A mixed boundary value problem for the prescribed mean curvature equation

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Abstract

We solve a mixed boundary value problem for the nonparametric prescribed mean curvature equation, prescribing continuous Dirichlet boundary values at some strictly convex boundary part and Neumann zero boundary values at the remaining part of the boundary. We assume that Dirichlet and Neumann boundary parts are some positive distance away from each other.

1. Introduction and the main result

For a given $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^2$ we consider the following mixed boundary value problem for the nonparametric prescribed mean curvature equation

$$\begin{aligned} &\operatorname{div} \frac{\nabla \zeta}{\sqrt{1+|\nabla \zeta|^2}} = 2H(x,y,\zeta) \quad \text{in } \Omega \\ &\zeta = g \quad \text{on } \Gamma_d \quad , \quad \frac{\partial \zeta}{\partial n} = 0 \quad \text{on } \Gamma_n \ , \end{aligned} \tag{1}$$

where n denotes the outer unit normal to $\partial\Omega$ and $H = H(x,y,z)$ is the prescribed mean curvature function. We prescribe Dirichlet boundary values g on some part $\Gamma_d \subset \partial\Omega$ of the boundary and Neumann zero boundary values on the other part $\Gamma_n := \partial\Omega \backslash \Gamma_d$. We will assume $\Gamma_d \neq \emptyset$ since otherwise one obtains a pure Neumann problem which is in general not solvable unless certain compatibility conditions are satisfied.

If we define a function b by $b_z(x,y,z) = 2H(x,y,z)$, then the mixed boundary value problem (1) is the Euler equation of the generalised nonparametric area functional

$$A(\eta) := \int_\Omega \Big(\sqrt{1+|\nabla \eta|^2} + b(x,y,\eta) \Big) dxdy$$

within the class of functions

$$\mathcal{C}(\Omega, \Gamma_d, g) := \Big\{ \eta \in C^2(\Omega, \mathbb{R}) \cap C^1(\overline{\Omega}, \mathbb{R}) \mid \eta = g \quad \text{on } \Gamma_d \Big\} \ .$$

As there are no boundary values prescribed on Γ_n within the class $\mathcal{C}$, the Neumann boundary condition $\frac{\partial \zeta}{\partial n} = 0$ appears as the natural boundary condition on Γ_n.

Using the continuity method we will construct a solution of (1) as in [16] (see also [17, §8 in chapter XII]), where the case $\Gamma_n = \emptyset$ is treated, i.e. pure Dirichlet problem. Two important assumptions were needed there: The strict convexity of the domain Ω and monotonocity condition $H_z \geq 0$ on the prescribed mean curvature. The second assumption is needed for uniqueness of solutions via the maximum principle but also for stability of solutions in the sence that a perturbation result can be proved.

We will also assume $H_z \geq 0$ in this work. However, the strict convexity is needed only at the Dirichlet boundary part Γ_d, i.e. $\kappa(x,y) > 0$ on Γ_d for the curvature $\kappa : \partial\Omega \to \mathbb{R}$ of $\partial\Omega$ w.r.t. the inner normal. Hence, no curvature assumption concering the Neumann boundary part Γ_n is needed. Next we have to assume $\text{dist}(\Gamma_d, \Gamma_n) > 0$, i.e. Dirichlet and Neumann boundary parts do not touch each other. In case $\Gamma_n \neq \emptyset$ this directly implies that the domain Ω cannot be simply connected. For the case of touching Dirichlet and Neumann boundary parts and simply connected domains the mixed boundary value problem was solved in [8] and [9] for the minimal surface case and in [11, Theorem 2] for the prescribed mean curvature case, where however a certain smallness condition on H is needed. An existence result for convex domains Ω in higher dimensions $n \geq 2$ can be found in [3].

A typical domain suitable in this paper is the following annular domain: Considering two simply connected $C^{2+\alpha}$-domains Ω_0, Ω_1 such that Ω_0 is strictly convex and $\overline{\Omega}_1 \subset \Omega_0$ we define $\Omega := \Omega_0 \backslash \overline{\Omega}_1$. Here, the Dirichlet boundary part is $\Gamma_d := \partial\Omega_0$ and the Neumann boundary part is $\Gamma_n := \partial\Omega_1$. More generally, we can also consider domains with finitely many holes rather than just one hole.

The main result of this paper is the following

Theorem 1: *Assumptions:*

a) Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2+\alpha}$-domain with the Dirichlet and Neumann boundary parts $\Gamma_d \neq \emptyset$ and Γ_n such that Γ_d is stricty convex and $dist(\Gamma_d, \Gamma_n) > 0$.

b) Let the prescribed mean curvature $H \in C^{1+\alpha}(\mathbb{R}^3, \mathbb{R})$ satisfy

$$H_z \geq 0 \quad \text{in } \Omega \times \mathbb{R} \quad \text{and} \quad 2|H(x,y,z)| < \kappa(x,y) \quad \text{for } (x,y,z) \in \Gamma_d \times \mathbb{R}$$

where $\kappa(x,y)$ denotes the curvature of $\partial\Omega$ w.r.t. the inner normal.

c) Assume that the mixed boundary value problem (1) has a solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ for some boundary values $g_0 \in C^{2+\alpha}(\Gamma_d, \mathbb{R})$.

