Locally compact groups built up from p-adic Lie groups, for p in a given set of primes

Helge Glöckner

Abstract. We analyze the structure of locally compact groups which can be built up from p-adic Lie groups, for p in a given set of primes. In particular, we calculate the scale function and determine tidy subgroups for such groups, and use them to recover the primes needed to build up the group.

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Introduction

While connected locally compact groups can be approximated by real Lie groups and hence can be described using real Lie theory ([20], [15]), the situation is more complicated in the case of a totally disconnected, locally compact group G. Here, we have a p-adic Lie theory available for each prime p, but it is not clear *a priori* which primes p will be needed to analyze the structure of G, nor whether p-adic Lie theory is useful at all in this context. Investigations in [8] indicate that indeed general locally compact groups are "too far away" from p-adic Lie groups (and from Lie groups over local fields) to expect meaningful applications of Lie theory. Therefore, it is essential to restrict attention to suitable classes of totally disconnected groups, which are "close enough" to p-adic Lie groups.

For example, we might consider the class of (locally compact) pro-p-adic Lie groups, viz. locally compact groups which can be approximated by p-adic Lie groups, for a fixed prime p (see [8], [14] for investigations of such groups). However, it is clearly very restrictive to use p-adic Lie theory for a single prime p only; it would be more natural to try to make use of p-adic Lie theory for variable primes p simultaneously. For instance, it should certainly be allowed to approximate a group also by finite products $\prod_{p\in\mathfrak{p}} G_p$ of p-adic Lie groups G_p (where $\mathfrak{p}$ is a finite set of primes), or by closed subgroups of such products.

Motivated by such considerations, given a non-empty subset $\mathfrak{p}$ of the set $\mathbb{P}$ of all primes, it was proposed in [8] to study the class $\mathbb{MIX}_{\mathfrak{p}}$ of all locally compact groups which can be manufactured from p-adic Lie groups with $p\in\mathfrak{p}$, by repeated application of the operations of forming cartesian products, closed subgroups, Hausdorff quotients, and passage to isomorphic topological groups.[1]

[1]Thus, technically speaking, $\mathbb{MIX}_{\mathfrak{p}}$ consists of all locally compact groups in the variety of Hausdorff groups generated by the class of topological groups which are p-adic Lie groups for some $p\in\mathfrak{p}$.

For example, consider a Hausdorff quotient S/N of a closed subgroup S of a finite product $\prod_{p \in F} G_p$ of p-adic Lie groups, with p in a finite set $F \subseteq \mathfrak{p}$, or a topological group isomorphic to S/N (such groups will be called $\mathbb{A}_{\mathfrak{p}}$*-groups*). Then S/N is a $\mathbb{MIX}_{\mathfrak{p}}$-group. Arbitrary $\mathbb{MIX}_{\mathfrak{p}}$-groups are not too far away from this example: a locally compact group G is a $\mathbb{MIX}_{\mathfrak{p}}$-group if and only if it is topologically isomorphic to a closed subgroup of a cartesian product $\prod_{i \in I} S_i/N_i$ of $\mathbb{A}_{\mathfrak{p}}$-groups (cf. (3)), by standard facts from the theory of varieties of topological groups ([6], [16], [21]).

This information alone would not be enough to analyze $\mathbb{MIX}_{\mathfrak{p}}$-groups via p-adic Lie theory. However, we can prove much more: Every $G \in \mathbb{MIX}_{\mathfrak{p}}$ can be approximated by $\mathbb{A}_{\mathfrak{p}}$-groups, in the sense that every identity neighbourhood of G contains a closed normal subgroup $K \subseteq G$ such that G/K is an $\mathbb{A}_{\mathfrak{p}}$-group (Remark 2.11). We can also show that every $\mathbb{A}_{\mathfrak{p}}$-group G contains an open subgroup which is a finite product $\prod_{p \in F} H_p$ of p-adic Lie groups (Corollary 2.5). Since every inner automorphism of G gives rise to local automorphisms of the factors H_p here, adapting techniques from [7] and [14] to the case of local automorphisms we are able to deduce very satisfactory results concerning the structure of $\mathbb{MIX}_{\mathfrak{p}}$-groups (including solutions to all open problems formulated in [8]). In particular, we obtain a clear picture of the "tidy subgroups" of a $\mathbb{MIX}_{\mathfrak{p}}$-group G and its "scale function" $s_G : G \to \mathbb{N}$, which are the essential structural features of G in the structure theory of totally disconnected groups initiated in ([25], [27]). We recall the definitions:

Definition (cf. [25], [27]). Let G be a totally disconnected, locally compact group and α be an automorphism of G. A compact, open subgroup U of G is called *tidy for* α if the following conditions are satisfied:

(T1) $U = U_+ U_-$, where $U_\pm := \bigcap_{n \in \mathbb{N}_0} \alpha^{\pm n}(U)$;

(T2) The subgroup $U_{++} := \bigcup_{n \in \mathbb{N}_0} \alpha^n(U_+)$ is closed in G.

It can be shown that compact, open subgroups tidy for α always exist, and that the index

$$r_G(\alpha) \;:=\; [\alpha(U_+) : U_+]$$

(called the "scale of α") is finite and independent of the choice of tidy subgroup U. Specialization to inner automorphisms $I_x : G \to G$, $I_x(y) := xyx^{-1}$ yields the *scale function* $s_G \colon G \to \mathbb{N}$, $s_G(x) := r_G(I_x)$ of G. We let $\mathbb{P}(G)$ be the set of all primes $p \in \mathbb{P}$ such that p divides $s_G(x)$ for some $x \in G$.

The main results. Writing $\mathbb{MIX}_{\emptyset}$ for the class of locally compact pro-discrete groups, we can summarize our main results as follows:

(a) For any sets of primes $\mathfrak{p}$ and $\mathfrak{q}$, we have $\mathbb{MIX}_{\mathfrak{p}} \cap \mathbb{MIX}_{\mathfrak{q}} = \mathbb{MIX}_{\mathfrak{p} \cap \mathfrak{q}}$ (Theorem 2.12).

(b) The scale function s_G of any $\mathbb{MIX}_{\mathfrak{p}}$-group G can be calculated by Lie-theoretic methods. Furthermore, a basis of compact, open subgroups tidy for x can be described explicitly for each $x \in G$, using Lie-theoretic methods (Theorem 3.4, Corollary 3.7).[2]

[2]This is new even for p-adic groups; in [7], s_G was calculated without formulas for tidy subgroups.

(c) For every $\mathbb{MIX}_{\mathfrak{p}}$-group G, the set $\mathbb{P}(G)$ of all prime divisors of the values of the scale function is a finite set, and $\mathbb{P}(G) \subseteq \mathfrak{p}$ (Corollary 3.7).

(d) If G is a compactly generated $\mathbb{MIX}_{\mathbb{P}}$-group and $\mathfrak{p}$ a set of primes, then $G \in \mathbb{MIX}_{\mathfrak{p}}$ if and only if $\mathbb{P}(G) \subseteq \mathfrak{p}$ (Theorem 4.2). In particular, finitely many primes suffice to build up G. Furthermore, every compactly generated, uniscalar $\mathbb{MIX}_{\mathbb{P}}$-group G is pro-discrete (Corollary 4.3).[3] Previously, this was only known for p-adic Lie groups (see [22] and [14]).

It is a natural idea that the set $\mathbb{P}(G)$ of all prime divisors of the values of s_G should tell us which kinds of p-adic Lie groups (which p) are needed to analyze a totally disconnected group G, at least in good cases.[4] Result (d) above shows that this general philosophy can be turned into a mathematical fact for the class of compactly generated $\mathbb{MIX}_{\mathbb{P}}$-groups.

We mention that most of the results carry over to the larger class $\mathbb{VSUB}_{\mathfrak{p}}$ (subsuming $\mathbb{MIX}_{\mathfrak{p}}$) of all locally compact groups in the variety of Hausdorff groups generated by topological groups having a direct product $\prod_{p\in F} H_p$ of p-adic Lie groups as an open subgroup, for p in a finite subset $F \subseteq \mathfrak{p}$. We therefore discuss such groups in parallel.

Although our studies may remind the reader of Adèle groups, closer inspection shows that the latter need not belong to $\mathbb{MIX}_{\mathbb{P}}$, nor $\mathbb{VSUB}_{\mathbb{P}}$ (see Remark 3.8).

Variants. Some results remain valid if p-adic Lie groups are replaced by *locally pro-p groups* (groups possessing a pro-p-group as a compact open subgroup): see Section 5.

Further results. Motivated by results in [26], in the final Section 6 we associate a set of primes $\mathbb{L}(G)$ to each totally disconnected, locally compact group G, which only depends on the local isomorphism type of G (the "local prime content of G"). Since $\mathbb{L}(G)$ contains all prime divisors of the scale function, it provides a means to deduce information concerning the global structure of G (its scale function) from the local structure of G. Using the local prime content, we show that for each $G \in \mathbb{MIX}_{\mathbb{P}}$, there exists a unique smallest set of primes $\mathfrak{p}$ such that $G \in \mathbb{MIX}_{\mathfrak{p}}$ (Theorem 6.7, Remark 6.9). If G is compactly generated, then simply $\mathfrak{p} = \mathbb{P}(G)$, as mentioned before. If G is not compactly generated, then $\mathfrak{p} \neq \mathbb{P}(G)$ in general. In this case, $\mathfrak{p}$ can still be determined in principle (it is the "intermediate prime content" of G, defined below), but it is a less tangible invariant.

In an appendix, which is of independent interest, we describe topological groups G whose normal subgroups N with G/N a real (resp. p-adic) Lie group do not form a filter basis.

The present paper uses (and generalizes) results and techniques from [7], [8], [14] and [26].

[3] Recall that a totally disconnected, locally compact group G is called *uniscalar* if $s_G \equiv 1$, which holds if and only if every $x \in G$ normalizes some compact, open subgroup of G.

[4] This idea was expressed by M. Stroppel (Stuttgart) in 1994.

1 Preliminaries and notation

1.1 Given a class of topological Hausdorff groups Ω, the variety of Hausdorff groups generated by Ω is the smallest class $\mathcal{V}(\Omega)$ of Hausdorff groups containing Ω and closed under the operations of formation of cartesian products "C," subgroups "S," Hausdorff quotients "$\overline{\mathrm{Q}}$," and passage to isomorphic topological groups (which is understood and suppressed in the notation). It is easy to see that

$$\mathcal{V}(\Omega) \;=\; \overline{\mathrm{Q}}\mathrm{SC}(\Omega) \tag{1}$$

here (cf. [6, Thm. 1] or [21, Thm. 6]), and it can be shown with more effort that

$$\mathcal{V}(\Omega) \;=\; \mathrm{SC}\overline{\mathrm{Q}}\,\overline{\mathrm{S}}\mathrm{P}(\Omega) \tag{2}$$

(see [6, Thm. 2], or [21, Thm. 7]), where "P" denotes the formation of all finite cartesian products, and "$\overline{\mathrm{S}}$" denotes formation of closed subgroups (or isomorphic copies thereof, as above). It is easy to see (cf. (1) above) that the class $\overline{\mathrm{Q}}\,\overline{\mathrm{S}}\mathrm{P}(\Omega)$ is closed under the formation of finite cartesian products, closed subgroups and Hausdorff quotients.

1.2 Throughout the following, $\mathbb{P}$ denotes the set of all primes. Given $p \in \mathbb{P}$, we let $\mathbb{LIE}_p$ be the class of p-adic Lie groups; given a non-empty subset $\mathfrak{p} \subseteq \mathbb{P}$, we set $\mathbb{LIE}_{\mathfrak{p}} := \bigcup_{p\in\mathfrak{p}} \mathbb{LIE}_p$. According to (2), the variety of Hausdorff groups generated by $\mathbb{LIE}_{\mathfrak{p}}$ is given by

$$\mathcal{V}(\mathbb{LIE}_{\mathfrak{p}}) \;=\; \mathrm{SC}(\mathbb{A}_{\mathfrak{p}})\,, \qquad \text{where} \qquad \mathbb{A}_{\mathfrak{p}} \;:=\; \overline{\mathrm{Q}}\,\overline{\mathrm{S}}\mathrm{P}(\mathbb{LIE}_{\mathfrak{p}}) \tag{3}$$

is the class of all topological groups isomorphic to a Hausdorff quotient S/N of a closed subgroup S of a product $\prod_{p\in F} G_p$, where $F \subseteq \mathfrak{p}$ is a finite subset and G_p a p-adic Lie group for each $p \in F$. For later use, we let $\mathbb{A}_\emptyset$ be the class of discrete groups. We define

$$\mathbb{MIX}_{\mathfrak{p}} \;:=\; \{G \in \mathcal{V}(\mathbb{LIE}_{\mathfrak{p}})\colon\; G \text{ is locally compact}\,\}\,.$$

Finally, we let $\mathbb{MIX}_\emptyset$ be the class of all *pro-discrete*, locally compact groups G, *i.e.*, locally compact groups G whose filter of identity neighbourhoods has a basis consisting of open, normal subgroups of G (see [8] for more information).

