Very Weak Solutions to the Stokes and Stokes-Resolvent Problem in Weighted Function Spaces

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Abstract

We investigate very weak solutions the stationary Stokes- and Stokes resolvent problem in function spaces with Muckenhoupt weights. The notion used here is similar but even more general than the one used in [1] or [12]. Consequently the class of solutions is enlarged. To describe boundary conditions we restrict ourselves to more regular data. We introduce a Banach space admitting a restriction operator and containing the solutions according to such data.

As a preparation we prove a weighted analogue to Bogowski's Theorem and extension theorems for functions defined on the boundary.

Keywords: Stokes equations, Muckenhoupt weights, very weak solutions, nonhomgeneous data

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1 Introduction

We consider the generalized Stokes resolvent problem on a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$

$$\lambda u - \Delta u + \nabla p = F, \quad \text{in } \Omega \tag{1.1}$$

$$\operatorname{div} u = K, \quad \text{in } \Omega \tag{1.2}$$

$$u|_{\partial\Omega} = g \tag{1.3}$$

for $\lambda \in \Sigma_\varepsilon \cup \{0\}$ where

$$\Sigma_\varepsilon := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \frac{\pi}{2} + \varepsilon\}, \quad 0 < \varepsilon < \frac{\pi}{2}.$$

Multiplication of (1.1) with a test function ϕ with $\operatorname{div} \phi = 0$ and $\phi|_{\partial\Omega} = 0$ and of (1.2) with a test function ψ and formal integration by parts yields

$$-\langle u, \Delta\phi\rangle = \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle \quad \text{and} \quad -\langle u, \nabla\psi\rangle = \langle K, \psi\rangle - \langle g, N\psi\rangle, \tag{1.4}$$

where N stands for the unit outer normal vector. These or similar equations have been used by in [1], [3] [4], [12] for the definition of very weak solutions.

One obtains a further generalization of this definition if one considers each right hand side of (1.4) as one functional

$$f = [\phi \mapsto \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle] \quad \text{and} \quad k = [\psi \mapsto \langle K, \psi\rangle - \langle g, N\psi\rangle].$$

Taking such f and k in appropriate spaces of functionals enlarges the class of solutions to the whole space $L^q_w(\Omega)$. However, the data is in general no longer given by distributions on Ω and $\partial\Omega$.

We consider the above resolvent problem in function spaces with general Muckenhoupt weights w. This is a large class of locally integrable weight-functions defined in (2.1). Their good properties concerning harmonic analysis [13], [17] where the base to treat the solvability of the Stokes and Navier Stokes equations [5], [8], [7], [9].

We consider the following spaces of functions and functionals:

$$Y := Y_{q',w'} := Y_{q',w'}(\Omega) := \{u \in W^{2,q'}_{w'}(\Omega) \mid u|_{\partial\Omega} = 0\} \ \text{ and } \ W^{-1,q}_{w,0}(\Omega) := (W^{1,q'}_{w'}(\Omega))'.$$

Then our main result on the very weak solutions to the Stokes equations is the following.

Theorem 1.1. *Let $f \in Y'_{q',w'}$, $k \in W^{-1,q}_{w,0}(\Omega)$ with $\langle k, 1\rangle = 0$ and let $\lambda \in \Sigma_\varepsilon \cup \{0\}$ with $0 < \varepsilon < \frac{\pi}{2}$. Then there exists a unique very weak solution $u \in L^q_w(\Omega)$ to the Stokes resolvent problem in the sense of Definition 5.2.2. It fulfills the a priori estimate*

$$\lambda\|u\|_{Y'_{q',w',\sigma}} + \|u\|_{q,w} \le c(\|f\|_{Y'_{q',w'}} + \|k\|_{W^{-1,q}_{w,0}(\Omega)}) \tag{1.5}$$

with $c = c(\Omega, q, w, \varepsilon)$ depending A_q consistently on w.

The outline of the paper is as follows.

Because needed in several steps of the theory in Section 3 we prove a weighted analogue to Bogowski's Theorem.

In Section 4 we establish an extension theorem for functions on the boundary. This theorem uses weaker assumptions to the regularity of the boundary than the well-known unweighted version in [14].

Section 5 is devoted to very weak solutions. We introduce the notion of the very weak solutions and give the proof of Theorem 1.1 in 5.1. Moreover, in 5.2 we prove regularity of the solution in the case of more regular data. In particular we obtain strong solutions to the Stokes resolvent problem with inhomogeneous boundary conditions and divergence. In Section 5.3 we return to data more regular data given by distributions on Ω and $\partial\Omega$. In this context we return to a situation similar to the one considered in [3], [12]. We show how the theory presented there is contained in the one presented in Section 5.1. To treat the boundary conditions we find a Banach space $\tilde{W}^{q,r}_{w,\tilde{w}}$ containing all very weak solutions with respect to appropriate data and a restriction operator $\mathrm{tr} : \tilde{W}^{q,r}_{w,\tilde{w}} \to T^{0,q}_w(\Omega)$ to the boundary which coincides with the usual trace on smooth functions.

2 Preliminaries

Let A_q, $1 < q < \infty$ be the set of Muckenhoupt weights which is given by all $0 \le w \in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} dx \right)^{q-1} < \infty \tag{2.1}$$

The supremum is taken over all cubes in $\mathbb{R}^n$ and $|Q|$ stands for the Lebesgue measure of Q. A constant $C = C(w)$ is called A_q-consistent if for every $c_0 > 0$ it can be chosen uniformly for all w with $A_q(w) < c_0$.

The the A_q consistence is of great importance when shoving the maximal regularity of an operator. (See [9] or [6] for details)

For $w \in A_q$ and an open set Ω we define

$$L^q_w(\Omega) := \{f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\, dx\right)^{\frac{1}{q}} < \infty\}.$$

It is easily seen that $(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and $w' = w^{-\frac{1}{q-1}}$. By [17] and [13], if $w \in A_q$ then the maximal operator

$$M : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n), \quad (Mf)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{|y|\le r} |f(x-y)| dy$$

is continuous.

Moreover, we introduce the weighted Sobolev spaces

$$W^{k,q}_w(\Omega) = \left\{u \in L^q_w(\Omega), \mid \|u\|_{k,q,w} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{q,w} < \infty\right\}$$

and $W^{k,q}_{w,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{k,q,w}}$ as well as its dual space $W^{-k,q}_w(\Omega) := (W^{k,q}_{w,0}(\Omega))'$.

Since for $k \ge 1$ one has $W^{k,q}_w(\Omega) \subset W^{k,1}_{loc}(\overline{\Omega})$ the restriction $u \mapsto u|_{\partial\Omega}$ is well-defined. Thus we may define $T^{k,q}_w(\partial\Omega) := (W^{k,q}_w(\Omega))|_{\partial\Omega}$ equipped with the norm $\|\cdot\|_{T^{k,q}_w}$ of the factor space

$$\|g\|_{T^{k,q}_w(\partial\Omega)} := \inf\{u \in W^{k,q}_w(\Omega) \mid u|_{\partial\Omega} = g\}.$$

Moreover, we set $T^{0,q}_w(\partial\Omega) = (T^{1,q}_w(\partial\Omega))'$. Then $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$, $W^{k,q}_{w,0}(\Omega)$ and $T^{k,q}_w(\partial\Omega)$ are reflexive Banach spaces in which $C^\infty(\overline{\Omega})$ ($C^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively) is dense.

We also use the divergence-free version of the spaces

$$W^{k,q}_{w,0,\sigma}(\Omega) := \{u \in W^{k,q}_{w,0}(\Omega) \mid \operatorname{div} u = 0\}$$

and $C^\infty_{0,\sigma}(\Omega)$, the space of smooth and divergence-free functions with compact support in Ω.

By [8] the following weighted analogue of the Poincaré inequality holds

$$\|u\|_{q,w} \le c\|\nabla u\|_{q,w} \text{ for every } u \text{ with } \int_\Omega u = 0 \tag{2.2}$$

3 The Problem $\operatorname{div} u = k$

Throughout this section let $1 < q < \infty$ and $w \in A_q$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded and locally lipschitzian domain. Assume $f \in W^{k,q}_{0,w}(\Omega)$ such that $\int f = 0$. Then there exists a function $u \in W^{k+1,q}_{w,0}(\Omega)$ such that*

$$\operatorname{div} u = f \quad \textit{and} \quad \|u\|_{k+1,q,w} \le c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k)$. Moreover u can be chosen such that it depends linearly on f and such that $u \in C_0^\infty(\Omega)$ if $f \in C_0^\infty(\Omega)$.

The proof follows the same lines as the unweighted case [11, chapter III.3]. It uses non-translation-invariant singular integral operators. Thus we apply the following theorem proved in [17, V.6.13] which ensures the continuity of a certain class of such operators.

Theorem 3.2. *Let T be a bounded operator from $L^2(\mathbb{R}^n)$ into itself that is associated to a kernel K in the sense that*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

for all compactly supported $f \in L^2(\mathbb{R}^n)$ and all x outside the support of f. Suppose that for some $\gamma > 0$ and some $A > 0$, K satisfies the inequalities

$$|K(x,y)| \leq A|x-y|^{-n} \tag{3.1}$$

and

$$|K(x,y) - K(x',y)| \leq A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}}, \text{ if } |x-x'| \leq \frac{1}{2}|x-y| \tag{3.2}$$

as well as the symmetric version of the second inequality in which the roles of x and y are interchanged. Writing

$$(T_\varepsilon f)(x) = \int_{|x-y|>\varepsilon} K(x,y) f(y) dy \text{ and } (T_* f)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)|,$$

we have that

$$\int [(T_* f)(x)]^q w(x) dx \leq c \int [(Mf)(x)]^q w(x) dx, \tag{3.3}$$

where f is bounded and has compact support, $w \in A_q$, and $1 < q < \infty$.