Then there exists a unique solution $\zeta \in C^{2+\alpha}(\Omega \cup \Gamma_n, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ for all boundary values $g \in C^0(\Gamma_d, \mathbb{R})$. In case $g \in C^{2+\alpha}(\Gamma_d, \mathbb{R})$, that solution belongs to the space $C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$.

An essential assumption of Theorem 1 is the existence of an initial solution for certain boundary values g_0. For the minimal surface case, i.e. $H \equiv 0$, such an initial solution always exists with $\zeta \equiv 0$ in Ω for $g_0 \equiv 0$ on Γ_d. The same applies to all prescribed mean curvatures H with the property $H(x,y,0) = 0$ for $(x,y) \in \Omega$.

On the other hand, note that under the assumptions $a)$ and $b)$ of Theorem 1 an initial solution need not necessarily exist. Consider e.g. the prescribed mean curvature $H = H(x,y) = 4 - x^2 - y^2$ and the disc $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$ with $\Gamma_d := \partial\Omega$ and $\Gamma_n = \emptyset$. Note that the assumption $2|H(x,y)| < \kappa(x,y)$ on $\partial\Omega$ of Theorem 1 is satisfied since $H = 0$ on $\partial\Omega$. However, no graph of prescribed mean curvature H over Ω exists due to $H(x,y) > 1$ for $x^2 + y^2 \leq 1$. Hence, we may also view Theorem 1 as a nonexistence theorem. Assuming that the mixed boundary value problem is not solvable for certain Dirichlet boundary values, it will not be solvable for any Dirichlet boundary values.

Now to obtain a solution of problem (1), we will employ the continuity method and study the following family $P(t)$ of mixed boundary value problems

$$\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = 2H(x,y,\zeta) \quad \text{in } \Omega \tag{2}$$
$$\zeta = t\,g + (1-t)g_0 \quad \text{on } \Gamma_d \quad , \quad \frac{\partial\zeta}{\partial n} = 0 \quad \text{on } \Gamma_n$$

with a parameter $t \in [0,1]$. Let $J \subset [0,1]$ be the set of all $t \in [0,1]$ for which $P(t)$ is solvable. Note that by assumption $c)$ of Theorem 1 we have $0 \in J$. It remains to show that J is both open and closed. The openness is basically a consequence of a perturbation result for graphs of prescribed mean curvature, which for the case $\Gamma_n = \emptyset$ is shown in [16, Proposition 2] or [17, Hilfssatz 4 in §8, chapter XII]. It can directly be generalized to the case $\Gamma_n \neq \emptyset$ using the assumption $\operatorname{dist}(\Gamma_d, \Gamma_n) > 0$. To show that J is closed a compactness result for graphs of prescribed mean curvature has to be proved. As the behavior at the Dirichlet boundary is basically known and studied we focus on the behavior at the Neumann boundary. In section 2 we start with an a priori $C^{2+\alpha}$-estimate up to the Neumann boundary for a conformal reparametrisation of the graph. We use it in section 3 to prove a compactness result for graphs which yields the closeness of the set J. Finally, we give the proof of Theorem 1 in section 4.

2. Local estimates at the Neumann boundary

We start with a solution $\zeta \in C^{2+\alpha}(\Omega \cup \Gamma_d, \mathbb{R})$ of problem (1) on a bounded $C^{2+\alpha}$-domain Ω. We now choose a simply connected part $T \subset \Gamma_n$ with its endpoint $(x_0, y_0), (x_2, y_2) \in T$ and some third point $(x_1, y_1) \in T$. Next, we choose a simply connected subset $C^{2+\alpha}$-domain $\Theta \subset \Omega$ with the properties $T \subset \partial\Omega \cap \partial\Theta$ and $\operatorname{dist}(\Theta, \Gamma_d) > 0$.

Defining the open half disc $B^+ := \{(u,v) \in \mathbb{R}^2 \,|\, u^2 + v^2 < 1 \ , \ v > 0\}$, we introduce conformal parameters on the graph $\zeta|_{\overline{\Theta}}$ (see [15]): There exists a positively oriented diffeomorphism

$$f : \overline{B^+} \to \overline{\Theta} \in C^0(\overline{B^+}, \mathbb{R}^2) \cap C^{2+\alpha}(\overline{B^+} \backslash \{(-1,0),(1,0)\}, \mathbb{R}^2) \tag{3}$$

satifying the three point condition

$$f(-1,0) = (x_0, y_0) \quad , \quad f(0,0) = (x_1, y_1) \quad \text{and} \quad f(1,0) = (x_2, y_2) \tag{4}$$

and having a positive Jakobi determinant $J_f := \det(f_u, f_v) > 0$ in B^+ such that the reparametrized graph

$$X(u,v) := (f(u,v), \zeta \circ f(u,v)) \tag{5}$$

is given in conformal parameters. The vector valued function X satisfies the prescribed mean curvature system together with the conformality relations

$$\triangle X = 2H(X) X_u \wedge X_v \quad \text{and} \quad |X_u|^2 - |X_v|^2 = X_u \cdot X_v = 0 \quad \text{in } B^+ .$$

From the three point condition we infer $f(I) \subset T \subset \partial\Omega$ for the interval $I := (-1,1) \times \{0\}$. We first need the following lemma, which shows how the Neumann condition for the graph ζ translates to the conformal reparametrisation.

