# Abelian topological groups with host algebras

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**Abstract.** The concept of a host algebra generalises that of a group $C^*$-algebra to groups which are not locally compact in the sense that its non-degenerate representations are in one-to-one correspondence with representations of the group under consideration. Here we consider the question of the existence of host algebras for abelian topological groups and also for multiplier representations. Our main negative result is essentially that a topological abelian group has a full host algebra (covering all its continuous unitary representations) if and only if it embeds into a locally compact group. On the positive side, we show that the canonical symplectic form on a countably dimensional complex vector space leads to an abelian group with multiplier for which a full host algebra exists. This provides a host algebra for the set of regular representations of the CCR algebra.

## Introduction

Group algebras and their generalizations (crossed products, groupoid algebras etc.) are important tools in the analysis of the continuous representation theory of locally compact groups and a range of related algebraic systems. Since non–locally compact groups (e.g., infinite dimensional Lie groups) regularly occur in physics and mathematics, there is a need to generalize the notion of a group algebra to topological groups which are not locally compact. Such a generalization, called a *full host algebra,* has been proposed and analyzed in [Gr05]. Briefly, it is a $C^*$-algebra $\mathcal{A}$ which has in its multiplier algebra $M(\mathcal{A})$ a unitary representation of the group $G$, such that the (unique) extension of the representation theory of $\mathcal{A}$ to $M(\mathcal{A})$ coincides exactly with the continuous (unitary) representation theory of $G$ through the copy of $G$ in $M(\mathcal{A})$. There is also an analogous concept for projective $\sigma$–representations where $\sigma$ is a continuous $\mathbb{T}$-valued 2-cocycle on $G$. Thus, given a full host algebra $\mathcal{A}$, the continuous representation theory of $G$ can be analyzed on $\mathcal{A}$. In [Gr05], a very general existence and uniqueness theorem for host algebras was obtained, which unfortunately is very hard to apply to concrete topological groups. Here we want to start a more detailed program with the aim of characterizing which classes of topological groups have host algebras for their continuous (projective) representation theory, and which do not.

We have two main results in this paper. Our first result is negative, and states that a topological abelian group $G$ (whose continuous unitary representations separate its points) has a full host algebra if and only if it embeds densely into a locally compact group such that its continuous representations factor through the embedding. As a corollary, we prove that a locally convex linear space, regarded as a topological group, has a full host algebra if and only if it is finite dimensional. Our second result is positive;- we give an explicit construction of a full host algebra for the $\sigma$–representations of an infinite dimensional topological linear space $S$, regarded as a group. Specifically, $S$ is the countably dimensional symplectic space with the (locally convex) inductive limit topology, and $\sigma(\cdot,\cdot) = \exp[iB(\cdot,\cdot)/2]$, where $B$ is the symplectic form. This example is important for physics, in that it provides a host algebra for the set of regular representations of the CCR algebra of $(S, B)$. Moreover, it demonstrates that the concept of a full host algebra is not a trivial extension of the concept of a (twisted) group algebra, in fact, we conclude that there are interesting pairs $(G,\sigma)$ of non-locally compact topological groups $G$ and continuous $\mathbb{T}$-valued 2-cocycles $\sigma$ for which full host algebras exist. The example $(S,\sigma)$ developed here, is the natural pair associated with a countably dimensional symplectic vector space.

From this point of view, it seems natural to conjecture the following: Let $(G,\sigma)$ be a (separable) abelian topological group with a (locally) continuous 2-cocycle and assume that this pair is non-degenerate in the sense that the center of the corresponding central extension $G_\sigma$ coincides with $\mathbb{T}$. Then the pair $(G,\sigma)$ has a full host algebra.

Another natural class of pairs to consider are given by $(G,\sigma)$, where $G$ is a direct limit of finite-dimensional Lie groups. When do these pairs have full host algebras? In this context, the class of restricted direct products seems to be a natural testing ground, and it includes the example treated below.

This paper is structured as follows;- in Sect. I we state the notation and definitions necessary for the subsequent material, and in Sect. II we state our main results in full. In Sect. III, we prove our first result concerning the embedding of an abelian group with host algebra in a locally compact group. In Sect IV, we construct the host algebra for the pair $(S,\sigma)$ mentioned above, and in the Appendix we add general results concerning host algebras and the strict topology which are required for our proofs. These results may be of independent interest.

## I. Definitions and notation

We will need the following notation and concepts for our main results.

- In the following, we write $M(\mathcal{A})$ for the multiplier algebra of a $C^*$-algebra $\mathcal{A}$ and, if $\mathcal{A}$ has a unit, $U(\mathcal{A})$ for its unitary group. We have an injective morphism of $C^*$-algebras $\iota_\mathcal{A}\colon \mathcal{A} \to M(\mathcal{A})$ and will just denote $\mathcal{A}$ for its image in $M(\mathcal{A})$. Then $\mathcal{A}$ is dense in $M(\mathcal{A})$ with respect to the *strict topology*, which is the locally convex topology defined by the seminorms
$$p_a(m) := \|m\cdot a\| + \|a\cdot m\|, \qquad a\in\mathcal{A},\ m\in M(\mathcal{A}).$$
- For a complex Hilbert space $\mathcal{H}$, we write $\mathrm{Rep}(\mathcal{A},\mathcal{H})$ for the set of non-degenerate representations of $\mathcal{A}$ on $\mathcal{H}$. Note that the collection $\mathrm{Rep}\,\mathcal{A}$ of all non-degenerate representations of $\mathcal{A}$ is not a set, but a (proper) class in the sense of von Neumann–Bernays–Gödel set theory, cf. [TZ75], and in this framework we can consistently manipulate the object $\mathrm{Rep}\,\mathcal{A}$. However, to avoid set–theoretical subtleties, we will express our results below concretely, i.e., in terms of $\mathrm{Rep}(\mathcal{A},\mathcal{H})$ for given Hilbert spaces $\mathcal{H}$. We have an injection
$$\mathrm{Rep}(\mathcal{A},\mathcal{H}) \hookrightarrow \mathrm{Rep}(M(\mathcal{A}),\mathcal{H}), \quad \pi\mapsto\widetilde{\pi} \quad \text{with} \quad \widetilde{\pi}\circ\iota_\mathcal{A} = \pi,$$
which identifies the non-degenerate representation $\pi$ of $\mathcal{A}$ with that representation $\widetilde{\pi}$ of its multiplier algebra which extends $\pi$ and is continuous with respect to the *strict topology* on $M(\mathcal{A})$ and the topology of pointwise convergence on $B(\mathcal{H})$.
- For topological groups $G$ and $H$ we write $\mathrm{Hom}(G,H)$ for the set of continuous group homomorphism $G\to H$. We also write $\mathrm{Rep}(G,\mathcal{H})$ for the set of all (strong operator) continuous unitary representations of $G$ on $\mathcal{H}$. Endowing $U(\mathcal{H})$ with the strong operator topology turns it into a topological group, denoted $U(\mathcal{H})_s$, so that $\mathrm{Rep}(G,\mathcal{H}) = \mathrm{Hom}(G,U(\mathcal{H})_s)$.
- Let $\mathbb{T}\subseteq\mathbb{C}^\times$ denote the unit circle, viewed as a multiplicative subgroup and $\sigma\colon G\times G\to\mathbb{T}$ be a continuous 2-cocycle, i.e.,
$$\sigma(\mathbf{1},x) = \sigma(x,\mathbf{1}) = 1, \quad \sigma(x,y)\sigma(xy,z) = \sigma(x,yz)\sigma(y,z) \quad \text{for} \quad x,y,z\in G.$$
We then form the topological group
$$G_\sigma := \mathbb{T}\times G, \quad (t,g)(t',g') := (tt'\sigma(g,g'),gg')$$

and note that the projection $q: G_\sigma \to G$ defines a central extension of $G$ by $\mathbb{T}$. A continuous unitary representation $(\pi, \mathcal{H})$ of $G_\sigma$ is called a *$\sigma$-representation of $G$* if $\pi(t, \mathbf{1}) = t\mathbf{1}$ holds for each $t \in \mathbb{T}$. Then

$$G \to U(\mathcal{H}), \quad g \mapsto \pi(1, g)$$

is continuous with respect to the strong operator topology, but

$$\pi(1, g)\pi(1, g') = \sigma(g, g')\pi(1, gg') \quad \text{for} \quad g, g' \in G.$$

We write $\mathrm{Rep}((G, \sigma), \mathcal{H})$ for the set of all continuous $\sigma$-representations of $G$ on $\mathcal{H}$.

**Definition I.1.** Let $\mathcal{A}$ be a $C^*$-algebra, $G$ a topological group and $\sigma$ a continuous 2-cocycle. A *host algebra for the pair* $(G, \sigma)$ is a pair $(\mathcal{A}, \eta)$, where $\eta: G_\sigma \to U(M(\mathcal{A}))$ is a homomorphism such that for each complex Hilbert space $\mathcal{H}$ the corresponding map

$$\eta^*: \mathrm{Rep}(\mathcal{A}, \mathcal{H}) \to \mathrm{Rep}((G, \sigma), \mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \circ \eta$$

is injective. We then write $\mathrm{Rep}(G, \mathcal{H})_\eta \subseteq \mathrm{Rep}(G, \mathcal{H})$ for the range of $\eta^*$. We say that $(G, \sigma)$ has a *full host algebra* if it has a host algebra for which $\eta^*$ is surjective for each Hilbert space $\mathcal{H}$.

In the case that $\sigma = 1$, we simply speak of a *host algebra for $G$*. In this case, $G_\sigma = G \times \mathbb{T}$ is a direct product, so that a host algebra for $G$ is a pair $(\mathcal{A}, \eta)$, where $\eta: G \to U(M(\mathcal{A}))$ is a homomorphism into the unitary group of $M(\mathcal{A})$ such that for each complex Hilbert space $\mathcal{H}$ the corresponding map

$$\eta^*: \mathrm{Rep}(\mathcal{A}, \mathcal{H}) \to \mathrm{Rep}(G, \mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \circ \eta$$

is injective. We then write $\mathrm{Rep}(G, \mathcal{H})_\eta \subseteq \mathrm{Rep}(G, \mathcal{H})$ for the range of $\eta^*$. We say that $G$ has a *full host algebra* if it has a host algebra for which $\eta^*$ is surjective for each Hilbert space $\mathcal{H}$. ■

It is well known that for each locally compact group $G$, the group $C^*$-algebra $C^*(G)$, and the natural map $\eta_G: G \to M(C^*(G))$ provide a full host algebra ([Dix64, Sect. 13.9]) and for each pair $(G, \sigma)$, where $G$ is locally compact, the corresponding twisted group $C^*$-algebra $C^*(G, \sigma)$, which is isomorphic to an ideal of $C^*(G_\sigma)$, is a full host algebra for the pair $(G, \sigma)$. This is most easily seen by decomposition of representations of $G_\sigma$ into isotypic summands with respect to the action of the central subgroup $\mathbb{T} \times \{\mathbf{1}\}$ (apply [BS70], [PR89] with $\mathcal{A} = \mathbb{C}$). The map $\eta_G : G \to M(C^*(G, \sigma))$ is continuous w.r.t. the strict topology of $M(C^*(G, \sigma))$.[1] In Section III below, we show a partial converse of these facts for the class of abelian groups.

## II. Main results

**Theorem II.2.** *For any abelian topological group $G$ the following assertions hold:*

(a) *If $G$ has a full host algebra, then there exists a locally compact abelian group $\widetilde{G}$ and a continuous homomorphism $\gamma_G: G \to \widetilde{G}$ with dense range for which each continuous unitary representation of $G$ factors through $\gamma_G$.*

(b) *If there exists a locally compact abelian group $\widetilde{G}$ and a continuous homomorphism $\gamma_G: G \to \widetilde{G}$ with dense range for which all continuous unitary representation of $G$ factor through $\gamma_G$, then $\eta_{\widetilde{G}} \circ \gamma_G: G \to M(C^*(\widetilde{G}))$ defines a full host algebra of $G$.*

(c) *Condition* (b) *is satisfied for any dense subgroup $G$ of a locally compact group $\widetilde{G}$.* ■

**Remark II.3.** (a) If we are interested in the continuous unitary representations of a topological group $G$, we can always mod out the common kernel $N \trianglelefteq G$ of all continuous unitary representations and endow $G/N$ with the coarsest topology for which all continuous unitary representations of $G$ lead to continuous representations of $G/N$. We write $G_u$ for the so obtained

[1] This is an easy consequence of the fact that $\mathrm{im}(\eta_G)$ is bounded and that the action on the corresponding $L^1$-algebra is continuous.

topological group. Then $G$ and $G_u$ have the same continuous unitary representations in the sense that for the natural homomorphism $q: G \to G_u$ the map $q^*: \mathrm{Rep}(G_u, \mathcal{H}) \to \mathrm{Rep}(G, \mathcal{H})$ is bijective for each complex Hilbert space $\mathcal{H}$.

(b) If $G$ is locally compact, then $N = \{\mathbf{1}\}$ and $G = G_u$. In fact, as we shall see below, the left regular representation of $G$ already leads to a topological embedding $G \hookrightarrow U(L^2(G))_s$ (Proposition III.2).

(c) For any abelian group $G$ with full host algebras we have $N = \ker \gamma_G$, and $G_u = \gamma_G(G)$ is a dense subgroup of the locally compact group $\widetilde{G}$, endowed with the subspace topology (Theorem II.2). ■

**Theorem II.4.** *A locally convex space, considered as an abelian topological group, has a full host algebra if and only if it is finite-dimensional.* ■

In the light of Proposition A.9 below, this implies that if a topological group has a quotient consisting of an infinite dimensional locally convex space, then it cannot have a full host algebra. In the context of these negative results, we have one positive result:

**Theorem II.5.** *For the locally convex space $S := \mathbb{C}^{(\mathbb{N})}$, endowed with the natural pre-Hilbert structure $(v, w) := \sum_j v_j \overline{w_j}$ and the cocycle $\sigma(v, w) := \exp[i \operatorname{Im}(v, w)/2]$, the pair $(S, \sigma)$ has a full host algebra.* ■

This particular example is of some importance for physics, in that it provides a host algebra for the regular representations of the $C^*$-algebra of the canonical commutation relations.

## III. Abelian groups with full host algebras

In this section, we prove Theorem II.2 and apply it to prove Theorem II.4.

