On flows of an incompressible fluid with discontinuous power-law-like rheology

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Abstract

We consider the model for blood flow, which takes into account the platelets activation, cf. [2]. Platelets are very sensitive to chemical and mechanical inputs, thus the viscosity of a material may change very rapidly. This phenomenon can be described with help of discontinuos Cauchy stress tensor. We will formulate the problem also in terms of maximal monotone operators.

1 Introduction

The list of non-Newtonian[1] phenomena exhibited by incompressible liquids usually includes: (i) shear thinning/shear thickening and/or pressure thickening (these are responses when the generalized viscosity decreases/ increases with increasing shear rate and/or increases with increasing pressure); (ii) the presence of normal stress differences at a simple shear flow (the response closely connected with the effects as rod-climbing, die swell, etc.), (iii) viscoelastic responses as stress relaxation and non-linear creep, and (iv) the presence of yield stress. The last of these responses can be described as follows:

$$\begin{aligned} &\text{if} \quad |\mathsf{T}| \le \tau^* \quad \text{then} \quad \mathsf{D}(\boldsymbol{v}) = 0, \\ &\text{if} \quad |\mathsf{T}| > \tau^* \quad \text{then} \quad \mathsf{D}(\boldsymbol{v}) \neq 0 \quad \text{and} \ \mathsf{T} = \mathbf{f}(\mathsf{D}(\boldsymbol{v})). \end{aligned} \tag{1.1}$$

Here, $\boldsymbol{v}$ is the velocity, $\mathsf{D}(\boldsymbol{v})$ the symmetric part of the velocity gradient $\nabla \boldsymbol{v}$, T denotes the Cauchy stress, τ^* is the threshold value for the magnitude of T and $\mathbf{f}$ stands for any constitutive equation. Note that we can alternatively rewrite (1.1) as

$$\begin{aligned} &\text{if} \quad \mathsf{D}(\boldsymbol{v}) = 0 \quad \text{then} \quad |\mathsf{T}| \le \tau^* \\ &\text{if} \quad \mathsf{D}(\boldsymbol{v}) \neq 0 \quad \text{then} \quad |\mathsf{T}| > \tau^* \quad \text{and} \quad \mathsf{T} = \mathbf{f}(\mathsf{D}(\boldsymbol{v})). \end{aligned} \tag{1.2}$$

The presence of yield stress is a controversial phenomenon since it contradicts to standard understanding of what is meant by a fluid, which is a material that cannot

[1] A fluid is said to be Newtonian if its behaviour is described by the Navier-Stokes equations.

sustain the shear stress. Thus the fluid, by its definition, is such a material that starts to flow immediately after the shear stress is applied while (1.2) requires that Cauchy stress oversees the critical value before the flow starts. We can however argue that for small magnitude of the stress, no flow is visible within the time scale of normal observation[2], consequently, we can view the model with the yield stress, which is also an example of models with discontinuous Cauchy stress, as a possible and reasonable approximation of more realistic fluid response. We refer to Málek and Rajagopal [14] for a discussion of these issues. In this article, we deal with the following "generalization" of the constitutive equation (1.2). For a given $d^* > 0$, we have

$$\begin{aligned} &\text{if} \quad |\mathbf{D}(\boldsymbol{v})| < d^* \quad \text{then} \quad \mathbf{T} = \mathbf{T}_1(\mathbf{D}(\boldsymbol{v})) = \nu_1(|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \\ &\text{if} \quad |\mathbf{D}(\boldsymbol{v})| > d^* \quad \text{then} \quad \mathbf{T} = \mathbf{T}_2(\mathbf{D}(\boldsymbol{v})) = \nu_2(|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \\ &\text{if} \quad |\mathbf{D}(\boldsymbol{v})| = d^* \quad \text{then} \quad \mathbf{T} = \nu^*\mathbf{D}(\boldsymbol{v}) \end{aligned} \tag{1.3}$$

where $\nu^* \in [\min\{\nu_1^-, \nu_2^+\}, \max\{\nu_1^-, \nu_2^+\}]$ with $\nu_1^- := \lim_{|\boldsymbol{\xi}| \to d^*-} \nu_1(|\boldsymbol{\xi}|^2)|\boldsymbol{\xi}|$ and $\nu_2^+ := \lim_{|\boldsymbol{\xi}| \to d^*+} \nu_2(|\boldsymbol{\xi}|^2)|\boldsymbol{\xi}|$.

We justify the model (1.3) using arguments similar to the yield stress phenomenon. Once the shear rate reaches a certain critical value d^*, this critical shear rate initiates series of chemical reactions that, within a very short time interval change the viscosity of the material dramatically. Since this change is significant and also very quick, it seems acceptable to capture this feature by the constitutive equation of the form (1.3). Note that if ν_i in (1.3) is of the form

$$\nu_i(|\boldsymbol{\xi}|^2) = \nu_{oi}|\boldsymbol{\xi}|^{r_i - 2}, \quad (i = 1, 2)$$

where $\nu_{oi} > 0$ and $r_i \in (1, \infty)$ are model characteristics, we talk about power-law fluid response, and (1.3) then describes the change of one power-law response to another. In this paper, we consider $\mathbf{T}_1, \mathbf{T}_2$ from (1.3) so that they generalize the power-law constitutive equations in the following sense. We assume that there are fixed parameters $r, q \in (1, \infty)$, positive constants c_1, c_2, c_4, c_5 and arbitrary constants c_3, c_6 such that for all $\boldsymbol{\xi} \in \mathbb{R}^{d^2}$ we have

$$\begin{aligned} |\mathbf{T}_1(\boldsymbol{\xi})| &\leq c_1(1 + |\boldsymbol{\xi}|)^{r-1}, \\ |\mathbf{T}_2(\boldsymbol{\xi})| &\leq c_4(1 + |\boldsymbol{\xi}|)^{q-1}, \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbf{T}_1(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\geq c_2|\boldsymbol{\xi}|^r - c_3, \\ \mathbf{T}_2(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\geq c_5|\boldsymbol{\xi}|^q - c_6. \end{aligned} \tag{1.4}$$

