Reconstruction of input signals in time-varying filters

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Abstract

We consider the problem of reconstruction of input signals u from output signals of time-varying filters of the form

$$(Au)(x) = \sum_{j \in \mathbb{Z}} a_j(x) u(x-j), \quad x \in \mathbb{Z},$$

under the assumption that $\sum_{j \in \mathbb{Z}} \|a_j\|_\infty < \infty$. The proposed algorithm of reconstruction of signals is based on the theory of band-dominated and pseudodifference operators as presented in the recent monograph [12] and on the finite sections method. The following classes of filters are considered this paper: slowly time-varying filters, perturbations of periodic time-varying filters, causal time-varying filters, and finite filters acting on signals with a finite number of values.

1 Introduction

Signal processing. We start with recalling some basic definitions and facts from signal processing theory. Standard references to this field are [6, 7, 1, 16], for instance.

A *digital complex signal* (DCS for short) is a two-sided sequence $u = \{u(x) : x \in \mathbb{Z}\}$ of complex numbers. In what follows we consider DCS with finite energy, that is we suppose that the DCS u belongs to the Hilbert space l^2 of complex-valued functions on $\mathbb{Z}$, provided with the scalar product

$$\langle u, v \rangle := \sum_{x \in \mathbb{Z}} u(x) \overline{v(x)}$$

and with associated norm

$$\|u\|_2 = \left(\sum_{x\in\mathbb{Z}} |u(x)|^2 \right)^{1/2} < \infty.$$

A *digital linear filter* is a linear mapping which transforms a DCS u, called the *input signal*, to a DCS $v = Au$, the *output signal*. We will exclusively consider linear filters, i.e., filters which act as a bounded linear operator on l^2.

An interesting but still easy to analyze class of digital filters is constituted by the *time-invariant* filters. They are characterized by their invariance with respect to shifts. More precisely, a digital filter A is said to be invariant with respect to shifts if

$$V_h A = A V_h \quad \text{for every } h \in \mathbb{Z},$$

where the shift operator $V_h : l^2 \to l^2$ acts via $(V_h u)(x) := u(x-h)$. One can show that, for each time-invariant digital filter A, there is a uniquely determined bounded sequence $(a_k)_{k\in\mathbb{Z}}$ of complex numbers, called the *coefficients* of the filter, such that

$$(Au)(x) = \sum_{k\in\mathbb{Z}} a_k u(x-k) \quad \text{for } x \in \mathbb{Z}. \tag{1}$$

In what follows we will suppose that the Wiener condition

$$\sum_{k\in\mathbb{Z}} |a_k| < \infty \tag{2}$$

holds. For time-invariant filters (1), the problem of reconstruction of the input signal from a given output is solved by means of the discrete Fourier transform

$$\hat{u}(\xi) = (Fu)(\xi) := \sum_{x\in\mathbb{Z}} u_k e^{-ix\xi}, \quad \xi \in \mathbb{R}.$$

Note that for $u \in l^2$, the Fourier transform $\hat{u}$ is a 2π-periodic function with

$$\int_0^{2\pi} |\hat{u}(\xi)|^2 \, d\xi < \infty,$$

that the inverse Fourier transform F^{-1} is given by

$$(F^{-1}u)(x) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) e^{ix\xi} \, d\xi,$$

and that Parseval's equality holds,

$$\int_0^{2\pi} |\hat{u}(\xi)|^2 \, d\xi = 2\pi \sum_{x \in \mathbb{Z}} |u(x)|^2.$$

If A is a time-invariant filter as in (1), set $\hat{a}(\xi) := \sum_{x \in \mathbb{Z}} a_k e^{-ix\xi}$ for $\xi \in [0, 2\pi]$ and suppose that

$$\inf_{\xi \in [0, 2\pi]} |\hat{a}(\xi)| > 0. \tag{3}$$

Then, indeed, the input signal u is obtained from the output $v = Au$ via

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\hat{v}(\xi)}{\hat{a}(\xi)} \, e^{ix\xi} \, d\xi, \quad x \in \mathbb{Z}. \tag{4}$$

A natural generalization of time-invariant filters are the *time-varying filters*. They are described by linear operators A of the form

$$(Au)(x) = \sum_{k \in \mathbb{Z}} a_k(x) u(x-k) = \sum_{k \in \mathbb{Z}} a_k(x)(V_k u)(x), \quad x \in \mathbb{Z}, \tag{5}$$

where now the coefficients a_k are sequences in l^∞, the Banach space of all bounded complex-valued functions on $\mathbb{Z}$ with norm

$$\|a\|_\infty := \sup_{x \in \mathbb{Z}} |a(x)|.$$

For time-varying filters, we will always suppose the Wiener type condition

$$\sum_{k \in \mathbb{Z}} \|a_k\|_\infty < \infty \tag{6}$$

which obviously generalizes (2).

Evidently, for time-varying filters, the problem of reconstruction of the input signal from the output signal is much more involved than for time-invariant filters. In particular, there is no explicit formula for the dependence of the input from the output signal. Thus, it is both necessary and natural to consider numerical methods to determine the input approximately. The present paper is devoted to this circle of problems.

Band-dominated operators. An operator of the form

$$(Au)(x) = \sum_{k=-N}^{N} a_k(x)(V_k u)(x), \quad x \in \mathbb{Z},$$

with coefficients $a_k \in l^\infty$ is also called a *band operator*. This name is motivated by the fact that the matrix of A with respect to the standard basis of l^2 has a finite number of non-zero diagonals only. The closure of the set of all band operators in the space $\mathcal{L}(l^2)$ of all bounded linear operators on l^2 is denoted by $\mathcal{A}_2$. The operators in $\mathcal{A}_2$ are called *band-dominated*. It is easy to see that the time-varying filters (5) with condition (6) and, in particular, the time-invariant filters (1) with condition (2) are band-dominated operators.

The Fredholm theory of band-dominated operators has been intensively studied in the papers [10, 11, 9]. A comprehensive account on this topic can be found in the recently published monograph [12]. The stability of certain projection methods (in particular, of the finite sections method) for the approximate solution of the equation $Au = v$ where A is a band-dominated operator is studied in [10, 11, 13] and in [12].

If each output $v \in l^2$ is generated by a uniquely determined input, then the operator A is invertible. Thus, the solution of the reconstruction problem for time-variable filters A requires conditions for the invertibility of A on the space l^2. Moreover, the invertibility of A is also a necessary condition for the applicability of projection methods for the solution of the equation $Au = v$, that is for the numerical reconstruction of input signals.

About this paper. The main aims of the paper are:

(a) to consider classes of band-dominated operators which are important for signal processing, and to give effective conditions of their invertibility,

(b) to derive conditions for the applicability of projection methods for the approximate solution of the problem of reconstructing input signals.

