Approximate Foveated Images and Reconstruction of their Uniform Pre-Images

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Abstract

Approximate foveated images can be obtained from uniform images via the approximation of some integral operators. In this paper it is shown that these operators belong to a well studied operator algebra, and the problem of restoration of the approximate uniform pre-images is considered. Under common assumptions on smoothness of the integral operator kernels, necessary and sufficient conditions are established for such procedure to be feasible.

1 Introduction

Foveated images are used in image processing to reduce information in the visual field while preserving resolution at a given point – the fovea. Technically, foveated images can be obtained from uniform images in different way, cf. [2, 3, 7, 8, 10]. One of the approaches proposed is based on the use of integral operators

$$(T\varphi)(x) = \int_{-\infty}^{+\infty} k(t,x)\varphi(t)\,dt \tag{1}$$

with special kernels k, see [3]. The function φ in (1) is the initial signal (uniform image) and $T\varphi$ represents its foveated image. To implement this method, Chang, Mallat and Yap [3] employed a wavelet approximation φ_n to the signal φ with subsequent approximation of $T\varphi_n$. More precisely, their approach can be characterized as follows. Let g be a function on the real line $\mathbb{R}$, and let $\omega = \omega_{\gamma,\beta}(x)$, $x \in \mathbb{R}$ denote the function

$$\omega_{\gamma,\beta}(x) := \beta|x-\gamma|, \quad \gamma \in \mathbb{R},\ \beta \in \mathbb{R}^+ \setminus \{0\}.$$

For a given positive number m, one considers the linear operator Q_m defined on $L^2(\mathbb{R})$ by

$$(Q_m f)(x) := \begin{cases} f(x) & \text{if } x \in [-m,\, m] \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let P_n denote the orthogonal projection onto a subspace of $L^2(\mathbb{R})$ generated by a wavelet basis. In [3], the following approximate foveated image ψ_n^m of the signal φ is considered:

$$\psi_n^m(x) = (P_n Q_m T Q_m P_n \varphi)(x) \tag{2}$$

where T is the operator (1) with kernel

$$k(x,t) := \frac{1}{\omega_{\gamma,\beta}(x)}\, g\left(\frac{t-x}{\omega_{\gamma,\beta}(x)}\right). \tag{3}$$

In this setting, the parameter γ represents the point of highest resolution and is called the fovea. The parameter β determines the speed at which the resolution falls off when the distance from the fovea grows, cf. [3].

Note that the operators $P_n Q_m T Q_m P_n$, $n \in \mathbb{N}$, can be viewed as Galerkin approximations for the operator $Q_m T Q_m$. Sequences of Galerkin approximations have been studied in connection with different problems of analysis and mathematical physics. If the kernel k has the form (3), then the corresponding operator T can be represented in a special way. Namely, if U_γ, $\gamma \in \mathbb{R}$, denotes the shift operator

$$(U_\gamma f)(x) = f(x-\gamma), \quad x \in \mathbb{R},$$

then T can be rewritten as

$$T = U_\gamma T_{g,\beta} U_{-\gamma} \tag{4}$$

where $T_{g,\beta}$ is the integral operator (1) with the fovea $\gamma = 0$, i.e.

$$(T_{g,\beta} f)(x) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\beta|x|} g\left(\frac{t-x}{\beta|x|}\right) dt. \tag{5}$$

Therefore, without loss of generality, we can restrict our attention to the operator $T_{g,\beta}$ only. Moreover, as we will see later, the operator $T_{g,\beta}$ belongs to a class of integral operators which is quite well understood. This enables us to examine the properties of the foveated image $T\varphi$ in more detail.

In the present paper we consider the following problems. Assume that an approximate foveated image ψ_n^m is known. Is it possible to reconstruct its approximate uniform pre-image $Q_m P_n \varphi$? It is obvious that, in general, the answer to this question is negative. However, one can try to find conditions which would make such a reconstruction feasible. Another relevant problem is that of the quality of the approximate uniform images one obtains. More precisely, if φ_n, $n \in \mathbb{N}$ is an approximation for φ, then what can be said about the errors $\varphi - \varphi_n$, at least for large n? In the present paper these problems are studied for the Galerkin approximation. For the sake of simplicity, piecewise constant splines are used to approximate the uniform and foveated images, although other approximation spaces can be used, too. Approximations based on splines of higher order are briefly discussed in the concluding part of the paper. It is also worth mentioning that the Galerkin scheme can be replaced by other approximation procedures, for example by collocation, quadrature or qualocation, which are often more convenient from a computational point of view.

Throughout this paper, we let p and α be real parameters with $1 < p < \infty$ and $0 < \alpha + 1/p < 1$ and, given an interval $I \subseteq \mathbb{R}$, we write $L^p(I, \alpha)$ for the Banach space of all Lebesgue measurable functions $f : I \to \mathbb{C}$ such that

$$\|f\|^p = \|f\|^p_{p,\,\alpha,\,I} := \int_I |f(t)|^p |t|^{\alpha p}\, dt < \infty. \tag{6}$$

As usual, the Banach dual $L^p(I,\,\alpha)^*$ of $L^p(I,\,\alpha)$ will be identified with $L^q(I,\,-\alpha)$ where $1/p + 1/q = 1$ with respect to the sesqui-linear form

$$\langle f,\, g\rangle := \int_I f(t)\overline{g(t)}\, dt.$$

Given a linear space X and a positive integer r, we denote by X_r the linear space of column vectors of length r with components from X, and we let $X^{r\times r}$ refer to the linear space of $r \times r$ matrices with entries from X. Further, $\mathfrak{B}(X)$ denotes the Banach algebra of all bounded linear operators on the Banach space X, and $\operatorname{im} A$ stands for the range of the operator $A \in \mathfrak{B}(X)$.

2 Integral Operators of Mellin type

In this section, we represent the operator $T_{g,\,\beta}$ defined by (5) in a special form which will prove to be helpful in what follows. In [3], $T_{g,\,\beta}$ has been studied on the space $L^2(\mathbb{R})$, whereas we will allow this operator to act on the weighted space $L^p(\mathbb{R}, \alpha)$.

