The Fredholm index of locally compact band-dominated operators on $L^p(\mathbb{R})$

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Abstract

We establish a necessary and sufficient criterion for the Fredholmness of a general locally compact band-dominated operator A on $L^p(\mathbb{R})$ and derive a formula for its Fredholm index in terms of the limit operators of A. The results are applied to operators of convolution type with almost periodic symbol.

1 Introduction

Throughout this paper, let $1 < p < \infty$, and for each Banach space X, let $L(X)$ stand for the Banach algebra of all bounded linear operators on X, $K(X)$ for the closed ideal of the compact operators, B_X for the closed unit ball of X, and X^* for the Banach dual space of X.

For each function $\varphi \in BUC$, the algebra of the bounded and uniformly continuous functions on the real line $\mathbb{R}$, and for each $t > 0$, set $\varphi_t(x) := \varphi(tx)$ and write φI for the operator on $L^p(\mathbb{R})$ of multiplication by φ. An operator $A \in L^p(\mathbb{R})$ is called *band-dominated* if

$$\lim_{t \to 0} \|A\varphi_t I - \varphi_t A\| = 0$$

for each function $\varphi \in BUC$. The set $\mathcal{B}_p$ of all band-dominated operators forms a closed subalgebra of $L^p(\mathbb{R})$. In this paper we will exclusively deal with band-dominated operators of the form $I + K$ where I is the identity operator and K is locally compact (which means that φA and $A\varphi I$ are compact for each function $\varphi \in BUC$ with bounded support). We write $\mathcal{L}_p$ for the set of all locally compact band-dominated operators on $L^p(\mathbb{R})$.

The announced Fredholm criterion and the index formula will be formulated in terms of limit operators. To introduce this notion, we will need the shift operators

$$U_k : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad (U_k f)(x) := f(x - k)$$

where $k \in \mathbb{Z}$. Given a sequence $h : \mathbb{N} \to \mathbb{Z}$ tending to infinity we call the operator $A_h \in L(L^p(\mathbb{R}))$ a *limit operator of* $A \in L(L^p(\mathbb{R}))$ *with respect to* h if

$$\lim_{m \to \infty} \|(U_{-h(m)} A U_{h(m)} - A_h)\varphi I\| = 0$$

and

$$\lim_{m\to\infty} \|\varphi(U_{-h(m)}AU_{h(m)} - A_h)\| = 0$$

for each function $\varphi \in BUC$ with bounded support. The set of all limit operators of a given operator $A \in L(L^p(\mathbb{R}))$ is called the *operator spectrum* of A and denoted by $\sigma_{op}(A)$. The operator spectrum splits into two components $\sigma_+(A) \cup \sigma_-(A)$ which collect the limit operators of A with respect to sequences h tending to $+\infty$ and to $-\infty$, respectively.

An operator $A \in L(L^p(\mathbb{R}))$ is said to be *rich* or *to possess a rich operator spectrum* if every sequence h tending to infinity possesses a subsequence g for which the limit operator A_g exists. The sets of all rich operators in $\mathcal{B}_p$ and $\mathcal{L}_p$ will be denoted by $\mathcal{B}_p^\$$ and $\mathcal{L}_p^\$$.

Let χ_+ and χ_- stand for the characteristic functions of the sets $\mathbb{R}_+$ and $\mathbb{R}_-$ of the non-negative and negative real numbers, respectively. The operators $\chi_+K\chi_-I$ and $\chi_-K\chi_+I$ are compact for each operator $K \in \mathcal{L}_p$. Indeed, let $\varepsilon > 0$ be arbitrarily given. Since K is band-dominated, there is a continuous function f which is 1 on $[0, \infty)$ and 0 on $(-\infty, -n_\varepsilon]$ with sufficiently large n_ε such that $\|fK - KfI\| < \varepsilon$. Thus,

$$\|\chi_+K\chi_-I - \chi_+Kf\chi_-I\| = \|\chi_+(fK - Kf)\chi_-I\| < \varepsilon.$$

The operator $\chi_+Kf\chi_-I$ is compact since $f\chi_-$ has a bounded support and K is locally compact. Since further ε can be chosen arbitrarily small, the compactness of $\chi_+K\chi_-I$ follows. The compactness of $\chi_-K\chi_+I$ can be checked analogously.

This simple observation implies that, for a Fredholm operator of the form $A = I + K$ with $K \in \mathcal{L}_p$, the operators $\chi_+A\chi_+I$ and $\chi_-A\chi_-I$, considered as acting on $L^p(\mathbb{R}_+)$ and $L^p(\mathbb{R}_-)$, are Fredholm operators again. We call

$$\text{ind}_+A := \text{ind}\,(\chi_+A\chi_+I) \quad \text{and} \quad \text{ind}_-A := \text{ind}\,(\chi_-A\chi_-I)$$

the *plus-* and the *minus-index of* A. Recall in this connection that a bounded linear operator A on a Banach space X is said to be *Fredholm* if its kernel $\ker A$ and its cokernel $\text{coker}\, A := X/\text{im}\, A$ are linear spaces of finite dimension, and that in this case the integer

$$\text{ind}\, A := \dim\ker A - \dim\text{coker}\, A$$

is called the *Fredholm index* of A.

Here is the main result of the present paper.

Theorem 1 *Let $A = I + K$ with $K \in \mathcal{L}_p^\$$.*

(*a*) *The operator A is Fredholm on $L^p(\mathbb{R})$ if and only if all limit operators of A are invertible and if the norms of their inverses are uniformly bounded.*

(*b*) *If A is Fredholm, then for arbitrary operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,*

$$\text{ind}_+B_+ = \text{ind}_+A \quad \textit{and} \quad \text{ind}_-B_- = \text{ind}_-A \tag{1}$$

and, consequently,

$$\text{ind}\, A = \text{ind}_+ B_+ + \text{ind}_- B_-. \qquad (2)$$

This result has a series of predecessors. One of the simplest classes of band-dominated and locally compact operators on $L^p(\mathbb{R})$ is constituted by the operators of convolution by $L^1(\mathbb{R})$-functions and by the restrictions of these operators to the half line, the classical Wiener-Hopf operators. The theory of the convolution type operators on the half line originates from the fundamental papers by Krein and Gohberg/Krein [7, 4] where the Fredholm theory for these operators is established and an index formula is derived. See also the monograph [3] by Gohberg/Feldman for an axiomatic approach to this circle of questions. For convolution type operators with variable coefficients which stabilize at infinite, a Fredholm criterion and an index formula have been obtained by Karapetiants/Samko in [5]; see also their monograph [6].

