On finite sections of band-dominated operators

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Abstract

In an earlier paper we showed that the sequence of the finite sections P_nAP_n of a band-dominated operator A on $l^p(\mathbb{Z})$ is stable if and only if the operator A is invertible, every limit operator of the sequence (P_nAP_n) is invertible, and if the norms of the inverses of the limit operators are uniformly bounded. The purpose of this short note is to show that the uniform boundedness condition is redundant.

1 Introduction

Let $1 < p < \infty$. We will work on the Banach space $l^p(\mathbb{Z}^+)$ of all sequences $(x_n)_{n=0}^\infty$ of complex numbers with $\sum |x_n|^p < \infty$. We provide this space with its standard basis which consists of all sequences $e_i := (0, \ldots, 0, 1, 0, \ldots)$ with the 1 standing at the ith position. Every bounded linear operator on $l^p(\mathbb{Z}^+)$ admits a matrix representation $(a_{ij})_{i,j\in\mathbb{Z}^+}$ with respect to the standard basis. We call an operator $A \in L(l^p(\mathbb{Z}^+))$ a band operator if the associated matrix is a band matrix, i.e., if there is a k such that $a_{ij} = 0$ whenever $|i - j| \geq k$. The operator A is said to be band-dominated if it is the norm limit of a sequence of band operators.

Let $n \in \mathbb{N}$. The nth finite section of an operator $A \in L(l^p(\mathbb{Z}^+))$ with matrix representation $(a_{ij})_{i,j\in\mathbb{Z}^+}$ is the $n \times n$-matrix $(a_{ij})_{i,j=0}^{n-1}$. We identify this matrix with the operator P_nAP_n where P_n is the projection

$$P_n : l^p(\mathbb{Z}^+) \to l^p(\mathbb{Z}^+), \quad (x_0, x_1, \ldots) \mapsto (x_0, \ldots, x_{n-1}, 0, 0, \ldots).$$

The sequence (P_nAP_n) of the finite sections of A is said to be stable if there is an n_0 such that the operators $P_nAP_n : \operatorname{im} P_n \to \operatorname{im} P_n$ are invertible for every $n \geq n_0$ and if the norms of their inverses are uniformly bounded.

There is an intimate relation between stability of sequences and Fredholmness of operators. For, we associate to the sequence $\mathbf{A} = (P_nAP_n)$ the block diagonal operator

$$\operatorname{Op}(\mathbf{A}) := \operatorname{diag}(P_1AP_1,\, P_2AP_2,\, P_3AP_3,\, \ldots) \tag{1}$$

*Supported by CONACYT project 43432.
†Supported by DFG Grant 444 MEX-112/2/05.

considered as acting on $l^p(\mathbb{Z}^+) = \operatorname{im} P_1 \oplus \operatorname{im} P_2 \oplus \operatorname{im} P_3 \oplus \ldots$. It is an easy exercise to show that the sequence $\mathbf{A}$ is stable if and only if the operator $\operatorname{Op}(\mathbf{A})$ is a Fredholm operator on $l^p(\mathbb{Z}^+)$, i.e., if its kernel and its cokernel have finite dimension. In general, the equivalence between stability and Fredholmness seems to be of less use. But if we start with the sequence $\mathbf{A} = (P_n A P_n)$ of the finite sections method of a band-dominated operator A, then we end up with a band-dominated operator $\operatorname{Op}(\mathbf{A})$ on $l^p(\mathbb{Z}^+)$ again. But for band-dominated operators on $l^p(\mathbb{Z}^+)$, there is a general Fredholm criterion which expresses the Fredholm property of a band-dominated operator in terms of its limit operators. To state this result, we need a few notations.

It will be convenient to work on the Banach space $l^p(\mathbb{Z})$ of the two-sided infinite sequences. The space $l^p(\mathbb{Z}^+)$ can be considered as a closed subspace of $l^p(\mathbb{Z})$ in a natural way. We let P denote the projection

$$P : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (x_n) \mapsto (\ldots, 0, 0, x_0, x_1, x_2, \ldots)$$

and write Q for the complementary projection $I - P$. Usually we will identify an operator A on $l^p(\mathbb{Z}^+)$ with the operator PAP acting on $l^p(\mathbb{Z})$.

For every $m \in \mathbb{Z}$, we introduce the shift operator

$$U_m : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (x_n) \mapsto (x_{n-m}).$$

Let further $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{N}$ which tend to infinity. An operator $A_h \in L(l^p(\mathbb{Z}))$ is called a *limit operator* of $A \in L(l^p(\mathbb{Z}^+))$ with respect to the sequence $h \in \mathcal{H}$ if $U_{-h(n)} PAPU_{h(n)}$ tends *-strongly to A_h as $n \to \infty$. Here, *-strong convergence means strong convergence of the sequence itself and of its adjoint sequence.