Lemma 1: *The Neumann boundary condition $\frac{\partial\zeta}{\partial n} = 0$ on T is equivalent to the boundary condition $X_v \cdot e_3 = 0$ on I where $e_3 := (0,0,1)$.*

Proof:

1.) We assume $\nabla\zeta\cdot n = 0$ and set $X(u,v) = (f(u,v), z(u,v))$ with $z(u,v) = \zeta\circ f(u,v)$. From $\det(f_u, f_v) > 0$ we first conclude $|f_u| > 0$ and $|f_v| > 0$. Using $f(I)\subset\partial\Omega$ we obtain that the vector f_u is perpendicular to n and hence $\nabla\zeta = \lambda f_u$ for some $\lambda\in\mathbb{R}$. Using the conformality relations we compute

$$\begin{aligned} 0 &= X_u\cdot X_v = f_u\cdot f_v + z_u z_v = f_u\cdot f_v + (\nabla\zeta\cdot f_u)(\nabla\zeta\cdot f_v) \\ &= f_u\cdot f_v + \lambda^2 (f_u\cdot f_u)(f_u\cdot f_v) = (1+\lambda^2|f_u|^2)(f_u\cdot f_v) \end{aligned}$$

concluding $f_u\cdot f_v = 0$. Together with $f_u\cdot n = 0$ and $|f_u| > 0$ we know that $f_v = \sigma n$ for some $\sigma\in\mathbb{R}$. Noting the assumption $\nabla\zeta\cdot n = 0$ this gives $X_v\cdot e_3 = z_v = \nabla\zeta\cdot f_v = \sigma\nabla\zeta\cdot n = 0$.

2.) Assume $X_v\cdot e_3 = z_v = 0$. From the conformality relation $X_u\cdot X_v = 0$ we conclude $f_u\cdot f_v = 0$. Since f_u is a nonzero tangent vector to $\partial\Omega$ we obtain that the vectors f_v and n are linearly dependent. From the relation $0 = z_v = \nabla\zeta\cdot f_v$ we conclude $\nabla\zeta\cdot n = 0$, using $|f_v| > 0$. $\square$

We now define $B_r^+ := B_r(0,0)\cap B^+ = \{(u,v)\in B^+\,|\,u^2+v^2 < r^2\}$ and show an a priori estimate of X in B_r^+.

Lemma 2: *Assume that the C^0-estimate*

$$|\zeta(x,y)| \le M \quad \text{in}\,\Omega \tag{6}$$

holds. Then the exist constants $r > 0$ and $C < \infty$ such that the conformal reparametrisation X of ζ in (5) satisfies the estimate

$$||X||_{C^{2+\alpha}(B_r^+)} \le C\ .$$

The constants r and C only depend on the data Ω, M, H, α and the modulus of continuity of X.

Proof:

1.) Setting

$$X(u,v) = (x(u,v), y(u,v), z(u,v))$$

we note that $f(u,v) = (x(u,v), y(u,v))$ for f from (5). Lemma 1 yields the Neumann boundary condition $X_v^3 = z_v = 0$ on I. Using the complex differential operator $\frac{\partial}{\partial w} := \frac{1}{2}\frac{\partial}{\partial u} - \frac{i}{2}\frac{\partial}{\partial v}$ the conformality relations for X can be written equivalently in the form

$$0 = X_w\cdot X_w = (x_w)^2 + (y_w)^2 + (z_w)^2\ .$$

Noting $|x_w^2| = |x_w|^2 = \frac{1}{4}|\nabla x|^2$ and similar formulas for y and z we deduce the inequalities

$$|\nabla x|^2 \le |\nabla y|^2 + |\nabla z|^2 \quad , \quad |\nabla y|^2 \le |\nabla x|^2 + |\nabla z|^2 \quad \text{and} \quad |\nabla z|^2 \le |\nabla x|^2 + |\nabla y|^2\ . \tag{7}$$

Using this together with $\triangle X = 2H(X)X_u\wedge X_v$ we now estimate

$$|\triangle X| \le 2c|X_u\wedge X_v| = c|\nabla X|^2 = c(|\nabla x|^2 + |\nabla y|^2 + |\nabla z|^2) \le 2c(|\nabla y|^2 + |\nabla z|^2)$$

which yields

$$|\triangle x| + |\triangle y| + |\triangle z| \le 3|\triangle X| \le 6c(|\nabla y|^2 + |\nabla z|^2)\ . \tag{8}$$

Here we have set

$$c := \sup_{(x,y)\in\Omega\,,\,|z|\le M} |H(x,y,z)| < \infty\ .$$

2.) We now prove an estimate for $|\nabla X|$. We consider the point $(x_1, y_1) = f(0,0)$ from the three point condition. After a suitable rotation in $\mathbb{R}^2$ we can locally represent $\partial\Omega$ in a neighborhood of (x_1, y_1) by a graph of some function β, more precisely: There exists a constant $d > 0$ and a function $\beta \in C^{2+\alpha}(J, \mathbb{R})$ with $J := [x_1 - d, x_1 + d]$ such that the following representation holds

$$\partial\Omega \cap B_d(x_1, y_1) \subset \{(x, \beta(x)) \in \mathbb{R}^2 \mid x \in J\} .$$

