Essential spectra of pseudodifferential operators and exponential decay of their solutions. Applications to Schrödinger operators.

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Abstract

The main aim of this paper is to study the relations between the location of the essential spectrum and the exponential decay of eigenfunctions of pseudodifferential operators on $L^p(\mathbb{R}^n)$ perturbed by singular potentials.

For a solution of this problem we apply the limit operators method. This method associates with each band-dominated operator A a family $op(A)$ of so-called limit operators which reflect the properties of A at infinity. Consider the compactification of $\mathbb{R}^n$ by the "infinitely distant" sphere S^{n-1}. Then the set $op(A)$ can be written as the union of its components $op_{\eta_\omega}(A)$ where ω runs through the points of S^{n-1} and where $op_{\eta_\omega}(A)$ collects all limit operators of A which reflect the properties of A one tends to infinity "in the direction of ω. Set $sp_{\eta_\omega} A := \cup_{A_h \in op_{\eta_\omega}(A)} sp\, A_h$.

We show that "the distance" of an eigenvalue $\lambda \notin sp_{ess} A$ to $sp_{\eta_\omega} A$ determines the exponential decay of the λ-eigenfunctions of A in the direction of ω. We apply these results to estimate the exponential decay of eigenfunctions of electromagnetic Schrödinger operators for a large class of electric potentials, in particular, for multiparticle Schrödinger operators and periodic Schrödinger operators perturbed by slowly oscillating at infinity potentials.

1 Introduction

The main aim of this paper is to study the relations between the location of the essential spectrum and the exponential decay of eigenfunctions of pseudodifferential operators perturbed by singular potentials. We are going to attack this problem by the limit operators method. This method was already used earlier to describe the location of essential spectra of perturbed pseudodifferential operators, which has found applications to electromagnetic Schrödinger operators, square-root Klein-Gordon operators and Dirac operators under quite general assumptions on the behavior of magnetic and electric potentials at infinity. By

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means of the limit operators method, also a simple and transparent proof of the celebrated Hunziker, van Winter, Zjislin (HWZ)-theorem for multi-particle Hamiltonians has been obtained. In [27, 29], the limit operators method was applied to study the location of the essential spectrum of discrete Schrödinger operators.

The basic idea of the limit operators method is as follows. For $1 < p < \infty$, we consider the Banach algebras $\mathcal{A}_p$ of all band-dominated operators on the Lebesgue spaces $L^p(\mathbb{R}^n)$ or $l^p(\mathbb{Z}^n)$. To each operator $A \in \mathcal{A}_p$, there is associated a family of so-called limit operators which are defined by sequences h tending to infinity. We denote this family by $op(A)$ and call it the operator spectrum of A. The results of [30, 31] (see also [32] for a comprehensive account) yield that

$$sp_{ess} A = \bigcup_{A_h \in op(A)} spA_h \tag{1}$$

for a large class of band-dominated operators on $\mathbb{R}^n$ or $\mathbb{Z}^n$. Since the limit operators of an operator are more simple objects than the operator itself, identity (1) provides an effective tool to study the essential spectra for large classes of operators. For instance, differential and pseudodifferential operators of order $m \in \mathbb{R}$ belong to the algebra $\mathcal{A}_p$ after multiplication by the operator $(I - \Delta)^{-m/2}$ of order reduction.

It should be noted that formulas similar to (1) have been obtained independently in [16] by means of admissible geometric methods. We also refer to the papers [13, 12, 20, 3] and the references therein where C^*-algebra techniques have been employed to study essential spectra of Schrödinger operators. The methods of [16, 13, 12, 20, 3] are applicable only for self-adjoint or normal operators acting on Hilbert spaces, whereas the limit operators approach allows one to consider non self-adjoint operators on L^p-type spaces, for example Schrödinger operators with complex potentials.

Let $\tilde{\mathbb{R}}^n$ be the compactification of $\mathbb{R}^n$ homeomorphic to the closed unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. Its boundary $\tilde{\mathbb{R}}^n \setminus \mathbb{R}^n$ is homeomorphic to the unit sphere S^{n-1}. We denote the point in $\tilde{\mathbb{R}}^n \setminus \mathbb{R}^n$ which corresponds to $\omega \in S^{n-1}$ by η_ω, and we write $op_{\eta_\omega}(A)$ for the set of all limit operators of A defined by sequences h tending to η_ω in the topology of the compactification. The set

$$sp_{\eta_\omega} A := \bigcup_{A_h \in op_{\eta_\omega}(A)} spA_h. \tag{2}$$

can be considered as the local essential spectrum of A at the point η_ω. Note that

$$sp_{ess} A = \bigcup_{\omega \in S^{n-1}} sp_{\eta_\omega} A.$$

In this paper we are going to show that the "distance" of an eigenvalue $\lambda \notin sp_{ess} A$ to the local essential spectrum $sp_{\eta_\omega} A$ determines the exponential decay of the λ-eigenfunction u_λ of A in the direction of η_ω. We apply this result to estimate the exponential decay of eigenfunctions for a large class of electromagnetic Schrödinger operators with electric potentials, in particular, for multiparticle Schrödinger operators and periodic Schrödinger operators perturbed by slowly oscillating potentials. We plan to examine further applications of the limit operators method to exponential decay estimates of eigenfunctions of other operators important in mathematical physics (including Dirac, Pauli, Klein-Gordon and Maxwell operators) in a forthcoming paper.

Exponential decay estimates are subject of an intensive research. We would like to mention the work of Agmon [1, 2] where estimates of eigenfunctions of second order elliptic operators have been obtained in terms of a special (Agmon)-metric. In contrast to Agmon's approach, we use pseudodifferential operators with analytical symbols and limit operators. Pseudodifferential operators with analytical symbols were employed earlier for exponential estimates of solutions of pseudodifferential equations in [18, 25, 27, 26, 24, 19] and for the tunnel effect in [22, 23, 24].

The paper is organized as follows. In Section 1 we collect some auxiliary material on the Banach algebra $\mathcal{A}_p(\mathbb{R}^n)$ of the band-dominated operators on $L^p(\mathbb{R}^n)$ and on a related Wiener algebra $W_p(\mathbb{R}^n)$, and we recall a criterion from [30, 31, 32] for operators in these algebras to be Fredholm on $L^p(\mathbb{R}^n)$. Then we apply this criterion to derive the equality (1) on which all further considerations are based. In Section 2 we employ this equality to study the essential spectrum of elliptic pseudodifferential operators in the class $OPS^m_{1,0}$ which are perturbed by measurable potentials. Section 3 is devoted to applications of limit operators to exponential estimates of solutions of pseudodifferential equations with analytical symbols perturbed by singular potentials. In Section 4 we consider the electromagnetic Schrödinger operator

$$(\mathcal{H}u)(x) := (i\partial_{x_j} - a_j(x))\rho^{jk}(x)(i\partial_{x_k} - a_k(x))u(x) + \Phi(x)u(x), \quad x \in \mathbb{R}^n,$$

as an unbounded operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ where $\mathbb{R}^n$ is equipped with a Riemann metric ρ_{jk} satisfying

$$\inf_{x\in\mathbb{R}^n,\,\omega\in S^{n-1}} \rho_{jk}(x)\omega^j\omega^k > 0. \tag{3}$$

Here we use the Einstein summation convention, and $\rho_{jk}(x)$ is the tensor inverse to $\rho^{jk}(x)$. We suppose that the functions ρ_{jk} and a_k are infinitely differentiable and bounded with all of their derivatives and that they are slowly oscillating at infinity. Under certain conditions on the electric potential, we prove that the limit operator of $\mathcal{H}$ defined by a sequence $g = (g_m)$ is unitarily equivalent to the operator

$$(\mathcal{H}_g u)(x) := -\rho_g^{jk}\partial_{x_j}\partial_{x_k}u(x) + \Phi_g(x)u(x),$$

where the numbers ρ_g^{jk} are independent of x and $\Phi_g(x)$ is the limit of the $\Phi(x + g_m)$ in the sense that

$$\text{s-lim}_{m\to\infty}\Phi(x + g_m)(1 - \Delta)^{-1} = \Phi_g(x)(1 - \Delta)^{-1}$$

with s-lim referring to the strong limit. Formula (1) then implies

$$sp_{ess}\mathcal{H} = \bigcup_{\mathcal{H}_g \in op(\mathcal{H})} sp\,\mathcal{H}_g. \tag{4}$$

Set

$$\mu_{\mathcal{H}}(\omega) := \inf_{\mathcal{H}_g \in op_{\eta_\omega}(\mathcal{H})} sp\,\mathcal{H}_g,$$

let g be a sequence tending to η_ω, and assume that $\rho_g^{jk} =: \rho_\omega^{jk}$ depends on ω only. The following theorem settles an exponential decay estimate for the λ-eigenfunctions u_λ of the operator $\mathcal{H}$.

Theorem 1 *Let the eigenfunction u_λ of $\mathcal{H}$ correspond to an eigenvalue*

$$\lambda < \hat{\mu}_{\mathcal{H}} := \inf_{\omega \in S^{n-1}} \mu_{\mathcal{H}}(\omega).$$

Then $e^{l(x/|x|)|x|}u_\lambda \in L^2(\mathbb{R}^n)$ where $l : S^{n-1} \to \mathbb{R}_+$ is an arbitrary smooth positive function such that

$$l(\omega) < \sqrt{\frac{\mu_{\mathcal{H}}(\omega) - \lambda}{(\rho_\omega \omega,\, \omega)}} \quad \textit{for all } \omega \in S^{n-1}. \tag{5}$$

Here, $(x,\, y) = \sum_{j=1}^n x_j y_j$ refers to the standard inner product on $\mathbb{R}^n$.

We apply Theorem 1 to estimate eigenfunctions associated with points in the discrete spectrum of multiparticle Schrödinger operators and of periodic Schrödinger operators perturbed by slowly oscillating potentials. In particular we will obtain an estimate of $\mu_{\mathcal{H}}(\omega)$ for a multiparticle Schrödinger operator $\mathcal{H}$, and then we apply Theorem 1 to get the exponential decay estimates for eigenfunctions of $\mathcal{H}$ corresponding to eigenvalues $\lambda < \hat{\mu}_{\mathcal{H}}$. Different forms of exponential decay estimates for eigenfunctions related with the discrete spectrum of multiparticle Schrödinger operators were obtained by Agmon [1], see also [9] and [10].

For periodic Schrödinger operators, it is well known that their spectrum has a band structure with at most countably many spectral gaps and that their discrete spectrum is empty (see, for instance, [33], XIII.16, and the excellent survey by Kuchment [14] on spectral theory of periodic operators). Consider perturbations $\mathcal{H} + \Phi I$ of a periodic Schrödinger operator $\mathcal{H}$ by potentials Φ for which $\Phi(I - \Delta)^{-1}$ is a compact operator in $L^2(\mathbb{R}^n)$. Perturbations of this kind do not influence essential spectra, but points of the discrete spectrum of $\mathcal{H} + \Phi I$ can emerge in spectral gaps of the operator $\mathcal{H}$. The structure of the discrete spectrum in spectral gaps of perturbed periodic operators is studied in [4, 5, 6].

Let the interval $(\alpha,\, \beta)$ be a spectral gap of the periodic Schrödinger operator $\mathcal{H}$ and $\lambda \in (\alpha,\, \beta)$ be an eigenvalue of $\mathcal{H} + \Phi I$. Then Theorem 1 implies the inclusion $e^{l|x|}u_\lambda \in L^2(\mathbb{R}^n)$ for every λ-eigenfunction u_λ of $\mathcal{H} + \Phi I$ where $l < \beta - \lambda$ is arbitrary positive number. Note that exponential estimates of eigenfunctions of periodic Schrödinger operators perturbed by decreasing potentials were considered in [21].

