STRONG L^p-SOLUTIONS TO THE NAVIER-STOKES FLOW PAST MOVING OBSTACLES: THE CASE OF SEVERAL OBSTACLES AND TIME DEPENDENT VELOCITY

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ABSTRACT. Consider the Navier-Stokes flow past several moving obstacles. It is shown that there exists a unique strong local solution in the L^p-setting, $1 < p < \infty$. Moreover, it is proved that the strong solution coincides with the known mild solution in the very weak sense.

1. INTRODUCTION

The mathematical description of the Navier-Stokes flow past rotating or moving obstacles gained quite some attention in the last years. The motion hereby is described by the equations of Navier-Stokes on an exterior domain depending on the time variable t. More precisely, consider the equation

$$\left\{\begin{array}{rcllll} \partial_t v - \Delta v + (v\cdot\nabla)v + \nabla q &=& f & \text{in} & \Omega(t), & t\in(0,T),\\ \operatorname{div} v &=& 0 & \text{in} & \Omega(t), & t\in(0,T),\\ v(x,t) &=& M_i(t)x & \text{on} & \Gamma_i(t),\ i=1,\dots,m, & t\in(0,T),\\ v(x,0) &=& v_0(x) & \text{in} & \Omega(0). & \end{array}\right. \tag{1.1}$$

Here $v = v(x,t)$ and $q = q(x,t)$ denote the velocity and the pressure of the fluid, respectively. The boundary condition on $\Gamma_i(t)$ is the usual no-slip condition. In this paper we consider time-dependent domains $\Omega(t)$ of the following form: let $\mathcal{O}_1,\dots,\mathcal{O}_m \subset \mathbb{R}^n$, $n \geq 2$, be compact sets with boundaries $\Gamma_1,\dots,\Gamma_m$ of class $C^{1,1}$. We denote by $\Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i$ the exterior domain. For time-dependent matrices

$$M_i \in C^\infty([0,T];\mathbb{R}^{n\times n}),$$

with tr $M_i(t) = 0$ for all $t \in [0,T]$, $i \in \{1,\dots,m\}$, we define the time dependent exterior domain

$$\Omega(t) := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i(t)$$

with $\mathcal{O}_i(t) := \{y = G_{(i)}(t)x, x \in \mathcal{O}_i\}$, $\Gamma_i(t) := \{y = G_{(i)}(t)x, x \in \Gamma_i\}$ for $t \in [0,T]$ and a suitably defined isomorphism $G_{(i)}(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$, for details see (2.5). As the obstacles shall not collide, we require

$$\operatorname{dist}\left(\overline{\mathcal{O}_i(t)}, \overline{\mathcal{O}_j(t)}\right) > 0, \qquad i \neq j,\ t \in [0,T].$$

Our aim is to construct a strong L^p-solution to (1.1).

It is interesting to compare our solution to problem (1.1) with the results which have been obtained recently by several different approaches. The situation of *one* obstacle rotating with *constant* angular velocity (i.e. M equals the rotation matrix) was considered first by Hishida [His99]. He proved the existence of a unique local *mild* solution to (1.1) in the context of L^2.

Strong solutions, again in the L^2-context and for one obstacle, were obtained by Galdi [Gal04], Galdi and Silvestre [GS05] by Galerkin methods as well as by Cumsille and Tucsnak [CT06].

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In the case of two dimensions, these strong solutions are even global in time under appropriate assumptions on the data.

The situation where the data belong to L^p for $1 < p < \infty$, was considered first in [GHH06a], where the existence of a unique *mild L^p-solution* to (1.1) was established. For a different approach in this setting we refer to the recent work of Hishida and Shibata [HS06].

In this paper, we consider the situation of *strong L^p-solutions*. We prove that the existence of a local, strong solution to (1.1) in L^p even for *several* non colliding obstacles which may rotate or move with a *time-dependent* angular velocity. One of the main tools in the proof of our results will be the maximal $L^p - L^q$-regularity of the Stokes operator in exterior domains.

It is a natural question to ask whether the strong solution to (1.1) obtained in Theorem 3.1 below coincides with the mild solution constructed in [GHH06a]. We give an affirmative answer to this question in Theorem 3.3 below. Of course, we need to explain first the meaning of *coincides*, since mild and strong solutions are defined on different spaces. In this section we make use of the concept of *very weak solutions* which was introduced in [FJR72] for $\mathbb{R}^n$ and in [Ama00] for domains.

For more information about the Navier-Stokes equation in the rotating framework of all of $\mathbb{R}^n$ or $\mathbb{R}^n_+$, we refer to the papers [CM97], [BMN99], [HS05], [GIMM04] and [GIMMS05] dealing in particular with data non decaying at infinity.

2. Preliminaries

We start by transforming equation (1.1) on the time-dependent domain $\Omega(t)$ to an equation on a fixed cylindrical domain. More precisely, following the approach introduced by Inoue and Wakimoto [IW77], we introduce a change of coordinates which coincides in the special case of pure rotation, i.e. M equals the rotation matrix, with the rotation in a neighborhood of the rotating obstacle, but equals the identity far away from the rotating body; see also [CT06].

For the time being, assume there is only one moving obstacle. We then make the following assumption.

(A1) Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geq 2$, be a compact set (the obstacle) with boundary $\Gamma := \partial\mathcal{O}$ of class $C^{1,1}$. Denote by $\Omega := \mathbb{R}^n \setminus \mathcal{O}$ the exterior domain corresponding to $\mathcal{O}$. For $M \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} M = 0$ define $\Omega(t), \Gamma(t), \mathcal{O}(t)$ as $\Omega(t) := \{y = e^{tM}x, x \in \Omega\}$, $\Gamma(t) := \{y = e^{tM}x, x \in \Gamma\}$, $\mathcal{O}(t) := \{y = e^{tM}x, x \in \mathcal{O}\}$.

We then consider a ball $B_r(0)$ of radius $r > 0$ such that $B_r(0) \supset \mathcal{O}(t)$ for all $t \in [0, T]$ for some $T > 0$. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ on $B_r(0)$ and $\eta = 0$ on $B_{2r}(0)^c$. Define $b : \mathbb{R}^n \to \mathbb{R}^n$ by

$$b(y) := \eta(y)My - B_K\left((\nabla\eta)M\cdot\right)(y), \tag{2.1}$$

where $K := \operatorname{supp}(\nabla\eta)$. Here B_K denotes the Bogovskiĭ operator; for details see [Bog79], [Gal94], [GHH06b]. Note that $b(y) = My$ for $y \in \overline{\mathcal{O}(t)}$. Moreover, since $\int_K (\nabla\eta)(y)My\,\mathrm{d}y = 0$ thanks to $\operatorname{tr} M = 0$, it follows from [Bog79] that $\operatorname{div} b = 0$ in $\mathbb{R}^n$ and $b \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Consider then the initial value problem

$$\left\{\begin{array}{rcll} \partial_t X(y,t) &=& b(X(y,t)), \quad t > 0, & y \in \mathbb{R}^n, \\ X(y,0) &=& y, & y \in \mathbb{R}^n. \end{array}\right. \tag{2.2}$$

Then, by standard theory of ODE's, there exists a unique vector field $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ satisfying (2.2). Moreover, $X(\cdot, t)$ is a C^∞-diffeomorphism from Ω onto $\Omega(t)$. Its inverse $Y(\cdot, t)$ is of the same class of regularity and satisfies the initial value problem

$$\left\{\begin{array}{rcll} \partial_t Y(x,t) &=& -b(Y(x,t)), \quad t > 0, & x \in \mathbb{R}^n, \\ Y(x,0) &=& x, & x \in \mathbb{R}^n. \end{array}\right. \tag{2.3}$$

In fact, we only need the restrictions of X and Y on $[0,T]$, nevertheless (2.2) and (2.3) even can be solved on the whole of $\mathbb{R}^n \times \mathbb{R}_+$.