Since the maximal operator $M : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$ is continuous, the inequality (3.3) guaranties the continuity of T_*.

However, to make use of the above theorem we have to modify the singular integral operator which appears in the proof of Lemma 3.3 outside the bounded set Ω such that it possesses the properties assumed in 3.2.

In the proof of the following Lemma the occurring integral operators have to be understood in the Cauchy principle value sense $\lim_{\varepsilon \to 0} T_\varepsilon f$.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be bounded and star-shaped with respect to every point of some ball B with $\overline{B} \subset \Omega$.*
Then for every $f \in W^{k,q}_{w,0}(\Omega)$ with $\int f = 0$ there exists a $v \in W^{k+1,q}_{w,0}(\Omega)$ with

$$\operatorname{div} v = f \quad \text{and} \quad \|v\|_{k+1,q,w} \leq c\|f\|_{k,q,w},$$

$c = c(\Omega, q, w, k)$, v depending linearly on f and $f \in C_0^\infty(\Omega)$ implies $v \in C_0^\infty(\Omega)$.

Proof. Without loss of generality we may assume, using a coordinate transformation, that $B = B_1(0)$.

First we assume that $f \in C_0^\infty(\Omega)$.

We choose $a \in C_0^\infty(B_1(0))$ such that $\int a = 1$ and define

$$v(x) := \int_\Omega f(y)(x-y) \left(\int_1^\infty a\left(y + \xi(x-y)\right) \xi^{n-1} d\xi \right) dy. \tag{3.4}$$

In the proof of [11, Lemma III.3.1] it is shown that $v \in C_0^\infty(\Omega)$ and $\operatorname{div} v = f$.

It thus remains to prove the weighted estimates. Therefore we use the following representation of $\partial_j v$ also shown in the proof of [11, Lemma III.3.1]:

$$\partial_j v_i(x) = \int_\Omega K_{i,j}(x, x-y) f(y) dy + f(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} a(y) dy =: F_1(x) + F_2(x), \tag{3.5}$$

where

$$\begin{aligned} K_{i,j}(x, x-y) = & \frac{\delta_{i,j}}{|x-y|^n} \int_0^\infty a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y| + r)^{n-1} dr \\ & + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y| + r)^n dr, \end{aligned} \tag{3.6}$$

for every $x, y \in \mathbb{R}^n$. To show the continuity of the integral operator $f \mapsto F_1$ its kernel must be modified. Set

$$E := \{ z \in \Omega \mid z = \lambda z_1 + (1-\lambda) z_2,\ z_1 \in \operatorname{supp} f,\ z_2 \in B_1(0),\ \lambda \in [0,1] \}.$$

Since Ω is star-shaped, E is a compact subset of Ω. For $x \notin E$ and $y \in \operatorname{supp} f$ we have

$$x + r\frac{x-y}{|x-y|} \notin \overline{B} \quad \text{for all } r > 0.$$

Thus, if we choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ on Ω and $\operatorname{supp} \psi \subset B_R(0)$ for some $R > 0$, and set $\varphi(x, y) = \psi(x)\psi(y)$ we obtain

$$f(y) K_{i,j}(x, x-y) = f(y) \varphi(x, y) K_{i,j}(x, x-y) =: f(y) \tilde{K}_{i,j}(x, x-y),$$

for $x, y \in \mathbb{R}^n$, if f is assumed to be extended by 0 to $\mathbb{R}^n$. Moreover, for $x \in B_R(0)$ we have $r > R+1 \Rightarrow a\left(x - r\frac{x-y}{|x-y|}\right) = 0$. Thus for $x \in \Omega$ one has

$$\begin{aligned} \int_\Omega f(y) K_{i,j}(x, x-y) dy = & \int_\Omega f(y) \tilde{K}_{i,j}(x, x-y) dy \\ = & \int_\Omega f(y) \varphi(x, y) \left[\frac{\delta_{i,j}}{|x-y|^n} \int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y| + r)^{n-1} dr \right. \\ & \left. + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y| + r)^n dr \right] dy, \end{aligned}$$

Now we have to prove that $\tilde{K}_{i,j}$ satisfies the assumptions of Theorem 3.2. By the proof of [11, Lemma III.3.1] and the Calderon-Zygmund Theorem [11, Theorem II.9.4] we find that

$$f \mapsto \int_\Omega K_{i,j}(x, x-y) f(y) dy : \quad L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is continuous. Since the multiplication M_ψ with the C_0^∞-function ψ is a continuous operator on $L^2(\mathbb{R}^n)$ we obtain the continuity of

$$f \mapsto \int_\Omega \tilde{K}_{i,j}(x, x-y) f(y) dy = M_\psi \int_\Omega K_{i,j}(x, x-y) M_\psi f(y) dy : \; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

It remains to prove the estimates (3.1) and (3.2). For (3.1) we may assume $|x|, |y| < R$. One has

$$\begin{aligned} |x-y|^n |\tilde{K}_{i,j}(x, x-y)| &= \Bigg| \varphi(x,y) \delta_{i,j} \int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^{n-1} dr \\ &\quad + \varphi(x,y) \frac{x_i - y_i}{|x-y|} \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^n dr \Bigg| \\ &\le c\left(\int_0^{R+1} (2R+r)^{n-1} dr + \int_0^{R+1} (2R+r)^n dr\right) = c. \end{aligned}$$

To prove (3.2) we take $x, x', y \in \mathbb{R}^n$ with $|x - x'| \le \frac{1}{2}|x-y|$. If $(x,y), (x',y) \notin \operatorname{supp}\varphi$ nothing is to prove. Thus, without loss of generality we may assume that $y \le R$ and $x \le 3R$. Then using the triangle inequality together with the fact that a, φ and $(|x-y|+r)^n$ are Lipschitz continuous on compact sets a straight forward calculation shows (3.2).

Combining the above and using Theorem 3.2 we obtain

$$\|F_1\|_{q,w} \le \|T^* f\|_{q,w} \le c\|Mf\|_{q,w} \le c\|f\|_{q,w}$$

where T^* is the operator given by Theorem 3.2 and associated to the kernel $\tilde{K}_{i,j}$. F_2 is easily estimated, since $\int_\Omega \frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2} a(y) dy$ is bounded. Thus using the Poincaré inequality (2.2) we obtain $\|v\|_{1,q,w} \le c\|f\|_{q,w}$. Now the general case with $f \in L^q_w(\Omega)$ follows easily, since we can approximate f by C_0^∞-functions (f_n) with $\int f_n = 0$.

It remains to prove the estimate in the spaces $W^{k,q}_w(\Omega)$. By [11, Remark III.3.2] we have

$$\partial^\alpha v(x) = \sum_{\beta \le \alpha} \binom{\alpha_n}{\beta_n} \int_\Omega N_\beta(x,y) \partial^{\alpha-\beta} f(x,y) dy,$$

where

$$N_\beta(x,y) = \int_1^\infty \partial^\beta a(y + r(x-y)) r^{n-1} dr.$$

Clearly $\partial^\beta a \in C_0^\infty(B_1(0))$. Hence the same proof as above yields $\|D^\alpha v\|_{1,q,w} \le c\|f\|_{q,w}$ for $f \in C_0^\infty(\Omega)$. Approximating an arbitrary $f \in W^{k,q}_{0,w}$ by C_0^∞-functions (f_n) with $\int f_n = 0$ finishes the proof. $\square$

Using a decomposition of the unity one obtain Theorem 3.1 from Lemma 3.3.

4 Extension Theorems

4.1 Appropriate Charts

A domain Ω is called a $C^{k,1}$-domain, if the boundary can be locally expressed as the graph of a $C^{k,1}$-function, i.e for every $x_0 \in \partial\Omega$ we can rotate the coordinate system such that in a neighborhood $U(x_0)$ of x_0 one has

$$\partial\Omega \cap U(x_0) = \{(x', a(x')) \mid x' \in V(x_0)\}, \tag{4.1}$$

where $V(x_0)$ is an appropriate ($(n-1)$-dimensional) neighborhood of 0 and $a : V(0) \to U(x_0) \subset \mathbb{R}$ is a $C^{k,1}$-function. The function a and the coordinate system can be chosen such that $\nabla a(0) = 0$.