It is convenient in what follows to identify the space $L^p(\mathbb{R}, \alpha)$ with the space $L^p_2(\mathbb{R}^+,\, \alpha)$ which consists of all pairs $(f_1,\, f_2)^T$ with $f_1,\, f_2 \in L^p(\mathbb{R}^+,\, \alpha)$. If we provide the space $L^p_2(\mathbb{R}^+,\, \alpha)$ with the norm

$$\|(f_1,\, f_2)^T\|^p := \|f_1\|^p_{p,\,\alpha,\,\mathbb{R}^+} + \|f_2\|^p_{p,\,\alpha,\,\mathbb{R}^+},$$

then the mapping

$$\eta : L^p(\mathbb{R},\, \alpha) \to L^p_2(\mathbb{R}^+,\, \alpha), \quad \eta(f) : s \mapsto (f(s),\, f(-s))^T \tag{7}$$

becomes an isometric bijection the inverse of which acts via

$$(\eta^{-1}[(f_1, f_2)^T])(s) = \begin{cases} f_1(s) & \text{if} \quad s \in \mathbb{R}^+ \\ f_2(-s) & \text{if} \quad s \in \mathbb{R}^-. \end{cases} \tag{8}$$

Thus, the mapping

$$\psi_\eta : \mathfrak{B}(L^p(\mathbb{R},\, \alpha)) \to \mathfrak{B}(L^p_2(\mathbb{R}^+,\, \alpha)), \quad A \mapsto \eta A \eta^{-1}$$

is an isometric algebra isomorphism and, therefore, the properties of the operator $\psi_\eta(A)$ completely reflect the corresponding properties of A and vice versa. There are, however, some instances where the operator $\psi_\eta(A)$ has a nicer structure than the operator A. In particular, it will turn out in a moment that the entries of the operator $\psi_\eta(A_{g,\beta})$ are Mellin convolution operators, a class of operators which is defined as follows.

Let M and M^{-1} denote the direct and inverse Mellin transform, respectively, i.e.

$$(Mf)(z) = \int_0^{+\infty} x^{1/p+\alpha+1-iz} f(x)\, dx, \quad z \in \mathbb{R}$$

and

$$(M^{-1}f)(x) = \frac{1}{2\pi} \int_0^{+\infty} x^{-1/p-\alpha+iz} f(z)\, dz, \quad x \in \mathbb{R}^+.$$

It is well known (see, e.g., [5], pp. 47-48) that if b is an $L^p(\mathbb{R})$-Fourier multiplier, then

$$\mathcal{M}(b) := MbM^{-1} \tag{9}$$

defines a bounded linear operator $\mathcal{M}(b)$ on $L^p(\mathbb{R}^+,\, \alpha)$, the so-called Mellin operator with symbol b. In case the kernel function $k := M^{-1}b$ belongs to $L^1(\mathbb{R})$ with respect to the measure ds/s, the Mellin operator (9) can be represented as the integral operator

$$(\mathcal{M}(b)f)(s) = \int_0^{+\infty} k(s/\sigma) f(\sigma)\, \frac{d\sigma}{\sigma}, \quad s \in \mathbb{R}^+. \tag{10}$$

Proposition 2.1 *Let the operator $T_{g,\beta} \in \mathfrak{B}(L^p(\mathbb{R},\, \alpha))$ be defined by (5). Then $\psi_\eta(T_{g,\beta})$ is the block Mellin operator $\mathcal{M}(B_{g,\beta})$ with symbol $B_{g,\beta} = M^{-1}G_{g,\beta}$ where*

$$G_{g,\beta}(t) = \frac{1}{\beta t} \begin{pmatrix} g\left(\frac{1}{\beta}\left(\frac{1}{t} - 1\right)\right) & g\left(\frac{1}{\beta}\left(-\frac{1}{t} - 1\right)\right) \\ g\left(\frac{1}{\beta}\left(\frac{1}{t} + 1\right)\right) & g\left(\frac{1}{\beta}\left(-\frac{1}{t} + 1\right)\right) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{11}$$

Proof. The operator $T_{g,\beta}\eta^{-1}$ acts as follows:

$$\begin{aligned}(T_{g,\beta}\eta^{-1}&[(f_1, f_2)^T])(s) \\ &= \frac{1}{\beta}\int_0^{+\infty} f_1(t)\, \frac{t}{|s|}\, g\left(\frac{t-s}{\beta|s|}\right) \frac{dt}{t} + \frac{1}{\beta}\int_{-\infty}^0 f_2(-t)\, \frac{t}{|s|}\, g\left(\frac{t-s}{\beta|s|}\right) \frac{dt}{t}. \end{aligned} \tag{12}$$

The second term on the right-hand side of this equality can be rewritten as

$$\frac{1}{\beta}\int_0^{+\infty} f_2(u)\,\frac{u}{|s|} g\left(\frac{-u-s}{\beta|s|}\right)\frac{du}{u}.$$

Hence, by (7),

$$\begin{aligned}&\eta\left(\frac{1}{\beta}\int_{-\infty}^{0} f_2(-t)\left(\frac{t}{|\cdot|}\right) g\left(\frac{t-\cdot}{\beta|\cdot|}\right)\frac{dt}{t}\right)(s)\\ &= \begin{pmatrix} \frac{1}{\beta}\int_0^{+\infty} f_2(u)\left(\frac{u}{s}\right) g\left(\frac{1}{\beta}\left(-\frac{u}{s}-1\right)\right)\frac{du}{u} \\ \frac{1}{\beta}\int_0^{+\infty} f_2(u)\left(\frac{u}{s}\right) g\left(\frac{1}{\beta}\left(-\frac{u}{s}+1\right)\right)\frac{du}{u}\end{pmatrix}, \quad s\in\mathbb{R}^+. \end{aligned} \tag{13}$$

Performing analogous transformations for the first term on the right-hand side of (12) we get

$$\eta T_{g,\beta}\eta^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix} \tag{14}$$

where each of the operators T_{rl} is a Mellin operator of the form (10) with a kernel defined by the corresponding entry of the matrix (11). ■

3 Galerkin approximations of the foveated images

Let $\overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ be the compactification of the real axis by the two points $\pm\infty$, and let $C(\overline{\mathbb{R}})$ refer to the algebra of all complex-valued functions f which are continuous on $\mathbb{R}$ and possess finite limits $f(\pm\infty)$ at $\pm\infty$. Further, let $\dot{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ refer to the one-point compactification of the real axis, and let $C(\dot{\mathbb{R}})$ consist of all complex-valued functions in $C(\overline{\mathbb{R}})$ for which $f(-\infty) = f(+\infty)$. We denote this common value by $f(\infty)$.