We also study perturbations $\mathcal{H}+\Phi I$ of the periodic operator $\mathcal{H}$ by bounded potentials Φ which are slowly oscillating. Let

$$m_\Phi(\omega) := \liminf_{x\to\eta_\omega} \Phi(x), \qquad M_\Phi(\omega) := \limsup_{x\to\eta_\omega} a(x),$$

$$m_\Phi := \inf_{\omega\in S^{n-1}} m_\Phi(\omega), \qquad M_\Phi := \inf_{\omega\in S^{n-1}} M_\Phi(\omega),$$

and let $[\gamma,\, \delta]$ be a spectral band of $\mathcal{H}$. Then formula (4) implies that $[\gamma + m_\Phi,\, \delta + M_\Phi]$ is a band of the *essential* spectrum of the operator $\mathcal{H}+\Phi I$. Hence, if the oscillation of the potential Φ at infinity is large enough, then the essential spectrum of $\mathcal{H}+\Phi I$ is the semi-axis $[\mu + m_\Phi,\, +\infty)$ where μ is the minimum of the spectrum of $\mathcal{H}$. If $M_\Phi - m_\Phi < \beta - \alpha$, then the perturbed operator $\mathcal{H}+\Phi I$ has a gap $(\alpha + M_\Phi,\, \beta + m_\Phi)$ in the *essential spectrum*. Moreover, if $\lambda \in (\alpha + M_\Phi,\, \beta + m_\Phi)$ is an eigenvalue of $\mathcal{H}+\Phi I$, then Theorem 1 implies that the associated λ-eigenfunction u_λ satisfies the inclusion $e^{l(x/|x|)|x|}u_\lambda \in L^2(\mathbb{R}^n)$ where l is a positive smooth function on the unit sphere S^{n-1} such that

$$l(\omega) < \beta - \lambda + m_\Phi(\omega)$$

for every $\omega \in S^{n-1}$.

2 Local invertibility at infinity and Fredholmness of band-dominated operators on $\mathbb{R}^n$

In what follows we thoroughly suppose that $p \in (1,\, \infty)$, and we use the following standard notations. For each Banach space X, let $\mathcal{L}(X)$ stand for the Banach algebra of all bounded linear operators on X and $\mathcal{K}(X)$ for the associated ideal of the compact operators. For $h \in \mathbb{R}^n$, consider the shift operator $(V_h u)(x) := u(x-h)$ which acts as an isometry on $L^p(\mathbb{R}^n)$. Further, let $C_b^u(\mathbb{R}^n)$ refer to the C^*-algebra of all bounded and uniformly continuous functions on $\mathbb{R}^n$, $C_0(\mathbb{R}^n)$ to its subalgebra consisting of all functions with compact support, $SO(\mathbb{R}^n)$ to the subalgebra of $C_b^u(\mathbb{R}^n)$ of all functions a which are slowly oscillating in the sense that

$$\lim_{x\to\infty} \sup_{y\in K} |a(x+y) - a(x)| = 0$$

for every compact $K \subset \mathbb{R}^n$, $C_b^\infty(\mathbb{R}^n)$ to the space of all infinitely differentiable functions which are bounded together with all of their derivatives, and finally $SO^\infty(\mathbb{R}^n)$ to the intersection $C_b^\infty(\mathbb{R}^n) \cap SO(\mathbb{R}^n)$.

Definition 2 *An operator $A \in \mathcal{L}(L^p(\mathbb{R}^n))$ is* band-dominated *if, for every function $\varphi \in C_b^u(\mathbb{R}^n)$,*

$$\lim_{t\to 0} \|\varphi_t A - A\varphi_t I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0$$

where $\varphi_t(x) := \varphi(t_1 x_1,\, \ldots,\, t_n x_n)$ for $t = (t_1,\, \ldots,\, t_n) \in \mathbb{R}^n$.

The set of all band-dominated operators is a Banach subalgebra of $\mathcal{L}(L^p(\mathbb{R}^n))$ which we denote by $\mathcal{A}_p(\mathbb{R}^n)$. This algebra is inverse closed in $\mathcal{L}(L^p(\mathbb{R}^n))$, and it contains the ideal $\mathcal{K}(L^p(\mathbb{R}^n))$.

Let $\mathcal{I}_p(\mathbb{R}^n)$ denote the set of all operators $A \in \mathcal{L}(L^p(\mathbb{R}^n))$ such that

$$\lim_{t\to 0} \|\varphi_t A\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = \lim_{t\to 0} \|A\varphi_t I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0$$

for every function $\varphi \in C_b^u(\mathbb{R}^n)$ with $\varphi(0) = 0$. One easily checks that $\mathcal{I}_p(\mathbb{R}^n) \subset \mathcal{A}_p(\mathbb{R}^n)$ and that $\mathcal{I}_p(\mathbb{R}^n)$ is a two sided ideal of $\mathcal{A}_p(\mathbb{R}^n)$.

For $R > 0$, let P_R denote the operator of multiplication by the characteristic function of the ball $\mathbb{B}_R := \{x \in \mathbb{R}^n : |x| < R\}$, which acts as a projection on $L^p(\mathbb{R}^n)$.

Definition 3 *Let $A \in \mathcal{L}(L^p(\mathbb{R}^n))$, and let $h = (h_m)$ be a sequence tending to infinity. The linear operator A_h is called a* limit operator *of A defined by the sequence h if, for every $R > 0$,*

$$\begin{aligned} &\lim_{m\to\infty} \|(A_h - V_{-h_m} A V_{h_m}) P_R\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \\ &\quad = \lim_{m\to\infty} \|P_R(A_h - V_{-h_m} A V_{h_m})\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0. \end{aligned} \tag{6}$$

Evidently, every operator A has at most one limit operator with respect to a given sequence h, which justifies the notation A_h. It is immediate from the definition that A_h is a bounded linear operator on $L^p(\mathbb{R}^n)$ and that $\|A_h\| \le \|A\|$. We denote the set of all limit operators of A by $op(A)$.

An operator $A \in \mathcal{L}(L^p(\mathbb{R}^n))$ is called *rich* if every sequence h tending to infinity has a subsequence g for which the limit operator A_g exists. The set of all rich operators in $\mathcal{A}_p(\mathbb{R}^n)$ is a closed subalgebra of $\mathcal{A}_p(\mathbb{R}^n)$ which we denote by $\mathcal{A}_p^\$(\mathbb{R}^n)$.

Theorem 4 *Let $A \in \mathcal{A}_p^\$(\mathbb{R}^n)$. Then the coset $A + \mathcal{I}_p(\mathbb{R}^n)$ is invertible in the quotient algebra $\mathcal{A}_p(\mathbb{R}^n)/\mathcal{I}_p(\mathbb{R}^n)$ if and only if all limit operators $A_h \in op(A)$ are invertible and if the norms of their inverses are uniformly bounded,*

$$\sup_{A_h \in op(A)} \|A_h^{-1}\| < \infty. \tag{7}$$

Note that the invertibility of the coset $A + \mathcal{I}_p(\mathbb{R}^n)$ in $\mathcal{A}_p(\mathbb{R}^n)/\mathcal{I}_p(\mathbb{R}^n)$ is equivalent to the existence of an operator $R \in \mathcal{A}_p(\mathbb{R}^n)$ such that

$$RA - I,\ AR - I \in \mathcal{I}_p(\mathbb{R}^n).$$

An operator $A \in \mathcal{A}_p(\mathbb{R}^n)$ which satisfies (one of) the equivalent conditions of Theorem 4 is said to be *locally invertible at infinity* on $L^p(\mathbb{R}^n)$.

Let χ_0 denote the characteristic function of the semi-open cube $(0, 1]^n$ and set $\chi_\alpha(x) := \chi_0(x - \alpha)$ for $\alpha \in \mathbb{Z}^n$.

Definition 5 *Let $W_p(\mathbb{R}^n)$ stand for the set of all operators $A \in \mathcal{L}(L^p(\mathbb{R}^n))$ with*

$$\begin{aligned} \|A\|_{W_p(\mathbb{R}^n)} &:= \sum_{\gamma \in \mathbb{Z}^n} \sup_{\alpha \in \mathbb{Z}^n} \|\chi_0 V_{-\alpha} A V_{\alpha-\gamma} \chi_0 I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \\ &= \sum_{\gamma \in \mathbb{Z}^n} \sup_{\alpha \in \mathbb{Z}^n} \|\chi_\alpha A \chi_{\alpha-\gamma} I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} < \infty. \end{aligned}$$

Provided with the operations inherited from $\mathcal{L}(L^p(\mathbb{R}^n))$ and with the above defined norm, the set $W_p(\mathbb{R}^n)$ becomes a Banach algebra, the so-called *Wiener algebra*. The importance of the Wiener algebra lies in the fact that the boundedness condition (7) in Theorem 4 is redundant for rich operators in $W_p(\mathbb{R}^n)$.

Theorem 6 *Let $A \in W_p^\$(\mathbb{R}^n) := W_p(\mathbb{R}^n) \cap \mathcal{A}_p^\$(\mathbb{R}^n)$. Then A is locally invertible at infinity on $L^p(\mathbb{R}^n)$ if and only if all limit operators of A are invertible on $L^p(\mathbb{R}^n)$.*

The local invertibility at infinity coincides with the common Fredholm property for operators which are locally compact.

Definition 7 *An operator $Q \in \mathcal{L}(L^p(\mathbb{R}^n))$ is* locally compact *if aQ and QaI are compact operators on $L^p(\mathbb{R}^n)$ for every function $a \in C_0(\mathbb{R}^n)$. We denote the set of all locally compact operators by $\mathcal{LC}_p(\mathbb{R}^n)$.*

Proposition 8 *$\mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$ is a closed ideal of $\mathcal{A}_p(\mathbb{R}^n)$.*

Proof. Let $Q \in \mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$, $A \in \mathcal{A}_p(\mathbb{R}^n)$, and $a \in C_0(\mathbb{R}^n)$. Then aQA is clearly a compact operator. We show that the operator $QAaI$ is compact, too. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function with $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$, and set $\varphi_R(x) := \varphi(x/R)$ for $R > 0$. Choose R_0 such that $\varphi_R a = a$ for all $R > R_0$. Since $A \in \mathcal{A}_p(\mathbb{R}^n)$,

$$QAaI = QA\varphi_R aI = Q\varphi_R AaI + T_R \quad \text{for } R > R_0$$

where the T_R are operators with $\lim_{R\to\infty} \|T_R\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0$. Since the operators $A\varphi_R I$ are compact for every $R > R_0$, the operator $QAaI$ is compact. ■

Note that the coset $A + (\mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n))$ is invertible in the quotient algebra $\mathcal{A}_p(\mathbb{R}^n)/(\mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n))$ if and only if there exist an operator $R \in \mathcal{A}_p(\mathbb{R}^n)$ and operators $Q_1,\, Q_2 \in \mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$ such that

$$RA = I + Q_1, \qquad AR = I + Q_2. \tag{8}$$

Theorem 9 *The operator $A \in \mathcal{A}_p(\mathbb{R}^n)$ is Fredholm on $L^p(\mathbb{R}^n)$ if and only if the following conditions hold:*

(i) A is invertible modulo the ideal $\mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$ of $\mathcal{A}_p(\mathbb{R}^n)$,

(ii) A is invertible modulo the ideal $\mathcal{I}_p(\mathbb{R}^n)$ of $\mathcal{A}_p(\mathbb{R}^n)$.