For the definition of the boundary values of a very weak solutions we need appropriate extension theorems. The proof of them requires a chart α for which one has $\frac{\partial}{\partial x_n}\alpha(x', 0) = -N(x')$, i.e. normals to the boundary of the half space are mapped to normals to $\partial\Omega$. The natural mapping with this property would be

$$x = (x', x_n) \mapsto \begin{pmatrix} x' \\ a(x') \end{pmatrix} + x_n \cdot N(x').$$

Such charts are used by Nečas [14]. However, if a is a $C^{k,1}$-function, then this chart is only of class $C^{k-1,1}$. For this reason we introduce a different chart which conserves the regularity and still has the mentioned property.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain. Then for every $x_0 \in \partial\Omega$ there exists a neighborhood U of x_0 and a neighborhood V of 0 and a bijective map $\alpha : V \to U$ such that*

$$\alpha(0) = x_0, \quad \alpha(V \cap \mathbb{R}^{n-1}) = U \cap \partial\Omega \quad \textit{and} \quad \alpha(V \cap \mathbb{R}_+) = U \cap \Omega$$

and with the following properties:

1. $\alpha \in C^{k,1}(V, U)$
2. $\frac{\partial}{\partial x_n}\alpha(x', 0) = -N(x')$ *and* $\left(\frac{\partial}{\partial x_n}\right)^j \alpha(x', 0) = 0, \quad (j = 2, ..., k).$

Proof. We use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Let $0 \leq \rho \in C_0^\infty(\mathbb{R}^{n-1})$ radial symmetric such that $\operatorname{supp}\rho \subset B_1(0)$ and $\int \rho = 1$. Set $\rho_t(x') = \frac{1}{t^{n-1}}\rho(\frac{x'}{t})$. We define the function α as follows:

$$\alpha(x', x_n) = \begin{pmatrix} x' \\ a(x') \end{pmatrix} - (x_n \rho_{x_n} * N)(x').$$

Then one has for every multi index $\gamma = (\gamma', \gamma_n)$, with $|\gamma| \leq k$ and $|\gamma'| < k$

$$\begin{aligned}
\partial^\gamma (x_n \rho_{x_n} * N)(x') =& \partial^{\gamma_n} (x_n \rho_{x_n} * N^{(\gamma')}(x')) \\
=& \partial^{\gamma_n} x_n \int \rho(\xi) N^{(\gamma')}(x' - x_n \xi) d\xi \\
=& \gamma_n (-1)^{\gamma_n - 1} \int \rho(\xi) \nabla^{\gamma_n - 1} N^{(\gamma')}(x' - x_n \xi) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi \\
& + x_n \frac{\partial}{\partial x_n} \left((-1)^{\gamma_n - 1} \int \rho(\xi) \nabla^{\gamma_n - 1} N^{(\gamma')}(x' - x_n \xi) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi \right) \\
=& (-1)^{\gamma_n - 1} \int ((-n+2)\rho(\xi) - \nabla \rho(\xi) \cdot \xi) \\
& \cdot \nabla^{\gamma_n - 1} N^{(\gamma')}(x' - \xi x_n) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi.
\end{aligned} \tag{4.2}$$

Still we have to consider the case $|\gamma'| = k$. Then the situation is easier:

$$\partial^\gamma (x_n \rho_{x_n} * N)(x') = \int \rho^{(\beta_1)}(\xi) N^{(\beta_2)}(x' - x_n \xi) d\xi \tag{4.3}$$

where $\gamma = \beta_1 + \beta_2$ and $|\beta_1| = 1$.

The map $x \mapsto \binom{x'}{a(x')}$ is of type $C^{k,1}$ because a is. It remains to show that $\partial^\gamma (x_n \rho_{x_n} * N(x'))$ is Lipschitz continuous for every $\gamma \in \mathbb{N}^n$, $|\gamma| \leq k$. This is an easy consequence of the representations (4.2) and (4.3) and of $N \in C^{k-1,1}$, e.g.

$$\begin{aligned}
& |\partial^\gamma (x_n \rho_{x_n} * N(x')) - \partial^\gamma (y_n \rho_{y_n} * N(y'))| \\
\leq & \int_{B_1(0)} |c_n \rho(\xi) - \nabla \rho(\xi) \cdot \xi| \, |\nabla^{\gamma_n - 1} N^{(\gamma')}(x' - \xi x_n) - \nabla^{\gamma_n - 1} N^{(\gamma')}(y' - \xi y_n)| d\xi \\
\leq & c L_\gamma \sup_{\xi \in B_1(0)} |x' - y' + \xi(x_n - y_n)| \leq c L_\gamma |x - y|.
\end{aligned}$$

A similar calculation shows that the expression in (4.3) is Lipschitz continuous.

The representation (4.2) and a straight forward calculation shows that for $1 < j \leq k$

$$\left(\frac{\partial}{\partial x_n} \right)^j \alpha(x', 0) = (-1)^{\gamma_n - 1} \nabla^{j-1} N(x') 2 \int \rho(\xi)(\xi, ..., \xi) d\xi = 0,$$

since ρ is rotation symmetric and $\xi \mapsto (\xi, ..., \xi)$ is an odd function. Moreover,

$$\frac{\partial}{\partial x_n} \alpha(x', 0) \quad = \quad -N(x') \left((2-n) \int \rho(\xi) d\xi - \sum_{i=1}^{n-1} \int \partial_i \rho(\xi) \xi_i d\xi \right) = -N(x').$$

This shows 2.

Using without loss of generality $\nabla a(0) = 0$ and the representation formulas (4.2) and (4.3) shows $\nabla \alpha(0) = \mathrm{id}$. Thus the Implicit Function Theorem shows that α is locally invertible and the proof is complete. $\square$

4.2 Extension of Normal Derivatives

Our next objective is to construct a linear extension operator. A good way to do this is to consider solutions to the resolvent problem of the Dirichlet-Laplacian.

The following Theorem is proved in exactly the same way as in the classical unweighted case. (see e.g. Evans [2] 6.3. Thm. 5). For the existence of weak solutions in weighted spaces see [9].

Theorem 4.2. (Regularity of the Dirichlet Problem)
Let $1 < q < \infty$, $k \in \mathbb{N}$, let $f \in W^{k,q}_w(\mathbb{R}^n_+)$ and $u \in W^{1,q}_w(\mathbb{R}^n_+)$ be the weak solution of

$$(1-\Delta)u = f \quad \textit{and} \quad u|_{\mathbb{R}^{n-1}} = 0.$$

Then $u \in W^{k+2,q}_w(\mathbb{R}^n_+)$ with $\|u\|_{k+2,q,w} \leq c\|f\|_{k,q,w}$.
The same is true for the solution u of $(1-\Delta)u = 0$, $u|_{\mathbb{R}^{n-1}} = g$, if $g \in T^{k+2,q}_w(\mathbb{R}^{n-1})$.

Theorem 4.3. *Let $1 < q < \infty$, $w \in A_q$ and $k \in \mathbb{N}$. Then there exists a continuous linear operator*

$$T : \prod_{j=0}^{k-1} T^{k-j,q}_w(\mathbb{R}^{n-1}) \to W^{k,q}_w(\mathbb{R}^n_+)$$

such that $\frac{\partial^j}{\partial x_n^j} T(g_0, ..., g_{k-1})|_{x_n=0} = g_j$.

Proof. It suffices to show that for every $g \in T^{k-j,q}_w(\mathbb{R}^{n-1})$, $j = 1, ..., k-1$ there exists a $u \in W^{k,q}_w(\mathbb{R}^n_+)$ depending continuously and linearly on g such that $\frac{\partial^j}{\partial x_n^j} u = g$ and $\frac{\partial^i}{\partial x_n^i} u = 0$ for every $i = 0, ..., j-1$.

To show this weaker assertion let $v \in W^{k-j,q}_w(\mathbb{R}^n_+)$ with $(1-\Delta)v = 0$ and $v|_{\mathbb{R}^{n-1}} = g$ which is uniquely defined by [9, Theorem 4.4.] and Theorem 4.2. Let $\zeta \in C^\infty(\mathbb{R}_+)$ be a cut-off function with $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We set

$$\phi(x) = \phi(x_n) = x_n^j \cdot \zeta(x_n) \quad \text{and} \quad u(x) = \phi(x)v(x).$$

We want to show that ϕu solves the problem. More precisely we prove the following:

If $\phi \in C^\infty(\overline{\mathbb{R}^n_+})$ with $\phi(x) = \phi(x_n)$, $\operatorname{supp}\phi \subset \mathbb{R}^{n-1} \times [0,2)$ and $(\frac{\partial}{\partial x_n})^m \phi|_{x_n=0} = 0$ for $m = 0, ..., l$ and $v \in W^{k,q}_w(\mathbb{R}^n_+)$ with $(1-\Delta)v = 0$ then $\phi v \in W^{k+l,q}_w(\Omega)$ with $\|\phi v\|_{k+l,q,w} \leq c\|v\|_{k,q,w}$.

To prove this we use mathematical induction with respect to l and assume that we already know the assertion is true for $l-1$, $l-2$ and all k.

Since $(1-\Delta)v = 0$ we obtain

$$(1-\Delta)(\phi v) = \Delta\phi v + 2\nabla v \nabla\phi. \tag{4.4}$$

As $(\frac{\partial}{\partial x_n})^m \Delta\phi|_{x_n=0} = 0$ for $m = 0, ..., l-2$, $(\frac{\partial}{\partial x_n})^m \nabla\phi|_{x_n=0} = 0$ for $m = 0, ..., l-1$ and $(1-\Delta)\nabla v = 0$, (4.4) and induction yields $(1-\Delta)(\phi v) \in W^{k+l-2,q}_w(\Omega)$. Thus and since $\phi v|_{\mathbb{R}^{n-1}} = 0$, one has $\phi v \in W^{k+l,q}_w(\Omega)$ by the regularity of the Laplace resolvent problem. Moreover

$$\|\phi v\|_{k+l,q,w} \leq c\|\Delta\phi v + 2\nabla v \nabla\phi\|_{k+l-2,q,w} \leq c(\|v\|_{k,q,w} + \|\nabla v\|_{k-1,q,w}) \leq c\|v\|_{k,q,w}.$$

For the start of induction we need the cases $l = 0$ and $l = 1$. However they are proved in the same way as the induction step.