In [12, 13], there is developed the limit operator approach to study Fredholm properties of general band-dominated operators on spaces l^p of vector-valued sequences. In [10] we demonstrated that this approach also applies to operators of convolution type acting on L^p spaces if a suitable discretization reducing L^p- to l^p-spaces is performed. (To be precisely: If the sequences in l^p take their values in an infinite dimensional Banach space, then we derived in [13] a criterion for a generalized form of Fredholmness, called $\mathcal{P}$-Fredholmness; see below. But the results of [10] refer to common Fredholmness.) The long standing problem to determine the Fredholm index of a band-dominated operator in terms of its limit operators, too, has been finally solved in [11] for band-dominated operators on the space l^2 with scalar-valued sequences. All mentioned results can be also found in the monograph [14]. The index formula has been generalized to l^p-spaces in [15]. In the present paper we will undertake a further generalization to band-dominated operators with compact entries acting on l^p-spaces of vector-valued functions. Thereby these results will get the right form to become applicable to locally compact band-dominated operators on L^p-spaces (and thus, to prove assertion (b) of the theorem).

The paper is organized as follows. We start with recalling some basic facts on sequences of compact operators. For the reader's convenience, the proofs are included. The main work will be done in Section 3 where we will derive the Fredholm criterion and the index formula for band-dominated operators on l^p with compact entries. In Section 4, these results will be applied to locally compact band-dominated operators on L^p which mainly requires to construct a suitable discretization mapping. Some applications will be discussed in the final section.

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2 Sequences of compact operators

Let X be a complex Banach space which enjoys the following *symmetric approximation property* (*sap*): There is a sequence $(\Pi_N)_{N\geq 1}$ of projections (= idempotents) $\Pi_N \in L(X)$ of finite rank such that $\Pi_N \to I$ and $\Pi_N^* \to I^*$ strongly as $N \to \infty$. Evident examples of Banach spaces with *sap* are the separable Hilbert spaces, the spaces $l^p(\mathbb{Z}^K)$ and the spaces $L^p[a, b]$. It is also clear that if X is a reflexive Banach space with *sap*, then X^* has *sap*, too, and the corresponding projections can be chosen as Π_N^*.

Definition 2 *A sequence (K_n) of operators in $L(X)$ is said to be*

(a) relatively compact *if the norm closure of $\{K_n : n \in \mathbb{N}\}$ is compact in $L(X)$;*

(b) collectively compact *if the set $\cup_{n\in\mathbb{N}} K_n B_X$ is relatively compact in X;*

(c) uniformly left (right, two-sided) approximable *if, for each $\varepsilon > 0$ there is an N_0 such that, for each $n \in \mathbb{N}$ and each $N \geq N_0$,*

$$\|K_n - \Pi_N K_n\| < \varepsilon \quad (\|K_n - K_n \Pi_N\| < \varepsilon, \quad \|K_n - \Pi_N K_n \Pi_N\| < \varepsilon).$$

Note that the uniform left approximability of (K_n) is equivalent to

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \|K_n - \Pi_N K_n\| = 0.$$

Proposition 3 *Let X be a Banach space with sap. The following conditions are equivalent for a sequence (K_n) of compact operators on X:*

(a) (K_n) is relatively compact;

(b) (K_n) and (K_n^*) are collectively compact;

(c) (K_n) is uniformly left and uniformly right approximable;

(d) (K_n) is uniformly two-sided approximable.

Proof. $(a) \Rightarrow (b)$: Let (x_n) be a sequence in $\cup_n K_n B_X$. For each $n \in \mathbb{N}$, choose $r(n) \in \mathbb{N}$ and $y_n \in B_X$ such that $x_n = K_{r(n)} y_n$. By hypothesis (a), the sequence $(K_{r(n)})$ has a convergent subsequence $(K_{r(n_k)})$. Let K denote the limit of that subsequence. Then

$$\|x_{n_k} - K y_{n_k}\| = \|K_{r(n_k)} y_{n_k} - K y_{n_k}\| \leq \|K_{r(n_k)} - K\| \to 0. \tag{3}$$

Since K is compact and $\|y_{n_k}\| \leq 1$, the sequence $(K_{r(n_k)})$ has a convergent subsequence. From (3) we conclude that then the sequence (x_{n_k}) (hence, the sequence (x_n)) has a convergent subsequence, too. This yields the collective compactness of the sequence (K_n). Since (K_n^*) is relatively compact whenever (K_n) is relatively compact, the collective compactness of (K_n^*) follows in the same way.

$(b) \Rightarrow (c)$: We will show that the collective compactness of (K_n) implies the uniform left approximability of that sequence. We will not make use of the strong convergence of Π_N to I^* in this part of the proof. So is becomes evident that then also the collective compactness of (K_n^*) implies the uniform left approximability of (K_n^*) with respect to the sequence (Π_N^*) which is equivalent to the uniform right approximability of (K_n).

Contrary to what we want to show, assume that (K_n) is not uniformly left approximable. Then there are an $\varepsilon > 0$, a monotonically increasing sequence $(N(r))_{r\geq 1}$ and operators $K_{n(r)} \in \{K_n : n \in \mathbb{N}\}$ such that

$$\|(I - \Pi_{N(r)})K_{n(r)}\| \geq \varepsilon \quad \text{for all } r \in \mathbb{N}.$$

Choose $x_{n(r)} \in B_X$ such that

$$\|(I - \Pi_{N(r)})K_{n(r)}x_{n(r)}\| \geq \varepsilon/2 \quad \text{for all } r \in \mathbb{N}. \tag{4}$$

By hypothesis (b), the sequence $(K_{n(r)}x_{n(r)})$ has a convergent subsequence. Let x_0 denote its limit. We conclude from (4) that $\|(I - \Pi_{N(r)})x_0\| \geq \varepsilon/4$ for all sufficiently large r. Letting r go to infinity, we arrive at a contradiction.

$(c) \Rightarrow (d)$: This implication follows immediately from

$$\begin{aligned}\|K_n - \Pi_N K_n \Pi_N\| &\leq \|K_n - \Pi_N K_n\| + \|\Pi_N K_n - \Pi_N K_n \Pi_N\| \\ &\leq \|K_n - \Pi_N K_n\| + \|\Pi_N\| \, \|K_n - K_n \Pi_N\|\end{aligned}$$

and from the uniform boundedness of the projections Π_N due to the Banach-Steinhaus theorem.

$(d) \Rightarrow (a)$: We consider a subsequence of (K_n) which we write as $(K_n)_{n\in\mathbb{N}_0}$ with an infinite subset $\mathbb{N}_0$ of $\mathbb{N}$. Since the projections Π_N have finite rank, there are an infinite subset $\mathbb{N}_1$ of $\mathbb{N}_0$ such that the sequence $(\Pi_1 K_n \Pi_1)_{n\in\mathbb{N}_1}$ converges, an infinite subset $\mathbb{N}_2$ of $\mathbb{N}_1$ such that the sequence $(\Pi_2 K_n \Pi_2)_{n\in\mathbb{N}_2}$ converges, etc. Thus, for each $N \geq 1$, one finds an infinite subset $\mathbb{N}_N$ of $\mathbb{N}_{N-1}$ such that the sequence $(\Pi_N K_n \Pi_N)_{n\in\mathbb{N}_N}$ converges. Let $k(n)$ denote the nth number in $\mathbb{N}_N$ (ordered with respect to the relation $<$) and set $\hat{K}_n := K_{k(n)}$. Clearly, $(\hat{K}_n)_{n\geq 1}$ is a subsequence of each of the sequences $(K_n)_{n\in\mathbb{N}_N}$ up to finitely many entries. Thus, for each $N \in \mathbb{N}$, the sequence $(\Pi_N \hat{K}_n \Pi_N)_{n\geq 1}$ converges. Now we have

$$\hat{K}_n - \hat{K}_m = (\hat{K}_n - \Pi_N \hat{K}_n \Pi_N) - (\hat{K}_m - \Pi_N \hat{K}_m \Pi_N) + \Pi_N(\hat{K}_n - \hat{K}_m)\Pi_N.$$