Notice that every operator can possess at most one limit operator with respect to a given sequence $h \in \mathcal{H}$. The set $\sigma_{op}(A)$ of all limit operators of a given operator A is the *operator spectrum* of A.

It is not hard to see that every limit operator of a Fredholm operator is invertible. A basic result of [2] claims that the operator spectrum of a *band-dominated operator* is rich enough in order to guarantee the reverse implications. Here is a summary of the results from [2] needed in what follows. A comprehensive treatment of this topic is in [4]; see also the references mentioned there.

Theorem 1 *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator. Then*

(*a*) *every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the limit operator A_g exists.*

(*b*) *the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.*

An elegant proof which also works for band-dominated operators on other discrete groups than $\mathbb{Z}$ is due to Roe [6].

Thus, and by the above mentioned equivalence between stability of the sequence $\mathbf{A}$ and Fredholmness of the associated operator $\text{Op}(\mathbf{A})$, one will get a stability criterion for $\mathbf{A}$ by computing all limit operators of $\text{Op}(\mathbf{A})$. This computation has been carried out in [2, 3, 5], see also Chapter 6 in [4]. Here is the result.

Theorem 2 *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator. Then the finite sections sequence $(P_n A P_n)_{n\geq 1}$ is stable if and only if the operator $PAP + Q$ and all operators*

$$QA_hQ + P \quad \textit{with} \quad A_h \in \sigma_{op}(A)$$

are invertible on $l^p(\mathbb{Z})$, and if the norms of their inverses are uniformly bounded.

The goal of this note is to show that the uniform boundedness condition in Theorem 2 can be removed.

2 Main result

For our goal, we will need a subsequence version of Theorem 2. We choose and fix a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$. Further, we write $\mathcal{H}_\eta$ for the set of all (infinite) subsequences of η and $\sigma_{op,\eta}(A)$ for the collection of all limit operators of A with respect to subsequences of η. Then we have the following version of Theorem 2.

Theorem 3 *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)} A P_{\eta(n)})_{n\geq 1}$ is stable if and only if the operator $PAP + Q$ and all operators*

$$QA_hQ + P \quad \textit{with} \quad A_h \in \sigma_{op,\eta}(A)$$

are invertible on $l^p(\mathbb{Z})$, and if the norms of their inverses are uniformly bounded.

Thus, instead of all limit operators of A with respect to monotonically increasing sequences h, only those with respect to subsequences of η are involved. The following proof of Theorem 3 is an adaptation of the proof of Theorem 2 given in [5].

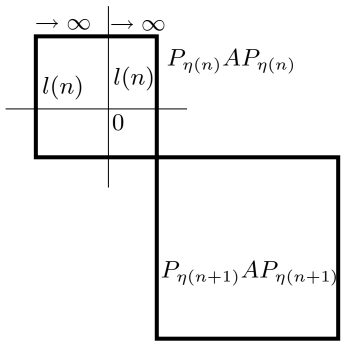
Proof. Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator, abbreviate $\mathbf{A}_\eta := (P_{\eta(n)} A P_{\eta(n)})$, and associate with the sequence $\mathbf{A}_\eta$ the block diagonal operator

$$\begin{aligned} \text{Op}(\mathbf{A}_\eta) &:= \text{diag}(P_{\eta(1)} A P_{\eta(1)},\, P_{\eta(2)} A P_{\eta(2)},\, P_{\eta(3)} A P_{\eta(3)},\, \ldots) \\ &= \sum_{n=1}^{\infty} U_{\mu(n)} P_{\eta(n)} A P_{\eta(n)} U_{-\mu(n)} \end{aligned}$$

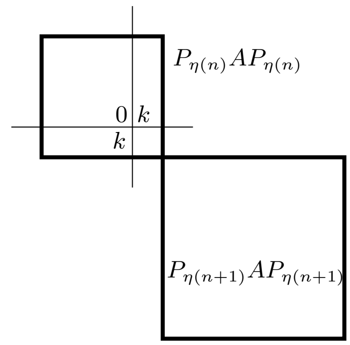
acting on $l^p(\mathbb{Z}^+)$ where $\mu(1) := 0$ and $\mu(n) := \eta(1) + \ldots \eta(n-1)$ for $n \geq 2$, and where the series converges in the strong operator topology. It is still true that

$\mathrm{Op}\,(\mathbf{A}_\eta)$ is a band-dominated operator on $l^p(\mathbb{Z}^+)$ and that the sequence $\mathbf{A}_\eta$ is stable if and only if the operator $\mathrm{Op}\,(\mathbf{A}_\eta)$ is Fredholm.