In addition, we assume that $\mathbf{T}_1, \mathbf{T}_2$ are strictly monotone, i.e., for $i = 1, 2$ we have

$$(\mathbf{T}_i(\boldsymbol{\xi}) - \mathbf{T}_i(\boldsymbol{\zeta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\zeta}) > 0 \quad \forall\, \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{R}^{d^2}, \ \boldsymbol{\xi} \neq \boldsymbol{\zeta}. \tag{1.5}$$

The motivation for considering the simplified cartoon given in (1.3) comes from the recent article [2], where Anand and Rajagopal discuss and model the influence of

[2] The flow of glacier, sand, or any other densely packed granular material (modeled as a single continuum) can serve as a good example.

platelets activation on the blood rheology. Despite the fact that platelets form only small amount of blood, they are extremely sensitive to chemicals and mechanical changes as well. At high shear rate (or high shear stresses) platelets release carried chemical species and a set of chemical reactions is triggered. It results in formation of platelets aggregates that exhibit significantly different characteristics than blood before the platelet activation process started. In [2] Anand and Rajagopal propose a constitutive equation for blood, in the framework of rate-type (viscoelastic) incompressible fluid-like materials, that is capable of incorporating platelet activation resulting into distinctly different material moduli (i.e. the viscosity, relaxation times, etc.) before and after the activation.

The constitutive equation (1.3) simplifies the model proposed by Anand and Rajagopal in several aspects. First of all, we completely neglect the elastic response exhibited by blood due to the presence of red blood cells, white blood cells, platelets and proteins in plasma. On the other hand, the model (1.3) includes shear thinning phenomenon exhibited by blood in particular in zones with platelets aggregates. Second, we eliminated the possibility of damaging the platelets aggregates in a later time instant. Finally, no chemical reactions that would take place around critical time are included into the model.

Our goal is to establish the mathematical theory for the steady and unsteady flows of fluids with discontinuous constitutive equation for the Cauchy stress of the form (1.3). In this article we provide the first approach to study such problems and using the tools as Young measures, maximal monotone operators, compact embeddings and energy equality we prove the existence of solution to the problem in consideration.

The scheme of the article is as follows: In Section 2 we formulate the governing equations, boundary conditions and the precise assumptions on the structure (properties) of the constitutive functions $\mathbf{T}_1, \mathbf{T}_2$ appearing in (1.3). We also discuss the relation of the problem to the problem of non-standard growth and we survey known mathematical literature on the related problems. Section 3 recalls various theorems and auxiliary assertions that are important in the analysis of the model performed in subsequent sections. In Section 4, we prove the existence of weak solutions in the steady case. Section 5 is devoted to measure-valued solutions in unsteady case. We also observe the solutions satisfy some kind of energy inequality and equality. All these results lead to Section 6, which contains the main result - the existence of weak solutions in unsteady case. Finally Section 7 provides the uniform integrability of a sequence of approximate solutions.

2 Assumptions, problem formulation and main results

It is convenient to reformulate the problem using the language of maximal monotone operators. For this purpose, we first introduce several notation. Set

$$U_1 = \{\boldsymbol{\eta} \in R^{d^2} \,:\, |\boldsymbol{\eta}| < d^*\}, \quad U_2 = \{\boldsymbol{\eta} \in R^{d^2} \,:\, |\boldsymbol{\eta}| > d^*\},$$

where d^* is the point of discontinuity appearing in the formulation (1.3). Next, we introduce $\mathbf{T} : \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ (see Figure 1) setting

$$\mathbf{T}(\boldsymbol{\xi}) = \begin{cases} \mathbf{T}_1(\boldsymbol{\xi}) & \text{for} \quad \boldsymbol{\xi} \in \overline{U}_1, \\ \mathbf{T}_2(\boldsymbol{\xi}) & \text{for} \quad \boldsymbol{\xi} \in U_2. \end{cases} \tag{2.6}$$

Note, that the coercivness and growth properties (1.4) of $\mathbf{T}_1$ and $\mathbf{T}_2$ imply

$$|\mathbf{T}(\boldsymbol{\xi})| \le \widetilde{c_1}(1 + |\boldsymbol{\xi}|)^{q-1} \quad \text{and} \quad \mathbf{T}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \ge \widetilde{c_2}|\boldsymbol{\xi}|^q - \widetilde{c_3}. \tag{2.7}$$

Indeed, $|\mathbf{T}_1(\boldsymbol{\xi})| \le c_1(1 + |\boldsymbol{\xi}|)^{r-1} \le c_1(1 + d^*)^{r-1} \le c_1(1 + d^*)^{r-1}(1 + |\boldsymbol{\xi}|)^q$ and $\mathbf{T}_1(\boldsymbol{\xi}) \cdot (\boldsymbol{\xi}) \ge c_2|\boldsymbol{\xi}|^r - c_3 \ge -c_3 \ge |\boldsymbol{\xi}|^q - (c_3 + d^*)^q$.