Denote by $J_X(\cdot,t)$ and $J_Y(\cdot,t)$ the Jacobian of $X(\cdot,t)$ and $Y(\cdot,t)$, respectively. Since div $b=0$, Liouville's theorem, see e. g. [Arn92], implies that

$$J_X(y,t)J_Y(X(y,t),t) = \text{id and } \det J_X(y,t) = \det J_Y(x,t) = 1 \tag{2.4}$$

for all $t \geq 0$ and $x,y \in \mathbb{R}^n$.

In the situation of several obstacles moving with time dependent velocity we make the following assumption:

(A2) Let $\mathcal{O}_1,\dots,\mathcal{O}_m \subset \mathbb{R}^n$, $n \geq 2$, be compact sets with boundaries $\Gamma_1,\dots,\Gamma_m$ of class $C^{1,1}$. We denote by $\Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i$ the exterior domain. For time-dependent matrices

$$M_i \in C^\infty([0,T];\mathbb{R}^{n\times n}),$$

with tr $M_i(t) = 0$ for all $t \in [0,T]$, $i \in \{1,\dots,m\}$, we define the time-dependent sets $\Omega(t), \Gamma_i(t)$ and $\mathcal{O}_i(t)$ for $t \in [0,T]$ by aid ot the unique solution $G_{(i)}(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$ of the following ODE:

$$\left\{ \begin{array}{rcll} G'_{(i)}(t)x &=& M_i(t)G_{(i)}(t)x, & x \in \overline{\mathcal{O}_i}, \quad t \in (0,T), \\ G_{(i)}(0)x &=& x, & x \in \overline{\mathcal{O}_i}. \end{array} \right. \tag{2.5}$$

Note, that $G_{(i)}(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$ is an isomorphism, as

$$\left\{ \begin{array}{rcll} [G'_{(i)}(t)x]_k &=& M_i(t)[G_{(i)}(t)x]_k, & x \in \overline{\mathcal{O}_i}, \quad t \in (0,T), \\ [G_{(i)}(0)x]_k &=& x_k, & x \in \overline{\mathcal{O}_i}, \end{array} \right.$$

with $1 \leq k \leq n$, $i \in \{1,\dots,m\}$, has a unique fundamental system of linear independent solutions $G_{(i)}(t) = \left(G^1_{(i)}(t),\dots,G^n_{(i)}(t)\right)$, due to the non-vanishing Wronski-determinant at zero. In particular, for the case of constant matrices $M_i \in \mathbb{R}^{n\times n}$, we are left with $G_{(i)}(t) = e^{tM_i}$ and inverse $G^{-1}_{(i)}(t) = e^{-tM_i}$.

As the obstacles shall not collide, we require

$$\text{dist } \left(\overline{\mathcal{O}_i(t)}, \overline{\mathcal{O}_j(t)}\right) > 0, \qquad i \neq j,\ t \in [0,T].$$

We now choose open sets $B_{1_i}, B_{2_i} \subset \mathbb{R}^n$, such that $\overline{\mathcal{O}_i} \subset B_{1_i} \subset \overline{B_{1_i}} \subset B_{2_i}$ and set

$$B_{k_i}(t) := \left\{ y = G_{(i)}(t)x, x \in B_{k_i} \right\}, \quad t \in [0,T],\, k=1,2,\, i \in \{1,\dots,m\}.$$

Then

$$\overline{\mathcal{O}_i(t)} \subset B_{1_i}(t) \subset \overline{B_{1_i}(t)} \subset B_{2_i}(t)$$

for $t \in [0,T]$. Moreover, we demand the sets B_{2_i} to be that small, that

$$\bigcap_{i=1}^m \overline{B_{2_i}(t)} = \emptyset, \qquad t \in [0,T].$$

Next, we introduce a time-dependent cut-off function $\eta \in C^\infty(\mathbb{R}^n \times [0,T])$, $0 \leq \eta \leq 1$, such that

$$\eta(y,t) := \left\{ \begin{array}{lll} 1 & \text{on} \quad \bigcup_{i=1}^m B_{1_i}(t), & t \in [0,T], \\ 0 & \text{on} \quad \mathbb{R}^n \setminus \bigcup_{i=1}^m B_{2_i}(t), & t \in [0,T]. \end{array} \right.$$

We define the sets

$$K_i(t) := (\text{supp} \nabla_y \eta(t,\cdot)) \cap \overline{B_{2_i}(t)}, \quad t \in [0,T], i \in \{1,\dots,m\},$$

and the time dependent vector field $b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ by

$$b(y,t) := \eta(y,t)\sum_{i=1}^m M_i(t)y - \sum_{i=1}^m B_{K_i(t)}\left((\nabla_y \eta)(\cdot,t)M_i(t)\cdot\right)(y).$$

Note, that due to tr $M_i(t) = 0$, $t \in [0,T]$, $i = 1, \dots, m$, and properties of the Bogovskiĭ operator, the vector field b is solenoidal for $t \in [0,T]$ on all of $\mathbb{R}^n$. Further, $b(y,t) = M_i(t)y$ for $y \in \overline{\mathcal{O}_i(t)}$, $t \in [0,T]$, $i \in \{1, \dots, m\}$.

Note that for fixed $t^* \in [0,T]$

$$b(y,t^*) \quad = \quad \eta(y,t^*) \sum_{i=1}^m M_i(t^*)y - \sum_{i=1}^m B_{K_i^*}\left((\nabla_y \eta)(\cdot,t^*)M_i(t^*)\cdot\right)(y).$$

Since η is smooth in the first variable, the mapping properties of the Bogovskiĭ operator imply that $b(\cdot,t^*) \in C_c^\infty(\mathbb{R}^n)$. Now, freeze $y^* \in \mathbb{R}^n$ and consider

$$b(y^*,t) \quad = \quad \eta(y^*,t) \sum_{i=1}^m M_i(t)y^* - \sum_{i=1}^m B_{K_i^*}\left((\nabla_y \eta)(\cdot,t)M_i(t)\cdot\right)(y^*).$$

The smoothness of the cut-off function η and the matrices $M_i(\cdot)$ inherits to b and we thus see that

$$b \in C_{c,\sigma}^\infty\left(\mathbb{R}^n \times [0,T]; \mathbb{R}^n\right).$$

In particular, b is uniformly Lipschitz continous with respect to the first variable and bounded on $[0,T] \times \mathbb{R}^n$.

Thus, the ordinary differential equation

$$\left\{ \begin{array}{rcll} \partial_t X(y,t) & = & b(X(y,t),t), \quad t > 0, & y \in \mathbb{R}^n, \\ X(y,0) & = & y, & y \in \mathbb{R}^n. \end{array} \right. \tag{2.6}$$

admits a unique solution by the Picard-Lindelöf theorem. Moreover, $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

As above, let $Y(\cdot,t)$ be the inverse of $X(\cdot,t)$. Then Y satisfies

$$\left\{ \begin{array}{rcll} \partial_t Y(x,t) & = & -b(Y(x,t),t), \quad t > 0, & x \in \mathbb{R}^n, \\ Y(x,0) & = & x, & x \in \mathbb{R}^n, \end{array} \right. \tag{2.7}$$

and (2.4). Again, only the restrictions of X, Y to $[0,T]$ will be relevant in the sequel.