Here, we may also assume that

$$\beta'(x_1) = 0 \quad , \quad |\beta'(t)| \leq \frac{1}{2} \quad \text{for } t \in J \quad \text{and} \quad ||\beta||_{C^{2+\alpha}(J)} \leq K . \tag{9}$$

Since Ω is a bounded $C^{2+\alpha}$-domain the constants $d = d(\Omega) > 0$ and $K = K(\Omega) < \infty$ can be choosen independently of the point (x_1, y_1). Due to $f(0,0) = (x_1, y_1)$ and the continuity of f we can choose some constant $r > 0$, depending on the modulus of continuity of f, such that $f(u,v) \in B_d(x_1, y_1)$ for $(u,v) \in \overline{B_r^+}$ holds. Hence, the following auxiliary function

$$h \in C^{2+\alpha}(\overline{B_r^+}, \mathbb{R}) \quad , \quad h(u,v) := y(u,v) - \beta \circ x(u,v) \quad \text{for } (u,v) \in \overline{B_r^+}$$

is welldefined. From $f(I) \subset \partial\Omega$ we obtain for h the boundary condition

$$h(u, 0) = 0 \quad \text{for } u \in (-r, r) . \tag{10}$$

Using (8) and (7) we estimate

$$\begin{aligned} |\triangle z| + |\triangle h| &= |\triangle z| + |\triangle y - \beta' \triangle x - \beta''(x_u^2 + x_v^2)| \\ &\leq |\triangle z| + |\triangle y| + |\triangle x| + K|\nabla x|^2 \\ &\leq 6c(|\nabla y|^2 + |\nabla z|^2) + K(|\nabla y|^2 + |\nabla z|^2) \\ &= (6c + K)(|\nabla y|^2 + |\nabla z|^2) . \end{aligned} \tag{11}$$

To obtain a bound for $|\nabla y|^2$ in terms of $|\nabla h|^2$ and $|\nabla z|^2$ we compute

$$\begin{aligned} |\nabla y|^2 &= |\nabla h + \beta' \nabla x|^2 \leq \Big(|\nabla h| + \frac{1}{2}|\nabla x|\Big)^2 \\ &\leq 2|\nabla h|^2 + \frac{1}{2}|\nabla x|^2 \leq 2|\nabla h|^2 + \frac{1}{2}|\nabla y|^2 + \frac{1}{2}|\nabla z|^2 \end{aligned}$$

which yields the inequality

$$|\nabla y|^2 \leq 4|\nabla h|^2 + |\nabla z|^2 . \tag{12}$$

Combining this estimate with (11) we obtain

$$|\triangle h| + |\triangle z| \leq (24c + 4K)|\nabla h|^2 + (12c + 2K)|\nabla z|^2 \quad \text{in } B_r^+ .$$

We now define another auxiliary function

$$\psi : B_r^+ \to \mathbb{R}^2 \quad , \quad \psi(u,v) := (h(u,v), z(u,v))$$

and note the differential inequality $|\triangle \psi| \leq (24c + 4K)|\nabla \psi|^2$. We want to apply the interior a priori estimates for systems with quadratic growth in the gradient (see [1, chapter 7.2] or [17, chapter XII, §2]). To obtain the required smallness condition for these estimates, we reduce the constant $r > 0$ such that

$$|\psi(u,v) - \psi(0,0)| \leq \frac{1}{2(24c + 4K)} \quad \text{in } B_{r'}^+$$

holds. Here, the choice of r' is determined by the modulus of continuity for ψ which is controlled by the modulus of continuity of X. We now reflect ψ across I by

$$\tilde{\psi}(u,v) := \begin{cases} (h(u,v), z(u,v)) & \text{if } (u,v) \in \overline{B^+_{r'}} \\ (-h(u,-v), z(u,-v)) & \text{if } (u,-v) \in B^+_{r'} \end{cases} \quad , \ (u,v) \in B_{r'} \ .$$

By the Dirichlet boundary condition (10) for h and the Neumann boundary condition $z_v = 0$ on I we conclude the regularity $\tilde{\psi} \in C^{1,1}(B_{r'}, \mathbb{R}^2)$, i.e. the first derivatives of $\tilde{\psi}$ exist in $B_{r'}$ and they are Lipschitz continuous. Furthermore, we have the inequality $|\triangle\tilde{\psi}| \leq (24c + 4K)|\nabla\tilde{\psi}|^2$ in $B_{r'}\backslash I$. Now the a priori estimates for systems with quadratic growth in the gradient, which in general hold only for C^2-functions, can also be applied to the function $\tilde{\psi}$ after a suitable approximation process by C^2-functions, e.g. by the Friedrich's mollifiers of $\tilde{\psi}$. For any $r < r'$ there exists a constant $C_1 = C_1(c, K, r, \alpha)$ such that

$$||z||_{C^{1+\alpha}(B^+_r)} + ||h||_{C^{1+\alpha}(B^+_r)} \leq C_1 \ . \tag{13}$$