The paper is organized as follows. In Section 2 we recall some auxiliary material from the theory of band-dominated operators. All cited facts can be found in [12]. In Section 3 we embark upon stable approximation procedures in order to reconstruct input signals of time-variable filters. In particular, we will derive conditions for the applicability of the finite sections method. Our approach to analyze these methods is is based on the calculus of pseudodifference operators developed in [8], see also [12], Chapter 5. Special emphasis is paid to periodic time-variable filters and to slowly varying perturbations of periodic filters. Section 4 is devoted to the problem of reconstruction of input signals in causal time-varying filters, and in the concluding Section 5 we consider the reconstruction of digital signals with a finite number of values for time-variable filters. For these results, we have to employ the calculus of pseudodifferential operators on the finite commutative (cyclic) group $\mathbb{Z}/d\mathbb{Z}$ where $d \in \mathbb{N}$.

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2 Auxiliary facts from the theory of band-dominated operators

2.1 Fredholm theory and index of band-dominated operators in the Wiener algebra W

Let $(a_\alpha)_{\alpha\in\mathbb{Z}}$ be a sequence of functions in l^∞ satisfying the Wiener condition

$$\sum_{\alpha\in\mathbb{Z}} \|a_\alpha\|_\infty < \infty. \tag{7}$$

Then the series $\sum_{\alpha\in\mathbb{Z}} a_\alpha V_\alpha$ converges in the norm of $\mathcal{L}(l^2)$ and, thus, it defines a bounded linear operator on l^2. We write W for the set of all operators obtained in this way. Provided with the usual operations of the addition and multiplication of operators and with the norm

$$\|A\|_W := \sum_{\alpha\in\mathbb{Z}} \|a_\alpha\|_\infty,$$

the set W becomes a Banach algebra, the so-called *Wiener algebra*. By construction, $W \subset \mathcal{L}(l^2)$. Moreover, it is easy to see that $W \subset \mathcal{A}_2$, the algebra of the band-dominated operators. One peculiarity of the algebra W is its property of inverse closedness, which is stated precisely as follows.

Proposition 1 . *Let the operator $A \in W$ be invertible as an operator on l^2. Then $A^{-1} \in W$.*

A bounded linear operator A acting on a Banach space X is called a *Fredholm operator* if both its kernel $\ker A := \{x \in X : Ax = 0\}$ and its cokernel $\operatorname{coker} A := X/A(X)$ are finite-dimensional linear spaces. Fredholmness of A means that the equations $Ax = 0$ and $A^*y = 0$ have only finitely many linearly independent solutions in the spaces X and X^*, respectively, and that the equation $Ax = f$ is solvable if and only if $y_j(f) = 0$ for $j = 1, \ldots, m$, where $\{y_1, \ldots, y_m\}$ is a basis of $\ker A^*$. The integer

$$\operatorname{ind} A := \dim \ker A - \dim \ker A^*$$

is called the *index* of the Fredholm operator A.

Let $h : \mathbb{N} \to \mathbb{Z}$ be a sequence which tends to infinity, and let

$$A = \sum_{\alpha \in \mathbb{Z}} a_\alpha V_\alpha \in W.$$

By using a Cantor diagonal argument, one can prove that the sequence h possesses a subsequence g such that the limits

$$\lim_{k \to \infty} a_\alpha(x + g_k) =: a_\alpha^g(x)$$

exist for all integers x and α. The operator

$$A^g := \sum_{\alpha \in \mathbb{Z}} a_\alpha^g V_\alpha$$

is called the *limit operator* of A defined by (or with respect to) the sequence g. It follows from $\|a_\alpha^g\|_\infty \leq \|a_\alpha\|_\infty$ that all limit operators of A belong to the Wiener algebra W again. Note also that A^g is the strong limit on l^2 of the operator-valued sequence $(V_{-g_k} A V_{g_k})_{k \in \mathbb{N}}$ for $k \to \infty$. We denote the set of all limit operators of $A \in W$ by $\sigma_{op}(A)$. Let furthermore $\sigma_{op}^{\pm}(A)$ refer to the set of all limit operators of A which correspond to sequences h tending to $\pm\infty$. Evidently, $\sigma_{op}(A) = \sigma_{op}^{+}(A) \cup \sigma_{op}^{-}(A)$.

The following theorem summarizes criteria for the Fredholmness of operators in W.

Theorem 2 *Let $A \in W$. Then the following assertions are equivalent:*

(a) A is a Fredholm operator on l^2.

(b) All limit operators of A are invertible on l^2.

(c) All limit operators of A are invertible on l^2, and the norms of their inverses are uniformly bounded.

Let P_+ denote the operator of multiplication by the characteristic function of the set $\mathbb{N}_0$ of the non-negative integers, and set $P_- := I - P_+$. Note that $P_\pm$ are orthogonal projections on l^2. We denote their ranges by $l^2_\pm := P_\pm(l^2)$.

Theorem 3 *Let the operator $A \in W$ satisfy one of the conditions of Theorem 2. Then*

(*a*) *the operators $P_+AP_+ : l^2_+ \to l^2_+$ and $P_-AP_- : l^2_- \to l^2_-$ are Fredholm;*

(*b*) *for arbitrary limit operators $A_h \in \sigma^+_{op}(A)$ and $A_g \in \sigma^-_{op}(A)$, the operators $P_+A_hP_+ : l^2_+ \to l^2_+$ and $P_-A_gP_- : l^2_- \to l^2_-$ are Fredholm;*

(*c*) *one has*

$$\begin{aligned} \operatorname{ind} A &= \operatorname{ind}(P_+AP_+) + \operatorname{ind}(P_-AP_-) \\ &= \operatorname{ind}(P_+A_hP_+) + \operatorname{ind}(P_-A_gP_-). \end{aligned}$$

Assertion (*a*) and the first equality in (*c*) follow easily from the (evident) compactness of the operators P_+AP_- and P_-AP_+ which implies the compactness of the operators

$$A - (P_+AP_+ + P_-)(P_-AP_- + P_+) \quad \text{and} \quad A - (P_-AP_- + P_+)(P_+AP_+ + P_-).$$

If A is Fredholm, then each limit operator of A is invertible by Theorem 2. Thus, assertion (*b*) is an immediate consequence of (*a*). The only serious result in the preceding theorem is the second equality in (*c*). This (in our eyes) surprising identity has been derived in [9].

A function $a \in l^\infty$ is called *slowly oscillating at infinity* if

$$\lim_{x\to\infty} (a(x+k) - a(x)) = 0$$

for every $k \in \mathbb{Z}$. We denote the subspace of l^∞ of all slowly oscillating at infinity functions by SO, and we write W^{SO} for the subalgebra of W which consists of all operators $A = \sum_{\alpha\in\mathbb{Z}} a_\alpha V_\alpha$ with coefficients a_α in SO.