Let $h \in \mathcal{H}$ be a sequence which tends to infinity and for which the limit operator $\mathrm{Op}\,(\mathbf{A}_\eta)_h$ exists. We call numbers of the form $\eta(1) + \eta(2) + \ldots + \eta(n)$ η-triangular and distinguish between two cases: Either there is a subsequence g of h such that the distance from $g(n)$ to the set of all η-triangular numbers tends to infinity as $n \to \infty$, or there are a $k \in \mathbb{Z}$ and a subsequence g of h such that $g(n)+k$ is η-triangular for all n. The figures below illustrate the shifted operator $U_{-g(n)}\mathrm{Op}\,(\mathbf{A}_\eta)U_{g(n)}$ in the neighborhood of its 00-entry (marked by 0).



Case 1



Case 2

In the first case, we let Δ_n denote the largest η-triangular number which is less than $g(n)$. Then $l(n) := g(n) - \Delta_n$ defines a sequence l which tends to infinity, and the limit operator $\mathrm{Op}\,(\mathbf{A}_\eta)_h = \mathrm{Op}\,(\mathbf{A}_\eta)_g$ of $\mathrm{Op}\,(\mathbf{A}_\eta)$ coincides with the limit operator A_l of A.

Let now g be a subsequence of h such that each $g(n) + k$ is η-triangular for some integer k. Then the sequence l defined by $l(n) := g(n)+k$ tends to infinity, the limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$ with respect to the sequence l exists, and

$$\mathrm{Op}\,(\mathbf{A}_\eta)_l = U_{-k}\mathrm{Op}\,(\mathbf{A}_\eta)_g U_k.$$

Let $d(n)$ be the (uniquely determined) positive integer such that

$$l(n) = \eta(1) + \eta(2) + \ldots + \eta(d(n)).$$

The sequence d is strongly monotonically increasing. Thus, the sequence $\eta \circ d$ is a subsequence of η and tends to infinity. Without loss we can assume that the limit operator of A with respect to the sequence $\eta \circ d$ exists (otherwise we pass to a suitable subsequence of d and, hence, of l and g). Then

$$\begin{aligned}\mathrm{Op}\,(\mathbf{A}_\eta)_h &= \mathrm{Op}\,(\mathbf{A}_\eta)_g \\ &= U_k\mathrm{Op}\,(\mathbf{A}_\eta)_l U_{-k} \\ &= U_k(QA_{\eta\circ d}Q + PAP)U_{-k}.\end{aligned}$$

Thus, each limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$ is either a limit operator of A or of the form

$$U_k(QA_{\eta\circ d}Q + PAP)U_{-k} \quad \text{with } k \in \mathbb{Z} \text{ and } A_{\eta\circ d} \in \sigma_{op,\eta}(A). \tag{2}$$

Next we are going to show that, conversely, each limit operator of A and each operator of the form (2) appears as a limit operator of $\mathrm{Op}\,(\mathbf{A}_\eta)$.

Let A_l be a limit operator of A with respect to a sequence $l \in \mathcal{H}$. Choose a strongly monotonically increasing sequence $d : \mathbb{N} \to \mathbb{N}$ such that $\eta(d(n)+1) - l(n) \to \infty$ and set

$$h(n) := (\eta(1) + \eta(2) + \ldots + \eta(d(n))) + l(n).$$

Then $h \in \mathcal{H}$, the limit operator $\mathrm{Op}\,(\mathbf{A}_\eta)_h$ exists, and it is equal to A_l.

Let now $d : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence such that the limit operator $A_{\eta\circ d}$ of A exists, and let $k \in \mathbb{Z}$. Consider

$$h(n) := (\eta(1) + \eta(2) + \ldots + \eta(d(n))) + k.$$

Again, $h \in \mathcal{H}$, the limit operator $\mathrm{Op}\,(\mathbf{A}_\eta)_h$ exists, and now this limit operator is equal to $U_{-k}(QA_{\eta\circ d}Q + PAP)U_k$. Thus,

$$\sigma_{op}(\mathrm{Op}\,(\mathbf{A}_\eta)) = \sigma_{op}(A) \cup \{U_{-k}(QA_hQ + PAP)U_k : k \in \mathbb{Z},\ A_h \in \sigma_{op,\eta}(A)\}.$$