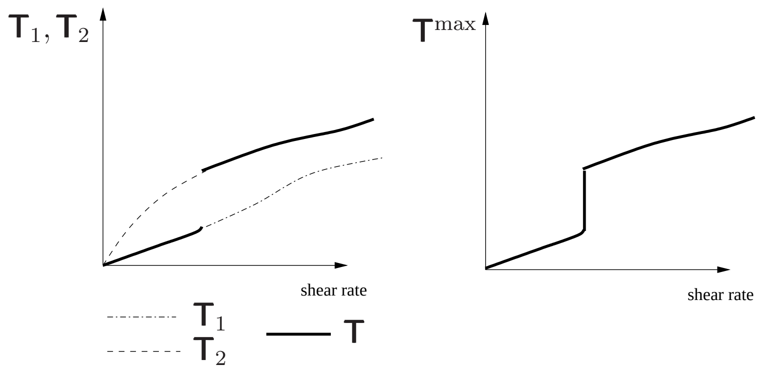

Figure 1: The graphs of $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}$ and $\mathbf{T}^{\max}$

The growth and coercivness properties (2.7) of $\mathbf{T}$ are sufficient to establish the existence of measure-valued solution for the considered problem. To prove existence of "weak" solution we require that $\mathbf{T}$ is in addition strictly monotone. This means we assume that

$$(\mathbf{T}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\zeta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\zeta}) > 0 \qquad \forall\, \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{R}^{d^2},\ \boldsymbol{\xi} \neq \boldsymbol{\zeta}. \tag{2.8}$$

Note, that if $\mathbf{T}_1 = (\varepsilon_1 + |\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}\mathbf{D}(\boldsymbol{v})$ and $\mathbf{T}_2 = (\varepsilon_2 + |\mathbf{D}(\boldsymbol{v})|^2)^{\frac{q-2}{2}}\mathbf{D}(\boldsymbol{v})$ then $\mathbf{T}$ constructed as in (2.6) is strictly monotone provided that $\mathbf{T}_1(\mathbf{D}) \le \mathbf{T}_2(\mathbf{D})$ for $\mathbf{D}$ satisfying $|\mathbf{D}| = d^*$.

It is useful to reformulate the problem expressed in terms of discontinuous functions as a set-valued problem. To do this, we assume that for $\mathbf{D}(\boldsymbol{v}) = d^*$ the value of the Cauchy stress tensor is not a single point but it takes values that can be parametrized by the interval $\left[\min\{\nu_1^-,\nu_2^+\},\max\{\nu_1^-,\nu_2^+\}\right]$. Thus, we introduce a set-valued operator defined as

$$\mathsf{T}^{\max}(\mathsf{D}(\boldsymbol{v})) = \begin{cases} \mathsf{T}(\mathsf{D}(\boldsymbol{v})) & \text{for} \quad |\mathsf{D}(\boldsymbol{v})| \neq d^*, \\ \left[\min\{\nu_1^-,\nu_2^+\},\max\{\nu_1^-,\nu_2^+\}\right] \mathsf{D}(\boldsymbol{v}) & \text{for} \quad |\mathsf{D}(\boldsymbol{v})| = d^*, \end{cases} \tag{2.9}$$

where T is defined above, cf. Figure 1.

Note that T is a selection of $\mathsf{T}^{\max}$ and thus if T is strictly monotone, then also $\mathsf{T}^{\max}$ is strictly monotone, i.e., then for all $\mathsf{D}_1, \mathsf{D}_2$ with $\mathsf{D}_1 \neq \mathsf{D}_2$ and all $\mathsf{S}_1 \in \mathsf{T}^{\max}(\mathsf{D}_1)$, $\mathsf{S}_2 \in \mathsf{T}^{\max}(\mathsf{D}_2)$,

$$(\mathsf{S}_2 - \mathsf{S}_1) \cdot (\mathsf{D}_2 - \mathsf{D}_1) > 0.$$

Let $\mathcal{T}$ denote the graph of $\mathsf{T}^{\max}$. We say that $\mathcal{T}$ is a graph of a maximal monotone operator if there is no other monotone operator, whose graph contains strictly $\mathcal{T}$. We say that $\mathcal{T}$ is strictly maximal monotone graph if for all $(\mathsf{D}_1, \mathsf{S}_1) \in \mathcal{T}$ and $(\mathsf{D}_2, \mathsf{S}_2) \in \mathcal{T}$ with $\mathsf{D}_1 \neq \mathsf{D}_2$ it holds

$$(\mathsf{S}_2 - \mathsf{S}_1) \cdot (\mathsf{D}_2 - \mathsf{D}_1) > 0.$$

The examples illustrating the difference between monotone and maximal monotone mappings are provided on Figure 2.

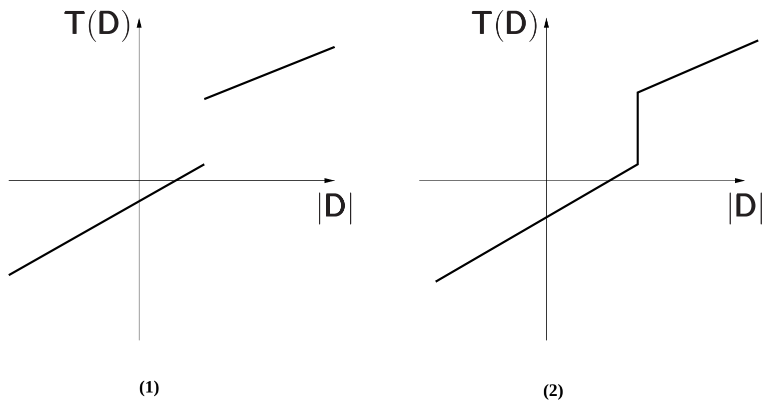

Figure 2: (1) - monotone map, (2) - maximal monotone map

Let us now consider what kind of behaviour of the viscosity leads to maximal strictly monotone graphs. Of course when the viscosity profile is strictly monotone as (1) in Figure 3, then surely T is strictly monotone. But we can admit also a decreasing viscosity satisfying the condition $\nu'(\xi) \cdot \xi + \nu(\xi) \geq 0$ for all $\xi \in \mathbb{R}_+$ and with a positive jump, i.e., $\nu_2^+ \geq \nu_1^+$.