Thus we have shown $u \in W_w^{k,q}(\Omega)$. Moreover

$$\frac{\partial^l}{\partial x_n^l} u(x', 0) = \sum_{\nu=0}^{l} \binom{l}{\nu} \frac{\partial^\nu}{\partial x_n^\nu} v \frac{\partial^{l-\nu}}{\partial x_n^{l-\nu}} \phi(x', 0) = \begin{cases} 0 & \text{if } l < j \\ g(x') & \text{if } l = j. \end{cases}$$

This shows the assertion about the boundary values. □

Lemma 4.4. *Let Ω and $\mathcal{O}$ be two domains in $\mathbb{R}^n$ and $\alpha : \overline{\mathcal{O}} \to \overline{\Omega}$ a bijective $C^{k-1,1}$-mapping, $k \geq 1$. Then the operator*

$$T : u \mapsto u \circ \alpha : \quad W_w^{k,q}(\Omega) \to W_{w\circ\alpha}^{k,q}(\mathcal{O})$$

is continuous with $\|Tu\|_{k,q,w\circ\alpha,\mathcal{O}} \leq c\|u\|_{k,q,w,\Omega}$, $c = c(k, q, \alpha)$.

The same is true for the operator

$$S : g \mapsto g \circ \alpha : \quad T_w^{k,q}(\partial\Omega) \to T_{w\circ\alpha}^{k,q}(\partial\mathcal{O}).$$

Proof. The case $k = 1$ has been proved in [7] Lemma 3.17. Assume $\alpha \in C^{k-1,1}(\overline{\mathcal{O}})$ and the asserted continuity holds for k replaced by j, $j < k$. Then

$$\|\nabla(u \circ \alpha)\|_{j,q,w\circ\alpha,\mathcal{O}} = \|((\nabla u) \circ \alpha) \cdot \nabla\alpha\|_{j,q,w\circ\alpha,\mathcal{O}} \leq c\|(\nabla u) \circ \alpha\|_{j,q,w\circ\alpha,\mathcal{O}} \leq c\|u\|_{j+1,q,w,\Omega}.$$

Thus $Tu \in W_{w\circ\alpha}^{j+1,q}(\mathcal{O})$ with $\|(u\circ\alpha)\|_{j+1,q,w\circ\alpha,\mathcal{O}} \leq c\|u\|_{j+1,q,w,\Omega}$. This proves the assertion. The second statement follows from the continuity of T and the identity $S(g) = T(u)|_{\partial\mathcal{O}}$, where $u \in W_w^{k,q}(\Omega)$ is an extension of g. □

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{k-1,1}$-domain, $k \geq 1$. Then there exists a continuous linear operator*

$$L : \prod_{j=0}^{k-1} T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega)$$

such that $\frac{\partial^j}{\partial N^j} L(g) = g_j$, where $g = (g_0, ..., g_{k-1})$.

Proof. We choose the collection of charts $(\alpha_i, V_i, U_i)_{i=1}^m$ according to Lemma 4.1 and a decomposition of the unity $(\phi_i)_{i=1}^m$ subordinate to the U_i.

Step 1: (Construction of L)

To simplify the notation we fix i and set $\gamma = \alpha_i$, $U = U_i$, $V = V_i$ and $\phi = \phi_i$. Moreover we set $\tilde{g}_j = (g_j \cdot \phi) \circ \gamma$ and $\tilde{g} = (\tilde{g}_0, ..., \tilde{g}_{k-1})$. By Lemma 4.4 we know $\tilde{g}_j \in T_w^{k-j,q}(\mathbb{R}^{n-1})$. Thus we may apply the operator T from Theorem 4.3 and set $v := v_i := L_i(g_0, ..., g_k) := \psi_i T(\tilde{g}_0, ..., \tilde{g}_k) \circ \gamma^{-1}$ and $L(g_0, ..., g_k) = \sum_{i=1}^{\infty} \psi_i L_i(g_0, ..., g_k)$, where $(\psi_i)_i \subset C_0^\infty(\overline{\Omega})$ with $\psi_i = 1$ in a neighborhood of $\operatorname{supp} \phi_i \cap \partial\Omega$ and $\operatorname{supp} \psi_i \subset U_i$.

Step 2: We show that $\frac{\partial^j}{\partial N^j} L(g_0, ..., g_{k-1}) = g_j$.

We have

$$\begin{aligned} \tilde{g}_j(x') &= \frac{\partial^j}{\partial x_n^j} T(\tilde{g})(x', 0) = \frac{\partial^j}{\partial x_n^j}(v \circ \gamma)(x', 0) = \frac{\partial^{j-1}}{\partial x_n^{j-1}}(\nabla v \circ \gamma)\partial_n\gamma(x', 0) \\ &= \nabla^j v \circ \gamma(\partial_n\gamma, ..., \partial_n\gamma)(x', 0) + \text{Terms containing } \partial_n^i\gamma,\ i \geq 2 \\ &= \nabla^j v(\gamma(x', 0)(\underbrace{N(x'), ..., N(x')}_{j}) = \frac{\partial^j}{\partial^j N} v(\gamma(x', 0)) \end{aligned}$$

by the choice of γ cording to Lemma 4.1. Finally we obtain

$$\frac{\partial^j}{\partial^j N} L(g_0, ..., g_{k-1}) = \sum_{i=1}^{\infty} \psi_i \frac{\partial^j}{\partial^j N} L_i(g_0, ..., g_{k-1}) = \sum_{i=1}^{\infty} \phi_i g_j = g_j.$$

Step 3: (Continuity of L)

$$\begin{aligned} \|L(g_0, ..., g_{k-1})\|_{k,q,w,\Omega}^q &= \|\sum_{i=1}^{m} \psi_i T(\phi_i g_0, ..., \phi_i g_{k-1}) \circ \gamma^{-1}\|_{k,q,w,\Omega}^q \\ &\leq c \sum_{i=1}^{m} \sum_{j=0}^{k-1} \|\phi g_j\|_{T_w^{k-j,q}(\partial\Omega)}^q \leq \sum_{j=0}^{k-1} \|g_j\|_{T_w^{k-j,q}(\partial\Omega)}^q \end{aligned}$$

using Lemma 4.4 and Theorem 4.3. □

5 The Stokes Problem with Irregular Data

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with $C^{1,1}$-boundary and let $1 < q < \infty$ and $w \in A_q$. The aim of this section is to find a class of solutions to the Stokes problem in weighted Lebesgue- and Sobolev-spaces, where the divergence and exterior force are so irregular that it is impossible to speak of boundary values. Moreover it will be shown that this class of solutions includes strong solutions.

In the case that the data is regular enough such that it can be described by distributions on Ω and on $\partial\Omega$ we describe in which sense boundary values can be explained.

5.1 Very Weak Solutions Concerning Non-Distributional Data

Let $w \in A_q$. We consider exterior forces f in $Y_{w',q'}(\Omega)'$, the dual space of

$$Y = Y_{w',q'}(\Omega) = \{u \in W_{w'}^{2,q'}(\Omega) \mid u|_{\partial\Omega} = 0\},$$

and assume the divergence k to be contained in the space

$$W_{w,0}^{-1,q}(\Omega) = (W_{w'}^{1,q'}(\Omega))'.$$

Lemma 5.1. *$C^\infty(\overline{\Omega})$ is dense in Y' and in $W_{w,0}^{-1,q}(\Omega)$.*

Proof. Y is reflexive being a closed subspace of the reflexive space $W_{w'}^{2,q'}(\Omega)$. Let $x \in Y'' = Y$ such that $\langle \phi, x \rangle = 0$ for all $\phi \in C^\infty(\overline{\Omega})$. This yields $x = 0$ and the assertion is proved. The assertion about $W_{w,0}^{-1,q}(\Omega)$ is proved in the same way. □

Note that these spaces do not consist of distributions on Ω since $C_0^\infty(\Omega)$ is neither dense in $Y_{q',w'}$ nor in $W_{w'}^{1,q'}(\Omega)$. This leads to some difficulties when talking about derivatives. However restricting f or k to test functions $\varphi \in C_0^\infty(\Omega)$ one obtains an element of $W_w^{-2,q}(\Omega)$ or $W_w^{-1,q}(\Omega)$, respectively. If we say that equations are fulfilled in the distributional sense, we consider these restrictions. Our space of test functions will be

$$Y_{q',w',\sigma} := Y_\sigma := \{\varphi \in Y_{w',q'}(\Omega) \mid \operatorname{div}\varphi = 0\},$$

which is by no coincidence equal to the domain of definition of the Stokes operator in $L^{q'}_{w'}(\Omega)$.