Let $G$ be an abelian topological group and $\mathcal{A}$ a full host algebra for $G$. We recall from [Gr05, Prop. 2.1(5)] that $\mathcal{A}$ is commutative and the canonical map

$$\eta_G^*: \widehat{\mathcal{A}} = \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\} \to \widehat{G} := \mathrm{Hom}(G, \mathbb{T}), \quad \chi \mapsto \widetilde{\chi} \circ \eta_G$$

is a bijection. The set $\widehat{\mathcal{A}}$ is a locally compact space with respect to the topology of pointwise convergence on the elements of $\mathcal{A}$. In the following, we endow the character group $\widehat{G}$ of $G$ with the locally compact topology for which the map $\eta_G^*$ is a homeomorphism.

**Proposition III.1.** *$\widehat{G}$ is a locally compact topological group w.r.t. pointwise multiplication.*

**Proof.** We have to show that multiplication and inversion of $\widehat{G}$ are continuous maps.

In view of Lemma A.6, the spatial tensor product $\mathcal{A} \otimes \mathcal{A}$ is a host algebra for the product group $G \times G$. Since commutative $C^*$-algebras are nuclear ([Fi96, Th. 7.4.1]), we have

$$\mathcal{A} \otimes \mathcal{A} \cong C_0(\widehat{\mathcal{A}}, \mathcal{A}) \cong C_0(\widehat{\mathcal{A}}, C_0(\widehat{\mathcal{A}})) \cong C_0(\widehat{\mathcal{A}} \times \widehat{\mathcal{A}})$$

([Fi96, 7.4.2]), and this implies that the topological product space $\widehat{G} \times \widehat{G}$ can be identified with the spectrum of $\mathcal{A} \otimes \mathcal{A}$.

Next we note that

$$\varphi: G \times G \to G \times G, \quad (g_1, g_2) \mapsto (g_1 g_2, g_2)$$

is an isomorphism of topological groups, hence induces an automorphism $\varphi_{\mathcal{A}}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of host algebras satisfying

$$M(\varphi_{\mathcal{A}}) \circ \eta_{G \times G} = \eta_{G \times G} \circ \varphi$$

(Corollary A.3). Since $\widehat{G} \times \widehat{G}$ coincides with the spectrum of $\mathcal{A} \otimes \mathcal{A}$, $\varphi_{\mathcal{A}}$ induces a homeomorphism $\widehat{\varphi}_{\mathcal{A}}$ of $\widehat{G} \times \widehat{G}$. For $g, h \in G$ and $\chi_1, \chi_2 \in \widehat{G}$ we then have

$$\begin{aligned}\widehat{\varphi}_{\mathcal{A}}(\chi_1, \chi_2)(g, h) &= (\chi_1, \chi_2)\Big(M(\varphi_{\mathcal{A}})(\eta_G(g), \eta_G(h))\Big) = (\chi_1, \chi_2)\Big(M(\varphi_{\mathcal{A}})(\eta_{G\times G}(g, h))\Big) \\ &= (\chi_1, \chi_2)(\eta_{G\times G}(\varphi(g, h))) = (\chi_1, \chi_2)(\eta_{G\times G}(gh, h)) \\ &= \chi_1(gh)\chi_2(h) = \chi_1(g)(\chi_2\chi_1)(h).\end{aligned}$$

We conclude that

$$\widehat{\varphi}_{\mathcal{A}}(\chi_1, \chi_2) = (\chi_1, \chi_2\chi_1),$$

and hence that this map is a homeomorphism of $\widehat{G} \times \widehat{G}$. This implies that the multiplication map of $\widehat{G}$ is continuous, and since $\widehat{\varphi}_{\mathcal{A}}^{-1}(\chi_1, \chi_2) = (\chi_1, \chi_2\chi_1^{-1})$ also is continuous, the inversion in $\widehat{G}$ also is continuous. ■

**Proposition III.2.** *If $G$ is a locally compact group, then the left regular representation induces a topological embedding $\pi: G \to U(L^2(G))_s$.*

**Proof.** Since the left regular representation is continuous, $\pi$ is a continuous group homomorphism ([Dix64, 13.3.6]).

Let $U$ be a compact symmetric $\mathbf{1}$-neighborhood of $G$ and $f \in C(G, \mathbb{R}^+)$ with $f = 0$ on $G \setminus U$ and $\int_G f^2\, dm_G = 1$, where $m_G$ is a left Haar measure on $G$.

Let $(g_i)$ be a net in $G$ with $\pi(g_i) \to \mathbf{1}$ in $U(L^2(G))_s$. Then

$$\langle \pi(g_i).f, f\rangle \to \langle f, f\rangle = \int_G f^2\, dm_G = 1, \quad \text{and} \quad \langle \pi(g_i).f, f\rangle = \int_G f(g_i^{-1}x)f(x)\, d\mu_G(x)$$

vanishes if $g_iU \cap U = \emptyset$, i.e., $g_i \notin U^2$. Since $U$ was arbitrary, we conclude that $g_i \to \mathbf{1}$, and hence that $\pi$ is a topological embedding. ■

Since $\widehat{G}$ is a locally compact group, its character group $\widehat{\widehat{G}}$ is a locally compact abelian group, if we endow it with the compact open topology ([HoMo98, Th. 7.7(ii)]). By definition, each element $g \in G$ defines a character $\widehat{g}$ of $\widehat{G}$, which leads to a natural homomorphism

$$\gamma_G: G \to \widehat{\widehat{G}}, \quad \gamma_G(\chi) := \chi(g).$$

**Lemma III.3.** *$\gamma_G$ is continuous with dense range, and each continuous unitary representation of $G$ factors through $\gamma_G$.*

**Proof.** (1) Density of $\mathrm{im}(\gamma_G)$: From the Pontrjagin–van Kampen duality theory of locally compact abelian groups, it follows that $\gamma_G(G)$ is dense in $\widehat{\widehat{G}}$ if and only if its annihilator in $\widehat{G}$ is trivial ([HoMo98, Th. 7.63]), but this is a consequence of the fact that $\widehat{G}$ consists of functions on $G$.

(2) Continuity of $\gamma_G$: To see that $\gamma_G: G \to H := \widehat{\widehat{G}}$ is continuous, we recall from the preceding proposition that the regular representation of $H$ yields a topological embedding. It therefore suffices to verify that for each continuous unitary representation $\pi: H \to U(\mathcal{H})$, the representation $\pi \circ \gamma_G$ is continuous.

Next we recall that Pontrjagin–van Kampen duality theory also implies that the natural map $\gamma_{\widehat{G}}: \widehat{G} \to \widehat{H}$ is an isomorphism of topological groups ([HoMo98, Th. 7.63]). We also note that the spectrum of the $C^*$-algebra $C^*(H)$ coincides with $\widehat{H} \cong \widehat{G}$, so that we obtain $C^*(H) \cong C_0(\widehat{H}) \cong C_0(\widehat{G}) = C_0(\widehat{\mathcal{A}}) \cong \mathcal{A}$. Therefore $\mathcal{A}$ is a host algebra of $H$ with respect to the inclusion map

$$\eta_H: H = \mathrm{Hom}(\widehat{G}, \mathbb{T}) \hookrightarrow C_b(\widehat{G}) = M(\mathcal{A}).$$

Now

$$\eta_H \circ \gamma_G: G \to C_b(\widehat{G}) \cong M(\mathcal{A})$$

(cf. [Bla98, Ex. 12.1.1(b)]) coincides with $\eta_G$. The host algebra properties of $\mathcal{A}$ w.r.t. $G$ and $H$ implies that for each continuous unitary representation $\pi$ of $H$ the corresponding representation $\widetilde{\pi}$ of $M(\mathcal{A})$ yields a continuous unitary representation

$$\widetilde{\pi} \circ \eta_G = \widetilde{\pi} \circ \eta_H \circ \gamma_G = \pi \circ \gamma_G$$

of $G$.

(3) Factorization of unitary representations: If $\pi: G \to U(\mathcal{H})$ is a continuous unitary representation, then the host algebra property of $\mathcal{A}$ implies the existence of a representation $\widetilde{\pi}: \mathcal{A} \to B(\mathcal{H})$ for which the corresponding representation $M(\widetilde{\pi})$ satisfies $M(\widetilde{\pi}) \circ \eta_G = \pi$.

Then the representation $\pi_H = M(\widetilde{\pi}) \circ \eta_H$ is also continuous, and we have

$$\pi_H \circ \gamma_G = M(\widetilde{\pi}) \circ \eta_H \circ \gamma_G = M(\widetilde{\pi}) \circ \eta_G = \pi. \qquad \blacksquare$$

**Proposition III.4.** *If $G$ is an abelian topological group, then each unitary representation of $G$ extends to a continuous unitary representation of its completion $\overline{G}$.*

**Proof.** Let $\varphi: G \to \mathbb{C}$ be a continuous positive definite function on $G$. Then there exists a continuous unitary representation $\pi: G \to U(\mathcal{H})$ and some $v \in \mathcal{H}$ with $\varphi(g) = \langle \pi(g).v, v\rangle$. We then have

$$|\varphi(g) - \varphi(h)| = |\langle (\pi(g) - \pi(h)).v, v\rangle| = |\langle \pi(h^{-1}g).v - v, \pi(h^{-1}).v\rangle| \leq \|\pi(h^{-1}g).v - v\| \cdot \|v\|,$$

showing that $\varphi$ is uniformly continuous, hence extends to a continuous function on $\overline{G}$.

Since each continuous positive definite function of $G$ extends to $\overline{G}$, the GNS construction implies the same for cyclic representations. As each representation is a direct sum of cyclic ones, the assertion follows. $\blacksquare$

**Corollary III.5.** *If $H$ is a locally compact abelian group and $G \subseteq H$ a dense subgroup, then each continuous unitary representation of $G$ extends to a continuous unitary representation of $H$. In particular, the morphism $\eta_G := \eta_H|_G: G \to M(C^*(H)) \cong C_b(\widehat{H})$ defines a full host algebra for $G$.* $\blacksquare$

**Proof of Theorem II.2:** Part (a) follows from Propositions III.1 and Lemma III.3. Part (b) is trivial, and Part (c) is an immediate consequence of Corollary III.5. $\blacksquare$

**Theorem III.6.** *A locally convex space, considered as an abelian topological group, has a full host algebra if and only if it is finite-dimensional.*

**Proof.** Let $E$ be a locally convex space and $G := (E, +)$ the underlying abelian topological group. Then its character group $\widehat{G}$ can be identified in a natural way with the topological dual space $E'$.

If $G$ is finite-dimensional, it is locally compact, so that $C^*(G)$ is a host algebra of $G$. Suppose, conversely, that $G$ has a host algebra and let $\widetilde{G}$ be as in Theorem II.2. Then the continuous homomorphism $\gamma_G: G \to \widetilde{G}$ has dense range, so that $\widetilde{G}$ is a connected locally compact abelian group, hence isomorphic to $\mathbb{R}^n \times C$ for some compact group $C$ ([HoMo98, Th. 7.57]). In this sense, we write $\gamma_G = (\gamma_1, \gamma_2)$. Then $\gamma_1: G \to \mathbb{R}^n$ is a continuous homomorphism with dense range, hence a surjective linear map.

Using the Hahn–Banach Theorem, we can split off a finite-dimensional subspace $G_1 \cong \mathbb{R}^n$ supplementing $G_2 := \ker \gamma_1$, such that $G \cong G_1 \times G_2$ is a product of locally convex spaces and $\gamma_1(g_1, g_2) = g_1$. Now $\varphi(v, c) := (v, \gamma_2(v)c)$ defines a topological automorphism of $\widetilde{G} \cong G_1 \times C$, and we have $\varphi^{-1} \circ \gamma = \mathrm{id}_{G_1} \times \gamma_C$. Here $\gamma_G: G_2 \to C$ is a continuous homomorphism with dense range into the compact abelian group $C$, and each continuous unitary representation of $G_2$ factors through $\gamma_C$. Since continuous unitary representations of compact groups decompose into direct sums of irreducibles, the same holds for $G_2$. If $G_2$ is non-zero, we have $G_2 \cong \mathbb{R} \times G_3$ for some locally convex space $G_3$. Then the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ yields a continuous unitary representation of $G_2$ which does not decompose into irreducibles, contradicting the properties of $C$. we conclude that $G_2 = \{0\}$, and hence that $G = G_1$ is finite-dimensional. $\blacksquare$

The analogous theorem concerning host algebras for $\sigma$-representations of a locally convex space does not hold, as our Theorem IV.1 below shows. However, Theorem III.6 provides a set of counterexamples for the general claim in [Gr97] that any inductive limit group of locally compact groups has a full host algebra.

From Proposition A.9, we get the following easy Corollary of Theorem III.6:

**Corollary III.7.** *If $G$ is a topological group and $N$ is a closed normal subgroup such that $G/N$ is isomorphic to an infinite dimensional locally convex space $S$ (regarded as a group), then $G$ does not have a full host algebra.* ■

## IV. An example of a full host algebra

Here we want to present an example of a host algebra for an infinite-dimensional group. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for an infinite-dimensional separable Hilbert space $\mathcal{H}$. Let $S := \operatorname{span}\{e_n | n \in \mathbb{N}\} \subset \mathcal{H}$. Then $S$ is clearly a dense subspace. We consider $S \cong \mathbb{C}^{(\mathbb{N})}$ as an inductive limit of the subspaces $S_n := \operatorname{span}\{e_1, \ldots, e_n\}$ and endow it with the inductive limit topology, which turns it into an abelian topological group with respect to addition (which is only true for countably dimensional spaces; cf. [Gl03]). Moreover, $S$ is a symplectic space with the symplectic form $B(v, w) := \operatorname{Im}(v, w)$ and hence $\sigma(v, w) := \exp[iB(v, w)/2]$ defines a group two-cocycle $\sigma$ on $S$. Let $S_\sigma$ denote the corresponding central extension of $S$ by $\mathbb{T}$ (cf. above Definition I.1). Since $S$ is a quotient of $S_\sigma$, it follows from Corollary III.7 that $S_\sigma$ does *not* have a full host algebra. However, we show in this section that the pair $(S, \sigma)$ does:

**Theorem IV.1.** *The pair $(S, \sigma)$ has a full host algebra.* ■

Let $\mathcal{A}$ denote the discrete twisted $\sigma$–group algebra of $S$, i.e. it is the unique (simple) $C^*$-algebra generated by a collection of unitaries $\{\delta_s \mid s \in S\}$ satisfying the (Weyl) relations $\delta_{s_1}\delta_{s_2} = \sigma(s_1, s_2)\,\delta_{s_1+s_2}$ ([BR97, Th. 5.2.8]). In physics, $\mathcal{A}$ is called the $C^*$-algebra of the canonical commutation relations of $(S, \sigma)$, and the representations important for physics are those for which the restrictions to all one-dimensional subspaces of $S$ are strongly continuous. Such representations are called *regular*, and we denote the set of regular representations on the Hilbert space $\mathcal{H}$ by $\mathcal{R}(\mathcal{H})$. Through the identification $\pi(s) := \pi(\delta_s)$, $\mathcal{R}(\mathcal{H})$ corresponds exactly with the $\sigma$-representations of $S$ on $\mathcal{H}$, i.e. with $\operatorname{Rep}((S, \sigma), \mathcal{H})$.

