On the classification of rational quantum tori and their automorphism groups

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Abstract. An n-dimensional quantum torus is a twisted group algebra of the group $\mathbb{Z}^n$. It is called rational if all invertible commutators are roots of unity. In the present note we classify all rational n-dimensional quantum tori over any field. Moreover, we show that for n=2 the natural exact sequence describing the automorphism group of the quantum torus splits over any field.
Keyword: Quantum torus, normal form, automorphisms of quantum tori
MSC: 16S35

Introduction

Let $\mathbb{K}$ be a field and Γ an abelian group. A Γ-*quantum torus* is a Γ-graded $\mathbb{K}$-algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, for which all grading spaces are one-dimensional and all non-zero elements in these spaces are invertible. For any basis $(\delta_\gamma)_{\gamma\in\Gamma}$ of such an algebra with $\delta_\gamma \in A_\gamma$, we have $\delta_\gamma\delta_{\gamma'} = f(\gamma,\gamma')\delta_{\gamma+\gamma'}$, where $f\colon\Gamma\times\Gamma\to\mathbb{K}^\times$ is a group cocycle. In this sense Γ-quantum tori are the same as *twisted group algebras* in the terminology of [OP95]. Quantum tori arise very naturally in non-commutative geometry as non-commutative algebras which are still very close to commutative ones (cf. [GVF01]).

For $\Gamma = \mathbb{Z}^n$, we also speak of n-dimensional quantum tori. Important special examples arise for $n = 2$ and $f(\gamma,\gamma') = q^{\gamma_1\gamma_2'}$, which leads to an algebra A_q with two generators $u_1 = \delta_{(1,0)}$ and $u_2 = \delta_{(0,1)}$, satisfying the commutator relation

$$u_1u_2 = qu_2u_1.$$

Finite-dimensional quantum tori and their Jordan analogs also play a key role in the structure theory of infinite-dimensional Lie algebras because they are the natural coordinate structures of extended affine Lie algebras ([BGK96], [AABGP97]).

The first problem we address in this note is the classification of the finite-dimensional *rational quantum tori*, i.e., quantum tori with grading group $\Gamma = \mathbb{Z}^n$, for which f takes values in the torsion group of $\mathbb{K}^\times$. This problem is solved completely in Section III, where we give a classification of rational quantum tori over arbitrary fields. We first show that any rational n-dimensional quantum torus A can be written as a tensor product

$$A \cong A_{q_1}\otimes\cdots\otimes A_{q_s}\otimes\mathbb{K}[\mathbb{Z}]^{n-2s}, \tag{1}$$

where the roots of unity $q_1,\ldots,q_s$ satisfy $1 < \mathrm{ord}(q_s) \leq \ldots \leq \mathrm{ord}(q_1)$. Two n-dimensional rational quantum tori given, as above, by the data $(q_1,\ldots,q_s)$ and $(q_1',\ldots,q_{s'}')$ are isomorphic if and only if $s = s'$ and $\mathrm{ord}(q_i) = \mathrm{ord}(q_i')$ for $i = 1,\ldots,s$. Under the assumption that the field $\mathbb{K}$ is algebraically closed of characteristic zero, the tensor product decomposition (1) has also been obtained in [ABFP05].

For any $\mathbb{Z}^n$-quantum torus A, its group of automorphisms is an abelian extension described by a short exact sequence

$$\mathbf{1}\to\mathrm{Hom}(\mathbb{Z}^n,\mathbb{K}^\times)\to\mathrm{Aut}(A)\to\mathrm{Aut}(\mathbb{Z}^n,\lambda)\to\mathbf{1}, \tag{2}$$

where $\lambda: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{K}^\times, (\gamma, \gamma') \mapsto \delta_\gamma \delta_{\gamma'} \delta_\gamma^{-1} \delta_{\gamma'}^{-1}$ is the alternating biadditive map determined by the commutator map of the unit group $A^\times$, and $\mathrm{Aut}(\mathbb{Z}^n, \lambda) \subseteq \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}^n)$ is the subgroup preserving λ. The second main result of this note is that for $n = 2$ the sequence (2) always splits. In this case $A \cong A_q$ for some $q \in \mathbb{K}^\times$, and $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \mathrm{GL}_2(\mathbb{Z})$ if $q^2 = 1$ and $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \mathrm{SL}_2(\mathbb{Z})$ otherwise. The statement of this result (in case q is not a root of unity) can also be found in [KPS94, Th. 1.5], but without any argument for the splitting of the exact sequence (2). According to [OP95, p.430], the determination of the automorphism groups of general quantum tori seems to be a hopeless problem, but we think that our splitting result stimulates some hope that more explicit descriptions might be possible if the range of the commutator map is sufficiently well-behaved.

We thank B. Allison and A. Pianzola for stimulating discussions on the subject matter of this paper and A. Pianzola for pointing out the reference [OP95].

Notation

Throughout this paper $\mathbb{K}$ denotes an arbitrary field. We write $A^\times$ for the unit group of a unital $\mathbb{K}$-algebra A.

Let Γ and Z be abelian groups, both written additively. A function $f: \Gamma \times \Gamma \to Z$ is called a 2-*cocycle* if

$$f(\gamma, \gamma') + f(\gamma + \gamma', \gamma'') = f(\gamma, \gamma' + \gamma'') + f(\gamma', \gamma'')$$

holds for $\gamma, \gamma', \gamma'' \in \Gamma$. The set of all 2-cocycles is an additive group $Z^2(\Gamma, Z)$ with respect to pointwise addition. The functions of the form $h(\gamma) - h(\gamma + \gamma') + h(\gamma')$ are called *coboundaries*. They form a subgroup $B^2(\Gamma, Z) \subseteq Z^2(\Gamma, Z)$, and the quotient group $H^2(\Gamma, Z) := Z^2(\Gamma, Z)/B^2(\Gamma, Z)$ is called the second cohomology group of Γ with values in Z. It classifies central extensions of Γ by Z up to equivalence. Here we assign to $f \in Z^2(\Gamma, Z)$ the central extension $Z \times_f \Gamma$, which is the set $Z \times \Gamma$, endowed with the group multiplication

$$(z, \gamma)(z', \gamma') = (z + z' + f(\gamma, \gamma'), \gamma + \gamma') \quad z, z' \in Z, \gamma, \gamma' \in \Gamma. \tag{0.1}$$

We also write $\mathrm{Ext}(\Gamma, Z) \cong H^2(\Gamma, Z)$ for the group of all central extensions of Γ by Z, and $\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z)$ for the subgroup corresponding to the abelian extensions of the group Γ by Z, which correspond to symmetric 2-cocycles.

We call a biadditive map $\Gamma \times \Gamma \to Z$ vanishing on the diagonal *alternating* and denote the set of these maps by $\mathrm{Alt}^2(\Gamma, Z)$. A function $q: \Gamma \to Z$ is called a *quadratic form* if the map

$$\beta_q: \Gamma \times \Gamma \to Z, \quad (\gamma, \gamma') \mapsto q(\gamma + \gamma') - q(\gamma) - q(\gamma')$$

is biadditive. Note that we do not require here that $q(n\gamma) = n^2 q(\gamma)$ holds for $n \in \mathbb{Z}$ and $\gamma \in \Gamma$.

For $n \in \mathbb{N}$ we write $Z[n] := \{z \in Z: nz = 0\}$ for the n-torsion subgroup of Z.

I. The correspondence between quantum tori and central extensions

Definition I.1. Let Γ be an abelian group. A unital associative $\mathbb{K}$-algebra A is said to be a Γ-*quantum torus* if it is Γ-graded, $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, with one-dimensional grading spaces A_γ, and each non-zero element of A_γ is invertible.*

For $\Gamma \cong \mathbb{Z}^d$ we call a Γ-quantum torus also a d-*dimensional quantum torus.* ■

* In [OP95], these algebras are called *twisted group algebras.*

Remark I.2. In each Γ-quantum torus A the set $A_h^\times := \bigcup_{\gamma\in\Gamma} \mathbb{K}^\times \delta_\gamma$ of homogeneous units (called *trivial units* in [OP95]) is a subgroup containing $\mathbb{K}^\times \mathbf{1} \cong \mathbb{K}^\times$ in its center. We thus obtain a central extension

$$\mathbf{1} \to \mathbb{K}^\times \to A_h^\times \to \Gamma \to \mathbf{1}$$

of abelian groups.