1.3 Given a set $\mathfrak{p}$ of primes, we let $\mathbb{SUB}_{\mathfrak{p}}$ be the class of all topological groups possessing an open subgroup isomorphic to $\prod_{p\in F} G_p$, where $F \subseteq \mathfrak{p}$ is finite and G_p a p-adic Lie group for each $p \in F$. We let

$$\mathbb{VSUB}_{\mathfrak{p}} \;:=\; \{G \in \mathcal{V}(\mathbb{SUB}_{\mathfrak{p}})\colon\; G \text{ is locally compact}\,\}\,.$$

In particular, $\mathbb{VSUB}_\emptyset$ is the class of locally compact, pro-discrete groups (cf. [8, Thm. 2.1]).

Note that all of the topological groups in $\mathbb{A}_{\mathfrak{p}}$, $\mathbb{SUB}_{\mathfrak{p}}$, $\mathbb{MIX}_{\mathfrak{p}}$ and $\mathbb{VSUB}_{\mathfrak{p}}$ are locally compact and totally disconnected. We shall see later that $\mathbb{A}_{\mathfrak{p}} \subseteq \mathbb{SUB}_{\mathfrak{p}}$ and thus $\mathbb{MIX}_{\mathfrak{p}} \subseteq \mathbb{VSUB}_{\mathfrak{p}}$.

1.4 A class $\mathbb{A}$ of Hausdorff topological groups which contains the trivial group and is closed under passage to isomorphic topological groups is called a *property of topological groups*; the elements of $\mathbb{A}$ are called $\mathbb{A}$-*groups*. If $\mathbb{A}$ is a property of topological groups, we say that the class $\mathbb{A}$ is *suitable for approximation* (or also: an "admissible property" of topological groups, in the terminology of [8]), if every $\mathbb{A}$-group is locally compact, $\mathbb{A}$ is closed under the formation of finite cartesian products, closed subgroups and Hausdorff quotients (which holds if and only if $\mathbb{A} = \overline{\mathrm{Q}}\,\overline{\mathrm{S}}\mathrm{P}(\mathbb{A})$), and if $G/\ker f$ is an $\mathbb{A}$-group, for every continuous homomorphism $f\colon G \to H$ from a locally compact group G to an $\mathbb{A}$-group H.

For example, the class of real Lie groups is suitable for approximation (cf. [16]), and so are the classes of p-adic Lie groups (see [8]), finite groups, finite p-groups, and finite nilpotent groups, respectively.

Quite a bit of work will be needed to see that the classes $\mathbb{A}_{\mathfrak{p}}$ and $\mathbb{SUB}_{\mathfrak{p}}$ are suitable for approximation. This information is very useful, because it is well understood which locally compact groups can be approximated by topological groups in a class of topological groups which is suitable for approximation. We recall [8, Thm. 2.1]:

Proposition 1.5 *Let $\mathbb{A}$ be a class of topological groups that is suitable for approximation, and G be a locally compact group. Then the following conditions are equivalent:*

(a) *G can be approximated by $\mathbb{A}$-groups, i.e., every identity neighbourhood U of G contains a closed normal subgroup N of G such that $G/N \in \mathbb{A}$.*

(b) *The set of all closed normal subgroups N as in* (a) *is a filter basis which converges to* 1 *in G.*

(c) *G is a pro-$\mathbb{A}$-group in the sense of* [8, 1.4].

(d) *G is a projective limit* (*in the category of topological groups*) *of a projective system of $\mathbb{A}$-groups and continuous homomorphisms.*

(e) *G is an element of the variety $\mathcal{V}(\mathbb{A})$ of Hausdorff groups generated by $\mathbb{A}$.* □

Choosing U compact, we see that N in (a) can always be assumed to be compact.

1.6 All topological groups considered in this article are Hausdorff. Open, surjective, continuous homomorphisms are called *quotient morphisms*. All isomorphisms or automorphisms of topological groups are, in particular, homeomorphisms. The automorphism group of a topological group G is denoted $\mathrm{Aut}(G)$. A *local isomorphism* between totally disconnected, locally compact groups G and H is an isomorphism from an open subgroup of G onto an open subgroup of H.

1.7 Our main sources for p-adic Lie theory are [5] and [23]. All Lie groups G considered here are finite-dimensional analytic Lie groups (unless we say otherwise explicitly). As usual, a p-adic Lie group will be identified with its underlying topological group. The

p-adic Lie algebra of G is denoted $L(G)$. All necessary background concerning pro-finite groups and pro-p-groups (in particular, the basics of Sylow theory needed here) can be found in [28].

1.8 Given a prime p, a topological group G is called *locally pro-p* if G has a compact, open subgroup U which is a pro-p-group. As an immediate consequence of the corresponding permanence properties of pro-p-groups, the class $\mathbb{LOC}_p$ of locally pro-p groups is closed under formation of finite direct products, closed subgroups, and Hausdorff quotients.

1.9 It is well known that every p-adic Lie group is locally pro-p, and so is every analytic Lie group G over a local field $\mathbb{K}$ whose residue field $\mathfrak{k}$ has characteristic p [23]. See also [10, Prop. 2.1 (h)] for a recent proof, which remains valid if G is not analytic but merely a C^1-Lie group (in the setting of [2]). While every p-adic C^k-Lie group admits a C^k-compatible analytic Lie group structure [10], for every local field of positive characteristic there exists a 1-dimensional smooth Lie group without an analytic Lie group structure compatible with its topological group structure, and C^k-Lie groups which are not C^{k+1} [9].

2 Relations between the various classes of groups

We first collect various simple facts.

Lemma 2.1 *Let $p \neq q$ be primes, G be a pro-p-group, H a pro-q-group and $f\colon G \to H$ be a continuous homomorphism. Then $f(x) = 1$ for all $x \in G$. In particular, every continuous homomorphism from a p-adic Lie group to a q-adic Lie group has open kernel.*

Proof. Since continuous homomorphisms to finite q-groups separate points on H, we may assume that H is a finite q-group. By [28, La. 1.2.6], $K := \ker f$ is open and hence $G/K \cong f(G)$ is a p-group (as a consequence of [28, Prop. 1.2.1]). Hence $f(G) = \{1\}$. The well-known final assertion (Cartan's Theorem) now follows with **1.9**. □

The following observation concerning closed subgroups of pro-nilpotent groups is the key to an understanding of $\mathbb{MIX}_\mathfrak{p}$-groups and $\mathbb{VSUB}_\mathfrak{p}$-groups.[5]

Proposition 2.2 *Let U_p be a pro-p-group for each $p \in \mathbb{P}$, and S be a closed subgroup of $U := \prod_{p\in\mathbb{P}} U_p$. Define $S_p := S \cap U_p$ for $p \in \mathbb{P}$, identifying U_p with $U_p \times \prod_{q\in\mathbb{P}\setminus\{p\}}\{1\} \subseteq U$. Then $S = \prod_{p\in\mathbb{P}} S_p$.*

Proof. Being a direct product of pro-p-groups, U is pro-nilpotent (cf. [28, Prop. 2.4.3]). As a consequence of [28, Thm. 1.2.3], also the closed subgroup S of U is pro-nilpotent. Hence S has a normal (and hence unique) p-Sylow subgroup $\tilde{S}_p$ for each $p \in \mathbb{P}$ (see [28, Prop. 2.4.3 (ii) and Prop. 2.2.2 (d)]). Since $\tilde{S}_p$ is contained in the unique p-Sylow subgroup U_p of U and contains S_p (see [28, Prop. 2.2.2 (c)]), we deduce that $\tilde{S}_p = S_p$. Hence $S = \prod_{p\in\mathbb{P}} S_p$ by [28, Prop. 2.4.3 (iii)]. □

[5] Recall from [28] that projective limits of nilpotent finite groups are called *pro-nilpotent*.

Corollary 2.3 *Let $\mathfrak{p}$ be a non-empty, finite set of primes, G_p be a locally pro-p group (resp., a p-adic Lie group) for $p \in \mathfrak{p}$, and S be a closed subgroup of $G := \prod_{p\in\mathfrak{p}} G_p$, which we consider as an internal direct product of the groups G_p. Then $S_p := S \cap G_p$ is a closed normal subgroup of S and locally pro-p (resp., a p-adic Lie group), being a closed subgroup of G_p. Furthermore, the product*

$$P := \prod_{p\in\mathfrak{p}} S_p$$

is an open subgroup of S.

Proof. For each $p \in \mathfrak{p}$, there exists an open pro-p subgroup $U_p \subseteq G_p$. Then $U := \prod_{p\in\mathfrak{p}} U_p$ is open in G and hence $U \cap S$ is open in S, where $U \cap S = \prod_{p\in\mathfrak{p}}(U_p \cap S)$ by Proposition 2.2. Since $U_p \cap S \subseteq S_p$, also P is open in S. □

We now focus on p-adic groups. Analogues for locally pro-p groups are outlined in Section 5.

Corollary 2.4 *Suppose that $\phi\colon G \to \prod_{p\in\mathfrak{p}} G_p$ is a continuous, injective homomorphism from a locally compact group G into a product of p-adic Lie groups G_p, for p in some finite set of primes $\mathfrak{p}$. Then G is a $\mathbb{SUB}_{\mathfrak{p}}$-group.*

Proof. Since ϕ is injective, G is totally disconnected. We choose a compact, open subgroup $U \subseteq G$; then U is isomorphic to the closed subgroup $S := \phi(U)$ of $\prod_{p\in\mathfrak{p}} G_p$. Thus Corollary 2.3 entails the claim. □

Corollary 2.5 *Let $\mathfrak{p}$ be a set of primes. Then $\mathbb{A}_{\mathfrak{p}} \subseteq \mathbb{SUB}_{\mathfrak{p}}$ and thus $\mathbb{MIX}_{\mathfrak{p}} \subseteq \mathbb{VSUB}_{\mathfrak{p}}$.*

Proof. Without loss of generality $\mathfrak{p} \neq \emptyset$, the omitted case being trivial. If G is an $\mathbb{A}_{\mathfrak{p}}$-group, then after passing to an isomorphic copy we may assume that $G = S/N$ where S is a closed subgroup of a product $P := \prod_{p\in F} G_p$ of p-adic Lie groups for p in some finite subset $F \subseteq \mathfrak{p}$, and $N \subseteq S$ a closed normal subgroup. Then $S_p := S \cap G_p$ is a closed subgroup of S, and $N_p := N \cap G_p$ is a closed normal subgroup of S_p, for each $p \in F$. By Corollary 2.3, $\widetilde{S} := \prod_{p\in F} S_p$ is an open subgroup of S and $\widetilde{N} := \prod_{p\in F} N_p$ an open subgroup of N (and it also is a closed normal subgroup of S). Hence $G = S/N$ is isomorphic to

$$\big(S/\widetilde{N}\big)\Big/\big(N/\widetilde{N}\big)\,. \tag{4}$$

But, $N/\widetilde{N}$ being discrete, the group in (4) is locally isomorphic to $S/\widetilde{N}$, which has $\widetilde{S}/\widetilde{N} \cong \prod_{p\in F}\big(S_p/N_p\big)$ as an open subgroup. Hence for all sufficiently small compact, open subgroups $U_p \subseteq S_p/N_p$, the product $\prod_{p\in F} U_p$ is isomorphic to a compact, open subgroup of G. Thus G is a $\mathbb{SUB}_{\mathfrak{p}}$-group. Hence $\mathbb{A}_{\mathfrak{p}} \subseteq \mathbb{SUB}_{\mathfrak{p}}$. The rest is obvious. □

Remark 2.6 The author does not know whether $\mathbb{MIX}_{\mathbb{P}}$ is a proper subclass of $\mathbb{VSUB}_{\mathbb{P}}$.

