Omega)$ such that*

$$\operatorname{div} u = f \quad \text{and} \quad \|u\|_{k+1,q,w} \leq c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k) > 0$ depends A_q-consistently on w. Moreover, u can be chosen such that it depends linearly on f and such that $u \in C^\infty_0(\Omega)$ if $f \in C^\infty_0(\Omega)$.

2 Weighted Function Spaces

In this section we collect the basic definitions of weight functions and function spaces, which are needed in this text. Moreover, we quote the main theorem about the boundedness of maximal operator.

Definition 2.1. Let A_q, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all $0 \leq w \in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{q-1}}dx\right)^{q-1} < \infty. \tag{2.1}$$

The supremum is taken over all cubes in $\mathbb{R}^n$. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

A constant $C = C(w)$ is called A_q-consistent if for every $c_0 > 0$ it can be chosen uniformly for all $w \in A_q$ with $A_q(w) < c_0$.

The A_q-consistence is of great importance since it is needed for the application of the Extrapolation Theorem [9, IV Lemma 5.18]. In particular this is used when showing the continuity of operator-valued Fourier multipliers and the maximal regularity of an operator; see e.g. [7] for details and applications.

We introduce some function spaces. First by $C^\infty_0(\Omega)$ we denote the space of smooth functions with compact support in Ω.

For $1 < q < \infty$, $w \in A_q$ and an open set Ω we define the weighted Lebesgue and Sobolev spaces as follows.

- The weighted Lebesgue spaces $L^q_w(\Omega)$ are given by

$$L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\,dx\right)^{\frac{1}{q}} < \infty \right\}.$$

- Assume in addition $k \in \mathbb{N}_0$, the set of nonnegative integers. The weighted Sobolev spaces are defined by

$$W^{k,q}_w(\Omega) = \left\{ u \in L^q_w(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{q,w} < \infty \right\}.$$

- Finally, we set
$$W^{k,q}_{w,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,q,w}}.$$

By [5], [7] and [3] the spaces $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$ and $W^{k,q}_{w,0}(\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$ or $C_0^\infty(\Omega)$, respectively, are dense.

For a locally integrable function f we define the maximal operator M by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(0)|} \int_{|y|\leq r} |f(x-y)|dy.$$

One has the following close connection between the Muckenhoupt class A_q and the maximal operator.

Theorem 2.2. *Let $1 < q < \infty$ and $w \in A_q$. Then the maximal operator M is continuous on $L^q_w(\mathbb{R}^n)$. More precisely, there exists an A_q-consistent constant c such that*

$$\|Mf\|_{q,w} \leq c\|f\|_{q,w} \quad \textit{for every} \;\; f \in L^q_w(\mathbb{R}^n).$$

Vice versa if μ is a nonnegative Borel measure and M is bounded on $L^q(\mathbb{R}^n, \mu)$, then μ is absolutely continuous and $d\mu = w\,dx$ for some $w \in A_q$.

Proof. See [9], Theorems 2.1 and 2.9. For the A_q-consistence of the constants one has to re-read the proof of [9], Theorem 2.9. The reverse inclusion can be found in [11, 2.2]. □

By [6] the following weighted analogue of the Poincaré inequality holds: there exists an A_q-consistent constant $c = c(q, w) > 0$ such that

$$\|u\|_{q,w} \leq c\|\nabla u\|_{q,w} \;\; \text{for every} \;\; u \in W^{1,q}_{w,0}(\Omega). \tag{2.2}$$

3 Proof of Theorem 1.1

Throughout this section let $1 < q < \infty$ and $w \in A_q$.

The proof follows the same lines as the unweighted case [1], [8, Chapter III.3]. It uses non-translation-invariant singular integral operators. Thus we apply the following theorem proved in [11, V.6.13] which ensures the continuity of a certain class of such operators.

Theorem 3.1. *Let T be a bounded operator from $L^2(\mathbb{R}^n)$ into itself that is associated to a kernel K in the sense that*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

for all compactly supported $f \in L^2(\mathbb{R}^n)$ and all x outside the support of f. Suppose that for some $\gamma > 0$ and some $A > 0$ the kernel K satisfies the inequalities

$$|K(x, y)| \leq A|x-y|^{-n} \tag{3.1}$$

and

$$|K(x, y) - K(x', y)| \leq A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \quad \textit{if } |x-x'| \leq \frac{1}{2}|x-y| \tag{3.2}$$

as well as the symmetric version of the second inequality in which the roles of x and y are interchanged. Writing

$$(T_\varepsilon f)(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)dy \quad \textit{and} \quad (T_* f)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)|,$$

we have that

$$\int [(T_* f)(x)]^q w(x)dx \leq c \int [(Mf)(x)]^q w(x)dx, \tag{3.3}$$

where f is bounded and has compact support, $w \in A_q$, and $1 < q < \infty$. The constant c depends A_q-consistently on w.