It is instructive to see how this can be made more explicit in terms of cocycles, which shows in particular that each central extension of Γ by $\mathbb{K}^\times$ arises as $A_h^\times$ for some Γ-quantum torus A.

Let A be a Γ-quantum torus and pick non-zero elements $\delta_\gamma \in A_\gamma$, so that $(\delta_\gamma)_{\gamma\in\Gamma}$ is a basis of A. Then each δ_γ is an invertible element of A, so that we get

$$\delta_\gamma\delta_{\gamma'} = f(\gamma,\gamma')\delta_{\gamma+\gamma'} \quad \text{for} \quad \gamma,\gamma' \in \Gamma, \tag{1.1}$$

where $f \in Z^2(\Gamma,\mathbb{K}^\times)$ is a 2-cocycle for which $A_h^\times \cong \mathbb{K}^\times \times_f \Gamma$ (cf. (0.1)).

Conversely, starting with a cocycle $f \in Z^2(\Gamma,\mathbb{K}^\times)$, we define a multiplication on the vector space $A := \bigoplus_{\gamma\in\Gamma} \mathbb{K}\delta_\gamma$ with basis $(\delta_\gamma)_{\gamma\in\Gamma}$ by $\delta_\gamma\delta_{\gamma'} := f(\gamma,\gamma')\delta_{\gamma+\gamma'}$. Then the cocycle property implies that we get a unital associative algebra, and it is clear from the construction that it is a Γ-quantum torus. ■

Definition I.3. There are two natural equivalence relations between quantum tori. The finest one is the notion of *graded equivalence:* Two Γ-quantum tori A and B are called *graded equivalent* if there is an algebra isomorphism $\varphi\colon A \to B$ with $\varphi(A_\gamma) = B_\gamma$ for all $\gamma \in \Gamma$.

A slightly weaker notion is *graded isomorphy:* Two Γ-quantum tori A and B are called *graded isomorphic* if there is an isomorphism $\varphi\colon A \to B$ and an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$ with $\varphi(A_\gamma) = B_{\varphi_\Gamma(\gamma)}$ for all $\gamma \in \Gamma$. ■

The following theorem reduces the corresponding classification problems to purely group theoretic ones.

Theorem I.4. *The graded equivalence classes of Γ-quantum tori are in one-to-one correspondence with the extensions of the group Γ by the multiplicative group $\mathbb{K}^\times$, hence parametrized by the cohomology group $H^2(\Gamma,\mathbb{K}^\times)$.*

The graded isomorphy classes of Γ-quantum tori are parametrized by the set

$$H^2(\Gamma,\mathbb{K}^\times)/\,\mathrm{Aut}(\Gamma)$$

of orbits of the group $\mathrm{Aut}(\Gamma)$ in the cohomology group $H^2(\Gamma,\mathbb{K}^\times)$, where the action is given on the level of cocycles by $\psi.f := (\psi^{-1})^* f = f \circ (\psi^{-1} \times \psi^{-1})$.

Proof. If $\varphi\colon A \to B$ is a graded equivalence of Γ-quantum tori, then the restriction to the group $A_h^\times$ of homogeneous units leads to the commutative diagram

$$\begin{array}{ccccc} \mathbb{K}^\times & \to & A_h^\times & \to & \Gamma \\ \big\downarrow \mathrm{id}_{\mathbb{K}^\times} & & \big\downarrow \varphi & & \big\downarrow \mathrm{id}_\Gamma \\ \mathbb{K}^\times & \to & B_h^\times & \to & \Gamma. \end{array}$$

This means that the central extensions $A_h^\times$ and $B_h^\times$ of Γ by $\mathbb{K}^\times$ are equivalent. If, conversely, these extensions are equivalent, then any equivalence $\varphi\colon A_h^\times \to B_h^\times$ extends linearly to a graded equivalence $A \to B$. Now the observation from Remark I.2 implies that the graded equivalence classes of Γ-quantum tori are parametrized by the cohomology group $H^2(\Gamma,\mathbb{K}^\times) \cong \mathrm{Ext}(\Gamma,\mathbb{K}^\times)$.

If $\varphi\colon A \to B$ is a graded isomorphism of Γ-quantum tori, then the diagram

$$\begin{array}{ccccc} \mathbb{K}^\times & \to & A_h^\times & \to & \Gamma \\ \big\downarrow \mathrm{id}_{\mathbb{K}^\times} & & \big\downarrow \varphi & & \big\downarrow \varphi_\Gamma \\ \mathbb{K}^\times & \to & B_h^\times & \to & \Gamma \end{array}$$

commutes, which means that the corresponding central extensions $A_h^\times$ and $B_h^\times$ are contained in the same orbit of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Ext}(\Gamma,\mathbb{K}^\times) \cong H^2(\Gamma,\mathbb{K}^\times)$ (we leave the easy verification to the reader). Conversely, any isomorphism $\varphi\colon A_h^\times \to B_h^\times$ of central extensions extends linearly to an isomorphism of algebras $A \to B$. ■

II. Central extensions of abelian groups

In this section Γ and Z are abelian groups, written additively. We shall derive some general facts on the set of equivalence classes $\mathrm{Ext}(\Gamma, Z) \cong H^2(\Gamma, Z)$ of central extensions of Γ by Z. In Sections III and IV below we shall apply these to the special case $Z = \mathbb{K}^\times$ for a field $\mathbb{K}$.

Remark II.1. Let $Z \hookrightarrow \widehat{\Gamma} \xrightarrow{q} \Gamma$ be a central extension of the abelian group Γ by the abelian group Z and

$$\widehat{\lambda}\colon \widehat{\Gamma} \times \widehat{\Gamma} \to Z, \quad (x,y) \mapsto [x,y] := xyx^{-1}y^{-1}$$

the commutator map of $\widehat{\Gamma}$. Its values lie in Z because Γ is abelian. We then have

$$-\widehat{\lambda}(x,y) = \widehat{\lambda}(x,y)^{-1} = \widehat{\lambda}(y,x)$$

and

$$\begin{aligned}\widehat{\lambda}(xx',y) &= xx'y(x')^{-1}x^{-1}y^{-1} = x \cdot (x'y(x')^{-1}y^{-1}) \cdot (yx^{-1}y^{-1}) = x \cdot \widehat{\lambda}(x',y) \cdot (yx^{-1}y^{-1}) \\ &= xyx^{-1}y^{-1}\widehat{\lambda}(x',y) = \widehat{\lambda}(x,y) + \widehat{\lambda}(x',y).\end{aligned}$$

We conclude that $\widehat{\lambda}$ is a skew-symmetric biadditive map (cf. [OP95, p.430]). Moreover, the commutator map is constant on the fibers of the map q, hence factors through a biadditive map $\lambda \in \mathrm{Alt}^2(\Gamma, Z)$.

Next we write $\widehat{\Gamma}$ as $Z \times_f \Gamma$ with a 2-cocycle $f \in Z^2(\Gamma, Z)$. For the map $\sigma\colon \Gamma \to \widehat{\Gamma}, \gamma \mapsto (0,\gamma)$ we then have $\sigma(\gamma)\sigma(\gamma') = \sigma(\gamma+\gamma')f(\gamma,\gamma')$, which leads to

$$\begin{aligned}f(\gamma,\gamma') &= \widehat{\lambda}(\sigma(\gamma),\sigma(\gamma')) = \sigma(\gamma)\sigma(\gamma')\big(\sigma(\gamma')\sigma(\gamma)\big)^{-1} \\ &= \sigma(\gamma+\gamma')f(\gamma,\gamma')\big(\sigma(\gamma+\gamma')f(\gamma',\gamma)\big)^{-1} = f(\gamma,\gamma')f(\gamma',\gamma)^{-1} = f(\gamma,\gamma') - f(\gamma',\gamma).\end{aligned}$$

Therefore the map $\lambda_f \in \mathrm{Alt}^2(\Gamma, Z)$ defined by

$$\lambda_f(\gamma,\gamma') := f(\gamma,\gamma') - f(\gamma',\gamma) \tag{2.1}$$

can be identified with the commutator map of $\widehat{\Gamma}$.