Definition 5.2. *Let $f \in Y_{w',q'}(\Omega)'$ and $k \in W^{-1,q}_{w,0}(\Omega)$. A function $u \in L^q_w(\Omega)$ is called*

1. *a very weak solution to the Stokes problem with respect to the data f and k if*

$$\begin{aligned} \langle f,\varphi\rangle &= -\langle u,\Delta\varphi\rangle, && \text{for all } \varphi \in Y_\sigma \text{ and} && (5.1)\\ \langle k,\psi\rangle &= -\langle u,\nabla\psi\rangle, && \text{for all } \psi \in W^{1,q'}_{w'}(\Omega). && (5.2)\end{aligned}$$

2. *a very weak solution to the Stokes resolvent problem with respect to the data f and k and $\lambda \in \mathbb{C}$, if*

$$\begin{aligned} \langle f,\varphi\rangle = {} & \langle \lambda u,\varphi\rangle - \langle u,\Delta\varphi\rangle, && \text{for all } \varphi \in Y_\sigma \text{ and} && (5.3)\\ \langle k,\psi\rangle = {} & -\langle u,\nabla\psi\rangle, && \text{for all } \psi \in W^{1,q'}_{w'}(\Omega). && (5.4)\end{aligned}$$

Setting $\psi = 1$ in (5.2) and (5.4) it follows that a necessary condition for the existence of such a very weak solution u is $\langle k, 1\rangle = 0$. This condition is the analogue to the compatibility condition $\langle k,1\rangle = \langle g,N\rangle_{\partial\Omega}$ between divergence and boundary values in the case of weak solutions.

Remark 5.3. *Two comments about the missing boundary values:*

1. *For every $u \in L^q_w(\Omega)$ one has $[\varphi \mapsto \langle u,\Delta\varphi\rangle] \in Y'$ and $[\psi \mapsto \langle u,\nabla\psi\rangle] \in W^{-1,q}_{w,0}(\Omega)$. Thus any $u \in L^q_w(\Omega)$ appears as a very weak solution to the Stokes problem with respect to appropriate data. However, since $C_0^\infty(\Omega)$ is dense in $L^q_w(\Omega)$, it is impossible to define boundary values for arbitrary L^q_w-functions in the sense of a continuous linear operator from $L^q_w(\Omega)$ into some boundary space which coincides with the usual trace on smooth functions.*

2. *Dealing with very weak solutions one can define boundary values as described in (1.4). This is done in [12, 3] in the case of more regular data. However one easily sees that if $g \in T^{0,q}_w(\partial\Omega)$ then*

$$G = [\varphi \mapsto \langle g, N\cdot\nabla\varphi\rangle_{\partial\Omega}] \in Y' \quad \text{and} \quad K = [\psi \mapsto \langle g, N\cdot\psi\rangle_{\partial\Omega}] \in W^{-1,q}_{w,0}(\Omega),$$

the spaces of exterior forces and divergences, respectively. This means

$$\begin{aligned} -\langle u,\Delta\varphi\rangle &= \langle f,\varphi\rangle + \langle g, N\cdot\nabla\varphi\rangle_{\partial\Omega} = \langle f+G,\varphi\rangle \quad \text{and}\\ -\langle u,\nabla\psi\rangle &= \langle k,\psi\rangle + \langle g, N\cdot\psi\rangle_{\partial\Omega} = \langle k+K,\psi\rangle.\end{aligned}$$

Hence, since the data is so irregular, it is impossible to distinguish between force or divergence and boundary value.

Proof of Theorem 1.1.
Step 1 Let $v \in L^{q'}_{w'}(\Omega)$. By the existence of strong solutions to the Stokes resolvent problem ([8, Theorem 3.3] in the case of weighted and [11, 16] in the case of un-weighted spaces) there are unique functions $\phi \in W^{2,q'}_{w'}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$ such that

$$\lambda\phi - \Delta\phi + \nabla\psi = v \quad \text{and} \quad \operatorname{div}\phi = 0 \quad \text{in } \Omega, \quad \phi|_{\partial\Omega} = 0 \quad \text{and} \quad \int \psi = 0. \qquad (5.5)$$

This solution satisfies $\lambda\|\phi\|_{q',w'} + \|\phi\|_{2,q',w'} + \|\psi\|_{1,q',w'} \le c\|v\|_{q',w'}$ with an A_q consistent constant c.
Step 2 (Existence and a priori estimates) Setting for $v \in L^{q'}_{w'}(\Omega)$

$$\langle u, v\rangle := \langle f, \phi\rangle - \langle k, \psi\rangle, \quad \text{with} \ \ (\phi, \psi) \ \ \text{as in} \ \ (5.5)$$

we obtain

$$\begin{aligned} |\langle u, v\rangle| &= |\langle f, \phi\rangle| + |\langle k, \psi\rangle| \le \|f\|_{Y'}\|\phi\|_{2,q',w'} + \|k\|_{W^{-1,q}_{w,0}(\Omega)}\|\psi\|_{1,q',w'} \\ &\le c(\|f\|_{Y'} + \|k\|_{W^{-1,q}_{w,0}(\Omega)})\|v\|_{q',w'}. \end{aligned}$$

Thus $u \in (L^{q'}_{w'}(\Omega))' = L^q_w(\Omega)$ and fulfills $\|u\|_{q,w} \le c(\|f\|_{Y'} + \|k\|_{W^{-1,q}_{w,0}})$ with c independent of λ and depending A_q-consistently on w.

We now show that u is a very weak solution to the Stokes problem with respect to f and k. Choose test-functions $\phi \in Y_\sigma$ and $\psi \in W^{1,q'}_{w'}(\Omega)$. Then setting $v = \lambda\phi - \Delta\phi + \nabla\psi$ we obtain from the uniqueness of strong solutions

$$\langle u, \lambda\phi - \Delta\phi + \nabla\psi\rangle = \langle u, v\rangle = \langle f, \phi\rangle - \langle k, \psi\rangle.$$

Since ϕ and ψ were chosen arbitrarily (5.3) and (5.4) are fulfilled.
Step 3 (Uniqueness) Assume $U \in L^q_w(\Omega)$ is a very weak solution to the Stokes resolvent problem with respect to f and k. As above for every $v \in L^{q'}_{w'}(\Omega)$ we find $\phi \in Y_\sigma$ and $\psi \in W^{1,q'}_{w'}(\Omega)$ such that $\lambda u - \Delta\phi + \nabla\psi = v$. If we add the equations (5.3) and (5.4) we obtain

$$\langle U, v\rangle = \langle U, \lambda\phi - \Delta\phi + \nabla\psi\rangle = \langle f, \phi\rangle - \langle k, \psi\rangle = \langle u, v\rangle.$$

Since $v \in L^{q'}_{w'}(\Omega)$ was arbitrary we obtain $u = U$. Moreover let $\phi \in Y_{q',w',\sigma}$. Then we obtain from the equation

$$\begin{aligned} |\langle \lambda u, \phi\rangle| &\le |\langle u, \Delta\phi\rangle| + |\langle f, \phi\rangle| \le (\|u\|_{q,w} + \|f\|_{Y'_{q',w'}})\|\phi\|_{2,q',w'} \\ &\le c(\|f\|_{Y'} + \|k\|_{W^{-1,q}_{w,0}(\Omega)})\|\phi\|_{2,q',w'}. \end{aligned}$$

This proves (1.5) $\square$

Theorem 5.4. *Let f and k be chosen as in Theorem 1.1 and let $u \in L^q_w(\Omega)$ be the associated very weak solution to the Stokes problem. Then there exists a unique pressure functional $p \in W^{-1,q}_{0,w}(\Omega)$ (unique modulo constants) such that (u, p) solves*

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \langle F, \phi\rangle \quad \textit{for all} \ \ \phi \in Y_{q',w'}.$$

In particular

$$-\Delta u + \nabla p|_{C_0^\infty(\Omega)} = f|_{C_0^\infty(\Omega)}$$

in the sense of distributions. The functions (u,p) *fulfill the inequality*

$$\|u\|_{q,w} + \|p\|_{W^{-1,q}_{w,0}} \le c(\|f\|_{Y'} + \|k\|_{W^{-1,q}_{w,0}}), \tag{5.6}$$

where $c = c(\Omega, q, w)$.

Proof. By Lemma 5.1 there exist sequences $(f_n)_n, (k_n)_n \subset C^\infty(\overline{\Omega})$ such that

$$f_n \xrightarrow{Y'_{q',w'}} f \quad \text{and} \quad k_n \xrightarrow{W^{-1,q}_{0,w}(\Omega)} k.$$

Then by [8, Theorem 3.3] there exist unique solutions $(u_n, p_n) \in W^{2,q}_w(\Omega) \times W^{1,q}_w(\Omega)$ such that

$$-\Delta u_n + \nabla p_n = f_n, \quad \operatorname{div} u_n = k_n, \quad u_n|_{\partial\Omega} = 0, \quad \int p_n = 0.$$

Integration by parts immediately yields that u_n is a very weak solution with respect to f_n, k_n. Now the a priori estimate (5.2) yields $u_n \xrightarrow{L^q_w(\Omega)} u$. For $\phi \in W^{1,q'}_{w'}(\Omega)$ with $\int \phi = 0$ let $\zeta \in Y_{q',w'}$ with $-\Delta\zeta + \nabla\pi = 0$ and $\operatorname{div}\zeta = \phi$ and $\|\zeta\|_{2,q',w} \le c\|\phi\|_{1,q',w'}$. Thus we obtain

$$\begin{aligned} |\langle p_n - p_m, \phi\rangle| &= |\langle p_n - p_m, \operatorname{div}\zeta\rangle| = |\langle \nabla(p_n - p_m), \zeta\rangle| \\ &\le |\langle \Delta(u_n - u_m), \zeta\rangle| + |\langle f_n - f_m, \zeta\rangle| \\ &\le c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{(Y_{q',w'})'})\|\phi\|_{1,q,w}. \end{aligned}$$

Thus $\|p_n - p_m\|_{-1,q,w,0} \le c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{Y'}) \xrightarrow{n,m\to\infty} 0$ and $(p_n)_n$ is a Cauchy sequence converging to some $p \in W^{-1,q}_{0,w}(\Omega)$. For this p

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \lim_{n\to\infty} (-\langle u_n, \Delta\phi\rangle - \langle p_n, \operatorname{div}\phi\rangle) = \lim_{n\to\infty} \langle f_n, \phi\rangle = \langle f, \phi\rangle$$

holds for every $\phi \in Y_{q',w'}$. The estimate (5.6) follows from the estimates for p_n. $\square$

5.2 Regularity

The following theorem describes how strong solutions fit into the context of very weak solutions considered in the previous section. Moreover it prepares further considerations about boundary values in the weakest possible sense.