We are in the position to give the precise formulation of the considered problems. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, open set. We say that the velocity field $\boldsymbol{v} = (v_1, ..., v_d)$

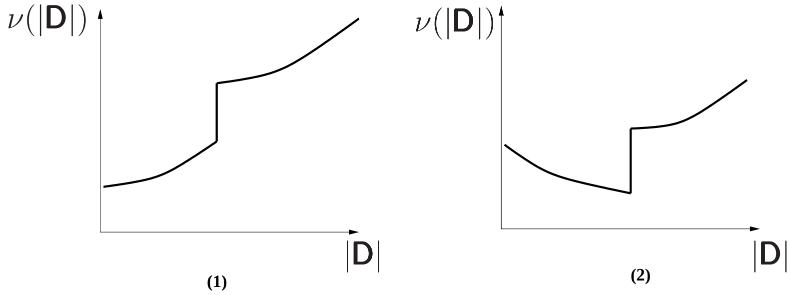

Figure 3: Possible viscosity profiles

and the pressure p describe steady flows of the incompressible fluid[3] obeying the constitutive equation (1.3) if

$$\begin{aligned} \operatorname{div} \mathsf{S} = \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \boldsymbol{b}, \quad \operatorname{div} \boldsymbol{v} = 0, \\ (\mathsf{D}(\boldsymbol{v}(x)), \mathsf{S}(x)) \in \mathcal{T} \quad \text{a.e. in } \Omega, \quad \boldsymbol{v}_{|\partial\Omega} = \mathbf{0}. \end{aligned} \tag{2.10}$$

Here, $\boldsymbol{b} = (b_1, ..., b_d)$ are given body forces, and $\boldsymbol{v} \otimes \boldsymbol{v}$ is the second order tensor (dyadic product) with the components $(\boldsymbol{v} \otimes \boldsymbol{v})_{ij} = v_i v_j$.

Similarly, $\boldsymbol{v}$ and p capture unsteady flows if

$$\begin{aligned} \operatorname{div} \mathsf{S} = \boldsymbol{v}_t + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \boldsymbol{b}, \ \operatorname{div} \boldsymbol{v} = 0, \\ (\mathsf{D}(\boldsymbol{v}(t,x)), \mathsf{S}(t,x)) \in \mathcal{T} \quad \text{a.e. in } Q_T, \\ \boldsymbol{v}(0,x) = \boldsymbol{v}_0, \ \boldsymbol{v}_{|\partial\Omega} = \mathbf{0}, \end{aligned} \tag{2.11}$$

where $\boldsymbol{v}_0 = (v_{01}, ..., v_{0d})$ is a given initial velocity; Q_T denotes $I \times \Omega$ with $I = (0, T)$, $T > 0$. Our goal is to establish the existence results for both problems. We will prove that if T is $q-$coercive and of $(q-1)-$growth, then there is a weak solution to both problems if q satisfies the following conditions

- $q > \frac{3d}{d+2}$ for time-independent problem,
- $q > \frac{3d+2}{d+2}$ for evolutionary problem.

We use Young measures as a convenient tool to show such results. As a by-product we obtain the existence of measure-valued solution; this step of the proof does not require to assume that T is monotone. For time-dependent problem we formulate this result separately (see Theoerem 5.1). One of the adavantages to use Young measures here consists in the fact that it allows to construct the solution directly for Galerkin approximations.

[3]We consider a homogeneous fluid with the uniform (constant) density $\rho^* > 0$. Eq. (2.10) represents the balance of linear momentum divided by ρ^*; S and p thus denotes the viscous part of the Cauchy stress and the pressure after this rescaling.

By $\mathcal{D}(\Omega)$ we will understand the space of all C^∞-functions with compact support in Ω and $\mathcal{V} = \{u : u \in \mathcal{D}(\Omega), \operatorname{div} u = 0\}$. By $W^{1,q}_{0,div}(\Omega)$ we mean the closure of $\mathcal{V}$ with respect to the norm $\|u\|_{1,q} = \left(\int_\Omega |\nabla u|^q dx\right)^{\frac{1}{q}}$, $L^2_{div}(\Omega)$ means the closure of $\mathcal{V}$ w.r.t. the standard L^2-norm and $W^{s,2}_{0,div}(\Omega)-$ the closure of $\mathcal{V}$ w.r.t. the $W^{s,2}-$ norm. Moroever, $\mathcal{D}(-\infty, T; \mathcal{V})$ is the space of all C^∞-functions with compact support from $(-\infty, T)$ to $\mathcal{V}$.

3 A generalization of Theorem on Young Measures

Consider T defined in (2.6) for explicit consideration in this section.[4] We use $B(x_0, r)$ to denote the ball of $\mathbb{R}^d$ with a centre in x_0 and radius r. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a radially symmetric function with support in $B(0,1)$ and $\int_{\mathbb{R}^d} \eta\, dx = 1$. We put $\eta^\varepsilon(x) = \frac{1}{\varepsilon^{d^2}} \eta\left(\frac{x}{\varepsilon}\right)$.