Let $\varepsilon > 0$. By hypothesis (d), there is an N such that

$$\|\hat{K}_n - \Pi_N \hat{K}_n \Pi_N\| < \varepsilon/3$$

for all $n \in \mathbb{N}$. Fix this N, and choose n_0 such that

$$\|\Pi_N(\hat{K}_n - \hat{K}_m)\Pi_N\| < \varepsilon/3$$

for all $m, n \geq n_0$ which is possible due to the convergence of the sequence $(\Pi_N \hat{K}_n \Pi_N)_{n\geq 1}$. Hence, $\|\hat{K}_n - \hat{K}_m\| < \varepsilon$ for all $m, n \geq n_0$. This implies the convergence of the sequence $(\hat{K}_n)$ and, thus, the relative compactness of (K_n). ∎

3 The Fredholm index of discrete band-dominated operators with compact entries

Let X be a complex Banach space with *sap*. By $E := l^p(\mathbb{Z}, X)$ we denote the Banach space of all sequences $x : \mathbb{Z} \to X$ with

$$\|x\|_E^p := \sum_{n\in\mathbb{Z}} \|x_n\|_X^p < \infty.$$

For $k \in \mathbb{Z}$, let $V_k : l^p(\mathbb{Z}, X) \to l^p(\mathbb{Z}, X)$ stand for the shift operator $(V_k x)_n := x_{n-k}$. In what follows, we will have to consider shift operators on different spaces $l^p(\mathbb{Z}, X)$. In order to indicate the underlying space we will sometimes also write $V_{k,X}$ for the shift operator V_k on $l^p(\mathbb{Z}, X)$. Further, for each non-negative integer n, let the projection operators $P_n : l^p(\mathbb{Z}, X) \to l^p(\mathbb{Z}, X)$ be defined by

$$(P_n x)_k := \begin{cases} x_k & \text{if} \quad |k| \le n \\ 0 & \text{if} \quad |k| > n, \end{cases}$$

and set $Q_n := I - P_n$ and $\mathcal{P} := (P_n)_{n\ge 0}$. Sometimes we will also write $P_{n,X}$ in place of P_n in order to indicate the underlying space.

Each operator $A \in L(E)$ can be represented in the obvious way by a two-sided infinite matrix with entries in $L(X)$ (in analogy with the representation of an operator on $l^p(\mathbb{Z}) := l^p(\mathbb{Z}, \mathbb{C})$ with respect to the standard basis). The operator $A \in L(E)$ is called a *band operator* if its matrix representation (A_{ij}) is a band matrix, i.e., if there is a $k \in \mathbb{N}$ such that $A_{ij} = 0$ if $|i - j| \ge k$. The closure of the set of all band operators on E is a closed subalgebra of $L(E)$ which we denote by $\mathcal{A}_E$. The elements of $\mathcal{A}_E$ will be called *band-dominated operators*. By $\mathcal{C}_E$ we denote the closed ideal of $\mathcal{A}_E$ which consists of all band-dominated operators which have only compact entries in their matrix representation.

Following the terminology introduced in [14], an operator $K \in L(E)$ is called $\mathcal{P}$-*compact* if

$$\lim_{n\to\infty} \|KQ_n\| = \lim_{n\to\infty} \|Q_n K\| = 0.$$

We denote the set of all $\mathcal{P}$-compact operators by $K(E, \mathcal{P})$, and we write $L(E, \mathcal{P})$ for the set of all operators $A \in L(E)$ for which both AK and KA are $\mathcal{P}$-compact whenever K is $\mathcal{P}$-compact. Then $L(E, \mathcal{P})$ is a closed subalgebra of $L(E)$ which contains $K(E, \mathcal{P})$ as a closed ideal.

Definition 4 *An operator $A \in L(E, \mathcal{P})$ is called* $\mathcal{P}$-Fredholm *if the coset $A + K(E, \mathcal{P})$ is invertible in the quotient algebra $L(E, \mathcal{P})/K(E, \mathcal{P})$, i.e., if there exist an operator $B \in L(E, \mathcal{P})$ and operators $K, L \in K(E, \mathcal{P})$ such that $BA = I + K$ and $AB = I + L$.*

This definition is equivalent to the following one: An operator $A \in L(E, \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if there exist an $m \in \mathbb{N}$ and operators $L_m, R_m \in L(E, \mathcal{P})$

such that

$$L_m A Q_m = Q_m A R_m = Q_m.$$

Thus, $\mathcal{P}$-Fredholmness is often referred to as *local invertibility at infinity.* If X has finite dimension, then the notions $\mathcal{P}$-Fredholmness and Fredholmness are synonymous.

All band-dominated operators belong to $L(E, \mathcal{P})$. This can be easily checked for the two basic types of band-dominated operators: the shift operators and the operators of multiplication by a function in $l^\infty(\mathbb{Z}, L(X))$, and it follows for general band-dominated operators since $L(E, \mathcal{P})$ is a closed algebra. Hence, it makes sense to speak about their $\mathcal{P}$-Fredholmness. A criterion for the $\mathcal{P}$-Fredholmness of a band-dominated operator A can be given in terms of the limit operators of A. These are, in analogy with the notions from Section 1, defined as follows. Let $A \in L(E)$, and let $h : \mathbb{N} \to \mathbb{Z}$ be a sequence which tends to infinity. An operator $A_h \in L(E)$ is called a *limit operator of A with respect to the sequence h* if

$$\lim_{n\to\infty} \|P_k(V_{-h(n)} A V_{h(n)} - A_h)\| = \lim_{n\to\infty} \|(V_{-h(n)} A V_{h(n)} - A_h) P_k\| = 0$$

for every $k \in \mathbb{N}$. The set of all limit operators of A will be denoted by $\sigma_{op}(A)$ and is called the *operator spectrum* of A again. An operator $A \in L(E)$ is said to be *rich* or to possess a *rich operator spectrum* if each sequence h which tends to infinity possesses a subsequence g for which the limit operator A_g exists. We refer to the rich operators in $\mathcal{A}_E$ as *rich* band-dominated operators and write $\mathcal{A}_E^\$$ and $\mathcal{C}_E^\$$ for the Banach algebra of the rich band-dominated operators and for its closed ideal consisting of the rich operators in $\mathcal{C}_E$.

The following is the main result on $\mathcal{P}$-Fredholmness of rich band-dominated operators. Its proof is in [14], Theorem 2.2.1.