Proof. Let the conditions (i) and (ii) be fulfilled. Then there exist operators R', $R'' \in \mathcal{A}_p(\mathbb{R}^n)$ such that

$$R'A = I + Q_1', \qquad AR' = I + Q_2', \tag{9}$$

$$R''A = I + Q_1'', \qquad AR'' = I + Q_2'' \tag{10}$$

where $Q_j' \in \mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$ and $Q_j'' \in \mathcal{I}_p(\mathbb{R}^n)$. Set $R := R' + R'' - R'AR''$. Then

$$\begin{aligned} RA - I &= R'A + R''A - R'AR''A - I \\ &= (I - R''A)\,(R'A - I) = -Q_1''Q_1'. \end{aligned}$$

In the same way one gets

$$AR - I = -Q_2'Q_2''.$$

The operators $Q_1''Q_1'$ and $Q_2'Q_2''$ are compact. Indeed, since $Q_1'' \in \mathcal{I}_p(\mathbb{R}^n)$,

$$\lim_{R\to\infty} \|Q_1''Q_1' - \varphi_R Q_1''Q_1'\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0.$$

Moreover, since $Q_1'' \in \mathcal{A}_p(\mathbb{R}^n)$,

$$\lim_{R\to\infty} \|Q_1''Q_1' - Q_1''\varphi_R Q_1'\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0.$$

Since, finally, $Q_1' \in \mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$, the operator $Q_1''\varphi_R Q_1'$ is compact. Hence, $Q_1''Q_1'$ is a compact operator. Analogously, the compactness of $Q_2'Q_2''$ can be checked. Thus, A is a Fredholm operator, and R is (one of its) inverses modulo compact operators.

Conversely, let $A \in \mathcal{A}_p(\mathbb{R}^n)$ be a Fredholm operator. Then there are operators $R \in \mathcal{A}_p(\mathbb{R}^n)$ and T_1, $T_2 \in \mathcal{K}(L^p(\mathbb{R}^n))$ such that

$$RA = I + T_1, \qquad AR = I + T_2. \tag{11}$$

Because of $\mathcal{K}(L^p(\mathbb{R}^n)) \subset \mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$ and $\mathcal{K}(L^p(\mathbb{R}^n)) \subset \mathcal{I}_p(\mathbb{R}^n)$, the equalities (11) imply conditions (i) and (ii). ■

Theorems 6 and 9 have the following consequences.

Theorem 10 *Let $A \in W_p^\$(\mathbb{R}^n)$. Then A is a Fredholm operator on $L^p(\mathbb{R}^n)$ if and only if*

(i) *A is invertible modulo the ideal $\mathcal{LC}_p(\mathbb{R}^n) \cap \mathcal{A}_p(\mathbb{R}^n)$ of $\mathcal{A}_p(\mathbb{R}^n)$,*

(ii) *all limit operators of A are invertible.*

Theorem 11 *Let $A \in W_p^\$(\mathbb{R}^n)$, and let condition (i) of Theorem 10 be satisfied. Then*

$$sp_{ess} A = \bigcup_{A_h \in op(A)} sp\, A_h. \tag{12}$$

Let $\tilde{\mathbb{R}}^n$ denote the compactification of $\mathbb{R}^n$ homeomorphic to the unit ball $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| \le 1\}$, and let $\eta_\omega \in \tilde{\mathbb{R}}^n \setminus \mathbb{R}^n$ be the infinitely distant point which corresponds to the point $\omega \in S^{n-1}$. We denote by $op_{\eta_\omega}(A)$ the set of the limit operators of A which correspond to sequences h tending to η_ω in the topology of $\tilde{\mathbb{R}}^n$. Then, under the conditions of Theorem 11, equality (12) can be written as

$$sp_{ess} A = \bigcup_{\omega \in S^{n-1}} \bigcup_{A_h \in op_{\eta_\omega}(A)} sp\, A_h. \tag{13}$$

3 Applications to pseudodifferential operators

3.1 Basics

We start with recalling some facts about pseudodifferential operators. Standard references are [37, 34, 36]. We will use the following standard notations. The n-tuple $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N} \cup \{0\}$, is a multi-index, $|\alpha| := \alpha_1 + \ldots + \alpha_n$ is its length, $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$ is the operator of αth partial derivative, and $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

We say that a C^∞-function a on $\mathbb{R}^n \times \mathbb{R}^n$ is a symbol in the Hörmander class $S^m_{1,\,0}$ if

$$|a|_N := \sum_{|\alpha|+|\beta| \le N} \sup_{(x,\,\xi) \in \mathbb{R}^n \times \mathbb{R}^n} \left| \partial_x^\beta \partial_\xi^\alpha a(x,\, \xi) \right| \langle \xi \rangle^{-m+|\alpha|} < \infty$$

for every $N \in \mathbb{N} \cup \{0\}$. The class of all pseudodifferential operators

$$(Op(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,\, \xi)\, e^{i(x-y,\, \xi)} u(y)\, dy\, d\xi, \quad u \in C_0^\infty(\mathbb{R}^n),$$

with symbols in $S^m_{1,\,0}$ is denoted by $OPS^m_{1,\,0}$. For $m \in \mathbb{R}$, we use the notation $\langle D \rangle^m$ to refer to the pseudodifferential operator with symbol $a(x,\, \xi) = \langle \xi \rangle^m$.

It is well known (see, for instance, [36], Chap. VI) that pseudodifferential operators with symbols in $S^0_{1,\,0}$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\, \infty)$ and there are a constant C_p and a number $N \in \mathbb{N}$ such that

$$\|Op(a)\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \le C_p |a|_N. \tag{14}$$

We will also have to work with pseudodifferential operators with double symbols a on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ which satisfy the estimates

$$\left| \partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,\, y,\, \xi) \right| \le C_{\alpha\beta\gamma} \langle \xi \rangle^{m-|\alpha|}$$

for every choice of multi-indices $\alpha,\, \beta, \gamma$. We denote the class of these symbols by $S^m_{1,\,0,\,0}$ and write $OPS^m_{1,\,0,\,0}$ for the associated class of pseudodifferential operators with double symbols. The latter act on $C_0^\infty(\mathbb{R}^n)$ via

$$(Op_d(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,\, y,\, \xi)\, e^{i(x-y,\, \xi)} u(y)\, dy\, d\xi. \tag{15}$$

One knows that $OPS^m_{1,0,0} \subset OPS^m_{1,0}$. More precisely, every pseudodifferential operator $Op_d(a)$ with double symbol can be viewed as a common pseudodifferential operator, $Op(b)$, with

$$b(x,\, \xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,\, x+y,\, \xi+\eta)\, e^{-i(y,\,\eta)}\, dy\, d\eta \tag{16}$$

a symbol in $S^m_{1,0}$. The double integral in (16) is understood as an oscillatory integral.

An important example of a pseudodifferential operator with double symbol in $S^m_{1,0,0}$ is the Weyl quantization operator $Op_{\mathcal{W}}(a)$ with $a \in S^m_{1,0}$ which acts on $C_0^\infty(\mathbb{R}^n)$ by

$$(Op_{\mathcal{W}}(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\, \xi\right) e^{i(x-y,\,\xi)}\, u(y)\, dy\, d\xi.$$

Theorem 12 *$OPS^0_{1,0} \subset W_p(\mathbb{R}^n)$ for every $p \in (1,\, \infty)$.*

Proof. Let $A = Op(a) \in OPS^m_{1,0}$. Then A acts on $C_0^\infty(\mathbb{R}^n)$ by

$$(Au)(x) = \int_{\mathbb{R}^n} k_A(x,\, x-y)\, u(y)\, dy, \tag{17}$$

and the kernel function $k_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ satisfies the estimates

$$\left|\partial_x^\beta \partial_z^\alpha k_A(x,\, z)\right| \le C|a|_M |z|^{-n-m-|\alpha|-N} \tag{18}$$

for all multi-indices α and β and for all $N \ge 0$ such that $n+m+|\alpha|+N > 0$. The constant $C > 0$ in (18) is independent of A, whereas M depends on n, m, α, β and N ([36], p. 241). We set

$$\varkappa^\gamma(A) := \sup_{\alpha \in \mathbb{Z}^n} \|\chi_0 V_{-\alpha} A V_{\alpha-\gamma} \chi_0 I\|_{\mathcal{L}(L^p(\mathbb{R}^n))}.$$

Hence, for the operator A in (17),

$$\varkappa^0(A) \le \|A\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \le C|a|_M,$$

whereas for $\gamma \ne 0$,

$$\varkappa^\gamma(A) \le \left(\int_{I_0} \left(\int_{I_0} |k_A(x-y+\gamma)|^{p'}\, dy\right)^{\frac{p}{p'}} dx\right)^{1/p}$$

with $1/p + 1/p' = 1$. Applying estimate (18) with $\alpha = \beta = 0$, $m = 0$ and $N = 1$ we obtain

$$\varkappa^\gamma(A) \le C|a|_M |\gamma|^{-n-1} \quad \text{for } \gamma \ne 0.$$

Hence,

$$\sum_{\gamma \in \mathbb{Z}^n} \sup_{\alpha \in \mathbb{Z}^n} \|\chi_0 V_{-\alpha} A V_{\alpha-\gamma} \chi_0 I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \le \sum_{\gamma \in \mathbb{Z}^n} \varkappa^\gamma(A) \le C|a|_M$$

for some $M \in \mathbb{N}$. ■

Definition 13 *Let* $S^0_{1,0}(\tilde{\mathbb{R}}^n)$ *(resp.* $S^0_{1,0,0}(\tilde{\mathbb{R}}^n)$*) denote the class of all symbols in* $S^0_{1,0}$ *(resp. in* $S^0_{1,0,0}$*) which can be extended to a continuous function on* $\mathbb{R}^n \times \tilde{\mathbb{R}}^n$ *(resp. on* $\mathbb{R}^n \times \mathbb{R}^n \times \tilde{\mathbb{R}}^n$*). We further say that* a *belongs to* $S^m_{1,0}(\tilde{\mathbb{R}}^n)$ *(resp. to* $S^m_{1,0,0}(\tilde{\mathbb{R}}^n)$*) if the function* $a\langle\xi\rangle^{-m}$ *lies in* $S^0_{1,0}(\tilde{\mathbb{R}}^n)$ *(resp. in* $S^0_{1,0,0}(\tilde{\mathbb{R}}^n)$*).*

Proposition 14 $OPS^0_{1,0}(\tilde{\mathbb{R}}^n) \subseteq W^\$_p(\mathbb{R}^n)$.

Proof. Let $a \in S^0_{1,0}(\tilde{\mathbb{R}}^n)$, and let $h = (h_j)$ be a sequence tending to infinity. Then

$$V_{-h_j} Op(a) V_{h_j} = Op(a(x + h_j,\, \xi)).$$

For every compact subset K of $\mathbb{R}^n$, the sequence of the functions $a(x+h_j,\, \xi)$, $j \in \mathbb{N}$, is uniformly bounded and equicontinuous on the compact subset $K \times \tilde{\mathbb{R}}^n$ of $\mathbb{R}^n \times \tilde{\mathbb{R}}^n$. By the Arcela-Ascoli theorem, there are a subsequence $g = (g_j)$ of h and a function $a_g \in C^\infty(\mathbb{R}^n \times \tilde{\mathbb{R}}^n)$ such that

$$\lim_{j\to\infty} \sup_{K\times\mathbb{R}^n} \left|\partial_x^\beta \partial_\xi^\alpha a(x+g_j,\, \xi) - \partial_x^\beta \partial_\xi^\alpha a_g(x,\, \xi)\right| = 0 \tag{19}$$

for all multi-indices α, β. The limit (19) implies that $a_g \in S^0_{1,0}$ and that $Op(a_g)$ is the limit operator of $Op(a)$ with respect to the sequence g. Hence, $OPS^0_{1,0}(\tilde{\mathbb{R}}^n) \subseteq W^\$_p(\mathbb{R}^n)$ for every $p \in (1,\, \infty)$. ∎

Let $a \in S^0_{1,0,0}(\tilde{\mathbb{R}}^n)$ and let $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence tending to infinity. Then there exists a subsequence $g = (g_j)$ of h such that

$$\lim_{j\to\infty} \sup_{K_1\times K_2\times\mathbb{R}^n} \left|\partial_x^\beta \partial_\xi^\alpha a(x+g_j,\, y+g_j,\, \xi) - \partial_x^\beta \partial_\xi^\alpha a_g(x,\, y,\, \xi)\right| = 0 \tag{20}$$

for every compact set $K_1 \times K_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and all multi-indices α, β. One can prove as above that $Op_d(a_g)$ is the limit operator of $Op_d(a)$ with respect to the sequence g.