Next, we set

$$\mathsf{T}^\varepsilon = \mathsf{T} * \eta^\varepsilon. \tag{3.12}$$

We show below that if T is strictly monotone, q-coercive and of $(q-1)$-growth then T^ε preserves these properties. More precisely, we have the following assertion

Lemma 3.1 *Let T^ε be from (3.12) and T fulfills (2.7), (2.8). Then*

(i) for every $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^{d^2}$, $\boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$ it holds

$$[\mathsf{T}^\varepsilon(\boldsymbol{\xi}_1) - \mathsf{T}^\varepsilon(\boldsymbol{\xi}_2)] \cdot [\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2] > 0,$$

and for all $\boldsymbol{\xi} \in \mathbb{R}^{d^2}$ there are positive constants c_1', c_2' and an arbitrary constant c_3' (all of them independent of $\varepsilon < 1$) such that

(ii) $$|\mathsf{T}^\varepsilon(\boldsymbol{\xi})| \le c_1'(1 + |\boldsymbol{\xi}|)^{q-1},$$

(iii) $$\mathsf{T}^\varepsilon(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \ge c_2'|\boldsymbol{\xi}|^q - c_3'.$$

Proof: (i)

$$[\mathsf{T}^\varepsilon(\boldsymbol{\xi}_1) - \mathsf{T}^\varepsilon(\boldsymbol{\xi}_2)] \cdot [\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2] = \int_{\mathbb{R}^{d^2}} [\mathsf{T}(\boldsymbol{\xi}_1 - \boldsymbol{\zeta}) - \mathsf{T}(\boldsymbol{\xi}_2 - \boldsymbol{\zeta})]\eta^\varepsilon(\boldsymbol{\zeta}) d\boldsymbol{\zeta} \cdot [\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2]$$
$$= \int_{\mathbb{R}^{d^2}} [\mathsf{T}(\boldsymbol{\xi}_1 - \boldsymbol{\zeta}) - \mathsf{T}(\boldsymbol{\xi}_2 - \boldsymbol{\zeta})] \cdot [(\boldsymbol{\xi}_1 - \boldsymbol{\zeta}) - (\boldsymbol{\xi}_2 - \boldsymbol{\zeta})]\eta^\varepsilon(\boldsymbol{\zeta}) d\boldsymbol{\zeta}.$$

[4] We could however take any selection, i.e., a single-valued function, $\mathbf{S} = \mathbf{S}(\mathsf{D}(\boldsymbol{v}))$ satisfying $(\mathsf{D}, \mathbf{S}(\mathsf{D})) \in \mathcal{T}$ for all $\mathsf{D} \in \mathbb{R}^{d^2}$ where $\mathcal{T}$ is a maximal monotone graph defined in Section 2.

Since T is strictly monotone, then the integral with respect to nonnegative probability measure is positive.
(*ii*)

$$
\begin{aligned}
|\mathsf{T}^\varepsilon(\boldsymbol{\xi})| = & |\int_{\mathbb{R}^{d^2}} \mathsf{T}(\boldsymbol{\zeta})\eta^\varepsilon(\boldsymbol{\xi}-\boldsymbol{\zeta})d\boldsymbol{\zeta}| \le \sup_{\boldsymbol{\zeta}\in B(\boldsymbol{\xi},\varepsilon)} |\mathsf{T}(\boldsymbol{\zeta})| \le c \sup_{\boldsymbol{\zeta}\in B(\boldsymbol{\xi},\varepsilon)} (1+|\boldsymbol{\zeta}|)^q \\
& \le c(1+\varepsilon+|\boldsymbol{\xi}|)^q \le \tilde{c}(1+|\boldsymbol{\xi}|)^q.
\end{aligned}
$$

(*iii*) First estimate

$$
\begin{aligned}
\mathsf{T}^\varepsilon(\boldsymbol{\xi})\cdot\boldsymbol{\xi} = & \int_{\mathbb{R}^{d^2}} \mathsf{T}(\boldsymbol{\xi}-\boldsymbol{\zeta})\eta^\varepsilon(\boldsymbol{\zeta})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta})d\boldsymbol{\zeta} + \int_{\mathbb{R}^{d^2}} \mathsf{T}(\boldsymbol{\xi}-\boldsymbol{\zeta})\eta^\varepsilon(\boldsymbol{\zeta})\cdot\boldsymbol{\zeta}d\boldsymbol{\zeta} \\
\ge & \int_{\mathbb{R}^{d^2}} (c_2|\boldsymbol{\xi}-\boldsymbol{\zeta}|^q - c_3)\eta^\varepsilon(\boldsymbol{\zeta})d\boldsymbol{\zeta} + I^\varepsilon \ge (c_2||\boldsymbol{\xi}|-|\varepsilon||^q - c_3) + I^\varepsilon.
\end{aligned} \tag{3.13}
$$

Also

$$
\begin{aligned}
|I^\varepsilon| \le & \int_{\mathbb{R}^{d^2}} c_1(1+|\boldsymbol{\xi}-\boldsymbol{\zeta}|)^{q-1}\eta^\varepsilon(\boldsymbol{\zeta})\cdot\boldsymbol{\zeta}d\boldsymbol{\zeta} \le \varepsilon c_1(1+|\boldsymbol{\xi}|)^{q-1} \le \varepsilon c_1(1+\frac{q}{q-1}|\boldsymbol{\xi}|)^{q-1} \\
\le & \varepsilon c_1(1+|\boldsymbol{\xi}|)^{\frac{q}{q-1}\cdot(q-1)} \le \varepsilon c_1 + \varepsilon c_1|\boldsymbol{\xi}|^q
\end{aligned} \tag{3.14}
$$