Proof. This Theorem is stated in [11, V.6.13]. The A_q-consistence of the constants is not explicitly mentioned there, however, it is established with the same arguments if one rereads the proof of Proposition 6 in [11, V.4.4]. □

Since the maximal operator $M : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$ is bounded, the inequality (3.3) guarantees that the sublinear operator T_* can be extended to a continuous sublinear operator $T_* : L^q_w(\Omega) \to L^q_w(\Omega)$.

However, to make use of the above theorem we have to modify the singular integral operator which appears in the proof of Lemma 3.2 below outside the bounded set Ω such that it possesses the properties assumed in Theorem 3.1.

In the proof of the following Lemma the occurring integral operators have to be understood in the Cauchy principle value sense $\lim_{\varepsilon\to 0} T_\varepsilon f$.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be bounded and star-shaped with respect to every point of some closed ball $\overline{B}$ with $\overline{B} \subset \Omega$.*

Then for every $f \in W^{k,q}_{w,0}(\Omega)$ with $\int_\Omega f = 0$ there exists a $v \in W^{k+1,q}_{w,0}(\Omega)$ with

$$\operatorname{div} v = f \quad \textit{and} \quad \|v\|_{k+1,q,w} \leq c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k) > 0$ depends A_q-consistently on w. The function v depends linearly on f and $f \in C^\infty_0(\Omega)$ implies $v \in C^\infty_0(\Omega)$.

Proof. Without loss of generality we may assume, using a coordinate transformation, that $B = B_1(0)$.

First we assume that $f \in C^\infty_0(\Omega)$.

We choose $a \in C^\infty_0(B_1(0))$ such that $\int a = 1$ and define

$$v(x) := \int_\Omega f(y)(x-y) \left(\int_1^\infty a\left(y + \xi(x-y)\right) \xi^{n-1} d\xi \right) dy. \tag{3.4}$$

In the proof of [8, Lemma III.3.1] it is shown that $v \in C^\infty_0(\Omega)$ and $\operatorname{div} v = f$.

It thus remains to prove the weighted estimates. To do this we use the following representation of $\partial_j v$ also shown in the proof of [8, Lemma III.3.1]:

$$\begin{aligned} \partial_j v_i(x) &= \int_\Omega K_{i,j}(x, x-y) f(y)dy + f(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} a(y)dy \\ &=: F_1(x) + F_2(x), \end{aligned} \tag{3.5}$$

where

$$
\begin{aligned}
K_{i,j}(x,x-y) = & \frac{\delta_{i,j}}{|x-y|^n} \int_0^\infty a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}dr \\
& + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n}dr,
\end{aligned}
\tag{3.6}
$$

for every $x, y \in \mathbb{R}^n$. To show the continuity of the integral operator $f \mapsto F_1$ its kernel must be modified. Set

$$
E := \left\{ z \in \Omega \mid z = \lambda z_1 + (1-\lambda) z_2,\ z_1 \in \operatorname{supp} f,\ z_2 \in \overline{B_1(0)},\ \lambda \in [0,1] \right\}.
$$

Since Ω is star-shaped with respect to $\overline{B_1(0)}$, E is a compact subset of Ω. For $x \notin E$ and $y \in \operatorname{supp} f$ we have

$$
x + r\frac{x-y}{|x-y|} \notin \overline{B} \quad \text{for all } r > 0,
$$

which means $K_{i,j}(x, x-y) = 0$. Thus, if we choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ on Ω and $\operatorname{supp} \psi \subset B_R(0)$ for some $R > 0$, and set $\varphi(x,y) = \psi(x)\psi(y)$ we obtain

$$
f(y) K_{i,j}(x, x-y) = f(y)\varphi(x,y) K_{i,j}(x,x-y) =: f(y)\tilde{K}_{i,j}(x,x-y),
$$

for $x, y \in \mathbb{R}^n$, if f is assumed to be extended by 0 to $\mathbb{R}^n$. Moreover, for $x \in B_R(0)$ we have

$$
r > R+1 \ \Rightarrow\ \left| x + r\frac{x-y}{|x-y|} \right| \geq r - |x| > 1 \ \Rightarrow\ a\left(x + r\frac{x-y}{|x-y|}\right) = 0.
$$