3.2 Essential spectra of elliptic pseudodifferential operators of zero order

A pseudodifferential operator $Op_d(a) \in OPS^m_{1,0,0}$ is said to be *uniformly elliptic* on $\mathbb{R}^n$ if

$$\lim_{R\to\infty} \inf_{(x,\,y)\in\mathbb{R}^n\times\mathbb{R}^n,\, |\xi|\geq R} |a(x,\, y,\, \xi)| > 0. \tag{21}$$

Theorem 15 *An operator* $Op_d(a) \in OPS^0_{1,0,0}$ *is Fredholm on* $L^p(\mathbb{R}^n)$ *if and only if it is*

(i) uniformly elliptic on $\mathbb{R}^n$*, and*

(ii) locally invertible at infinity.

Proof. Let the ellipticity condition (21) hold. It is easy to show that then there is a double symbol $b \in S^0_{1,0,0}$ such that

$$Op_d(b)Op_d(a) = I + Op_d(t_1), \qquad Op_d(a)Op_d(b) = I + Op_d(t_2)$$

where the operators $Op_d(t_1)$ and $Op_d(t_2)$ belong to $OPS^{-1}_{1,0,0}$ and are therefore locally compact on $L^p(\mathbb{R}^n)$. Hence, by Theorem 10, $Op_d(a)$ is a Fredholm operator on $L^p(\mathbb{R}^n)$.

Conversely, it is well known that the Fredholmness of an operator $Op_d(a)$ in $OPS^0_{1,\,0,\,0}$ implies its uniform ellipticity (21). Since $\mathcal{K}(L^p(\mathbb{R}^n)) \subset \mathcal{I}_p(\mathbb{R}^n)$, the Fredholmness of $Op_d(a)$ also implies the local invertibility at infinity of $Op_d(a)$. ■

The previous results imply the following.

Theorem 16 *Let $A = Op(a) \in OPS^0_{1,\,0,\,0}(\tilde{\mathbb{R}}^n)$ be a uniformly elliptic pseudodifferential operator. Then A is a Fredholm operator on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ if and only if all limit operators of A are invertible. Moreover,*

$$sp_{ess}(A : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)) = \bigcup_{A_h \in op(A)} sp\,(A_h : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)). \qquad (22)$$

3.3 Essential spectra of perturbed elliptic pseudodifferential operators of nonzero order

Here we consider a class of pseudodifferential operators of order $m \geq 0$ perturbed by a singular potential Φ, i.e., operators of the form $A = B + \Phi I$ where

(**A**) $B = Op_d(b) \in OPS^m_{1,\,0,\,0}$ and $b\langle\xi\rangle^{-m} \in S^0_{1,\,0,\,0}(\tilde{\mathbb{R}}^n)$;

(**B**) the operator $\Phi\langle D\rangle^{-m}$ is locally compact on $L^p(\mathbb{R}^n)$, and it belongs to $W^{\$}_p(\mathbb{R}^n)$.

We consider A as a bounded operator from $H^{m,\,p}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ where $H^{m,\,p}(\mathbb{R}^n)$ is the Sobolev space with norm

$$\|u\|_{H^{m,\,p}(\mathbb{R}^n)} := \|\langle D\rangle^m u\|_{L^p(\mathbb{R}^n)}.$$

Theorem 15 implies the following.

Theorem 17 *Let the operator $A = Op_d(b) + \Phi I$ satisfy conditions* (**A**) *and* (**B**). *Then A, considered as acting from $H^{m,\,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, is a Fredholm operator if and only if*

(i) the operator $Op_d(b)$ is uniformly elliptic on $\mathbb{R}^n$, and

(ii) all limit operators of the operator $A\langle D\rangle^{-m}$ are invertible on $L^p(\mathbb{R}^n)$.

Let A satisfy conditions (**A**) and (**B**), and let $m \geq 0$. Then A can be considered as an unbounded closed operator on $L^p(\mathbb{R}^n)$ with domain $H^{m,\,p}(\mathbb{R}^n)$. A point $\lambda \in \mathbb{C}$ is said to belong to the *essential spectrum* of A if the operator $A - \lambda I$ is not a Fredholm operator on $L^p(\mathbb{R}^n)$ in the sense of unbounded operators. We denote the spectrum and the essential spectrum of A acting on $L^p(\mathbb{R}^n)$ by $sp^p A$ and $sp^p_{ess} A$, respectively. Further we agree upon the following notations. Let $A = Op_d(b) + \Phi I$ be of order $m \geq 0$, and let $\tilde{A}_h$ be a limit operator of

$\tilde{A} := A\langle D\rangle^{-m}$ with respect to a sequence h. Then we call $A_h := \tilde{A}_h\langle D\rangle^m$ the *limit operator* of A defined by the sequence h. In general, A_h is an unbounded closed operator on $L^p(\mathbb{R}^n)$ with domain $H^{m,\,p}(\mathbb{R}^n)$. We write $op(A)$ for the set of all limit operators of A defined in this way. The following is a consequence of Theorem 17.

Theorem 18 *Let the operator $A = Op_d(b) + \Phi I$ satisfy conditions* (**A**) *and* (**B**), *and let the operator $Op_d(b)$ be uniformly elliptic on $\mathbb{R}^n$. Then*

$$sp^p_{ess} A = \bigcup_{A_h \in op(A)} sp^p\, A_h. \tag{23}$$

The next proposition states some sufficient conditions for a potential Φ to be subject of condition (**B**). In case $p = 2$, similar results have been obtained [8], p. 62 – 71.

Proposition 19 (*i*) *If $m > 0$ and $\Phi \in L^\infty(\mathbb{R}^n)$, then the operator $\Phi\langle D\rangle^{-m}$ belongs to $W_p(\mathbb{R}^n)$ and is locally compact.*

(*ii*) *If $m < n$ and $\Phi \in L^{n/m}(\mathbb{R}^n)$, then the operator $\Phi\langle D\rangle^{-m}$ belongs to $W_p(\mathbb{R}^n)$ and is compact on $L^p(\mathbb{R}^n)$.*

(*iii*) *If $m > n$ and $\Phi \in L^p(\mathbb{R}^n)$, then the operator $\Phi\langle D\rangle^{-m}$ belongs to $W_p(\mathbb{R}^n)$ and is compact on $L^p(\mathbb{R}^n)$.*

Proof. (*i*) One has

$$\varkappa^\gamma(\Phi\langle D\rangle^{-m}) \le \|\Phi\|_{L^\infty(\mathbb{R}^n)}\langle\gamma\rangle^{-n-1},$$

which implies $\Phi\langle D\rangle^{-m} \in W_p(\mathbb{R}^n)$. The local compactness of this operator follows since m is positive.

(*ii*) The operator $\langle D\rangle^{-m}$ acts boundedly from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ if $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$. Indeed, $\langle\xi\rangle^{-m}$ is an L_p-Fourier multiplier for every $p \in (1, \infty)$. Write

$$\langle\xi\rangle^{-m} = |\xi|^{-m}\left(|\xi|^m\langle\xi\rangle^{-m}\right).$$

The operator $|\xi|^m\langle\xi\rangle^{-m}$ is an L_p-Fourier multiplier, too, whereas $Op(|\xi|^{-m})$ is a Riesz potential which is bounded as an operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ by the Hardy-Littlewood-Sobolev theorem (see, for instance, [36], Chap. V).

The generalized Hölder inequality

$$\|uv\|_{L^s(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)}\|v\|_{L^q(\mathbb{R}^n)}$$

holding if $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ implies

$$\|\Phi\langle D\rangle^{-m}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \le C_p\,\|\Phi\|_{L^{n/m}(\mathbb{R}^n)}. \tag{24}$$

Further, for $x \in I_0$,

$$\chi_0 V_{-\alpha}\Phi\langle D\rangle^{-m}V_{\alpha-\gamma}\chi_0 v(x) = \Phi(x+\alpha)\int_{I_0} k_{\langle D\rangle^{-m}}(x-y+\gamma)\,v(y)\,dy.$$

In case $\gamma = 0$ we conclude from the equality $\|\Phi(\cdot + \alpha)\|_{L^{n/m}(\mathbb{R}^n)} = \|\Phi\|_{L^{n/m}(\mathbb{R}^n)}$ and from estimate (24) that

$$\|\chi_0 V_{-\alpha} \Phi \langle D \rangle^{-m} V_\alpha \chi_0 I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq \|\Phi\|_{L^{n/m}(\mathbb{R}^n)}.$$

Let now $\gamma \neq 0$. Then, for $u \in L^p(\mathbb{R}^n)$,

$$\|\chi_0 V_{-\alpha} \Phi \langle D \rangle^{-m} V_{\alpha-\gamma} \chi_0 u\|_{L^p(\mathbb{R}^n)} \leq \|\Phi\|_{L^{n/m}(\mathbb{R}^n)} \|\chi_0 V_{-\alpha} \langle D \rangle^{-m} V_{\alpha-\gamma} \chi_0 u\|_{L^q(\mathbb{R}^n)}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{m}{n}$. Further, with $\frac{1}{p} + \frac{1}{p'} = 1$, one has

$$\begin{aligned} &\|\chi_0 V_{-\alpha} \langle D \rangle^{-m} V_{\alpha-\gamma} \chi_0 u\|_{L^q(\mathbb{R}^n)} \\ &\quad \leq \left(\int_{I_0} \left(\int_{I_0} \left| k_{\langle D \rangle^{-m}}(x - y + \gamma) \right|^{p'} dy \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}} \|u\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Applying estimates (18) and (24) we obtain

$$\sup_{\alpha \in \mathbb{Z}^n} \|\chi_0 V_{-\alpha} \Phi \langle D \rangle^{-m} V_{\alpha-\gamma} \chi_0 I\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq C_0 \|\Phi\|_{L^{n/m}(\mathbb{R}^n)} \langle \gamma \rangle^{-n-1}.$$

Hence, $\Phi \langle D \rangle^{-m} \in W_p(\mathbb{R}^n)$.

In order to prove the compactness of $\Phi \langle D \rangle^{-m}$, choose a sequence of functions Φ_k in $C_0^\infty(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \|\Phi - \Phi_k\|_{L^{n/m}(\mathbb{R}^n)} = 0$. Employing estimate (24) once again we find

$$\lim_{k\to\infty} \|\Phi \langle D \rangle^{-m} - \Phi_k \langle D \rangle^{-m}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = 0.$$

Since the operators $\Phi_k \langle D \rangle^{-m}$ are compact on $L^p(\mathbb{R}^n)$, so is $\Phi \langle D \rangle^{-m}$.