Together with (12) and (7) we can find a constant C_2 such that

$$|\nabla x| + |\nabla y| + |\nabla z| \leq C_2 \quad \text{in } B^+_r \ .$$

3.) In the next step we show a Hölder estimate for ∇x and ∇y. To do this, we first derive a boundary condition for x_v on I. By the conformality relations, we have

$$0 = x_u x_v + y_u y_v + z_u z_v = x_u x_v + y_u y_v \quad \text{on } I$$

using $z_v = 0$ on I. Putting $y = h + \beta \circ x$ into this equation yields

$$\begin{aligned} 0 &= x_u x_v + (h_u + \beta' x_u)(h_v + \beta' x_v) = x_u x_v + (h_v + \beta' x_v)\beta' x_u \\ &= x_u x_v (1 + (\beta')^2) + h_v \beta' x_u \quad \text{on } I \end{aligned} \tag{14}$$

taking $h_u = 0$ on I into account. We now claim $x_u \neq 0$ on I. Otherwhise we would have $0 = h_u = y_u - \beta' x_u$ and so $y_u = 0$, contradicting $\det(f_u, f_v) = x_u y_v - x_v y_u > 0$ which holds because f is a positively oriented diffeomorphism. Dividing by x_u in (14) we obtain for x the Neumann boundary condition

$$x_v = -\frac{\beta'(x) h_v}{1 + (\beta')^2} \quad \text{on } I \ . \tag{15}$$

By (13) there exists a C^α-bound on the right hand side of this equation. Noting (8) potentialtheoretic estimates give a constant C_3 such that

$$||x||_{C^{1+\alpha}(B^+_r)} \leq C_3 \ .$$

Using $y = h + \beta \circ x$ we can then also bound the $C^{1+\alpha}(B^+_r)$-norm of y.

4.) Finally we will show the $C^{2+\alpha}$-estimate for x, y and z. Due to $\triangle z = 2H(X)(x_u y_v - x_v y_v)$ and the boundary condition $z_v = 0$ on I, we can first find a constant C_4 such that

$$||z||_{C^{2+\alpha}(B^+_r)} \leq C_4$$

holds, employing Schauder a priori estimates for Neumann boundary values. Next, using the differential equation

$$\begin{aligned} \triangle h &= \triangle y - \beta' \triangle x - \beta'' |\nabla x|^2 \\ &= 2H(X)(z_u x_v - z_v x_u) - 2\beta' H(X)(y_u z_v - z_u y_v) - \beta'' |\nabla x|^2 \end{aligned}$$

together with the Dirichlet boundary condition (10) we can give a bound for the $C^{2+\alpha}(B_r^+)$-norm of h. The Neumann condition (15) together the differential equation $\triangle x = 2H(X)(y_u z_v - z_u y_v)$ then yield the estimate

$$||x||_{C^{2+\alpha}(B_r^+)} \leq C_5$$

for some constant C_5. Finally, using $y = h + \beta \circ x$ we can find a bound for the $C^{2+\alpha}(B_r^+)$-norm of y. $\square$

Remarks:

1.) Some of the arguments of the proof are similar to those of the boundary regularity theorem [1, Theorem 2 in section 7.3] for surfaces of prescribed mean curvature with a Plateau boundary (see also [7]).

2.) The proof of this result does not need the graph property of the solution. Hence it can also be applied to any conformally parametrized H-surface X meeting the boundary of cylinder $Z_\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \Omega\}$ orthogonally on I. Such surfaces appear as critical points of the parametric functional

$$E_Q(X) := \int\limits_{B^+} \Big(|\nabla X|^2 + Q(X) \cdot X_u \wedge X_v\Big) du dv$$

within a certain class of functions. The vector field Q must satisfy $\operatorname{div} Q(x, y, z) = 2H(x, y, z)$ and meet the boundary of the cylinder Z_Ω orthogonally (see [11] for more details). When considering critical points of the functional E_Q, the crucial part is first to obtain a modulus of continuity of X, as our a priori estimate depends on it.

3.) As we are dealing with graphs, it is relatively easy to obtain a modulus of continuity for the conformal reparametrisation X of the graph ζ (see the proof of Theorem 2).

3. Two compactness results

We start this section with the following result.

Theorem 2*:*
Assumptions:

a) A bounded $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^2$ is given with the Dirichlet boundary part $\Gamma_d \subset \partial\Omega$ and the Neumann boundary part $\Gamma_n := \partial\Omega \backslash \Gamma_d$.

b) The prescribed mean curvature $H \in C^{1+\alpha}(\mathbb{R}^3, \mathbb{R})$ satisfies

$$\frac{\partial}{\partial z} H = H_z \geq 0 \quad \mathit{in}\, \Omega \times \mathbb{R}\,.$$

c) A given sequence of boundary values $g^n \in C^{2+\alpha}(\Gamma_d, \mathbb{R})$ converges uniformly to some limit function $g \in C^0(\Gamma_d, \mathbb{R})$.

d) Let $\zeta^n \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of problem (1) for Dirichlet boundary values g^n on Γ_d.

Then problem (1) has a solution $\zeta \in C^{2+\alpha}(\Omega \cup \Gamma_n, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ for Dirichlet boundary boundary values g on Γ_d.