We set

$$\begin{array}{rcll} U(y,t) & := & J_Y\left(X(y,t),t\right) v\left(X(y,t),t\right), & y \in \Omega, \quad t \in [0,T], \\ \pi(y,t) & := & q(X(y,t),t), & y \in \Omega, \quad t \in [0,T]. \end{array}$$

Then, (similarly to [IW77], [CT06, Prop.3.5]), a function

$$v \in L^p(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0,T; L^q(\Omega(\cdot))), \quad q \in L^p(0,T; \hat{W}^{1,q}(\Omega(\cdot)))$$

is a solution to (1.1) if and only if

$$U \in L^p(0,T; W^{2,q}(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)), \quad \pi \in L^p(0,T; \hat{W}^{1,q}(\Omega))$$

and (U,π) satisfies the following set of equations

$$\left\{ \begin{array}{rcll} \partial_t U + (\mathcal{M} - \mathcal{L})\, U & = & f - \mathcal{N}U - \mathcal{G}\pi, & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} U & = & 0, & \text{in} \quad \Omega \times (0,T), \\ U & = & M_i(t)x, & \text{on} \quad \Gamma_i \times (0,T), \; i = 1, \dots, m, \\ U(0) & = & v_0, & \text{in} \quad \Omega. \end{array} \right. \tag{2.8}$$

Here

$$
\begin{aligned}
(\mathcal{L}U)_i &:= \sum_{j,k=1}^{n} \partial_j(g^{jk}\partial_k U_i) + 2\sum_{j,k,l=1}^{n} g^{kl}\Gamma^i_{jk}\partial_l U_j + \sum_{j,k,l=1}^{n}\left(\partial_k(g^{kl}\Gamma^i_{jl}) + \sum_{m=1}^{n} g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right)U_j,\\
(\mathcal{N}U)_i &:= \sum_{j=1}^{n} U_j\partial_j U_i + \sum_{j,k=1}^{n}\Gamma^i_{jk}U_jU_k,\\
(\mathcal{M}U)_i &:= \sum_{j=1}^{n}\dot{Y}_j\partial_j U_i + \sum_{j,k=1}^{n}\left(\Gamma^i_{jk}\dot{Y}_k + (\partial_k Y_i)(\partial_j\dot{X}_k)\right)U_j,\\
(\mathcal{G}\pi)_i &:= \sum_{j=1}^{n} g^{ij}\partial_j\pi,
\end{aligned}
$$

with the metric contravariant tensor

$$g^{ij} = \sum_{k=1}^{n}(\partial_k Y_i)(\partial_k Y_j),$$

the metric covariant tensor

$$g_{ij} = \sum_{k=1}^{n}(\partial_i X_k)(\partial_j X_k)$$

and the Christoffel's symbol

$$\Gamma^k_{ij} = 1/2\sum_{l=1}^{n} g^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right).$$

Note that $\mathcal{L}$ is the transformed Stokes operator, while $\mathcal{M}$ arises from transforming the time derivative. The nonlinearity $\mathcal{N}$ and modified gradient $\mathcal{G}$ correspond to $(v\cdot\nabla)v$ and ∇, respectively.

Setting $u(y,t) := U(y,t) - b(y,t)$, we see that

$$U \in L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)), \quad \pi \in L^p(0,T;\hat{W}^{1,q}(\Omega))$$

is a solution of (2.8) if and only if

$$u \in L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)), \quad \pi \in L^p(0,T;\hat{W}^{1,q}(\Omega))$$

and (u,π) solves

$$
(2.9)\quad \left\{
\begin{array}{rcll}
\partial_t u - \Delta u + \nabla\pi &=& F - \mathcal{N}u - \mathcal{B}u + (\mathcal{L}-\Delta)u & \\
&& \qquad - \mathcal{M}u + (\nabla - \mathcal{G})\pi & \text{in } \ \Omega\times(0,T),\\
\operatorname{div} u &=& 0 & \text{in } \ \Omega\times(0,T),\\
u &=& 0 & \text{on } \ \Gamma_i\times(0,T),\ i=1,\ldots,m,\\
u(0) &=& v_0 - b(0) & \text{in } \ \Omega,
\end{array}
\right.
$$

with

$$(\mathcal{B}u)_i = \sum_{j=1}^{n}(u_j\partial_j b_i + b_j\partial_j u_i) + 2\sum_{j,k=1}^{n}\Gamma^i_{jk}b_k u_j, \quad F = f - \partial_t b - (\mathcal{M}-\mathcal{L})b - \mathcal{N}b.$$

In the sequel, maximal L^p-regularity of the Stokes operator in $L^q_\sigma(\Omega)$ plays an important role. More precisely, for $1 < q < \infty$, we define the Stokes operator A_q in $L^q_\sigma(\Omega)$ by

$$
\left\{
\begin{array}{rcl}
A_q u &:=& P_q\Delta u,\\
D(A_q) &:=& W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega).
\end{array}
\right.
$$

As usual, P_q denotes the Helmholtz projection on $L^q(\Omega)$.

Let now $T_0 > 0$, $1 < p < \infty$, $T \in (0, T_0)$, $f \in L^p(0, T; L^q_\sigma(\Omega))$ and $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$. Then it follows from a classical result of Solonnikov [Sol77] (also see [Gig81] and [Frö02]) that there exists a unique solution

$$u \in X^T_{p,q} := W^{1,p}(0, T; L^q_\sigma(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega))$$

of the instationary Stokes problem

$$\begin{cases} u'(t) - A_q u(t) &= f(t), \qquad t \in (0, T), \\ u(0) &= u_0. \end{cases}$$

Moreover, there exists $C > 0$, independent of T, f and u_0, such that

$$\|u\|_{X^T_{p,q}} \le C(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}}).$$

Setting $\nabla\pi := (\mathrm{Id} - P_q)\Delta u$, we see that (u, π) is a solution to

$$\begin{cases} u'(t) - \Delta u(t) - \nabla\pi(t) &= f(t), \qquad t \in (0, T), \\ u(0) &= u_0, \end{cases}$$

satisfying

$$\|u\|_{X^T_{p,q}} + \|\pi\|_{Y^T_{p,q}} \le C(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{(L^q_\sigma,(\Omega) D(A_q))_{1-\frac{1}{p},p}}), \tag{2.10}$$

where $Y^T_{p,q} := L^p(0, T; \hat{W}^{1,q}(\Omega))$ and $C > 0$ is a constant independent of $T \in (0, T_0)$, f and u_0.

For the rest of this section, assume that $M = -M^t$, (A1) holds and consider

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla)v + \nabla q &= 0 \quad \text{in} \quad \Omega(t), \quad t \in (0, T), \\ \operatorname{div} v &= 0 \quad \text{in} \quad \Omega(t), \quad t \in (0, T), \\ v(x, t) &= Mx \quad \text{on} \quad \Gamma(t), \quad t \in (0, T), \\ v(x, 0) &= v_0(x) \quad \text{in} \quad \Omega(0). \end{cases} \tag{2.11}$$

It was shown in [GHH06a] that (2.11) admits a unique, local, mild solution w. In order to compare the solutions obtained in [GHH06a] and in Corollary 3.2 we introduce the notion of very weak solutions to equation (2.11). To this end, let $1 < p, q < \infty$. A function $u \in L^p(0, T_0; L^q_\sigma(\Omega(\cdot)))$ is called a *very weak solution* to (2.11) if

$$\begin{aligned} < v_0, \varphi >_\Omega \quad &- \int_0^{T_0} < u(t), \varphi'(t) + \Delta\varphi(t) + (u(t) \cdot \nabla)\varphi(t) >_{\Omega(t)} \, \mathrm{d}t \\ &+ \int_0^{T_0} < Mx, \nabla\varphi(t) >_{\Gamma(t)} \, \mathrm{d}t = 0, \end{aligned}$$

for $\varphi \in \mathrm{D}$, where

$$\mathrm{D} := \left\{ \varphi \in C^1\left([0, T_0]; C^\infty_c(\overline{\Omega(\cdot)})\right) : \varphi(T_0) = 0, \ \operatorname{div} \varphi(t) = 0, \ \varphi(t)|_{\Gamma(t)} = 0 \text{ for all } t \in [0, T_0] \right\}.$$

We say that two very weak solutions $u, v \in L^p(0, T_0; L^q_\sigma(\Omega(\cdot)))$ to (2.11) with initial value $v_0 \in L^q_\sigma(\Omega)$ coincide in the *very weak sense* if there exists $T \in (0, T_0)$ such that $u(t) = v(t)$ for a.a. $t \in (0, T)$.