We may assume that $\mathrm{pr}_p(S)$ is dense in G_p in the preceding proof (where $\mathrm{pr}_p\colon P \to G_p$ is the coordinate projection), entailing that the closed subgroup $N_p := N \cap G_p$ of G_p is a normal subgroup of G_p. Hence $S/\widetilde{N}$ is a closed subgroup of $(\prod_{p\in F} G_p)/\widetilde{N} \cong \prod_{p\in F}(G_p/N_p)$, where G_p/N_p is a p-adic Lie group. Combining this with (4), we get:

Corollary 2.7 *If $\mathfrak{p} \neq \emptyset$, then every $\mathbb{A}_{\mathfrak{p}}$-group is topologically isomorphic to a quotient S/D, where S is a closed subgroup of a product $\prod_{p\in F} G_p$ of p-adic Lie groups G_p for p in a finite set $F \subseteq \mathfrak{p}$, and D is a* **discrete** *normal subgroup of S.* $\square$

Proposition 2.8 *Let $\mathfrak{p}$ be a set of primes. Then we have:*

(a) *The class $\mathbb{SUB}_{\mathfrak{p}}$ is suitable for approximation.*

(b) *The class $\mathbb{A}_{\mathfrak{p}}$ is suitable for approximation.*

Proof. The case $\mathfrak{p} = \emptyset$ being trivial, we may assume that $\mathfrak{p}$ is non-empty.

(a) Every $\mathbb{SUB}_{\mathfrak{p}}$-group is locally compact. Let us show that the class $\mathbb{SUB}_{\mathfrak{p}}$ is closed under the formation of finite direct products, closed subgroups, and Hausdorff quotients. It is obvious that finite products of $\mathbb{SUB}_{\mathfrak{p}}$-groups are $\mathbb{SUB}_{\mathfrak{p}}$-groups. Let G be a $\mathbb{SUB}_{\mathfrak{p}}$-group now and S a closed subgroup. Let $U \subseteq G$ be an open subgroup which is a product $\prod_{p\in F} U_p$ of p-adic Lie groups U_p for p in a finite subset $F \subseteq \mathfrak{p}$. Then $S \cap U$ is a closed subgroup of $U = \prod_{p\in F} U_p$, whence S is a $\mathbb{SUB}_F$-group (and thus *a fortiori* a $\mathbb{SUB}_{\mathfrak{p}}$-group), by Corollary 2.3. If G and U are as before and N is a closed normal subgroup of G, then G/N has an open subgroup isomorphic to $U/(U \cap N)$, which is an $\mathbb{A}_{\mathfrak{p}}$-group and hence a $\mathbb{SUB}_{\mathfrak{p}}$-group by Corollary 2.5. Hence also G/N is a $\mathbb{SUB}_{\mathfrak{p}}$-group.

Finally, suppose that $f : G \to H$ is a continuous homomorphism from a locally compact group G to a $\mathbb{SUB}_{\mathfrak{p}}$-group H. Since H is totally disconnected, the connected identity component G_0 of G is contained in the kernel of f, entailing that $Q := G/\ker f$ is totally disconnected. Let $\overline{f} : Q \to H$ be the injective continuous homomorphism determined by $\overline{f} \circ q = f$, where $q\colon G \to Q$ is the quotient map. Let $W \subseteq H$ be an open subgroup which is a finite product of p-adic Lie groups (for certain $p \in \mathfrak{p}$). Being totally disconnected and locally compact, Q has a compact, open subgroup U contained in $\overline{f}^{-1}(W)$. Then U is a $\mathbb{SUB}_{\mathfrak{p}}$-group by Corollary 2.4, and hence so is $Q = G/\ker f$. The proof of (a) is complete.

(b) It is obvious that $\mathbb{A}_{\mathfrak{p}}$ is closed under the formation of closed subgroups, Hausdorff quotients, and finite cartesian products. Given a continuous homomorphism $\phi\colon G \to H$ from a locally compact group G to an $\mathbb{A}_{\mathfrak{p}}$-group H, the locally compact group $Q := G/\ker\phi$ is totally disconnected. There is a unique continuous injective homomorphism $\overline{\phi}\colon Q \to H$ such that $\overline{\phi} \circ \kappa = \phi$, where $\kappa : G \to Q$ is the quotient map. Let us show that Q is an $\mathbb{A}_{\mathfrak{p}}$-group. For convenience of notation, after replacing G with Q and ϕ with $\overline{\phi}$, we may assume without loss of generality that $\phi\colon G \to H$ is injective. Furthermore, in view of Corollary 2.7, we may assume that $H = S/D$ for some closed subgroup S of a product $\prod_{p\in F} G_p$ of p-adic Lie groups G_p for p in a finite subset $F \subseteq \mathfrak{p}$, and some discrete normal subgroup D of S. We let $\rho\colon S \to S/D = H$ be the canonical quotient map. Our goal is to equip $S' := \rho^{-1}(\phi(G))$ with a finer topology which turns this group into a closed subgroup of another, suitable chosen product of p-adic Lie groups, and such that $S'/D \cong G$ as a topological group.

To this end, we choose a compact, open subgroup U of G. Then $W := \rho^{-1}(\phi(U))$ is a closed subgroup of S and hence also of $\prod_{p\in F} G_p$. Let $W_p := W \cap G_p$; by Corollary 2.3, $\prod_{p\in F} W_p$ is an open, normal subgroup of W. We claim that, for each $y \in S'$, the inner automorphism $I_y\colon S \to S$, $I_y(s) := ysy^{-1}$ restricts to a local automorphism of W. Indeed, there exists $x \in G$ such that $\rho(y) = \phi(x)$, and a compact, open subgroup $A \subseteq G$ such that $xAx^{-1} \subseteq U$. Then, $\phi|_U^{\phi(U)}$ being a homeomorphism as U is compact, $\phi(A)$ is an open, compact subgroup of $\phi(U)$, such that $I_{\phi(x)}(\phi(A)) = \phi(I_x(A)) \subseteq \phi(U)$. Hence, $\rho|_W^{\phi(U)}$ being continuous, $B := \rho^{-1}(\phi(A))$ is an open subgroup of W, and from $\rho(I_y(B)) = I_{\rho(y)}(\rho(B)) = I_{\phi(x)}(\phi(A)) \subseteq \phi(U)$ we deduce that $I_y(B) \subseteq W$. The continuity of $I_y : S \to S$ and its inverse entails that $I_y|_B^W$ is an isomorphism from B onto an open subgroup of W.

As a consequence, there is a uniquely determined topology on S' making it a topological group, and which makes W an open subgroup and induces on it the given locally compact topology. Throughout the following, S' will be equipped with this locally compact topology, which is finer than the topology induced by S. Given $p \in F$, let pr_p be the canonical projection of $\prod_{q\in F} G_q$ onto G_p. Pick a compact, open subgroup C of W. Considerations very similar to the preceding ones show that there is a uniquely determined group topology on $H_p := \mathrm{pr}_p(S')$ such that $\mathrm{pr}_p(C)$, equipped with the compact topology induced by G_p (which makes it a p-adic Lie group) is a compact, open subgroup of H_p. Then H_p is a p-adic Lie group, and $\mathrm{pr}_p|_C^{\mathrm{pr}_p(C)}$ being continuous and open, we see that $\mathrm{pr}_p|_{S'}^{H_p} : S' \to H_p$ is a quotient map. The topology on C being induced by the maps $\mathrm{pr}_p|_C^{\mathrm{pr}_p(C)}$, where $\mathrm{pr}_p(C)$ is open in H_p, we easily see that $\prod_{p\in F} H_p$ induces the given locally compact topology on S'. Note that D, being discrete in S, is *a fortiori* a discrete (and hence closed) normal subgroup of S' (whose topology is finer). Hence $X := \phi(G) \cong S'/D$ is an $\mathbb{A}_{\mathfrak{p}}$-group, where we equip X now with the topology making $\rho|_{S'}^X$ a quotient morphism. In order that G be an $\mathbb{A}_{\mathfrak{p}}$-group, it only remains to show that $\theta := \phi|^X : G \to X$ is an isomorphism of topological groups. But $\rho|_W^{\rho(W)} = \rho|_W^{\phi(U)}$ is a quotient morphism with respect to the new topologies on domain and range. Since the topology on the domain W coincides with the old topology, we deduce that so does the topology on the image $\phi(U)$. Now $\phi|_U^{\phi(U)}$ being an isomorphism and $\phi(U) = \rho(W)$ being open in X, we see that θ is an isomorphism. □

Remark 2.9 Suppose that G is an $\mathbb{A}_{\mathfrak{p}}$-group, say $G = S/D$ as in Corollary 2.7. Applying the construction from the proof of Proposition 2.8 (b) to $\phi := \mathrm{id}_G$, we see that $G = S'/D$ where S' is a closed subgroup of a product $P := \prod_{p\in F} H_p$ of p-adic Lie groups for some finite subset $F \subseteq \mathfrak{p}$, D is a discrete normal subgroup of S', and furthermore all of the coordinate projections $\mathrm{pr}_p\colon P \to H_p$ restrict to *quotient morphisms* $\mathrm{pr}_p|_{S'}\colon S' \to H_p$.

Remark 2.10 Let $\mathfrak{p}$ be a finite set of primes, S a closed subgroup of a product $\prod_{p\in\mathfrak{p}} G_p$ of p-adic Lie groups, and $f\colon G \to S$ be a continuous homomorphism from a locally compact group to S. Repeating the proof of Proposition 2.8 (b) with $D := \{1\}$, we see that $G/\ker f$ is isomorphic to a closed subgroup of a product $\prod_{p\in\mathfrak{p}} H_p$ of p-adic Lie groups.

Remark 2.11 Proposition 2.8 allows us to apply Proposition 1.5 to the cases $\mathbb{A} := \mathbb{A}_{\mathfrak{p}}$ and $\mathbb{A} := \mathbb{SUB}_{\mathfrak{p}}$. We deduce, in particular, that every $\mathbb{MIX}_{\mathfrak{p}}$-group (resp., $\mathbb{VSUB}_{\mathfrak{p}}$-group) is a pro-$\mathbb{A}_{\mathfrak{p}}$-group (resp., a pro-$\mathbb{SUB}_{\mathfrak{p}}$-group), whence it is a projective limit of a projective system of $\mathbb{A}_{\mathfrak{p}}$-groups (resp., $\mathbb{SUB}_{\mathfrak{p}}$-groups), such that all bonding maps and all limit maps are quotient morphisms. We also deduce the useful fact that a locally compact group is a $\mathbb{MIX}_{\mathfrak{p}}$-group (resp., a $\mathbb{VSUB}_{\mathfrak{p}}$-group) if and only if it can be approximated by $\mathbb{A}_{\mathfrak{p}}$-groups (resp., by $\mathbb{SUB}_{\mathfrak{p}}$-groups).

Theorem 2.12 *For any sets of primes* $\mathfrak{p}$ *and* $\mathfrak{q}$*, we have*

(a) $\mathbb{VSUB}_{\mathfrak{p}} \cap \mathbb{VSUB}_{\mathfrak{q}} = \mathbb{VSUB}_{\mathfrak{p}\cap\mathfrak{q}}$;

(b) $\mathbb{VSUB}_{\mathfrak{p}} \cap \mathbb{MIX}_{\mathfrak{q}} = \mathbb{MIX}_{\mathfrak{p}\cap\mathfrak{q}}$;

(c) $\mathbb{MIX}_{\mathfrak{p}} \cap \mathbb{MIX}_{\mathfrak{q}} = \mathbb{MIX}_{\mathfrak{p}\cap\mathfrak{q}}$.

Proof. We may assume that $\mathfrak{p}, \mathfrak{q} \neq \emptyset$, the excluded case being trivial.