To continue estimate (3.13) we consider two cases. Let first $|\boldsymbol{\xi}| \ge 2$, then since $\varepsilon \le 1$

$$
(c_2(|\boldsymbol{\xi}|-|\varepsilon|)^q - c_3) \ge (c_2(|\boldsymbol{\xi}|-1)^q - c_3) \ge \frac{c_2}{2^q}|\boldsymbol{\xi}|^q - c_3. \tag{3.15}
$$

In the case $|\boldsymbol{\xi}| < 2$ we have to notice that $(|\boldsymbol{\xi}|-1)^q$ is bounded from below thus we can adjust the constant c_3 such that

$$
(c_2(|\boldsymbol{\xi}|-1)^q - c_3) \ge c\cdot 2^q - \tilde{c}_3 \ge c|\boldsymbol{\xi}|^q - \tilde{c}_3. \tag{3.16}
$$

Combining (3.14)-(3.16) yields the assertion. ■

We recall without proofs the following fact, which is the special case of Theorem 2.1 from [12], that can be considered as a generalization of the so-called Fundamental Theorem on Young Measures (see [3, 13]) to discontinuous nonlinearities.

Theorem 3.2 *Assume $\Omega \subset \mathbb{R}^m$ to be an open set of a finite measure. Let $U_l \subset \mathbb{R}^k$, where $l \in J-$ the finite set of indeces, be a family of open sets such that*

$$
\mathbb{R}^k = \bigcup_{l\in J} \overline{U_l}, \quad U_n \cap U_l = \emptyset \quad \text{for } n \ne l.
$$

Let $\boldsymbol{z}^\varepsilon : \Omega \to \mathbb{R}^k$ be a sequence of measurable functions, and

$$
\nu_x^{\varepsilon,l} = (\delta_{\boldsymbol{z}^\varepsilon(x)} * \eta^\varepsilon)_{|\overline{U}_l}.
$$

Then there exists a subsequence (still denoted) by $\boldsymbol{z}^\varepsilon$ and a family of weak -∗ measurable maps $\nu^l : \Omega \to \mathcal{M}(\mathbb{R}^k)$, such that the measure $\nu_x = \sum_l \nu_x^l$ is nonnegative, $\mathrm{supp}(\nu_x^l) \subset \overline{U}_l$, *and* *i)* $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^k)} \leq 1$ *for almost all* $x \in \Omega$*;*

ii) $\nu^{\varepsilon,l} \overset{*}{\rightharpoonup} \nu^l$ *in* $L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^k))$*.*

iii) If for some measurable $E \subset \Omega$ *and some* $1 < p < \infty$ *the sequence* $(|\boldsymbol{z}^\varepsilon|^p)$ *is relatively weakly compact in* $L^1(E)$*, then*

$$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1 \quad a.e.\ in\ E;$$

iv) if (iii) holds, then for every function $\boldsymbol{F} : \mathbb{R}^k \to \mathbb{R}^n$ *satisfying growth condition*

$$|\boldsymbol{F}(\boldsymbol{\xi})| \leq C(1 + |\boldsymbol{\xi}|^p) \ \ for\ \ all\ \ \boldsymbol{\xi} \in \mathbb{R}^k$$

and such that $\boldsymbol{F}_{|U_l}$ *has for every* l *a Carathéodory extension on the set* $\overline{U_l}$ *denoted by* $\boldsymbol{F}_l$*, we have:*

$$(\eta^\varepsilon * \boldsymbol{F})(\boldsymbol{z}^\varepsilon(\cdot)) \rightharpoonup \overline{\boldsymbol{F}}, \quad \overline{\boldsymbol{F}}(x) = \sum_{l\in J} \int_{\overline{U}_l} \boldsymbol{F}_l(\boldsymbol{\xi})\, \mathrm{d}\nu^l_x(\boldsymbol{\xi})$$

in $L^1(E)$*.*

v) if $\nu_x = \delta_{\overline{z}(x)}$ *a.e. in* E*, then* $\boldsymbol{z}^\varepsilon(x) \to \overline{\boldsymbol{z}}(x)$ *in measure on* E*.*

We will also need the lower semicontinuity condition.

Lemma 3.3 *Let the assumptions of Theorem 3.2 be satisfied. Then for every function* $h : \mathbb{R}^k \to \mathbb{R}_+$ *such that* $h_{|U_l}$ *has for every* l *a continuous extension on the set* $\overline{U}_l$ *denoted by* h_l*, it holds*

$$\liminf_{\varepsilon\to 0} \int_\Omega (\eta^\varepsilon * h)(\boldsymbol{z}^\varepsilon(x))\, dx \geq \int_\Omega \sum_{l\in J} \int_{\overline{U}_l} h_l(\boldsymbol{\xi}) d\nu^l_x(\boldsymbol{\xi}) dx.$$