Thus for $x \in \Omega$ one has

$$
\begin{aligned}
\int_{\mathbb{R}^n} f(y) K_{i,j}(x,x-y)dy &= \int_{\mathbb{R}^n} f(y)\tilde{K}_{i,j}(x,x-y)dy \\
&= \int_{\mathbb{R}^n} f(y)\varphi(x,y) \left[\frac{\delta_{i,j}}{|x-y|^n} \int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}dr \right. \\
&\quad \left. + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n}dr \right] dy.
\end{aligned}
$$

Now we have to prove that $\tilde{K}_{i,j}$ satisfies the assumptions of Theorem 3.1. By the Calderón-Zygmund Theorem [2] we find that

$$
f \mapsto \int_{\mathbb{R}^n} \psi(x) K_{i,j}(x,x-y) f(y) dy : \quad L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)
$$

is continuous. Since the multiplication M_ψ with the C_0^∞-function ψ is a continuous operator on $L^2(\mathbb{R}^n)$ we obtain the continuity of

$$
\begin{aligned}
f \mapsto & \int_{\mathbb{R}^n} \tilde{K}_{i,j}(x,x-y) f(y) dy \\
& = \int_{\mathbb{R}^n} \psi(x) K_{i,j}(x,x-y) M_\psi f(y) dy : \ L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).
\end{aligned}
$$

It remains to prove the estimates (3.1) and (3.2). For (3.1) we may assume $|x|, |y| < R$. One has

$$
\begin{aligned}
&|x-y|^n |\tilde{K}_{i,j}(x, x-y)| \\
&= \Bigg| \varphi(x,y)\delta_{i,j} \int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1} dr \\
&\quad + \varphi(x,y)\frac{x_i - y_i}{|x-y|} \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^n dr \Bigg| \\
&\le c\left(\int_0^{R+1} (2R+r)^{n-1} dr + \int_0^{R+1} (2R+r)^n dr \right) = c'.
\end{aligned}
$$

To prove (3.2) we take $x, x', y \in \mathbb{R}^n$ with $|x - x'| \le \frac{1}{2}|x-y|$. If $(x,y), (x',y) \notin \operatorname{supp}\varphi$ nothing is to prove. Thus, without loss of generality we may assume that $y \le R$ and $x \le 3R$, since if $y \le R$ and $x \ge 3R$ then

$$
|x'| \ge |x| - |x - x'| \ge |x| - \frac{1}{2}(|x| + |y|) \ge \frac{3}{2}R - \frac{1}{2}R = R.
$$

Then using the triangle inequality together with the fact that a, φ and $(|x-y|+r)^n$ are Lipschitz continuous on compact sets we can estimate

$$
\begin{aligned}
&\Bigg| \frac{x_i - y_i}{|x-y|^{n+1}} \varphi(x,y) \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^n dr \\
&- \frac{x'_i - y_i}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \partial_j a\left(x' + r\frac{x'-y}{|x'-y|}\right)(|x'-y|+r)^n dr \Bigg| \\
&\le \left| \left(\frac{x_i - y_i}{|x-y|^{n+1}} - \frac{x'_i - y_i}{|x'-y|^{n+1}} \right) \varphi(x,y) \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^n dr \right| \\
&+ \left| \frac{x'_i - y_i}{|x'-y|^{n+1}} (\varphi(x,y) - \varphi(x',y)) \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^n dr \right| \\
&+ \frac{|x'_i - y_i|}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \left| \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) - \partial_j a\left(x' + r\frac{x'-y}{|x'-y|}\right) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (|x-y|+r)^n dr \\
&+ \frac{|x'_i - y_i|}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \left| \partial_j a\left(x' + r\frac{x'-y}{|x'-y|}\right) \right| |(|x-y|+r)^n - (|x'-y|+r)^n| dr \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Using the Lipschitz continuity of $\partial_i a$ and $|x-y| \le 4R$ we obtain

$$
I_3 \le \frac{c}{|x-y|^n} \int_0^{R+1} L\left(|x-x'| + \frac{|x-x'|}{|x-y|}\right)(|x-y|+r)^n dr \le c\frac{|x-x'|}{|x-y|^{n+1}}.
$$