In [GHH06a], problem (2.11) is transformed into the equation

$$\begin{cases} \hat{u}' - A_{\Omega,b}\hat{u} + (\hat{u} \cdot \nabla\hat{u}) &= F_2, \qquad t \in (0, T), \\ \hat{u}(0) &= v_0 - b, \end{cases} \tag{2.12}$$

where $\hat{u}(x, t) := \mathrm{e}^{-tM} v(\mathrm{e}^{tM}x, t) - b(x)$. Here,

$$A_{\Omega,b}\hat{u} := P_q(\Delta\hat{u} + Mx \cdot \nabla\hat{u} - M\hat{u} - b \cdot \nabla\hat{u} - \hat{u} \cdot \nabla b)$$

with $D(A_{\Omega,b}) := \left\{\hat{u} \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^p_\sigma(\Omega) : Mx \cdot \nabla \hat{u} \in L^q(\Omega)\right\}$ and $F_2 := \Delta b + Mx \cdot \nabla b - Mb - b \cdot \nabla b$. It is proved that, for $v_0 - b \in L^r_\sigma(\Omega)$ with $r \geq n$, there exists a *mild solution* $\hat{u}$ to (2.12), i.e., $\hat{u} \in C([0,T]; L^r_\sigma(\Omega))$ satisfies the integral equation

$$\hat{u}(t) = T_{\Omega,b}(v_0 - b) - \int_0^t T_{\Omega,b}(t-s) P_q(\hat{u} \cdot \nabla \hat{u})(s)\, \mathrm{d}s + \int_0^t T_{\Omega,b} P_q F_2(s)\, \mathrm{d}s, \quad t \in (0,T). \tag{2.13}$$

Here, $(T_{\Omega,b}(t))_{t\geq 0} := \left(e^{tA_{\Omega,b}}\right)_{t\geq 0}$ is the semigroup generated by $(A_{\Omega,b}, D(A_{\Omega,b}))$ in $L^r_\sigma(\Omega)$. Moreover, this solution $\hat{u}$ satisfies:

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} \hat{u}(\cdot) \in C([0,T]; L^q_\sigma(\Omega)), \tag{2.14}$$

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)+\frac{1}{2}} \nabla \hat{u}(\cdot) \in C([0,T]; L^q(\Omega)). \tag{2.15}$$

We show in Section 4 that w given by

$$w(x,t) := \mathrm{e}^{tM} \hat{u}(\mathrm{e}^{-tM}x, t) + \mathrm{e}^{tM} b(\mathrm{e}^{-tM}x), \tag{2.16}$$

with $\hat{u}$ given by the variation-of-constants formula (2.13) is a very weak solution to (2.11) for suitable choices of p, q and n. We are in the position to state our main results.

3. Main Results

Our existence and uniqueness result for equation (1.1) reads as follows.

Theorem 3.1. *Assume (A2) and let $p, q \in (1,\infty)$ such that $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Assume that*

a) $f \in L^p(0,T; L^q_\sigma(\Omega(\cdot)))$,
b) $v_0 - b(\cdot,0) \in (L^q_\sigma(\Omega), W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega) \cap L^q_\sigma(\Omega))_{1-1/p,p}$.

Then there exists $T > 0$ such that the problem (1.1) *admits a unique strong solution*

$$v \in L^p(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0,T; L^q_\sigma(\Omega(\cdot))), \quad q \in L^p(0,T; \hat{W}^{1,q}(\Omega(\cdot))).$$

Moreover, we can choose $T > 0$ sucht that either $T = +\infty$ or the function

$$t \mapsto \|v(t)\|_{\left(L^q_\sigma(\Omega(t)), W_0^{1,q}(\Omega(t)) \cap W^{2,q}(\Omega(t) \cap L^q_\sigma(\Omega(t)))\right)_{1-1/p,p}}$$

is unbounded on its maximal interval of existence $[0,T)$.

Corollary 3.2. *Assume (A1) and let $p, q \in (1,\infty)$ such that $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Assume that*

a) $f \in L^p(0,T; L^q_\sigma(\Omega(\cdot)))$,
b) $v_0 - b \in (L^q_\sigma(\Omega), W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega) \cap L^q_\sigma(\Omega))_{1-1/p,p}$.

Then there exists $T > 0$ such that the problem (2.11) *admits a unique strong solution*

$$v \in L^p(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0,T; L^q_\sigma(\Omega(\cdot))), \quad q \in L^p(0,T; \hat{W}^{1,q}(\Omega(\cdot))).$$

We have either $T = +\infty$ or

$$t \mapsto \|v(t)\|_{\left(L^q_\sigma(\Omega(t)), W_0^{1,q}(\Omega(t)) \cap W^{2,q}(\Omega(t) \cap L^q_\sigma(\Omega(t)))\right)_{1-1/p,p}}$$

is unbounded on its maximal interval of existence $[0,T)$.

The following theorem says that the two solutions v and w coincide in the very weak sense. More precisely, the following holds true.

Theorem 3.3. *Let $1 < p, q < \infty$ such that $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Assume that*

a) $v_0 - b \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$,
b) $v_0 - b \in L^r_\sigma(\Omega)$ *for some $r > n$ provided $\frac{n}{2q} + \frac{1}{p} = \frac{3}{2}$.*

Then w given by (2.16) *and v given in Corollary 3.2 coincide in the very weak sense.*

4. Proof of the first main result

Note that in order to prove our main result it suffices to construct a unique solution (u, π) to (2.9). The strategy of the proof is as follows: First, we derive estimates on the coefficients of the operators $\mathcal{N}, \mathcal{B}, \mathcal{L}, \mathcal{M}$ and $\mathcal{G}$ in the problem (2.9). Then, a fixed point argument in a suitable closed subspace of $X^T_{p,q} \times Y^T_{p,q}$ yields a unique solution $(u, \pi) \in X^T_{p,q} \times Y^T_{p,q}$ via maximal L^p-regularity of the Stokes operator.

Observe, that for a multi-index α and $k \in \mathbb{N}$ there is some constant $K_{|\alpha|,k,T} > 0$ such that

$$\left\|\partial_y^\alpha \partial_t^k b\right\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq K_{|\alpha|,k,T}, \qquad |\alpha| + k > 0.$$

The following lemma yields estimates of the transformation mappings X and Y, respectively, that are defined by (2.6) and (2.7). Clearly, the assertions remain true in the case of b being independent from the time variable and X, Y defined as in (2.2) and (2.3).

Lemma 4.1. *Let $T_0 > 0$, $k \in \mathbb{N}$ and α a multi-index satisfying $|\alpha| + k > 0$. Then there exists $C_{|\alpha|,k,T_0} > 0$ such that*

$$\|\partial_y^\alpha \partial_t^k X\|_{L^\infty(\mathbb{R}^n \times [0,T_0])} \leq C_{|\alpha|,k,T_0}.$$

The above estimates remain valid when $X(\cdot, t)$ is replaced by its inverse $Y(\cdot, t)$.