Proof:
Notice first that according to Theorem 3.2 (ii) $\nu^{\varepsilon,l} \overset{*}{\rightharpoonup} \nu^l$ in $L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^k))$. Furthermore, if (g^M) is an increasing sequence of cut-off functions from $C^\infty_0(\mathbb{R}^{d^2})$ such that $g^M \to 1$ as $|M| \to \infty$, Theorem 3.2 and monotone convergence theorem imply

$$\begin{aligned}
\int_\Omega (\eta^\varepsilon * h)(\boldsymbol{z}^\varepsilon(x))\, dx &= \int_\Omega \sum_{l\in J} \int_{\overline{U}_l} h_l(\boldsymbol{\xi}) d\nu^{\varepsilon,l}_x(\boldsymbol{\xi})\, dx \\
&\geq \int_\Omega \sum_{l\in J} \int_{\overline{U}_l} h_l(\boldsymbol{\xi}) g^M(\boldsymbol{\xi}) d\nu^{\varepsilon,l}_x(\boldsymbol{\xi})\, dx \xrightarrow{\varepsilon\to 0} \int_\Omega \sum_{l\in J} \int_{\overline{U}_l} h_l(\boldsymbol{\xi}) g^M(\boldsymbol{\xi}) d\nu^l_x(\boldsymbol{\xi})\, dx \\
&\xrightarrow{|M|\to\infty} \int_\Omega \sum_{l\in J} \int_{\overline{U}_l} h_l(\boldsymbol{\xi}) d\nu^l_x(\boldsymbol{\xi})\, dx.
\end{aligned}$$

4 Steady solutions

Theorem 4.1 *Let $q \geq \frac{3d}{d+2}$. Given $\boldsymbol{b} \in (W^{1,q}_{0,div}(\Omega))^*$ there exists a function $\boldsymbol{v} \in W^{1,q}_{0,div}(\Omega)$ and a measurable selection $\mathsf{S} \in L^{q'}(\Omega)$ such that*

1. $(\mathsf{D}(\boldsymbol{v}(x)), \mathsf{S}(x)) \in \mathcal{T}$ *a.e. in* Ω.
2. *For all* $\boldsymbol{\varphi} \in \mathcal{V}$:

$$\int_\Omega \mathsf{S} \cdot \mathsf{D}(\boldsymbol{\varphi})\, dx = \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} + \boldsymbol{b} \cdot \boldsymbol{\varphi})\, dx. \tag{4.17}$$

4.1 Proof of Theorem 4.1

The approximation follows analogously

$$\begin{aligned} &\operatorname{div} \mathsf{T}^\varepsilon = \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) + \nabla p^\varepsilon - \boldsymbol{b}, \\ &\operatorname{div} \boldsymbol{v}^\varepsilon = 0, \quad \boldsymbol{v}^\varepsilon_{|\partial\Omega} = \boldsymbol{0}. \end{aligned} \tag{4.18}$$

In the next step we do Galerkin approximation. Let $\{\boldsymbol{\omega}_r\}_{r=1}^\infty$ be an orthonormal basis of $L^2_{div}(\Omega)$. We define $\boldsymbol{v}^n = \sum_{r=1}^n c_r^n \boldsymbol{\omega}_r$, $\boldsymbol{v}^n \in V^n = \operatorname{span}\{\boldsymbol{\omega}_1, ..., \boldsymbol{\omega}_n\}$, $c_r^n \in \mathbb{R}$, as a solution to

$$\int_\Omega (\mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n)) \cdot \mathsf{D}(\boldsymbol{\omega}_r) + \boldsymbol{v}^n \otimes \boldsymbol{v}^n \cdot \nabla \boldsymbol{\omega}_r) dx = \langle \boldsymbol{b}, \boldsymbol{\omega}_r \rangle \tag{4.19}$$

for all $1 \leq r \leq n$, where we have chosen $\varepsilon(n) = \frac{1}{n}$. Existence of approximated solutions follows from the corollary of Brouwer's Fixed Point Theorem, cf. [9, p. 493]. Multiplying equations (4.19) by c_r^n and summing over r we obtain

$$\int_\Omega \mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n)) \cdot \mathsf{D}(\boldsymbol{v}^n)\, dx = \langle \boldsymbol{b}, \boldsymbol{v}^n \rangle. \tag{4.20}$$

The growth conditions and Korn's inequality (cf. [13, p. 196]) imply

$$\|\boldsymbol{v}^n\|_{1,q}^q \leq c(\|\boldsymbol{b}\|_{-1,q'}^{q'} + |\Omega|). \tag{4.21}$$

Letting $n \to \infty$ at least for a subsequence it holds

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \quad \text{in} \quad W^{1,q}_{0,div}(\Omega). \tag{4.22}$$

Moreover, notice that $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ if $q > \frac{2d}{d+2}$. This provides that

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{in} \quad L^2(\Omega). \tag{4.23}$$

The strong convergence yields

$$\int_\Omega \boldsymbol{v}^n \otimes \boldsymbol{v}^n \cdot \nabla \boldsymbol{\varphi} \to \int_\Omega \boldsymbol{v} \otimes \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} \quad \text{for all} \quad \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{4.24}$$

Moreover,

$$\int_\Omega |\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n))|^{q'} dx \le c_1 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v}^n)|)^{(q-1)q'} dx = c_1 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v}^n)|)^q.$$

Hence we conclude existence of a subsequence and some $\mathbf{S} \in L^{q'}(\Omega)$ such that

$$\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)) \rightharpoonup \mathbf{S} \quad \text{in} \quad L^{q'}(\Omega). \tag{4.25}$$

For the moment we can state the limit identity for all $\boldsymbol{\varphi} \in \mathcal{V}$

$$\int_\Omega \mathbf{S} \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx = \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} - \boldsymbol{b} \cdot \boldsymbol{\varphi})\ dx. \tag{4.26}$$

Notice that, since $\mathcal{V}$ is dense in $W^{1,q}_{0,div}(\Omega)$ and for $q \ge \frac{3d}{d+2}$, all the integrals are well defined with $\boldsymbol{\varphi} \in W^{1,q}_{0,div}(\Omega)$, thus for these q the limit identity (4.26) also holds for all $\boldsymbol{\varphi} \in W^{1,q}_{0,div}(\Omega)$.