I_2 and I_4 can be estimated analogously. For I_1 we estimate

$$\begin{aligned}\left|\frac{x_i-y_i}{|x-y|^{n+1}}-\frac{x_i'-y_i}{|x'-y|^{n+1}}\right| &\leq \frac{|x_i-x_i'|}{|x-y|^{n+1}}+\left|\frac{1}{|x-y|^{n+1}}-\frac{1}{|x'-y|^{n+1}}\right||x_i'-y_i| \\ &\leq \frac{|x-x'|}{|x-y|^{n+1}}+\left|\frac{|x'-y|^{n+1}-|x-y|^{n+1}}{|x-y|^{n+1}|x'-y|^{n+1}}\right||x_i'-y_i| \\ &\leq \frac{|x-x'|}{|x-y|^{n+1}}+c\frac{\big||x'-y|-|x-y|\big|\cdot|x-y|^n}{|x-y|^{2n+2}}|x_i'-y_i| \\ &\leq c\frac{|x-x'|}{|x-y|^{n+1}},\end{aligned}$$

where we used that $|x'-y|\geq\frac{1}{2}|x-y|$. The estimate $\big||x'-y|^{n+1}-|x-y|^{n+1}\big|\leq c\big||x'-y|-|x-y|\big|\cdot|x-y|^n$ follows from an elementary induction with respect to n.

The first summand in (3.6) can treated in the same way. Moreover, interchanging the roles of x and y the same kind of estimates can be done.

Combining the above and using Theorem 3.1 we obtain

$$\|F_1\|_{q,w}\leq\|T^*f\|_{q,w}\leq c\|Mf\|_{q,w}\leq c\|f\|_{q,w}$$

where T^* is the operator given by Theorem 3.1 and associated to the kernel $\tilde{K}_{i,j}$. The function F_2 appearing in (3.5) is easily estimated since

$$\int_\Omega\frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}a(y)dy$$

is bounded. Thus using the Poincaré inequality (2.2) we obtain $\|v\|_{1,q,w}\leq c\|f\|_{q,w}$.

Now the general case with $f\in L^q_w(\Omega)$ follows easily, since we can approximate f by C_0^∞-functions (f_n) with $\int f_n=0$.

It remains to prove the estimate in the spaces $W^{k,q}_w(\Omega)$. Using Leibniz' formula one can show (see [8, Remark III.3.2])

$$\partial^\alpha v(x)=\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\int_\Omega N_\beta(x,y)\partial^{\alpha-\beta}f(y)dy,$$

where

$$N_\beta(x,y)=(x-y)\int_1^\infty\partial^\beta a(y+r(x-y))r^{n-1}dr.$$

Clearly $\partial^\beta a\in C_0^\infty(B_1(0))$. Hence the same proof as above yields

$$\|\partial^\alpha v\|_{1,q,w}\leq c\|f\|_{k,q,w}$$

for $f\in C_0^\infty(\Omega)$ and every α with $|\alpha|\leq k$. Approximating an arbitrary $f\in W^{k,q}_{w,0}(\Omega)$ with $\int f=0$ by C_0^∞-functions (f_n) with $\int f_n=0$ finishes the proof. □

The following Lemma is the weighted analogue to [8, Lemma III.3.4.]. Its proof works in exactly the same way as in the case of unweighted function spaces.

Lemma 3.3. *Let Ω be a bounded and locally lipschitzian domain.*

1. *There exist open sets $\Omega_1, ..., \Omega_m$ with $\Omega = \bigcup_{i=1}^m \Omega_i$ such that each Ω_i is star-shaped with respect to an open ball B_i with $\overline{B}_i \subset \Omega_i$.*

2. *For every $f \in C_0^\infty(\Omega)$ with $\int_\Omega = 0$ there exist $f_i \in C_0^\infty(\Omega_i)$, $i = 1, ..., m$, with $f = \sum_{i=1}^m f_i$, $\int f_i = 0$ and $\|f_i\|_{k,q,w} \le c\|f\|_{k,q,w}$ for every $k \in \mathbb{N}_0$ and $q \ge 1$ and an A_q-consistent constant $c = c(k, q, w, \Omega)$.*

Proof of Theorem 1.1. Let $f \in C_0^\infty(\Omega)$ with $\int f = 0$ and take Ω_i, f_i, $i = 1, ..., m$, as in Lemma 3.3. We denote by v_i the solution to $\operatorname{div} v_i = f_i$ given by Lemma 3.2. Then we have

$$\|v_i\|_{k+1,q,w} \le c\|f_i\|_{k,q,w} \le c\|f\|_{k,q,w}.$$

Then $v = \sum_{i=1}^m v_i$ solves $\operatorname{div} v = f$ with $\|v\|_{k+1,q,w} \le c\|f\|_{k,q,w}$. For arbitrary $f \in W_{w,0}^{k,q}(\Omega)$ with $\int f = 0$ use again approximations with C_0^∞-functions. □

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