It is a remarkable property of operators A in W^{SO} that their limit operators A^h are necessarily of the form

$$A^h = \sum_{\alpha\in\mathbb{Z}} a^h_\alpha V_\alpha \tag{8}$$

with *constant* coefficients $a^h_\alpha \in \mathbb{C}$. One easily checks that the operator (8) is unitarily equivalent to the operator of multiplication by the function

$$\hat{A}_h(\xi) := \sum_{\alpha\in\mathbb{Z}} a^h_\alpha e^{-i\xi\alpha}, \quad \xi \in [0,\, 2\pi].$$

Thus, the operator (8) is invertible on l^2 is and only if

$$\inf_{\xi \in [0,\, 2\pi]} |\hat{A}_h(\xi)| > 0. \tag{9}$$

Moreover, condition (9) implies that the operators $P_\pm A_h P_\pm : l^2_\pm \to l^2_\pm$ are Fredholm and that

$$\text{ind}\,(P_+ A^h P_+) = -\text{wind}\,(\hat{A}_h), \quad \text{ind}\,(P_- A^h P_-) = \text{wind}\,(\hat{A}_h)$$

where the winding number of the 2π-periodic and non-vanishing function $\hat{A}_h$ is defined by

$$\text{wind}\,(\hat{A}_h) := \frac{1}{2\pi} [\arg \hat{A}_h(\xi)]_{\xi=0}^{2\pi}.$$

Thus, specifying Theorems 2 and 3 to operators with slowly oscillating coefficients yields the following.

Theorem 4 *Let $A \in W^{SO}$. Then $A : l^2 \to l^2$ is a Fredholm operator if and only if condition* (9) *holds for every limit operator A_h. In this case, the Fredholm index of the A is given by*

$$\text{ind}\, A = -\text{wind}\,(\hat{A}_h / \hat{A}_g),$$

where A_h is an arbitrary operator in $\sigma^+_{op}(A)$ and A_g an arbitrary operator in $\sigma^-_{op}(A)$.

2.2 The finite sections method

Let $A \in \mathcal{L}(l^2)$ be an invertible operator. Then the equation $Au = v$ has a unique solution u for every function $v \in l^2$. Let P_N be the operator of multiplication by the characteristic function of the discrete segment $[-N,\, N]_{\mathbb{Z}} := \{k \in \mathbb{Z} : |k| \leq N\}$. Together with the equation $Au = v$, we consider the sequence of its finite sections

$$P_N A P_N u_N = P_N v, \quad N \in \mathbb{N}, \tag{10}$$

the solutions u_N of which are sought in $\text{im}\, P_N$. The crucial questions are whether the equations (10) possess unique solutions for sufficiently large N and whether the sequence (u_N) of these solutions converges (in the norm of l^2) to the solution u of $Au = v$. If the answer to both questions is *yes*, then the finite sections method is said to be *applicable* to the operator A.

It is well-known (see for instance [2, 3, 4, 12]) that the finite sections method is applicable to the operator $A : l^2 \to l^2$ if and only if this operator is invertible and if the sequence $(P_N A P_N)_{N \in \mathbb{N}}$ is stable. The latter means that the operators $P_N A P_N : \operatorname{im} P_N \to \operatorname{im} P_N$ are invertible for sufficiently large N and that the norms of their inverses $(P_N A P_N)^{-1}$ are uniformly bounded. This fact is usually referred to as Polski's theorem.

The following theorem provides a necessary and sufficient criterion for the applicability of the finite sections method to operators in W.

Theorem 5 *Consider $A \in W$ as an operator on l^2. The finite sections method is applicable to A if and only if*

(a) the operator A is invertible on l^2;

(b) for every limit operator $A_h \in \sigma_{op}^{+}(A)$, the operators $P_- A_h P_- : l_-^2 \to l_-^2$ are invertible;

(c) for every limit operator $A_h \in \sigma_{op}^{-}(A)$, the operators $P_+ A_h P_+ : l_+^2 \to l_+^2$ are invertible.

Theorem 5 takes a more simple form for operators in the algebra W^{SO}.

Theorem 6 *Let $A \in W^{SO}$. The finite sections method is applicable to A if and only if:*

(a) the operator A is invertible on l^2, and

(b) the plus-index $\operatorname{ind}(P_+ A P_+)$ *of A is equal to zero.*

Note that

$$\operatorname{ind}(P_+ A P_+) = -\operatorname{wind}(\hat{A}_h)$$

where A_h is an arbitrary limit operator of A in $\sigma_{op}^{+}(A)$. Thus, condition (b) in Theorem 6 can be effectively verified in many instances. For proofs of Theorems 5 and 6, consult [5, 13].

3 Reconstruction of signals in time-varying filters

3.1 Auxiliary facts from the theory of pseudodifference operators

We still have to recall some facts about pseudodifferential operators. Standard references to the theory of pseudodifferential operators are [14, 15], for instance.

Definition 7 *A function* $a : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ *belongs to the class* $\mathcal{S}$ *if it is* 2π*-periodic with respect to the second (real) variable and if*

$$|a|_N := \sup_{(x,\xi)\in\mathbb{Z}\times[0,2\pi],\, m\le N} \left| \frac{d^m a(x,\,\xi)}{d\xi^m} \right| < \infty \tag{11}$$

for all $N \ge 0$*. With each function* $a \in \mathcal{S}$*, there is associated a* pseudodifference operator $A = Op(a)$ *which acts on* l^2 *via*

$$(Au)(x) = \frac{1}{2\pi} \sum_{y\in\mathbb{Z}} \int_0^{2\pi} a(x,\,\xi)\, \hat{u}(\xi)\, e^{ix\xi}\, d\xi, \quad x \in \mathbb{Z}. \tag{12}$$

The class of all operators of this form is denoted by $OP\mathcal{S}$*.*

The operator $A = Op(a)$ is also called the *pseudodifference operator generated by* a, and the function $\sigma_A := a$ is called the *symbol* of this operator. Pseudodifference operators can be considered as discrete analogs of the classical pseudodifferential operators on $\mathbb{R}$; in fact, they *are* pseudodifferential operators related with the discrete group $\mathbb{Z}$.

Note that every time-varying filter A of the form

$$(Au)(x) = \sum_{j\in\mathbb{Z}} a_j(x) u(x-j), \quad x \in \mathbb{Z}, \tag{13}$$

where the $a_j \in l^\infty$ satisfy

$$\sum_{j\in\mathbb{Z}} |j|^k \|a_j\|_\infty < \infty \quad \text{for each } k \in \mathbb{N}_0 \tag{14}$$

is a pseudodifference operator with symbol $a \in \mathcal{S}$ defined by

$$a(x,\,\xi) := \sum_{j\in\mathbb{Z}} a_j(x) e^{-ij\xi}.$$

It is easy to prove that an arbitrary operator $A \in OP\mathcal{S}$ has a representation (13) such that condition (14) is satisfied.

The Fredholm properties of pseudodifference operators have been studied in [8] in terms of their limit operators; see also [12], Chapter 5. We recall some facts from [8] which will be used below.