Theorem 5 *An operator $A \in \mathcal{A}_E^\$$ is $\mathcal{P}$-Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded, i.e.,*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_{op}(A)\} < \infty.$$

In case $X = \mathbb{C}$, $\mathcal{P}$-Fredholmness coincides with common Fredholmness. In this case one can also express the Fredholm index of a Fredholm band-dominated operator in terms of the (local) indices of its limit operators. To cite these results from [11, 15], let $P : l^p(\mathbb{Z}, X) \to l^p(\mathbb{Z}, X)$ refer to the projection operator

$$(Px)_k := \begin{cases} x_k & \text{if} \quad k \ge 0 \\ 0 & \text{if} \quad k < 0, \end{cases}$$

and set $Q := I - P$. If necessary, we will write also P_X in place of P. Then, for each band-dominated operator on $l^p(\mathbb{Z}, \mathbb{C})$, the operators PAQ and QAP are compact. This is obvious for band operators in which case PAQ and QAP are of finite rank, and it follows for general band-dominated operators by an obvious approximation argument. Consequently, the operators $A-(PAP+Q)(P+QAQ)$ and $A-(P+QAQ)(PAP+Q)$ are compact, which implies that a band-dominated operator on $l^p(\mathbb{Z}, \mathbb{C})$ is Fredholm if and only if both operators $PAP+Q$ and $P+QAQ$ are Fredholm and that

$$\operatorname{ind} A = \operatorname{ind}(PAP+Q) + \operatorname{ind}(P+QAQ).$$

In this case we call

$$\operatorname{ind}_+ A := \operatorname{ind}(PAP+Q) \quad \text{and} \quad \operatorname{ind}_- A := \operatorname{ind}(P+QAQ)$$

the *plus-* and the *minus-index of* A. Finally, let $\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$, the latter components collecting the limit operators of A with respect to sequences h tending to $+\infty$ and to $-\infty$, respectively, and note that in case $X = \mathbb{C}$ all band-dominated operators are rich.

Theorem 6 *Let $X = \mathbb{C}$, and let A be a Fredholm band-dominated operator on $l^p(\mathbb{Z})$. Then, for arbitrary operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,*

$$\operatorname{ind}_+ B_+ = \operatorname{ind}_+ A \quad \textit{and} \quad \operatorname{ind}_- B_- = \operatorname{ind}_- A \tag{5}$$

and, consequently,

$$\operatorname{ind} A = \operatorname{ind}_+ B_+ + \operatorname{ind}_- B_-. \tag{6}$$

In particular, all operators in $\sigma_+(A)$ have the same plus-index, and all operators in $\sigma_-(A)$ have the same minus-index.

It is the goal of the present section to generalize the assertion of Theorem 6 to operators acting on $E = l^p(\mathbb{Z}, X)$ with a general Banach space X with *sap* which are of the form $I+K$ with $K \in \mathcal{C}_E^\$$. A first observation is that for these operators $\mathcal{P}$-Fredholmness and common Fredholmness coincide.

Proposition 7 *An operator in $I+\mathcal{C}_E$ is Fredholm if and only if it is $\mathcal{P}$-Fredholm.*

Proof. We claim that

$$\mathcal{C}_E \cap K(E, \mathcal{P}) = K(E). \tag{7}$$

The inclusion $K(E) \subseteq \mathcal{C}_E$ is evident, and the inclusion $K(E) \subseteq K(E, \mathcal{P})$ holds since the projections P_n and P_n^* converge strongly to the identity operators on E and E^*, respectively. Thus, $K(E) \subseteq \mathcal{C}_E \cap K(E, \mathcal{P})$. For the reverse inclusion, let $K \in \mathcal{C}_E \cap K(E, \mathcal{P})$. Since $K \in \mathcal{C}_E$, one has $P_n K \in K(E)$ for every n, and since $K \in K(E, \mathcal{P})$, one has $\|K - P_n K\| \to 0$. Thus, $K \in K(E)$, which verifies (7).

Since $K(E) \subseteq K(E, \mathcal{P})$ by (7), every Fredholm operator in $L(E, \mathcal{P})$ is $\mathcal{P}$-Fredholm. For the reverse implication, let $A := I + K$ with $K \in \mathcal{C}_E$ be a $\mathcal{P}$-Fredholm operator. Then there are operators $B \in L(E, \mathcal{P})$ and $L \in K(E, \mathcal{P})$ such that $BA = I - L$. Set $R := I - KB$. Then

$$RA - I = A - I - KBA = K - KBA = K(I - BA) = KL.$$

Since $KL \in \mathcal{C}_E \cap K(E, \mathcal{P})$ is compact by (7), the operator R is a left Fredholm regularizer for A. Similarly one checks that A possesses a right Fredholm regularizer. Thus, the operator A is Fredholm. ■

Combining Proposition 7 with Theorem 5 one gets the following.

Corollary 8 *Let $A := I + K$ with $K \in \mathcal{C}_E^\$$. Then the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.*

We will make use of the following lemma several times.

Lemma 9 *Every band-dominated operator in $\mathcal{C}_E$ (resp. in $\mathcal{C}_E^\$$) is the norm limit of a sequence of band operators $\mathcal{C}_E$ (resp. in $\mathcal{C}_E^\$$).*

This can be proved in exactly the way as we derived Theorem 2.1.18 in [14] which states that every rich band-dominated operators is the norm limit of a sequence of rich band operators. ■

As a first consequence of the $\mathcal{C}_E$-version of Lemma 9 we conclude that PAQ and QAP are compact operators for each operator $A \in I + \mathcal{C}_E$. Indeed, this is obvious for A being a band operators in which case PAQ and QAP have only a finite number of non-vanishing entries, and these are compact. The case of general A follows by an obvious approximation argument. Consequently, the operators $A - (PAP + Q)(P + QAQ)$ and $A - (P + QAQ)(PAP + Q)$ are compact, which implies that an operator $A \in I + \mathcal{C}_E$ is Fredholm if and only if both operators $PAP + Q$ and $P + QAQ$ are Fredholm. In this case, the integers

$$\text{ind}_+ A := \text{ind}\,(PAP + Q) \quad \text{and} \quad \text{ind}_- A := \text{ind}\,(P + QAQ)$$

are called the *plus-* and the *minus-index of* A. Clearly,

$$\text{ind}\, A = \text{ind}_+ A + \text{ind}_- A. \tag{8}$$

Finally, let $\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$ in analogy with the case $X = \mathbb{C}$.

Here is the announced result for the indices of Fredholm operators in $I + \mathcal{C}_E^\$$.

Theorem 10 *Let $A \in I + \mathcal{C}_E^\$$ be a Fredholm operator. Then, for arbitrary operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,*

$$\text{ind}_+ B_+ = \text{ind}_+ A \quad \textit{and} \quad \text{ind}_- B_- = \text{ind}_- A \tag{9}$$

and, consequently,

$$\text{ind}\, A = \text{ind}_+ B_+ + \text{ind}_- B_-. \tag{10}$$

The remainder of this section is devoted to the proof of Theorem 10. We will verify this theorem by reducing its assertion step by step until we will arrive at operators on $l^p(\mathbb{Z}, \mathbb{C})$ (with scalar entries) for which the result is known (Theorem 6). The first step of the reduction procedure is based on the following observation.