Note that the commutator map λ_f only depends on the cohomology class $[f] \in H^2(\Gamma, Z)$. We thus obtain a group homomorphism

$$\Phi\colon H^2(\Gamma, Z) \to \mathrm{Alt}^2(\Gamma, Z), \quad [f] \mapsto \lambda_f. \qquad \blacksquare$$

Remark II.2. Each biadditive map $f\colon \Gamma \times \Gamma \to Z$ is a cocycle, but not each cohomology class in $H^2(\Gamma, Z)$ has a biadditive representative. A typical examples is the class corresponding to the exact sequence $\mathbf{0} \to m\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to \mathbf{0}$. ■

Proposition II.3. *For abelian groups Γ and Z we have an exact sequence*

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \to \mathrm{Ext}(\Gamma, Z) \cong H^2(\Gamma, Z) \xrightarrow{\Phi} \mathrm{Alt}^2(\Gamma, Z),$$

describing the kernel of the map Φ. The cokernel of Φ is an elementary abelian 2-group.

Proof. For the exactness of the sequence, we only have to observe that an extension $\widehat{\Gamma}$ of Γ by Z is an abelian group if and only if the commutator map of $\widehat{\Gamma}$ is trivial (cf. Remark II.1).

To see that the cokernel of Φ is an elementary abelian 2-group, we note that each element $f \in \mathrm{Alt}^2(\Gamma, Z)$ is biadditive, hence in particular a cocycle (Remark II.2), and with (2.1) we see that $\Phi([f]) = \lambda_f = 2f$. This shows that $2\,\mathrm{Alt}^2(\Gamma, Z) \subseteq \mathrm{im}(\Phi)$, i.e., that $\mathrm{coker}(\Phi)$ is an elementary abelian 2-group. ■

For the following proposition we recall that, as a consequence of the Well-Ordering Theorem, each set I carries a total order.

Proposition II.4. *Let $\Gamma = \bigoplus_{i \in I} \Gamma_i$ be a direct sum of cyclic groups $\Gamma_i \cong \mathbb{Z}/m_i\mathbb{Z}$, $m_i \in \mathbb{N}_0$. Further let $\leq$ be a total order on I. Then the map*

$$\Phi: H^2(\Gamma, Z) \to \mathrm{Alt}^2(\Gamma, Z), \quad [f] \mapsto \lambda_f$$

is surjective and splits, so that

$$H^2(\Gamma, Z) \cong \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \oplus \mathrm{Alt}^2(\Gamma, Z) \cong \prod_{i<j} Z[\mathrm{lcm}(m_i, m_j)] \oplus \prod_{|m_i|<\infty} Z/m_iZ, \tag{2.2}$$

where we put $\mathrm{lcm}(m, 0) := m$ for $m \in \mathbb{N}_0$.

If, in addition, Γ is free, then Φ is an isomorphism, $H^2(\Gamma, Z) \cong Z^{\{(i,j) \in I^2 : i<j\}}$, and each cohomology class has a biadditive representative.

Proof. To see that Φ is surjective, let $\eta \in \mathrm{Alt}^2(\Gamma, Z)$. If γ_i is a generator of Γ_i, we have $\eta(n\gamma_i, m\gamma_i) = nm\eta(\gamma_i, \gamma_i) = 0$ for $n, m \in \mathbb{Z}$, so that η vanishes on $\Gamma_i \times \Gamma_i$. We define a biadditive map $f_\eta: \Gamma \times \Gamma \to Z$ by

$$f_\eta(\gamma_i, \gamma_j) := \begin{cases} \eta(\gamma_i, \gamma_j) & \text{for } i > j,\ \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j, \\ 0 & \text{for } i \leq j,\ \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j. \end{cases}$$

Then f_η is biadditive, hence a 2-cocycle (Remark II.2), and $\Phi(f_\eta) = \eta$.

Clearly, the assignment $\eta \mapsto f_\eta$ defines an injective homomorphism $\mathrm{Alt}^2(\Gamma, Z) \to H^2(\Gamma, Z)$, splitting Φ. We know from Proposition II.3, that $\ker \Phi = \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z)$.

We next observe that

$$\mathrm{Alt}(\Gamma, Z) \cong \prod_{i<j} \mathrm{Hom}(\Gamma_i \otimes \Gamma_j, Z),$$

and $\Gamma_i \otimes \Gamma_j \cong \mathbb{Z}/\mathrm{lcm}(m_i, m_j)\mathbb{Z}$, which leads to

$$\mathrm{Hom}(\Gamma_i \otimes \Gamma_j, Z) \cong Z[\mathrm{lcm}(m_i, m_j)].$$

On the other hand,

$$\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \cong \prod_{i \in I} \mathrm{Ext}_{\mathrm{ab}}(\Gamma_i, Z) \cong \prod_{|m_i|<\infty} Z/m_iZ$$

(cf. [Fu70]), which leads to (2.2).

If, in addition, Γ is free, then $m_i = 0$ for each $i \in I$, and the assertion follows from $\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) = \mathbf{0}$. ■

Problem II. Find a pair (Γ, Z) of abelian groups for which the map $\Phi: H^2(\Gamma, Z) \to \mathrm{Alt}^2(\Gamma, Z)$ is not surjective. ■

III. The Normal form of rational quantum tori

In this section we write $\Gamma := \mathbb{Z}^n$ for the free abelian group of rank n. For an abelian group Z we write $\mathrm{Alt}_n(Z)$ for the set of *alternating* $(n \times n)$*-matrices with entries in* Z, i.e., $a_{ii} = 0$ for each i and $a_{ij} = -a_{ji}$ for $i \neq j$. This is an abelian group with respect to matrix addition.

Clearly the map $\mathrm{Alt}^2(\Gamma, Z) \to \mathrm{Alt}_n(Z), f \mapsto (f(e_i, e_j))_{i,j=1,\ldots,n}$ is an isomorphism of abelian groups, so that $\mathrm{Alt}_n(Z) \cong H^2(\Gamma, Z)$ by Proposition II.4. Writing $\lambda_A \in \mathrm{Alt}^2(\Gamma, Z)$ for the alternating form $\lambda_A(\alpha, \beta) := \beta^\top A\alpha$ determined by the alternating matrix A, we have for $g \in \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\Gamma)$ the relation

$$\lambda_A(g.\alpha, g.\beta) = \beta g^\top A g\alpha,$$

so that the orbits of the natural action of $\mathrm{Aut}(\Gamma) \cong \mathrm{GL}_n(\mathbb{Z})$ on the set of alternating forms correspond to the orbits of the action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathrm{Alt}_n(Z)$ by

$$g.A := gAg^\top, \tag{3.1}$$

where we we multiply matrices in $M_n(\mathbb{Z})$ with matrices in $M_n(Z)$ in the obvious fashion. We conclude that

$$H^2(\Gamma, Z)/\,\mathrm{Aut}(\Gamma) \cong \mathrm{Alt}_n(Z)/\,\mathrm{GL}_n(\mathbb{Z}) \tag{3.2}$$

can be identified with the set of $\mathrm{GL}_n(\mathbb{Z})$-orbits in $\mathrm{Alt}_n(Z)$.

If $n = n_1 + \ldots + n_r$ is a partition of n and $A_i \in M_{n_i}(Z)$, then we write

$$A_1 \oplus A_2 \oplus \ldots \oplus A_r := \mathrm{diag}(A_1, \ldots, A_r),$$

for the block diagonal matrix with entries $A_1, \ldots, A_r$.

The following theorem classifies the orbits of $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{Alt}_n(Z)$ for cyclic groups Z. Note that each cyclic group Z has a ring structure, so that we may write $a|b$ for $bZ \subseteq aZ$.