(iii) Let $m > n$. Then the function $k_{\langle D \rangle^{-m}}$ is continuous at the point 0, and it satisfies estimate (18). We claim that then $\langle D \rangle^{-m}$ is a bounded operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For $u \in L^p(\mathbb{R}^n)$, let

$$v(x) := \int_{\mathbb{R}^n} k_{\langle D \rangle^{-m}}(x - y + \gamma)\, u(y)\, dy.$$

By the Hölder inequality,

$$|v(x)| \leq \|k_{\langle D \rangle^{-m}}\|_{L^{p'}(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}.$$

Thus, if $\Phi \in L^p(\mathbb{R}^n)$, then

$$\|\Phi \langle D \rangle^{-m}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq C_p \|\Phi\|_{L^p(\mathbb{R}^n)},$$

verifying the claim. The compactness of $\Phi \langle D \rangle^{-m}$ on $L^p(\mathbb{R}^n)$ for $p < \infty$ can be shown as in (ii). ∎

4 Exponential estimates for perturbed pseudodifferential equations

4.1 Weights

Let w be a positive measurable function on $\mathbb{R}^n$ which we call a weight. We denote by $L^p(\mathbb{R}^n, w)$ the space of all measurable functions u on $\mathbb{R}^n$ for which

$$\|u\|_{L^p(\mathbb{R}^n,\, w)} := \|wu\|_{L^p(\mathbb{R}^n)} < \infty.$$

Let $\mathcal{D}$ be a convex domain in $\mathbb{R}^n$ which contains the point $0 \in \mathbb{R}^n$. In what follows we consider weights of the form $w(x) := \exp v(x)$ where v is a function satisfying $\partial_{x_j} v \in C_b^\infty(\mathbb{R}^n)$ for $j = 1, \ldots, n$ and $\nabla v(x) \in \mathcal{D}$ for every $x \in \mathbb{R}^n$. We denote the class of these weights by $\mathcal{R}(\mathcal{D})$, and we write $\mathcal{R}_{sl}(\mathcal{D})$ for the class of all weights in $\mathcal{R}(\mathcal{D})$ such that

$$\lim_{x\to\infty} \partial^2_{x_i x_j} v(x) = 0 \tag{25}$$

for $1 \le i, j \le n$. The weights in $\mathcal{R}_{sl}(\mathcal{D})$ are called *slowly oscillating*.

Let $l : S^{n-1} \to \mathbb{R}$ be a positive C^∞-function. We associate with l the weight

$$w_l(x) := \exp v_l(x) \quad \text{where} \quad v_l(x) := l(x/|x|)\,|x| \tag{26}$$

and consider the open star-like domain

$$\Omega_l := \{x \in \mathbb{R}^n : x = tl(\omega)\omega,\ t \in [0, 1),\ \omega \in S^{n-1}\}.$$

One can show that

$$v_l(x) = \max_{y\in\overline{\Omega_l}} (x, y) \quad \text{for all } x \in \mathbb{R}^n$$

where $(x, y) = \sum_{j=1}^n x_j y_j$ is a standard scalar product on $\mathbb{R}^n$.

Proposition 20 (i) *The function v_l is positively homogeneous, that is $v_l(tx) = tv_l(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Moreover, $v_l \in C^\infty(\mathbb{R}^n \setminus \{0\})$.*
(ii) *$\nabla v_l(x) \in \overline{\Omega_l}$ for every $x \in \mathbb{R}^n$ and $\nabla v_l(\omega) = l(\omega)\omega \in \partial\Omega_l$ for every point $\omega \in S^{n-1}$.*

Proof. Statement (i) is evident. To prove (ii), fix $x \in \mathbb{R}^n$. The continuous function $y \mapsto (x, y)$ attains its maximum over $\overline{\Omega_l}$ at some point $\xi(x) \in \overline{\Omega_l}$. Thus, $v_l(x) = (x, \xi(x))$. In particular, $\xi(x)$ is a stationary point of the function $y \mapsto (x, y)$ on $\overline{\Omega_l}$, that is, $\frac{\partial(x,\xi(x))}{\partial y_i} = 0$ for $i = 1, \ldots, n$, whence

$$\frac{\partial v_l(x)}{\partial x_j} = \xi_j(x) + \sum_{i=1}^n \frac{\partial(x,\, \xi(x))}{\partial y_i} \frac{\partial \xi_i(x)}{\partial x_j} = \xi_j(x).$$

This shows that $\nabla v_l(x) = \xi(x) \in \overline{\Omega_l}$ for every $x \in \mathbb{R}^n$. It is further evident that ξ is a homogeneous function of order zero. Hence, $\nabla v_l(\omega) = \xi(\omega) = l(\omega)\omega$. ∎

Let $\tilde{v}_l$ be a C^∞-function on $\mathbb{R}^n$ which coincides with v_l outside a certain neighborhood of the origin. Then the weight $\tilde{w}_l := \exp \tilde{v}_l$ belongs to the class $\mathcal{R}_{sl}(\mathcal{D})$. Moreover,

$$\lim_{x \to \eta_\omega} \nabla \tilde{v}_{\Omega_l}(x) = \nabla v_{\Omega_l}(\omega) = l(\omega)\omega. \tag{27}$$

4.2 ΨDO with analytical symbols

Let $\mathcal{D} \subset \mathbb{R}^n$ be a convex domain, and abbreviate $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. We say that a symbol a belongs to $S^m_{1,0,0}(\mathcal{D})$ if the function $\xi \mapsto a(x, y, \xi)$ extends analytically into the tube domain $\mathbb{R}^n + i\mathcal{D}$ and if

$$|\partial_x^\beta \partial_y^\beta \partial_\xi^\alpha a(x, y, \xi + i\eta)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}$$

for all triples $(x, y, \xi + i\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^n + i\mathcal{D})$. For proofs of the following propositions see Section 4.5 in [32] and page 308 in [11], respectively.

Proposition 21 *Let a be a symbol in $S^m_{1,0,0}(\mathcal{D})$ and w a weight in $\mathcal{R}(\mathcal{D})$. Then*

$$w^{-1} Op(a) wI = Op_d(a(x, y, \xi + i\theta_w(x, y))) \tag{28}$$

is a pseudodifferential operator in $OPS^m_{1,0,0}$, and

$$\theta_w(x, y) = \int_0^1 (\nabla v)((1-t)x + ty)\, dt.$$

Proposition 22 *Let Banach spaces X_1 and Y_1 be densely embedded into Banach spaces X_2 and Y_2, respectively. Further let $A : X_2 \to Y_2$ and $A|_{X_1} : X_1 \to Y_1$ be Fredholm operators with the same index,*

$$\operatorname{ind}(A : X_2 \to Y_2) = \operatorname{ind}(A|_{X_1} : X_1 \to Y_1).$$

Then every solution $u \in X_2$ of the equation $Au = f$ with $f \in Y_1$ belongs already to X_1.

Theorem 23 *Let w be a weight in $\mathcal{R}(\mathcal{D})$ with $\lim_{x \to \infty} w(x) = \infty$, and let $A := Op_d(b) + \Phi I$ be an operator which satisfies the following conditions:*
(i) $b \in S^m_{1,0,0}(\mathcal{D})$, $b\langle \xi \rangle^{-m} \in S^0_{1,0,0}(\tilde{\mathbb{R}}^n)$, and conditions (**A**) *and* (**B**) *hold;*
(ii) $Op(b)$ is an elliptic operator on $\mathbb{R}^n$;
(iii) for every $t \in [-1, 1]$, all limit operators $A^h_{tw} : H^{m,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ of the operator

$$A_{tw} := Op(a(x, y, \xi + it\theta_w(x, y))) + \Phi I$$

are invertible.
If u is a function in $H^{m,p}(\mathbb{R}^n, w^{-1})$ for which $Op_d(a)u \in L^p(\mathbb{R}^n, w)$, then u belongs to $H^{m,p}(\mathbb{R}^n, w)$.

Proof. Note that $A : H^{m,\,p}(\mathbb{R}^n,\, w^t) \to L^p(\mathbb{R}^n,\, w^t)$ is a Fredholm operator if and only if $w^{-t}Aw^tI : H^{m,\,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a Fredholm operator, and that the Fredholm indices of these operator coincide. The conditions of the theorem guarantee that $w^{-t}Aw^tI : H^{m,\,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a Fredholm operator for every $t \in [-1,\, 1]$. The representation (28) and the estimate (14) for the norm of pseudodifferential operators imply that the mapping

$$[-1,\, 1] \ni t \to w^{-t}Aw^tI : H^{m,\,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is norm continuous. Thus, the Fredholm index of the operators $w^{-t}Aw^tI : H^{m,\,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is independent of $t \in [-1,\, 1]$, and so is the index of the operators $A : H^{m,\,p}(\mathbb{R}^n,\, w^t) \to L^p(\mathbb{R}^n,\, w^t)$. Hence,

$$\begin{aligned} &\operatorname{ind}(A : H^{m,\,p}(\mathbb{R}^n,\, w^{-1}) \to L^p(\mathbb{R}^n,\, w^{-1})) \\ &\quad = \operatorname{ind}(A : H^{m,\,p}(\mathbb{R}^n,\, w) \to L^p(\mathbb{R}^n,\, w)). \end{aligned}$$

Moreover, the embedding of $H^{m,\,p}(\mathbb{R}^n,\, w)$ into $H^{m,\,p}(\mathbb{R}^n,\, w^{-1})$ is dense. Proposition 22 implies that all solutions of the equation $Au = f$ with $f \in L^p(\mathbb{R}^n,\, w)$, which a priori lie in $H^{m,\,p}(\mathbb{R}^n,\, w^{-1})$, already belong to $H^{m,\,p}(\mathbb{R}^n,\, w)$. ■

Corollary 24 *Let the conditions of Theorem 23 hold for every $t \in [0,\, 1]$. If $u \in L^p(\mathbb{R}^n)$ is a solution of the equation $Au = 0$ then $u \in H^{m,\,p}(\mathbb{R}^n,\, w)$.*

The next theorem follows from Corollary 24 if one takes into account that

$$\lim_{g_k \to \eta_\omega} \nabla v_l(x + g_k) = l(\omega)\omega$$

for every sequence g tending to η_ω.

Theorem 25 *Let the operator $A = Op_d(b) + \Phi I$ be such that $b \in S^m_{1,\,0,\,0}(\mathcal{D})$, $b\langle\xi\rangle^{-m} \in S^0_{1,\,0,\,0}(\tilde{\mathbb{R}}^n)$, and $Op_d(b)$ is uniformly elliptic. Let further conditions* (**A**) *and* (**B**) *be satisfied. Moreover, let $Op_d(b^g) + \Phi^g I$ be a limit operator of A defined by the sequence g, and let l be a positive C^∞-function on the unit sphere S^{n-1}. If $\lambda \in sp^p_{dis}A$, but*

$$\lambda \notin sp^p(Op_d(b^g(x,\, y,\, \xi + itl(\omega)\omega)) + \Phi^g I)$$

for every $\omega \in S^{n-1}$, every $t \in [0,\, 1]$ and every sequence g tending to η_ω, then the λ-eigenfunction u_λ of A lies in $H^{m,\,p}(\mathbb{R}^n,\, e^{l(\frac{x}{|x|})|x|})$.

4.3 ΨDO with slowly oscillating symbols

Here we introduce a class of pseudodifferential operators for which the statements of Theorems 23 – 25 can be formulated more explicitly. We say that a double symbol $a \in S^m_{1,\,0,\,0}$ is slowly oscillating if

$$\lim_{x \to \infty} \sup_{\xi \in \mathbb{R}^n,\, y \in K} |\partial_x^\beta \partial_\xi^\alpha a(x,\, x + y,\, \xi)|\langle\xi\rangle^{-m} = 0$$

for all multi-indices α, $\beta \neq 0$ and for every compact subset K of $\mathbb{R}^n$. We denote the class of these symbols by $SO^m_{1,0,0}$ and write $OPSO^m_{1,0,0}$ for the corresponding class of pseudodifferential operators. Further we set $OPSO^m_{1,0,0}(\tilde{\mathbb{R}}^n) := OPSO^m_{1,0,0} \cap OPS^m_{1,0,0}(\tilde{\mathbb{R}}^n)$.

We consider perturbed pseudodifferential operators of the form

$$A = Op_d(b) + \Phi I$$

where $Op_d(b) \in OPSO^m_{1,0,0}(\tilde{\mathbb{R}}^n)$ for some $m > 0$ and where $\Phi := \Phi_1 + \Phi_2$ with $\Phi_1 \langle D \rangle^{-m}$ a compact operator and $\Phi_2 \in SO(\mathbb{R}^n)$.