(a) Let $G \in \mathbb{VSUB}_{\mathfrak{p}} \cap \mathbb{VSUB}_{\mathfrak{q}}$. By Remark 2.11, in order that $G \in \mathbb{VSUB}_{\mathfrak{p}\cap\mathfrak{q}}$, we only need to show that G can be approximated by $\mathbb{SUB}_{\mathfrak{p}\cap\mathfrak{q}}$-groups. To verify the latter, let U be an identity neighbourhood of G; after shrinking U, we may assume that U is a compact, open subgroup of G. Since $G \in \mathbb{VSUB}_{\mathfrak{p}}$, there exists a closed normal subgroup $K \subseteq U$ of G such that $G/K \in \mathbb{SUB}_{\mathfrak{p}}$ (Remark 2.11), and thus $G/K \in \mathbb{SUB}_F$ for some finite subset $F \subseteq \mathfrak{p}$. Now $G \in \mathbb{VSUB}_{\mathfrak{q}}$ entails that $G/K \in \mathbb{VSUB}_{\mathfrak{q}}$, whence the identity neighbourhood U/K contains a closed normal subgroup N of G/K such that $(G/K)/N$ has an open subgroup of the form $H = \prod_{q\in E} H_q$, where H_q is a q-adic Lie group for q in a finite subset $E \subseteq \mathfrak{q}$. The class $\mathbb{SUB}_F$ being closed under the formation of Hausdorff quotients and closed subgroups, we see that H is a $\mathbb{SUB}_F$-group. Hence H has an open subgroup of the form $W = \prod_{p\in F} W_p$ for certain p-adic Lie groups W_p; we may assume that W_p is a pro-p-group. Given $p \in F$, for each $q \in E \setminus \{p\}$, the continuous homomorphism $\mathrm{pr}_q|_{W_p} : W_p \to H_q$ (where $\mathrm{pr}_q : H \to H_q$ is the coordinate projection) has kernel W_p, by Lemma 2.1. Hence $\mathrm{pr}_q(W_p) = \{1\}$ for each $q \in E \setminus \{p\}$. Then $W_p = \{1\}$ for all $p \in F \setminus E$. Consequently, $W = \prod_{p\in E\cap F} W_p$, showing that H and thus also $(G/K)/N$ is a $\mathbb{SUB}_{\mathfrak{p}\cap\mathfrak{q}}$-group. Let $\rho\colon G \to G/K$ be the quotient map. Then $\rho^{-1}(N) \subseteq U$ and this is a closed normal subgroup of G such that $G/\rho^{-1}(N) \cong (G/K)/N \in \mathbb{SUB}_{\mathfrak{p}\cap\mathfrak{q}}$. Thus G can be approximated by $\mathbb{SUB}_{\mathfrak{p}\cap\mathfrak{q}}$-groups.

(b) Suppose that $G \in \mathbb{VSUB}_{\mathfrak{p}} \cap \mathbb{MIX}_{\mathfrak{q}}$. By Remark 2.11, in order that $G \in \mathbb{MIX}_{\mathfrak{p}\cap\mathfrak{q}}$, we only need to show that G can be approximated by $\mathbb{A}_{\mathfrak{p}\cap\mathfrak{q}}$-groups. To verify this, let U be a compact, open subgroup of G. Since $G \in \mathbb{VSUB}_{\mathfrak{p}}$, there exists a closed normal subgroup $K \subseteq U$ of G such that $G/K \in \mathbb{SUB}_F$ for some finite, non-empty subset $F \subseteq \mathfrak{p}$. Now $G \in \mathbb{MIX}_{\mathfrak{q}}$ entails that $G/K \in \mathbb{MIX}_{\mathfrak{q}}$, whence the identity neighbourhood U/K contains a closed normal subgroup N of G/K such that $(G/K)/N$ is an $\mathbb{A}_{\mathfrak{q}}$-group (Remark 2.11). Hence, in view of Remark 2.9, there are q-adic Lie groups H_q for q in a non-empty finite subset $E \subseteq \mathfrak{q}$, a closed subgroup $S \subseteq \prod_{q\in E} H_q =: H$, and a quotient morphism $\rho\colon S \to (G/K)/N$, with discrete kernel D, such that $\mathrm{pr}_q|_S\colon S \to H_q$ is a quotient morphism for each q, where $\mathrm{pr}_q : H \to H_q$ is the coordinate projection. Now, D being discrete, the groups S and $(G/K)/N$ are locally isomorphic. The group $(G/K)/N$ is a $\mathbb{SUB}_F$-group as a quotient of the $\mathbb{SUB}_F$-group G/K (Proposition 2.8 (a)), whence every identity neighbourhood of $(G/K)/N$ contains a compact, open subgroup which is a product of p-adic Lie groups, with $p \in F$. Hence also S has a compact, open subgroup of the form $V = \prod_{p\in F} V_p$, where V_p is a p-adic Lie group for each $p \in F$. Let $q \in E \setminus F$. By Lemma 2.1, $\mathrm{pr}_q|_{V_p}$ has open kernel for each $p \in F$, whence $\mathrm{pr}_q|_V$ has open kernel. The group V being compact, $\ker \mathrm{pr}_q|_V$ has finite index in V, entailing that $\mathrm{pr}_q(V)$ is finite. The latter set being open in H_q (since $\mathrm{pr}_q|_S$ is a quotient morphism), we deduce that H_q is a discrete group (and hence a p-adic Lie group for each p). Thus $H = D' \times \prod_{q\in E\cap F} H_q$, where $D' := \prod_{q\in E\setminus F} H_q$ is discrete. Therefore $(G/K)/N \cong S/D$ is an $\mathbb{A}_{E\cap F}$-group and hence an $\mathbb{A}_{\mathfrak{p}\cap\mathfrak{q}}$-group. The kernel of the natural quotient map $G \to (G/K)/N$ being contained in U, we see that G can be approximated by $\mathbb{A}_{\mathfrak{p}\cap\mathfrak{q}}$-groups, as required.

(c) Since $\mathbb{MIX}_{\mathfrak{p}} \subseteq \mathbb{VSUB}_{\mathfrak{p}}$, assertion (c) is a trivial consequence of (b). □

Note that $\mathbb{Z}_p^{\mathbb{N}}$ is a $\mathbb{MIX}_{\mathbb{P}}$-group without open subgroups satisfying an ascending chain condition on closed subgroups.

3 Tidy subgroups and the scale function for $\mathbb{SUB}_{\mathfrak{p}}$-groups and $\mathbb{VSUB}_{\mathfrak{p}}$-groups

We now describe tidy subgroups and calculate the scale function for $\mathbb{SUB}_{\mathfrak{p}}$-groups and then, passing to projective limits, for $\mathbb{VSUB}_{\mathfrak{p}}$-groups. For this purpose, we require a slight generalization of the notion of the module of an automorphism.

3.1 If G is a locally compact group, with Haar measure λ, and $\alpha\colon H \to G$ an injective, continuous homomorphism from an open subgroup $H \subseteq G$ onto an open subgroup $S := \alpha(H)$ of G, then, due to uniqueness of Haar measure on S up to a multiplicative constant, there is a positive real number $\Delta_G(\alpha)$ (also written $\Delta(\alpha)$ when G is understood), the *module of* α, such that $\lambda|_S = \Delta_G(\alpha)\,\alpha(\lambda|_H)$. Thus $\Delta_G(\alpha) = \frac{\lambda(\alpha(U))}{\lambda(U)}$, for every non-empty open subset $U \subseteq H$ of finite measure. If G is a p-adic Lie group here, identifying $L(H)$ and $L(S)$ with $L(G)$, we have

$$\Delta_G(\alpha) \;=\; \Delta_{L(G)}(L(\alpha)) \;=\; |\det L(\alpha)|_p\,, \tag{5}$$

using the natural absolute value $|.|_p$ on $\mathbb{Q}_p$. In fact, the proof of [5, Ch. 3, §3.16, Prop. 55] (treating only étale endomorphisms) directly generalizes to the present situation.

It is also useful to know that every $\mathbb{SUB}_{\mathbb{P}}$-group has an open subgroup satisfying the ascending chain condition on closed subgroups, because this property ensures that a compact, open subgroup satisfying condition (T1) of tidiness (as described in the Introduction) automatically satisfies condition (T2) as well [1, Thm. 3.32 and Rem. 3.33 (2)].

Proposition 3.2 *Every $G \in \mathbb{SUB}_{\mathbb{P}}$ has a compact, open subgroup U such that U satisfies the ascending chain condition on closed subgroups.*

Proof. Since $G \in \mathbb{SUB}_{\mathbb{P}}$, there exists a finite set $F \subseteq \mathbb{P}$ and an open subgroup U of G such that $U = \prod_{p\in F} U_p$ for certain p-adic Lie groups U_p. After shrinking U_p, we may assume that U_p also is a pro-p-group. We may furthermore assume that each U_p satisfies the ascending chain condition on closed subgroups, because every p-adic Lie group has an open subgroup with this property [24, proof of Prop. 3.5]. Now let $S_1 \subseteq S_2 \subseteq \cdots$ be an ascending sequence of closed subgroups of U. Then $(S_n \cap U_p)_{n \in \mathbb{N}}$ becomes stationary for each $p \in F$, and hence so does $(S_n)_{n\in\mathbb{N}}$, as $S_n = \prod_{p\in F}(S_n \cap U_p)$ by Proposition 2.2. □

The following fact is essential for the calculation of the scale function and tidy subgroups.

3.3 If E is a finite-dimensional $\mathbb{Q}_p$-vector space and α a linear automorphism of E, then

$$\begin{aligned} E &= E_{\mathrm{p}} \oplus E_0 \oplus E_{\mathrm{m}}\,, \quad \text{where} \\ E_{\mathrm{m}} &:= \{x \in E\colon \alpha^n(x) \to 0 \text{ as } n \to \infty\}\,, \\ E_{\mathrm{p}} &:= \{x \in E\colon \alpha^{-n}(x) \to 0 \text{ as } n \to \infty\} \quad \text{and} \\ E_0 &:= \{x \in E\colon \alpha^{\mathbb{Z}}(x) \text{ is relatively compact}\}\,; \end{aligned} \tag{6}$$

see [24, La. 3.4] or [7, La. 3.3] (cf. [18, pp. 80–83] for more refined information). We call (6) the *contraction decomposition of E with respect to α.* It is known (see, e.g., [7, La. 3.3] and its proof) that there is an ultrametric norm $\|.\|$ on E which is *adapted* to the decomposition (6) in the sense that $\|\alpha(x)\| = \|x\|$ for all $x \in E_0$ and, for suitable $\theta > 1$,

$$\|\alpha(x)\| \;\geq\; \theta\,\|x\| \quad \text{for all } x \in E_{\mathrm{p}} \quad \text{and} \quad \|\alpha(x)\| \;\leq\; \theta^{-1}\|x\| \quad \text{for all } x \in E_{\mathrm{m}}\,.$$

If $\mathfrak{g}$ is a finite-dimensional p-adic Lie algebra, there exists a compact, open submodule $V \subseteq \mathfrak{g}$ such that the Campbell-Hausdorff series converges on $V \times V$ to a function $*\colon V \times V \to V$ making V a p-adic Lie group (see [5, Ch. II, §8.3, Prop. 3]). We call V a *CH-group*.

Theorem 3.4 *Let $\mathfrak{p} \neq \emptyset$ be a finite set of primes, G be a $\mathbb{SUB}_{\mathfrak{p}}$-group, and $\alpha\colon G \to G$ be an automorphism (e.g., $\alpha := I_x : y \mapsto xyx^{-1}$ for some $x \in G$). Let H be an open subgroup of G of the form $\prod_{p\in\mathfrak{p}} H_p$, where H_p is a p-adic Lie group for $p \in \mathfrak{p}$. Then we have:*

(a) *For each $p \in \mathfrak{p}$, there exists an open subgroup U_p of H_p such that $\alpha(U_p) \subseteq H_p$.*

(b) *Identifying $L(U_p)$ with $L(H_p)$, we consider $\beta_p := L(\alpha|_{U_p}^{H_p})$ as a Lie algebra automorphism of $L(H_p)$. We let $L(H_p) = L(H_p)_{\mathrm{p}} \oplus L(H_p)_0 \oplus L(H_p)_{\mathrm{m}}$ be the contraction decomposition of $L(H_p)$ with respect to β_p, and abbreviate $L(H_p)_+ := L(H_p)_{\mathrm{p}} \oplus L(H_p)_0$ and $L(H_p)_- := L(H_p)_0 \oplus L(H_p)_{\mathrm{m}}$. Then*

$$r_G(\alpha) \;=\; \prod_{p\in\mathfrak{p}} \Delta\big(\beta_p|_{L(H_p)_+}^{L(H_p)_+}\big) \;=\; \prod_{p\in\mathfrak{p}} \prod_{\substack{i\in\{1,\ldots,\dim L(H_p)\} \\ |\lambda_{p,i}|_p \geq 1}} |\lambda_{p,i}|_p \;, \tag{7}$$

where $\lambda_{p,1}, \ldots, \lambda_{p,\dim L(H_p)}$ are the eigenvalues of β_p in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ (repeated according to their algebraic multiplicities), and $|.|_p$ is the unique extension of the usual absolute value on $\mathbb{Q}_p$ to an absolute value on $\overline{\mathbb{Q}}_p$. In particular,

$$\operatorname{im} s_G \;\subseteq\; \prod_{p\in\mathfrak{p}} p^{\mathbb{N}_0}\,, \quad \text{i.e.,} \quad \mathbb{P}(G) \;\subseteq\; \mathfrak{p}\,. \tag{8}$$

(c) *For each p, let $\phi_p : V_p \to W_p$ be a topological isomorphism from a CH-group $V_p \subseteq L(H_p)$ onto an open subgroup W_p of H_p, such that $L(\phi_p) = \mathrm{id}_{L(H_p)}$. Let $\|.\|_p : L(H_p) \to [0, \infty[$ be an ultrametric norm on $L(H_p)$ adapted to the contraction decomposition. Then there exists $\varepsilon_0 > 0$ such that $B_{p,\varepsilon_0} := \{y \in L(H_p) \colon \|y\|_p < \varepsilon_0\} \subseteq V_p$ for each $p \in \mathfrak{p}$, and such that*

$$B_\varepsilon := \prod_{p \in \mathfrak{p}} \phi_p(B_{p,\varepsilon}) \tag{9}$$

is a compact, open subgroup of G which is tidy for α, for each $\varepsilon \in]0, \varepsilon_0]$.