Proof. Let $T_0 > 0$, $k \in \mathbb{N}$ and α a multi-index satisfying $|\alpha| + k > 0$. By a direct calculation, we see that $X(t, y) = y$ for $y \notin \operatorname{supp} b$. Hence,

$$\|\partial_y^\alpha \partial_t^k X\|_{L^\infty((\operatorname{supp}\ b)^c \times [0,T_0])} \leq 1.$$

Since supp b is compact and $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, there exists $C_{|\alpha|,k,T_0}$, such that

$$\|\partial_y^\alpha \partial_t^k X\|_{L^\infty((\operatorname{supp}\ b) \times [0,T_0])} \leq C_{|\alpha|,k,T_0}.$$

□

It follows from the definition of g^{ij}, g_{ij} and Γ^k_{ij} and the previous lemma that all coefficients of $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{B}$ and $\mathcal{G}$ are smooth and bounded on finite time intervals $[0, T_0]$ for any $T_0 > 0$. Moreover, by the mean value theorem, for $x \in \mathbb{R}^n$ we have $g^{ij}(x, t) - \delta_{ij} = t\partial_\tau g^{ij}(x, \tau)$ for some $\tau \in (0, t)$. Hence, it follows from Lemma 4.1 that

$$\|g^{ij} - \delta_{ij}\|_{L^\infty(\mathbb{R}^n \times [0,T_0])} \leq t\|\partial_t g^{ij}\|_{L^\infty(\mathbb{R}^n \times [0,T_0])} \leq Ct, \quad t \in (0, T_0). \tag{4.1}$$

The following embedding of $X^T_{p,q}$ in $L^k(0, T; W^{s,m}(\Omega))$ is needed to cope with the gradient terms. It mainly relies on the mixed derivatives theorem. Precisely, the following lemma holds:

Lemma 4.2. *Let $1 < p, q < \infty$, $T_0 > 0$ and $s = 0$ or $s = 1$. Assume that $k, m \in (1, \infty)$ obey $\frac{2-s}{2} + \frac{n}{2m} - \frac{n}{2q} \geq \frac{1}{p} - \frac{1}{k}$. Then $X^T_{p,q}$ is continously embedded in $L^k(0, T; W^{s,m}(\Omega))$ for any $T \in (0, T_0)$. Moreover, there exists $C_{T_0} > 0$ such that*

$$\|u\|_{L^k(0,T;W^{s,m}(\Omega))} \leq C_{T_0}\|u\|_{X^T_{p,q}}, \qquad T \in (0, T_0),\ u \in X^T_{p,q},\ u(0) = 0.$$

Proof. By the mixed derivative theorem (see [Sob75], [DHP05]), for $\theta \in (0, 1)$ there exists $C > 0$ such that

$$\|u\|_{H^{\theta,p}(0,T_0;H^{2-2\theta,q}(\Omega))} \leq C\|u\|_{X^{T_0}_{p,q}}, \quad u \in X^{T_0}_{p,q}.$$

It then follows from Sobolev embeddings that

$$\|u\|_{L^k(0,T;W^{s,m}(\Omega))} \leq C_{T_0}\|u\|_{X^{T_0}_{p,q}}, \qquad u \in X^{T_0}_{p,q}.$$

Let $S : \left\{u \in X^T_{p,q} : u(0) = 0\right\} \to X^{T_0}_{p,q}$ be defined by

$$Su(t) := \begin{cases} u(t - (T_0 - T)), & t \geq T_0 - T, \\ 0, & 0 \leq t < T_0 - T. \end{cases}$$

Then $\|Su\|_{X^{T_0}_{p,q}} = \|u\|_{X^T_{p,q}}$ and $\|Su\|_{L^k(0,T;W^{s,m}(\Omega))} = \|u\|_{L^k(0,T_0;W^{s,m}(\Omega))}$. We thus obtain

$$\|u\|_{L^k(0,T_0;W^{s,m}(\Omega))} = \|Su\|_{L^k(0,T;W^{s,m}(\Omega))} \le C_{T_0}\|Su\|_{X^{T_0}_{p,q}} = C_{T_0}\|u\|_{X^T_{p,q}}$$

for all $u \in X^T_{p,q}$ satisfying $u(0) = 0$. □

Next we prove estimates for the terms on the right-hand side of (2.9).

Lemma 4.3. *Let $T_0 > 0$. Then there exists $C > 0$ such that for $T \in (0, T_0)$ and $(v_1, q_1), (v_2, q_2) \in X^T_{p,q} \times Y^T_{p,q}$ satisfying $v_1(0) = v_2(0) = 0$*

(a) $\|\mathcal{N}v_1 - \mathcal{N}v_2\|_{L^p(0,T;L^q(\Omega))} \le C\left(\|v_1\|_{X^T_{p,q}} + \|v_2\|_{X^T_{p,q}}\right)\|v_1 - v_2\|_{X^T_{p,q}}$,

(b) $\|\mathcal{B}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \le CT^{\frac{1}{3p}}\|v_1 - v_2\|_{X^T_{p,q}}$,

(c) $\|\mathcal{M}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \le CT^{\frac{1}{3p}}\|v_1 - v_2\|_{X^T_{p,q}}$,

(d) $\|(\mathcal{L} - \Delta)(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \le C(T + T^{\frac{1}{3p}})\|v_1 - v_2\|_{X^T_{p,q}}$,

(e) $\|(\nabla - \mathcal{G})(q_1 - q_2)\|_{L^p(0,T;L^q(\Omega))} \le CT\|q_1 - q_2\|_{Y^T_{p,q}}$.

The estimates above are valid even if $v_1(0) \ne 0$ and $v_2(0) \ne 0$. However, in this case, C depends on T as well.

Proof. Set $k = 3p$, $k' = 3p/2$, $m = 3q$ and $m' = 3q/2$. By Hölder's inequality, we obtain for the first term of $\mathcal{N}v_1 - \mathcal{N}v_2$

$$\begin{aligned}
\|v_1 \cdot \nabla v_1 - v_2 \cdot \nabla v_2\|_{L^p(0,T;L^q(\Omega))} &\le \|(v_1 - v_2)\cdot\nabla v_1 + v_2 \cdot (\nabla v_1 - \nabla v_2)\|_{L^p(0,T;L^q(\Omega))} \\
&\le \|v_1\|_{L^{k'}(0,T;W^{1,m'}(\Omega))}\|v_1 - v_2\|_{L^k(0,T;L^m(\Omega))} \\
&\quad + \|v_2\|_{L^k(0,T;L^m(\Omega))}\|v_1 - v_2\|_{L^{k'}(0,T;W^{1,m'}(\Omega))}.
\end{aligned}$$

Hence, by Lemma 4.2, there exists $C > 0$, independent of $T \in (0, T_0)$ and v_1, v_2, such that

$$\|v_1 \cdot \nabla v_1 - v_2 \cdot \nabla v_2\|_{L^p(0,T;L^q(\Omega))} \le C\left(\|v_1\|_{X^T_{p,q}} + \|v_2\|_{X^T_{p,q}}\right)\|v_1 - v_2\|_{X^T_{p,q}}.$$

Since the second term of $\mathcal{N}v_1 - \mathcal{N}v_2$ can be estimated similary, assertion (a) follows.