Theorem III.1. *Suppose that Z is a cyclic group and $A \in \mathrm{Alt}_n(Z)$. Then the $\mathrm{GL}_n(\mathbb{Z})$-orbit of A contains a unique matrix of the skew normal form*

$$\begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & h_2 \\ -h_2 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & h_s \\ -h_s & 0 \end{pmatrix} \oplus \mathbf{0}_{n-2s},$$

where $h_1|h_2|\cdots|h_s$.

Proof. Let $q\colon \mathbb{Z} \to Z$ be a surjective homomorphism and $q_n\colon M_n(\mathbb{Z}) \to M_n(Z)$ the induced surjective homomorphism of additive matrix groups which is equivariant with respect to the action (3.1) of $\mathrm{GL}_n(\mathbb{Z})$ on both groups. Since $A \in M_n(Z)$ is a matrix with vanishing diagonal and $a_{ij} = -a_{ji}$, there exists a matrix $\widetilde{A} \in \mathrm{Alt}_n(\mathbb{Z})$ with $q_n(\widetilde{A}) = A$.

As $\mathbb{Z}$ is a principal ideal ring, the Theorem on the Skew Normal Form ([New72, Thms. IV.1,IV.2]) implies the existence of $g \in \mathrm{GL}_n(\mathbb{Z})$ with

$$g^\top \widetilde{A} g = \begin{pmatrix} 0 & \widetilde{h}_1 \\ -\widetilde{h}_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \widetilde{h}_2 \\ -\widetilde{h}_2 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & \widetilde{h}_t \\ -\widetilde{h}_t & 0 \end{pmatrix} \oplus \mathbf{0}_{n-2t}$$

and $\widetilde{h}_1|\widetilde{h}_2|\cdots|\widetilde{h}_t$. We then have

$$g.A = q_n(g^\top \widetilde{A} g) = \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & h_2 \\ -h_2 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & h_s \\ -h_s & 0 \end{pmatrix} \oplus \mathbf{0}_{n-2s},$$

where $h_j := q(\widetilde{h}_j)$ satisfies $h_1|h_2|\cdots|h_s$ and s is maximal with $h_s \neq 0$. Note that this implies that $h_j \neq 0$ for all $j \leq s$.

For $B \in M_n(Z)$ and $g \in \mathrm{GL}_n(\mathbb{Z})$ we have $q_n(g) \in \mathrm{GL}_n(Z)$ and $g.B = q_n(g)Bq_n(g)^\top$, so that all matrices in the same $\mathrm{GL}_n(\mathbb{Z})$-orbit are equivalent in the sense that they are contained in the same double cosets of $\mathrm{GL}_n(Z)$ in $M_n(Z)$. For $1 \leq j \leq n$ the determinantal divisor $d_j(B)$ is defined as the greatest common divisor of all minors of size j of B; considered as an orbit of the multiplication action of the unit group $Z^\times$ of $(Z, \cdot)$ on Z. According to [New72, Th. II.8], the determinantal divisors d_j are constant on the $\mathrm{GL}_n(Z)$-double cosets in $M_n(Z)$, hence invariants of the $\mathrm{GL}_n(\mathbb{Z})$-action on $\mathrm{Alt}^2(Z)$. Now the assertion follows from

$$h_1 = d_1(B) = d_2(B)/d_1(B), \ldots, h_s = d_{2s-1}(B)/d_{2s-2}(B) = d_{2s}(B)/d_{2s-1}(B)$$

and $d_j(B) = 0$ for $j > 2s$. ■

Definition III.2. (a) We call a Γ-quantum torus *rational* if the set of all commutators in $A^\times = A_h^\times$ (cf. Proposition A.1) consists of roots of unity in $\mathbb{K}$.

(b) For each $q \in \mathbb{K}^\times$ we write A_q for the $\mathbb{Z}^2$-quantum torus corresponding to the biadditive cocycle $f\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{K}^\times$ determined by

$$f(e_1, e_1) = f(e_2, e_2) = f(e_2, e_1) = 1 \quad \text{and} \quad f(e_1, e_2) = q.$$

Then the algebra A_q has generators $u_1 = \delta_{e_1}$ and $u_2 = \delta_{e_2}$ satisfying

$$u_1 u_2 = q u_2 u_1. \tag{3.3}$$

The quantum torus A_q is rational if and only if q is a root of unity. ■

Theorem III.3. (Classification of rational quantum tori) *Let $\mathbb{K}$ be any field. For each rational $\mathbb{Z}^n$-quantum torus over $\mathbb{K}$ there exists an $s \in \mathbb{N}_0$ with $2s \leq n$ and roots of unity $q_1, \ldots, q_s \in \mathbb{K}^\times$ with $\mathrm{ord}(q_s) \leq \ldots \leq \mathrm{ord}(q_1)$, such that*

$$A \cong A_{q_1} \otimes A_{q_2} \otimes \ldots \otimes A_{q_s} \otimes \mathbb{K}[\mathbb{Z}^{n-2s}].$$

Two n-dimensional rational quantum tori given, as above, by the data $(q_1, \ldots, q_s)$ and $(q_1', \ldots, q_{s'}')$ are isomorphic if and only if $s = s'$ and $\mathrm{ord}(q_i) = \mathrm{ord}(q_i')$ for $i = 1, \ldots, s$.

Proof. We know from Theorem I.4 and (3.2) that the Γ-quantum tori over $\mathbb{K}$ are classified by the orbits of $\mathrm{Aut}(\Gamma) \cong \mathrm{GL}_n(\mathbb{Z})$ in $H^2(\Gamma, \mathbb{K}^\times) \cong \mathrm{Alt}^2(\Gamma, \mathbb{K}^\times)$. In this picture the rational quantum tori correspond to alternating forms $f \in \mathrm{Alt}^2(\Gamma, \mathbb{K}^\times)$ on Γ whose values are roots of unity. Since the group Z generated by the image of f is generated by the finite set $f(e_i, e_j)$, $i, j = 1, \ldots, n$, it is a finite subgroup of $\mathbb{K}^\times$, hence cyclic (cf. [La93, Th. IV.1.9]). Therefore Theorem III.1 applies, and we see A is isomorphic to a quantum torus defined by a biadditive cocycle $f\colon \Gamma \times \Gamma \to Z \subseteq \mathbb{K}^\times$ satisfying

$$f(e_1, e_2) = q_1, \quad f(e_3, e_4) = q_2 \quad \text{and} \quad f(e_{2s-1}, e_{2s}) = q_s$$

and $f(e_i, e_j) = 1$ for all other pairs (i, j), where $q_1 | q_2 | \ldots | q_s$ holds in the cyclic group Z, viewed as a ring. This means that $\langle q_s \rangle \subseteq \ldots \subseteq \langle q_1 \rangle$, or, equivalently, $\mathrm{ord}(q_s) \leq \ldots \leq \mathrm{ord}(q_1)$. The quantum torus $A_f \cong A$ defined by f then satisfies

$$A_f \cong A_{q_1} \otimes A_{q_2} \otimes \ldots \otimes A_{q_s} \otimes \mathbb{K}[\mathbb{Z}^{n-2s}].$$

That two such quantum tori are isomorphic if and only if $s = s'$ and $\mathrm{ord}(q_i) = \mathrm{ord}(q_i')$ for $i = 1, \ldots, s$, follows from Theorem I.4, combined with Theorem III.1, because the order of an element $q \in Z$ determines the subgroup $\langle q \rangle$ it generates uniquely, and vice versa. ■

IV. Graded automorphisms of quantum tori

In this section we briefly discuss the group of automorphisms of a general quantum torus, but our main result only concerns the 2-dimensional case: For $A = A_q$ and the corresponding alternating form λ on $\mathbb{Z}^2$, the group $\mathrm{Aut}(A)$ it is a semi-direct product $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times) \rtimes \mathrm{Aut}(\mathbb{Z}^2, \lambda)$.

Definition IV.1. Let A be a Γ-quantum torus. We write $\mathrm{Aut}_{\mathrm{gr}}(A)$ for the group of *graded automorphisms of* A, i.e., all those automorphisms $\varphi \in \mathrm{Aut}(A)$ for which there exists an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$ with $\varphi(A_\gamma) = A_{\varphi_\Gamma(\gamma)}$ for all $\gamma \in \Gamma$. ■

Note that Proposition A.1 in the appendix implies that if Γ is torsion free, then all units are homogeneous, which implies that each automorphism of A is graded.