**Lemma IV.2.** *With the notation above, we have $\mathcal{A} = \bigotimes_{n=1}^{\infty} \mathcal{A}_n$ with the spatial (minimal) tensor norms, where $\mathcal{A}_n := \mathrm{C}^*\{\delta_{ze_n} \mid z \in \mathbb{C}\}$.*

**Proof.** This follows directly from Proposition 11.4.3 of Kadison and Ringrose [KR83], we only need to verify that its conditions hold in the present context. For this, observe that $\mathcal{A} = \mathrm{C}^*\{\bigcup_{n=1}^{\infty} \mathcal{A}_n\}$, $\mathbb{1} \in \mathcal{A}_n$, $[\mathcal{A}_n, \mathcal{A}_m] = 0$ when $n \neq m$. Moreover, the linear maps

$$\psi_k : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k \to \mathcal{A} \qquad \text{by} \qquad \psi_k\big(A_1 \otimes \cdots \otimes A_k\big) := A_1 A_2 \cdots A_k$$

are *–monomorphisms because each image subalgebra $\mathrm{C}^*\{\bigcup_{n=1}^{k} \mathcal{A}_n\}$ is the unique $C^*$-algebra generated by the unitaries $\{\delta_{ze_i} \mid z \in \mathbb{C},\ i = 1, \ldots, k\}$, and this is also true for $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k$. This is enough to apply the proposition loc. cit. ■

Observe that each $\mathcal{A}_n$ is just the discrete $\sigma$-group algebra of the subgroup $\mathbb{C}e_n \subset S$, and as the latter is locally compact, we can construct its $\sigma$–twisted group algebra which we denote by $\mathcal{L}_n$ (recall that $\mathcal{L}_n$ is just the enveloping $C^*$-algebra of $L^1(\mathbb{C})$, equipped with $\sigma$-twisted convolution). It is well-known that $\mathcal{L}_n \cong \mathcal{K}(L^2(\mathbb{R}))$ (cf. Segal [Se67]). Note that for each finite set $F \subset \mathbb{N}$, the algebra $\bigotimes_{n\in F} \mathcal{L}_n \cong \mathcal{K}(\bigotimes_{n\in F} L^2(\mathbb{R})) \cong \mathcal{K}(L^2(\mathbb{R}^F))$ is a host algebra for the regular representations of $\bigotimes_{n\in F}\mathcal{A}_n = \mathrm{C}^*\big\{\delta_{ze_n}\,\big|\, z\in\mathbb{C},\ n\in F\big\}$, i.e. for the $\sigma$–representations of $\mathrm{span}\{e_n\,|\,n\in F\}\subset S$.

It is natural to try some infinite tensor product $\bigotimes_{n=1}^{\infty}\mathcal{L}_n$ for a host algebra, but because the algebras $\mathcal{L}_n$ are non-unital, the definition of the infinite tensor product needs some care [Bla77]. For each $n\in\mathbb{N}$, choose a nonzero projection $P_n\in\mathcal{L}_n\cong\mathcal{K}(\mathcal{H})$ and define $C^*$-embeddings

$$\Psi_{\ell k}:\mathcal{L}^{(k)}\to\mathcal{L}^{(\ell)}\qquad\text{by}\qquad \Psi_{\ell k}(A_1\otimes\cdots\otimes A_k):=A_1\otimes\cdots\otimes A_k\otimes P_{k+1}\otimes\cdots\otimes P_\ell,$$

where $k<\ell$ and $\mathcal{L}^{(k)}:=\bigotimes_{n=1}^{k}\mathcal{L}_n$. Then the inductive limit makes sense, so we define

$$\mathcal{L}:=\bigotimes_{n=1}^{\infty}\mathcal{L}_n:=\varinjlim\big\{\mathcal{L}^{(n)},\,\Psi_{\ell k}\big\}$$

and write $\Psi_k\colon\mathcal{L}^{(k)}\to\mathcal{L}$ for the corresponding embeddings, satisfying $\Psi_k\circ\Psi_{kj}=\Psi_j$ for $j\le k$. Since each $\mathcal{L}_n$ is simple, so are the finite tensor products $\mathcal{L}^{(k)}$ ([WO93], Prop. T.6.25), and as inductive limits of simple $C^*$-algebras are simple ([KR83], Prop. 11.4.2), so is $\mathcal{L}$. It is also clear that $\mathcal{L}$ is separable.

Since $\Psi_{k+n,k}(L_k)=L_k\otimes P_{k+1}\otimes\cdots\otimes P_{k+n}$, where $L_k\in\mathcal{L}^{(k)}$, this means that we can consider $\mathcal{L}$ to be built up out of elementary tensors of the form

$$\Psi_k(L_1\otimes\cdots\otimes L_k)=L_1\otimes L_2\otimes\cdots\otimes L_k\otimes P_{k+1}\otimes P_{k+2}\otimes\cdots\ ,\qquad\text{where}\qquad L_i\in\mathcal{L}_i\ ,\quad(4.1)$$

i.e. eventually they are of the form $\cdots\otimes P_k\otimes P_{k+1}\otimes\cdots$. We will use this picture below, and generally will not indicate the maps $\Psi_k$.

**Lemma IV.3.**

(i) *With respect to componentwise multiplication, we have an inclusion*

$$\mathcal{A}=\bigotimes_{n=1}^{\infty}\mathcal{A}_n\subset M(\mathcal{L})=M\Big(\bigotimes_{n=1}^{\infty}\mathcal{L}_n\Big)\ .$$

(ii) *There is a natural embedding $\iota_n\colon M(\mathcal{L}^{(n)})\hookrightarrow M(\mathcal{L})$. This is a topological embedding on each bounded subset of $M(\mathcal{L}^{(n)})$. Moreover, $\mathcal{L}^{(n)}$ is dense in $M(\mathcal{L}^{(n)})$ with respect to the restriction of the strict topology of $M(\mathcal{L})$.*

(iii) *Let $\pi\in\mathrm{Rep}(\mathcal{L},\mathcal{H})$, and let $\pi_n$ denote the unique representation which it induces on $\mathcal{L}^{(n)}\subset M(\mathcal{L}^{(n)})\subset M(\mathcal{L})$ by strict extension. Then*

$$\pi(L_1\otimes L_2\otimes\cdots)=\mathrm{s}\!-\!\lim_{n\to\infty}\pi_n(L_1\otimes\cdots\otimes L_n)$$

*for all $L_1\otimes L_2\otimes\cdots\in\mathcal{L}$ as in* (4.1).

**Proof.** (i) For each $k$ we obtain a homomorphism $\Theta_k:\bigotimes_{n=1}^{k}\mathcal{A}_n\to M(\mathcal{L})$ by componentwise multiplication in the first $k$ entries of $\mathcal{L}$, leaving all entries further up invariant. By simplicity of its domain, each $\Theta_k$ is a monomorphism. From $\Theta_k\big(\bigotimes_{n=1}^{k}\mathcal{A}_n\big)\subset M(\mathcal{L})$ for each $k\in\mathbb{N}$, we obtain all the generating unitaries $\delta_s$ in $M(\mathcal{L})$, then they generate $\mathcal{A}$ in $M(\mathcal{L})$ by uniqueness of the $C^*$-algebra of the canonical commutation relations.

(ii) Now $\mathcal{L}=\mathcal{L}^{(n)}\otimes\mathcal{B}$ for a $C^*$-algebra $\mathcal{B}$ (cf. Blackadar [Bla77, p. 315]), and $M(\mathcal{L}^{(n)})$ embeds in $M(\mathcal{L})$ as $M(\mathcal{L}^{(n)})\otimes\mathbb{1}$. Therefore (ii) follows from Lemma A.4.

(iii) Note that $U_n:=\Psi_n(\mathbb{1})=\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes P_{n+1}\otimes P_{n+2}\otimes\cdots\in\mathcal{L}\subseteq M(\mathcal{L})$ converges strictly to $\mathbb{1}$. Recall that $L=L_1\otimes L_2\otimes\cdots\in\mathcal{L}$ as in (4.1) is of the form

$$A_1\otimes A_2\otimes\cdots\otimes A_k\otimes P_{k+1}\otimes P_{k+2}\otimes\cdots\ ,$$

where $A_i \in \mathcal{L}_i$, so for $n \geq k$ we get for all $\psi \in \mathcal{H}_\pi$ that for the strictly continuous extension $\widetilde{\pi}$ of $\pi$ to $M(\mathcal{L})$:

$$\begin{aligned}\big\|\widetilde{\pi}(L - L_1 \otimes \cdots \otimes L_n \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots)\psi\big\| &= \Big\|\widetilde{\pi}\big(L_1 \otimes \cdots \otimes L_n \otimes (P_{n+1} \otimes P_{n+2} \otimes \cdots - \mathbb{1})\big)\psi\Big\| \\ &= \Big\|\widetilde{\pi}\big(L_1 \otimes \cdots \otimes L_n \otimes \mathbb{1} \otimes \cdots\big) \cdot \widetilde{\pi}(U_n - \mathbb{1})\psi\Big\| \\ &\leq C \cdot \big\|\pi(U_n - \mathbb{1})\psi\big\| \to 0\end{aligned}$$

as $n \to \infty$, where $C > 0$ is chosen such that $\|L_1 \otimes \cdots \otimes L_n\| < C$ for all $n$, and this is possible because $\|P_{k+1} \otimes P_{k+2} \otimes \cdots\| = 1$. But this is exactly the claim we needed to prove. ■

Let $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$ be regular. Observe that $\pi$ is regular on all $\mathcal{A}_n$, hence there are unique $\widehat{\pi}_n \in \mathrm{Rep}(\mathcal{L}_n, \mathcal{H})$ which extend (on $\mathcal{H}$) to $\pi\restriction\mathcal{A}_n$ by the host algebra property of $\mathcal{L}_n$. For the distinguished projections $P_n \in \mathcal{L}_n$, we simplify the notation to $\pi(P_n) := \widehat{\pi}_n(P_n)$. Observe that the projections $\pi(P_j)$ all commute, and so the strong limit

$$\mathbb{P}_k := \operatorname*{s-lim}_{n\to\infty} \pi(P_k)\cdots\pi(P_n)$$

exists, and it is the projection onto the intersection of the ranges of of all $\pi(P_j)$, $j \geq k$. Since $\mathbb{P}_k = \pi(P_k)\,\mathbb{P}_{k+1}$ we note that $\mathbb{P}_{k+1} \geq \mathbb{P}_k$ and so also $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k \leq \mathbb{1}$ exists.

We will use the notation $\mathcal{A}^{(n)} := \bigotimes_{j=1}^{n} \mathcal{A}_j$ below.

**Proposition IV.4.** *Define a monomorphism $\eta : S_\sigma \to U(M(\mathcal{L}))$ by $\eta((s,t)) := t\delta_s \in \mathcal{A} \subset M(\mathcal{L})$ (by Lemma IV.3(i)). Then $\eta$ is continuous with respect to the strict topology on $M(\mathcal{L})$ and $\mathcal{L}$ is a host algebra of $(S,\sigma)$, i.e., the maps $\eta^* : \mathrm{Rep}(\mathcal{L},\mathcal{H}) \to \mathrm{Rep}((S,\sigma),\mathcal{H})$ are injective. The range of $\eta^*$ consists of those $\pi \in \mathrm{Rep}\big((S,\sigma),\mathcal{H}\big)$ such that $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k = \mathbb{1}$.*

**Proof.** Let $\pi$ be a representation of $\mathcal{L}$ and $\widetilde{\pi}$ its strictly continuous extension to $M(\mathcal{L})$. To see that the representation $\eta^*\widetilde{\pi}$ of $S_\sigma$ is continuous, we show that $\eta$ is continuous with respect to the strict topology on $M(\mathcal{L})$. Since $S_\sigma$ is a topological direct limit of the subgroups $S_{m,\sigma}$, where $S_m = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_m\}$, it suffices to show that $\eta$ is continuous on each subgroup $S_{m,\sigma}$. Recall that the twisted group algebra $C^*(S_m, \sigma) \cong \mathcal{L}^{(m)}$ is a full host algebra for $(S_m, \sigma)$ and that the corresponding strictly continuous homomorphism $\eta_m \colon S_{m,\sigma} \to M(\mathcal{L}^{(m)})$ is compatible with the embedding $\iota_m \colon M(\mathcal{L}^{(m)}) \hookrightarrow M(\mathcal{L})$ in the sense that $\eta|_{S_{m,\sigma}} = \iota_m \circ \eta_m$. Since $\iota_m$ restricts to an embedding on the unitary group (Lemma IV.3(ii)), the continuity of $\eta_m$ implies the continuity of $\eta$ on $S_{m,\sigma}$, which in turn implies the continuity of $\eta$. As a consequence, $\widetilde{\pi} \circ \eta$ is a continuous unitary representation of $S_\sigma$ for each strictly continuous representation $\widetilde{\pi}$ of $M(\mathcal{L})$.

To see that $\eta^*$ is injective, we have to show that two representations $\pi_1, \pi_2$ of $\mathcal{L}$ for which $\eta^*\pi_1 = \eta^*\pi_2$ coincide are equal. If $\eta^*\pi_1 = \eta^*\pi_2$, then we obtain for each $m \in \mathbb{N}$ the relation $\eta_m^*\pi_1 = \eta_m^*\pi_2$ on $S_{m,\sigma}$. This means that the corresponding unitary representations of the group $S_{m,\sigma}$ coincide. In view of Lemma IV.3(iii), it suffices to argue that the two non-degenerate representations $\pi_{1,m}$ and $\pi_{2,m}$ of $\mathcal{L}^{(m)}$ coincide (cf. Lemma A.5 for the non-degeneracy), which in turn follows from the host algebra property of $\mathcal{L}^{(m)}$ for $S_{m,\sigma}$.