Similarly, (b) and (c) follow from the estimates

$$\|\mathcal{B}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \le CT^{\frac{1}{k}}\|v_1 - v_2\|_{L^{k'}(0,T;W^{1,q}(\Omega))} \le CT^{\frac{1}{k}}\|v_1 - v_2\|_{X^T_{p,q}}$$

and

$$\|\mathcal{M}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \le CT^{\frac{1}{k}}\|v_1 - v_2\|_{L^{k'}(0,T;W^{1,q}(\Omega))} \le CT^{\frac{1}{k}}\|v_1 - v_2\|_{X^T_{p,q}},$$

respectively, where $C > 0$ is independent of $T \in (0, T_0)$ and v_1, v_2.

As all coefficients of $\mathcal{L}$ are smooth we may rewrite it in non-divergence form and, by (4.1), we obtain

$$\begin{aligned}
&\|(\mathcal{L} - \Delta)(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \\
&\quad \le C\|g^{jk} - \delta_{jk}\|_{L^\infty(\mathbb{R}^n \times (0,T))}\|D^2(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} + CT^{\frac{1}{k}}\|v_1 - v_2\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\quad \le C(T + T^{\frac{1}{k}})\|v_1 - v_2\|_{X^T_{p,q}}, \quad T \in (0, T_0),\ v_1, v_2 \in X^T_{p,q},\ v_1(0) = 0,\ v_2(0) = 0.
\end{aligned}$$

Hence (d) follows. Assertion (e) similarly follows from (4.1). □

Let $T_0 > 0$, $T \in (0, T_0)$ and $p, q \in (1,\infty)$ satisfy $\frac{n}{2q} + \frac{1}{p} \le \frac{3}{2}$. Next, we introduce the space for the fixed point argument. In order to do this, consider

$$\left\{ \begin{array}{rcll} \partial_t u^* - \Delta u^* + \nabla \pi^* &=& \hat{F}(0,0) & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} u^* &=& 0 & \text{in} \quad \Omega \times (0,T), \\ u^* &=& 0 & \text{on} \quad \Gamma \times (0,T), \\ u^*(0) &=& u_0 & \text{in} \quad \Omega, \end{array} \right.$$

where $\hat{F} : X^{T_0}_{p,q} \times Y^{T_0}_{p,q} \to L^p(0, T_0; L^q(\Omega))$ is defined by

$$\hat{F}(v, q) := F - \mathcal{N}v - \mathcal{B}v + (\mathcal{L} - \Delta)v - \mathcal{M}v + (\nabla - \mathcal{G})q.$$

Note that $\hat{F}$ is well-defined by Lemma 4.3 thanks to $\frac{n}{2q} + \frac{1}{p} \le \frac{3}{2}$. Hence, by (2.10), there exists a unique solution $(u^*, \pi^*) \in X^T_{p,q} \times Y^T_{p,q}$.

Furthermore, let $(u^{**}, \pi^{**}) \in X^T_{p,q} \times Y^T_{p,q}$ denote the unique solution to

$$\left\{ \begin{array}{rcll} \partial_t u^{**} - \Delta u^{**} + \nabla \pi^{**} &=& \hat{F}(u^*, \pi^*) & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} u^{**} &=& 0 & \text{in} \quad \Omega \times (0,T), \\ u^{**} &=& 0 & \text{on} \quad \Gamma \times (0,T), \\ u^{**}(0) &=& u_0 & \text{in} \quad \Omega. \end{array} \right.$$

We rewrite (2.9) in terms of a fixed point problem equivalently by

$$\left\{ \begin{array}{rcll} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{\pi} &=& \hat{F}(v,q) - \hat{F}(u^*, \pi^*) & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} \tilde{u} &=& 0 & \text{in} \quad \Omega \times (0,T), \\ \tilde{u} &=& 0 & \text{on} \quad \Gamma \times (0,T), \\ \tilde{u}(0) &=& 0 & \text{in} \quad \Omega, \end{array} \right. \tag{4.2}$$

where we used the notation

$$\tilde{u} := u - u^{**}, \qquad \tilde{\pi} := \pi - \pi^{**}.$$

In view of the Banach fixed point theorem we define for a given radius $R > 0$ and $T \in (0, T_0)$ the closed set

$$\mathcal{K}_{R,T} := \Big\{ (v, q) \in X^T_{p,q} \times Y^T_{p,q} : v(0) = 0 \text{ and } \|v\|_{X^T_{p,q}} + \|q\|_{Y^T_{p,q}} \le R \Big\}$$

and a mapping

$$\Phi_{R,T} : \left\{ \begin{array}{lcl} \mathcal{K}_{R,T} & \to & X^T_{p,q} \times Y^T_{p,q}, \\ (v,q) & \mapsto & (\tilde{u}, \tilde{\pi}) \qquad \text{such that (4.2) holds.} \end{array} \right.$$

In order to apply the Banach fixed point theorem to $\Phi_{R,T}$ we have to show that the mapping is well-defined, maps $\mathcal{K}_{R,T}$ into itself and is a contraction.

While $\phi_{R,T}$ is well-defined due to Lemma 4.3 and (2.10), the other two outstanding debits are shown in the two lemmata given below.

The next lemma shows that for suitable choices of $R > 0$ and $T > 0$ the closed set $\mathcal{K}_{R,T}$ is mapped by $\Phi_{R,T}$ into itself.

Lemma 4.4. *There exist $R > 0$ and $T_1 > 0$, such that $\Phi_{R,T} : \mathcal{K}_{R,T} \to \mathcal{K}_{R,T}$ for all $T \in (0, T_1)$.*

Proof. By (2.10) and Lemma 4.3, we obtain

$$\begin{aligned} \|\Phi_{R,T}(v,q)\|_{X^T_{p,q} \times Y^T_{p,q}} &\le C \|\hat{F}(v,q)\|_{L^p(0,T;L^q(\Omega))} \\ &\le C\big(\|F\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{N}v\|_{L^p(0,T;L^q(\Omega))} \\ &\qquad + \|\mathcal{B}v\|_{L^p(0,T;L^q(\Omega))} + \|(\mathcal{L} - \Delta)v\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{M}v\|_{L^p(0,T;L^q(\Omega))} \\ &\qquad + \|(\nabla - \mathcal{G})q\|_{L^p(0,T;L^q(\Omega))} \big) \\ &\le C \Big(\|F\|_{L^p(0,T;L^q(\Omega))} + R^2 + T^{\frac{1}{3p}} R + TR \Big), \quad (v,q) \in \mathcal{K}_{R,T}. \end{aligned}$$

Since $\lim_{T\to 0}\|F\|_{L^p(0,T;L^q(\Omega))}=0$, we obtain $\|\Phi_{R,T}(v,q)\|_{X^T_{p,q}\times Y^T_{p,q}}\le R$ provided R and T are small enough. $\square$

Lemma 4.5. *There exists $T_0>0$ such that $\Phi_{R,T}:\mathcal{K}_{R,T}\to\mathcal{K}_{R,T}$ is a contraction for all $T\in(0,T_0)$.*

Proof. Again, by (2.10) and Lemma 4.3, we obtain

$$\begin{aligned}
&\|\Phi_{R,T}(v_1,q_1)-\Phi_{R,T}(v_2,q_2)\|_{X^T_{p,q}\times Y^T_{p,q}}\\
\le\ & C\Big(\|\mathcal{N}v_1-\mathcal{N}v_2\|_{L^p(0,T;L^q(\Omega))}+\|\mathcal{B}(v_1-v_2)\|_{L^p(0,T;L^q(\Omega))}\\
&+\|(\mathcal{L}-\Delta)(v_1-v_2)\|_{L^p(0,T;L^q(\Omega))}+\|\mathcal{M}(v_1-v_2)\|_{L^p(0,T;L^q(\Omega))}\\
&+\|(\nabla-\mathcal{G})(q_1-q_2)\|_{L^p(0,T;L^q(\Omega))}\Big)\\
\le\ & C(R+T+T^{\frac{1}{3p}})\,\|(v_1,q_1)-(v_2,q_2)\|_{X^T_{p,q}\times Y^T_{p,q}},\quad (v_1,q_1),(v_2,q_2)\in\mathcal{K}_{R,T}.
\end{aligned}$$