Let $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence tending to infinity. There is a subsequence g of h and a function b^g on $\mathbb{R}^n$ such that

$$b(x + g_k,\, y + g_k,\, \xi) \to b^g(\xi) \tag{29}$$

in the topology of the space $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ (see [32], p. 228, for details). Since the values $b^g(\xi)$ of the limit function are independent of x and y, the limit operators of $Op_d(b) \in OPSO^m_{1,0,0}(\tilde{\mathbb{R}}^n)$ have symbols which are independent of x and y, too. Similarly, the limit operators of the potential Φ are operators of multiplication by the constants

$$\Phi_2^g := \lim_{k\to\infty} \Phi_2(x + g_k) = \lim_{k\to\infty} \Phi_2(g_k). \tag{30}$$

Thus, all limit operators $A^g := Op(b^g(\xi)) + \Phi_2^g$ of A are invariant with respect to shifts. One can prove that then the spectrum of the limit operator A^g, considered as an unbounded operator on $L^p(\mathbb{R}^n)$ with domain $H^{m,p}(\mathbb{R}^n)$, is independent of $p \in (1, \infty)$ and that

$$spA^g = \{\lambda \in \mathbb{C} : \lambda = b^g(\xi) + \Phi_2^g,\, \xi \in \mathbb{R}^n\}.$$

This implies the following result.

Theorem 26 *Let $A = Op_d(b) + \Phi I$ where $Op_d(b) \in OPSO^m_{1,0,0}(\tilde{\mathbb{R}}^n)$ for some $m > 0$ is a uniformly elliptic operator, $\Phi := \Phi_1 + \Phi_2$ with $\Phi_1 \langle D \rangle^{-m}$ being compact on $L^p(\mathbb{R}^n)$ and $\Phi_2 \in SO(\mathbb{R}^n)$. Then the essential spectrum of A, the latter considered as an unbounded operator on $L^p(\mathbb{R}^n)$ with domain $H^{m,p}(\mathbb{R}^n)$, is independent of $p \in (1, \infty)$, and*

$$sp_{ess}A = \bigcup_{A^g \in op(A)} \{\lambda \in \mathbb{C} : \lambda = b^g(\xi) + \Phi_2^g,\, \xi \in \mathbb{R}^n\} \tag{31}$$

where the constant functions Φ_2^g are defined by (30).

Our next goal are estimates of the exponential decay of eigenfunctions of pseudodifferential operators with slowly oscillating symbols acting on spaces with slowly oscillating weights.

Theorem 27 *Let $A = Op_d(b) + \Phi I$ where $b \in SO^m_{1,0,0} \cap S^m_{1,0,0}(\mathcal{D})$, $b\langle\xi\rangle^{-m} \in S^0_{1,0,0}(\tilde{\mathbb{R}}^n)$, and $Op_d(b)$ is a uniformly elliptic operator. Let further $w = \exp v$ be a weight in $\mathcal{R}_{sl}(\mathcal{D})$. Finally, let $\lambda \in sp^p_{dis}A$ and u_λ be a λ-eigenfunction of A. If*

$$\inf_{\xi \in \mathbb{R}^n} |b^g(\xi + it\theta^g_w) + \Phi^g_2 - \lambda| > 0$$

for every sequence g tending to infinity such that the limits (29), (30) and the limit

$$\theta^g_W := \lim_{k\to\infty} \nabla v(x + g_k)$$

exist, then u_λ lies in $L^p(\mathbb{R}^n, w_l)$.

Let l be a positive C^∞-function on the unit sphere S^{n-1} such that $\overline{\Omega_l} \subset \mathcal{D}$, and let $w_l := \exp v_l$ where $v_l(x) = l(\frac{x}{|x|})|x|$. Modifying w_l in a neighborhood of the origin, we obtain a weight in $\mathcal{R}_{sl}(\mathcal{D})$. Then Theorem 27 gives the following result.

Theorem 28 *Let $A = Op_d(b) + \Phi I$ where $b \in SO^m_{1,0,0} \cap S^m_{1,0,0}(\mathcal{D})$, $b\langle\xi\rangle^{-m} \in S^0_{1,0,0}(\tilde{\mathbb{R}}^n)$, and $Op_d(b)$ is uniformly elliptic. Let $\lambda \in sp^p_{dis}A$, and let u_λ be a λ-eigenfunction of A. For every $t \in [0, 1]$, every $\omega \in S^{n-1}$, and for every sequence g tending to η_ω such that the limits (29) and (30) exist, let*

$$\inf_{\xi \in \mathbb{R}^n} |b^g(\xi + itl(\omega)\omega) + \Phi^g_2 - \lambda| > 0.$$

Then $u_\lambda \in L^p(\mathbb{R}^n, w_l)$.

5 Schrödinger operators

5.1 Essential spectra

Consider the electromagnetic Schrödinger operator

$$\mathcal{H} := (i\partial_{x_j} - a_j)\rho^{jk}(i\partial_{x_k} - a_k) + \Phi I$$

on $\mathbb{R}^n$ equipped with the Riemann metric ρ_{jk} such that

$$\inf_{x\in\mathbb{R}^n,\, \omega\in S^{n-1}} \rho_{jk}(x)\omega^j\omega^k > 0. \tag{32}$$

Here we use again the Einstein summation convention, and $\rho_{jk}(x)$ refers to the tensor inverse to $\rho^{jk}(x)$.

We suppose that the functions a_j and $\rho^{jk} \in SO^\infty(\mathbb{R}^n)$ are real-valued, whereas Φ is a complex-valued electric potential such that $\Phi\langle D\rangle^{-2}$ is a locally compact operator on $L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$ which, moreover, belongs to $W^\$_p(\mathbb{R}^n)$. We consider $\mathcal{H}$ as a closed unbounded operator on $L^p(\mathbb{R}^n)$ with domain $H^{2,p}(\mathbb{R}^n)$. Then it follows from Theorem 18 that

$$sp^p_{ess}\mathcal{H} = \bigcup_{\mathcal{H}_g \in op(\mathcal{H})} sp^p\mathcal{H}_g \tag{33}$$

where

$$\mathcal{H}_g = (i\partial_{x_j} - a_j^g I)\rho_g^{jk}(i\partial_{x_k} - a_k^g) + \Phi^g I,$$

$$\rho_g^{jk} := \lim_{l\to\infty} \rho^{jk}(g_l) \in \mathbb{R}, \qquad a_k^g := \lim_{l\to\infty} a_k(g_l) \in \mathbb{R}, \tag{34}$$

and where $\Phi^g I$ is the limit operator of the operator ΦI with respect to g. It is not hard to show that

$$\text{s-lim}\, \Phi(x + g_l)\langle D\rangle^{-2} = \Phi^g(x)\langle D\rangle^{-2} \tag{35}$$

where the s-lim refers to the strong limit.

For $a^g = (a_1^g, \ldots, a_n^g) \in \mathbb{R}^n$, set $(T_{a^g}u)(x) := e^{i(a^g,\, x)}u(x)$. The so-defined operator T_{a^g} acts as an isometry on $L^p(\mathbb{R}^n)$, and

$$T_{a^g}\mathcal{H}_g T_{a^g}^{-1} = -\partial_{x_j}\rho_g^{jk}\partial_{x_k} + \Phi_1^g.$$

Consequently, the limit operator $\mathcal{H}_g$ is isometrically equivalent to the operator

$$\mathcal{H}'_g := -\rho_g^{jk}\partial_{x_j}\partial_{x_k} + \Phi^g I$$

without magnetic field. This shows that

$$sp_{ess}^p\mathcal{H} = \bigcup_{\mathcal{H}_g \in op(\mathcal{H})} sp^p\mathcal{H}'_g$$

and that the essential spectrum of $\mathcal{H}$ is independent of the magnetic field.

Let $\Phi = \Phi_1 + \Phi_2$ where $\Phi_2\langle D\rangle^{-2}$ is a compact operator on $L^p(\mathbb{R}^n)$ operator and where $\Phi_1 \in SO(\mathbb{R}^n)$. Then the limit operators $\mathcal{H}'_g = -\rho_g^{jk}\partial_{x_j}\partial_{x_k} + \Phi^g I$ are operators with constant coefficients. Hence, the essential spectrum

$$sp_{ess}^p\mathcal{H} = \bigcup_g \{\lambda \in \mathbb{R} : \lambda = -\rho_g^{jk}\xi_j\xi_k + \Phi_1^g,\, \xi \in \mathbb{R}^n\} \tag{36}$$

is independent of $p \in (1, \infty)$.

Let $\Gamma_{\Phi_1} \subset \mathbb{C}$ denote the set of all partial limits at infinity of the function Φ_1. It turns out that Γ_{Φ_1} is a closed connected set of the complex plane. Identity (36) implies the following.

Theorem 29 *(i)* $sp_{ess}^p\mathcal{H} = \{\lambda \in \mathbb{C} : \lambda = \lambda_1 + i\lambda_2,\, \lambda_1 \in [0, \infty),\, \lambda_2 \in \Gamma_{\Phi_1}\}$*, and* $sp_{ess}^p\mathcal{H}$ *does not depend on* p.
(ii) If the function Φ_1 *is real-valued, then* $sp_{ess}^p\mathcal{H}$ *is the interval*

$$sp_{ess}^p\mathcal{H} = [\liminf_{x\to\infty} \Phi_1(x),\, +\infty).$$

5.2 Exponential estimates of eigenfunctions of the discrete spectrum

In what follows we restrict our attention to Schrödinger operators acting on $L^2(\mathbb{R}^n)$. We suppose that the functions ρ^{jk} and a_j as well as the weight w are slowly oscillating, that is, condition (25) holds. The limit operators $(w^{-1}\mathcal{H}w)_g$ of $w^{-1}\mathcal{H}w$ are unitarily equivalent to the operators

$$\mathcal{H}_w^g := \rho_g^{jk}(i\partial_{x_j} + i(\nabla v)_j^g)(i\partial_{x_k} + i(\nabla v)_k^g) + \Phi^g I$$

where

$$(\nabla v)^g := \lim_{k\to\infty} (\nabla v)(g_k) \in \mathbb{R}^n. \tag{37}$$

Below we will need the following evident observation.

Proposition 30 *Let $A = A_1 + iA_2$ be a closed and densely defined unbounded operator on a Hilbert space H where $A_1 = \Re A = (1/2)(A + A^*)$ and $A_2 = \Im A = (1/2i)(A - A^*)$ are symmetric operators on H. Let λ be a real number which is not in $sp\, A_1$. Then $\lambda \notin sp\, A$.*

Note that

$$\Re\mathcal{H}_w^g = -\rho_g^{jk}\partial_{x_j}\partial_{x_k} - |(\nabla v)^g|_{\rho_g}^2 + \Phi^g I \tag{38}$$

where

$$|(\nabla v)^g(x)|_{\rho_g}^2 = \rho_g^{jk}(\nabla v)_j^g(\nabla v)_k^g = (\rho_g(\nabla v)^g,\, (\nabla v)^g)$$

and $\rho_g = (\rho_g^{jk})_{j,\,k=1}^n$. From Theorem 23 we conclude the following.