Remark IV.2. We fix a basis $(\delta_\gamma)_{\gamma\in\Gamma}$ of A and suppose that $f \in Z^2(\Gamma, Z)$ is the corresponding cocycle determined by (1.1). Then for each graded automorphism φ of A there is an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$ and a function $\chi\colon \Gamma \to \mathbb{K}^\times$ such

$$\varphi(\delta_\gamma) = \chi(\gamma)\delta_{\varphi_\Gamma(\gamma)}, \quad \gamma \in \Gamma. \tag{4.1}$$

Conversely, for a pair (χ, φ_Γ) of a function $\chi\colon \Gamma \to \mathbb{K}^\times$ and an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$ the prescription $\varphi(\delta_\gamma) := \chi(\gamma)\delta_{\varphi_\Gamma(\gamma)}$ defines an automorphism of A if and only if

$$\frac{(\varphi_\Gamma^* f)(\gamma,\gamma')}{f(\gamma,\gamma')} = \frac{\chi(\gamma+\gamma')}{\chi(\gamma)\chi(\gamma')} \quad \text{for all} \quad \gamma,\gamma' \in \Gamma. \tag{4.2}$$

Note that if f is biadditive, then $\varphi_\Gamma^* f/f$ is biadditive, so that χ is a corresponding $\mathbb{K}^\times$-valued quadratic form. If f and φ_Γ are given, then a χ satisfying (4.2) exists if and only if $[\varphi_\Gamma^* f] = [f]$ holds in $H^2(\Gamma, Z)$. ■

Lemma IV.3. *The image of the map*

$$Q\colon \mathrm{Aut}_{\mathrm{gr}}(A) \to \mathrm{Aut}(\Gamma), \quad \varphi \mapsto \varphi_\Gamma$$

is the group

$$\mathrm{Aut}(\Gamma)_{[f]} := \{\psi \in \mathrm{Aut}(\Gamma)\colon [\psi^* f] = [f]\},$$

which is contained in

$$\mathrm{Aut}(\Gamma, \lambda_f) := \{\psi \in \mathrm{Aut}(\Gamma)\colon \psi^*\lambda_f = \lambda_f\},$$

where $\lambda_f(\gamma,\gamma') = \frac{f(\gamma,\gamma')}{f(\gamma',\gamma)}$. *If, in addition,* Γ *is free, then* $\mathrm{Aut}(\Gamma)_{[f]} = \mathrm{Aut}(\Gamma, \lambda_f)$.

Proof. Let $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$. In view of Remark IV.2, the existence of $\varphi \in \mathrm{Aut}_{\mathrm{gr}}(A)$ with $Q(\varphi) = \varphi_\Gamma$ is equivalent to the existence of χ satisfying (4.2), which is equivalent to $[\varphi_\Gamma^* f] = [f]$ in $H^2(\Gamma, \mathbb{K}^\times)$. Since (4.2) implies that $\varphi_\Gamma^* f/f$ is symmetric, we have $\varphi_\Gamma^*\lambda_f = \lambda_{\varphi_\Gamma^* f} = \lambda_f$.

If, in addition, Γ is free, then Proposition II.4 entails that $\varphi_\Gamma^*\lambda_f = \lambda_f$ is equivalent to $[\varphi_\Gamma^* f] = [f]$ in $H^2(\Gamma, \mathbb{K}^\times)$ (cf. [OP95, Lemma 3.3(iii)]). ■

From (4.2) we derive in particular that $(\chi, \mathbf{1})$ defines an automorphism of A if and only if $\chi \in \mathrm{Hom}(\Gamma, \mathbb{K}^\times)$, so that we obtain the exact sequence

$$\mathbf{1} \to \mathrm{Hom}(\Gamma, \mathbb{K}^\times) \to \mathrm{Aut}_{\mathrm{gr}}(A) \to \mathrm{Aut}(\Gamma)_{[f]} \to \mathbf{1} \tag{4.3}$$

(cf. [OP95, Lemma 3.3(iii)]). We call the automorphisms of the form $(\chi, \mathbf{1})$ *scalar*.

Remark IV.4. If the map Φ from Proposition II.4 is not injective, then the groups $\mathrm{Aut}(\Gamma, \lambda_f)$ and $\mathrm{Aut}(\Gamma)_{[f]}$ need not coincide, but with Proposition II.3 we obtain a 1-cocycle

$$I\colon \mathrm{Aut}(\Gamma, \lambda_f) \to \mathrm{Ext}_{\mathrm{ab}}(\Gamma, \mathbb{K}^\times), \quad \psi \mapsto [\psi^* f - f]$$

satisfying $\mathrm{Aut}(\Gamma)_{[f]} = I^{-1}(0)$. ■

In the remainder of this section we restrict our attention to the case, where $\Gamma = \mathbb{Z}^n$ is a free abelian group of rank n, which implies that $\mathrm{Aut}(\Gamma)_{[f]} = \mathrm{Aut}(\Gamma, \lambda_f)$ and that $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$ (Corollary A.2).

Remark IV.5. (a) For $n = 1$, each alternating biadditive map λ on Γ vanishes, so that $\mathrm{Aut}(\Gamma, \lambda) = \mathrm{Aut}(\Gamma) \cong \{\pm \mathrm{id}_\Gamma\}$.

(b) For each alternating form $\lambda\colon \Gamma \times \Gamma \to \mathbb{K}^\times$ we have $-\mathrm{id}_\Gamma \in \mathrm{Aut}(\Gamma, \lambda)$.

(c) In [OP95] it is shown that if the subgroup $\langle \mathrm{im}(\lambda) \rangle$ of $\mathbb{K}^\times$ generated by the image of λ is free of rank $\binom{n}{2}$, then $\mathrm{Aut}(\Gamma, \lambda_f) = \{\pm \mathrm{id}_\Gamma\}$.

Moreover, for $n = 3$ and $\langle \mathrm{im}(\lambda) \rangle$ free of rank 2, [OP95, Prop. 3.7] implies the existence of a basis $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ with $\lambda(\gamma_1, \gamma_2) = 1$ and

$$\begin{aligned}\mathrm{Aut}(\Gamma, \lambda) &\cong \{\sigma \in \mathrm{Aut}(\Gamma)\colon (\exists a, b \in \mathbb{Z}, \varepsilon \in \{\pm 1\})\ \sigma(\gamma_1) = \gamma_1^\varepsilon, \sigma(\gamma_2) = \gamma_2^\varepsilon, \sigma(\gamma_3) = \gamma_1^a \gamma_2^b \gamma_3^\varepsilon\} \\ &\cong \mathbb{Z}^2 \rtimes \{\pm \mathrm{id}_{\mathbb{Z}^2}\}.\end{aligned}$$

■

We now take a closer look at the case $n = 2$. Any alternating form $\lambda \in \mathrm{Alt}^2(\mathbb{Z}^2, \mathbb{K}^\times)$ is uniquely determined by $q := \lambda(e_1, e_2)$, which implies $\lambda(\gamma, \gamma') = q^{\gamma_1\gamma_2' - \gamma_2\gamma_1'}$. We may therefore assume that a corresponding bimultiplicative cocycle f satisfies $f(\gamma, \gamma') = q^{\gamma_1\gamma_2'}$, which leads to the quantum torus A_q with two generators $u_i = \delta_{e_i}$ satisfying $u_1u_2 = qu_2u_1$, as defined in the introduction.