Choosing T and R small enough, we obtain $C(R+T+T^{\frac{1}{3p}})<1$. $\square$

Proof of Theorem 3.1. The existence of a unique strong solution now follows from Lemma 4.4, Lemma 4.5 and the Banach fixed point theorem. Now, the theorem follows in a standard way from the fact, that $T>0$ is uniform with respect to v_0, provided

$$\|v_0\|_{\left(L^q_\sigma(\Omega),W^{1,q}_0(\Omega)\cap W^{2,q}(\Omega)\cap L^q_\sigma(\Omega)\right)_{1-1/p,p}}<C_0,$$

and the continuous embedding

$$X^T_{p,q}\hookrightarrow C\left(0,T;\left(W^{1,q}_0(\Omega)\cap W^{2,q}(\Omega)\cap L^q_\sigma(\Omega),L^q_\sigma(\Omega)\right)_{1-1/p,p}\right).$$

$\square$

5. Comparison of strong and mild solutions

In this section we prove Theorem 3.3. For the notion of *very weak* solutions we refer back to Section 2. In a first step, we show that a mild solution is a very weak solution.

Lemma 5.1. *Let $(v_0-b)\in L^q_\sigma(\Omega)$ for some $q\ge n$ and denote the mild solution to (2.12) on $[0,T]$ for some $T>0$ by $\hat u$ (for a representation by the variation-of-constants formula see (2.13)). Then w, defined by (2.16), is a very weak solution to (2.11).*

Proof. By (2.14) and (2.15), we obtain $\hat u\in C([0,T];L^q_\sigma(\Omega))$ and $\nabla\hat u\in L^1(0,T;L^q(\Omega))$. Choose $\hat u^1_n\in C^1([0,T];C^\infty_{c,\sigma}(\Omega))$ and $\hat u^2_n\in C^1([0,T];C^\infty_c(\overline{\Omega}))$ such that

$$\lim_{n\to\infty}\|\hat u^1_n-\hat u\|_{C([0,T];L^q(\Omega))}=0 \text{ and } \lim_{n\to\infty}\|\nabla\hat u^2_n-\nabla\hat u\|_{L^1(0,T,L^q(\Omega))}=0. \tag{5.1}$$

Then, by [Paz83, Chapter 4, Theorem 2.9], there exists a solution $\hat v_n\in L^1(0,T;D(A_{\Omega,b}))\cap W^{1,1}(0,T;L^q_\sigma(\Omega))$ satisfying

$$\left\{\begin{array}{rcll}\hat v_n'(t)-A_{\Omega,b}\hat v_n(t)+P_q\left((\hat u^1_n\cdot\nabla\hat u^2_n)(t)\right)&=&P_qF_2(t), & t\in(0,T),\\ \hat v_n(0)&=&\hat u^1_n(0).&\end{array}\right.$$

By the representation of $\hat v_n$ via variation-of-constants formula, we obtain

$$\lim_{n\to\infty}\|\hat v_n-\hat u\|_{L^1(0,T;L^{\frac{q}{2}}(K))}=0 \tag{5.2}$$

for any compact $K \subset \overline{\Omega}$. Setting

$$\begin{aligned}\nabla \hat{\pi}_n &= (\mathrm{Id} - P_q)[F_2(t) + \hat{u}_n^1 \cdot \nabla \hat{u}_n^2(t) - \Delta \hat{v}_n(t) - Mx \cdot \nabla \hat{v}_n(t) \\ &\qquad + M\hat{v}_n(t) + b \cdot \nabla \hat{v}_n(t) + \hat{v}_n(t) \cdot \nabla b]\end{aligned}$$

for $t \in (0,T)$, we see that $(\hat{v}_n, \hat{\pi}_n)$ solves

$$\left\{\begin{array}{rcll} \hat{v}_n' - \Delta \hat{v}_n - Mx \cdot \nabla \hat{v}_n + M\hat{v}_n & & & \\ + b \cdot \nabla \hat{v}_n + \hat{v}_n \cdot \nabla b + \hat{u}_n^1 \cdot \nabla \hat{u}_n^2 + \nabla \hat{\pi}_n &=& F_2 & \text{in } \Omega, \quad t \in (0,T), \\ \operatorname{div} v_n &=& 0 & \text{in } \Omega, \quad t \in (0,T), \\ \hat{v}_n &=& 0 & \text{on } \Gamma, \quad t \in (0,T), \\ \hat{v}_n(0) &=& \hat{u}_n^1(0) & \text{in } \Omega. \end{array}\right.$$

Hence, $(v_n, \pi_n) := (\mathrm{e}^{tM}\hat{v}_n(\mathrm{e}^{-tM}x, t) + \mathrm{e}^{tM} b(\mathrm{e}^{-tM}x), \hat{\pi}(\mathrm{e}^{-tM}x, t))$ is a solution to

$$\left\{\begin{array}{rcll} \partial_t v_n - \Delta v_n + u_n^1 \cdot \nabla u_n^2 + (b \cdot \nabla)(v_n - u_n^2) & & & \\ + \left((v_n - u_n^1) \cdot \nabla\right) b + \nabla \pi_n &=& 0 & \text{in } \Omega(t), \quad t \in (0,T) \\ \operatorname{div} v_n &=& 0 & \text{in } \Omega(t), \quad t \in (0,T) \\ v_n &=& Mx & \text{on } \Gamma(t), \quad t \in (0,T) \\ v_n(0) &=& u_n^1(0) & \text{in } \Omega, \end{array}\right.$$

where $u_n^1 := \mathrm{e}^{tM}\hat{u}_n^1(\mathrm{e}^{-tM}x, t) + \mathrm{e}^{tM} b(\mathrm{e}^{-tM}x)$ and $u_n^2 := \mathrm{e}^{tM}\hat{u}_n^2(\mathrm{e}^{-tM}x, t) + \mathrm{e}^{tM} b(\mathrm{e}^{-tM}x)$. Moreover,

$$(v_n, \pi_n) \in \left(L^1(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,1}(0,T; L^q(\Omega(\cdot)))\right) \times L^1(0,T; \hat{W}^{1,q}(\Omega(\cdot)))$$

thanks to $\hat{v}_n \in L^1(0,T; D(A_{\Omega,b})) \cap W^{1,1}(0,T; L^q(\Omega))$. Therefore, integration by parts yields

$$\begin{aligned} < v_n(0), \varphi >_\Omega &- \int_0^T < v_n(t), \varphi'(t) + \Delta\varphi(t) > + < u_n^2(t), (u_n^1(t) \cdot \nabla)\varphi(t) >_{\Omega(t)} \;\mathrm{d}t \\ &+ \int_0^T < (b \cdot \nabla)(v_n - u_n^2) + \left((v_n - u_n^1) \cdot \nabla\right) b, \varphi >_{\Omega(t)} \;\mathrm{d}t \\ &+ \int_0^T < Mx, \nabla\varphi(t) >_{\Gamma(t)} \;\mathrm{d}t = 0, \quad \varphi \in \mathrm{D}. \end{aligned}$$

Letting $n \to \infty$, it follows from (5.1) and (5.2) that u is a very weak solution. $\square$

In a second step, the following lemma shows uniqueness of the very weak solution for suitable values of p and q. Especially, we will conclude coincidence of the mild and strong notion of solutions, as stated in Theorem 3.3.