To characterize the range of $\eta^*$, let $\pi \in \mathrm{Rep}(\mathcal{A},\mathcal{H})$ be the strictly continuous extension of a $\pi_0 \in \mathrm{Rep}(\mathcal{L},\mathcal{H})$. Then, by Lemma IV.3(iii), it must satisfy

$$\pi_0(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi_n(L_1 \otimes \cdots \otimes L_n)$$

for all $L_1 \otimes L_2 \otimes \cdots \in \mathcal{L}$. Now we have

$$\pi_n(L_1 \otimes \cdots \otimes L_{n-1} \otimes P_n) = \widetilde{\pi}_n(L_1 \otimes \cdots \otimes L_{n-1} \otimes \mathbb{1})\,\widetilde{\pi}_n(\mathbb{1} \otimes \cdots \mathbb{1} \otimes P_n)$$

where $\widetilde{\pi}_n$ denotes the strictly continuous extension to $M(\mathcal{L}^{(n)})$, and it is obvious that these two operators commute. From the algebra relations $\mathcal{A}^{(n)} \supset \mathcal{A}^{(n-1)} \subset M(\mathcal{L}^{(n-1)}) \subset M(\mathcal{L}^{(n)})$, and the host algebra properties we get that $\widetilde{\pi}_n(L_1\otimes\cdots\otimes L_{n-1}\otimes\mathbb{1}) = \pi_{n-1}(L_1\otimes\cdots\otimes L_{n-1})$ and $\widetilde{\pi}_n(\mathbb{1}\otimes\cdots\mathbb{1}\otimes P_n) = \pi(P_n)$, so

$$\pi_n(L_1\otimes\cdots\otimes L_{n-1}\otimes P_n) = \pi_{n-1}(L_1\otimes\cdots\otimes L_{n-1})\,\pi(P_n)\ .$$

Thus, for

$$\begin{aligned}&L = L_1\otimes L_2\otimes\cdots = A_1\otimes A_2\otimes\cdots\otimes A_k\otimes P_{k+1}\otimes P_{k+2}\otimes\cdots\in\mathcal{L},\quad \text{we get for } n>k:\\ &\pi_n(L_1\otimes\cdots\otimes L_n) = \pi_k(A_1\otimes\cdots\otimes A_k)\,\pi(P_{k+1})\cdots\pi(P_n)\ .\end{aligned}$$

Using the fact that the projections $\pi(P_j)$ all commute,

$$\pi_0(L_1\otimes L_2\otimes\cdots) = \operatorname{s-}\lim_{n\to\infty}\pi_n(L_1\otimes\cdots\otimes L_n) = \pi_k(A_1\otimes\cdots\otimes A_k)\,\mathbb{P}_{k+1}\ .$$

Since $\pi_0$ is non-degenerate, and all $\pi_k\restriction\mathcal{L}^{(k)}$ are non-degenerate, it follows that $\operatorname{s-}\lim_{k\to\infty}\mathbb{P}_k = \mathbb{1}$.

Conversely, if we start from a regular representation $\pi$ of $\mathcal{A}$ which satisfies $\operatorname{s-}\lim_{k\to\infty}\mathbb{P}_k = \mathbb{1}$, we will *define* a representation $\pi_0$ on $\mathcal{L}$ by

$$\pi_0(L) =: \pi_k(A_1\otimes\cdots\otimes A_k)\,\mathbb{P}_{k+1}\quad\text{for}\quad L = A_1\otimes A_2\otimes\cdots\otimes A_k\otimes P_{k+1}\otimes P_{k+2}\otimes\cdots$$

where $\pi_k\in\operatorname{Rep}\mathcal{L}^{(k)}$ is obtained from $\pi\restriction\mathcal{A}^{(k)}$, using the host algebra property of $\mathcal{L}^{(k)}$. To see that this can be done, note that for $A\in\mathcal{L}^{(k)}$ we have

$$\pi_k(A)\mathbb{P}_{k+1} = \pi_{k+1}(\Psi_{k+1,k}(A))\mathbb{P}_{k+2}.$$

Therefore the universal property of the direct limit algebra $\mathcal{L}$ implies the existence of a representation $\pi_0$ of $\mathcal{L}$, satisfying

$$\pi_0(\Psi_k(A)) = \pi_k(A)\mathbb{P}_{k+1}\quad\text{for}\quad A\in\mathcal{L}^{(k)}.$$

That it is non-degenerate follows from the fact that each $\pi_k$ is non-degenerate, and that $\operatorname{s-}\lim_{k\to\infty}\mathbb{P}_k = \mathbb{1}$. To see that $\widetilde{\pi}_0\restriction\mathcal{A} = \pi$, recall that $\pi_k$ is the representation obtained from from $\pi\restriction\mathcal{A}^{(k)}$, using the host algebra property of $\mathcal{L}^{(k)}$. Let $B\in\mathcal{A}^{(k)}$, then for $A\in\mathcal{L}^{(k)}$ we have

$$\widetilde{\pi}_0(B)\,\pi_0(\Psi_k(A)) = \pi_0(B\cdot\Psi_k(A)) = \pi_k(B\cdot A)\mathbb{P}_{k+1} = \pi(B)\,\pi_k(A)\mathbb{P}_{k+1} = \pi(B)\,\pi_0(\Psi_k(A))$$

from which it follows that $\widetilde{\pi}_0\restriction\mathcal{A} = \pi$. ∎

Thus for every family of projections $P_k\in\mathcal{L}_k$ we get a host algebra. Now recall that $\mathcal{L}_k\cong\mathcal{K}(\ell^2(\mathbb{N}))$, and that there is a (countable) approximate identity $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(\ell^2(\mathbb{N}))$ consisting of a strictly increasing sequence of projections $E_n$ with $\dim(E_n\ell^2(\mathbb{N})) = n$. For each $k$, choose such an approximate identity $(E^{(k)}_n)\subset\mathcal{L}_k$, then for each sequence $\mathbf{n} = (n_1, n_2, \ldots)\in\mathbb{N}^\infty$, we have a sequence of projections $\big(E^{(1)}_{n_1}, E^{(2)}_{n_2}, \ldots\big)$ from which we can construct an infinite tensor product as above, and we will denote it by $\mathcal{L}[\mathbf{n}]$. For the elementary tensors, we streamline the notation to: $A_1\otimes\cdots\otimes A_k\otimes E[\mathbf{n}]_{k+1} := A_1\otimes\cdots\otimes A_k\otimes E^{(k+1)}_{n_{k+1}}\otimes E^{(k+2)}_{n_{k+2}}\otimes\cdots\in\mathcal{L}[\mathbf{n}]$, where $A_i\in\mathcal{L}_i$, and their closed span is the simple $C^*$-algebra $\mathcal{L}[\mathbf{n}]$. Note that we can multiply these, in fact, since for componentwise multiplication, the sequences give:

$$\big(E^{(1)}_{n_1}, E^{(2)}_{n_2}, \ldots\big)\cdot\big(E^{(1)}_{m_1}, E^{(2)}_{m_2}, \ldots\big) = \big(E^{(1)}_{p_1}, E^{(2)}_{p_2}, \ldots\big)$$

where $p_j := \min(n_j, m_j)$, we can consistently define a product between the equivalence classes which define the inductive limit algebras $\mathcal{L}[\mathbf{n}]$ and $\mathcal{L}[\mathbf{m}]$ to get one in $\mathcal{L}[\mathbf{p}]$, i.e. we have that

$$\mathcal{L}[\mathbf{n}]\cdot\mathcal{L}[\mathbf{m}]\subseteq\mathcal{L}[\mathbf{p}], \tag{4.3}$$

and in fact

$$\mathcal{L}[\mathbf{n}] \subset M(\mathcal{L}[\mathbf{p}]) \supset \mathcal{L}[\mathbf{m}]. \tag{4.4}$$

Thus $\mathcal{L}[\mathbf{n}] \subseteq M(\mathcal{L}[\mathbf{1}])$ for all $\mathbf{n}$, where $\mathbf{1} := (1, 1, \ldots)$ and so we can form the $C^*$-algebra in $M(\mathcal{L}[\mathbf{1}])$ generated by all $\mathcal{L}[\mathbf{n}]$, and denote it by $\mathcal{L}[E]$. By (4.3), this is just the closed span of all $\mathcal{L}[\mathbf{n}]$ and hence the closure of the dense *-subalgebra $\mathcal{L}_0 \subset \mathcal{L}[E]$, where:

$$\mathcal{L}_0 := \sum_{\mathbf{n}\in\mathbb{N}^\infty} \mathcal{L}[\mathbf{n}]_0 \quad \text{and} \quad \mathcal{L}[\mathbf{n}]_0 := \bigcup_{k\in\mathbb{N}} \mathcal{L}^{(k)} \otimes E[\mathbf{n}]_{k+1}.$$

We still have $\mathcal{A} \subset M(\mathcal{L}[E]) \supset \mathcal{L}^{(n)}$ for each $n \in \mathbb{N}$. Note that if two sequences $\mathbf{n}$ and $\mathbf{m}$ differ only in a finite number of entries, then $\mathcal{L}[\mathbf{n}] = \mathcal{L}[\mathbf{m}]$, and hence we actually have that the correct index set for the algebras $\mathcal{L}[\mathbf{n}]$ is not the sequences $\mathbb{N}^\infty$, but the set of equivalence classes $\mathbb{N}^\infty/\!\sim$ where $\mathbf{n} \sim \mathbf{m}$ if they differ only in finitely many entries. Some of the structures of $\mathbb{N}^\infty$ will factor through to $\mathbb{N}^\infty/\!\sim$, e.g. we have a partial ordering of equivalence classes defined by $[\mathbf{n}] \geq [\mathbf{m}]$ if for any representatives $\mathbf{n}$ and $\mathbf{m}$ resp., we have that there is an $N$ (depending on the representatives) such that $n_k \geq m_k$ for all $k > N$. In particular, we note that products reduce sequences, i.e., we have $\mathcal{L}[\mathbf{n}] \cdot \mathcal{L}[\mathbf{p}] \subseteq \mathcal{L}[\mathbf{q}]$ for $q_i = \min(n_i, p_i)$, so $[\mathbf{n}] \geq [\mathbf{q}] \leq [\mathbf{p}]$.

Let $\varphi : \mathbb{N}^\infty/\!\sim \to \mathbb{N}^\infty$ be a section of the factor map. Then $\mathcal{L}[E]$ is the $C^*$-algebra generated in $M(\mathcal{L}[\mathbf{1}])$ by $\{\mathcal{L}[\varphi(\gamma)] \,|\, \gamma \in \mathbb{N}^\infty/\!\sim\}$, and it is the closure of the span of the elementary tensors in this generating set.

Below we will prove that $\mathcal{L}[E]$ is a full host algebra for $(S, \sigma)$, and so it is of some interest to explore its algebraic structure. From the reducing property of products, we already know that $\mathcal{L}[E]$ has the ideal $\mathcal{L}[\mathbf{1}]$ (we will show that it is proper), hence that it is not simple. However, it has in fact infinitely many proper ideals and each of the generating algebras $\mathcal{L}[\mathbf{n}]$ is contained in such an ideal:

**Proposition IV.5.** *For the $C^*$–algebra $\mathcal{L}[E]$ we have the following:*

(i) *$\mathcal{L}[E]$ is nonseparable,*

(ii) *Define $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ to be the closed span of*

$$\{\mathcal{L}[\mathbf{q}]_0 \ | \ [\mathbf{q}] \leq [\mathbf{n}_\ell] \text{ for some } \ell = 1, \ldots, k\}.$$

*Let $[\mathbf{p}] > [\mathbf{n}_\ell]$ strictly for all $\ell \in \{1, \ldots, k\}$, then $\mathcal{L}[\mathbf{p}] \cap \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k] = \{0\}$.*

(iii) *$\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is a proper closed two sided ideal of $\mathcal{L}[E]$.*

(iv) *Define $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] := C^*(\mathcal{L}[\mathbf{n}_1] \cup \cdots \cup \mathcal{L}[\mathbf{n}_k])$. Then $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \subset \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ and*

$$C^*(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]) \subseteq \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k], \quad \text{where:} \quad (\mathbf{q}_j)_\ell = \min\big((\mathbf{n}_j)_\ell, (\mathbf{n}_{k+1})_\ell\big).$$

**Proof.** (i) $\mathcal{L}[E] \supset Q := \{E[\mathbf{n}]_1 := E^{(1)}_{n_1} \otimes E^{(2)}_{n_2} \otimes \cdots \ | \ \mathbf{n} \in \mathbb{N}^\infty\}$. If $\mathbf{n} \neq \mathbf{p}$, there is some $k$ for which $E^{(k)}_{n_k} \neq E^{(k)}_{p_k}$ and as the approximate identity is linearly increasing, one of these must be larger than the other, so take $E^{(k)}_{n_k} > E^{(k)}_{p_k}$ strictly. Group the remaining parts of the tensor product together, i.e., write

$$E[\mathbf{n}]_1 = E^{(k)}_{n_k} \otimes A \quad \text{and} \quad E[\mathbf{p}]_1 = E^{(k)}_{p_k} \otimes B,$$

where $A$ and $B$ are projections, then choose a product representation $\pi = \pi_1 \otimes \pi_2$ in which $\pi_1$ is faithful on $\mathcal{L}_k$ and $\pi_2$ is faithful on the $C^*$-algebra generated by $A$ and $B$. Thus there is a unit vector $\psi \in \mathcal{H}_1$ such that $\|\pi_1(E^{(k)}_{n_k})\psi\| = 1$ and $\pi_1(E^{(k)}_{p_k})\psi = 0$. For any unit vector $\varphi \in \mathcal{H}_2$ we get

$$\begin{aligned}\|E[\mathbf{n}]_1 - E[\mathbf{p}]_1\| &\geq \Big\|(\pi_1 \otimes \pi_2)\big(E^{(k)}_{n_k} \otimes A - E^{(k)}_{p_k} \otimes B\big)(\psi \otimes \varphi)\Big\| \\ &= \|\pi_1(E^{(k)}_{n_k})\psi \otimes \pi_2(A)\varphi\| = \|\pi_1(E^{(k)}_{n_k})\psi\| \cdot \|\pi_2(A)\varphi\| = \|\pi_2(A)\varphi\|\end{aligned}$$

and by letting $\varphi$ range over the unit ball we get that $\big\|E[\mathbf{n}]_1 - E[\mathbf{p}]_1\big\| \geq \|A\| = 1$. Thus, since $Q$ is uncountable and its elements far apart, $\mathcal{L}[E]$ cannot be separable.