Proposition 8 *Let $a \in \mathcal{S}$. Then $A := Op(a)$ is a bounded operator on l^2, and there is a constant C independent of a such that*

$$\|Au\|_2 \le C|a|_2 \|u\|_2 \quad \textit{for every}\, u \in l^2.$$

Proposition 9 *Let A and B be pseudodifference operators with symbols a and b in $\mathcal{S}$. Then AB is a pseudodifference operator in $OP\mathcal{S}$, and $AB = Op(c)$ where*

$$c(x,\, \xi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_0^{2\pi} a(x,\, \xi + \eta)\, b(x + y,\, \xi)\, e^{-iy\eta}\, d\eta. \tag{15}$$

Note that the series in (15) does not converge in the common sense. It converges after a regularization by means of integration by parts,

$$c(x,\, \xi) = \frac{1}{2\pi} \sum_{y \in \mathbb{Z}} (1 + y^2)^{-1} \int_0^{2\pi} \left[(1 + \frac{d^2}{d\eta^2})\, a(x,\, \xi + \eta) \right] b(x + y,\, \xi)\, e^{-iy\eta}\, d\eta.$$

This formula implies the estimate

$$|c|_N \le C|a|_{N+2}\, |b|_N \quad \text{for } N \in \mathbb{N}_0 \tag{16}$$

with a certain constant C independent of a and b.

It has been proved in [8] that a pseudodifference operator $A \in OP\mathcal{S}$ is a bounded operator on l^∞ and that its symbol σ_A can be obtained by

$$\sigma_A(x,\, \xi) = e^{-ix\xi} A(e^{ix\xi}). \tag{17}$$

Thus, a time-varying filter A can be completely reconstructed if its outputs are known for all input signals of the form $x \mapsto e^{ix\xi}$ with $\xi \in [0,\, 2\pi]$.

3.2 Slowly time-varying filters

Let $a \in \mathcal{S}$ and $k \in \mathbb{N}_0$. We introduce the *oscillation* $\omega_k^1(a)$ of a with respect to the discrete (first) variable x by

$$\omega_k^1(a) := \sup_{x \in \mathbb{Z},\, y \in \mathbb{Z} \setminus \{0\},\, \xi \in [0,\, 2\pi]} \sum_{j=0}^{k} \left| \partial_\xi^j\, a(x + y,\, \xi) - \partial_\xi^j\, a(x,\, \xi) \right|\, |y|^{-1}.$$

Theorem 10 *Let $a \in \mathcal{S}$ with*

$$\inf_{x\in\mathbb{Z},\xi\in[0,2\pi]} |a(x,\,\xi)| > 0. \tag{18}$$

It the oscillation $\omega_2^1(a)$ is small enough, then the operator $Op(a)$ is invertible.

Proof. Condition (18) implies that $a^{-1} \in \mathcal{S}$. Hence, the operator $B := Op(a^{-1})$ is well defined, and it belongs to $OP\mathcal{S}$. Due to (15), the operator BA is equal to $Op(c)$ with c given by

$$\begin{aligned} c(x,\,\xi) &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_0^{2\pi} a^{-1}(x,\,\xi+\eta)\,a(x+y,\,\xi)\,e^{-iy\eta}\,d\eta \\ &\quad +\frac{a(x,\,\xi)}{2\pi}\sum_{y\in\mathbb{Z}}\int_0^{2\pi} a^{-1}(x,\,\xi+\eta)\,e^{-iy\eta}\,d\eta \\ &\quad +\frac{1}{2\pi}\sum_{y\in\mathbb{Z}}\int_0^{2\pi} a^{-1}(x,\,\xi+\eta)\,(a(x+y,\,\xi)-a(x,\,\xi))\,e^{-iy\eta}\,d\eta \\ &=: 1 + r(x,\,\xi) \end{aligned}$$

where

$$r(x,\,\xi) = \frac{1}{2\pi}\sum_{y\in\mathbb{Z}}\int_0^{2\pi} a^{-1}(x,\,\xi+\eta)\,(a(x+y,\,\xi)-a(x,\,\xi))\,e^{-iy\eta}\,d\eta \tag{19}$$

satisfies the estimate

$$|r(x,\,\xi)| \le \frac{\omega_0^1(a)}{2\pi}\sum_{y\in\mathbb{Z}}\frac{|y|}{(1+y^2)^2}\int_0^{2\pi}\int_0^1\left|\left(1+\frac{d^2}{d\eta^2}\right)^2 a^{-1}(x,\,\xi+\eta)\right|\,d\eta\,d\theta.$$

This estimate implies that

$$|r(x,\,\xi)| \le C\omega_0^1(a)|a|_4$$

with a constant C independent of a. In the same way one can prove that

$$|r|_2 \le C\omega_2^1(a)|a|_6.$$

It follows from Proposition 8 that $\|Op(r)\|_{\mathcal{L}(l^2)} < 1$ if the oscillation $\omega_2^1(a)$ is sufficiently small. Thus, by Neumann series, the operator

$$A^{-1} := (I + Op(r))^{-1}Op(a^{-1}) \tag{20}$$

is a left inverse for $A = Op(a)$. In the same way one gets that A is invertible form the right-hand side for sufficiently small $\omega_2^1(a)$. This proves the assertion. ■

Hence, we have reduced the problem of reconstruction of the input signal $u \in l^2$ from the output $Au = v \in l^2$ to the solution of the equation

$$u + Op(r)u = Op(a^{-1})v. \tag{21}$$

The unique solution of this equation can be obtained by successive approximations: Set $u_0 := 0$ and define

$$u_{n+1} := -Op(r)u_n + Op(a^{-1})v \quad \text{for } n \in \mathbb{N}_0 .$$

Then the sequence (u_n) tends to the input u in the norm of l^2. The reconstructed signal u is *stable in* l^2 in the sense that small (with respect to the norm in l^2) variations of the output signal correspond to small variations of the input signal. Formula (20) is an extension to slowly time-varying filters of the well-known formula (4) for the reconstruction of input signals for time-invariant filters.

Let us next consider the applicability of the finite sections method to the reconstruction of signals in filters $A \in OP\mathcal{S}$ which satisfy the conditions of Theorem 10 and which are slowly oscillating at infinity.

Theorem 11 *Let $A \in OP\mathcal{S} \cap W^{SO}$ be an operator for which the conditions of Theorem* 10 *hold. Moreover, let* $\operatorname{wind}(\hat{A}_h) = 0$ *for a certain limit operator $A_h \in \sigma_{op}^+(A)$ (and, consequently, for all limit operators of A in $\sigma_{op}^+(A)$). Then the finite sections method*

$$P_N A P_N u_N = P_N v, \quad N \in \mathbb{N}_0 ,$$

for the reconstruction of the input signal $u \in l^2$ from the output signal $v = Au$ is stable.

This follows immediately from Theorem 6 and from the invertibility of A which is guaranteed by Theorem 10.