Proof. Let $H = \prod_{p\in\mathfrak{p}} H_p$ be as described in the theorem. After shrinking H_p, we may assume that H_p is a pro-p-group and that there exists an isomorphism $\phi_p : V_p \to H_p$ with $L(\phi_p) = \mathrm{id}_{L(H_p)}$, for some compact, open submodule $V_p \subseteq L(H_p)$, equipped with the Campbell-Hausdorff multiplication. There exists a compact, open subgroup $W_p \subseteq H_p$ such that $\alpha(W_p) \subseteq H$ and $\alpha^{-1}(W_p) \subseteq H$. Then $\mathrm{pr}_q \circ \alpha|_{W_p}^{H} = 1$ and $\mathrm{pr}_q \circ \alpha^{-1}|_{W_p}^{H} = 1$ for $q \neq p$, by Lemma 2.1, where $\mathrm{pr}_q \colon H \to H_q$ is the coordinate projection. Hence

$$\alpha(W_p) \subseteq H_p \quad \text{and} \quad \alpha^{-1}(W_p) \subseteq H_p\,, \quad \text{for each } p \in \mathfrak{p}. \tag{10}$$

Shrinking W_p further, we may assume that $\alpha|_{W_p}^{H_p}$ is linear in exponential coordinates, viz.

$$\phi_p(L(\alpha|_{W_p}^{H_p}).y) = \alpha(\phi_p(y)) \qquad \text{for all } \; y \in \phi_p^{-1}(W_p), \tag{11}$$

and likewise for α^{-1}. In the following, we identify H_p with $V_p \subseteq L(H_p)$ by means of the isomorphism ϕ_p^{-1}, for convenience. Then (11) and its analogue for α^{-1} take the form

$$\alpha(y) = L(\alpha|_{W_p}^{H_p}).y \quad \text{and} \quad \alpha^{-1}(y) = L(\alpha|_{W_p}^{H_p})^{-1}.y \quad \text{for all } y \in W_p. \tag{12}$$

For each $p \in \mathfrak{p}$, we choose an ultrametric norm $\|.\|_p$ on $L(H_p)$ adapted to the contraction decomposition $L(H_p) = L(H_p)_{\mathrm{p}} \oplus L(H_p)_0 \oplus L(H_p)_{\mathrm{m}}$ of $L(H_p)$ with respect to the Lie algebra automorphism $\beta_p := L(\alpha|_{W_p}^{H_p})$. Choose $\delta > 0$ such that $B_{p,\delta} := \{y \in L(H_p) \colon \|y\|_p < \delta\} \subseteq H_p$ for each $p \in \mathfrak{p}$. Next, choose $\varepsilon_0 \in]0, \delta]$ such that

$$B_{p,\varepsilon_0} \subseteq W_p \quad \text{and} \quad \alpha(B_{p,\varepsilon_0}) \subseteq B_{p,\delta}, \quad \text{for all } p \in \mathfrak{p},$$

and such that $B_{p,\varepsilon}$ is a subgroup of $(H_p, *)$ (and hence of G), for all $\varepsilon \in]0, \varepsilon_0]$; the latter is possible by [5, Ch. III, §4.2, La. 3 (iii)]. For $p \in \mathfrak{p}$, consider the map

$$f_p \colon (W_p \cap L(H_p)_+) \times (W_p \cap L(H_p)_{\mathrm{m}}) \to H_p \subseteq L(H_p), \quad (u, v) \mapsto uv$$

(product in G). Then f_p is analytic (and hence strictly differentiable by [4, 4.2.3 & 3.2.4]) and its differential at 0 is the identity map $\mathrm{id}_{L(H_p)}$. By the Inverse Function Theorem in the form [12, Prop. 7.1 (b)$'$], after shrinking ε_0 we can achieve that

$$f_p\big((B_{p,\varepsilon} \cap L(H_p)_+) \times (B_{p,\varepsilon} \cap L(H_p)_{\mathrm{m}})\big) \;=\; B_{p,\varepsilon}\,, \quad \text{for each } \varepsilon \in]0, \varepsilon_0]. \tag{13}$$

Fix $\varepsilon \in \,]0, \varepsilon_0]$; we claim that the compact, open subgroup $B_\varepsilon := \prod_{p\in\mathfrak{p}} B_{p,\varepsilon}$ of G is tidy for α. To this end, note that

$$\alpha^{-1}(B_{p,\varepsilon} \cap L(H_p)_+) \;=\; \beta_p^{-1}.(B_{p,\varepsilon} \cap L(H_p)_+) \;\subseteq\; B_{p,\varepsilon} \cap L(H_p)_+$$

by choice of the norm $\|.\|_p$, and thus $\alpha^{-n}(B_{p,\varepsilon} \cap L(H_p)_+) \subseteq B_{p,\varepsilon}$ for each $n \in \mathbb{N}_0$, entailing that $B_{p,\varepsilon} \cap L(H_p)_+ \subseteq (B_\varepsilon)_+$ and thus

$$\prod_{p\in\mathfrak{p}} (B_{p,\varepsilon} \cap L(H_p)_+) \;\subseteq\; (B_\varepsilon)_+,$$

where $(B_\varepsilon)_\pm := \bigcap_{n\in\mathbb{N}_0} \alpha^{\pm n}(B_\varepsilon)$. If $y = (y_p)_{p\in\mathfrak{p}} \in B_\varepsilon$ and $y_q \notin L(H_q)_+$ for some $q \in \mathfrak{p}$, then $\|(\beta_q)^{-n}.y_q\|_q \geq \|(\beta_q)^{-n}.(y_q)_{\mathrm{m}}\|_q \geq \theta^n \|(y_q)_{\mathrm{m}}\|_q \to \infty$ as $n \to \infty$, where $(y_q)_{\mathrm{m}}$ is the component of y_q in $L(H_q)_{\mathrm{m}}$ and $\theta > 1$ is as in **3.3**, the definition of an adapted norm. This entails that there is $m \in \mathbb{N}$ such that $(\beta_p)^{-n}(y_p) \in B_{p,\varepsilon}$ for all $n \in \{0, \ldots, m-1\}$ and all $p \in \mathfrak{p}$, and an element $q \in \mathfrak{p}$ such that

$$(\beta_q)^{-m}.y_q \;\notin\; B_{q,\varepsilon}\,. \tag{14}$$

Then $\beta_p^{-n}.y_p = \alpha^{-n}(y_p)$ for all $n \in \{1, \ldots, m\}$ and all $p \in \mathfrak{p}$, by (12). Since $\alpha^{-m+1}(y_p) \in B_{p,\varepsilon}$, we have $\alpha^{-m}(y_p) \in H_p$ for each $p \in \mathfrak{p}$. Hence (14) entails that $\alpha^{-m}(y) = (\beta_p^{-m}.y_p)_{p\in\mathfrak{p}} \notin B_\varepsilon$, whence $y \notin (B_\varepsilon)_+$. Summing up, we have shown that

$$(B_\varepsilon)_+ \;=\; \prod_{p\in\mathfrak{p}} (B_{p,\varepsilon} \cap L(H_p)_+)\,,$$

and an analogous argument gives $(B_\varepsilon)_- = \prod_{p\in\mathfrak{p}} (B_{p,\varepsilon} \cap L(H_p)_-)$. Using (13), we see that

$$\begin{aligned} B_\varepsilon \;&\supseteq\; (B_\varepsilon)_+(B_\varepsilon)_- \;=\; \prod_{p\in\mathfrak{p}} (B_{p,\varepsilon} \cap L(H_p)_+)(B_{p,\varepsilon} \cap L(H_p)_-) \\ &\supseteq\; \prod_{p\in\mathfrak{p}} (B_{p,\varepsilon} \cap L(H_p)_+)(B_{p,\varepsilon} \cap L(H_p)_{\mathrm{m}}) \;=\; \prod_{p\in\mathfrak{p}} B_{p,\varepsilon} \;=\; B_\varepsilon \end{aligned}$$

and hence $B_\varepsilon = (B_\varepsilon)_+(B_\varepsilon)_-$, *i.e.*, B_ε satisfies condition (T1) of tidiness. As a consequence of Proposition 3.2, B_ε also satisfies (T2) and thus B_ε is tidy for α.

To calculate $r_G(\alpha)$, we choose $\varepsilon \in \,]0, \varepsilon_0]$ and obtain

$$\begin{aligned} r_G(\alpha) \;&=\; [\alpha((B_\varepsilon)_+) : (B_\varepsilon)_+] \;=\; \prod_{p\in\mathfrak{p}} [\alpha(B_{p,\varepsilon} \cap L(H_p)_+) \;:\; B_{p,\varepsilon} \cap L(H_p)_+] \\ &=\; \prod_{p\in\mathfrak{p}} \Delta(\alpha|_{B_{p,\varepsilon}\cap L(H_p)_+}^{H_p\cap L(H_p)_+}) \;\stackrel{(5)}{=}\; \prod_{p\in\mathfrak{p}} \Delta(\beta_p|_{L(H_p)_+}) \;=\; \prod_{p\in\mathfrak{p}} |\det(\beta_p|_{L(H_p)_+})|_p \\ &=\; \prod_{p\in\mathfrak{p}} \prod_{\substack{i\in\{1,\ldots,\dim L(H_p)\} \\ |\lambda_{p,i}|_p \geq 1}} |\lambda_{p,i}|_p\,, \end{aligned}$$

with $\lambda_{p,i}$ as described in the theorem. Here, we used [7, La. 3.4] to pass to the third line. Since $|\mathbb{Q}_p^\times|_p = p^{\mathbb{N}_0}$, the second term in the second line shows that $r_G(\alpha) \in \prod_{p\in\mathfrak{p}} p^{\mathbb{N}_0}$. Hence $\operatorname{im} s_G \subseteq \prod_{p\in\mathfrak{p}} p^{\mathbb{N}_0}$ in particular. □

Remark 3.5 In the special case where G is a p-adic Lie group, Theorem 3.4 provides a self-contained, explicit calculation of the scale function, and a basis of subgroups tidy for $x \in G$. The earlier calculation of s_G in [7] relied on a result from [24], and tidy subgroups could not be described explicitly in that paper.

Lemma 3.6 *Let G be a totally disconnected, locally compact group, $K \subseteq G$ be a compact, normal subgroup, $q\colon G \to G/K$ be the quotient map, and $x \in G$. Then $q^{-1}(U)$ is tidy for x, for every compact, open subgroup $U \subseteq G/K$ which is tidy for xK, and $s_G(x) = s_{G/K}(xK)$.*

Proof. Let $y := xK$ and $V := q^{-1}(U)$. Since $I_x^n(V)$ is K-saturated for each $n \in \mathbb{Z}$, and $q^{-1}(I_y^n(U)) = I_x^n(V)$, we easily see that $V_\pm = q^{-1}(U_\pm)$, $V = q^{-1}(U_+U_-) = V_+V_-$, and $V_{++} = q^{-1}(U_{++})$, which is closed. Thus V is tidy for x, and $s_G(x) = [I_x(V_+) : V_+] = [q^{-1}(I_y(U_+)) : q^{-1}(U_+)] = [I_y(U_+) : U_+] = s_{G/K}(y)$. □

Combining Theorem 3.4 and Lemma 3.6, we obtain:

Corollary 3.7 *Let $\mathfrak{p} \neq \emptyset$ be a set of primes, and $G \in \mathbb{VSUB}_{\mathfrak{p}}$. Then we have:*

(a) *Let $K \subseteq G$ be a compact, normal subgroup such that G/K is a $\mathbb{SUB}_{\mathfrak{p}}$-group; then G/K is a $\mathbb{SUB}_F$-group for some finite subset $F \subseteq \mathfrak{p}$. We have*

$$s_G(x) = s_{G/K}(xK) \qquad \text{for each } x \in G, \tag{15}$$

where $s_{G/K}(xK)$ can be calculated explicitly as described in Theorem 3.4. In particular, $\mathbb{P}(G) \subseteq F$, whence $\mathbb{P}(G)$ is a finite subset of $\mathfrak{p}$.