Proposition 11 *Let $\mathcal{F}$ be a dense subset of the set of all Fredholm operators in $I + \mathcal{C}_E^\$$. If the assertion of Theorem* 10 *holds for all operators in $\mathcal{F}$, then it holds for all Fredholm operators in $I + \mathcal{C}_E^\$$.*

Proof. Let A be a Fredholm operator in $I + \mathcal{C}_E^\$$, and let $B \in \sigma_+(A)$. We will show that

$$\text{ind}_+ B = \text{ind}_+ A, \tag{11}$$

which settles the plus-assertion of (9). The minus-assertion follows similarly, and (9) implies (10) via (8).

To prove (11), choose a sequence (A_n) of operators in $\mathcal{F}$ which converges to A in the operator norm, and let h be a sequence tending to $+\infty$ such that $B = A_h$. Employing Cantor's diagonal method, we construct a subsequence g of h for which all limit operators $(A_n)_g$ exist. For the details of this construction, consult the proof of Proposition 1.2.6 in [14]. From Proposition 1.2.2 (e) in [14] we conclude that $\|B - (A_n)_g\| = \|A_g - (A_n)_g\| \to 0$. Now one has

$$\text{ind}_+ (A_n)_g = \text{ind}_+ A_n \quad \text{for all } n \in \mathbb{N}$$

and this implies (11) by letting n go to infinity due to the continuity of the index.
■

Our choice of the set $\mathcal{F}$ is as follows. The $\mathcal{C}_E^\$$-version of Lemma 9 allows one to approximate each band-dominated operator $A = I + K$ with $K \in \mathcal{C}_E^\$$ by a sequence of band operators $A_n := I + K_n$ with $K_n \in \mathcal{C}_E^\$$. Each band operator K_n can be written as a sum

$$K_n = \sum_{k \in \mathbb{Z}} K_n^{(k)} V_k \tag{12}$$

with only finitely many non-vanishing items. The coefficients $K_n^{(k)}$ in (12) are operators of multiplication by sequences of compact operators on X, and these multiplication operators are rich whenever K_n is rich. From Theorem 2.1.16 in [14] we know that a multiplication operator is rich if and only if the set of its entries is relatively compact in $L(X)$. So we conclude from the equivalence between (a) and (d) in Proposition 3 that each coefficient $K_n^{(k)}$ in (12) can be approximated as closely as desired by a sequence $(K_{n,N}^{(k)})_{N \in \mathbb{N}}$ of multiplication operators the entries of which map $\text{im}\, \Pi_N$ into itself and act on $\text{im}\,(I_X - \Pi_N)$ as the zero operator. Thus, one can approximate the operator $A = I + K$ as closely as desired by band operators $A_{n,N} = I + K_{n,N}$ where the entries of $K_{n,N}$ map $\text{im}\, \Pi_N$ into itself and act as the zero operator on $\text{im}\,(I_X - \Pi_N)$. We denote the set of all operators $K_{n,N}$ of this form by $\mathcal{C}_{E,N}$. Note that the operators in $\mathcal{C}_{E,N}$ are automatically rich.

Further, if $A = I + K$ is a Fredholm operator, then the operators $A_{n,N} = I + K_{n,N}$ are Fredholm for all sufficiently large n and N. Thus, we can choose $\mathcal{F}$

as the set of all Fredholm operators $I + K_{n,N}$ with $K_{n,N} \in \mathcal{C}_{E,N}$. By Proposition 11, it remains to prove Theorem 10 for these operators.

We agree upon writing X_N in place of $\operatorname{im} \Pi_N$ if we want to consider $\operatorname{im} \Pi_N$ as a Banach space in its own right, not as a subspace of X. Further we introduce the mappings

$$R : l^p(\mathbb{Z}, X) \to l^p(\mathbb{Z}, X_N), \quad (x_n) \mapsto (\Pi_N x_n)$$

where $\Pi_N x_n$ is considered as an element of X_N, and

$$L : l^p(\mathbb{Z}, X_N) \to l^p(\mathbb{Z}, X), \quad (x_n) \mapsto (x_n)$$

where the x_n on the right-hand side are considered as elements of X. Clearly, RL is the identity operator on $l^p(\mathbb{Z}, X_N)$, whereas LR is the projection

$$\Pi : l^p(\mathbb{Z}, X) \to l^p(\mathbb{Z}, X), \quad (x_n) \mapsto (\Pi_N x_n),$$

now with the $\Pi_N x_n$ being considered as elements of X. We are going to show that the operators $A = I + K_{n,N}$ as well as their limit operators behave well under the mapping $A \mapsto RAL$.

Proposition 12 *Let $A = I + K_{n,N}$ with $K_{n,N} \in \mathcal{C}_{E,N}$.*
(a) If A is a Fredholm operator on $l^p(\mathbb{Z}, X)$, then RAL is a Fredholm operator on $l^p(\mathbb{Z}, X_N)$, and the Fredholm indices of A and RAL coincide.
(b) If the limit operator of A with respect to a sequence $h : \mathbb{N} \to \mathbb{Z}$ exists, then the limit operator of RAL with respect to h exists, too, and $(RAL)_h = RA_hL$.

Proof. Since A is Fredholm, there are operators B, T on $l^p(\mathbb{Z}, X)$ with T compact such that

$$BA = I + T. \tag{13}$$

For $x \in \ker A$ one gets $x + Tx = 0$, whence $x \in \operatorname{im} T$. Hence, $\dim \ker A \le \operatorname{rank} T$ for each pair (B, T) such that (13) holds. One can choose the pair (B, T) even in such a way that $\dim \ker A = \operatorname{rank} T$. For write X as a direct sum $\ker A \oplus X_0$ and let $P_{\ker A}$ refer to the projection from X onto $\ker A$ parallel to X_0. Then

$$A(I - P_{\ker A}) : \operatorname{im} (I - P_{\ker A}) \to \operatorname{im} A$$

is an invertible operator. Let B denote its inverse. Then $BA(I - P_{\ker A}) = I - P_{\ker A}$ and

$$BA = I - P_{\ker A} + BAP_{\ker A} = I - (I - BA)P_{\ker A}.$$

Clearly, $\operatorname{rank} (I - BA)P_{\ker A} \le \dim \ker A$. Thus, one can indeed assume that (13) holds with $\dim \ker A = \operatorname{rank} T$. From (13) we get

$$RBAL = RL + RTL = I + RTL,$$

and since $L = \Pi L$ and A commutes with Π, we obtain

$$RBL\,RAL = I + RTL. \tag{14}$$

In the same way, $AB = I+T'$ with T' compact implies that $RAL\,RBL = I+RT'L$ with $RT'L$ compact. Hence, RAL is Fredholm, and (14) moreover shows that

$$\dim\ker RAL \le \operatorname{rank} RTL \le \operatorname{rank} T = \dim\ker A.$$

For the reverse estimate, let B, T be operators on $l^p(\mathbb{Z}, X_N)$ with $BRAL = I+T$ and $\dim\ker RAL = \operatorname{rank} T$. Then $LBRALR = LR + LTR$, whence

$$(LBR\Pi + I - \Pi)A = I + LTR$$

(take into account that $A\Pi = \Pi A = A - (I - \Pi)$). This identity shows that

$$\dim\ker A \le \operatorname{rank} LTR \le \operatorname{rank} T = \dim\ker RAL,$$

whence finally $\dim\ker A = \dim\ker RAL$. In the same way one gets $\dim\ker A^* = \dim\ker (RAL)^*$. Since $\dim\ker A^* = \dim\operatorname{im} A$ for each Fredholm operator A, we arrive at assertion (a).