3.3 Periodic time-varying filters

Let $d \in \mathbb{N}$. We consider d-periodic time-varying filters, that is, filters in W of the form

$$(Au)(x) = \sum_{j \in \mathbb{Z}} a_j(x)\, u(x - j), \quad x \in \mathbb{Z}, \tag{22}$$

with d-periodic coefficients $a_j \in l^\infty$,

$$a_j(x+d) = a_j(x) \quad \text{for every } x \in \mathbb{Z}.$$

Thus, d-periodic time-varying filters can be viewed of as band-dominated operators in W with periodic coefficients. These operators form a closed subalgebra W_d^{per} of W. Let further l_d^2 denote the Hilbert space of all vector-valued functions $u = (u_1, u_2, \ldots, u_d)$ on $\mathbb{Z}$ with values in $\mathbb{C}^d$, provided with the norm

$$\|u\|_{l_d^2} := \left(\sum_{j=1}^{d} \|u_j\|_2 \right)^{1/2}.$$

Consider the mapping

$$T_d : l^2 \to l_d^2, \quad u \mapsto (u_1, u_2, \ldots, u_d)$$

where

$$u_j(y) := u(dy + j - 1) \quad \text{for } y \in \mathbb{Z} \text{ and } 1 \le j \le d.$$

For $1 \le j \le d$, let $\mathbb{Z}_j := d\mathbb{Z} + j - 1$ denote the jth residue class modulo d. Since $\mathbb{Z}_j \cap \mathbb{Z}_k = \emptyset$ for $j \ne k$ and $\mathbb{Z} = \cup_{j=1}^d \mathbb{Z}_j$, the operator $T_d : l^2 \to l_d^2$ is a bijective isometry, i.e., a unitary operator.

We examine the operator $T_d B T_d^{-1}$ for several operators B on l^2. If $B = aI$ is the operator of multiplication by a d-periodic function a, then $T_d a T_d^{-1}$ is the operator of multiplication by the constant diagonal matrix $\operatorname{diag}(a(1), \ldots, a(d))$. Next let $B = V_{-1}$. Writing $T_d u =: (u_1, \ldots, u_d)$ for $u \in l^2$, one gets

$$T_d V_{-1} u = (u_2, u_3, \ldots, u_d, V_{-1} u_1).$$

Thus, the operator $T_d V_{-1} T_d^{-1}$ acts on l_d^2 as the matrix operator

$$T_d V^{-1} T_d^{-1} = \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ V_{-1} & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Consequently, if A is the operator given by (22), then the operator $T_d A T_d^{-1}$

acts as

$$T_d A T_d^{-1} = \sum_{j \in \mathbb{Z}} \operatorname{diag}(a_j(1), \ldots, a_j(d)) \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ V_{-1} & \cdot & \cdot & \cdot & 0 \end{pmatrix}^{-j}.$$

The discrete Fourier transform maps this operator to the operator of multiplication by the 2π-periodic $d \times d$-matrix valued function $\mathcal{A} : \mathbb{R} \to \mathcal{L}(\mathbb{C}^d)$ given by

$$\mathcal{A}(\xi) := \sum_{j \in \mathbb{Z}} \operatorname{diag}(a_j(1), \ldots, a_j(d)) \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ e^{i\xi} & \cdot & \cdot & \cdot & 0 \end{pmatrix}^{-j}.$$

So we finally arrive at the following theorem.

Theorem 12 *Let $A = Op(a) \in W_d^{per}$. Then A is an invertible operator on l^2 if and only if*

$$\det \mathcal{A}(\xi) \neq 0 \quad \textit{for each } \xi \in [0, 2\pi]. \tag{23}$$

Thus, for a periodic filter A, there is an evident and effective way to reconstruct the input signal u from a given output v by

$$u = T_d^{-1} F_{\xi \to y}^{-1} \mathcal{A}^{-1}(\xi) F_{y \to \xi} T_d v.$$

3.4 Slowly oscillating perturbations of periodic time-varying filters

Now we consider filters of the form (22) where the coefficients a_j depend on two variables x and y in such a way that the dependence on x is d-periodic whereas that on y is slowly oscillating. More precisely, let

$$(Au)(x) = \sum_{j \in \mathbb{Z}} a_j(x)\, u(x - j), \quad x \in \mathbb{Z}, \tag{24}$$

with coefficients of the form

$$a_j(x) := \tilde{a}_j(x, y)|_{y=x}, \quad x \in \mathbb{Z},$$

where each $\tilde{a}_j : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is a bounded function with $\tilde{a}_j(x+d,\, y) = \tilde{a}_j(x,\, y)$ for each pair $x,\, y \in \mathbb{Z}$. Moreover, we assume that

$$\sum_{j \in \mathbb{Z}} |j|^k \|a_j\|_\infty < \infty$$

for every $k \in \mathbb{N}_0$. We associate the symbol

$$\tilde{a} : \mathbb{Z} \times \mathbb{Z} \times [0,\, 2\pi] \to \mathbb{C}, \quad (x,\, y,\, \xi) \mapsto \sum_{j \in \mathbb{Z}} \tilde{a}_j(x,\, y)\, e^{ix\xi}$$

with the operator A in (24), and we denote the oscillation of a with respect to the (second) variable y by $\omega_k^2(a)$, that is,

$$\omega_k^2(a) := \sup_{x,\, y \in \mathbb{Z},\, z \in \mathbb{Z}\setminus\{0\},\, \xi \in [0,\, 2\pi]} \sum_{j=0}^{k} \left| \partial_\xi^j a(x,\, y+z,\, \xi) - \partial_\xi^j a(x,\, y,\, \xi) \right| \, |z|^{-1}.$$

Note that the operator $T_d A T_d^{-1}$ has the matrix representation

$$\sum_{j \in \mathbb{Z}} \mathrm{diag}\,(a_j(1,\, x),\, \ldots,\, a_j(d,\, x+d)) \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ V_{-1} & \cdot & \cdot & \cdot & 0 \end{pmatrix}^{-j} \tag{25}$$

and that $T_d A T_d^{-1}$ is a matrix pseudodifference operator with symbol $\mathcal{A}$ given by

$$\mathcal{A}(x,\, \xi) = \sum_{j \in \mathbb{Z}} \mathrm{diag}\,(a_j(1,\, x),\, \ldots,\, a_j(d,\, x+d)) \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ e^{-i\xi} & \cdot & \cdot & \cdot & 0 \end{pmatrix}^{-j}.$$

A slight modification of Theorem 10 for the case of matrix-valued pseudodifference operators yields the following result.