(ii) Here we adapt the argument in (i) as follows. It suffices to show that for $\mathbf{q}_1, \ldots, \mathbf{q}_d$ with $\mathbf{q}_i \leq \mathbf{n}_j$ for some $j$, the norm distance between $\sum\limits_{i=1}^{d} \mathcal{L}[\mathbf{q}_i]_0$ and any $C \in \mathcal{L}[\mathbf{p}]_0$ is always $\geq \|C\|$. Let $C \in \mathcal{L}[\mathbf{p}]_0$ be nonzero and consider a sum $\sum\limits_{i=1}^{d} C_i$ with $C_i \in \mathcal{L}[\mathbf{q}_i]_0$ and $[\mathbf{p}] > [\mathbf{n}_j]$ for all $j$, which implies $[\mathbf{p}] > [\mathbf{q}_i]$ for all $i$. Choose an $M > 0$ large enough so that all $C$ and $C_i$ can be expressed in the form:

$$C_i = C_i^{(0)} \otimes E[\mathbf{n}_i]_M\,, \quad \text{for} \quad C_i^{(0)} \in \mathcal{L}^{(M-1)}\,.$$

Then by $[\mathbf{p}] > [\mathbf{q}_i]$ there is an entry of the tensor products, say for $j > M$, which consist only of elements of the approximate identity $(E_n^{(j)})_{n=1}^{\infty} \subset \mathcal{L}_j$ and for which $B > B_i$ for all $i$, where $B$ (resp. $B_i$) is the $j^{\text{th}}$ entry of $C$ (resp. $C_i$). Denote the remaining parts of the tensor products by $A$ (resp. $A_i$), i.e.,

$$C = A \otimes B, \quad C_i = A_i \otimes B_i, \quad \text{where} \quad B > B_i \;\forall\, i$$

and $B,\ B_i$ consist of commuting projections. Then $\|C - \sum\limits_{i=1}^{d} C_i\| = \big\|A \otimes B - \sum\limits_{i=1}^{d} (A_i \otimes B_i)\big\|$. Choose a product representation $\pi = \pi_1 \otimes \pi_2$ such that $\pi_1$ is faithful on $\mathcal{L}[\mathbf{p}]$ and $\pi_2$ is faithful on the $C^*$-algebra generated by $(E_n^{(j)})_{n=1}^{\infty} \subset \mathcal{L}_j$. Thus there is a unit vector $\varphi \in \mathcal{H}_{\pi_2}$ such that $\|\pi_2(B)\varphi\| = 1$ and $\pi_2(B_i)\varphi = 0$ for all $i$ (which exists because $B > B_i$ for all $i$). Then we have for any unit vector $\psi \in \mathcal{H}_{\pi_1}$ that

$$\begin{aligned}\|C - \sum_{i=1}^{d} C_i\| &\geq \Big\|(\pi_1 \otimes \pi_2)\Big(A \otimes B - \sum_{i=1}^{d} A_i \otimes B_i\Big)(\psi \otimes \varphi)\Big\| \\ &= \|\pi_1(A)\psi \otimes \pi_2(B)\varphi\| = \|\pi_1(A)\psi\| \cdot \|\pi_2(B)\varphi\| = \|\pi_1(A)\psi\|\end{aligned}$$

and by letting $\psi$ range over the unit ball of $\mathcal{H}_{\pi_1}$, we find that $\|C - \sum_{i=1}^{d} C_i\| \geq \|A\| = \|C\|$ since $\|B\| = 1$. This establishes the claim.

(iii) It is obvious from the reduction property $\mathcal{L}[\mathbf{n}] \cdot \mathcal{L}[\mathbf{p}] \subseteq \mathcal{L}[\mathbf{q}]$ for $q_j = \min(n_j, p_j)$, that $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is a two–sided ideal (hence a *–algebra). To see that it is proper, note that $[\mathbf{p}] > [\mathbf{n}_i]$ strictly for all $i$ where $p_j = \max((\mathbf{n}_1)_j, \ldots, (\mathbf{n}_k)_j) + 1$. Thus, by (ii) we see that $\mathcal{L}[\mathbf{p}] \cap \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k] = \{0\}$ and hence that $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is proper.

(iv) $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \subset \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ because $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is a $C^*$-algebra which contains all the generating elements $\mathcal{L}[\mathbf{n}_i]$ of $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$. Next we need to prove that

$$C^*\left(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]\right) \subseteq \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k], \quad \text{where:} \quad (\mathbf{q}_j)_\ell = \min\big((\mathbf{n}_j)_\ell,\, (\mathbf{n}_{k+1})_\ell\big)\,.$$

By definition, $C^*\left(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]\right)$ is the closed linear span of monomials $\prod\limits_{i=1}^{N} L_i$, where $L_i$ can be either of the form $A_iB_i$ or $B_iA_i$, where $A_i \in \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ and $B_i \in \mathcal{L}[\mathbf{n}_{k+1}]$. So it suffices to show that

$$AB \in \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k] \quad \text{for} \quad A \in \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \quad \text{and} \quad B \in \mathcal{L}[\mathbf{n}_{k+1}]$$

(since then $BA \in \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k]$ by involution). Since $\mathcal{L}[\mathbf{n}]_0$ is dense in $\mathcal{L}[\mathbf{n}]$, it suffices to prove this for $A = A_1A_2 \ldots A_p$ where $A_i = C_i \otimes E[\mathbf{n}_{k_i}]_{r_i+1}$ and $C_i \in \mathcal{L}^{(r_i)}$, $k_i \in \{1, \ldots, k\}$, and $B = D \otimes E[\mathbf{n}_{k+1}]_{r+1}$, where $D \in \mathcal{L}^{(r)}$. Now

$$A_pB = F \otimes E[\mathbf{q}_{k_p}]_{s+1} \in \mathcal{L}[\mathbf{q}_{k_p}]$$

for some $F \in \mathcal{L}^{(s)}$, $s \geq \max(r_p, r)$. Then

$$A_{p-1}A_pB = \big(C_{p-1} \otimes E[\mathbf{n}_{k_{p-1}}]_{r_{p-1}+1}\big)\big(F \otimes E[\mathbf{q}_{k_p}]_{s+1}\big) = G \otimes E[\mathbf{m}]_{t+1},$$

where $t \geq \max(r_{p-1}, s)$ and

$$\begin{aligned} m_i &= \min\big((\mathbf{n}_{k_{p-1}})_i,\, (\mathbf{q}_{k_p})_i)\big) = \min\Big((\mathbf{n}_{k_{p-1}})_i,\, \min\big((\mathbf{n}_{k_p})_i,\, (\mathbf{n}_{k+1})_i\big)\Big) \\ &= \min\Big(\min\big((\mathbf{n}_{k_{p-1}})_i,\, (\mathbf{n}_{k+1})_i\big),\, \min\big((\mathbf{n}_{k_p})_i,\, (\mathbf{n}_{k+1})_i\big)\Big) = \min\big((\mathbf{q}_{k_{p-1}})_i,\, (\mathbf{q}_{k_p})_i\big) \end{aligned}$$

and so we have in fact that

$$A_{p-1}A_pB = \big(\widetilde{C} \otimes E[\mathbf{q}_{k_{p-1}}]_{t+1}\big)\big(\widetilde{F} \otimes E[\mathbf{q}_{k_p}]_{t+1}\big) \in \mathcal{L}[\mathbf{q}_{k_{p-1}}] \cdot \mathcal{L}[\mathbf{q}_{k_p}]$$

where $\widetilde{C}, \widetilde{F} \in \mathcal{L}^{(t)}$. Hence $A_{p-1}A_pB \in \mathcal{L}[\mathbf{q}_{k_{p-1}},\, \mathbf{q}_{k_p}]$. We continue the process to get $AB = A_1A_2 \ldots A_pB \in \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k]$. ■

For each strictly increasing sequence $\big([\mathbf{n}_1], [\mathbf{n}_2], \ldots\big) \subset \mathbb{N}^\infty/\sim$ we get from part (ii) a strictly increasing chain of proper ideals $\mathcal{J}_k := \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$.

Now we want to prove our main theorem in this section.

**Theorem IV.6.** *The monomorphism $\eta : S_\sigma \to U(M(\mathcal{L}[E]))$ from above, defined by*

$$\eta((s,t)) := t\delta_s \in \mathcal{A} \subset M(\mathcal{L}[E]),$$

*is continuous with respect to the strict topology on $M(\mathcal{L}[E])$ and $\mathcal{L}[E]$ is a host algebra, i.e., the map*

$$\eta^* : \mathrm{Rep}(\mathcal{L}[E], \mathcal{H}) \to \mathrm{Rep}((S, \sigma), \mathcal{H})$$

*is injective. The range of $\eta^*$ is exactly $\mathcal{R}(\mathcal{H})$.*

**Proof.** First we show that $\eta$ is continuous with respect to the strict topology on $M(\mathcal{L}[E])$. This implies that for each $\pi \in \mathrm{Rep}(\mathcal{L}[E], \mathcal{H})$ the representation $\widetilde{\pi} \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$ is regular, hence

$$\eta^*\big(\mathrm{Rep}(\mathcal{L}[E], \mathcal{H})\big) \subseteq \mathcal{R}(\mathcal{H}).$$

Since $\mathrm{im}(\eta)$ is bounded, it suffices to show that the set

$$\big\{L \in \mathcal{L}[E] \mid g \mapsto \eta(g)L \quad \text{is norm continuous in} \quad g \in S_\sigma\big\}$$

spans a dense subspace of $\mathcal{L}[E]$. This reduces the assertion to the corresponding result for the action of $S_\sigma$ on $\mathcal{L}[\mathbf{n}]$ for each $\mathbf{n}$, which follows from the continuity of the corresponding map $S_\sigma \to M(\mathcal{L}[\mathbf{n}])$ (Proposition IV.4).

To prove that $\eta^*$ is injective we show that $\mathcal{A}$ separates $\mathrm{Rep}(\mathcal{L}[E], \mathcal{H})$ for all $\mathcal{H}$. Let $\pi \in \mathrm{Rep}(\mathcal{L}[E], \mathcal{H})$, then by Proposition IV.4 we know that the values which $\widetilde{\pi}(\mathcal{A})$ takes on $\mathcal{H}_\mathbf{n}$ uniquely determine the values of $\pi\big(\mathcal{L}[\mathbf{n}]\big)$ on $\mathcal{H}_\mathbf{n}$, hence on all $\mathcal{H}$, as $\pi\big(\mathcal{L}[\mathbf{n}]\big)$ is zero on the orthogonal complement of $\mathcal{H}_\mathbf{n}$. This holds for all $\mathbf{n}$, hence $\widetilde{\pi}(\mathcal{A})$ uniquely determines the values of $\pi$ on $\mathcal{L}[E]$, i.e., $\eta^*$ is injective.

It remains to prove that $\eta^*\big(\mathrm{Rep}(\mathcal{L}, \mathcal{H})\big) = \mathcal{R}(\mathcal{H})$. Start from a $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$ which is regular. Then we have to show how to obtain a $\pi_0 \in \mathrm{Rep}\,\mathcal{L}[E]$ such that $\widetilde{\pi}_0 \restriction \mathcal{A} = \pi$. Observe that $\pi$ is regular on all $\mathcal{A}^{(n)}$, hence there are unique $\pi_n \in \mathrm{Rep}(\mathcal{L}^{(n)}, \mathcal{H})$ which extend (on $\mathcal{H}$) to coincide with $\pi \restriction \mathcal{A}^{(n)}$ by the host algebra property of $\mathcal{L}^{(n)}$. For each $\mathbf{n}$ define the projections

$$\mathbb{E}^\mathbf{n}_k := \mathrm{s-}\lim_{m\to\infty} \pi(E^{(k)}_{n_k}) \cdots \pi(E^{(m)}_{n_m}) \quad \text{and} \quad \mathbb{E}^\mathbf{n} := \mathrm{s-}\lim_{k\to\infty} \mathbb{E}^\mathbf{n}_k\,.$$

Now each $\pi_n(\mathcal{L}^{(n)})$ commutes with the projections $\mathbb{E}^\mathbf{n}_k$ for $k > n$, and in particular preserves the space $\mathcal{H}^\mathbf{n} := \mathbb{E}^\mathbf{n}\mathcal{H}$, and hence so does $\pi(\mathcal{A}^{(n)})$. Then by Proposition IV.4 we know that we can define a (non-degenerate) representation $\pi^\mathbf{n}_0 : \mathcal{L}[\mathbf{n}] \to \mathcal{B}(\mathcal{H}^\mathbf{n})$ by

$$\pi^\mathbf{n}_0(L) = \pi_k(A_1 \otimes \cdots \otimes A_k)\,\mathbb{E}^\mathbf{n}_{k+1}$$

for $L = A_1 \otimes \cdots \otimes A_k \otimes E^{(k+1)}_{n_{k+1}} \otimes E^{(k+2)}_{n_{k+2}} \otimes \cdots \in \mathcal{L}[\mathbf{n}]$ such that $\widetilde{\pi}^{\mathbf{n}}_0 \restriction \mathcal{A}$ is $\pi(\mathcal{A})$, restricted to $\mathcal{H}^{\mathbf{n}}$. We extend $\pi^{\mathbf{n}}_0$ to all of $\mathcal{H}$, by putting it to zero on the orthogonal complement of $\mathcal{H}^{\mathbf{n}}$. Note that

$$\mathbf{n} \leq \mathbf{m} \quad \Rightarrow \quad \mathcal{H}^{\mathbf{n}} \subseteq \mathcal{H}^{\mathbf{m}}.$$

We now argue that these representations $\pi^{\mathbf{n}}_0$ combine into a single representation of $\mathcal{L}[E]$. First, we want to extend by linearity the maps $\pi^{\mathbf{n}}_0 : \mathcal{L}[\mathbf{n}] \to \mathcal{B}(\mathcal{H})$ to define a linear map $\pi_0$ from the dense $*$-subalgebra $\mathcal{L}_0 \subset \mathcal{L}[E]$ to $\mathcal{B}(\mathcal{H})$, where we recall that $\mathcal{L}_0 := \sum\limits_{\mathbf{n}\in\mathbb{N}^\infty} \mathcal{L}[\mathbf{n}]_0$.