Proof:

1.) Each ζ^n is solution of the quasilinear, elliptic equation (1). Then the difference function $\tilde{\zeta} := \zeta^n - \zeta^m$ for fixed n,m is the solution of $\mathcal{M}(\tilde{\zeta}) = 0$ in Ω for some linear, elliptic differential operator $\mathcal{M}$ (see [17, chapter VI, §2]) which by the assumption $H_z \geq 0$ is subject to the maximum principle. The maximum principle together with Hopf's boundary point lemma then gives the estimate

$$\begin{aligned} ||\zeta^m - \zeta^n||_{C^0(\Omega)} &\leq ||\zeta^m - \zeta^n||_{C^0(\partial\Omega)} \leq ||\zeta^m - \zeta^n||_{C^0(\Gamma_d)} \\ &= ||g^m - g^n||_{C^0(\Gamma_d)} \to 0 \quad \text{for } m,n \to \infty \,. \end{aligned}$$

Thus the sequence ζ^n converges uniformly to some limit function $\zeta \in C^0(\overline{\Omega}, \mathbb{R})$ with $\zeta = g$ on Γ_d. Furthermore, we can find a constant $M \in (0, +\infty)$ independent of n such that $||\zeta^n||_{C^0(\Omega)} \leq M$ holds.

2.) By interior estimates for graphs of prescribed mean curvature (see e.g. [14]) there is a constant $C_1 = C_1(r)$ for each $r > 0$ such that

$$||\zeta^n||_{C^{2+\alpha}(\Omega_r)} \leq C_1$$

holds where $\Omega_r := \{(x,y) \in \Omega \,|\, \mathrm{dist}((x,y), \partial\Omega) \geq r\}$. We conclude that $\zeta \in C^{2+\alpha}(\Omega, \mathbb{R})$ and that ζ satisfies the differential equation (1) in Ω.

3.) We now show that ζ is smooth up to the Neumann boundary Γ_n and that the Neumann boundary condition is satisfied. To do this, we introduce conformal parameters

$$X^n(u,v) = (f^n(u,v), \zeta^n \circ f^n(u,v)) \quad \text{for } (u,v) \in \overline{B^+}$$

on each graph $\zeta^n|_{\overline{\Theta}}$ for some simply connected domain $\Theta \subset \Omega$ having the properties described in the beginning of section 2. Employing the area estimate for graphs of bounded mean curvature (see [13, Hilfssatz 11])

$$A(\zeta^n) = \int\limits_{\Omega} \sqrt{1 + |\nabla\zeta^n|^2} dxdy \leq C_2(\Omega, H, M)$$

we first obtain a bound on the Dirichlet integral

$$D(f^n) = \int\limits_{B^+} |\nabla f^n|^2 dudv \leq \int\limits_{B^+} |\nabla X^n|^2 dudv = 2 \int\limits_{B^+} |X_u^n \wedge X_v^n| dudv = 2A(X^n) \leq 2C_2$$

using the conformal parametrization of X^n. Similarly to [6, Lemma 16], one can now derive a uniform modulus of continuity for f^n with the Courant-Lebesgue-Lemma using the three point condition and the fact that f^n is injective. Noting $X^n = (f^n, \zeta^n \circ f^n)$ and the uniform convergence of ζ^n, we can derive a uniform modulus of continuity for the sequence X^n. Hence, by Lemma 2 there are constants $r > 0$ and $C_3 < \infty$ independent of n such that

$$||X||_{C^{2+\alpha}(B_r^+)} \leq C_3$$

holds. After extracting a convergent subsequence we have the convergence $X^n \to X$ in $C^2(B_r^+, \mathbb{R}^3)$ with some limit function $X \in C^{2+\alpha}(\overline{B_r^+}, \mathbb{R}^3)$. We set

$$X(u,v) = (x(u,v), y(u,v), z(u,v)) \quad \text{and} \quad f(u,v) = (x(u,v), y(u,v))$$

and obtain $z(u,v)=\zeta\circ f(u,v)$ in B_r^+. Due to the C^2-convergence X satisfies the prescribed mean curvature system as well as the conformality relations

$$\triangle X = 2H(X)X_u \wedge X_v \quad , \quad |X_u|^2-|X_v|^2 = 0 = X_u\cdot X_v \quad \text{in } B_r^+ \ .$$

Using an interior lower bound of the area element $W^n=|X_u^n\wedge X_v^n|$ (see [17, Satz 1 in chapter XII, §9]) one can exclude interior branchs point for X, i.e. $|X_u\wedge X_v|>0$ in B_r^+. Defining the interval $\gamma:=(-r,r)\times\{0\}$ we conclude from $f^n(B^+)\subset\overline{\Omega}$ and $f^n(\gamma)\subset\partial\Omega$ that

$$f(B_r^+)\subset\overline{\Omega} \quad \text{and} \quad f(\gamma)\subset\partial\Omega$$