Theorem 31 *Let $\lambda \in sp_{dis}\mathcal{H}$, and let the above conditions for the metric ρ, the vector potential a and the weight $w = \exp v$ hold. Further assume that*

$$\lambda + t|(\nabla v)^g|_{\rho_g}^2 \notin sp\,(-\rho_g^{jk}\partial_{x_j}\partial_{x_k} + \Phi^g I) \tag{39}$$

for every $t \in [0,\, 1]$ and every sequence g for which the limits (34), (35) *and* (37) *exist. Then the λ-eigenfunction u_λ of $\mathcal{H}$ belongs to $H^2(\mathbb{R}^n,\, w)$.*

Corollary 32 *Let $\lambda \in sp_{dis}\mathcal{H}$ and let the above conditions for the metric ρ, the vector potential a and the weight $w = \exp v$ hold. Further let $\Phi = \Phi_1 + \Phi_2$ be a real-valued function for which $\Phi_1 \in SO(\mathbb{R}^n)$ and $\Phi_2\langle D\rangle^{-2}$ is a compact operator on $L^2(\mathbb{R}^n)$, and assume that*

$$|(\nabla v)^g|_{\rho_g} < \sqrt{\Phi_1^g - \lambda} \tag{40}$$

for every sequence g as in the previous theorem. Then every λ-eigenfunction u_λ of $\mathcal{H}$ belongs to $H^2(\mathbb{R}^n,\, w)$.

Next we introduce a function which describes the local distribution of the essential spectrum of the operator $\mathcal{H}$ by

$$\mu_{\mathcal{H}}(\omega) := \inf_{\mathcal{H}_g \in op_{\eta_\omega}(\mathcal{H})} sp\,\mathcal{H}_g.$$

Clearly,

$$\hat{\mu}_{\mathcal{H}} := \inf_{\omega \in S^{n-1}} \mu_{\mathcal{H}}(\omega)$$

is the infimum of the essential spectrum of $\mathcal{H}$.

Theorem 33 *Let the above conditions for the metric ρ and the vector potential a hold. Moreover, assume that the limit $\lim_{x \to \eta_\omega} \rho^{jk}(x) =: \rho_\omega^{jk}$ exists for every $\omega \in S^{n-1}$ and set $\rho_\omega := (\rho_\omega^{jk})_{j,\,k=1}^n$. Let $\lambda < \hat{\mu}_{\mathcal{H}}$ be an eigenvalue of $\mathcal{H}$ and u_λ an associated λ-eigenfunction. Let finally l be a positive C^∞-function on S^{n-1} for which*

$$l(\omega) < \sqrt{\frac{\mu_{\mathcal{H}}(\omega) - \lambda}{(\rho_\omega \omega,\, \omega)}} \quad \textit{for all}\, \omega \in S^{n-1}. \tag{41}$$

Then $u_\lambda \in H^2(\mathbb{R}^n,\, w_l)$ where $w_l(x) := \exp(l(x/|x|)|x|)$.

Proof. Let the sequence g tend to η_ω. According to formula (38),

$$\Re \mathcal{H}^g_{w_l} = -\rho_\omega^{jk} \partial_{x_j} \partial_{x_k} - |(\nabla v)(\omega)|^2_{\rho_\omega} + \Phi^g I.$$

By means of Proposition 20 (ii) we obtain further that

$$|(\nabla v)(\omega)|^2_{\rho_\omega} = (\rho_\omega (\nabla v)(\omega), \qquad (\nabla v)(\omega)) = l^2(\omega)(\rho_\omega \omega,\, \omega).$$

Thus, condition (41) implies condition (39) of Theorem 31. By the latter theorem, u_λ is indeed in $H^2(\mathbb{R}^n,\, w_l)$. ■

Theorem 33 associates to each eigenvalue $\lambda(< \hat{\mu}_{\mathcal{H}})$ and to the distribution function $\mu_{\mathcal{H}}$ a weight $w_l(x) = \exp(l(x/|x|)|x|)$ such that $u_\lambda \in H^2(\mathbb{R}^n,\, w_l)$. If only the infimum $\hat{\mu}_{\mathcal{H}}$ of the essential spectrum of $\mathcal{H}$ is available, then one still gets a rough "isotropic" exponential decay estimate of the λ-eigenfunction u_λ, namely

$$u_\lambda \in H^2(\mathbb{R}^n,\, e^{l|x|}) \quad \text{where} \quad 0 < l < \sqrt{\frac{\hat{\mu}_{\mathcal{H}} - \lambda}{\inf_{\omega \in S^{n-1}} (\rho_\omega \omega,\, \omega)}}.$$

5.3 Multiparticle Schrödinger operators

We consider an atomic type system of $N+1$ particles with coordinates $x_i \in \mathbb{R}^\nu$, $0 \le i \le N$, and interacting real-valued potentials φ_{ij}, $0 \le i < j \le N$, defined on $\mathbb{R}^n$. The particle x_0 (considered as the "nucleus") has infinite mass and is fixed at $x_0 = 0$. We assume that the functions φ_{ij} are subject to the following conditions:

- (i) the functions φ_{ij} are measurable on $\mathbb{R}^\nu$, and the operators $\varphi_{ij} \langle D \rangle^{-2}$ are compact on $L^2(\mathbb{R}^\nu)$,
- (ii) $\varphi_{ij} \ge 0$ for $0 \le i < j \le N$.

The configuration space of the system consists of the product of N copies of $\mathbb{R}^\nu$ which we identify with $\mathbb{R}^{N\nu}$ and denote by X. The generic point in X is $x = (x^1, \ldots, x^N)$, where the $x^i = (x^i_1, \ldots, x^i_\nu)$ are the coordinates of the particles. The Schrödinger operator of the system is the elliptic operator $\mathcal{H}$ on X defined by

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_i - \sum_{i=1}^{N} \varphi_{0j}(x^i) + \sum_{1\le i<j\le N} \varphi_{ij}(x^i - x^j) \tag{42}$$

where Δ_i is the usual Laplace operator with respect to the variable x^i and the m_i are positive numbers representing the mass of the particles. We consider potentials Φ which are functions on X of the form

$$\Phi(x) = \sum_{0\le i<j\le N} \Phi_{ij}(x)$$

where

$$\Phi_{ij}(x) = \begin{cases} \varphi_{0j}(x^i) & \text{if} \quad 1 \le j \le N, \\ \varphi_{ij}(x^i - x^j) & \text{if} \quad 1 \le i < j \le N. \end{cases}$$

Let $r \le N$. For each r-tuple $(i_1, \ldots, i_r)$ of integers $1 \le i_1 < \ldots < i_r \le N$, we let $\Sigma_{(i_1, \ldots, i_r)}$ stand for the set of all $\omega \in S^{\nu N-1}$ with $\omega^k \neq 0$ if $k \in \{i_1, \ldots, i_r\}$ and $\omega^k = 0$ for $k \notin \{i_1, \ldots, i_r\}$.

Proposition 34 *(i) Let $1 \le r \le N-1$ and $\omega \in \Sigma_{(i_1 \ldots, i_r)}$. Then*

$$\mu_{\mathcal{H}}(\omega) = \inf sp\, \mathcal{A}_{i_1, i_2, \ldots, i_r} \quad \textit{and} \quad \mu_{\mathcal{H}}(\omega) \le 0$$

where

$$\begin{aligned} \mathcal{A}_{i_1, \ldots, i_r} \quad &= \quad - \sum_{1\le j\le N,\, j\neq i_1, \ldots, i_r} \left(\frac{1}{2m_j}\Delta_j - \varphi_{0j}(x^j) \right) \\ &\qquad + \sum_{1\le i<j\le N,\, i\neq i_1, \ldots, i_r,\, j\neq i_1, \ldots i_r} \varphi_{ij}(x^i - x^j). \end{aligned}$$

(ii) Let $r = N$ and $\omega \in \Sigma_{(1, \ldots, N)}$. Then $\mu_{\mathcal{H}}(\omega) = 0$.

Proof. (i) Let $h = (h_m) = (h^1_m, \ldots, h^N_m)$ be a sequence in $\mathbb{Z}^{N\nu}$ which tends to η_ω. We distinguish between two cases.

Case (a): The coordinate sequences $(h^{i_1}_m), \ldots (h^{i_r}_m)$ tend to infinity, whereas all other coordinate sequences (h^j_m) with $j \notin \{i_1, \ldots, i_r\}$ remain bounded.

Denote by $\mathcal{L}_{i_1, \ldots, i_r}$ the set of all pairs (i_μ, i_ν) with $i_\mu, i_\nu \in \{i_1, \ldots, i_r\}$ for which the sequence $(h^{i_\mu}_m - h^{i_\nu}_m)$ is bounded. We can assume that then $\lim_{m\to\infty}(h^{i_\mu}_m - h^{i_\nu}_m) = \infty$ for all pairs $(i_\mu, i_\nu) \notin \mathcal{L}_{i_1, \ldots, i_r}$ (otherwise we pass to a suitable subsequence of h). Then there is a subsequence of h, for which the limit operator of $\mathcal{H}$ exists and is unitarily equivalent to the operator

$$\begin{aligned}\mathcal{H}_h &= -\sum_{1\le j\le N} \frac{1}{2m_j}\Delta_{x^j} - \sum_{1\le j\le N,\, j\ne i_1,\,\dots,\, i_r} \varphi_{0j}(x^j) \\ &+ \sum_{1\le i<j\le N,\, i\ne i_1,\,\dots,\,i_r,\, j\ne i_1,\,\dots,\,i_r} \varphi_{ij}(x^i - x^j) \\ &+ \sum_{(i,\,j)\in\mathcal{L}_{i_1,\,\dots,\,i_r}} \varphi_{ij}(x^i - x^j).\end{aligned}$$

The non-negativity of the potentials φ_{ij} implies that the spectrum of $\mathcal{H}_h$ is a subset of the spectrum of the operator $\mathcal{H}'_h$ defined by

$$\begin{aligned}\mathcal{H}'_h &= -\sum_{1\le j\le N} \frac{1}{2m_j}\Delta_{x^j} - \sum_{1\le j\le N,\, j\ne i_1,\,\dots,\, i_r} \varphi_{0j}(x^j) \\ &+ \sum_{1\le i<j\le N,\, i\ne i_1,\,\dots,\,i_r,\, j\ne i_1,\,\dots,\,i_r} \varphi_{ij}(x^i - x^j).\end{aligned}$$

For the reverse inclusion of spectra, note that there exists a limit operator $\tilde{\mathcal{H}}_g$ of $\mathcal{H}_h$ which equals $\mathcal{H}'_h$. Theorem 18 implies that $sp\,\mathcal{H}_h \supseteq sp_{ess}\,\mathcal{H}_h \supseteq sp\,\tilde{\mathcal{H}}_g = sp\,\mathcal{H}'_h$. Hence, the spectra of $\mathcal{H}_h$ and $\mathcal{H}'_h$ coincide.

Case (b): Again we assume that the coordinate sequences $(h_m^{i_1}), \dots (h_m^{i_r})$ tend to infinity, but now we also allow that the sequences $(h_m^{j_1}), \dots (h_m^{j_l})$ tend to infinity for some indices $j_1, \dots, j_l \notin \{i_1, \dots, i_r\}$.