(b) Let A_h be a limit operator of A. Then, by definition,

$$\|(A_h - V_{-h(n),\,X} A V_{h(n),\,X}) P_{k,\,X}\| \to 0 \quad \text{for each } k \in \mathbb{N}.$$

Thus,

$$\|R(A_h - V_{-h(n),\,X} A V_{h(n),\,X}) P_{k,\,X} L\| \to 0 \quad \text{for each } k \in \mathbb{N}.$$

Since the projection Π commutes with each of the operators $P_{k,\,X}$, $V_{h(n),\,X}$ and A, and since

$$RV_{h(n),\,X} L = V_{h(n),\,X_N} \quad \text{and} \quad RP_{k,\,X} L = P_{k,\,X_N},$$

one concludes that

$$\|(RA_hL - V_{-h(n),\,X_N} RAL\, V_{h(n),\,X_N}) P_{k,\,X_N}\| \to 0 \quad \text{for each } k \in \mathbb{N}.$$

Similarly one obtains

$$\|P_{k,\,X_N}(RA_hL - V_{-h(n),\,X_N} RAL\, V_{h(n),\,X_N})\| \to 0 \quad \text{for each } k \in \mathbb{N}.$$

Thus, RA_hL is the limit operator of RAL with respect to the sequence h. ■

Since the projections P and Π also commute, it is an immediate consequence of the preceding proposition and its proof that

$$\operatorname{ind}_+ A = \operatorname{ind}_+ RAL$$

and

$$\operatorname{ind}_+ A_h = \operatorname{ind}_+ RA_hL = \operatorname{ind}_+ (RAL)_h$$

for each limit operator $A_h \in \sigma_+(A)$. Thus, the assertion of Theorem 6 will follow once we have proved this theorem for band-dominated operators on $l^p(\mathbb{Z}, X_N)$ in place of $l^p(\mathbb{Z}, X)$.

Proposition 13 *The assertion of Theorem* 10 *holds for all Fredholm band-dominated operators on* $l^p(\mathbb{Z}, X_N)$ *(with fixed* $N \in \mathbb{N}$*).*

Proof. Let $d < \infty$ be the dimension of X_N, and let $e_1, \ldots, e_d$ be a basis of X_N. Then there are positive constants C_1, C_2 such that

$$C_1 \|(x_1, \ldots, x_d)\|_{l^p} \le \|x_1 e_1 + \ldots + x_d e_d\|_{X_N} \le C_2 \|(x_1, \ldots, x_d)\|_{l^p} \tag{15}$$

for each vector $(x_1, \ldots, x_d) \in \mathbb{C}^d$. Define $J : l^p(\mathbb{Z}, X_N) \to l^p(\mathbb{Z}, \mathbb{C})$ by

$$(Jx)_{nd+r} := (x_n)_r, \qquad 0 \le r \le d-1$$

where $(x_n)_r$ refers to the rth coordinate of the nth entry $x_n \in X_N$ of the sequence x. It follows from (15) that

$$C_1 \|Jx\|_{l^p(\mathbb{Z},\mathbb{C})} \le \|x\|_{l^p(\mathbb{Z},X_N)} \le C_2 \|Jx\|_{l^p(\mathbb{Z},\mathbb{C})},$$

i.e., J is a topological isomorphism from $l^p(\mathbb{Z}, X_N)$ onto $l^p(\mathbb{Z}, \mathbb{C})$. The definition of J implies that if A is a Fredholm band operator on $l^p(\mathbb{Z}, X_N)$, then JAJ^{-1} is a Fredholm band operator on $l^p(\mathbb{Z}, \mathbb{C})$, and conversely. Moreover, $\operatorname{ind} A = \operatorname{ind} JAJ^{-1}$ in this case. This identity holds for the plus- and minus-indices as well, since $JP_{X_N}J^{-1} = P_{\mathbb{C}}$. Moreover, one has

$$JV_{n, X_N} J^{-1} = V_{dn,\mathbb{C}} \quad \text{and} \quad JP_{k, X_N} J^{-1} = P_{dk,\mathbb{C}}$$

for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. These equalities imply that if A_h is the limit operator of the band-dominated operator $A \in l^p(\mathbb{Z}, X_N)$ with respect to the sequence h, then JA_hJ^{-1} is the limit operator of JAJ^{-1} with respect to the sequence $dh : \mathbb{N} \to \mathbb{Z}$, $m \mapsto dh(m)$, i.e.,

$$(JAJ^{-1})_{dh} = JA_hJ^{-1}.$$

Summarizing, we obtain

$$\operatorname{ind}_+ A = \operatorname{ind}_+ JAJ^{-1}$$

and

$$\operatorname{ind}_+ A_h = \operatorname{ind}_+ JA_hJ^{-1} = \operatorname{ind}_+ (JAJ^{-1})_{dh}$$

for each Fredholm band-dominated operator A on $l^p(\mathbb{Z}, X_N)$ and for each of its limit operators $A_h \in \sigma_+(A)$. Since dh tends to $+\infty$ whenever h does, one has $(JAJ^{-1})_{dh} \in \sigma_+(JAJ^{-1})$, and from Theorem 6 we infer that $\operatorname{ind}_+ JAJ^{-1} = \operatorname{ind}_+ (JAJ^{-1})_{dh}$. Thus, $\operatorname{ind}_+ A = \operatorname{ind}_+ A_h$ for each Fredholm band-dominated operator A on $l^p(\mathbb{Z}, X_N)$ and for each of its limit operators $A_h \in \sigma_+(A)$. The minus-counterpart of this assertion follows analogously. This proves the proposition and finishes the proof of Theorem 10. ∎

4 The Fredholm index of locally compact band-dominated operators on $L^p(\mathbb{R})$

This section is devoted to the proof of Theorem 1. As in the discrete case, the limit operators approach provides us with a criterion for the $\hat{\mathcal{P}}$-Fredholmness of an operator rather than for its common Fredholmness. Here, $\hat{\mathcal{P}} = (\hat{P}_n)_{n \geq 0}$ where $\hat{P}_n : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is the operator of multiplication by the characteristic function of the interval $[-n, n]$, i.e.,

$$(\hat{P}_n f)(x) = \begin{cases} f(x) & \text{if } x \in [-n, n] \\ 0 & \text{else,} \end{cases}$$

and $\hat{\mathcal{P}}$-compactness and $\hat{\mathcal{P}}$-Fredholmness are defined literally as in the discrete case. The following proposition can be proved as its discrete counterpart Proposition 7.