Theorem 13 *Let the symbol $\mathcal{A}$ of A satisfy*

$$\inf_{x \in \mathbb{Z},\, \xi \in [0,\, 2\pi]} |\det \mathcal{A}(x,\, \xi)| > 0.$$

If the oscillation $\omega_2^2(a)$ is small enough, then the operator A is invertible on l^2, and its inverse admits a representation

$$A^{-1} = T_d^{-1}(I + Op(r))^{-1} Op(\mathcal{A}^{-1}))T_d$$

with an operator $Op(r)$ with $\|Op(r)\|_{\mathcal{L}(l_d^2)} < 1$.

Theorem 13 offers a way to reconstruct input signals for slowly oscillating perturbations of quasi time-invariant periodic filters.

The following theorem establishes necessary and sufficient conditions for the stability of finite sections method with respect to the sequence of projectors $(P_{dN})_{N\in\mathbb{N}}$.

Theorem 14 *Let A satisfy the conditions of Theorem 13. Then the finite sections method*

$$P_{dN} A P_{dN} u_{dN} = P_{dN} v, \quad N \in \mathbb{N}, \tag{26}$$

is stable if and only if:

(a) for every limit operator $(T_d A T_d^{-1})_h \in \sigma_{op}^{+}(T_d A T_d^{-1})$, the operator

$$P_+(T_d A T_d^{-1})_h P_+ : P_+(l_d^2) \to P_+(l_d^2)$$

is invertible;

(b) for every limit operator $(T_d A T_d^{-1})_h \in \sigma_{op}^{-}(T_d A T_d^{-1})$, the operator

$$P_-(T_d A T_d^{-1})_h P_- : P_-(l_d^2) \to P_-(l_d^2)$$

is invertible.

4 Reconstruction of signals in causal filters

In this section we consider discrete signals $u : \mathbb{N}_0 \to \mathbb{C}$ which are defined for non-negative values of time x. As usual, we call A a *causal filter* if $(Au)(x) = 0$ for $x < 0$. We will describe a class of causal filters and consider the problem of reconstruction of input signals for them.

4.1 Invertibility of causal band-dominated operators

Let $l^2(\mathbb{N}_0)$ denote the Hilbert space of all $u : \mathbb{N}_0 \to \mathbb{C}$ with norm

$$\|u\|_{l^2(\mathbb{N}_0)} := \left(\sum_{x \in \mathbb{N}_0} |u(x)|^2 \right)^{1/2} < \infty,$$

and write $l^\infty(\mathbb{N}_0)$ for the Banach space of all bounded complex-valued functions on $\mathbb{N}_0$ with norm

$$\|u\|_{l^\infty(\mathbb{N}_0)} = \sup_{x \in \mathbb{N}_0} |u(x)|.$$

For $\eta \in \mathbb{R}$, let $e_\eta : \mathbb{N}_0 \to \mathbb{R}$ denote the mapping $x \mapsto e^{-\eta x}$, and consider the weighted Hilbert space $l^2(\mathbb{N}_0, e_\eta)$ of all functions u for which $e_\eta u \in l^2(\mathbb{N}_0)$. This space is provided with the norm

$$\|u\|_{l^2(\mathbb{N}_0, e_\eta)} = \|e_\eta u\|_{l^2(\mathbb{N}_0)}.$$

We study band-dominated operators in W of the form

$$(Au)(x) = a_0(x)u(x) + \sum_{j=1}^{x} a_j(x)\, u(x-j), \quad x \in \mathbb{N}_0 \tag{27}$$

where $a_j \in l^\infty(\mathbb{N}_0)$ and

$$\|A\|_{W(\mathbb{N}_0)} := \sum_{j=0}^{\infty} \|a_j\|_{l^\infty(\mathbb{N}_0)} < \infty. \tag{28}$$

The class of all operators of this form is denoted by $W(\mathbb{N}_0)$.

Proposition 15 *Let $A \in W(\mathbb{N}_0)$ and $\eta \geq 0$. Then A is a bounded operator on $l^2(\mathbb{N}_0, e_\eta)$, and*

$$\|A\|_{\mathcal{L}(l^2(\mathbb{N}_0, e_\eta))} \leq \|a_0\|_{l^\infty(\mathbb{N}_0)} + e^{-\eta} M_A \tag{29}$$

where

$$M_A := \sum_{j=1}^{\infty} \|a_j\|_{l^\infty(\mathbb{N}_0)}.$$

Proof. Since A is causal, one has $Au(x) = 0$ for $x \leq 0$ if $u(x) = 0$ for $x \leq 0$. It is also evident that $\|A\|_{\mathcal{L}(l^2(\mathbb{N}_0, e_\eta))} = \|e_\eta A e_{-\eta} I\|_{\mathcal{L}(l^2(\mathbb{N}_0))}$ and that

$$e_\eta A e_{-\eta} I = a_0 I + \sum_{j=1}^{\infty} e^{-\eta j} a_j V_j.$$

Hence,

$$\|e_\eta A e_{-\eta} I\|_{\mathcal{L}(l^2(\mathbb{N}_0))} \leq \|a_0\|_{l^\infty(\mathbb{N}_0)} + e^{-\eta} M_A,$$

which implies estimate (29). ■

Theorem 16 *Let $A \in W(\mathbb{N}_0)$ and*

$$\inf_{x \in \mathbb{N}_0} |a_0(x)| > 0. \tag{30}$$

Then there exists an $\eta_0 > 0$ such that the operator A is invertible on each of the spaces $l^2(\mathbb{N}_0, e_\eta)$ with $\eta \geq \eta_0$.

Proof. Set $B := \sum_{j=1}^{\infty} a_j a_0^{-1} V_j$. It follows from estimate (29) that

$$\|B\|_{\mathcal{L}(l^2(\mathbb{N}_0, e_\eta))} \leq e^{-\eta} \sum_{j=1}^{\infty} \|a_j a_0^{-1}\|.$$

Hence, there exists an $\eta_0 > 0$ such that $\|B\|_{\mathcal{L}(l^2(\mathbb{N}_0, e_\eta))} < 1$ for all $\eta \geq \eta_0$. Because of $A = a_0(I + B)$, this implies

$$A^{-1} = (I + B)^{-1} a_0^{-1} I = \sum_{k=0}^{\infty} B^k a_0^{-1} I \tag{31}$$

via Neumann series. ■

Definition 17 *A function $a \in l^\infty(\mathbb{N}_0)$ is called* slowly oscillating at $+\infty$ *if*

$$\lim_{x \to +\infty} (a(x + y) - a(x)) = 0 \quad \textit{for each } y \in \mathbb{N}_0.$$

Further we say that an operator $A \in W(\mathbb{N}_0)$ of the form (27) belongs to the class $W^{SO}(\mathbb{N}_0)$ if all of its coefficients a_j, $j \in \mathbb{N}_0$, are slowly oscillating at $+\infty$.