Lemma 5.2. *Let $1 < p, q < \infty$ satisfy $\frac{n}{2q} + \frac{1}{p} \le \frac{1}{2}$ and let $v_1, v_2 \in L^p(0,T_0; L^q(\Omega(\cdot)))$ be two very weak solutions to (2.11) for some $T_0 > 0$ with initial value $v_0 \in L^q(\Omega)$. Assume that $v_1 - v_2 \in L^p(0,T_0; L^q_\sigma(\Omega(\cdot)))$. Then there exists $T \in (0,T_0)$ such that $v_1(t) = v_2(t)$ for a.e. $t \in (0,T)$.*

Proof. For the time being let us assume that for some $T \in (0,T_0)$ and all $f \in L^{p'}(0,T; L^{q'}(\Omega(\cdot)))$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, there exists a solution

$$(\varphi, \pi) \in \mathrm{D}_{\mathrm{ext}} := \Big(W^{1,p'}(0,T; L^{q'}_\sigma(\Omega(\cdot))) \cap L^{p'}(0,T; W^{2,q'}(\Omega(\cdot)))\Big) \times L^{p'}(0,T; \hat{W}^{1,q'}(\Omega(\cdot)))$$

to the dual backward problem

$$
\left\{\begin{array}{rcll} -\partial_t\varphi - \Delta\varphi + \nabla\pi - (v_1+v_2)\cdot\nabla\varphi &=& f & \text{in } \Omega(t), \quad t\in(0,T),\\ \operatorname{div}\varphi &=& 0 & \text{in } \Omega(t), \quad t\in(0,T),\\ \varphi &=& 0 & \text{on } \Gamma(t), \quad t\in(0,T),\\ \varphi(T) &=& 0 & \text{in } \Omega. \end{array}\right. \tag{5.3}
$$

Then, we obtain for all $f\in L^{p'}(0,T;L^{q'}(\Omega(\cdot)))$

$$
\begin{aligned}
&\int_0^T < v_1(t)-v_2(t), f(t) >_{\Omega(t)} \ \mathrm{d}t\\
=&\int_0^T < v_1(t)-v_2(t), \left(-\partial_t\varphi-\Delta\varphi+\nabla\pi-(v_1+v_2)\cdot\nabla\varphi\right)(t) >_{\Omega(t)} \ \mathrm{d}t\\
=&\int_0^T < v_1(t)-v_2(t), \left(-\partial_t\varphi-\Delta\varphi-(v_1+v_2)\cdot\nabla\varphi\right)(t) >_{\Omega(t)} \ \mathrm{d}t = 0.
\end{aligned}
$$

Here, we have used that φ can be approximated by functions in D. This implies $v_1 = v_2$ in $(0,T)$.

It thus remains to show that for $f\in L^{p'}(0,T;L^{q'}(\Omega(\cdot)))$, there exists a solution $(\varphi,\pi)\in \mathrm{D}_{\mathrm{ext}}$ to (5.3). In order to do so, we first consider the forward problem

$$
\left\{\begin{array}{rcll} \partial_t\varphi - \Delta\varphi + \nabla\pi - (v_1^{\mathrm{T}}+v_2^{\mathrm{T}})\cdot\nabla\varphi &=& f^{\mathrm{T}} & \text{in } \Omega(t), \quad t\in(0,T),\\ \operatorname{div}\varphi &=& 0 & \text{in } \Omega(t), \quad t\in(0,T),\\ \varphi &=& 0 & \text{on } \Gamma(t), \quad t\in(0,T),\\ \varphi(0) &=& 0 & \text{in } \Omega, \end{array}\right. \tag{5.4}
$$

where $v_1^{\mathrm{T}}(t)=v_1(T-t)$, $v_2^{\mathrm{T}}(t)=v_2(T-t)$ and $f^{\mathrm{T}}(t)=f(T-t)$ for $t\in(0,T)$. Note first, that, by a scaling argument, we may assume that $\|f^T\|_{L^{p'}(0,T;L^{q'}(\Omega(\cdot)))}$ is arbitrary small. Then, similar to the proof of Theorem 3.1, it follows that there exists $T>0$, independent of f^T and a solution $(\varphi,\pi)\in\mathrm{D}_{\mathrm{ext}}$ to (5.3). Indeed, we have an additional term coming from f^T, which is no problem since f^T is arbitrary small. Moreover, the nonlinear term has to be replaced by a term coming from $(v_1^{\mathrm{T}}+v_2^{\mathrm{T}})\cdot\nabla\varphi$. For convenience of the reader, we will give the estimates for $(\tilde v_1^{\mathrm{T}}+\tilde v_2^{\mathrm{T}})\cdot\nabla\tilde\varphi$. We choose $r,s\in(1,\infty)$ such that $\frac{1}{p'}=\frac{1}{p}+\frac{1}{r}$ and $\frac{1}{q'}=\frac{1}{q}+\frac{1}{s}$. Then, it follows from Hölder's inequality and Lemma 4.2

$$
\begin{aligned}
\|(\tilde v_1^{\mathrm{T}}-\tilde v_2^{\mathrm{T}})\cdot\nabla\tilde\varphi\|_{L^{p'}(0,T;L^{q'}(\Omega))} &\le \|\tilde v_1^{\mathrm{T}}-\tilde v_2^{\mathrm{T}}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\tilde\varphi\|_{L^r(0,T;L^s(\Omega))}\\
&\le C\|\tilde v_1^{\mathrm{T}}-\tilde v_2^{\mathrm{T}}\|_{L^p(0,T;L^q(\Omega))}\|\tilde\varphi\|_{X^T_{p',q'}}.
\end{aligned}
$$

Since $\|\tilde v_1^{\mathrm{T}}-\tilde v_2^{\mathrm{T}}\|_{L^p(0,T;L^q(\Omega))}\to 0$ for $T\to 0$, we get a solution $(\varphi^{\mathrm{T}},\pi^{\mathrm{T}})$ to (5.4) for some $T>0$. Finally, (φ,π), where $\varphi(t):=\varphi^{\mathrm{T}}(T-t)$ and $\pi(t):=\pi^{\mathrm{T}}(T-t)$ for $t\in(0,T)$, is a solution to (5.3). □

We finally prove Theorem 3.3.

Proof of Theorem 3.3. Let us first assume that $\frac{n}{2q}+\frac{1}{p}<\frac{3}{2}$. Then, by Sobolev embeddings,

$$
(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}\hookrightarrow L^{\tilde r}_\sigma(\Omega)
$$

for some $\tilde r>n$. Hence, $v,w\in C([0,T_0];L^{\tilde r}(\Omega)$ for some $T_0>0$. In particular, v,w satisfy the assumption of Lemma 5.2. Hence, the assertion follows from iteration in this case.

Let us now assume that $\frac{n}{2q}+\frac{1}{p}=\frac{3}{2}$ and $v_0-b\in L^r_\sigma(\Omega)$ for some $r>n$. By Sobolev embeddings, we have

$$
v\in L^s(0,T;L^{\tilde s}(\Omega))
$$

for some $\tilde{s} \in (n, r)$ and $1 < s < \infty$ satisfying $\frac{n}{2\tilde{s}} + \frac{1}{s} \leq \frac{1}{2}$. Moreover, since $v_0 - b \in L^{\tilde{s}}_\sigma(\Omega)$, we have $v \in C([0,T]; L^{\tilde{s}}(\Omega))$. Hence, the assertion follows similar to above in this case. $\square$

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