Proposition 14 *An operator $A \in L(L^p(\mathbb{R}))$ of the form $A = I + K$ with $K \in \mathcal{L}_p$ is Fredholm if and only if it is $\hat{\mathcal{P}}$-Fredholm.*

We will now prove Theorem 1 via a suitable discretization. Let χ_0 denote the characteristic function of the interval $[0, 1]$. The mapping $G : L^p(\mathbb{R}) \to l^p(\mathbb{Z}, L^p[0, 1])$ which sends the function $f \in L^p(\mathbb{R})$ to the sequence

$$Gf = ((Gf)_n)_{n \in \mathbb{Z}} \quad \text{where} \quad (Gf)_n := \chi_0 U_{-n} f$$

is a bijective isometry the inverse of which maps the sequence $x = (x_n)_{n \in \mathbb{Z}}$ to the function

$$G^{-1} x = \sum_{n \in \mathbb{Z}} U_n x_n \chi_0,$$

the series converging in $L^p(\mathbb{R})$. Thus, the mapping

$$\Gamma : L(L^p(\mathbb{R})) \to L(l^p(\mathbb{Z}, L^p[0, 1])), \quad A \mapsto GAG^{-1}$$

is an isometric algebra isomorphism. It is shown in Proposition 3.1.4 in [14] that

$$\Gamma(A_h) = (\Gamma(A))_h$$

for each limit operator A_h of an operator $A \in \mathcal{B}_p$, whereas Proposition 3.1.6 in [14] states that Γ maps $\mathcal{B}_p^\$$ onto $\mathcal{A}_E^\$$ with $E = l^p(\mathbb{Z}, L^p[0, 1])$. Further, if $A \in L^p(\mathbb{R})$ is a locally compact operator, then the entries of the matrix representation of its discretization $\Gamma(A)$ are compact operators. Thus, Γ maps $\mathcal{L}_p^\$$ into $I + \mathcal{C}_E^\$$. Finally, one evidently has

$$\text{ind}\, A = \text{ind}\, \Gamma(A)$$

for each operator $A \in L(L^p(\mathbb{R}))$, and the Banach space $L^p[0, 1]$ has the *sap* as already mentioned. Thus, the assertions of Theorem 1 follow immediately from their discrete counterparts Corollary 8 and Theorem 10. ■

5 Applications

As an application of Theorem 1, we are going to examine the Fredholm properties of operators of the form $I + K$ with $K \in \mathcal{K}_p(BUC)$. The latter stands for the smallest closed subalgebra of $L(L^p(\mathbb{R}))$ which contains all operators of the form $aCbI$ where $a, b \in BUC$ and where C is a Fourier convolution operator with L^1-kernel k. Thus,

$$(Cf)(x) = (k * f)(x) = \int_{\mathbb{R}} k(x-y)f(y)\,dy, \quad x \in \mathbb{R}.$$

In Proposition 3.3.6 in [14] it is verified that

$$\mathcal{K}_p(BUC) \subseteq \mathcal{L}_p^{\$}.$$

Hence, Theorem 1 applies to operators in $\mathcal{K}_p(BUC)$, and it yields the following.

Theorem 15 *Let $A \in L(L^p(\mathbb{R}))$ be a convolution type operator of the form $I+K$ with $K \in \mathcal{K}_p(BUC)$. Then*

(a) A is Fredholm if and only if all of its limit operators are invertible, and if the norms of their inverses are uniformly bounded.

(b) if A is Fredholm then, for arbitrary limit operators $B_\pm \in \sigma_\pm(A)$,

$$\operatorname{ind} A = \operatorname{ind}_+ B_+ + \operatorname{ind}_- B_-.$$

One cannot say much about the limit operators of a general operator $A \in I + \mathcal{K}_p(BUC)$. It is only clear that they belong to $I + \mathcal{K}_p(BUC)$ again. Thus, the computation of the plus- and minus indices of the limit operators of convolution type operators will remain a serious problem in general. In what follows we will discuss some instances where this computation can be easily done (slowly oscillating coefficients) or is at least manageable (slowly oscillating plus periodic coefficients).

Let SO stand for the set of all functions $f \in BUC$ which are *slowly oscillating* in the sense that

$$\lim_{t\to\pm\infty} \sup_{h\in[0,\,1]} |f(t) - f(t+h)| = 0.$$

This set forms a C^*-subalgebra of BUC. Let $\mathcal{K}_p(SO)$ stand for the smallest closed subalgebra of $\mathcal{K}_p(BUC)$ which contains all operators of the form $aCbI$ where $a, b \in SO$ and where C is a Fourier convolution with L^1-kernel. Further, we write PER for the C^*-subalgebra of BUC which consists of all continuous functions of period 1 on $\mathbb{R}$. By $\mathcal{K}_p(PER, SO)$ we denote the smallest closed subalgebra of $\mathcal{K}_p(BUC)$ which contains all operators of the form $aCbI$ where now $a, b \in PER + SO$ and where C is again a Fourier convolution with L^1-kernel. Similarly, $\mathcal{K}_p(PER)$ refers to the smallest closed subalgebra of $\mathcal{K}_p(BUC)$ which contains all operators $aCbI$ with $a, b \in PER$ and with a Fourier convolution C with L^1-kernel.

Lemma 16 *The limit operators of operators in $\mathcal{K}_p(SO)$ are operators of Fourier convolution with L^1-kernel, and all limit operators of operators in $\mathcal{K}_p(PER,\, SO)$ belong to $\mathcal{K}_p(PER)$.*

Proof. Operators of convolution are shift invariant with respect to arbitrary shifts, and operators of multiplications by functions in PER are invariant with respect to integer shifts. Hence, operators of this form as well as there sums and products possess exactly one limit operator, namely the operator itself. Further, as it has been pointed out in Proposition 3.3.9 in [14], all limit operators of operators of multiplication by slowly oscillating functions are constant multiples of the identity operator, whence the assertion. ■

Hence, the determination of the index of a Fredholm operator in $I + \mathcal{K}_p(SO)$ requires the computation of the plus- and the minus index of an operator of the form $I + C$ where C is a Fourier convolution with kernel $k \in L^1(\mathbb{R})$. Equivalently, one has to determine the common Fredholm index of operators of the form $I + \chi_\pm C \chi_\pm I$. The operator $I + \chi_+ C \chi_+ I$ is the Wiener-Hopf operator with generating function $1+a$ where a is the Fourier transform of the kernel k of C. After reflection at the origin, the operator $I + \chi_- C \chi_- I$ also becomes a Wiener-Hopf operator.

The Fredholm property of Wiener-Hopf operators of this type is well understood (see [1, 3, 7]). Since

$$\lim_{x\to+\infty} a(x) = \lim_{x\to-\infty} a(x) = 0,$$

one can consider $1 + a$ as a continuous function on the one-point compactification $\dot{\mathbb{R}}$ of the real line, which is also called the *symbol* of the operator. It turns out that the Wiener-Hopf operator with symbol $1 + a$ is Fredholm if and only if the function $1 + a$ does not vanish on $\dot{\mathbb{R}}$, and that in this case its Fredholm index is the negative winding number of the closed curve $1 + a(\dot{\mathbb{R}})$ around the origin. This solves the problem of computing the Fredholm index of an operator in $I + \mathcal{K}_p(SO)$ completely and in an easy way.