(b) *Let $x \in G$. For every K as in* (a) *and compact, open subgroup $U \subseteq G/K$ tidy for xK, the subgroup $q_K^{-1}(U) \subseteq G$ is tidy for x, where $q_K : G \to G/K$ is the quotient map. The set of subgroups $q_K^{-1}(U)$ tidy for x, for all possible K and U as before, is a basis for the filter of identity neighbourhoods of G. Hence G has small tidy subgroups.* □

Remark 3.8 If G is an Adèle group then $\mathbb{P}(G/G_0)$ typically is an infinite set. In this case, G/G_0 is not a $\mathbb{MIX}_{\mathbb{P}}$-group (nor a $\mathbb{VSUB}_{\mathbb{P}}$-group), by Corollary 3.7 (a). For example, we have $\mathbb{P}(G/G_0) = \mathbb{P}$ for $G := \varinjlim_F \big(\mathrm{SL}_n(\mathbb{R}) \times \prod_{p\in F} \mathrm{SL}_n(\mathbb{Q}_p) \times \prod_{\mathbb{P}\setminus F} \mathrm{SL}_n(\mathbb{Z}_p)\big)$, where F ranges through the set of finite subsets of $\mathbb{P}$, and $n \geq 2$ (cf. [7, Thm. 5.1]).

4 The minimal set of primes needed to build up a compactly generated $\mathbb{MIX}_{\mathbb{P}}$-group, or $\mathbb{VSUB}_{\mathbb{P}}$-group

As a tool, we introduce an analogue of the adjoint action of Lie groups for $\mathbb{SUB}_{\mathbb{P}}$-groups.

4.1 Suppose that G is a $\mathbb{SUB}_{\mathbb{P}}$-group and $H \subseteq G$ an open subgroup of the form $H = \prod_{p\in F} H_p$, where F is a finite set of primes and H_p is a p-adic Lie group, for each $p \in F$. Given $x \in G$, consider the inner automorphism $I_x\colon G \to G$, $I_x(y) := xyx^{-1}$. Given x, for each $p \in F$, there is a compact, open subgroup $U_p \subseteq H_p$ such that $I_x(U_p) \subseteq H_p$. Identifying $L(U_p)$ with $L(H_p)$ by means of the isomorphism of Lie algebras $L(i_p)$, where $i_p : U_p \to H_p$ is the inclusion map, we may consider $\mathrm{Ad}_p(x) := L(I_x|_{U_p}^{H_p})$ as an automorphism of the Lie algebra $L(H_p)$. It is easy to see that

$$\mathrm{Ad}_p : G \to \mathrm{Aut}(L(H_p)), \quad x \mapsto \mathrm{Ad}_p(x)$$

is a homomorphism. Since $\mathrm{Ad}_p(x) = \mathrm{Ad}(\mathrm{pr}_p(x))$ for $x \in H$, where $\mathrm{pr}_p : H \to H_p$ is the coordinate projection and $\mathrm{Ad} : H_p \to \mathrm{Aut}(L(H_p))$ is continuous, we see that the homomorphism Ad_p is continuous on the open subgroup H and hence continuous.

Theorem 4.2 *Let $\mathfrak{p}$ be a set of primes.*

(a) *If $G \in \mathbb{VSUB}_{\mathbb{P}}$ is compactly generated, then $G \in \mathbb{VSUB}_{\mathfrak{p}}$ if and only if $\mathbb{P}(G) \subseteq \mathfrak{p}$.*

(b) *If $G \in \mathbb{MIX}_{\mathbb{P}}$ is compactly generated, then $G \in \mathbb{MIX}_{\mathfrak{p}}$ if and only if $\mathbb{P}(G) \subseteq \mathfrak{p}$.*

Proof. By Corollary 3.7, $G \in \mathbb{VSUB}_{\mathfrak{p}}$ entails $\mathbb{P}(G) \subseteq \mathfrak{p}$ for $\mathfrak{p}$ non-empty, and apparently $\mathbb{P}(G) \subseteq \mathfrak{p}$ also holds if $\mathfrak{p} = \emptyset$, as every pro-discrete group is uniscalar.

(a) Let $G \in \mathbb{VSUB}_{\mathbb{P}}$ be compactly generated; we want to show that $G \in \mathbb{VSUB}_{\mathbb{P}(G)}$. In view of Remark 2.11, we only need to show that G can be approximated by $\mathbb{SUB}_{\mathbb{P}(G)}$-groups. Now, given a compact, open subgroup $U \subseteq G$, there exists a compact normal subgroup $N \subseteq U$ of G such that G/N is a $\mathbb{SUB}_{\mathfrak{q}}$-group for some finite subset $\mathfrak{q} \subseteq \mathbb{P}$. Then G/N is compactly generated, and $\mathbb{P}(G) = \mathbb{P}(G/N)$ (Lemma 3.6), whence $\mathbb{P}(G) \subseteq \mathfrak{q}$, by what has already been shown. Let $\rho : G \to G/N$ be the quotient map. If we can show that $G/N \in \mathbb{VSUB}_{\mathbb{P}(G)}$, then we can find a compact normal subgroup $Z \subseteq \rho(U)$ of G/N such that $(G/N)/Z \cong G/\rho^{-1}(Z)$ is a $\mathbb{SUB}_{\mathbb{P}(G)}$-group, where $\rho^{-1}(Z) \subseteq U$, whence indeed G can be approximated by $\mathbb{SUB}_{\mathbb{P}(G)}$-groups.

Replacing G with G/N, we may therefore assume that $G \in \mathbb{SUB}_{\mathfrak{q}}$ for some finite set of primes $\mathfrak{q}$. Let $U \subseteq G$ be as before. Then G has an open subgroup $H \subseteq U$ of the form $H = \prod_{p \in \mathfrak{q}} H_p$, where H_p is a p-adic Lie group. Given $x \in G$, define $I_x : G \to G$, $I_x(y) := xyx^{-1}$. After shrinking H_p, we may identify H_p with a compact, open $\mathbb{Z}_p$-submodule of $L(H_p)$, equipped with the CH-multiplication (as in the proof of Theorem 3.4 (a)) and may assume that $I_x(y) = \mathrm{Ad}(x).y$ for all $x, y \in H_p$. We let $\emptyset \neq K$ be a compact, symmetric generating set for G. Then $K \subseteq FH$ for some finite subset $F \subseteq K$. For each $p \in \mathfrak{q}$ and $x \in F$, there exists a compact, open subgroup $V_p(x) \subseteq H_p$ such that $I_x(V_p(x)) \subseteq H_p$ and $I_x(y) = L(I_x|_{V_p(x)}^{H_p}).y$ for all $y \in V_p(x)$; we set $V_p := \bigcap_{x \in F} V_p(x)$. Let W_p be a compact, open, normal subgroup of H_p such that $W_p \subseteq V_p$. Since $K \subseteq FH$, where $I_x|_{H_q} \equiv 1$ for $x \in H_p$ with $p \neq q \in \mathfrak{q}$, we conclude that $I_x(W_p) \subseteq H_p$ for all $x \in K$, and $I_x(y) = L(I_x|_{W_p}^{H_p}).y = \mathrm{Ad}_p(x).y$ for all $x \in K$ and $y \in W_p$. Now let $\mathfrak{r} := \mathfrak{q} \setminus \mathbb{P}(G)$. Given $p \in \mathfrak{r}$ and $x \in G$ we deduce from (7) and the fact that p neither divides $s_G(x)$ nor $s_G(x^{-1})$ that all eigenvalues of $L(I_x|_{W_p}^{H_p})$ in $\overline{\mathbb{Q}_p}$ have modulus 1. Repeating the arguments used to prove "1)⇒4)" of Prop. 3.1 in [14], we find that $\mathrm{Ad}_p(x) = L(I_x|_{W_p}^{H_p})$ is a compact element of

$\mathrm{Aut}(L(H_p))$, for each $x \in G$ and each $p \in \mathfrak{r}$. The homomorphism $\mathrm{Ad}_p \colon G \to \mathrm{Aut}(L(H_p))$ being continuous (see **4.1**), we deduce that the subgroup $R_p := \mathrm{Ad}_p(G) \subseteq \mathrm{Aut}(L(H_p))$ is compactly generated. Being compactly generated and periodic, R_p is relatively compact in $\mathrm{Aut}(L(H_p))$ (see [22]). As a consequence of [23, Part II, Appendix 1, Thm. 1], there exists a compact, open $\mathbb{Z}_p$-submodule $M_p \subseteq W_p$ of $L(H_p)$ which is invariant under R_p. Then the subgroup $C_p := \langle M_p \rangle$ of W_p generated by M_p is open in W_p and compact, and it is a normal subgroup of G as it is normalized by each $x \in K$, where K generates G (here we use that $I_x(C_p) = \langle I_x(M_p) \rangle = \langle \mathrm{Ad}_p(x).M_p \rangle = \langle M_p \rangle = C_p$). As a consequence, $C := \prod_{p \in \mathfrak{r}} C_p \subseteq U$ is a compact, normal subgroup of G such that G/C contains

$$H/C \;\cong\; \prod_{p \in \mathbb{P}(G)} H_p \;\times\; \prod_{p \in \mathfrak{r}} H_p/C_p$$

as an open subgroup, where $\prod_{p \in \mathfrak{r}} H_p/C_q$ is discrete. Thus $G/C \in \mathbb{SUB}_{\mathbb{P}(G)}$, showing that G can be approximated by $\mathbb{SUB}_{\mathbb{P}(G)}$-groups, which completes the proof of (a).

(b) Now suppose that $G \in \mathbb{MIX}_{\mathbb{P}}$ is compactly generated. Then $G \in \mathbb{VSUB}_{\mathbb{P}}$ *a fortiori* and hence $G \in \mathbb{VSUB}_{\mathbb{P}(G)}$, by Part (a). Thus $G \in \mathbb{MIX}_{\mathbb{P}} \cap \mathbb{VSUB}_{\mathbb{P}(G)} = \mathbb{MIX}_{\mathbb{P} \cap \mathbb{P}(G)} = \mathbb{MIX}_{\mathbb{P}(G)}$, using Theorem 2.12 (b). □

Generalizing [22, Cor. 5] and [14, Thm. 5.2], we obtain:

Corollary 4.3 *Every compactly generated, uniscalar $\mathbb{MIX}_{\mathbb{P}}$-group (or $\mathbb{VSUB}_{\mathbb{P}}$-group) G is pro-discrete.*

Proof. Since $\mathbb{P}(G) = \emptyset$, Theorem 4.2 shows that $G \in \mathbb{VSUB}_{\emptyset}$. □

Theorem 4.2 and Corollary 4.3 become false for groups that are not compactly generated, as there is a uniscalar p-adic Lie group without a compact, open, normal subgroup [14, § 6].

5 Variants based on locally pro-p groups

Variants of some of our results can be obtained when p-adic Lie groups are replaced with locally pro-p groups. We also provide counterexamples for results which do not carry over.

5.1 In **1.8**, we introduced the class $\mathbb{LOC}_p$ of locally pro-p groups. Given $\emptyset \neq \mathfrak{p} \subseteq \mathbb{P}$, we set $\mathbb{LOC}_{\mathfrak{p}} := \bigcup_{p \in \mathfrak{p}} \mathbb{LOC}_p$. Then $\mathcal{V}(\mathbb{LOC}_{\mathfrak{p}}) = \mathrm{SC}(\mathbb{A}_{\mathfrak{p}}^{\vee})$, where $\mathbb{A}_{\mathfrak{p}}^{\vee} := \overline{\mathrm{Q}}\,\overline{\mathrm{S}}\mathrm{P}(\mathbb{LOC}_{\mathfrak{p}})$. We define

$$\mathbb{MIX}_{\mathfrak{p}}^{\vee} \;:=\; \{G \in \mathcal{V}(\mathbb{LOC}_{\mathfrak{p}}) \colon \, G \text{ is locally compact}\},$$

$\mathbb{MIX}_{\emptyset}^{\vee} := \mathbb{MIX}_{\emptyset}$, and $\mathbb{A}_{\emptyset}^{\vee} := \mathbb{A}_{\emptyset}$. Given $\mathfrak{p} \subseteq \mathbb{P}$, we let $\mathbb{SUB}_{\mathfrak{p}}^{\vee}$ be the class of all topological groups possessing an open subgroup isomorphic to $\prod_{p \in F} G_p$, where $F \subseteq \mathfrak{p}$ is finite and G_p a pro-p-group for each $p \in F$. We define

$$\mathbb{VSUB}_{\mathfrak{p}}^{\vee} \;:=\; \{G \in \mathcal{V}(\mathbb{SUB}_{\mathfrak{p}}^{\vee}) \colon \, G \text{ is locally compact}\}.$$

In particular, $\mathbb{VSUB}_{\emptyset}^{\vee} = \mathbb{VSUB}_{\emptyset}$.