We start with two simple observations:

Lemma IV.6. $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \begin{cases} \mathrm{SL}_2(\mathbb{Z}) & \text{for } q^2 \neq 1 \\ \mathrm{GL}_2(\mathbb{Z}) & \text{for } q^2 = 1. \end{cases}$

Proof. Clearly $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{Aut}(\mathbb{Z}^2, \lambda) \subseteq \mathrm{GL}_2(\mathbb{Z})$. The map $g_0(\gamma) = (\gamma_2, \gamma_1)$ satisfies $\mathrm{GL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}) \rtimes \langle g_0 \rangle$, and we have

$$\frac{g_0^*\lambda(e_1, e_2)}{\lambda(e_1, e_2)} = \frac{\lambda(e_2, e_1)}{\lambda(e_1, e_2)} = q^{-2}. \qquad \blacksquare$$

Example IV.7. (a) On $\mathbb{Z}^2$ the map $\chi(\gamma) := \gamma_1\gamma_2$ is a quadratic form with

$$\chi(\gamma + \gamma') - \chi(\gamma) - \chi(\gamma') = \gamma_1\gamma_2' + \gamma_2\gamma_1'.$$

(b) On $\mathbb{Z}$ the map $\chi(n) := \binom{n}{2}$ is a quadratic form with

$$\chi(n + n') - \chi(n) - \chi(n') = \frac{(n + n')(n + n' - 1) - n(n - 1) - n'(n' - 1)}{2} = \frac{nn' + n'n}{2} = nn'.$$

$\blacksquare$

From $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{Aut}(\mathbb{Z}^2, \lambda)$, it follows in particular that each matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

can be lifted to an automorphism of A_q. To determine a corresponding quadratic form $\chi\colon \mathbb{Z}^2 \to \mathbb{K}^\times$, we have to solve the equation (4.2):

$$\frac{(g^*f)(\gamma, \gamma')}{f(\gamma, \gamma')} = \frac{\chi(\gamma + \gamma')}{\chi(\gamma)\chi(\gamma')}.$$

The form g^*f/f is determined by its values on the pairs $(e_1, e_1), (e_1, e_2)$ and (e_2, e_2):

$$(g^*f/f)(e_1, e_1) = f(g.e_1, g.e_1) = q^{ac}, \quad (g^*f/f)(e_1, e_2) = f(g.e_1, g.e_2)q^{-1} = q^{ad-1}$$

and

$$(g^*f/f)(e_2, e_2) = f(g.e_2, g.e_2) = q^{bd}.$$

This means that

$$(g^*f/f)(\gamma, \gamma') = q^{ac\gamma_1\gamma_1' + (ad-1)(\gamma_1\gamma_2' + \gamma_1'\gamma_2) + bd\gamma_2\gamma_2'}.$$

Before we turn to lifting the full groups $\mathrm{Aut}(\mathbb{Z}^2, \lambda)$ to an automorphism group of A, we discuss certain specific elements of finite order separately.

Remark IV.8. (a) For the central element $z = -\mathbf{1} \in \mathrm{SL}_2(\mathbb{Z})$, any lift $\widehat{z} \in \mathrm{Aut}(A_q)$ is of the form

$$\widehat{z}.\delta_\gamma = r^{\gamma_1}s^{\gamma_2} \cdot \delta_{-\gamma} \quad \text{for some} \quad r, s \in \mathbb{K}^\times,$$

and any such element satisfies $\widehat{z}^2.\delta_\gamma = r^{\gamma_1}s^{\gamma_2} \cdot \widehat{z}.\delta_{-\gamma} = r^{\gamma_1 - \gamma_1}s^{\gamma_2 - \gamma_2} \cdot \delta_\gamma = \delta_\gamma$. Hence each lift $\widehat{z}$ of z is an element of order 2.

(b) The matrices

$$g_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad g_2 := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfy $g_1^2 = z = g_2^3$, which leads to $\mathrm{ord}(g_1) = 4$ and $\mathrm{ord}(g_2) = 6$. From the preceding paragraph we conclude that for any lift $\widehat{g}_j$ of g_j, $j = 1, 2$, we have $\widehat{g}_1^4 = \mathbf{1} = \widehat{g}_2^6$.

In view of

$$(g_1^* f/f)(\gamma, \gamma') = q^{-(\gamma_1\gamma_2' + \gamma_1'\gamma_2)},$$

a lift $\widetilde{g}_1$ of g_1 is given by $\widetilde{g}_1.\delta_\gamma = q^{-\gamma_1\gamma_2}\delta_{g_1.\gamma}$ (Example IV.7(a)). We then have

$$\widetilde{g}_1^2.\delta_\gamma = q^{-\gamma_1\gamma_2}\widetilde{g}_1.\delta_{(\gamma_2,-\gamma_1)} = q^{-\gamma_1\gamma_2}q^{\gamma_2\gamma_1}.\delta_{-\gamma} = \delta_{-\gamma}.$$

Any other lift $\widehat{g}_1$ of g_1 is of the form

$$\widehat{g}_1.\delta_g = r_1^{\gamma_1} s_1^{\gamma_2} q^{-\gamma_1\gamma_2}\delta_{g_1.\gamma}$$

for two elements $r_1, s_1 \in \mathbb{K}^\times$. The square of this element is given by

$$\widehat{g}_1^2.\delta_g = r_1^{\gamma_1} s_1^{\gamma_2}\widehat{g}_1\widetilde{g}_1.\delta_\gamma = r_1^{\gamma_1+\gamma_2} s_1^{\gamma_2-\gamma_1}\widetilde{g}_1^2.\delta_\gamma = \Big(\frac{r_1}{s_1}\Big)^{\gamma_1}(r_1 s_1)^{\gamma_2}\cdot\delta_{-\gamma}. \tag{4.4}$$

For the matrix g_2 we have

$$(g_2^* f/f)(\gamma, \gamma') = q^{-\gamma_1\gamma_1' - (\gamma_1\gamma_2' + \gamma_1'\gamma_2)},$$

so that we obtain a lift $\widetilde{g}_2$ of g_2 by $\widetilde{g}_2.\delta_\gamma = q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}\delta_{(\gamma_1+\gamma_2,-\gamma_1)}$ (Example IV.7(b)). Hence each lift $\widehat{g}_2$ of g_2 is of the form

$$\widehat{g}_2.\delta_g = r_2^{\gamma_1} s_2^{\gamma_2} q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}\delta_{(\gamma_1+\gamma_2,-\gamma_1)},$$

for some $r_2, s_2 \in \mathbb{K}^\times$. In view of $g_2^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, we get with Example IV.7(b):

$$\begin{aligned}\widetilde{g}_2^3.\delta_\gamma &= q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}\widetilde{g}_2^2.\delta_{\gamma_1+\gamma_2,-\gamma_1} = q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}q^{-\binom{\gamma_1+\gamma_2}{2}+(\gamma_1+\gamma_2)\gamma_1}\widetilde{g}_2.\delta_{\gamma_2,-\gamma_1-\gamma_2}\\ &= q^{-2\binom{\gamma_1}{2}-\binom{\gamma_2}{2}-\gamma_1\gamma_2+\gamma_1^2}q^{-\binom{\gamma_2}{2}+(\gamma_1+\gamma_2)\gamma_2}\delta_{-\gamma} = q^{-\gamma_1(\gamma_1-1)-\gamma_2(\gamma_2-1)+\gamma_1^2+\gamma_2^2}\delta_{-\gamma} = q^{\gamma_1+\gamma_2}\delta_{-\gamma}.\end{aligned}$$

This further leads to

$$\begin{aligned}\widehat{g}_2^3.\delta_\gamma &= r_2^{\gamma_1}s_2^{\gamma_2}\widehat{g}_2^2\widetilde{g}_2.\delta_\gamma = r_2^{2\gamma_1+\gamma_2}s_2^{-\gamma_1+\gamma_2}\widehat{g}_2\widetilde{g}_2^2.\delta_\gamma = r_2^{2\gamma_1+2\gamma_2}s_2^{-2\gamma_1}\widetilde{g}_2^3.\delta_\gamma\\ &= r_2^{2(\gamma_1+\gamma_2)}s_2^{-2\gamma_1}q^{\gamma_1+\gamma_2}.\delta_{-\gamma} = \Big(\frac{r_2^2}{s_2^2}q\Big)^{\gamma_1}(r_2^2 q)^{\gamma_2}\delta_{-\gamma}.\end{aligned} \tag{4.5}$$