From now on we assume that the entries of the matrix $B_{g,\beta}$ have finite total variation on $\mathbb{R}$ and belong to $C(\overline{\mathbb{R}})$. These conditions imply the boundedness of the operator $T_{g,\beta}$ on $L^p(\mathbb{R},\,\alpha)$, cf. [5]. If, in addition, $\det B_{g,\beta}(t)\neq 0$ for all $t\in\overline{\mathbb{R}}$, then the operator $T_{g,\beta}$ is invertible, and its inverse is the Mellin operator $\mathcal{M}(B_{g,\beta}^{-1})$. Note that the invertibility of the operator $T_{g,\beta}$ does not play any role if one only considers the approximate foveated image ψ_n^m. However, this condition cannot be avoided if we want to restore the approximate uniform image $Q_m P_n\varphi$.

Let $\chi:\mathbb{R}\to\{0,\,1\}$ be the characteristic function of the interval $[0,1)$, and let I denote one of the sets $\mathbb{R}$ or $\mathbb{R}^+$. For each fixed natural number n, we consider the functions

$$\varphi_{nj}(t) := \chi(nt-j), \quad j\in\mathbb{Z},$$

and we denote by $S_n(I)$ the smallest closed subspace of $L^p(I,\,\alpha)$ which contains all functions φ_{nj}, $j \in \mathbb{Z}$, which have their support in I. Further we introduce the Galerkin projections $P_n^I : L^p(I,\,\alpha) \to S_n(I)$ by

$$P_n^I f := n \sum_{k \in \mathbb{Z} \cap I} \langle f,\, \varphi_{nk} \rangle \varphi_{nk}.$$

To simplify notations, we abbreviate $P_n^{\mathbb{R}}$ to P_n and $P_n^{\mathbb{R}^+}$ to P_n^+.

What we are interested in is approximations ψ_n^m of the foveated image of the signal $\varphi \in L^p(\mathbb{R},\,\alpha)$ of the form

$$\psi_n^m := P_n Q_m T_{g,\beta} Q_m P_n \varphi. \tag{15}$$

As already mentioned, such kind of approximate foveated images has been considered in [3] (based on a projection onto a space of wavelets in place of the spline projection P_n). Assume that ψ_n^m is known. Are there any conditions which allow us to restore the approximate uniform image $Q_m P_n \varphi$ (provided the parameter β and the smoothing function g are known)?

To put these questions into an appropriate context, we have to recall some notions from numerical analysis. Let X be a Banach space, and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of projections on X which converges strongly to the identity operator. As usual, strong convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ to an operator $A \in \mathfrak{B}(X)$ means that $\lim_{n\to\infty} A_n x = Ax$ for every $x \in X$. Consider an operator equation

$$Ax = y, \quad x,\, y \in X, \quad A \in \mathfrak{B}(X) \tag{16}$$

and a sequence of its approximations

$$A_n L_n x_n = L_n y, \quad x_n \in \operatorname{im} L_n, \quad A_n \in \mathfrak{B}(\operatorname{im} L_n). \tag{17}$$

Regarding the approximation operators A_n, one usually assumes that equations (17) are consistent with equation (16) in the sense that the sequence $(A_n L_n)$ converges strongly to the operator A.

Definition 3.1 *The approximation method* (17) *is* stable *if there is a number n_0 such that the operators $A_n : \operatorname{im} L_n \to \operatorname{im} L_n$ are invertible for all $n \geq n_0$ and if*

$$M := \sup_{n \geq n_0} \|A_n^{-1}\| < \infty. \tag{18}$$

Let the operator A be invertible and the approximation method (17) be stable, and let x^* and x_n^*, $n \geq n_0$, denote the solutions of equations (16) and (17), respectively. Then one easily gets the error estimate

$$\|x^* - x_n^*\| \leq M\|Ax^* - A_n L_n x^*\| + \|x^* - L_n x^*\| \tag{19}$$

where M is defined by (18).

4 Invertibility of $P_n T_{g,\beta} P_n$

We start our considerations with studying the stability of the following approximation method

$$P_n T_{g,\beta} P_n \varphi_n = P_n \psi, \quad n \in \mathbb{N}, \quad \varphi_n \in \operatorname{im} P_n \tag{20}$$

where P_n are the above defined projections, and ψ is the foveated image of the initial signal φ. Here we assume that the user knows $P_n\psi$ and wants to restore φ_n.

Proposition 4.1 *Let $B_{g,\beta} \in C^{2\times 2}(\overline{\mathbb{R}})$, and let the entries of $B_{g,\beta}$ have finite total variation on $\mathbb{R}$. Then the approximation method* (20) *is stable if and only if the operator $P_1 T_{g,\beta} P_1 : S_1(\mathbb{R}) \to S_1(\mathbb{R})$ is invertible.*

Proof. Let $l^p(\mathbb{Z}, \alpha)$ refer to the set of all sequences $(\xi_j)_{j\in\mathbb{Z}}$ of complex numbers such that

$$\|(\xi_j)\|^p := \sum_{j\in\mathbb{Z}} |\xi_j|^p (1 + |j|)^{\alpha p} < \infty.$$

For every natural number n, we consider the operators

$$E_n : l^p(\mathbb{Z}, \alpha) \to S_n(\mathbb{R}), \quad (\xi_j)_{j\in\mathbb{Z}} \to \sum_{j\in\mathbb{Z}} \xi_j \varphi_{nj}.$$