holds. Now with the same reasoning as in the proof of [11, Lemma 5] one can exclude boundary branch points on γ and show that $J_f(u,v)=\det(f_u,f_v)>0$ on γ holds. By the inverse function theorem then f is locally invertible. Noting that both z and f are of regularity class $C^{2+\alpha}$ then the relation $\zeta(x,y)=z\circ f^{-1}(x,y)$ gives $\zeta\in C^{2+\alpha}(\overline{\Omega}\cap U,\mathbb{R})$ for some open neighborhood $U=U(x_1,y_1)$ of the point $f(0,0)=(x_1,y_1)$. Since the point $(x_1,y_1)\in\Gamma_n$ from the three point condition can be choosen arbitrarily, we get $\zeta\in C^{2+\alpha}(\Omega\cup\Gamma_n,\mathbb{R})$. Finally, noting the Neumann boundary condition $X_v^n\cdot e_3=0$ on γ we obtain for $n\to\infty$ the condition $X_v\cdot e_3=0$ on γ and from Lemma 1 we deduce the Neumann boundary condition

$$\frac{\partial\zeta}{\partial n}=0 \quad \text{on} \quad \Gamma_n \ ,$$

ending the proof. □

The next result treats the boundary regularity at the Dirichlet boundary part Γ_d.

Corollary 1: *Let the assumptions of Theorem 2 be satified and additionally $g\in C^{2+\alpha}(\Gamma_d,\mathbb{R})$ as well as*

$$2|H(x,y,z)|<\kappa(x,y) \quad \textit{for } (x,y,z)\in\Gamma_d\times\mathbb{R} \tag{16}$$

where $\kappa:\partial\Omega\to\mathbb{R}$ denotes the curvature of $\partial\Omega$ w.r.t. the inner normal.
Then problem (1) has a solution $\zeta\in C^{2+\alpha}(\overline{\Omega},\mathbb{R})$ for Dirichlet boundary values g on Γ_d.

Proof:
By Theorem 1, there exists a solution $\zeta\in C^{2+\alpha}(\Omega\cup\Gamma_d,\mathbb{R})\cap C^0(\overline{\Omega},\mathbb{R})$ and we only have to show that ζ is smooth up to the Dirichlet boundary Γ_d. We choose a simply connected subset $T\subset\Gamma_d$ with its endpoint (x_0,y_0) and (x_2,y_2) and some other point $(x_1,y_1)\in T$. Next we choose a simply connected domain $\Theta\subset\Omega$ such that $T\subset\partial\Omega\cap\partial\Theta$. We introduce conformal parameters

$$X^n(u,v)=(f^n(u,v),\zeta^n\circ f^n(u,v)) \quad \text{for } (u,v)\in\overline{B^+}$$

on each graph $\zeta^n|_{\overline{\Theta}}$ with f^n satisfying (3) and the three point condition (4), which yields the following Plateau-type boundary condition

$$X^n(I)\subset\Sigma^n:=\{(x,y,g^n(x,y))\in\mathbb{R}^3 \mid (x,y)\in\Gamma_d\} \ . \tag{17}$$

As in the proof of Theorem 2, we first derive a uniform modulus of continuity for the sequence X^n and obtain a limit mapping $X\in C^0(\overline{B^+},\mathbb{R}^3)$. Due to interior estimates for systems with quadratic growth in gradient we have the regularity $X\in C^{2+\alpha}(B^+,\mathbb{R}^3)$. Furthermore, from (17) we derive the boundary condition

$$X(I)\subset\Sigma:=\{(x,y,g(x,y))\in\mathbb{R}^3 \mid (x,y)\in\Gamma_d\}$$

with the $C^{2+\alpha}$-curve Σ. The boundary regularity theorem [1, Theorem 2 in section 7.3] yields $X \in C^{2+\alpha}(B^+ \cup I, \mathbb{R}^3)$. We now define the cylinder

$$Z := \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \overline{\Omega}\}$$

and note $X(B^+) \subset Z$ and $X(I) \subset \partial Z$. Defining $\tilde{H} : \partial Z \to \mathbb{R}$ to be the mean curvature of ∂Z w.r.t. the inner normal we have the relation $\tilde{H}(x,y,z) = \frac{1}{2}\kappa(x,y)$ and by assumption (16) we obtain

$$|H(x,y,z)| < \tilde{H}(x,y,z) \quad \text{for } (x,y,z) \in \Gamma_d \times \mathbb{R} .$$

Using the cylinder Z as barrier, we can derive the condition of transversality $(X_u \wedge X_v) \cdot e_3 = \det(f_u, f_v) > 0$ on I (see [13, Satz 2] or [5, §2]). The same arguments as in the end of proof of Theorem 2 yield the boundary regularity $\zeta \in C^{2+\alpha}(\overline{\Omega} \cap U, \mathbb{R})$ for some open neighborhood $U = U(x_1, y_1)$. As $(x_1, y_1) \in \Gamma_d$ was choosen arbitrarily, we obtain $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$. □

4. The proof of Theorem 1

1.) We first prove Theorem 1 for Dirichlet boundary values $g \in C^{2+\alpha}(\Gamma_d, \mathbb{R})$. For a parameter $t \in [0,1]$ we consider the family $P(t)$ of mixed boundary value problems (2) and set

$$J := \{t \in [0,1] \mid P(t) \text{ has a solution } \zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})\} .$$

Note that by assumption $c)$ of Theorem 1 we have $0 \in J$. Hence it remains to show that J is both open and closed.