Consider the operator $A \in W^{SO}(\mathbb{N}_0)$ as acting on the space $l^2(\mathbb{N}_0,\, e_\eta)$ for a certain $\eta \geq 0$. We associate with A its symbol

$$\sigma_A(x,\, \xi + i\eta) := \sum_{j=0}^{\infty} a_j(x)\, e^{ij(\xi+i\eta)}$$

which is defined for $x \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$. Set $\zeta := \xi + i\eta$. It is evident that the function $(x,\, \zeta) \mapsto \sigma_A(x,\, \zeta)$ depends analytically on ζ in the upper complex half-plane $\Im\zeta > 0$, whereas it is 2π-periodic and continuous with respect to $\xi \in \mathbb{R}$.

Theorem 18 *Let $A \in W^{SO}(\mathbb{N}_0)$. If condition* (30) *holds, and if*

$$\lim_{R\to+\infty} \inf_{x>R,\, \xi\in[0,\, 2\pi]} |\sigma_A(x,\, \xi + i\eta)| > 0 \tag{32}$$

for every $\eta \geq 0$, then the operator $A : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ is invertible.

Proof. Consider the family of operators

$$A_\eta := e_\eta A e_{-\eta} I = \sum_{j=0}^{\infty} a_j\, e^{-\eta j}\, V_j$$

where $\eta \geq 0$. It is not hard to check that $\|A_\eta\|_{W(\mathbb{N}_0)} \leq \|A\|_{W(\mathbb{N}_0)}$. Hence, the operators A_η belong to $W^{SO}(\mathbb{N}_0)$. Moreover, $A_\eta P_+ = P_+ A_\eta P_+$ due to causality. With $A_\eta : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$, we associate the band-dominated operator $B_\eta = P_+ A_\eta P_+ + P_- : l^2 \to l^2$. It is evident that the operators A_η and B_η are Fredholm operators only simultaneously. Condition (32) implies the invertibility of all limit operators in $\sigma^+_{op}(B_\eta)$ for every $\eta \geq 0$. Hence, A_η is a Fredholm operator on $l^2(\mathbb{N}_0)$ for every $\eta \geq 0$. Moreover, the family of operators A_η depends continuously on $\eta \in [0,\, \infty)$, and condition (30) implies the invertibility of the operators A_η for η large enough. Since the index is a continuous (and integer-valued) function on the set of all Fredholm operators, the index of A_η is zero for all $\eta \geq 0$.

Next we will verify that $A : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ has a trivial kernel. Indeed, consider the equation $Au = 0$ for $u \in l^2(\mathbb{N}_0)$. Then

$$0 = e_\eta A u = e_\eta A e_{-\eta}\, e_\eta u = A_\eta e_\eta u.$$

Choose $\eta > 0$ large enough such that A_η becomes an invertible operator on $l^2(\mathbb{N}_0)$. Note that $e_\eta u \in l^2(\mathbb{N}_0)$ since $u \in l^2(\mathbb{N}_0)$ and $\eta > 0$. Hence, $e_\eta u = 0$, which implies $u = 0$. Thus, the operator A considered as acting on $l^2(\mathbb{N}_0)$ is a Fredholm operator with index zero and with the trivial kernel. Hence, A is invertible. ■

4.2 Finite sections of causal time-varying filters

We will use the notation $[0, N]_{\mathbb{Z}} := \{n \in \mathbb{N}_0 : n \le N\}$ for $N \in \mathbb{N}$. Let $P_N : l^2(\mathbb{N}_0) \to \operatorname{im} P_N$ be the projection

$$(P_N u)(x) := \begin{cases} u(x) & \text{if} \quad x \in [0, N]_{\mathbb{Z}}, \\ 0 & \text{if} \quad x > N. \end{cases}$$

Let further A be an operator of the form (27), and suppose that condition (28) holds. We consider the problem of reconstructing the first N values of the input signal from the first N values of the output signal. In other words, we consider the solution of the system of linear equations $P_N A P_N u_N = P_N v$ with u_N sought in $\operatorname{im} P_N$, which is equivalent to the solution of the system

$$a_0(x) u_N(x) + \sum_{j=1}^{x} a_j(x)\, u_N(x-j) = v(x) \quad \text{for } x \in [0, N]_{\mathbb{Z}}. \tag{33}$$

The system (33) is triangular. Thus, under the condition $\inf_{x \in \mathbb{N}_0} |a_0(x)| > 0$, this system has a unique solution u_N for every right-hand side $v \in l^2(\mathbb{N}_0)$ and every $N \in \mathbb{N}$, which can be obtained by means of elementary elimination.

The operators $P_N A P_N \in \mathcal{L}(\operatorname{im} P_N)$ are invertible for every $N \in \mathbb{N}$, and

$$(P_N A P_N)^{-1} = P_N A^{-1} P_N. \tag{34}$$

For the latter note that $P_N A = P_N A P_N$ as a consequence of causality, whence

$$(P_N A P_N)\,(P_N A^{-1} P_N) = P_N A A^{-1} P_N = P_N.$$

Thus, $P_N A^{-1} P_N$ is a right inverse for $P_N A P_N \in \mathcal{L}(\operatorname{im} P_N)$. Since $P_N A P_N$ acts on a finite-dimensional space, the operator $P_N A^{-1} P_N$ is also a left inverse for $P_N A P_N$. Identity (34) implies

$$\sup_{N \in \mathbb{N}} \|(P_N A P_N)^{-1}\|_{\mathcal{L}(\operatorname{im} P_N)} = \sup_{N \in \mathbb{N}} \|P_N A^{-1} P_N\|_{\mathcal{L}(\operatorname{im} P_N)} \le \|A^{-1}\|_{\mathcal{L}(l^2(\mathbb{N}_0))}.$$

This estimate yields the applicability of the finite sections method for the reconstruction of the input signal in causal filters (see for instance [3, 2, 12]). Thus,

$$\lim_{N \to \infty} \|u_N - u\|_{l^2(\mathbb{N}_0)} = 0$$

where u_N is the solution of equation (33) extended by zero outside the discrete interval $[0, N]_{\mathbb{Z}}$.

5 Reconstruction of digital finite signals in time-varying filters

This concluding section is devoted to the problem of reconstruction of finitely supported digital input signals from known finitely supported output signals. We will consider such signals as periodic sequences $(u(x))_{x\in\mathbb{Z}}$ with a period $d \in \mathbb{N}$, i.e., we identify $l^2([1,\, d]_{\mathbb{Z}})$ with $l^2(\mathbb{Z}/(d\mathbb{Z}))$. Let A be a time-varying filter of the form

$$(Au)(x) = \sum_{j=0}^{d-1} a_j(x)\,(V_j u)(x), \quad x \in \mathbb{Z}, \tag{35}$$

where the a_j are d-periodic functions and the V_j are the shifts $(V_j u)(x) = u(x-j)$, $x \in \mathbb{Z}$, acting cyclically on d-periodic sequences. We introduce the discrete Fourier transform of a periodic sequence u by

$$\hat{u}(\xi) = (Fu)(\xi) := \sum_{x=0}^{d-1} u(x)\,\gamma_d^{-x\xi}, \quad \xi \in \mathbb{Z}, \tag{36}$$