It is well-known (see, e.g., [1]) that these operators possess continuous inverses $E_{-n} := E_n^{-1} : S_n(\mathbb{R}) \to l^p(\mathbb{Z}, \alpha)$ and that there is a constant C such that

$$\|E_n\| \le C\, n^{-(1/p+\alpha)} \quad \text{and} \quad \|E_{-n}\| \le C\, n^{1/p+\alpha}. \tag{21}$$

Hence, the operators

$$P_n T_{g,\beta} P_n : \quad S_n(\mathbb{R}) \to S_n(\mathbb{R})$$

are invertible if and only if the corresponding operators

$$E_n P_n T_{g,\beta} P_n E_{-n} : \quad l^p(\mathbb{Z}, \alpha) \to l^p(\mathbb{Z}, \alpha)$$

are so. Consider the matrix representation $A_n = (A_{jk})_{j,k\in\mathbb{Z}}$ of the operator $E_n P_n T_{g,\beta} P_n E_{-n}$ with respect to the standard basis of $l^p(\mathbb{Z}, \alpha)$. Straightforward calculations yield

$$\begin{aligned} A_{jk} &= \frac{n}{\beta} \int_{\mathbb{R}} \chi(nx - j) \int_{\mathbb{R}} \frac{1}{|x|} g\left(\frac{1}{\beta}\left(\frac{t-x}{|x|}\right)\right) \chi(nt - k)\, dt\, dx \\ &= \frac{1}{\beta} \int_j^{j+1} \int_k^{k+1} \frac{1}{|x|} g\left(\frac{1}{\beta}\left(\frac{t-x}{|x|}\right)\right) dt\, dx. \end{aligned} \tag{22}$$

Thus, the entries of the matrices A_n are independent of n. Therefore, the approximation method (20) is stable if and only if the operator $E_1 P_1 T_{g,\beta} P_1 E_{-1}$ is continuously invertible. Taking into account the estimates (21), we get the claim. ■

Now we are going to consider the operator $E_1 P_1 T_{g,\beta} P_1 E_{-1}$ in more detail. As was already mentioned, $\eta T_{g,\beta} \eta^{-1} = \mathcal{M}(B_{g,\beta}) = (T_{rl})_{r,l=1}^2$ where every T_{rl} is a Mellin convolution operator on $L^p(\mathbb{R}^+, \alpha)$ with symbol from $C(\overline{\mathbb{R}})$. It is easy to check that for every n,

$$\eta P_n \eta^{-1} = \operatorname{diag}(P_n^+, P_n^+).$$

Hence, $E_1 P_1 T_{g,\beta} P_1 E_{-1}$ can be identified with an operator $D_{g,\beta} = (D_{rl})_{r,l=1}^2$ where the operators D_{rl} act on $l_2^p(\mathbb{N}, \alpha)$. Moreover, a detailed analysis yields that each operator D_{rl} belongs to an algebra which is generated by Toeplitz operators.

So let us recall briefly what a Toeplitz operator is. Write $\mathbb{T}$ for the complex unit circle, let $a \in L^\infty(\mathbb{T})$, and denote by a_k the k th Fourier coefficient of a,

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta} \, d\theta, \quad k \in \mathbb{Z}.$$

If the function a is piecewise continuous on $\mathbb{T}$ and has a finite total variation, then the operator which acts on the finitely supported sequences in $l^p(\mathbb{N}, \alpha)$ via

$$(x_n)_{n\in\mathbb{N}} \mapsto (y_n)_{n\in\mathbb{N}} \quad \text{with} \quad y_n := \sum_{k\in\mathbb{N}} a_{n-k} x_k$$

extends by continuity to a bounded linear operator $T(a)$ acting on all of $l^p(\mathbb{N}, \alpha)$. Thus, the matrix representation of $T(a)$ with respect to the standard basis of $l^p(\mathbb{N}, \alpha)$ is given by

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We let $\mathcal{T}^p(\alpha)$ stand for the smallest closed subalgebra of $\mathfrak{B}(l^p(\mathbb{N}, \alpha))$ which contains all Toeplitz operators $T(a)$ with a generating function a having finite total variation on $\mathbb{T}$, being continuous on $\mathbb{T}\setminus\{1\}$, and possessing finite one-sided limits at $1 \in \mathbb{T}$. Thus, the precise formulation of the above vague statement on $D_{g,\beta}$ is that this operator belongs to $\mathcal{T}^p(\alpha)^{2\times 2}$. A proof of this fact is in [5], Sections 2.2.3 and 2.4.3.

It is a serious problem to decide whether an operator in $\mathcal{T}^p(\alpha)$ is invertible. But there is a very comfortable criterion for the Fredholmness of operators in $\mathcal{T}^p(\alpha)$ which we will recall next. To each Toeplitz operator $A = T(a)$ where a is as above (i.e. it has finite total variation, is continuous on $\mathbb{T}\backslash\{1\}$, and possesses finite one-sided limits $a(1 \pm 0)$ taken with respect to the counter-clockwise orientation of $\mathbb{T}$) we associate the function $A^\sharp : \mathbb{T} \times \overline{\mathbb{R}} \to \mathbb{C}$ which maps $(t,\, z)$ into $a(t)$ if $t \neq 1$ and into

$$\frac{a(1+0)+a(1-0)}{2} - \frac{a(1+0)-a(1-0)}{2} \coth \pi (z + i(1/p + \alpha))$$

if $t = 1$. Thus, one makes the range of a to a closed curve in $\mathbb{C}$ by joining the points $a(1 \pm 0)$ by a certain circular arc depending on the parameter $1/p + \alpha$. If now $A \in \mathcal{T}^p(\alpha)$ is a finite sum of products of Toeplitz operators A_{ij}, then we define

$$A^\sharp = (\sum \prod A_{ij})^\sharp := \sum \prod A_{ij}^\sharp.$$

The mapping $A \mapsto A^\sharp$ is correctly defined, and it extends by continuity onto all of $\mathcal{T}^p(\alpha)$. The function $A^\sharp$ is also called the symbol of the operator A. The relevance of the symbol $A^\sharp$ for the purpose of Fredholmness is as follows: The operator $A \in \mathcal{T}^p(\alpha)$ is Fredholm if and only if the point 0 does not belong to the range of $A^\sharp$. Moreover, if one provides the curve $A^\sharp(\mathbb{T} \times \overline{\mathbb{R}})$ with the orientation inherited by the counter-clockwise orientation of $\mathbb{T}$, then the Fredholm index of $A \in \mathcal{T}^p(\alpha)$ is equal to the negative winding number of the curve $A^\sharp(\mathbb{T} \times \overline{\mathbb{R}})$ with respect to the origin.