This equality shows that the conditions of the theorem are necessary. They are also sufficient since the invertibility of A implies those of all limit operators of A, and if both A and $QA_hQ + P$ are invertible then the operator $U_{-k}(QA_hQ + PAP)U_k$ is invertible for every integer k. ∎

Corollary 4 *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator, and let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence for which the limit operator A_η exists. Then the sequence $(P_{\eta(n)}AP_{\eta(n)})_{n\geq 1}$ is stable if and only if the operators $PAP + Q$ and $QA_\eta Q + P$ are invertible on $l^p(\mathbb{Z})$.*

Indeed, under the conditions of the corollary, the set $\sigma_{op,\eta}(A)$ is a singleton. Here is the announced main result of the present paper.

Theorem 5 *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)}AP_{\eta(n)})_{n\geq 1}$ is stable if and only if the operator $PAP + Q$ and all operators*

$$QA_hQ + P \quad \textit{with} \quad A_h \in \sigma_{op,\eta}(A)$$

are invertible on $l^p(\mathbb{Z})$.

Proof. The necessity of invertibility of the mentioned operators follows from Theorem 3. Conversely, let $PAP+Q$ and all operators QA_hQ+P with $A_h \in \sigma_{op,\eta}(A)$ be invertible on $l^p(\mathbb{Z})$. Contrary to what we want to show, assume that the sequence $\mathbf{A}_\eta = (P_{\eta(n)}AP_{\eta(n)})$ fails to be stable. Then there is a strongly monotonically increasing sequence $d : \mathbb{N} \to \mathbb{N}$ such that

$$\|(P_{\eta(d(n))}AP_{\eta(d(n))})^{-1}\| \geq n \quad \text{for all } n \in \mathbb{N}$$

where we agree upon writing $\|A_n^{-1}\| = \infty$ if the matrix A_n fails to be invertible. Thus, no subsequence of the sequence $\mathbf{A}_{\eta \circ d}$ is stable.

Let g be a subsequence of $\eta \circ d$ for which the limit operator A_g exists. (The existence of a sequence d with these properties follows from Theorem 1 (a).) Then $A_g \in \sigma_{op,\eta}(A)$, and the operators $PAP+Q$ and QA_gQ+P are invertible by hypothesis. Corollary 4 implies that the subsequence $\mathbf{A}_g$ of $\mathbf{A}_{\eta \circ d}$ is stable. Contradiction. ∎

There is also a version of Theorem 5 for band-dominated operators on $l^p(\mathbb{Z})$ which we will briefly sketch. For $n \in \mathbb{N}$, consider the projections

$$R_n : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (R_nx)(m) := \begin{cases} x(m) & \text{if } -n \leq m < n \\ 0 & \text{otherwise.} \end{cases}$$

The finite sections sequence for an operator A on $l^p(\mathbb{Z})$ is the sequence (R_nAR_n) where the R_nAR_n are viewed of as operators on $\operatorname{im} R_n$, provided with the norm induced by the norm on $L(l^p(\mathbb{Z}))$. The stability of the sequence (R_nAR_n) as well as the notion of a band-dominated operator on $l^p(\mathbb{Z})$ are defined as above, with obvious modifications.

Let again $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence. In $\sigma_{+,\eta}(A)$ and $\sigma_{-,\eta}(A)$, we collect all limit operators of A with respect to subsequences of η and of $-\eta$, tending to $+\infty$ and $-\infty$, respectively. The following can be proved in the same vein as Theorem 5.

Theorem 6 *Let $A \in L(l^p(\mathbb{Z}))$ be a band-dominated operator. Then the finite sections sequence $(R_nAR_n)_{n\geq 1}$ is stable if and only if the operator A, all operators*

$$QA_hQ+P \quad \textit{with} \quad A_h \in \sigma_+(A)$$

and all operators

$$PA_hP+Q \quad \textit{with} \quad A_h \in \sigma_-(A)$$

are invertible on $l^p(\mathbb{Z})$.

We would like to mention that the stability of the finite sections sequence for band-dominated operators on l^∞ can be studied as well. This involves some technical subtleties (when working with adjoint sequences, for instance), but it is easier with respect to the concern of the present paper: Indeed, for $p = \infty$, the uniform boundedness condition in Theorem 1 (b) is already redundant. For much more on this topic, we refer to the recent textbook [1].

It remains an open question whether the uniform boundedness condition in Theorem 1 (b) is redundant for $p \in (1, \infty)$ or at least for $p = 2$.

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