Since the potentials $\varphi_{ij}(x^j)$ are non-negative, the inclusion $sp\,\mathcal{H}'_h \subseteq sp\,\mathcal{H}_h$ follows as in case (a). Applying the Fourier transform with respect to the variables x^k, $k = i_1, \dots, i_r$ we obtain that the operator $\mathcal{H}'_h$ is unitarily equivalent to the operator of multiplication by operator-valued function which is defined at $(\xi_{i_1}, \dots, \xi_{i_r}) \in \mathbb{R}^r$ by

$$\widehat{\mathcal{H}'_h}(\xi_{i_1}, \dots, \xi_{i_r}) := \frac{1}{2m_{i_1}}\xi_{i_1}^2 + \dots + \frac{1}{2m_{i_r}}\xi_{i_r}^2 + \mathcal{A}_{i_1,\dots,i_r}. \tag{43}$$

It follows from (43) that

$$sp\,\mathcal{H}_h = sp\,\mathcal{H}'_h = [\inf sp\,\mathcal{A}_{i_1,\dots,i_r},\, +\infty)$$

for $1 \le r \le N-1$. Note that the spectrum of the operator $\mathcal{H}'_h$ contains the semi axis $[0, +\infty)$. Indeed, there exists a limit operator $-\sum_{1\le j\le q} \frac{1}{2m_j}\Delta_{x^j}$ of $\mathcal{H}'_h$ the spectrum of which is $[0, +\infty)$, and by Theorem 18, $sp\,\mathcal{H}'_h \supseteq sp_{ess}\,\mathcal{H}'_h \supset [0, +\infty)$. Thus,

$$\mu_{\mathcal{H}}(\omega) = \inf sp\,\mathcal{A}_{i_1,\dots,i_r} \le 0.$$

(ii) Let $h = (h_m) = (h_m^1, \dots, h_m^N)$ be a sequence in $\mathbb{Z}^{N\nu}$ which tends to η_ω for some point $\omega \in \Sigma_{(1,\dots,N)}$. Then each of the sequences $(h_m^1), \dots, (h_m^N)$ tends to infinity. Let U be the set of all pairs for which the sequence $(h_m^j - h_m^k)$ is

bounded. Then there exists a subsequence of h for which the limit operator of $\mathcal{H}$ exists and is unitarily equivalent to the operator

$$B_U := -\sum_{1\leq j\leq N} \frac{1}{2m_j}\Delta_{x^j} + \sum_{(i,\,j)\in U} \varphi_{ij}(x^i - x^j).$$

The non-negativity of the potentials φ_{ij} implies that $sp\, B_U \subseteq [0, +\infty)$. For the reverse inclusion, note that there exists a limit operator of B_U which is equal to $-\sum_{1\leq j\leq q} \frac{1}{2m_j}\Delta_{x^j}$ and which, thus, has the interval $[0, +\infty)$ as its spectrum. Hence,

$$sp\, B_U = [0, +\infty) \tag{44}$$

which implies in particular that $\mu_{\mathcal{H}}(\omega) = 0$. ∎

As a corollary we obtain the Hunziker, van Winter, Zjislin theorem (see, for instance, [7], 3.3.3) on the location of the essential spectrum of the multiparticle Schrödinger operators.

Theorem 35 (HWZ theorem) *The infimum $\hat{\mu}_{\mathcal{H}}$ of the essential spectrum of the multiparticle Schrödinger operator $\mathcal{H}$ is*

$$\hat{\mu}_{\mathcal{H}} = \inf_{1\leq i_1<\ldots<i_r\leq N,\, 1\leq r\leq N-1} \inf sp\, \mathcal{A}_{i_1,\,\ldots,\,i_r}.$$

Theorem 36 *Let $\lambda < \hat{\mu}_{\mathcal{H}} < 0$ be an eigenvalue of $\mathcal{H}$ and $l : S^{n-1} \to \mathbb{R}$ be a positive C^∞-function such that*

$$l(\omega) < \sqrt{\frac{\mu_{\mathcal{H}}(\omega) - \lambda}{(\rho\omega,\, \omega)}}$$

where

$$\rho = diag\left\{\frac{1}{2m_1}, \ldots, \frac{1}{2m_1}, \ldots, \frac{1}{2m_N}, \ldots, \frac{1}{2m_N}\right\}$$

is a diagonal matrix with each value occurring ν times, and $\mu_{\mathcal{H}}(\omega)$ is given in Proposition 34. Then each λ-eigenfunction u_λ of $\mathcal{H}$ belongs to $L^2(\mathbb{R}^n,\, e^{l(\frac{x}{|x|})|x|})$.

5.4 Perturbed periodic Schrödinger operators

In this concluding subsection we consider Schrödinger operators

$$\mathcal{H}_\Phi = -\Delta + \Phi I$$

with potential $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ where $\Phi_1 \in L^\infty(\mathbb{R}^n)$ is a real-valued function which is periodic with respect to a lattice Γ of periods (or Γ-periodic for short), Φ_2 is a real-valued measurable function such that the operator $\Phi_2\langle D\rangle^{-2}$ is compact on $L^2(\mathbb{R}^n)$, and Φ_3 is a real-valued function in $SO(\mathbb{R}^n)$.

First we consider the periodic operator

$$\mathcal{H}_{\Phi_1} := -\Delta + \Phi_1 I = P_1(x,\, D).$$

For simplicity we suppose that Φ_1 is a periodic function with respect to the lattice $\mathbb{Z}^n$. We shall need the Floquet transform

$$(\mathcal{F}f)(x,\, k) := \sum_{\alpha \in \mathbb{Z}^n} V_\alpha(f e^{-i(x,\, k)}) \quad \text{for } x,\, k \in \mathbb{R}^n.$$

The function $(x,\, k) \mapsto \mathcal{F}f(x,\, k)$ is $\mathbb{Z}^n$-periodic with respect to x and satisfies the following cyclic condition with respect to the quasi-impuls k,

$$(\mathcal{F}f)(x,\, k+\gamma) = e^{-i(\gamma,\, x)}(\mathcal{F}f)(x,\, k)$$

for all $\gamma \in 2\pi\mathbb{Z}^n$ and $k \in \mathbb{R}^n$. It is well known that $\mathcal{F}$ acts as a unitary operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{T}^*,\, L^2(\mathbb{T}))$ where $\mathbb{T}^* = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is the so-called *Brillouin zone* and where $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$. The inverse Floquet transform is given by

$$(\mathcal{F}^{-1}v)(x) = \frac{1}{(2\pi)^n} \int_{[0,\, 2\pi]^n} v(x,\, k) e^{i(x,\, k)}\, dk$$

where the function $v(x,\, k) \in L^2(\mathbb{T}^*,\, L^2(\mathbb{T}))$ is continued to a periodic function with respect to the variable $x \in \mathbb{R}^n$. A straightforward calculation yields

$$(\mathcal{F}P_1(x,\, D)\mathcal{F}^{-1}f)(x,\, k) = P_1(x,\, D_x + k)f(x,\, k).$$

Hence, the operator $P_1(x,\, D)$, considered as an unbounded operator on $L^2(\mathbb{R}^n)$, is unitarily equivalent to the "direct integral"

$$\int_{\mathbb{T}^*} P_1(x,\, D_x + k)\, dk$$

of the unbounded self-adjoint operators $P_1(x,\, D_x + k)$ on $L^2(\mathbb{T})$ with domain $C^2(\mathbb{T})$. The operators $P_1(x,\, D_x + k)$ have a discrete and real spectrum for every $k \in \mathbb{T}^*$. For $k \in \mathbb{T}^*$, let $\lambda_j(k)$, $j \geq 1$, refer to the eigenvalues of the operator $P_1(x,\, D_x + k)$.

A basic property of spectra of periodic Schrödinger operators is that

$$sp\, \mathcal{H}_1 = sp_{ess}\, \mathcal{H}_1 = \bigcup_{j=1}^{\infty} [\nu_j,\, \mu_j]$$

where

$$\nu_j := \min_{k \in [0,\, 2\pi]^n} \lambda_j(k), \qquad \mu_j := \max_{k \in [0,\, 2\pi]^n} \lambda_j(k), \tag{45}$$

and

$$\min_{k \in [0,\, 2\pi]^n} \lambda_j(k) \to \infty \quad \text{as } j \to \infty$$

(see, for instance, Section XIII.16 in [33] and [14]). Hence, the spectrum (= the essential spectrum) of a periodic Schrödinger operator is the union of intervals $[\nu_j,\, \mu_j]$ which are also called spectral bands.

The existence of spectral gaps, i.e., of intervals of the real axis which do not intersect any of the spectral bands is what makes a crystal a semi-conductor. Note that there is also a semi-infinite gap $(-\infty, \inf_j \nu_j)$ left of the spectrum.

In the one-dimensional setting, it is known that generically there exist infinitely many spectral gaps. The long standing Bethe-Sommerfeld conjecture claims that in dimensions 2 and 3 only finitely many gaps occur. This conjecture has been verified in the affirmative for Schrödinger operators with periodic electric potentials.

Next we consider potentials $\Phi = \Phi_1 + \Phi_2$ where Φ_1 is as above and Φ_2 is such that the operator $\Phi_2 \langle D \rangle^{-2}$ is compact on $L^2(\mathbb{R}^n)$. Then, clearly,

$$sp_{ess}(-\Delta + (\Phi_1 + \Phi_2)I) = sp_{ess}(-\Delta + \Phi_1 I),$$

but points of the discrete spectrum of $-\Delta + (\Phi_1 + \Phi_2)I$ can appear in spectral gaps of the operator $-\Delta + \Phi_1 I$.

Theorem 37 *Let (μ_j, ν_{j+1}) be a spectral gap of the operator $-\Delta + \Phi_1 I$, and let $\lambda \in (\mu_j, \nu_{j+1})$ be an eigenvalue of $-\Delta + (\Phi_1 + \Phi_2)I$ and u_λ an associated eigenfunction. Then $e^{l|x|} u_\lambda \in L^2(\mathbb{R}^n)$ for each positive number $l < \nu_{j+1} - \lambda$.*

Finally we consider the general case of potentials $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ where Φ_1 and Φ_2 are as above and Φ_3 is a real-valued function in $SO(\mathbb{R}^n)$. For $a \in L^\infty(\mathbb{R}^n)$, set

$$m_a(\omega) := \liminf_{x \to \eta_\omega} a(x), \qquad M_a(\omega) := \limsup_{x \to \eta_\omega} a(x),$$

$$m_a := \inf_{\omega \in S^{n-1}} m_a(\omega), \qquad M_a := \inf_{\omega \in S^{n-1}} M_a(\omega).$$

Theorem 38 *Let $\mathcal{H}_\Phi = -\Delta + \Phi I$ with $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ where $\Phi_1 \in L^\infty(\mathbb{R}^n)$ is real-valued and $\mathbb{Z}^n$-periodic, Φ_2 is a real-valued measurable function for which the operator $\Phi_2 \langle D \rangle^{-2}$ is compact on $L^2(\mathbb{R}^n)$, and Φ_3 is a real-valued function in $SO(\mathbb{R}^n)$. Then*

$$sp_{ess} \mathcal{H}_\Phi = \bigcup_{j=1}^{\infty} [\nu_j + m_{\Phi_3},\, \mu_j + M_{\Phi_3}]$$

where the μ_j and ν_j are given by (45).

Corollary 39 *Let (μ_j, ν_{j+1}) be a spectral gap of the operator $\mathcal{H}_{\Phi_1}$. If*

$$\nu_{j+1} - \mu_j > M_{\Phi_3} - m_{\Phi_3}$$

then the interval $(\mu_j + M_{\Phi_3}, \nu_{j+1} + m_{\Phi_3})$ is a spectral gap in the essential spectrum of $\mathcal{H}_\Phi$. In the opposite case

$$\nu_{j+1} - \mu_j \leq M_{\Phi_3} - m_{\Phi_3},$$

the spectral gap (μ_j, ν_{j+1}) of the unperturbed operator is contained in the spectrum of the perturbed operator (thus, the gap closes).

Note that

$$sp_{ess}\mathcal{H}_\Phi = sp_{ess}\mathcal{H}_{\Phi_1+\Phi_3}.$$

Theorem 40 *Suppose that the spectral gap* $(\mu_j + M_{\Phi_3},\, \nu_{j+1} + m_{\Phi_3})$ *is not empty, and let* $\lambda \in (\mu_j + M_{\Phi_3},\, \nu_{j+1} + m_{\Phi_3})$ *be an eigenvalue of* $\mathcal{H}_\Phi$*. Further let* $l : S^{n-1} \to \mathbb{R}$ *be a positive smooth function with*

$$l(\omega) < \nu_{j+1} - \lambda + m_{\Phi_2}(\omega) \qquad \textit{for every } \omega \in S^{n-1}.$$

Then the λ*-eigenfunction* u_λ *of* $\mathcal{H}_\Phi$ *belongs to* $H^2(\mathbb{R}^n,\, \exp(l(x/|x|)|x|))$.

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