In order to apply these results to the discretized operator $D_{g,\beta}$ we still need another property of the algebra $\mathcal{T}^p(\alpha)$, namela it is commutative modulo compact operators. From this commutativity we conclude that the operator $D_{g,\beta} = (D_{rl}) \in \mathcal{T}^p(\alpha)^{2\times 2}$ is Fredholm if and only the operator

$$\det(D_{rl}) := D_{11}D_{22} - D_{12}D_{21} \in \mathcal{T}^p(\alpha)$$

is Fredholm, and that their indices coincide. Thus, $D_{g,\beta}$ is a Fredholm operator if and only if 0 does not lie on the curve $(D_{11}^\sharp D_{22}^\sharp - D_{12}^\sharp D_{21}^\sharp)(\mathbb{T} \times \overline{\mathbb{R}})$, and in this case the index of $D_{g,\beta}$ is minus the winding number of that curve.

It remains to compute the symbols of the operators $D_{rl} = E_1 P_1 \mathcal{M}(b_{rl}) P_1 E_{-1}$ where b_{rl} is the rlth component of the function $B_{g,\beta}$. We write

$$b_{rl}(z) = \mu_{rl} + \nu_{rl} \coth \pi (z + i(1/p + \alpha)) + n_{rl}(z)$$

where $\mu_{rl},\, \nu_{rl} \in \mathbb{C}$ are chosen such that $n_{rl}(\pm\infty) = 0$. (Recall that the limits of the coth-function at infinity are ± 1.) Then the symbol of D_{rl} is equal to

$$(t,\, z) \mapsto \begin{cases} \mu_{rl} + \nu_{rl}\, \sigma(t) & \text{if} \quad t \neq 1 \\ \mu_{rl} - \nu_{rl} \, \coth \pi (z + i(1/p + \alpha)) + n_{rl}(z) & \text{if} \quad t = 1 \end{cases}$$

where

$$\sigma(e^{2\pi i y}) := -\frac{\sin^2 \pi y}{\pi^2} \sum_{m \in \mathbb{Z}} \frac{\operatorname{sgn}(y+m)}{(y+m)^2} \tag{23}$$

for $y \in (0, 1)$. A detailed computation of these functions can be found in [5], Sections 2.2.3, 2.4 and 2.5.2. Geometrically, this condition is quite simple again since the range of the restriction of this mapping onto $(\mathbb{T} \setminus \{1\}) \times \overline{\mathbb{R}}$ is just the interval $(\mu_{rl} + \nu_{rl},\, \mu_{rl} - \nu_{rl})$.

5 Invertibility of $P_n Q_m T_{g,\beta} Q_m P_n$

We will now apply the methodology developed in the previous section to obtain results on the stability of the following approximation method

$$P_n Q_m T_{g,\beta} Q_m P_n \varphi_n^m = P_n Q_m \psi, \quad m,\, n \in \mathbb{N}, \quad \varphi_n^m \in \operatorname{im} P_n \tag{24}$$

where again P_n and Q_m are the above defined projections and where $\psi := T_{g,\beta}\varphi$ is the foveated image of the initial signal. Thus, we assume $P_n Q_m \psi$ to be known to the user who wants to restore φ_n^m. Observe that the right hand sides of the equations (24) can be replaced by the approximate foveated images ψ_n^m defined by (2) without changing the asymptotic solvability properties of these equations. This follows simply from the fact that the sequences (P_n) and (Q_m) of projections converge strongly to the identity operator. Thus, the norms $\|P_n Q_m \psi - \psi_n^m\|$ become as small as desired if m and n are chosen large enough.

It turn out that the problem of stability of the approximation method (24) can be reduced to the stability of a finite section method for the operator $D_{g,\beta}$ which belongs to the Toeplitz algebra $\mathcal{T}^p(\alpha)^{2\times 2}$. Towards this end we provide the space $l_2^p(\mathbb{N},\, \alpha)$ with the norm $\|(f,\, g)\|^p := \|f\|^p + \|g\|^p$. Then the mapping $(f,\, g) \mapsto h$ with

$$h(n) := \begin{cases} f(n) & \text{if} \quad n \geq 0 \\ g(-1-n) & \text{if} \quad n < 0 \end{cases}$$

is an isometry from $l_2^p(\mathbb{N},\, \alpha)$ onto $l^p(\mathbb{Z},\, \alpha)$. Analogously, the space $L_2^p([0,\, 1],\, \alpha)$ is identified with $L^p([-1,\, 1],\, \alpha)$

For $l \in \mathbb{N}$, define projections R_l on $l^p(\mathbb{N},\, \alpha)$ by

$$R_l : (x_n)_{n\in\mathbb{N}} \mapsto (y_n)_{n\in\mathbb{N}} \quad \text{with} \quad y_n := \begin{cases} x_n & \text{if} \quad n < l \\ 0 & \text{if} \quad n \geq l. \end{cases}$$