This linear extension $\pi_0$ is possible if the sum of the spaces $\mathcal{L}[\mathbf{n}]_0$ is direct for different $\mathbf{n} \in \varphi\big(\mathbb{N}^\infty/{\sim}\big)$, i.e., if $0 = \sum\limits_{k=1}^m B_k$ for $B_k \in \mathcal{L}[\mathbf{n}_k]_0$, where $\mathbf{n}_k \not\sim \mathbf{n}_\ell$ if $k \neq \ell$ implies that $B_k = 0$ for all $k$. Let us prove this implication, so assume $0 = \sum\limits_{k=1}^m B_k$ as above. Choose an $M > 0$ large enough so that for all $k$, the $B_k$ can be expressed in the form $B_k = A_k \otimes E[\mathbf{n}_k]_M$ for $A_k \in \mathcal{L}^{(M-1)}$, define the projections $P_k := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes E[\mathbf{1}]_k$ (there are $k-1$ factors of $\mathbb{1}$), and note that $P_\ell$ commutes with all $B_k$ for $\ell \geq M$. In fact, for $B_k$ as above, we have (simplifying notation to $\mathbf{n}_k = \mathbf{n}$):

$$B_k P_\ell = A_k \otimes E^{(M)}_{n_M} \otimes \cdots \otimes E^{(\ell-1)}_{n_{\ell-1}} \otimes E[\mathbf{1}]_\ell \in \mathcal{L}^{(\ell-1)} \otimes E[\mathbf{1}]_\ell$$

and so multiplication by $P_\ell$ for $\ell \geq M$ maps the $B_k$ to elementary tensors of the form $A_k \otimes E^{(M)}_{n_M} \otimes \cdots \otimes E^{(\ell-1)}_{n_{\ell-1}}$ in $\mathcal{L}^{(\ell-1)}$ (after identifying $\mathcal{L}^{(\ell-1)} \otimes E[\mathbf{1}]_\ell$ with $\mathcal{L}^{(\ell-1)}$). Now a set of elementary tensors (in a finite tensor product) will be linearly independent if the entries in a fixed slot are linearly independent so it suffices to find $\ell > M$ such that the pieces $E^{(M)}_{n_M} \otimes \cdots \otimes E^{(\ell-1)}_{n_{\ell-1}}$ are linearly independent for $\mathbf{n} \in N := \big\{\mathbf{n}_k \mid k = 1, \ldots, m\big\}$. Since the approximate identities $(E^{(k)}_n)_{n=1}^\infty \subset \mathcal{L}_k$ consist of strictly increasing projections, their terms are linearly independent from which it follows that tensor products of these with distinct entries are linearly independent. Thus we only have to identify an $\ell$ large enough so that the portions of the sequences $\mathbf{n}_k$ between the entries $M$ and $\ell$ can distinguish all the sequences in $N$, and this is always possible since the $\mathbf{n}_k$ are representatives of distinct equivalence classes in $\mathbb{N}^\infty/{\sim}$. Thus $\{B_1 P_\ell, \ldots, B_m P_\ell\}$ is linearly independent for this $\ell$, so $0 = \sum\limits_{k=1}^m B_k P_\ell$ implies that all $B_k = 0$. We conclude that the linear extension $\pi_0$ exists.

That $\pi_0$ respects involution is clear. To see that it is a homomorphism, consider the elementary tensors

$$L = A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} \in \mathcal{L}[\mathbf{n}] \quad \text{and} \quad M = B_1 \otimes B_2 \otimes \cdots \otimes B_m \otimes E[\mathbf{p}]_{m+1} \in \mathcal{L}[\mathbf{p}]$$

where $m > k$ and $\mathbf{n} \not\sim \mathbf{p} \in \mathbb{N}^\infty$. Then

$$\begin{aligned}
\pi_0(L)\,\pi_0(M) &= \pi_k(A_1 \otimes \cdots \otimes A_k)\,\mathbb{E}^{\mathbf{n}}_{k+1} \pi_m(B_1 \otimes \cdots \otimes B_m)\,\mathbb{E}^{\mathbf{p}}_{m+1} \\
&= \pi_m\big(A_1 \otimes \cdots \otimes A_k \otimes E^{(k+1)}_{n_{k+1}} \otimes \cdots \otimes E^{(m)}_{n_m}\big)\,\mathbb{E}^{\mathbf{n}}_{m+1} \pi_m(B_1 \otimes \cdots \otimes B_m)\,\mathbb{E}^{\mathbf{p}}_{m+1} \\
&= \pi_m\big(A_1 B_1 \otimes \cdots \otimes A_k B_k \otimes E^{(k+1)}_{n_{k+1}} B_{k+1} \cdots \otimes E^{(m)}_{n_m} B_m\big)\,\mathbb{E}^{\mathbf{n}}_{m+1}\,\mathbb{E}^{\mathbf{p}}_{m+1}\,.
\end{aligned}$$

Now recall that the operator product is jointly continuous on bounded sets in the strong operator topology, hence

$$\begin{aligned}
\mathbb{E}^{\mathbf{n}}_k\,\mathbb{E}^{\mathbf{p}}_k &= \text{s--}\lim_{m\to\infty} \pi(E^{(k)}_{n_k}) \cdots \pi(E^{(m)}_{n_m}) \cdot \text{s--}\lim_{r\to\infty} \pi(E^{(k)}_{p_k}) \cdots \pi(E^{(r)}_{p_r}) \\
&= \text{s--}\lim_{m\to\infty} \pi(E^{(k)}_{n_k}) \cdots \pi(E^{(m)}_{n_m}) \pi(E^{(k)}_{p_k}) \cdots \pi(E^{(m)}_{p_m}) \\
&= \text{s--}\lim_{m\to\infty} \pi(E^{(k)}_{q_k}) \cdots \pi(E^{(m)}_{q_m}) = \mathbb{E}^{\mathbf{q}}_k
\end{aligned}$$

where $q_j := \min(n_j, p_j)$. Thus we get exactly that $\pi_0(L)\,\pi_0(M) = \pi_0(LM)$.

We now verify that $\pi_0$ is bounded. For this, we first need to prove the following:

**Claim:** *Recall that* $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k] = C^*\left(\mathcal{L}[\mathbf{n}_1]\cup\cdots\cup\mathcal{L}[\mathbf{n}_k]\right)$. *Then for each* $k\geq 1$ *and* $k$*-tuple* $(\mathbf{n}_1,\ldots,\mathbf{n}_k)$ *such that* $\mathbf{n}_k\not\sim\mathbf{n}_\ell$ *if* $k\neq\ell$ *the map* $\pi_0$ *on* $\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$ *extends to a representation of the* $C^*$*-algebra* $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$.

**Proof:** Note that the claim implies the compatibility of the representations, i.e., on intersections $\mathcal{L}[\mathbf{p}_1,\ldots,\mathbf{p}_\ell]\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$, the representations produced by $\pi_0$ on $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$ and $\mathcal{L}[\mathbf{p}_1,\ldots,\mathbf{p}_\ell]$ coincide. This is because $\pi_0$ is given as a consistent map on the dense space $\mathcal{L}_0$. We now prove the claim by induction on $k$. We already have by definition that $\pi_0$ is the representation $\pi^{\mathbf{n}}$ on $\mathcal{L}[\mathbf{n}]$ for each $\mathbf{n}$, hence the claim is true for $k=1$. Assume the claim is true for all values of $k$ up to a fixed $k\geq 1$, then we now prove it for $k+1$. Observe that $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$ contains the closed two–sided ideals

$$\mathcal{J}_1 := C^*\left(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\cdot\mathcal{L}[\mathbf{n}_{k+1}]\right)\subset\mathcal{J}_2\cap\mathcal{J}_3,$$

where:

$$\mathcal{J}_2 := \mathcal{J}_1+\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\quad\text{and}\quad\mathcal{J}_3 := \mathcal{J}_1+\mathcal{L}[\mathbf{n}_{k+1}]$$

and that $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}] = \mathcal{J}_2+\mathcal{J}_3$. We will prove below that $\mathcal{J}_1$ is proper (hence that the ideal structure above is nontrivial). Consider the factorization $\xi:\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\to\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]/\mathcal{J}_1$. Then

$$\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big) = \xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)+\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$$

and $\xi(\mathcal{J}_2)\cdot\xi(\mathcal{J}_3)=0$. If $\mathcal{J}_1$ is *not* proper, then

$$\mathcal{L}[\mathbf{n}_{k+1}]\subset\mathcal{J}_1\supset\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k].$$

By Proposition IV.5(iv), we have that

$$\mathcal{J}_1\subset\mathcal{L}[\mathbf{q}_1,\ldots,\mathbf{q}_k]\subset\mathcal{I}[\mathbf{q}_1,\ldots,\mathbf{q}_k]\quad\text{for}\quad(\mathbf{q}_j)_\ell=\min\big((\mathbf{n}_j)_\ell,\,(\mathbf{n}_{k+1})_\ell\big),$$

and hence $\mathcal{L}[\mathbf{n}_{k+1}]\subset\mathcal{J}_1\subset\mathcal{I}[\mathbf{q}_1,\ldots,\mathbf{q}_k]$. Thus, by Proposition IV.5(ii) we conclude that $[\mathbf{n}_{k+1}]$ cannot be strictly greater than all the $[\mathbf{q}_i]$, i.e., there is one member of the set $\{\mathbf{q}_1,\ldots,\mathbf{q}_k\}$, say $\mathbf{q}_j$, which satisfies $[\mathbf{q}_j]=[\mathbf{n}_{k+1}]$, and so by definition of $\mathbf{q}_j$, we have that eventually $(\mathbf{n}_{k+1})_\ell=\min\big((\mathbf{n}_j)_\ell,\,(\mathbf{n}_{k+1})_\ell\big)$, i.e., $[\mathbf{n}_j]\geq[\mathbf{n}_{k+1}]$.

Likewise, the inclusion $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\subset C^*\left(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\cdot\mathcal{L}[\mathbf{n}_{k+1}]\right)=\mathcal{J}_1$ implies that no $\mathbf{n}_j$, $j=1,\ldots,k$, is reduced through multiplication by $\mathbf{n}_{k+1}$, i.e., eventually $(\mathbf{n}_j)_\ell=\min\big((\mathbf{n}_j)_\ell,\,(\mathbf{n}_{k+1})_\ell\big)$ for all $j$, i.e., $[\mathbf{n}_j]\leq[\mathbf{n}_{k+1}]$. So, together with the previous inequality, we see that there must be a $j\in\{1,\ldots,k\}$ such that $[\mathbf{n}_j]=[\mathbf{n}_{k+1}]$. This contradicts the initial assumption that all $[\mathbf{n}_\ell]$ are distinct, and so $\mathcal{J}_1$ must be proper.

Now consider $\pi_0$ on $\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$. By the induction assumption, $\pi_0$ on

$$\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$$

is the restriction of a representation on $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$,- we denote the projection onto its essential subspace by $\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$. Note that $\mathbb{E}[\mathbf{n}_{k+1}]$ commutes with $\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$ because it commutes with all the generating elements $\pi_0(L_i)=\pi^{\mathbf{n}_i}(L_i)$, $L_i\in\mathcal{L}[\mathbf{n}_i]$. Thus we have an orthogonal decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4$, where

$$\begin{aligned}\mathcal{H}_1 &:= \mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\mathbb{E}[\mathbf{n}_{k+1}]\mathcal{H}, &\qquad \mathcal{H}_2 &:= \mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big(\mathbb{1}-\mathbb{E}[\mathbf{n}_{k+1}]\big)\mathcal{H}\\ \mathcal{H}_3 &:= \mathbb{E}[\mathbf{n}_{k+1}]\big(\mathbb{1}-\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)\mathcal{H}, &\qquad \mathcal{H}_4 &:= \big(\mathbb{1}-\mathbb{E}[\mathbf{n}_{k+1}]\big)\big(\mathbb{1}-\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)\mathcal{H}\end{aligned}$$

and $\pi_0$ preserves these subspaces. Now by Proposition IV.5(iv) and the induction assumption, $\pi_0$ extends from the $\mathcal{L}_0\cap\mathcal{J}_1$ to a representation on $\mathcal{J}_1$, and as $\mathcal{J}_1=C^*\left(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\cdot\mathcal{L}[\mathbf{n}_{k+1}]\right)$, the essential projection for $\pi_0\restriction\mathcal{J}_1$ is $\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\mathbb{E}[\mathbf{n}_{k+1}]$, i.e., its essential subspace is $\mathcal{H}_1$. But since $\mathcal{J}_1$ is a closed two–sided ideal of $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$, its non-degenerate representations extend uniquely to $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$. Thus on $\mathcal{H}_1$, $\pi_0$ extends from $\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$ to a representation on $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$.