Let us now turn over to the setting of operators in $I + \mathcal{K}_p(PER,\, SO)$. Here we are left with the problem to determine the Fredholm index of operators of the form $\chi_+(I + K)\chi_+ I$ on $L^p(\mathbb{R}^+)$ where $K \in \mathcal{K}_p(PER)$. The proofs of Theorems 1 and 10 given above offer a way to perform this calculation. The decisive point is that, due to the periodicity, the operator

$$\Gamma(\chi_+(I + K)\chi_+ I) \in L(l^p(\mathbb{Z}^+,\, L^p[0,\, 1])) \tag{16}$$

is a band-dominated *Toeplitz operator* the entries of which are of the form *I + compact* if they are located on the main diagonal, whereas they are compact when located outside the main diagonal. Recall that a Toeplitz operator on $l^p(\mathbb{Z}^+,\, X)$ is an operator with matrix representation $(A_{i-j})_{i,j\in\mathbb{Z}^+}$, i.e., the entries of the matrix are constant along each diagonal which is parallel to the main diagonal.

If now $I + K$ is Fredholm on $L^p(\mathbb{R})$, then the Toeplitz operator (16) is Fredholm, too, and it has the same index. Employing the reduction procedure used in the proof of Theorem 10, one can further approximate the Toeplitz operator (16) by a Toeplitz operator on $l^p(\mathbb{Z}^+, \mathbb{C}^N)$ with band structure which is also Fredholm and has the same index as the original operator $I+K$. Thus, we are left with the determination of the index of a common Toeplitz operator $T(g)$ on $l^p(\mathbb{Z}^+, \mathbb{C}^N)$ where each entry g_{ij} of the generating function $g : \mathbb{T} \to \mathbb{C}^{N\times N}$ is a trigonometric polynomial. This operator can be identified with an operator matrix $(T(g_{ij}))_{i,j=1}^N$ where each $T(g_{ij})$ is a Toeplitz band operator on $l^p(\mathbb{Z}^+, \mathbb{C}) = l^p(\mathbb{Z}^+)$. As it is well known (see, e.g., Theorem 6.12 in [1]), this operator is Fredholm if and only if the common Toeplitz operator (with scalar-valued polynomial generating function) $T(\det g)$ is Fredholm, and the indices of these operators coincide. Moreover, the index of $T(\det g)$ is equal to the negative winding number of the function $\det g$ with respect to the origin.

For a general account on matrix functions and the Toeplitz and Wiener-Hopf operators generated by them, we refer to the monographs [2] and [9]. For general results about relations between the Fredholmness of a block operator and its determinant one should consult Chapter 1 in [8].

A similar approach is possible for operators in $I + \mathcal{K}_p(PER_{\mathbb{Z}}, SO)$ where $PER_{\mathbb{Z}}$ stands for the set of all functions with integer period. After discretization and approximation as above, one finally arrives at a block Toeplitz operator in place of (16) which again can be reduced to a matrix of Toeplitz operators on $l^p(\mathbb{Z}^+)$.

The results of Theorems 1, 10 and 15 can be completed by an observation made in [15] for the case of band-dominated operators on $l^p(\mathbb{Z}, \mathbb{C})$. This observation concerns the independence of the Fredholm index on p. To make this statement precise we have to explain what is meant by a band-dominated operator which acts on different l^p-spaces (notice that the class of all band operators is independent of p whereas the algebra $\mathcal{A}_E$ of all band-dominated operators depends on the parameter p of $E = l^p(\mathbb{Z}, X)$ heavily).

Every infinite matrix $(a_{ij})_{i,j\in\mathbb{Z}}$ induces an operator A on the Banach space $c_{00}(\mathbb{Z}, X)$ of all functions $x : \mathbb{Z} \to X$ with compact support by

$$i \mapsto (Ax)_i := \sum_{j\in\mathbb{Z}} a_{ij} x_j.$$

We say that A *extends to a bounded linear operator on* $l^p(\mathbb{Z}, X)$ or that A *acts on* $l^p(\mathbb{Z}, X)$ if $Ax \in l^p(\mathbb{Z}, X)$ for each $x \in c_{00}(\mathbb{Z}, X)$ and if there is a constant C such that $\|Ax\|_p \le C\|x\|_p$ for each $x \in c_{00}(\mathbb{Z}, X)$. If A extends to a band-dominated operator on both $l^p(\mathbb{Z}, X)$ and $l^r(\mathbb{Z}, X)$, then we say that A is a *band-dominated operator on* $l^p(\mathbb{Z}, X)$ *and* $l^r(\mathbb{Z}, X)$. Otherwise stated: we consider two band-dominated operators B and C acting on $l^p(\mathbb{Z}, X)$ and $l^r(\mathbb{Z}, X)$, respectively, as identical, and we denote them by the same letter, if their matrix representations coincide.

Proposition 17 *Let* $A \in I + \mathcal{C}_E^{\$}$ *be a Fredholm band-dominated operator both on* $E = l^p(\mathbb{Z}, X)$ *and on* $E = l^r(\mathbb{Z}, X)$ *with* $1 < r < p < \infty$. *Then* A *is a Fredholm band-dominated operator on each space* $l^s(\mathbb{Z}, X)$ *with* $r < s < p$, *and the Fredholm index* $\mathrm{ind}_s A$ *of* A, *considered as an operator on* $l^s(\mathbb{Z}, X)$, *is independent of* $s \in [r, p]$.

The proof follows exactly the line of the proof of Theorem 10, finally reducing the assertion of the proposition to the case $X = \mathbb{C}$ which is treated in [15]. It should be also mentioned that Proposition 17 remains valid for band-dominated operators on $L^p(\mathbb{Z}^N, X)$ with N a positive integer which also follows from [15].

In combination with Theorems 1 and 15 one gets the following corollary.

Corollary 18 *(a) Let* $A \in I + \mathcal{L}_q^{\$}$ *be a Fredholm band-dominated operator both for* $q = p$ *and for* $q = r$ *with* $1 < r < p < \infty$. *Then* A *is a Fredholm band-dominated operator on each space* $L^s(\mathbb{R})$ *with* $r < s < p$, *and the Fredholm index* $\mathrm{ind}_s A$ *of* A, *considered as an operator on* $L^s(\mathbb{R})$, *is independent of* $s \in [r, p]$.

(b) Let $A \in I + \mathcal{K}_q(BUC)$ *be a Fredholm convolution type operator both for* $q = p$ *and for* $q = r$ *with* $1 < r < p < \infty$. *Then* A *is a Fredholm convolution type operator on each space* $L^s(\mathbb{R})$ *with* $r < s < p$, *and the Fredholm index* $\mathrm{ind}_s A$ *of* A, *considered as an operator on* $L^s(\mathbb{R})$, *is independent of* $s \in [r, p]$.

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