Since $P_n Q_m = Q_m P_n$, the calculations from the previous section immediately yield that the operator $P_n Q_m T_{g,\beta} Q_m P_n$ is invertible if and only if the operator

$$\begin{aligned} E_n P_n Q_m T_{g,\beta} Q_m P_n E_{-n} &= E_n Q_m E_{-n} E_n P_n T_{g,\beta} P_n E_{-n} E_n Q_m E_{-n} \\ &= E_n Q_m E_{-n} E_1 P_1 T_{g,\beta} P_1 E_{-1} E_n Q_m E_{-n} \end{aligned}$$

and, thus, the operator

$$\begin{pmatrix} R_{mn} & 0 \\ 0 & R_{mn} \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} R_{mn} & 0 \\ 0 & R_{mn} \end{pmatrix} = \begin{pmatrix} R_{mn} D_{11} R_{mn} & R_{mn} D_{12} R_{mn} \\ R_{mn} D_{21} R_{mn} & R_{mn} D_{22} R_{mn} \end{pmatrix} \tag{25}$$

is invertible. Consequently, the stability of the approximation method (24) is equivalent to the stability of the finite section method for the operator $D_{g,\beta} \in \mathcal{T}^p(\alpha)^{2\times 2}$ which has

$$\begin{pmatrix} R_l & 0 \\ 0 & R_l \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} R_l & 0 \\ 0 & R_l \end{pmatrix} \tag{26}$$

as its system matrices. The finite section method for a large class of operators including the operators (D_{rl}) from (26) has been studied in [5], Sections 4.1.1 – 4.1.3. A characterization of the stability of the approximation method (26) can be deduced from these general results. Note that formally, we only get a subsequence of the sequence of the finite section method. But, as has been shown in [6], the sequence formed by the matrices in (26) and its subsequence formed by the matrices in (25) are simultaneously stable or not. Moreover, the finite section method for operators in the Toeplitz algebra is an example of a fractal approximation method; roughly speaking fractality means that every infinite subsequence of the sequence of the approximation matrices allows one to restore the complete sequence up to a sequence which tends to zero in the norm. Thus, every subsequence contains the same "asymptotic information" as the whole sequence. For more facettes of this fascinating topic see [6].

Summarizing these results we arrive at the following theorem.

Theorem 5.1 *Let $B_{g,\beta} \in C^{2\times 2}(\overline{\mathbb{R}})$, and let the entries of $B_{g,\beta}$ have finite total variation on $\mathbb{R}$. The approximation method (24) stable if and only if*

(a) the operator $D_{g,\beta}$ is invertible on $l_2^p(\mathbb{N},\, \alpha)$,

(b) the operator $(T(\mu_{rl} - \nu_{rl}\sigma))_{r,\,l=1}^2$ is invertible on $l_2^p(\mathbb{N},\, 0)$, and

(c) the operator $(\chi_{[0,\,1]}T_{rl}\chi_{[0,\,1]}I)_{r,\,l=1}^2$ is invertible on $L_2^p([0,\,1],\, \alpha)$.

6 An example

We will illustrate the obtained results by an example of a very special kind. Let the function g be given by

$$g(t) = \frac{1}{\pi i t}, \quad t \in \mathbb{R}. \tag{27}$$

We are aware of the fact that this is not a smoothing function as considered in [3]. But for this function, several conditions mentioned previously take a simple and effective form. Thus, it seems to be a good candidate to illustrate our approach to foveation operators.

If g is specified as above, then the operator $T_{g,\beta}$ does no longer depend on β, and we denote it by T_g. It turns out that $T_g : L^p(\mathbb{R},\, \alpha) \to L^p(\mathbb{R},\, \alpha)$ is just the singular integral operator $S_{\mathbb{R}}$ acting by

$$(S_{\mathbb{R}} f)(x) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t-x}dt.$$

For good (say Hölder continuous) functions, this integral exists in the Cauchy principal value sense, and it can be extended by continuity to all of $L^p(\mathbb{R}, \alpha)$. The corresponding matrix G_g has the form

$$G_g(t) = \frac{1}{\pi i} \begin{pmatrix} \dfrac{1}{1-t} & -\dfrac{1}{1+t} \\ \dfrac{1}{1+t} & \dfrac{1}{t-1} \end{pmatrix}, \quad t \in \mathbb{R}^+.$$

Consider the matrix $B_g(z) := MG_g(z)$. We determine its entries. Using formulae 3.238.1 and 3.238.2 of [4] one obtains for $z \in \mathbb{R}$

$$b_{11}(z) = \frac{1}{\pi i} \int_0^{+\infty} \frac{x^{1/p+\alpha-1-iz}}{1-x} dz = \coth(\pi(z + i(1/p+\alpha)).$$

Analogously, by 3.194.4 of [4],

$$b_{21}(z) = \frac{1}{\pi i} \int_0^{+\infty} \frac{x^{1/p+\alpha-1-iz}}{1+x} dz = \frac{1}{\sinh(\pi(z + i(1/p+\alpha))}.$$

Hence,

$$B_g(z) = \begin{pmatrix} \coth(\pi(z+i(1/p+\alpha)) & -1/\sinh(\pi(z+i(1/p+\alpha)) \\ 1/\sinh(\pi(z+i(1/p+\alpha)) & -\coth(\pi(z+i(1/p+\alpha)) \end{pmatrix}. \quad (28)$$

From this representation, some basic properties of the singular integral operator can be derived almost at once. For example, the entries of the matrix B_g are continuous and have finite total variation on $\mathbb{R}$, and their limits at $\pm\infty$ are

$$B_g(+\infty) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_g(-\infty) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $S_{\mathbb{R}}$ is a bounded operator on $L^p(\mathbb{R}, \alpha)$. Notice in this connection that the continuity conditions for the operator T_g given in [3], Theorem 1, are too restrictive. The function g used above is neither bounded nor belongs to $L^1(\mathbb{R})$ as required in [3], but nevertheless the operator $S_{\mathbb{R}} = T_g$ is bounded on $L^p(\mathbb{R}, \alpha)$.