(c) If, in addition, $q^2 = 1$, then $\mathrm{Aut}(\Gamma, \lambda_f) = \mathrm{Aut}(\Gamma) \cong \mathrm{GL}_2(\mathbb{Z})$ (Remark IV.8). For the involution

$$g_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we have $\mathrm{GL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \rtimes \langle g_0\rangle$, and the elements g_0, g_1, g_2 satisfy

$$g_0 g_1 g_0 = g_1^{-1} = g^3 \quad \text{and} \quad g_0 g_2 g_0 = g_2^5 = g_2^{-1}. \tag{4.6}$$

To lift g_0 to an automorphism of A_q, we first note that $q^2 = 1$ implies that

$$(g_0^* f/f)(\gamma, \gamma') = q^{\gamma_2\gamma_1' - \gamma_1\gamma_2'}, = q^{\gamma_2\gamma_1' + \gamma_1\gamma_2'},$$

which shows that each lift $\widehat{g}_0$ of g_0 is of the form $\widehat{g}_0.\delta_\gamma = r_0^{\gamma_1}s_0^{\gamma_2}q^{\gamma_1\gamma_2}\delta_{(\gamma_2,\gamma_1)}$ for some $r_0, s_0 \in \mathbb{K}^\times$. In view of

$$\widehat{g}_0^2.\delta_\gamma = r_0^{\gamma_1}s_0^{\gamma_2}q^{\gamma_1\gamma_2}\widehat{g}_0.\delta_{(\gamma_2,\gamma_1)} = r_0^{\gamma_1+\gamma_2}s_0^{\gamma_2+\gamma_1}q^{2\gamma_1\gamma_2}\delta_\gamma = (r_0 s_0)^{\gamma_1+\gamma_2}\delta_\gamma,$$

$\widehat{g}_0^2 = \mathbf{1}$ is equivalent to $r_0 s_0 = 1$. If this condition is satisfied, then $\widehat{g}_0.\delta_\gamma = r_0^{\gamma_1-\gamma_2}q^{\gamma_1\gamma_2}\delta_{(\gamma_2,\gamma_1)}$.■

Before we state the following theorem, we recall that for any split abelian extension

$$\mathbf{1} \to A \to \widehat{G} \xrightarrow{q} G \to \mathbf{1}$$

of a group G by some (abelian) G-module A, the set of all splittings is parametrized by the group

$$Z^1(G, A) = \{f\colon G \to A\colon (\forall x, y \in G)\ f(xy) = f(x) + x.f(y)\}$$

of A-valued 1-cocycles. This parametrization is obtained by choosing a homomorphic section $\sigma_0\colon G \to \widehat{G}$ and then observing that any other section $\sigma\colon G \to \widehat{G}$ is of the form $\sigma = f \cdot \sigma_0$, where $f \in Z^1(G, A)$.

Theorem IV.9. *For each element $q \in \mathbb{K}^\times$ and $\lambda(\gamma, \gamma') = q^{\gamma_1\gamma_2' - \gamma_2\gamma_1'}$ the exact sequence*

$$\mathbf{1} \to \mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times) \to \mathrm{Aut}(A_q) \to \mathrm{Aut}(\mathbb{Z}^2, \lambda) \to \mathbf{1}$$

splits. For $q^2 = 1$, the homomorphisms $\sigma\colon \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ splitting the sequence are parametrized by the abelian group

$$Z^1(\mathrm{GL}_2(\mathbb{Z}), \mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times)) \cong \{(r_0, r_1, r_2) \in (\mathbb{K}^\times)^3 \colon r_4^2 r_0^2 = r_1^2\},$$

and for $q^2 \neq 1$, the homomorphisms $\sigma\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ splitting the sequence are parametrized by

$$Z^1(\mathrm{SL}_2(\mathbb{Z}), \mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times)) \cong (\mathbb{K}^\times)^2 \times \{z \in \mathbb{K}^\times : z^2 = 1\}.$$

Proof. First we consider the case $q^2 \neq 1$, where $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \mathrm{SL}_2(\mathbb{Z})$ (Remark IV.8). We shall use the description of the lifts of g_1, g_2 given in Remark IV.8. Since $\mathrm{SL}_2(\mathbb{Z})$ is presented by the relations

$$g_1^4 = g_2^6 = \mathbf{1}, \quad g_1^2 = g_2^3$$

([Ha00, p.51]), Remark IV.8 implies that a pair of elements $(\widehat{g}_1, \widehat{g}_2)$ lifting (g_1, g_2) leads to a lift $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ if and only if $\widehat{g}_1^2 = \widehat{g}_2^3$. Comparing (4.4) and (4.5), we see that $\widehat{g}_1^2 = \widehat{g}_2^3$ is equivalent to

$$\frac{r_1}{s_1} = \frac{r_2^2}{s_2^2} q \quad \text{and} \quad r_1 s_1 = r_2^2 q,$$

which is equivalent to

$$s_1^2 = s_2^2 \quad \text{and} \quad s_1 = \frac{r_2^2 q}{r_1}, \tag{4.8}$$

These equations have the simple solution $r_1 = q, r_2 = s_1 = s_2 = 1$, showing that the action of the group $\mathrm{SL}_2(\mathbb{Z})$ on Γ lifts to an action on A_q. Moreover, for each pair (r_1, r_2), the set of all solutions is determined by the choice of sign in $s_2 := \pm s_1$, which is vacuous if $\mathrm{char}(\mathbb{K}) = 2$.

Next we consider the case $q^2 = 1$. We assume that the lift $\widehat{g}_0$ of g_0 satisfies $\widehat{g}_0^2 = \mathbf{1}$ (cf. Remark IV.8(c)). Now the relation $\widehat{g}_0 \widehat{g}_1 \widehat{g}_0 = \widehat{g}_1^{-1}$ is equivalent to $(\widehat{g}_0 \widehat{g}_1)^2 = \mathbf{1}$. We calculate

$$\widehat{g}_0 \widehat{g}_1.\delta_\gamma = r_1^{\gamma_1} s_1^{\gamma_2} q^{-\gamma_1\gamma_2} \widehat{g}_0.\delta_{(\gamma_2, -\gamma_1)} = (r_0 r_1)^{\gamma_1} (r_0 s_1)^{\gamma_2} \delta_{(-\gamma_1, \gamma_2)}$$

to get

$$(\widehat{g}_0 \widehat{g}_1)^2.\delta_\gamma = (r_0 r_1)^{\gamma_1} (r_0 s_1)^{\gamma_2} \widehat{g}_0 \widehat{g}_1.\delta_{(-\gamma_1, \gamma_2)} = (r_0 s_1)^{2\gamma_2} \delta_\gamma.$$

Hence $\widehat{g}_0 \widehat{g}_1 \widehat{g}_0 = \widehat{g}_1^{-1}$ is equivalent to

$$r_0^2 s_1^2 = 1. \tag{4.9}$$

To see when $\widehat{g}_0 \widehat{g}_2 \widehat{g}_0 = \widehat{g}_2^{-1}$ holds, we first observe that

$$\widehat{g}_2^{-1}.\delta_\gamma = r_2^{\gamma_2} s_2^{-\gamma_1-\gamma_2} q^{\binom{-\gamma_2}{2} - \gamma_2(\gamma_1+\gamma_2)} \delta_{(-\gamma_2, \gamma_1+\gamma_2)}.$$

Further

$$\begin{aligned}
\widehat{g}_0 \widehat{g}_2.\delta_\gamma &= r_2^{\gamma_1} s_2^{\gamma_2} q^{-\binom{\gamma_1}{2} - \gamma_1\gamma_2} \widehat{g}_0.\delta_{(\gamma_1+\gamma_2, -\gamma_1)} = (r_0^2 r_2)^{\gamma_1} (r_0 s_2)^{\gamma_2} q^{\binom{\gamma_1}{2} + \gamma_1\gamma_2 + (\gamma_1+\gamma_2)\gamma_1} \cdot \delta_{(-\gamma_1, \gamma_1+\gamma_2)} \\
&= (r_0^2 r_2)^{\gamma_1} (r_0 s_2)^{\gamma_2} q^{\binom{\gamma_1}{2} + \gamma_1^2} \cdot \delta_{(-\gamma_1, \gamma_1+\gamma_2)} = (r_0^2 r_2 q)^{\gamma_1} (r_0 s_2)^{\gamma_2} q^{\binom{\gamma_1}{2}} \cdot \delta_{(-\gamma_1, \gamma_1+\gamma_2)}
\end{aligned}$$

because $q^2 = 1$ implies $q^{n^2} = q^n = q^{-n}$ for each $n \in \mathbb{Z}$.