From now on we assume the following situation: Let $1 < r < \infty$ and $\tilde{w} \in A(r)$ such that

$$L^{r'}_{\tilde{w}'}(\Omega) \hookrightarrow L^{q'}_{w'}(\Omega). \tag{5.7}$$

Then by duality it follows

$$L^r_{\tilde{w}}(\Omega) \hookrightarrow W^{-1,q}_{w,0}(\Omega) \quad \text{and} \quad W^{-1,r}_{\tilde{w}}(\Omega) \hookrightarrow Y'_{q',w'}.$$

The reason why we require these embeddings is that Sobolev-like inequalities in weighted spaces need strong assumptions on the weight-functions. In [10] sufficient conditions for such embeddings are shown using the continuity of singular integral operators shown in [15].

Theorem 5.5. *Assume that $f \in Y'_{q',w'}$ and $k \in W^{-1,q}_{w,0}(\Omega)$ allow a decomposition into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for all } \phi \in Y_{q',w'}, \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega} && \text{for all } \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned} \tag{5.8}$$

with $g \in T^{0,q}_w(\partial\Omega), F \in W^{-1,r}_{\tilde w}(\Omega), K \in L^r_{\tilde w}(\Omega)$, where $1 < r < \infty$ and $\tilde w \in A(r)$ are chosen according to (5.7). Then one has

1. *Such a decomposition is uniquely defined by f and k.*
2. *For $\lambda \in \Sigma_\varepsilon \cup \{0\}$ every strong solution u to the Stokes resolvent problem corresponding to the data $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ and $K \in W^{1,q}_w(\Omega)$ is a very weak solution corresponding to the data f and k with the notation of (5.8).*
3. *If $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ and $K \in W^{1,q}_w(\Omega)$, then the very weak solution u to the Stokes resolvent problem with respect to f and k is a strong solution with respect to F, K and g. In particular $u \in W^{2,q}_w(\Omega)$ and*

$$|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \le c(\|F\|_{q,w} + \|K\|_{1,q,w} + \|\lambda K\|_{W^{-1,q}_{w,0}} + \|g\|_{T^{2,q}_w(\partial\Omega)} + \|\lambda g\|_{T^{0,q}_w}). \tag{5.9}$$

Proof. 1. Let $\langle f, \phi\rangle = \langle F_i, \phi\rangle - \langle g_i, N\cdot\nabla\phi\rangle_{\partial\Omega}$ for $i = 1, 2$. This means

$$\langle F_1 - F_2, \phi\rangle = \langle g_1 - g_2, N\cdot\nabla\phi\rangle_{\partial\Omega} \quad \text{for } \phi \in Y_{q,w}.$$

The latter functional vanishes on $C_0^\infty(\Omega)$ and since $F_1 - F_2$ is a distribution on Ω, it follows that $F_1 - F_2 = 0$ and hence $\langle g_1 - g_2, N\cdot\nabla\phi\rangle = 0$ for every $\phi \in Y_{q',w'}$. By Theorem 4.3 the mapping

$$\phi \mapsto N\cdot\nabla\phi : \; Y_{q',w'} \to T^{1,q'}_{w'}(\partial\Omega)$$

is surjective, hence $g_1 = g_2$. Analogously for the divergence.
2. Follows immediately from Green's Theorem.
3. By Theorem 4.3 there exists $v_1 \in W^{2,q}_w(\Omega)$ with $v_1|_{\partial\Omega} = g$ and $\|v_1\|_{2,q,w} \le c\|g\|_{T^{2,q}_w}$ and one has

$$\langle K - \operatorname{div} v_1, 1\rangle = \langle K, 1\rangle - \langle g, N\rangle_{\partial\Omega} = \langle k, 1\rangle = 0,$$

since $\langle k, 1\rangle = 0$ is a necessary condition for the existence of a solution.

Hence, by [8, Theorem 3.3] there exists a strong solution $v_2 \in W^{2,q}_w(\Omega)$ with respect to the exterior force $f - \lambda v_1 + \Delta v_1$ and divergence $K - \operatorname{div} v_1$. It fulfills the estimate

$$\begin{aligned} &|\lambda|\|v_2\|_{q,w} + \|v_2\|_{2,q,w} \\ &\le c(\|F\|_{q,w} + \|\Delta v_1\|_{q,w} + |\lambda|\|v_1\|_{q,w} + \|K - \operatorname{div} v_1\|_{1,q,w} + |\lambda|\|K - \operatorname{div} v_1\|_{W^{-1,q}_{w,0}}). \\ &\le c(\|F\|_{q,w} + |\lambda|\|v_1\|_{q,w} + \|K\|_{1,q,w} + |\lambda|\|K - \operatorname{div} v_1\|_{W^{-1,q}_{w,0}} + \|g\|_{T^{2,q}_w}). \end{aligned} \tag{5.10}$$

Thus $u = v_1 + v_2$ is a strong solution to the Stokes resolvent problem with respect to the given data. Moreover, in the case $\lambda = 0$, also the estimate is proved.

Now we repeat the above arguments with v_1 replaced by the solution to the Stokes problem

$$-\Delta v_1 + \nabla p = 0, \quad \operatorname{div} v_1 = 0 \ \text{ and } \ v_1|_{\partial\Omega} = g.$$

Then v_1 fulfills the estimate $\|v_1\| \le c\|g\|_{2,q,w}$. In addition, by 2. we know that v_1 is also a very weak solution with respect to the data

$$\tilde{f} = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle] \ \text{ and } \ \tilde{k} = [\psi \mapsto \langle g, N \cdot \psi\rangle].$$

Thus we obtain the estimate

$$\|v_1\|_{q,w} \le c(\|\tilde{f}\|_{Y'_{q',w'}} + \|\tilde{k}\|_{W_w^{-1,q}}) \le c\|g\|_{T_w^{0,q}}.$$

Inserting this in (5.10) we obtain (5.9).

Thus there exists a strong solution to the Stokes resolvent problem with respect to the given data which fulfills the estimate.

The uniqueness of very weak solutions proved in Theorem 1.1 together with 2. yields that u coincides with the very weak solution. In particular the very weak solution is regular according to the data. □

Remark 5.6. *If there exist decompositions for the data f and k as in (5.8) even with smooth functions F, K, g this does not mean that f and k are smooth. The reason is, that if $g \neq 0$, then $\phi \mapsto \langle g, N \cdot \nabla\phi\rangle$ is never more regular than $Y'_{q',w'}$ since it is a functional supported by the boundary and depending on derivatives.*

Vice versa, if f and k are regular, e.g. $f \in W_w^{-1,q}(\Omega)$ and $k \in L_w^q(\Omega)$ allowing a decomposition according to (5.8), then we automatically obtain $g = 0$, which means that the very weak solution with respect to f and k has zero boundary values.

5.3 Boundary Values in Case of More Regular Data

Our next aim is to define boundary values for very weak solutions to the Stokes problem presumed the data is sufficiently regular. To this aim we find a Banach space containing all these solutions and a continuous linear operator on this space coinciding with the usual trace on $C^\infty(\overline{\Omega})$.

From now on let $1 < r < \infty$, $\tilde{w} \in A_r$ such that (5.7) is fulfilled. As a large space of functions in which the definition of tangential boundary conditions is possible we define

$$\tilde{W}_{w,\tilde{w}}^{q,r}(\Omega) := \left\{u \in L_w^q(\Omega) \middle| (\Delta u)|_{C_{0,\sigma}^\infty(\Omega)} \text{ can be extended to an element of } (W_{\tilde{w}',0,\sigma}^{1,r'}(\Omega))'\right\}.$$

We will omit the Ω and write $\tilde{W}_{w,\tilde{w}}^{q,r}$ if no confusion can occur.