2.) To show that J is open, we need to generalize the perturbation result for surfaces of prescribed mean curvature of [17, Hilfssatz 4 in §9 of chapter XII] or [16, Proposition 2] to the case of mixed boundary values. As this is straightforward, we will only give the basic ideas. Given a solution ζ of the problem $P(t)$ for some $t \in [0,1]$, we want to find a function $\delta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ such that $\tilde{\zeta} := \zeta + \delta$ is a solution of the problem $P(t^*)$ for some $t^* \in [0,1]$. We put $\tilde{\zeta} = \zeta + \delta$ into the prescribed mean curvature equation and develop in terms of δ. Putting all linear terms in δ on the left side and all quadratic or higher order terms on the right side, we obtain for δ the following mixed boundary value problem

$$\mathcal{L}(\delta) = \Phi(\delta) \quad \text{in } \Omega \quad , \quad \delta = (t^* - t)(g - g_0) \quad \text{on } \Gamma_d \quad \text{and} \quad \frac{\partial \delta}{\partial n} = 0 \quad \text{on } \Gamma_n . \tag{18}$$

Here, $\mathcal{L}(\delta)$ is some linear, uniformly elliptic differential operator, which by the assumption $H_z \geq 0$ is subject to the maximum principle. The expression $\Phi(\delta)$ contains all terms of order higher than linear in δ and hence satisfies a contraction condition of the following type

$$\begin{aligned} &||\Phi(\delta_1) - \Phi(\delta_2)||_{C^\alpha(\overline{\Omega})} \leq C(r)||\delta_1 - \delta_2||_{C^{2+\alpha}(\overline{\Omega})} \\ &\text{whenever } ||\delta_1||_{C^{2+\alpha}(\overline{\Omega})} + ||\delta_2||_{C^{2+\alpha}(\overline{\Omega})} \leq r \end{aligned} \tag{19}$$

for some constant $C(r)$ with $C(r) \to 0$ for $r \to 0$.
We now consider the following mixed boundary value problem

$$\begin{aligned} &\delta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}) \quad , \quad \mathcal{L}(\delta) = f \quad \text{in } \Omega \\ &\delta = (t^* - t)(g - g_0) \quad \text{on } \Gamma_d \quad \text{and} \quad \frac{\partial \delta}{\partial n} = 0 \quad \text{on } \Gamma_n \end{aligned}$$

which is uniquely solvable due to Schauder theory for all $f \in C^{\alpha}(\overline{\Omega}, \mathbb{R})$. The solution of this problem we denote by $\mathcal{L}^{-1}(f) := \delta$. Combining interior Schauder estimates with local Schauder estimates at the Dirichlet boundary (see [2, chapter 6.2]) and at the Neumann boundary (see [2, chapter 6.7]), the following estimate can be shown

$$||\mathcal{L}^{-1}(f)||_{C^{2+\alpha}(\Omega)} \leq M\Big(||f||_{C^{\alpha}(\Omega)} + ||(t^* - t)(g - g_0)||_{C^{2+\alpha}(\Gamma_d)}\Big) \tag{20}$$

for some constant M, if we use the assumption $\mathrm{dist}(\Gamma_d, \Gamma_n) > 0$. Now problem (18) is equivalent to the fixed point equation $\mathcal{L}^{-1} \circ \Phi(\delta) = \delta$. Using the contraction condition (19) together with the estimate (20) a fixed point of $\mathcal{L}^{-1} \circ \Phi$ can be constructed using the contraction mapping principle. To make the mapping $\mathcal{L}^{-1} \circ \Phi$ a contraction from a set into itself one has to assume $|t^* - t| \leq \varepsilon$ for sufficiently small $\varepsilon > 0$.

3.) To prove that J is closed, let ζ^n be a sequence of solutions for of $P(t^n)$ with $t^n \in J$ and assume that $t^n \to t^*$ for $n \to \infty$. Then by Theorem 2 together with Corollary 1 there exists a solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ of problem $P(t^*)$ showing the closeness of the set J. We conclude $J = [0, 1]$ and for $t = 1$ we obtain the desired solution of problem (1).

4.) Finally, we solve problem (1) for continuous Dirichlet boundary values $g \in C^0(\Gamma_d, \mathbb{R})$. Let $g^n \in C^{2+\alpha}(\Gamma_d, \mathbb{R})$ be a sequence of boundary values converging uniformly on Γ_d to g. By solving the mixed boundary value problem for g^n and applying Theorem 2, we obtain a solution $\zeta \in C^{2+\alpha}(\Omega \cup \Gamma_d, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ for the Dirichlet boundary values g. □

Remark: Using methods from the calculus of variations it is relatively easy to show that the solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ for Dirichlet boundary values $g \in C^{2+\alpha}(\Gamma_d, \mathbb{R})$ of Theorem 1 is the unique minimizer of the generalized nonparametric area functional

$$A(\eta) := \int_{\Omega} \Big(\sqrt{1 + |\nabla \eta|^2} + b(x, y, \eta)\Big) dx dy$$

within the class of functions

$$\mathcal{C}(\Omega, \Gamma_d, g) := \Big\{\eta \in C^2(\Omega, \mathbb{R}) \cap C^1(\overline{\Omega}, \mathbb{R}) \mid \eta = g \quad \text{on } \Gamma_d\Big\}\,.$$

Here, we have to choose $b = b(x, y, z)$ such that $b_z = 2H$ holds.

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