5.2 Then $\mathbb{A}_\mathfrak{p} \subseteq \mathbb{A}_\mathfrak{p}^\vee$ and $\mathbb{SUB}_\mathfrak{p} \subseteq \mathbb{SUB}_\mathfrak{p}^\vee$ and thus $\mathbb{MIX}_\mathfrak{p} \subseteq \mathbb{MIX}_\mathfrak{p}^\vee$ and $\mathbb{VSUB}_\mathfrak{p} \subseteq \mathbb{VSUB}_\mathfrak{p}^\vee$.

Theorem 5.3 *All of Corollary 2.4–Theorem 2.12 remain valid if p-adic Lie groups are replaced with locally pro-p groups, $\mathbb{A}_\mathfrak{p}$ by $\mathbb{A}_\mathfrak{p}^\vee$, $\mathbb{MIX}_\mathfrak{p}$ by $\mathbb{MIX}_\mathfrak{p}^\vee$, $\mathbb{SUB}_\mathfrak{p}$ by $\mathbb{SUB}_\mathfrak{p}^\vee$, and $\mathbb{VSUB}_\mathfrak{p}$ by $\mathbb{VSUB}_\mathfrak{p}^\vee$. With analogous replacements, also Corollary 3.7 (a) carries over.*[6]

Proof. The proofs of Corollary 2.4–Theorem 2.12 can be repeated verbatim in the new situation, making the replacements described in the theorem. Also the adaptation of Corollary 3.7 (a) is immediate in view of the Proposition 5.5 below. □

Remark 5.4 In particular, $\mathbb{A}_\mathfrak{p}^\vee$ and $\mathbb{SUB}_\mathfrak{p}^\vee$ are suitable for approximation. Hence a locally compact group belongs to $\mathbb{MIX}_\mathfrak{p}^\vee$ and $\mathbb{VSUB}_\mathfrak{p}^\vee$ if and only if it can be approximated by $\mathbb{A}_\mathfrak{p}^\vee$-groups (resp., by $\mathbb{SUB}_\mathfrak{p}^\vee$-groups).

Proposition 5.5 *Let G be a locally pro-p group. Then*

$$r_G(\alpha) \in p^{\mathbb{N}_0}$$

for each $\alpha \in \mathrm{Aut}(G)$ and thus $\mathbb{P}(G) \subseteq \{p\}$, i.e., the scale function s_G takes its values in $p^{\mathbb{N}_0}$. In particular, the conclusions apply if G is an analytic Lie group (or C^1-Lie group) over a local field $\mathbb{K}$, and $p := \mathrm{char}(\mathfrak{k})$ the characteristic of the residue field $\mathfrak{k}$ of $\mathbb{K}$.

Proof. Let U be an open, pro-p subgroup of G. Since U cannot contain an infinite q-Sylow subgroup for any $q \neq p$, [26, end of p. 173] entails that $r_G(\alpha) \in p^{\mathbb{N}_0}$ (see also Section 6). Since every C^1-Lie group over $\mathbb{K}$ is locally pro-p (see **1.8**), the final assertion follows. □

Remark 5.6 Of course, [7, Thm. 3.5] (and Theorem 3.4 above) provide much more information in the special case where G is a Lie group over a local field of characteristic 0.

Remark 5.7 If G is a Lie group over local field of positive characteristic, then not every topological group automorphism of G needs to be analytic. It is interesting that Proposition 5.5 provides some information also on these non-analytic automorphisms. The scale of analytic automorphisms has been studied in [13]. While

$$r_G(\alpha) = r_{L(G)}(L(\alpha)) \tag{16}$$

always holds in characteristic 0, surprisingly this equation becomes false in general if $\mathrm{char}(\mathbb{K}) > 0$. Closer inspection reveals that (16) holds if and only if G has small subgroups tidy for α (see [13]). This natural property (which is not always satisfied) was first explored in [1] in the context of contraction groups and has been exploited further in [11].

A straightforward adaptation of the proof of Theorem 2.12 (a) also shows:

[6] Except for the reference to Theorem 3.4 for the explicit calculation of $s_{G/K}(xK)$.

Proposition 5.8 $\mathbb{VSUB}_{\mathfrak{p}} \cap \mathbb{VSUB}_{\mathfrak{q}}^{\vee} = \mathbb{VSUB}_{\mathfrak{p}\cap\mathfrak{q}}$ *holds, for all subsets* $\mathfrak{p}, \mathfrak{q} \subseteq \mathbb{P}$. □

Here are some differences. First of all, $\mathbb{SUB}_{\mathbb{P}}^{\vee}$-groups need not have an open subgroup satisfying an ascending chain condition on closed subgroups, as the pro-p group $\mathbb{Z}_p^{\mathbb{N}}$ shows. Next, Theorem 4.2 and Corollary 4.3 do not carry over to $\mathbb{MIX}_{\mathfrak{p}}^{\vee}$-groups and $\mathbb{VSUB}_{\mathfrak{p}}^{\vee}$-groups.

Example 5.9 If we start with a non-trivial, finite p-group K, the construction described in [3, p. 269, lines 2–5] (based on an ansatz from [17]) outputs a totally disconnected, locally compact group G which is compactly generated and uniscalar, but does not possess a compact open normal subgroup. Since, by construction, G contains an open subgroup topologically isomorphic to $K^{\mathbb{N}}$, we see that G is locally pro-p.

Finally, we observe that $\mathbb{MIX}_{\{p\}}^{\vee}$-groups need not be locally pro-p. Indeed: Given any prime $q \neq p$ and non-trivial finite q-group K, the group $G := K^{\mathbb{N}}$ is pro-discrete and thus $G \in \mathbb{MIX}_{\{p\}}^{\vee}$. However, G is not locally pro-p.

6 The minimal set of primes in the general case

In this section, we show that also for groups $G \in \mathbb{MIX}_{\mathbb{P}}$ that are not compactly generated, there always is a smallest set of primes $\mathfrak{p}$ such that $G \in \mathbb{MIX}_{\mathfrak{p}}$. As a tool to find $\mathfrak{p}$, we associate certain sets of primes to totally disconnected, locally compact groups G, which only depend on the local isomorphism type of G.

Definition 6.1 Given a totally disconnected, locally compact group G, we let $\mathbb{L}(G) \subseteq \mathbb{P}$ be the set of all primes p such that, for every compact, open subgroup $U \subseteq G$, the element p divides the index $[U : V]$ for some compact, open subgroup $V \subseteq U$. The set $\mathbb{L}(G)$ is called the *local prime content* of G. The *reduced prime content* of G is defined as

$$\mathbb{L}_r(G) := \bigcap_K \mathbb{L}(G/K),$$

where K runs through the set of all compact, normal subgroups of G. Then $\mathbb{L}_r(G) \subseteq \mathbb{L}(G)$.

Standard arguments from Sylow theory show that $p \in \mathbb{L}(G)$ if and only if some (and hence any) compact, open subgroup U of G has an infinite p-Sylow subgroup. In [26, end of p. 173], it has been noted that this is the case if $p \in \mathbb{P}(G)$. More generally, we observe:

Proposition 6.2 *Let G be a totally disconnected, locally compact group. Then we have:*

(a) *For every $\alpha \in \mathrm{Aut}(G)$, the set of prime divisors of $r_G(\alpha)$ is a subset of $\mathbb{L}(G)$.*

(b) $\mathbb{P}(G) \subseteq \mathbb{L}(G)$ *holds, and indeed* $\mathbb{P}(G) \subseteq \mathbb{L}_r(G)$.

Proof. (a) If $p \in \mathbb{P}\setminus\mathbb{L}(G)$, then there exists a compact, open subgroup U of G such that p and $[U : V]$ are coprime, for every compact, open subgroup $V \subseteq U$. Now $U_N := \bigcap_{n=0}^{N} \alpha^n(U)$ satisfies condition (T1) of tidiness for α, for some $N \in \mathbb{N}_0$ (see [25, La. 1]). By [26, La. 2.2], $r_G(\alpha)$ divides $[U_N : U_N \cap \alpha^{-1}(U_N)]$, which in turn divides $[U : U_N \cap \alpha^{-1}(U_N)]$, because

$[U : U_N \cap \alpha^{-1}(U_N)] = [U : U_N] \cdot [U_N : U_N \cap \alpha^{-1}(U_N)]$. Since p and $[U, U_N \cap \alpha^{-1}(U_N)]$ are coprime, we deduce that p does not divide $r_G(\alpha)$.

(b) Is immediate from (a) and Lemma 3.6. $\square$

We shall use the following simple observations:

Lemma 6.3 (a) *If G is a totally disconnected, locally compact group and $U \subseteq G$ an open subgroup, then $\mathbb{L}(G) = \mathbb{L}(U)$.*

(b) *If $G := \prod_{j=1}^n G_j$ is a finite product of totally disconnected, locally compact groups G_j, then $\mathbb{L}(G) = \bigcup_{j=1}^n \mathbb{L}(G_j)$.*

Proof. The proof is obvious from the definition of the local prime contents, the definition of the product topology and the fact that $[U : W] = [U : V] \cdot [V : W]$ for any compact, open subgroups $W \subseteq V \subseteq U$. $\square$

Proposition 5.5 can be rephrased as follows:

Proposition 6.4 *Let G be a locally pro-p group (e.g., a C^1-Lie group over a local field whose residue field has characteristic p). Then $\mathbb{L}(G) \subseteq \{p\}$, and $\mathbb{L}(G) = \{p\}$ if and only if G is non-discrete. In particular, $\mathbb{P}(G) \subseteq \{p\}$.*

Proof. Let $U \subseteq G$ be a compact, open subgroup which is pro-p. Then $\mathbb{L}(G) = \mathbb{L}(U) \subseteq \{p\}$, where $\mathbb{L}(U) = \emptyset$ if and only if U is discrete. The rest follows from Proposition 6.2 (b). $\square$

Our next aim is to analyze $\mathbb{VSUB}_{\mathbb{P}}^{\vee}$-groups by means of their "intermediate" prime content:

Definition 6.5 If G is a $\mathbb{VSUB}_{\mathbb{P}}^{\vee}$-group, we define its *intermediate prime content* via

$$\mathbb{L}_i(G) \;:=\; \bigcup_{K' \in J(G)} \; \bigcap_{K' \supseteq K \in J(G)} \mathbb{L}(G/K)\,,$$

where $J(G)$ is the set of all compact, normal subgroups K of G such that $G/K \in \mathbb{SUB}_{\mathbb{P}}^{\vee}$.

Note that $J(G)$ is a filter basis in G converging to 1, by Proposition 1.5 (b) and Remark 5.4.

The following lemma explains the terminology "intermediate":

Lemma 6.6 *If G is a $\mathbb{VSUB}_{\mathbb{P}}^{\vee}$-group, then*

$$\mathbb{L}_r(G) = \bigcap_{K \in J(G)} \mathbb{L}(G/K)\,, \tag{17}$$

where $J(G)$ is as in Definition 6.5. Furthermore, $\mathbb{L}_r(G) \subseteq \mathbb{L}_i(G) \subseteq \mathbb{L}(G)$.