Next observe that on $\mathcal{H}_1^\perp=\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4$ we have $\{0\}=\pi_0(\mathcal{J}_1)$. We show that one can define a consistent representation of $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)$ by $\rho(\xi(A)):=\pi_0(A)\restriction\mathcal{H}_1^\perp$, for $A\in\mathcal{L}[\mathbf{n}_{k+1}]+\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$, using the structure of $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)$ above. First observe that $\rho$ is well-defined on $\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$ and $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)$ separately, because if $A_1-A_2\in\mathcal{J}_1$, then $\pi_0(A_1-A_2)\restriction\mathcal{H}_1^\perp=0$. Next, $\rho$ is well-defined on the set $\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]+\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)$ by the induction assumption, and the consistency of the extensions of $\pi_0$. To see that $\rho$ is well-defined on the algebra $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)=\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)+\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$, it suffices by the direct sum decomposition to check it on $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ separately. On $\mathcal{H}_2$, $\pi_0$ vanishes on $\mathcal{L}[\mathbf{n}_{k+1}]$, so since $\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$ is an ideal of $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)$ (and $\xi(\mathcal{J}_2)\cdot\xi(\mathcal{J}_3)=\{0\}$), it follows that we can extend $\rho(\xi(A))\restriction\mathcal{H}_2$ by linearity, i.e., $\rho(\xi(A)+\xi(B))=\rho(\xi(A))$ for $A\in\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$, $B\in\mathcal{L}[\mathbf{n}_{k+1}]$ to define a representation on $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)$. Likewise, on $\mathcal{H}_3$, $\pi_0$ vanishes on $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$, so we can show $\rho$ defines a representation of $\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)$ and on $\mathcal{H}_4$, $\rho$ is zero. Then $\rho$ lifts to a representation of $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$ on $\mathcal{H}_1^\perp$ which coincides with $\pi_0$ on $\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$. Taking the direct sum of this with the representation we obtained on $\mathcal{H}_1$, produces a representation of $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$ on all $\mathcal{H}$ which coincides with $\pi_0$ on $\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$. Thus, we have proven the claim for $k+1$, which completes the induction. ▼

That $\pi_0$ is bounded on $\mathcal{L}_0$ now follows immediately from the claim, because any $A\in\mathcal{L}_0$ is of the form $A=\sum_{k=1}^m B_k$ for $B_k\in\mathcal{L}[\mathbf{n}_k]_0$, where $\mathbf{n}_k\not\sim\mathbf{n}_\ell$ if $k\neq\ell$. But this is an element of $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_m]$ and by the claim $\pi_0$ extends as a representation to it, hence $\|\pi_0(A)\|\leq\|A\|$. We conclude that $\pi_0$ is a bounded representation, hence extends to all of $\mathcal{L}[E]$. To see that $\pi_0$ is non-degenerate, recall that $\{E_n^{(k)}\}\subset\mathcal{L}_k$ is an approximate identity of increasing projections. Thus we can find a sequence $\mathbf{n}$ such that $\text{s-}\lim_{m\to\infty}\pi(E_{n_m}^{(m)})=\mathbb{1}$, and hence $\mathbb{E}^{\mathbf{n}}=\mathbb{1}$ by $\pi(E_{n_m}^{(m)})\leq\mathbb{E}^{\mathbf{n}}\leq\mathbb{1}$ for all $m$. Since the essential subspace of $\pi_0\restriction\mathcal{L}[\mathbf{n}]$ is $\mathbb{E}^{\mathbf{n}}\mathcal{H}$, it follows that $\pi_0$ is non-degenerate. It then follows from Proposition IV.4 applied to $\mathcal{L}[\mathbf{n}]$ that $\widetilde{\pi}_0\restriction\mathcal{A}=\pi$. ■

It now follows from the uniqueness of host algebras (Theorem A.2) that up to isomorphism, $\mathcal{L}[E]$ is independent of the initial choice of approximate identities.

The following lemma shows that our seemingly special choice of $(S,B)$ covers all countably dimensional symplectic vector spaces:

**Lemma IV.7.** *In each countably dimensional symplectic vector space $(V,\Omega)$, there exists a basis $(p_n,q_n)_{n\in\mathbb{N}}$ with*

$$\Omega(p_n,q_m)=\delta_{nm}\quad\text{and}\quad\Omega(p_n,p_m)=\Omega(q_n,q_m)=0\quad\text{for}\quad n,m\in\mathbb{N}.$$

*Then $Ip_n:=-q_n$ and $Iq_n=p_n$ defines a complex structure on $V$ for which $(v,w):=\Omega(Iv,w)$ is positive definite and extends to a positive definite hermitian form on $(V,I)$ with*

$$\Omega(v,w)=-\operatorname{Re}(Iv,w)=\operatorname{Im}(v,w).$$

**Proof.** Let $(e_n)_{n\in\mathbb{N}}$ be a linear basis of $V$. We construct the basis elements $p_n,q_n$ inductively as follows. If $p_1,\ldots,p_k$ and $q_1,\ldots,q_k$ are already chosen, pick a minimal $m$ with $e_m\notin\operatorname{span}\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ and put

$$p_{k+1}:=e_m-\sum_{i=1}^k\big(B(e_m,q_i)p_i+B(p_i,e_m)q_i\big)$$

to ensure that this element is $B$-orthogonal to all previous ones. Then pick $\ell$ minimal, such that $B(p_{k+1},e_\ell)\neq0$, put

$$\widetilde{q}_{k+1}:=e_\ell-\sum_{i=1}^k\big(B(e_\ell,q_i)p_i+B(p_i,e_\ell)q_i\big)$$

and pick $q_{k+1}\in\mathbb{R}\widetilde{q}_{k+1}$ with $B(p_{k+1},q_{k+1})=1$. This process can be repeated ad infinitum and produces the required bases of $V$ because for each $k$, the span of $p_1,\ldots,p_k,q_1,\ldots,q_k$ contains at least $e_1,\ldots,e_k$. ■

## Appendix. Host algebras and the strict topology

**Lemma A.1.** *Let $X$ be a locally compact space.*

(a) *On each bounded subset of $M(C_0(X)) \cong C_b(X)$ the strict topology coincides with the topology of compact convergence, i.e., the compact open topology. This holds in particular for the subgroup $C(X, \mathbb{T}) \cong U(C_b(X))$.*

(b) *A unital $*$-subalgebra $S \subseteq C_b(X)$ is strictly dense if and only if it separates the points of $X$.*

**Proof.** (a) ([Bl98, Ex. 12.1.1(b)]) Let $\mathcal{B} \subseteq C_b(X)$ be a bounded subset with $\|f\| \leq C$ for each $f \in \mathcal{B}$.

For each $\varphi \in C_0(X)$ and $\varepsilon > 0$ we now find a compact subset $K \subseteq X$ with $|\varphi| \leq \varepsilon$ outside $K$. For $f_i \to f$ in $\mathcal{B}$ with respect to the compact open topology, we then have

$$\|(f - f_i)\varphi\| \leq \|(f - f_i)|_K\| \|\varphi\| + \varepsilon \|f - f_i\| \leq \varepsilon \|\varphi\| + 2\varepsilon C$$

for sufficiently large $i$. Therefore the maps $\mathcal{B} \to C_0(X), f \mapsto f\varphi$ are continuous if $\mathcal{B}$ carries the compact open topology. This means that the strict topology on $\mathcal{B}$ is coarser than the compact open topology.

If, conversely, $K \subseteq X$ is a compact subset and $h \in C_0(X)$ with $h|_K = 1$, then

$$\|(f - f_i)|_K\| \leq \|(f - f_i)h\|$$

shows that the strict topology on $C_b(X)$ is finer than the compact open topology. This proves (a).

(b) If $S$ is strictly dense, then it obviously separates the points of $X$ because the point evaluations are strictly continuous.

Suppose, conversely, that $S$ separates the points of $X$. Replacing $S$ by its norm closure, we may w.l.o.g. assume that $S$ is norm closed. Let $K \subseteq X$ be compact. Since $S$ separates the points of $K$, the Stone-Weierstraß Theorem implies that $S|_K = C(K)$. For any $f \in C_b(X)$ we therefore find some $f_K \in S$ with $\|f_K\| \leq 2\|f\|$ and $f_K|_K = f|_K$ because the restriction map is a quotient morphism of $C^*$-algebras. Since the net $(f_K)$ is bounded and converges to $f$ in the compact open topology, (a) implies that it also converges in the strict topology. Therefore $S$ is strictly dense in $C_b(X)$. ■

**Theorem A.2.** (Uniqueness of host algebras) *Suppose that $(\mathcal{A}, \eta_G)$ and $(\mathcal{B}, \eta'_G)$ are both host algebras of the group $G$ for the same set of representations. Then there exists a unique isomorphism of $C^*$-algebras $\varphi: \mathcal{A} \to \mathcal{B}$ such that the corresponding homomorphism $M(\varphi): M(\mathcal{A}) \to M(\mathcal{B})$ satisfies $M(\varphi) \circ \eta_G = \eta'_G$.*

**Proof.** From [Gr05, Th. 3.4], we obtain the existence of some $\varphi$ with the required properties. The uniqueness follows from the fact that $M(\varphi)$ is continuous with respect to the strict topology on $M(\mathcal{A})$ and $\eta_G(G)$ spans a strictly dense subspace of $M(\mathcal{A})$ ([Gr05, Prop. 2.1]). ■

**Corollary A.3.** *If $(\mathcal{A}, \eta_G)$ and $(\mathcal{B}, \eta_H)$ are host algebras of the groups $G$, resp., $H$ and $\varphi: G \to H$ is an isomorphism of topological groups, then there exists a unique isomorphism $\varphi_{\mathcal{A}}: \mathcal{A} \to \mathcal{B}$ with*

$$M(\varphi_{\mathcal{A}}) \circ \eta_G = \eta_H \circ \varphi.$$

**Proof.** Apply Theorem A.2 with $\eta'_G := \eta_H \circ \varphi$. ■

## Tensor products of $C^*$-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and $\mathcal{A}\otimes\mathcal{B}$ their spatial $C^*$-tensor product (defined by the minimal cross norm) ([Fi96]), which is a suitable completion of the algebraic tensor product $\mathcal{A}\otimes\mathcal{B}$, turning it into a $C^*$-algebra. We then have homomorphisms

$$i_{\mathcal{A}}\colon M(\mathcal{A})\to M(\mathcal{A}\otimes\mathcal{B}),\quad i_{\mathcal{B}}\colon M(\mathcal{B})\to M(\mathcal{A}\otimes\mathcal{B}),$$

uniquely determined by

$$i_{\mathcal{A}}(\varphi)(A\otimes B)=(\varphi\cdot A)\otimes B,\quad i_{\mathcal{B}}(\varphi)(A\otimes B)=A\otimes(\varphi\cdot B).$$

Moreover, for each complex Hilbert space $\mathcal{H}$, we have

$$\operatorname{Rep}(\mathcal{A}\otimes\mathcal{B},\mathcal{H})\cong\{(\alpha,\beta)\in\operatorname{Rep}(\mathcal{A},\mathcal{H})\times\operatorname{Rep}(\mathcal{B},\mathcal{H})\colon[\alpha(\mathcal{A}),\beta(\mathcal{B})]=\{0\}\}.$$

This correspondence is established by assigning to each pair $(\alpha,\beta)$ with commuting range the representation

$$\pi:=\alpha\otimes\beta\colon\mathcal{A}\otimes\mathcal{B}\to B(\mathcal{H}),\quad a\otimes b\mapsto\alpha(a)\beta(b).$$

Note that this representation of $\mathcal{A}\otimes\mathcal{B}$ is non-degenerate if $\alpha$ and $\beta$ are non-degenerate.

**Lemma A.4.** *The following assertions hold for the embedding* $i_{\mathcal{A}}\colon M(\mathcal{A})\to M(\mathcal{A}\otimes\mathcal{B})$*:*

(1) *The map*

$$i_{\mathcal{A}}^{-1}\colon M(\mathcal{A})\otimes\mathbf{1}\to M(\mathcal{A}),\quad m\otimes\mathrm{id}_{\mathcal{B}}\mapsto m$$

*is continuous with respect to the strict topology on its domain obtained from* $\mathcal{A}\otimes\mathcal{B}$ *and the strict topology on its range obtained from* $\mathcal{A}$.

(2) *Its restriction to bounded subsets is a homeomorphism.*

(3) $i_{\mathcal{A}}(\mathcal{A})$ *is dense in* $M(\mathcal{A})\otimes\mathbf{1}$ *with respect to the strict topology on* $M(\mathcal{A}\otimes\mathcal{B})$.

**Proof.** (1) The strict topology on $M(\mathcal{A})$ is defined by the seminorms

$$p_a(m)=\|m\cdot a\|+\|a\cdot m\|,$$

satisfying $p_a\circ i_{\mathcal{A}}^{-1}=p_{a\otimes\mathbf{1}}$, which shows immediately that $i_{\mathcal{A}}^{-1}$ is continuous.

(2) Since the embedding $i_{\mathcal{A}}$ is isometric, it suffices to show that for each bounded subset $\mathcal{M}\subseteq M(\mathcal{A})$, the restriction of $i_{\mathcal{A}}$ to $\mathcal{M}$ is continuous. Since $i_{\mathcal{A}}$ is linear, it suffices to show that for each bounded net $(M_\nu)$ with $\lim M_\nu=0$ in the strict topology of $M(\mathcal{A})$, we also have $\lim i_{\mathcal{A}}(M_\nu)=0$ in $M(\mathcal{A}\otimes\mathcal{B})$. For $A\in\mathcal{A}$ and $B\in\mathcal{B}$ we have

$$\|M_\nu(A\otimes B)\|=\|M_\nu A\otimes B\|=\|M_\nu A\|\|B\|\to 0$$

and likewise $(A\otimes B)M_\nu\to 0$. Since the elementary tensors span a dense subset of $A\otimes B$, the boundedness of the net $(M_\nu)$ implies that $i_{\mathcal{A}}(M_\nu)\to 0$ holds in the strict topology of $M(\mathcal{A}\otimes\mathcal{B})$ (cf. Wegge–Olsen [WO93], Lemma 2.3.6).

(3) Let $\{E_\alpha\}$ be any approximate identity of $\mathcal{A}$, satisfying $\|E_\alpha\|\leq 1$. Then for any $A\in M(\mathcal{A})$, the net $\{AE_\alpha\}\subset M(\mathcal{A})$ is bounded by $\|A\|$ and converges to $A$ in the strict topology of $M(\mathcal{A})$, and hence in the strict topology of $M(\mathcal{A}\otimes\mathcal{B})$ by (2). This proves (3). ∎

**Lemma A.5.** *For each non-degenerate representation* $\pi \in \mathrm{Rep}(\mathcal{A} \otimes \mathcal{B}, \mathcal{H})$ *the representations* $\pi_1(a) := \widetilde{\pi}(a \otimes \mathbf{1})$ *and* $\pi_2(b) := \widetilde{\pi}(\mathbf{1} \otimes b)$ *are non-degenerate, where* $\widetilde{\pi}$ *denotes the unique extension of* $\pi$ *from* $\mathcal{A} \otimes \mathcal{B}$ *to* $M(\mathcal{A} \otimes \mathcal{B})$*. Moreover, the corresponding extensions* $\widetilde{\pi}_1 \in \mathrm{Rep}(M(\mathcal{A}), \mathcal{H})$ *and* $\widetilde{\pi}_2 \in \mathrm{Rep}(M(\mathcal{B}), \mathcal{H})$ *from* $\pi_1$, $\pi_2$ *on* $\mathcal{A}$, $\mathcal{B}$ *resp., satisfy*

$$\widetilde{\pi}_1 = \widetilde{\pi} \circ i_{\mathcal{A}} \quad \text{and} \quad \widetilde{\pi}_2 = \widetilde{\pi} \circ i_{\mathcal{B}}.$$

*In particular, the representations* $\widetilde{\pi} \circ i_{\mathcal{A}}$ *and* $\widetilde{\pi} \circ i_{\mathcal{B}}$ *are continuous with respect to the strict topology on* $M(\mathcal{A})$, $M(\mathcal{B})$ *resp., and the the topology of pointwise convergence on* $B(\mathcal{H})$*.*

**Proof.** To see that $\pi_1$ is non-degenerate, we observe that for $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$ we have $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$, so that any vector annihilated by $\pi_1(\mathcal{A})$ is also annihilated by $\mathcal{A} \otimes \mathcal{B}$, hence zero. The same argument proves non-degeneracy of $\pi_2$.