Since $\Delta : L_w^q(\Omega) \to W_w^{-2,q}(\Omega)$ is continuous, it follows that $\tilde{W}_{w,\tilde{w}}^{q,r}$ is a Banach space equipped with the norm

$$\|u\|_{\tilde{W}_{w,\tilde{w}}^{q,r}} = \|u\|_{q,w} + \|\Delta u|_{C_0^\infty(\Omega)}\|_{(W_{\tilde{w}',0,\sigma}^{1,r'}(\Omega))'}.$$

Lemma 5.7. *Let $f \in (Y_{q',w',\sigma})'$ with $\langle f, \phi\rangle = 0$ for every $\phi \in C_{0,\sigma}^\infty(\Omega)$. Then there exists an extension $F \in (Y_{q',w'})'$ such that $\langle F, \phi\rangle = 0$ for every $\phi \in C_0^\infty(\Omega)$ and with $\|F\|_{(Y_{q',w'})'} \le c\|f\|_{(Y_{q',w',\sigma})'}$.*

Proof. First we show that $\tilde{f}$ defined by

$$\langle \tilde{f}, \phi \rangle = \begin{cases} \langle f, \phi \rangle & \text{if } \phi \in Y_{q',w',\sigma} \\ 0 & \text{if } \phi \in W^{2,q}_{w,0}(\Omega) \end{cases}$$

is a continuous functional on $Y_{q',w',\sigma} + W^{2,q}_{w,0}(\Omega)$. Well-definedness and linearity is clear since $\langle f, \phi \rangle = 0$ on $Y_{q',w',\sigma} \cap W^{2,q'}_{w',0}(\Omega) = W^{2,q'}_{w',0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{2,q',w'}$.

Thus it remains to prove continuity. By Theorem 3.1 there exists a continuous linear operator $T : \{v \in W^{1,q'}_{w',0}(\Omega) \mid \int v = 0\} \to W^{2,q'}_{w',0}(\Omega)$ such that $\operatorname{div}(Tv) = v$.

Let $\phi \in Y_{q',w',\sigma} + W^{2,q'}_{w',0}(\Omega)$. Then $\operatorname{div} \phi \in W^{1,q'}_{w',0}(\Omega)$, $\int \operatorname{div} \phi = 0$ and we may write $\phi = (\phi - T \operatorname{div} \phi) + T(\operatorname{div} \phi)$. Thus we obtain

$$|\langle \tilde{f}, \phi \rangle| = |\langle \tilde{f}, \phi - T(\operatorname{div} \phi) \rangle + \langle \tilde{f}, T(\operatorname{div} \phi) \rangle| = |\langle f, \phi - T(\operatorname{div} \phi) \rangle| \le c \|f\|_{(Y_{q',w,\sigma})'} \|\phi\|_{2,q',w,\sigma}$$

and $\|\tilde{f}\|_{(Y_{q',w',\sigma} + W^{2,q}_{w,0}(\Omega))'} \le c \|f\|_{(Y_{q',w,\sigma})'}$.

By the Hahn-Banach Theorem we may extend $\tilde{f}$ to an element $F \in (Y_{q',w'})'$ with $\|F\|_{(Y_{q',w'})'} = \|\tilde{f}\|_{(Y_{q',w',\sigma} + W^{2,q}_{w,0}(\Omega))'}$. This finishes the proof. ☐

The following Lemma is crucial when proving the well-definedness of the tangential component of the trace on $\tilde{W}^{q,r}_{w,\tilde{w}}$.

Lemma 5.8. $C^\infty(\overline{\Omega})$ *is dense in* $\tilde{W}^{q,r}_{w,\tilde{w}}$.

Proof. Let $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$. Then by the definition of $W^{q,r}_{w,\tilde{w}}$ we have $\Delta u|_{C^\infty_{0,\sigma}} \in (W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))' \hookrightarrow Y'_{q',w',\sigma}$. The Hahn-Banach-theorem yields the existence of some $f \in (W^{1,r'}_{\tilde{w}',0}(\Omega))'$ such that

$$\langle f, \phi \rangle = \langle \Delta u, \phi \rangle \quad \text{for all} \quad \phi \in C^\infty_{0,\sigma}(\Omega).$$

By Lemma 5.7 there exists an extension $F \in Y'_{q',w'}$ of $(\langle u, \Delta \cdot \rangle - f)|_{Y_{q',w',\sigma}}$ vanishing on $C^\infty_0(\Omega)$. Analogously to the Stokes equations but easier one can prove existence and uniqueness of very weak solutions to the Laplace equation. Thus there exists a $v \in L^q_w(\Omega)$ such that

$$\langle v, \Delta \phi \rangle = \langle F, \phi \rangle \quad \text{for all} \quad \phi \in Y_{q',w'}.$$

This v is harmonic on Ω because $\langle F, \phi \rangle = 0$ for all $\phi \in C^\infty_0(\Omega)$.
Now we assume temporarily that Ω is star-shaped with respect to some ball $B_r(0)$ with center 0 and radius r. So we may set $v_\lambda(x) := v(\lambda x)$, where $\lambda \in (0,1)$. We show that $v_\lambda \xrightarrow{\lambda \to 1} v$ in $L^q_w(\Omega)$.

Note that the following argumentation strongly relies on the fact that v is harmonic. For arbitrary $u \in L^q_w(\Omega)$ the conclusion $u(\lambda \cdot) \in L^q_w(\Omega)$ is in general wrong.

Let $d = \sup_{x \in \Omega} |x|$ and $K < \frac{r}{d}$. Then for every λ with $\frac{1}{2} < \lambda < 1$ one has $B_{K(1-\lambda)|x|}(\lambda x) \subset \Omega$ for every $x \in \Omega$.

Now let $\tilde{v}$ be the extension of v by 0 to the whole of $\mathbb{R}^n$. Take $x \in \Omega$ and $\lambda < 1$ fixed. Since v is harmonic we have by the mean value property

$$\begin{aligned}
|v_\lambda(x)| &= |v(\lambda x)| = \frac{1}{|B_{K(1-\lambda)|x|}(\lambda x)|}\left|\int_{B_{K(1-\lambda)|x|}(\lambda x)} v(x)dt\right| \\
&\leq \frac{1}{|B_{K(1-\lambda)|x|}(\lambda x)|}\int_{B_{|x|((1-\lambda)+K(1-\lambda))}(x)} |\tilde{v}(t)|dt \\
&\leq \frac{(K+1)^3}{K^3}\frac{1}{|B_{(K+1)(1-\lambda)|x|}(x)|}\int_{B_{|x|(1-\lambda)+K(1-\lambda))}(x)} |\tilde{v}(t)|dt \leq cM\tilde{v}(x).
\end{aligned}$$

Since M, the maximal operator in $L^q_w(\Omega)$ is continuous, one has $M\tilde{v} \in L^q_w(\mathbb{R}^n)$. Thus, we have found a majorant. Moreover, since the harmonic function $v \in C^\infty(\Omega)$, the convergence $v_\lambda \to v$ is point-wise. By Lebesgue's Theorem we find $v_\lambda \to v$ in $L^q_w(\Omega)$.

For a general bounded $C^{1,1}$-domain we use decomposition of the unity.

Moreover, since every v_λ is harmonic we have $\Delta v_\lambda - \Delta v = 0$ for all $\lambda \in (0,1)$ which yields the convergence in $\tilde{W}^{q,r}_{w,\tilde{w}}$.

Moreover we have

$$\begin{aligned}
\langle u-v, \Delta\phi\rangle &= \langle f,\phi\rangle + \langle F,\phi\rangle - \langle F,\phi\rangle = \langle f,\phi\rangle && \text{for } \phi \in Y_{q',\sigma,w'} \\
\langle u-v, \nabla\psi\rangle &=: \langle k,\psi\rangle && \text{for } \psi \in W^{1,q'}_{w'}(\Omega).
\end{aligned}$$

Let $(f_n)_n, (k_n)_n \subset C^\infty(\overline{\Omega})$ such that $f_n \xrightarrow{n\to\infty} f$ in $W^{-1,r}_{\tilde{w}}(\Omega)$ and $k_n \xrightarrow{n\to\infty} k$ in $W^{-1,q}_{0,w}(\Omega)$. The embedding $W^{-1,r}_{\tilde{w}}(\Omega) \hookrightarrow Y'_{q',w'}$ and the a priori estimate for very weak solutions to the Stokes equations (1.5) yields that the sequence of very weak solutions $(u_n)_n$ to the Stokes problem w.r.t. f_n and k_n converges to $u-v$ in $L^q_w(\Omega)$. By the regularity of the data and of the boundary (Theorem 5.5) one has $u_n \in W^{2,q}_w(\Omega)$.

We show that u_n tends to $u-v$ in $\tilde{W}^{q,r}_{w,\tilde{w}}$. The convergence in $L^q_w(\Omega)$ is already shown. Moreover for $\phi \in C^\infty_{0,\sigma}(\Omega)$ one has

$$\langle u_n, \Delta\phi\rangle = \langle f_n,\phi\rangle \xrightarrow{n\to\infty} \langle f,\phi\rangle.$$

Thus the sequence $(u_n + v_{\lambda_n})_n \subset W^{2,q}_w(\Omega)$ approximates u in the norm of $\tilde{W}^{q,r}_{w,\tilde{w}}$ where $(\lambda_n) \subset (0,1)$ is a sequence converging to 1. However, since $C^\infty(\overline{\Omega})$ is dense in $W^{2,q}_w(\Omega)$, the assertion is proved. □

It is not difficult to see that if $\phi \in W^{2,q}_w(\Omega)$ with $\phi|_{\partial\Omega} = 0$, then $N\cdot\nabla\phi$ is purely tangential. The next Lemma shows that vice versa every purely tangential function on the boundary is a normal derivative of such a function. This ensures that the amount of test functions is sufficient.