As another application of (28), we examine the invertibility of $S_{\mathbb{R}}$. As already mentioned, this operator is invertible if and only if the determinant of B_g does not vanish on $\mathbb{R}$. Since

$$\det B_g(z) = \frac{1 - \cosh^2(\pi(z + i(1/p+\alpha))}{\sinh^2(\pi(z + i(1/p+\alpha))} = -1,$$

the operator $S_{\mathbb{R}}$ is invertible on each of the spaces $L^p(\mathbb{R}, \alpha)$ with $1 < p < \infty$ and $0 < 1/p + \alpha < 1$. Its inverse operator can be written as a block Mellin operator the Mellin symbol at $z \in \mathbb{R}$ of which is equal to

$$\frac{1}{\det B_g(z)} \begin{pmatrix} -\coth(\pi(z+i(1/p+\alpha)) & 1/\sinh(\pi(z+i(1/p+\alpha)) \\ -1/\sinh(\pi(z+i(1/p+\alpha)) & \coth(\pi(z+i(1/p+\alpha)) \end{pmatrix}$$

$$= \begin{pmatrix} \coth(\pi(z+i(1/p+\alpha)) & -1/\sinh(\pi(z+i(1/p+\alpha)) \\ 1/\sinh(\pi(z+i(1/p+\alpha)) & -\coth(\pi(z+i(1/p+\alpha)) \end{pmatrix}. \quad (29)$$

Comparing (28) and (29), we obtain

$$S_{\mathbb{R}}^{-1} = S_{\mathbb{R}}. \quad (30)$$

Of course, all these results are well known and can be proved without having recourse to Mellin techniques. But they illustrate these techniques quite well.

Due to the simple structure of the operator $T_g = S_{\mathbb{R}}$, it is more convenient to study the invertibility of the Galerkin approximations $P_n S_{\mathbb{R}} P_n$ and $P_n Q_m S_{\mathbb{R}} Q_m P_n$ directly and without doubling the dimension. The point is that the operator $E_n P_n S_{\mathbb{R}} P_n E_{-n}$ is again independent of n and that this operator coincides with the Laurent operator $L(\sigma)$ on $l^p(\mathbb{Z}, \alpha)$. By definition, the Laurent operator $L(a)$ with generating function $a \in L^\infty(\mathbb{T})$ is given via its matrix representation

$$L(a) = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & a_{-2} & a_{-3} & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_2 & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & a_3 & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

with respect to the standard basis of $l^p(\mathbb{Z}, \alpha)$. As in the Toeplitz operator case, a sufficient condition for the boundedness of this operator is that the function a is piecewise continuous and has a finite total variation. A basic difference between Toeplitz and Laurent operators with piecewise continuous generating functions is that $L(a)L(b) = L(ab)$ (whereas the corresponding result for Toeplitz operators is definitely wrong in general). Consequently, the Laurent operators generate a commutative algebra, whereas the Toeplitz operator algebra $\mathcal{T}^p(\alpha)$ is merely commutative modulo compact operators. This implies that the Laurent operator $L(a)$ is invertible if and only if the function a is invertible in $L^\infty(\mathbb{T})$.

Since the essential range of σ is the interval $[-1, 1]$, the operators $P_n S_{\mathbb{R}} P_n = E_{-n} L(\sigma) E_n$ cannot be invertible. For that reason, we replace the kernel function (27) by

$$g(t) = a + b\frac{1}{\pi i t}, \quad t \in \mathbb{R} \quad (31)$$

with complex constants a and b. Then $T_g = aI + bS_\mathbb{R}$ and $E_n P_n S_\mathbb{R} P_n E_{-n} = L(a + b\sigma)$, and the latter operator becomes invertible if a and b are suitably chosen. At this point, it is sufficient to require that 0 does not lie on the segment joining $a - b$ to $a + b$.

Proposition 6.1 *Let the kernel function g be given by* (31). *Then the sequence* $(P_n T_g P_n)$ *is stable if and only if* $0 \notin [a - b,\, a + b]$.

What happens with the operators $P_n Q_m T_g Q_m P_n$? Their invertibility corresponds to the stability of the finite sections sequence $R'_n L(a) R'_n$ where the projections R'_m on $l^p(\mathbb{Z},\, \alpha)$ are defined by

$$R'_m : (x_n)_{n\in\mathbb{Z}} \mapsto (y_n)_{n\in\mathbb{Z}} \quad \text{with} \quad y_n := \begin{cases} x_n & \text{if } -m \le n \le m-1 \\ 0 & \text{else.} \end{cases}$$

Evidently, the matrix $R'_n L(a) R'_n$ can be identified with $R_{2n} T(a) R_{2n}$. Thus, the finite section method for the Laurent operator $L(a)$ corresponds to (a subsequence of) the finite section method for the Toeplitz operator $T(a)$. The stability of the finite section method for that Toeplitz operator is well understood. The following result is a corollary of a more general theorem of [5]. A direct proof which works in case the generating function of the Toeplitz operator has exactly one point of discontinuity can be found in [9].

Proposition 6.2 *Let c be a piecewise continuous function with finite total variation. Then the finite section method applies to the Toeplitz operator $T(c)$ on $l^p(\mathbb{N},\, \alpha)$ if and only if the operator $T(c)$ is invertible on $l^p(\mathbb{N},\, \alpha)$ and if the operator $T(\tilde{c})$ with $\tilde{c}(y) := c(1/y)$ is invertible on $l^p(\mathbb{N},\, 0)$.*

For the kernel function g as in (31), the applicability of the finite section method to $T(a+b\sigma)$ is equivalent to the invertibility of $T(a+b\sigma)$ on $l^p(\mathbb{N},\, \alpha)$ and of $T(a-b\sigma)$ on $l^p(\mathbb{N},\, 0)$. The invertibility of the first mentioned operator is equivalent to the fact that the point 0 does not lie in the region which is bounded by

$$[a + b,\, a - b] \cup \{a + b \coth \pi(z + i(1/p + \alpha)) : z \in \overline{\mathbb{R}}\},$$

whereas the invertibility of the second operator is equivalent to the fact that 0 is not contained in the region bounded by

$$[a - b,\, a + b] \cup \{a - b \coth \pi(z + i/p) : z \in \overline{\mathbb{R}}\}.$$

Both regions are bounded by a union of a straight line with a circular arc.

7 Splines of Higher Order

The Galerkin approximations considered so far have been based on the first order splines, viz. the splines generated by the characteristic function $N^{(1)} := \chi_{[0,1)}$.