For $m \in M(\mathcal{A})$ we have

$$\widetilde{\pi}(m \otimes \mathbf{1})\pi_1(a) = \widetilde{\pi}(m)\widetilde{\pi}(a \otimes \mathbf{1}) = \widetilde{\pi}(ma \otimes \mathbf{1}) = \pi_1(ma) = \widetilde{\pi}_1(m)\widetilde{\pi}_1(a),$$

so that the non-degeneracy of $\pi_1$ implies $\widetilde{\pi} \circ i_{\mathcal{A}} = \widetilde{\pi}_1$, and likewise $\widetilde{\pi} \circ i_{\mathcal{B}} = \widetilde{\pi}_2$.

The last assertion follows from the general fact that for a non-degenerate representation of $\mathcal{A}$, the corresponding extension to $M(\mathcal{A})$ is continuous with respect to the strict topology on $M(\mathcal{A})$ and the topology of pointwise convergence on $B(\mathcal{H})$; similary for $\mathcal{B}$. ■

**Lemma A.6.** *Let* $G_1, G_2$ *be topological groups and suppose that* $(\mathcal{A}_1, \eta_1)$*, resp.,* $(\mathcal{A}_2, \eta_2)$ *are full host algebras for* $G_1$*, resp.,* $G_2$*. Then*

$$\eta \colon G_1 \times G_2 \to M(\mathcal{A}_1 \otimes \mathcal{A}_2), \quad (g_1, g_2) \mapsto i_{\mathcal{A}_1}(\eta_1(g_1)) i_{\mathcal{A}_2}(\eta_2(g_2))$$

*defines a full host algebra of* $G_1 \times G_2$*.*

**Proof.** This follows from the observation that unitary representations of the direct product group $G := G_1 \times G_2$ can be viewed as pairs of commuting representations $\pi_j \colon G_j \to U(\mathcal{H})$, and we have the same picture on the level of non-degenerate representations of $C^*$-algebras. We only have to observe that both pictures are compatible. In fact, let $\pi_j$ be commuting unitary representations of $G_j$, $j = 1, 2$, and $\widetilde{\pi}_j$ the corresponding representations of the host algebras $\mathcal{A}_j$. Then we have

$$\big(\eta^*(\widetilde{\pi}_1 \otimes \widetilde{\pi}_2)\big)(g_1, g_2) = (\widetilde{\pi}_1 \otimes \widetilde{\pi}_2)(\eta_1(g_1) \otimes \eta_2(g_2)) = \widetilde{\pi}_1(\eta_1(g_1))\widetilde{\pi}_2(\eta_2(g_2)) = \pi_1(g_1)\pi_2(g_2). \quad ■$$

Corollary A.10 below provides a converse to this lemma.

## Ideals of multiplier algebras

Let $\mathcal{A}$ be a $C^*$-algebra and $M(\mathcal{A})$ its multiplier algebra. We are interested in the relation between the ideals of $\mathcal{A}$ and $M(\mathcal{A})$.

**Lemma A.7.** (a) *Each strictly closed ideal* $J \subseteq M(\mathcal{A})$ *coincides with the strict closure of the ideal* $J \cap \mathcal{A}$ *of* $\mathcal{A}$*, which is norm-closed.*

(b) *For each norm closed ideal* $I \trianglelefteq \mathcal{A}$ *its strict closure* $\widetilde{I}$ *satisfies* $\widetilde{I} \cap \mathcal{A} = I$*.*

(c) *The map* $J \mapsto J \cap \mathcal{A}$ *induces a bijection from the set of strictly closed ideals of* $M(\mathcal{A})$ *onto the set of norm-closed ideals of* $\mathcal{A}$*.*

**Proof.** (a) Let $(u_i)_{i \in I}$ be an approximate identity in $\mathcal{A}$ and $\mu \in J$. Then $\mu.u_i \in J \cap \mathcal{A}$ converges to $\mu$ in the strict topology, and the assertion follows. Since on $\mathcal{A}$ the norm topology is finer than the strict topology, the ideal $J \cap \mathcal{A}$ of $\mathcal{A}$ is norm-closed.

(b) The ideal $I$ is automatically $*$-invariant, so that $\mathcal{A}/I$ is a $C^*$-algebra. Let $q \colon \mathcal{A} \to \mathcal{A}/I$ denote the quotient homomorphism. The existence of an approximate identity in $\mathcal{A}$ implies that $I$ is invariant under the left and right action of the multiplier algebra, so that we obtain a natural homomorphism

$$M(q) \colon M(\mathcal{A}) \to M(\mathcal{A}/I),$$

which is strictly continuous ([Bu68, Prop. 3.8]). Then $\widetilde{I} := \ker M(q) \trianglelefteq M(\mathcal{A})$ is a strictly closed ideal satisfying $\widetilde{I} \cap \mathcal{A} = I$, and (a) implies that $\widetilde{I}$ is the strict closure of $I$.

(c) follows from (a) and (b). ■

The following proposition shows that for each closed normal subgroup $N$ of a topological group $G$ with a host algebra, the quotient group $G/N$ also has a host algebra.

**Proposition A.9.** *Let $G$ be a topological group and suppose that $\mathcal{A}$ is a host algebra for $G$ with respect to the homomorphism*

$$\eta_G\colon G \to M(\mathcal{A}).$$

*Let $N \trianglelefteq G$ be a closed normal subgroup, $\widetilde{I}_N \trianglelefteq M(\mathcal{A})$ the strictly closed ideal generated by $\eta_G(N) - \mathbf{1}$, and $I_N := \mathcal{A} \cap \widetilde{I}_N$. Then $\eta_G$ factors through a homomorphism*

$$\eta_{G/N}\colon G/N \to M(\mathcal{A}/I_N),$$

*turning $\mathcal{A}/I_N$ into a host algebra for the quotient group $G/N$. If, in addition, $\mathcal{A}$ is a full host algebra of $G$, then $\mathcal{A}/I_N$ is a full host algebra of $G/N$.*

**Proof.** If $\pi$ is a unitary representation of $G$, then we write $\pi_{\mathcal{A}}$ for the corresponding representation of $\mathcal{A}$ and $\widetilde{\pi}_{\mathcal{A}}$ for the extension to $M(\mathcal{A})$ with $\widetilde{\pi}_{\mathcal{A}} \circ \eta_G = \pi$. Further, let $q_G\colon G \to G/N$ denote the quotient map.

We consider the $C^*$-algebra $\mathcal{B} := \mathcal{A}/I_N$ and recall that the quotient morphism $q\colon \mathcal{A} \to \mathcal{B}$ induces a strictly continuous morphism $M(q)\colon M(\mathcal{A}) \to M(\mathcal{B})$ ([Bu68, Prop. 3.8]). In view of $I_N = \ker q = \ker M(q) \cap \mathcal{A}$, Lemma A.7 implies that $\ker M(q) = \widetilde{I}_N$.

Next we observe that $\eta_G(N) - \mathrm{id}_{\mathcal{A}} \subseteq \widetilde{I}_N$ implies that $N$ acts by trivial multipliers on the algebra $\mathcal{B} = \mathcal{A}/I_N$. We therefore obtain a group homomorphism

$$\eta_{G/N}\colon G/N \to U(M(\mathcal{B})) \quad \text{with} \quad \eta_{G/N} \circ q_G = M(q) \circ \eta_G.$$

To see that $\eta_{G/N}$ turns $\mathcal{B}$ into a host algebra for the quotient group $G/N$, we first note that every non-degenerate representation $\pi\colon \mathcal{B} \to B(H)$ can be viewed as a non-degenerate representation $\pi_{\mathcal{A}}\colon \mathcal{A} \to B(H)$ with $\pi_{\mathcal{A}} := \pi \circ q$. The corresponding representations of the multiplier algebras satisfy

$$\widetilde{\pi} \circ M(q) = \widetilde{\pi}_{\mathcal{A}}\colon M(\mathcal{A}) \to B(H).$$

This leads to

$$\widetilde{\pi} \circ \eta_{G/N} \circ q_G = \widetilde{\pi} \circ M(q) \circ \eta_G = \widetilde{\pi}_{\mathcal{A}} \circ \eta_G,$$

showing that the unitary representation of $\widetilde{\pi} \circ \eta_{G/N}$ of $G/N$ is continuous. We thus obtain a map

$$\eta_{G/N}^*\colon \mathrm{Rep}(\mathcal{B}) \to \mathrm{Rep}(G/N), \quad \pi \mapsto \widetilde{\pi} \circ \eta_{G/N}.$$

If two representations $\pi$ and $\gamma$ of $\mathcal{B}$ lead to the same representation of $G/N$, i.e.,

$$\eta_{G/N}^*(\pi) = \widetilde{\pi} \circ \eta_{G/N} = \widetilde{\gamma} \circ \eta_{G/N} = \eta_{G/N}^*(\gamma),$$

then the corresponding representations on $G$ coincide, i.e., $\widetilde{\pi}_{\mathcal{A}} \circ \eta_G = \widetilde{\gamma}_{\mathcal{A}} \circ \eta_G$ but since $\mathcal{A}$ is a host algebra for $G$, we have $\pi_{\mathcal{A}} = \gamma_{\mathcal{A}}$ i.e., $\pi \circ q = \gamma \circ q$ and as $q$ is surjective, we get $\pi = \gamma$.

If, in addition, $\eta_G^*$ is surjective, then every continuous unitary representation $\pi$ of $G/N$ pulls back to a continuous unitary representation of $G$ which defines a unique representation $\rho_{\mathcal{A}}$ of $\mathcal{A}$ which in turn extends to the representation $\widetilde{\rho}_{\mathcal{A}}$ of $M(\mathcal{A})$ satisfying $\widetilde{\rho}_{\mathcal{A}} \circ \eta_G = \pi \circ q_G$. Further, $\widetilde{I}_N \subseteq \ker \widetilde{\rho}_{\mathcal{A}}$ implies $I_N \subseteq \ker \rho_{\mathcal{A}}$, so that $\widetilde{\rho}_{\mathcal{A}}$ factors via $M(q)\colon M(\mathcal{A}) \to M(\mathcal{B})$ through a strictly continuous representation $\widetilde{\pi}_{\mathcal{B}}$ of $M(\mathcal{B})$, satisfying $\widetilde{\pi}_{\mathcal{B}} \circ \eta_{G/N} = \pi$. This implies that $\eta_{G/N}^*$ is also surjective. ■

**Corollary A.10.** *Let $G_1, G_2$ be topological groups and $G := G_1 \times G_2$. If $G$ has a full host algebra $(\mathcal{A}, \eta)$, then $G_1$ and $G_2$ have full host algebras $(\mathcal{A}_1, \eta_1)$ and $(\mathcal{A}_2, \eta_2)$ with $\mathcal{A} \cong \mathcal{A}_1 \otimes \mathcal{A}_2$.*

**Proof.** The existence of host algebras of $G_1 \cong G/(\{\mathbf{1}\} \times G_2)$ and $G_2 \cong G/(G_1 \times \{\mathbf{1}\})$ follows directly from the last statement in Proposition A.9. Now Lemma A.6 applies. ■

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## References

[Bla77] Blackadar, B., *Infinite tensor products of $C^*$-algebras*, Pacific J. Math. **77** (1977), 313–334.

[Bla98] —, "$K$-theory for Operator Algebras," 2nd Ed., Cambridge Univ. Press, 1998.

[BR97] Bratteli, O., and D. W. Robinson, "Operator Algebras and Quantum Statistical Mechanis 2," 2nd ed., Texts and Monographs in Physics, Springer-Verlag, 1997.

[Bu68] Busby, R. C., *Double centralizers and extensions of $C^*$-algebras*, Transactions of the Amer. Math. Soc. **132** (1968), 79–99.

[BS70] Busby, R. C., Smith, H. A.: *Representations of Twisted Group Algebras* Trans. Amer. Math. Soc. **149:2** (1970), 503–537.

[Dix64] Dixmier, J., "Les $C^*$-algèbres et leurs représentations," Gauthier-Villars, Paris, 1964.

[Fi96] Fillmore, P. A., "A User's Guide to Operator Algebras," Wiley, New York, 1996.

[Gl03] Glöckner, H., *Direct limit Lie groups and manifolds*, J. Math. Kyoto Univ. **43** (2003), 1–26.

[Gr97] Grundling, H., *A group algebra for inductive limit groups. Continuity problems of the canonical commutation relations*, Acta Appl. Math. **46** (1997), 107–145.

[Gr05] —, *Generalizing group algebras*, J. London Math. Soc. **72** (2005), 742–762.

[HoMo98] Hofmann, K. H., and S. A. Morris, "The Structure of Compact Groups," Studies in Math., de Gruyter, Berlin, 1998.

[KR83] Kadison, R. V., and Ringrose, J. R., "Fundamentals of the Theory of Operator Algebras II," New York: Academic Press 1983.

[Ne99] Neeb, K.-H., "Holomorphy and Convexity in Lie Theory," Expositions in Mathematics **28**, de Gruyter Verlag, 1999.

[PR89] Packer, J., and Raeburn, I., *Twisted crossed products of $C^*$-algebras*, Math. Proc. Camb. Phil. Soc. **106** (1989) 293–311.

[Se67] Segal, I. E., *Representations of the canonical commutation relations,* Cargése lectures in theoretical physics, 107–170, Gordon and Breach, 1967.

[TZ75] Takeuti, G., Zaring, W. M., "Introduction to Axiomatic Set Theory," Berlin: Springer-Verlag, 1975.

[WO93] Wegge-Olsen, N. E., "K–theory and $C^*$-algebras," Oxford Science Publications, 1993.

[Wo95] Woronowicz, S. L., *$C^*$-algebras generated by unbounded elements*, Rev. Math. Phys. **7** (1995), 481–521.