Lemma 5.9. *Let Ω be a bounded $C^{1,1}$-domain, $1 < q < \infty$ and $w \in A_q$. For every $h \in T^{1,q}_w(\partial\Omega)$ with $N\cdot h = 0$ there exists a function $\varphi_h \in W^{2,q}_w(\Omega)$ such that*

$$\varphi_h|_{\partial\Omega} = 0,\ N\cdot\nabla\varphi_h = h \text{ and } \operatorname{div}\varphi_h = 0.$$

Moreover φ_h can be chosen depending linearly on h and fulfilling the estimate

$$\|\varphi_h\|_{2,q,w} \leq c\|h\|_{T^{1,q}_w(\partial\Omega)}$$

with a constant $c = c(\Omega,q,w)$.

Proof. For $h \in T^{1,q}_w(\partial\Omega)$ there exists by Theorem 4.3 a function $\psi_h \in W^{2,q}_w(\Omega)$ depending linearly on h such that

$$\psi_h|_{\partial\Omega} = 0, \;\; N \cdot \nabla\psi_h = h \text{ and } \|\psi_h\|_{2,q,w} \le c\|h\|_{T^{1,q}_w(\partial\Omega)}.$$

Since $h = N \cdot \nabla\psi_h$ is purely tangential, one can show (see [12]) that $\operatorname{div}\psi_h \in W^{1,q}_{0,w}(\Omega)$. Thus by Theorem 3.1 there exists a function $\zeta \in W^{2,q}_{w,0}(\Omega)$ with $\operatorname{div}\zeta = \operatorname{div}\psi_h$ depending linearly on ψ_h and satisfying the estimate $\|\zeta\|_{2,q,w} \le c\|\operatorname{div}\psi_h\|_{1,q,w} \le c\|\psi_h\|_{2,q,w}$.

Now $\varphi_h := \psi_h - \zeta$ solves the problem. $\square$

Using this lemma we define the tangential component of $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ on the boundary as follows.

Theorem 5.10. *There exists a continuous linear operator*

$$\begin{array}{lll} \gamma : \tilde{W}^{q,r}_{w,\tilde{w}} & \to T^{0,q}_w(\partial\Omega), & \textit{such that} \\ \langle\gamma(u), h\rangle_{\partial\Omega} & = \langle u, \Delta\varphi_h\rangle - \langle\Delta u, \varphi_h\rangle & \textit{if } N \cdot h = 0, \\ \langle\gamma(u), h\rangle_{\partial\Omega} & = 0 & \textit{if } N \times h = 0 \end{array} \tag{5.11}$$

for $h \in T^{1,q'}_w(\partial\Omega)$ where φ_h is given by Lemma 5.9. Moreover this tangential trace is independent of the choice of the extension φ_h and coincides with the tangential component of the usual restriction if $u \in C^\infty(\overline{\Omega})$.

Proof. Assume that γ is defined by (5.11). Let $m \in T^{1,q'}_w(\partial\Omega)$. The function m can be decomposed into normal and tangential component

$$m = (N \cdot m)N + (N \times m) \times N = (N \cdot m)N + h$$

with $\|h\|_{T^{1,q'}_{w'}(\partial\Omega)} \le c\|m\|_{T^{1,q'}_{w'}(\partial\Omega)}$. Then one obtains

$$\begin{aligned} |\langle\gamma(u), m\rangle_{\partial\Omega}| &= |\langle\gamma(u), h\rangle_{\partial\Omega}| = |\langle u, \Delta\varphi_h\rangle - \langle\Delta u, \varphi_h\rangle| \\ &\le \|u\|_{q,w}\|\varphi_h\|_{2,q',w'} + \|\Delta u\|_{(W^{1,r'}_{\tilde{w}',0,\sigma})'}\|\varphi_h\|_{1,r',\tilde{w}'} \le c\|u\|_{\tilde{W}^{q,r}_{w,\tilde{w}}}\|m\|_{T^{1,q'}_{w'}(\partial\Omega)}. \end{aligned}$$

Thus γ is continuous.

By Gauss' Theorem we know that for $u \in C^\infty(\overline{\Omega})$ we have $\gamma(u)$ is equal to $(N \times u|_{\partial\Omega}) \times N$, the tangential component of $u|_{\partial\Omega}$ which is in particular independent of the extension of h. Since by Lemma 5.8 the space $C^\infty(\overline{\Omega})$ is dense in $\tilde{W}^{q,r}_{w,\tilde{w}}$ the same is true for $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$. $\square$

The definition of tangential traces is easier. If

$$u \in E^{q,r}_{w,\tilde{w}} := \{v \in L^q_w(\Omega) \mid \operatorname{div} v \in L^r_{\tilde{w}}(\Omega)\}$$

then we can define the normal trace as in unweighted spaces: Using convolutions one shows that $C^\infty(\overline{\Omega})$ is dense in $E^{q,r}_{w,\tilde{w}}$ and we can define the normal trace $u \mapsto N \cdot u|_{\partial\Omega}$ using Green's formula by

$$\langle N \cdot u|_{\partial\Omega}, Nv\rangle_{\partial\Omega} := \langle\operatorname{div} u, v\rangle + \langle u, \nabla v\rangle \;\text{ for all }\; v \in W^{1,q'}_{w'}(\Omega). \tag{5.12}$$

Using the above theorem we say that $u|_{\partial\Omega} = g$ if $\langle\gamma(u), h\rangle_{\partial\Omega} = \langle g, h\rangle_{\partial\Omega}$ for $h \cdot N = 0$ and $u \cdot N|_{\partial\Omega} = g \cdot N$.

Proposition 5.11. *Let u be a very weak solution to the Stokes problem corresponding to the data $\langle f,\phi\rangle = \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$ and $\langle k,\psi\rangle = \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}$ with $F\in W^{-1,r}_{\tilde w}(\Omega)$, $K\in L^r_{\tilde w}(\Omega)$, $g\in T^{0,q}_w(\partial\Omega)$.*
Then $u\in \tilde W^{q,r}_{w,\tilde w}$ and $u|_{\partial\Omega}=g$.

Proof. u is the solution of

$$\begin{aligned} -\langle u,\Delta\phi\rangle &= \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}, \quad \text{for all } \phi\in Y_{q',w',\sigma} \text{ and}\\ -\langle u,\nabla\psi\rangle &= \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}, \quad \text{for all } \psi\in W^{1,q}_{w'}(\Omega). \end{aligned}$$

Inserting $\phi\in C^\infty_{0,\sigma}(\Omega)$ into the first equation we obtain that $\phi\mapsto[\langle\Delta u,\phi\rangle = -\langle F,\phi\rangle]$ is extendable to an element of $(W^{1,r'}_{\tilde w,0,\sigma}(\Omega))'$. Thus $u\in\tilde W^{q,r}_{w,\tilde w}$ and by the definition of the tangential trace we have

$$\langle\gamma(u), N\cdot\nabla\phi\rangle_{\partial\Omega} = \langle u,\Delta\phi\rangle - \langle\Delta u,\phi\rangle = \langle u,\Delta\phi\rangle + \langle F,\phi\rangle = \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$$

for all $\phi\in Y_{q',w',\sigma}$. Using the second equation one shows that $N\cdot u|_{\partial\Omega} = N\cdot g$. $\square$

Remark 5.12. *1. It is not difficult to see that the space $\tilde W^{q,r}_{w,\tilde w}$ is equal to the space of very weak solutions to the Stokes problem with respect to data $f=[\phi\mapsto\langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle]$ with $F\in W^{-1,r}_{\tilde w}(\Omega)$ and $g\in T^{0,q}_w(\partial\Omega)$ and $k\in W^{-1,q}_{w,0}(\Omega)$. Indeed let $u\in\tilde W^{q,r}_{w,\tilde w}$ and let $F\in W^{-1,r}_{\tilde w}(\Omega)$ be an extension of $-\Delta u|_{C^\infty_{0,\sigma}(\Omega)}$. Then $g:=\operatorname{tr} u\in T^{0,q}_w(\Omega)$ and by definition*

$$-\langle u,\Delta\phi\rangle = \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle \quad \textit{for every} \quad \phi\in Y_{q',w',\sigma}.$$

2. In [12] the space in which the traces are well-defined is given by

$$\widehat W^{1,q}(\Omega) := \overline{W^{1,q}(\Omega)}^{\|\cdot\|_{\widehat W^{1,q}(\Omega)}} \quad \textit{where} \quad \|u\|_{\widehat W^{1,q}(\Omega)} := \|u\|_q + \|\mathcal A_r^{-\frac12} P_r\Delta u\|_r,$$

where $\mathcal A_r$ stands for the Stokes operator in $L^r(\Omega)$, $\frac1r\le\frac1n+\frac1q$. For $u\in C^\infty(\overline\Omega)$ one has

$$\|\Delta u\|_{(W^{1,r'}_{0,\sigma})'} = \sup_{\phi\in C^\infty_{0,\sigma},\|\phi\|_{1,r'}=1}\langle\Delta u,\phi\rangle \sim \sup_{\phi\in C^\infty_{0,\sigma},\|\phi\|_{r'}=1}\langle P\Delta u, \mathcal A_{r'}^{-\frac12}\phi\rangle = \|\mathcal A_r^{-\frac12} P\Delta u\|_r$$

Thus in the unweighted case these norms are equivalent and the spaces are equal.

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