where $\gamma_d := \exp(2\pi i/d)$ is a primitive root of unit of degree d. It is well-known that the inverse discrete Fourier transform acts via

$$u(x) = (F^{-1}\hat{u})(x) = \frac{1}{d}\sum_{\xi=0}^{d-1} \hat{u}(\xi)\,\gamma_d^{x\xi}, \quad x \in \mathbb{Z}, \tag{37}$$

and that Parseval's equality

$$\sum_{\xi=0}^{d-1} |\hat{u}(\xi)|^2 = d\sum_{x=0}^{d-1} |u(x)|^2$$

holds (see, for instance, [1]). Formulas (36) and (37) imply that

$$u(x) = \frac{1}{d}\sum_{y=0}^{d-1}\sum_{\xi=0}^{d-1} u(x+y)\,\gamma_d^{-y\xi} \tag{38}$$

and that

$$\widehat{V_j u}(\xi) = \gamma_d^{-j\xi}\,\hat{u}(\xi), \quad \xi \in \mathbb{Z},\, j = 0,\, 1,\, \ldots,\, d-1.$$

Let $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a function which is d-periodic with respect to both variables. With a, we associate the d-periodic pseudodifference operator (which *is*, in fact, a pseudodifferential operator on the cyclic group $\mathbb{Z}_d := \mathbb{Z}/(d\mathbb{Z})$)

$$(Au)(x) = (Op(a)u)(x) := \sum_{\xi=0}^{d-1} a(x,\,\xi)\,\hat{u}(\xi)\,\gamma_d^{x\xi}, \quad x \in \mathbb{Z}, \tag{39}$$

which is defined on the d-periodic functions. The function $\sigma_A := a$ is called the symbol of the operator A.

Note that the time-varying filter (35) can be represented as a d-periodic pseudodifference operator with symbol

$$\sigma_A(x,\,\xi) = \sum_{j=0}^{d-1} a(x)\,\gamma_d^{-j\xi}, \quad (x,\,\xi) \in \mathbb{Z} \times \mathbb{Z}.$$

Let l_d^2 denote the space of all d-periodic functions u on $\mathbb{Z}$ with norm

$$\|u\|_{l_d^2} := \left(\sum_{x=0}^{d-1} |u(x)|^2 \right)^{1/2}.$$

Proposition 19 *Let $A = Op(a)$ be a d-periodic pseudodifference operator. Then*

$$\|A\|_{\mathcal{L}(l_d^2)} \le \left(d \sum_{x=0}^{d-1} \sum_{\xi=0}^{d-1} |a(x,\,\xi)|^2 \right)^{1/2} \le d^{3/2} \max_{x,\,\xi \in [0,\,d-1]_{\mathbb{Z}}} |a(x,\,\xi)|. \tag{40}$$

Estimate (40) is obtained by a direct calculation using Parseval's equality.

Proposition 20 *Let $A = Op(a)$ and $B = Op(b)$ be d-periodic pseudodifference operators. Then AB is a d-periodic pseudodifference operator with symbol*

$$\sigma_{AB}(x,\,\xi) = \frac{1}{d} \sum_{y=0}^{d-1} \sum_{\eta=0}^{d-1} a(x,\,\xi+\eta)\,b(x+y,\,\xi)\,\gamma_d^{-y\eta}, \quad x,\,\xi \in \mathbb{Z}. \tag{41}$$

Formula (41) for the symbol of the product of two d-periodic pseudodifference operators follows by straightforward calculation.

We denote by $\omega_1(\sigma_A)$ the oscillation of the function $(x, \xi) \mapsto \sigma_A(x, \xi)$ with respect to the (first) variable $x \in [0, d-1]_{\mathbb{Z}}$, that is,

$$\omega_1(\sigma_A) = \max_{x,y,\xi \in [0,d-1]_{\mathbb{Z}}} |\sigma_A(x, \xi) - \sigma_A(y, \xi)|.$$

Theorem 21 *Let the following conditions hold for the symbol of the d-periodic pseudodifference operator A:*

$$\sigma_A(x, \xi) \neq 0 \quad \textit{for all } x, \xi \in [0, d-1]_{\mathbb{Z}} \tag{42}$$

and

$$\omega_1(\sigma_A) \max_{x \in [0,d-1]_{\mathbb{Z}}} \sum_{\eta=0}^{d-1} |a^{-1}(x, \eta)| < d^{-3/2}. \tag{43}$$

Then A is an invertible operator on l^2_d, and

$$A^{-1} = (I+T)^{-1} Op(a^{-1}) \tag{44}$$

where T is an operator with $\|T\|_{\mathcal{L}(l^2_d)} < 1$.

Proof. Let $B := Op(a^{-1})$. Employing (38), we obtain

$$\begin{aligned} \sigma_{BA}(x, \xi) &= \frac{1}{d} \sum_{y=0}^{d-1} \sum_{\eta=0}^{d-1} a^{-1}(x, \xi+\eta)\, a(x+y, \xi)\, \gamma_d^{-y\eta} \\ &= \frac{a(x, \xi)}{d} \sum_{y=0}^{d-1} \sum_{\eta=0}^{d-1} a^{-1}(x, \xi+\eta)\, \gamma_d^{-y\eta} + t(x, \xi) =: 1 + t(x, \xi), \end{aligned}$$

where

$$t(x, \xi) = \frac{1}{d} \sum_{y=0}^{d-1} \sum_{\eta=0}^{d-1} a^{-1}(x, \xi+\eta)\, (a(x+y, \xi) - a(x, \xi))\, \gamma_d^{-y\eta}.$$

From estimate (40), we further conclude

$$\|T\|_{\mathcal{L}(l^2_d)} = \|Op(t)\|_{\mathcal{L}(l^2_d)} \leq d^{3/2} \omega_1(\sigma_A) \max_{x \in [0,d-1]_{\mathbb{Z}}} \sum_{\eta=0}^{d-1} |a^{-1}(x, \eta)|.$$

Hence, $BA = I + T$ with $\|T\|_{\mathcal{L}(l_d^2)} < 1$ by condition (43), and the operator $A^{-1} = (I+T)^{-1}Op(a^{-1})$ is a left inverse of A. Since l_d^2 is a finite-dimensional space, A^{-1} is also a right inverse A, whence the invertibility of A. ■

Thus, we've obtained an effective algorithm for the reconstruction of input signals. It follows from (44) that the input signal u is a solution of the equation

$$u + Tu = Op(a^{-1})v$$

where the function $Op(a^{-1})v$ can be calculated by means of fast Fourier transform algorithms (see for instance [1]). Then u is calculated by means of successive approximations, that is, we set $u_0 := 0$ and define $u_{n+1} := -Tu_n + Op(a^{-1})v$ for $n \in \mathbb{N}_0$ to obtain a sequence (u_n) which converges to the solution of the equation $Op(a)u = v$.

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