Proof. The inclusion "$\subseteq$" in (17) is obvious. To see the converse inclusion, let $p \in \mathbb{P} \setminus \mathbb{L}_r(G)$. Then there exists a compact, normal subgroup K of G such that $p \notin \mathbb{L}(G/K)$. Hence, there exists a compact, open subgroup $U \subseteq G/K$ such that p and the index $[U : V]$ are coprime, for every compact, open subgroup V of U. Let $\pi\colon G \to G/K$ be the quotient map. Since G/K is a $\mathbb{VSUB}_{\mathbb{P}}^{\vee}$-group, there exists a compact, normal subgroup N of G/K such that $H := (G/K)/N$ is a $\mathbb{SUB}_{\mathbb{P}}^{\vee}$-group (Remark 5.4). Let $q\colon G/K \to H$ be the quotient morphism. Then $U' := q(U)$ is a compact, open subgroup of H such that $[U' : V] = [q^{-1}(U') : q^{-1}(V)] = [U : q^{-1}(V)]$ is not divisible by p, for any compact, open subgroup V of H, and thus $p \notin \mathbb{L}(H)$. Since $G/M \cong H \in \mathbb{SUB}_{\mathbb{P}}^{\vee}$, where $M := \pi^{-1}(N)$ is a compact normal subgroup of G, we see that p is not contained in the right hand side of (17). Thus (17) is established.

It is obvious that $\mathbb{L}_r(G) \subseteq \bigcap_{K' \supseteq K \in J(G)} \mathbb{L}(G/K)$ for all $K' \in J(G)$, and thus $\mathbb{L}_r(G) \subseteq \mathbb{L}_i(G)$. To see that $\mathbb{L}_i(G) \subseteq \mathbb{L}(G)$, let $p \in \mathbb{L}_i(G)$. Then there exists $K' \in J(G)$ such that $p \in \bigcap_{K' \supseteq K \in J(G)} \mathbb{L}(G/K)$. For every compact, open subgroup $U \subseteq G$, there exists $K'' \in J(G)$ such that $K'' \subseteq U$ (see Remark 5.4). Hence, $J(G)$ being a filter basis, we find $K \in J(G)$ such that $K \subseteq K' \cap K'' \subseteq U$. Let $\rho : G \to G/K$ be the quotient map. Since $p \in \mathbb{L}(G/K)$, there exists a compact, open subgroup $V \subseteq \rho(U)$ such that $[\rho(U) : V] = [U : \rho^{-1}(V)]$ is divisible by p. The subgroup $\rho^{-1}(V)$ of U being compact and open, we deduce that $p \in \mathbb{L}(G)$. □

Theorem 6.7 *Let $\mathfrak{p}$ be a set of primes. Then the following holds:*

(a) *If $G \in \mathbb{SUB}_{\mathfrak{p}}^{\vee}$, then $\mathbb{L}(G) \subseteq \mathfrak{p}$.*

(b) *If $G \in \mathbb{VSUB}_{\mathbb{P}}^{\vee}$, then $G \in \mathbb{VSUB}_{\mathfrak{p}}^{\vee}$ if and only if $\mathbb{L}_i(G) \subseteq \mathfrak{p}$.*

(c) *If $G \in \mathbb{MIX}_{\mathbb{P}}^{\vee}$, then $G \in \mathbb{MIX}_{\mathfrak{p}}^{\vee}$ if and only if $\mathbb{L}_i(G) \subseteq \mathfrak{p}$.*

(d) *If $G \in \mathbb{VSUB}_{\mathbb{P}}$, then $G \in \mathbb{VSUB}_{\mathfrak{p}}$ if and only if $\mathbb{L}_i(G) \subseteq \mathfrak{p}$.*

(e) *If $G \in \mathbb{MIX}_{\mathbb{P}}$, then $G \in \mathbb{MIX}_{\mathfrak{p}}$ if and only if $\mathbb{L}_i(G) \subseteq \mathfrak{p}$.*

Proof. (a) There is a finite subset $F \subseteq \mathfrak{p}$ such that G has an open subgroup of the form $U = \prod_{p \in \mathfrak{p}} U_p$, where U_p is a non-discrete pro-p-group for $p \in F$. Hence $\mathbb{L}(G) = F \subseteq \mathfrak{p}$, by Lemma 6.3 (a), (b) and Proposition 6.4.

(b) If $G \in \mathbb{VSUB}_{\mathfrak{p}}^{\vee}$, then $\mathbb{L}(G/K) \subseteq \mathfrak{p}$ for each $K \in J(G)$, by (a). Hence $\mathbb{L}_i(G) \subseteq \mathfrak{p}$. Let us show now that $G \in \mathbb{VSUB}_{\mathbb{L}_i(G)}^{\vee}$, for each $G \in \mathbb{VSUB}_{\mathbb{P}}^{\vee}$. It suffices to show that G can be approximated by $\mathbb{SUB}_{\mathbb{L}_i(G)}^{\vee}$-groups. Thus, let $U \subseteq G$ be a compact, open subgroup of G. There exists a compact, normal subgroup $K' \subseteq U$ of G such that $G/K' \in \mathbb{SUB}_{\mathbb{P}}^{\vee}$. Hence, there exists a finite set of primes F such that G/K' has an open subgroup of the form $V = \prod_{p \in F} V_p$, where V_p is a non-discrete pro-p-group for each $p \in F$. We claim that $F \subseteq \mathbb{L}_i(G)$ (whence indeed G can be approximated by $\mathbb{SUB}_{\mathbb{L}_i(G)}^{\vee}$-groups). To see this, suppose to the contrary that there exists some $\bar{p} \in F \setminus \mathbb{L}_i(G)$. Then $\bar{p} \notin \bigcap_{K' \supseteq K \in J(G)} \mathbb{L}(G/K)$, whence there exists $K \in J(G)$ such that $K \subseteq K'$ and $\bar{p} \notin \mathbb{L}(G/K)$. Let $\rho\colon G/K \to G/K'$ be the natural map, $\rho(gK) := gK'$. Then G/K has an open subgroup $W \subseteq \rho^{-1}(V)$ of the form $W = \prod_{p \in E} W_p$ for some finite set of primes E, where W_p is a non-discrete pro-p-group for each $p \in E$. Since $\mathbb{L}(G/K) = E$ by the proof of (a), we see that $\bar{p} \notin E$, whence $\mathrm{pr}_{\bar{p}} \circ \rho|_{W_p}^{V} = 1$ for each $p \in E$ by Lemma 2.1, where $\mathrm{pr}_{\bar{p}}\colon V \to V_{\bar{p}}$ is the coordinate projection. Hence $\mathrm{pr}_{\overline{p}}(\rho(W)) = \{1\}$. The latter being an open subset of $V_{\bar{p}}$, we deduce that $V_{\bar{p}}$ is discrete. We have reached a contradiction.

(c)–(e): If $G \in \mathbb{MIX}_{\mathbb{P}}^{\vee}$, then $G \in \mathbb{MIX}_{\mathfrak{p}}^{\vee}$ if and only if $G \in \mathbb{VSUB}_{\mathfrak{p}}^{\vee}$ (by the analogue of Theorem 2.12 (b) subsumed by Theorem 5.3). Similarly, Proposition 5.8 entails that $G \in \mathbb{VSUB}_{\mathbb{P}}$ belongs to $\mathbb{VSUB}_{\mathfrak{p}}$ if and only if $G \in \mathbb{VSUB}_{\mathfrak{p}}^{\vee}$, and hence $G \in \mathbb{MIX}_{\mathbb{P}}$ belongs to $\mathbb{MIX}_{\mathfrak{p}}$ if and only if $G \in \mathbb{VSUB}_{\mathfrak{p}}^{\vee}$, by Theorem 2.12 (b). Thus (d)–(e) follow from (b). □

Remark 6.8 The argument used to prove Part (b) of the preceding theorem shows that $\mathbb{L}(G/K) \supseteq \mathbb{L}(G/K')$ for all $K, K' \in J(G)$ such that $K \subseteq K'$. Hence $\mathbb{L}_i(G)$ is in fact given by the simpler formula $\mathbb{L}_i(G) = \bigcup_{K\in J(G)} \mathbb{L}(G/K)$, for each $\mathbb{VSUB}_{\mathbb{P}}^{\vee}$-group G.

Remark 6.9 Given $G \in \mathbb{VSUB}_{\mathbb{P}}^{\vee}$, Theorem 6.7 (a) shows that $\mathfrak{p} := \mathbb{L}_i(G)$ is the smallest set of primes such that $G \in \mathbb{VSUB}_{\mathfrak{p}}^{\vee}$. Parts (b)–(e) have analogous interpretations. Strengthening Theorem 2.12 (a), (c) and its analogue in Theorem 5.3, we deduce that

$$\bigcap_{i\in I} \mathbb{VSUB}_{\mathfrak{p}_i}^{\vee} = \mathbb{VSUB}_{\mathfrak{p}}^{\vee} \qquad \text{with } \mathfrak{p} := \textstyle\bigcap_{i\in I} \mathfrak{p}_i \,,$$

for any family $(\mathfrak{p}_i)_{i\in I}$ of sets $\mathfrak{p}_i \subseteq \mathbb{P}$. Analogous formulas hold for $\mathbb{VSUB}_{\mathfrak{p}}$, $\mathbb{MIX}_{\mathfrak{p}}$ and $\mathbb{MIX}_{\mathfrak{p}}^{\vee}$.

Remark 6.10 If $G \in \mathbb{VSUB}_{\mathbb{P}}$, instead of $J(G)$ we can use the set $\overline{J}(G)$ of all compact, normal subgroups $K \subseteq G$ such that $G/K \in \mathbb{SUB}_{\mathbb{P}}$ to define a set $\overline{\mathbb{L}}_i(G)$ analogous to $\mathbb{L}_i(G)$. Repeating the preceding proofs with $\mathbb{VSUB}_{\mathbb{P}}$ instead of $\mathbb{VSUB}_{\mathbb{P}}^{\vee}$, we see that $\mathfrak{p} := \overline{\mathbb{L}}_i(G)$ is the smallest set of primes with $G \in \mathbb{VSUB}_{\mathfrak{p}}$ and thus $\overline{\mathbb{L}}_i(G) = \mathbb{L}_i(G)$.

$\mathbb{VSUB}_{\emptyset}^{\vee}$ being the class of locally compact, pro-discrete groups, Theorem 6.7 (b) implies:

Corollary 6.11 *A group $G \in \mathbb{VSUB}_{\mathbb{P}}^{\vee}$ is pro-discrete if and only if $\mathbb{L}_i(G) = \emptyset$.* □

The following corollary is immediate from Theorem 4.2 and Theorem 6.7 (d):

Corollary 6.12 *If G is a compactly generated $\mathbb{VSUB}_{\mathbb{P}}$-group, then $\mathbb{P}(G) = \mathbb{L}_i(G)$.* □

A The set of normal subgroups with Lie quotients need not be a filter basis

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}_p$ for some p. Given a topological group G, let $N_{\mathbb{K}}(G)$ be the set of all closed normal subgroups $N \subseteq G$ such that G/N is a $\mathbb{K}$-Lie group. We describe a complete abelian topological group G such that $N_{\mathbb{K}}(G)$ is not a filter basis. Examples for such behaviour had not been known before. For locally compact G, the pathology cannot occur, the class of $\mathbb{K}$-Lie groups being suitable for approximation (see [8, 1.7]; cf. [16]).

Construction of G. The topology induced by $\mathbb{K}$ on $\mathbb{Q}$ can be refined to a topology τ which makes $\mathbb{Q}$ a non-discrete, complete topological group [19]. We write $H := (\mathbb{Q}, \tau)$ and define $G := \mathbb{K}\times H$. Then H is not a $\mathbb{K}$-Lie group, as it is countable but non-discrete. Hence G is not a $\mathbb{K}$-Lie group either, since otherwise $H \cong G/(\mathbb{K} \times \{0\})$ would be a $\mathbb{K}$-Lie group. The first coordinate projection $G \to \mathbb{K}$, $(x, y) \mapsto x$ is a quotient homomorphism, with kernel $N := \{0\} \times H$. On the other hand, the inclusion map $\iota\colon H \to \mathbb{K}$ being continuous,

$$q\colon G \to \mathbb{K}\,, \quad (x, y) \mapsto x + \iota(y) = x + y$$

is a continuous homomorphism. Apparently q is surjective. Furthermore, q is open since, for any 0-neighbourhoods $U \subseteq \mathbb{K}$ and $V \subseteq H$, we have $U \subseteq q(U \times V)$. We set $M := \ker q = \{(x, -x) : x \in \mathbb{Q}\}$. Then $M \cap N = \{0\}$, G is not a $\mathbb{K}$-Lie group, and both G/M and G/N are topologically isomorphic to $\mathbb{K}$. □

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Helge Glöckner, TU Darmstadt, FB Mathematik AG 5, Schlossgartenstr. 7, 64289 Darmstadt, Germany.
E-Mail: gloeckner@mathematik.tu-darmstadt.de