The use of higher order splines might give a better approximation for both uniform and foveated signals. Of course the replacement of the basis functions leads to different approximation operators, so the stability problem must be studied once again. Thereby, it turns out that the operators which arise if one is using splines of higher order to approximate the foveated images again belong to the Toeplitz algebra $\mathcal{T}^p(\alpha)^{2\times 2}$ we already met. Therefore, the study of the stability problem for such operator sequences one can again based upon the approach presented in sections 4 and 5. Let us briefly comment on the amendments which have to be made in this situation.

For a given $d \geq 2$, the d-th order cardinal B-spline $N^{(d)}$ is defined recursively by

$$N^{(d)}(t) := \int_0^1 N^{(d-1)}(t-s)\, ds.$$

The following important properties of the cardinal splines $N^{(d)}$ are well known:

1. The support of $N^{(d)}$ is the interval $[0, d]$.
2. $N^{(d)}(t) > 0$ for all $t \in (0, d)$.
3. $N^{(d)}(-t+d) = N^{(d)}(t)$ for every $t \in \mathbb{R}$.

Let us fix a positive integer n and introduce functions φ_{nj}, $j \in \mathbb{Z}$ by

$$\varphi_{nj}(t) := \begin{cases} N^{(d)}(nt-j) & \text{if} \quad j \geq 0 \\ N^{(d)}(nt-j-d+1) & \text{if} \quad j < 0. \end{cases}$$

Let $S_n^d(\mathbb{R}^+)$ be the smallest closed subspace of $L^p(\mathbb{R}^+, \alpha)$ which contains all functions φ_{nj}, $j = 0, 1, \ldots$, and let $S_n^d(\mathbb{R})$ be the smallest closed subspace of $L^p(\mathbb{R}, \alpha)$ which contains all functions φ_{nj}, $j \in \mathbb{Z}$. In analogy to the previous analysis in case $d = 1$, one can introduce the Galerkin projections $\widetilde{P}_n^I : L^p(I, \alpha) \to S_n^d(I)$ by

$$\widetilde{P}_n^I f := n \sum_{k \in \mathbb{Z}} \langle f, \varphi_{nk} \rangle \varphi_{nk}, \tag{32}$$

and rewrite equation (15) with the projections onto the spaces $S_n(I) = S_n^{(1)}(I)$ replaced by projections $\widetilde{P}_n^I$. Again, the operators

$$\widetilde{E}_n : l^p(\mathbb{Z}, \alpha) \to S_n^d(\mathbb{R}), \quad (\xi_j)_{j \in \mathbb{Z}} \to \sum_{j \in \mathbb{Z}} \xi_j \varphi_{nj}$$

are continuously invertible with inverses $\widetilde{E}_{-n}$, and

$$\|\widetilde{E}_n\| \leq C\, n^{-(1/p+\alpha)}, \qquad \|\widetilde{E}_{-n}\| \leq C\, n^{1/p+\alpha}$$

with a certain constant C. Repeating and modifying the arguments of Section 4 one obtains that the approximation method which is the analog of the method (15) but based on the spline $N^{(d)}$ with $d \geq 2$, is stable if and only if the operator

$$\widetilde{E}_1 \widetilde{P}_1 T_{g,\beta} \widetilde{P}_1 \widetilde{E}_{-1} : \quad l^p(\mathbb{Z},\, \alpha) \to l^p(\mathbb{Z},\, \alpha)$$

is invertible. The entries $\widetilde{A}_{jk}$ of the matrix representation of that operator with respect to the standard basis of $l^p(\mathbb{Z}, \alpha)$ can be calculated by formulas similar to (22). For example, if $j \geq 0$ and $k \geq 0$, then

$$\widetilde{A}_{jk} = \frac{1}{\beta} \int_j^{j+d} N^{(d)}(x-j) \int_k^{k+d} \frac{1}{|x|} g\left(\frac{1}{\beta}\left(\frac{t-x}{|x|}\right)\right) N^{(d)}(t-k)\, dt\, dx.$$

An additional analysis shows that the operator $\widetilde{E}_1 \widetilde{P}_1 T_{g,\beta} \widetilde{P}_1 \widetilde{E}_{-1}$ can be identified with an operator $\widetilde{D}_{g,\beta} = (\widetilde{D}_{rl})_{r,l=1}^2$ in the algebra $\mathcal{T}^p(\alpha)^{2\times 2}$ where the symbol of the operator $\widetilde{D}_{rl}$ is

$$(t,\, z) \mapsto \begin{cases} \mu_{rl} + \nu_{rl}\, \widetilde{\sigma}(t) & \text{if} \quad t \neq 1 \\ \mu_{rl} - \nu_{rl} \, \coth \pi(z + i(1/p + \alpha)) + c(d) n_{rl}(z) & \text{if} \quad t = 1 \end{cases}$$

with μ_{rl} and ν_{rl} as before,

$$\widetilde{\sigma}(e^{2\pi i y}) := -\frac{\sin^{2d} \pi y}{\pi^{2d}} \sum_{m \in \mathbb{Z}} \frac{\mathrm{sgn}\,(y+m)}{(y+m)^{2d}} \quad \text{for } y \in (0,\, 1)$$

and

$$c(d) = \left(\int_0^d N^{(d)}(t)\, dt\right)^2.$$

Further steps lead to results analogous to those of Section 5. Thus Theorem 5.1 can be reformulated with the corresponding replacement of the operator $D_{g,\beta}$ by $\widetilde{D}_{g,\beta}$.

Concluding Remarks

Our analysis shows that the use of the operator (1) does not only allow to create and transmit foveated images. It offers also the possibility to restore their approximate uniform pre-images. Moreover, if the kernel k of the integral operator (1) gives rise to a stable approximation method, then the quality of such restored uniform pre-images can be quite satisfactory as estimate (19) shows.

As was also pointed out, one might use other numerical procedures to approximate foveated images. For example, one can apply procedures based on meshes having higher density around the points of interest. Another possibility is to use wavelets instead of splines. The Galerkin procedures considered in the present paper can also be replaced by other numerical methods, for example by quadrature or by collocation methods. Of course, each new approache will require an additional analysis for the stability of the employed numerical procedures. However, in many cases, these approximation procedures for the operator (1) can be associated with well understood operator algebras, which allows one to find criteria of their stability.

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