On the other hand, we have

$$\begin{aligned}
\widehat{g}_2^{-1} \widehat{g}_0.\delta_\gamma &= r_0^{\gamma_1} r_0^{-\gamma_2} q^{\gamma_1\gamma_2} \widehat{g}_2^{-1}.\delta_{(\gamma_2, \gamma_1)} = r_0^{\gamma_1} r_0^{-\gamma_2} q^{\gamma_1\gamma_2} r_2^{\gamma_1} s_2^{-\gamma_2-\gamma_1} q^{\binom{-\gamma_1}{2} - \gamma_1(\gamma_1+\gamma_2)} \delta_{(-\gamma_1, \gamma_1+\gamma_2)} \\
&= (r_0 r_2 s_2^{-1})^{\gamma_1} (r_0 s_2)^{-\gamma_2} q^{\binom{-\gamma_1}{2} - \gamma_1^2} \delta_{(-\gamma_1, \gamma_1+\gamma_2)} = (r_0 r_2 s_2^{-1})^{\gamma_1} (r_0 s_2)^{-\gamma_2} q^{-\binom{\gamma_1}{2}} \delta_{(-\gamma_1, \gamma_1+\gamma_2)} \\
&= (r_0 r_2 s_2^{-1})^{\gamma_1} (r_0 s_2)^{-\gamma_2} q^{\binom{\gamma_1}{2}} \delta_{(-\gamma_1, \gamma_1+\gamma_2)}.
\end{aligned}$$

Therefore $\widehat{g}_0\widehat{g}_2\widehat{g}_0 = \widehat{g}_2^{-1}$ is equivalent to

$$r_0 r_2 s_2^{-1} = r_0^2 r_2 q \quad \text{and } (r_0 s_2)^2 = 1,$$

which is equivalent to

$$r_0 s_2 = q, \tag{4.10}$$

because this relation implies $(r_0 s_2)^2 = q^2 = 1$.

We conclude that the numbers r_0, r_1, r_2, s_1, s_2 which determine $\widehat{g}_0, \widehat{g}_1, \widehat{g}_2$ define a lift of $\mathrm{GL}_2(\mathbb{Z})$ to $\mathrm{Aut}(A_q)$ if and only if the equations (4.8), (4.9) and (4.10) are satisfied:

$$s_1^2 = s_2^2, \quad s_1 = \frac{r_2^2 q}{r_1}, \quad r_0^2 s_1^2 = 1, \quad \text{and} \quad r_0 s_2 = q.$$

If r_0, r_1 and r_2 are given, we determine s_1 and s_2 by $s_1 := \frac{r_2^2 q}{r_1}$ and $s_2 := \frac{q}{r_0}$. Then

$$\frac{s_1^2}{s_2^2} = \frac{r_2^4 r_0^2}{r_1^2} = r_0^2 s_1^2,$$

so that we obtain only the relation $r_4^2 r_0^2 = r_1^2$ for r_0, r_1, r_2. This completes the proof. ■

Remark IV.10. (a) From the proof of the preceding theorem, we see that we obtain the particularly simple solution

$$r_0 = r_1 = r_2 = 1, \quad s_1 = s_2 = q.$$

(b) For $\mathrm{char}\,\mathbb{K} = 2$ the equation $q^2 = 1$ has the unique solution $q = 1$, so that $A_q \cong \mathbb{K}[\mathbb{Z}^2]$, and the action of $\mathrm{GL}_2(\mathbb{Z})$ has a canonical lift to an action on A_q. ■

Problem IV.1. Does the sequence (4.3) always split? We have seen above, that this is true for $\Gamma = \mathbb{Z}^2$. If the answer is no, it would be of some interest to understand the cohomology groups

$$H^2(\mathrm{Aut}(\Gamma)_{[f]}, \mathrm{Hom}(\Gamma, \mathbb{K}^\times))$$

parametrizing the possible abelian extensions of $\mathrm{Aut}(\Gamma)_{[f]}$ by the module $\mathrm{Hom}(\Gamma, \mathbb{K}^\times)$. ■

Problem IV.2. Let $\lambda \in \mathrm{Alt}^2(\mathbb{Z}^n, Z)$, where Z is a cyclic group. Determine the structure of the group $\mathrm{Aut}(\mathbb{Z}^n, \lambda)$. It should have a semidirect product structure, where the normal subgroup is something like a Heisenberg group and the quotient is the automorphism group of $\mathbb{Z}^n/\mathrm{rad}(\lambda)$, endowed with the induced non-degenerate form. Can this group be described in a conventient way by generators and relations? Maybe the results in [Is03] can be used to deal with degenerate cocycles. ■

A. The group of units if Γ is torsion free

The following result is used in [OP95, Lemma 3.1] without reference. Here we provide a detailed proof.

Proposition A.1. *If the group Γ is torsion free and A a Γ-quantum torus, then $A^\times = A_h^\times$, i.e., each unit of A is graded.*

Proof. Let $a \in A^\times$ be a unit and write $a = \sum_\gamma a_\gamma \delta_\gamma$ in terms of some graded basis. We do the same with its inverse $a^{-1} = \sum_\gamma (a^{-1})_\gamma \delta_\gamma$, and observe that the set

$$\mathrm{supp}(a) := \{\gamma \in \Gamma : a_\gamma \neq 0\}$$

is finite. The same holds for $\mathrm{supp}(a^{-1})$, so that both sets generate a free subgroup F of Γ. Then $A_F := \mathrm{span}\{\delta_\gamma\colon \gamma \in F\}$ is an F-quantum torus with $a \in A_F^\times$. We may therefore assume that $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}_0$.

We prove by induction on $k \in \{0,\ldots,d\}$ that the subalgebra

$$A_k := \mathrm{span}\{\delta_\gamma\colon \gamma \in \mathbb{Z}^k \times \{0\}\}$$

has no zero-divisors and that all its units are homogeneous. This holds trivially for $k = 0$.

Let $u_i := \delta_{e_i}$, where $e_1,\ldots,e_d$ is the canonical basis of $\mathbb{Z}^d$. We write $0 \neq x \in A$ as a finite sum $\sum_{k=k_0}^{k_1} x_k u_d^k$ with $x_k \in A_{d-1}$ and x_{k_0} and x_{k_1} non-zero. Likewise we write $0 \neq y \in A$ as $\sum_{m=m_0}^{m_1} y_m u_d^m$ with $y_m \in A_{d-1}$ and y_{m_0} and y_{m_1} non-zero. Then the lowest degree term with respect to u_d in xy is

$$x_{k_0} u_d^{k_0} y_{m_0} u_d^{m_0} = x_{k_0}\big(u_d^{k_0} y_{m_0} u_d^{-k_0}\big) u_d^{k_0+m_0},$$

and the induction hypothesis implies $x_{k_0} u_d^{k_0} y_{m_0} u_d^{-k_0} \neq 0$ because conjugation with u_d preserves the subalgebra A_{d-1}. This implies that $xy \neq 0$.

Now assume that $x \in A$ is a unit and $y = x^{-1}$. Since A_{d-1} has no zero-divisors,

$$x_{k_0} u_d^{k_0} y_{m_0} u_d^{-k_0} \in A_{d-1} \setminus \{0\}$$

leads to $k_0 + m_0 = 0$. A similar consideration for the highest order term implies $k_1 + m_1 = 0$, which leads to $k_0 = k_1$ and $m_0 = m_1$. Now we can argue by induction. ■

Corollary A.2. ([OP95, Lemma 3.1]) *If the group Γ is torsion free, then each automorphism of A is graded, i.e.,* $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$. (cf. Def. IV.1) ■

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