For later use, let $n \to \infty$ in (4.20). Then

$$\lim_{n\to\infty} \int_\Omega \mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}^n) dx = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{v}\, dx.$$

Using now (4.26) tested with $\boldsymbol{\varphi} = \boldsymbol{v}$ we claim

$$\lim_{n\to\infty} \int_\Omega \mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}^n) dx = \int_\Omega \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v}) dx. \tag{4.27}$$

For the limit passage in the Cauchy stress tensor, consider the Young measure μ_x associated with the sequence $(\nabla \boldsymbol{v}^n)$. By $\mu_x^i : \Omega \to \mathcal{M}(\mathbb{R}^{d^2})$ we will understand the measures generated by $(\nabla \boldsymbol{v}^n)$ for the sets U_i, $i = 1, 2$, namely the weak* limits of the sequences $(\delta_{\{\nabla \boldsymbol{v}^n(x)\}} * \eta^{\frac{1}{n}})_{|\overline{U}_i}$.

Set for $i = 1, 2$ and for $\boldsymbol{\xi}, \mathbf{D}(\boldsymbol{v}) \in \mathbb{R}^{d^2}$

$$h^i(\boldsymbol{\xi}) = \left[\mathbf{T}_i\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right) - \mathbf{T}(\mathbf{D}(\boldsymbol{v}))\right] \cdot \left[\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2} - \mathbf{D}(\boldsymbol{v})\right].$$

Since μ_x^i are nonnegative, the monotonicity of $\mathbf{T}$ provides also that

$$\int_\Omega \left[\int_{\overline{U}_1} h^1(x,\boldsymbol{\xi}) d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} h^2(x,\boldsymbol{\xi}) d\mu_x^2(\boldsymbol{\xi})\right] dx \ge 0. \tag{4.28}$$

By Theorem 3.2 $\mu_x = \mu_x^1 + \mu_x^2$. The sequence $(\nabla \boldsymbol{v}^n)$ is bounded in $L^q(\Omega)$, thus it is weakly relatively compact in $L^1(\Omega)$, which provides that μ_x is a probability measure, compare Theorem 3.2 (iv). This allows to conclude

$$\mathbf{D}(\boldsymbol{v}(x)) \stackrel{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2} d\mu_x(\boldsymbol{\xi}) \tag{4.29}$$

and to compute the above integral

$$\begin{aligned}
&\int_\Omega \left[\int_{\overline{U}_1} h^1(x,\boldsymbol{\xi})d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} h^2(x,\boldsymbol{\xi})d\mu_x^2(\boldsymbol{\xi})\right] dx \\
&= \int_\Omega \left[\int_{\overline{U}_1} \mathsf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} \mathsf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^2(\boldsymbol{\xi})\right] dx \\
&- \int_\Omega \left[\int_{\overline{U}_1} \mathsf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} \mathsf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_x^2(\boldsymbol{\xi})\right]\cdot\mathsf{D}(\boldsymbol{v})dx \\
&- \int_\Omega \mathsf{T}(\mathsf{D}(\boldsymbol{v}))\cdot\left(\int_{\mathbb{R}^{d^2}} \frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x(\boldsymbol{\xi}) - \mathsf{D}(\boldsymbol{v})\right)dx.
\end{aligned} \tag{4.30}$$

The latter term vanishes thanks to (4.29). Theorem 3.2 (*iv*) and (4.25) provide that

$$\mathsf{S} = \int_{\overline{U}_1} \mathsf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} \mathsf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_x^2(\boldsymbol{\xi}). \tag{4.31}$$

Combining (4.28), (4.30) and (4.31) yields

$$\begin{aligned}
&\int_\Omega \left[\int_{\overline{U}_1} \mathsf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} \mathsf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^2(\boldsymbol{\xi})\right] dx \\
&\qquad \geq \int_\Omega \mathsf{S}\cdot\mathsf{D}(\boldsymbol{v})\,dx.
\end{aligned} \tag{4.32}$$

In the following we will show the opposite inequality, namely

$$\begin{aligned}
&\int_\Omega \left[\int_{\overline{U}_1} \mathsf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} \mathsf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^2(\boldsymbol{\xi})\right] dx \\
&\qquad \leq \int_\Omega \mathsf{S}\cdot\mathsf{D}(\boldsymbol{v})\,dx.
\end{aligned} \tag{4.33}$$

The monotonicity of T implies $\mathsf{T}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n)$ is nonnegative, thus Lemma 3.3 can be applied to conclude

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega (\eta^{\frac{1}{n}} * (\mathsf{T}\cdot\mathrm{Id}))(\mathsf{D}(\boldsymbol{v}^n))dx \\
&\geq \int_\Omega \left[\int_{\overline{U}_1} \mathsf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^1(\boldsymbol{\xi}) + \int_{\overline{U}_2} \mathsf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^2(\boldsymbol{\xi})\right] dx.
\end{aligned}$$

To use this information first we need to show that

$$\liminf_{n\to\infty} \int_\Omega (\eta^{\frac{1}{n}} * (\mathsf{T}\cdot\mathrm{Id}))(\mathsf{D}(\boldsymbol{v}^n))dx = \liminf_{n\to\infty} \int_\Omega \mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n)dx. \tag{4.34}$$

To see this we first observe (compare with commutator estimate introduced by Lions and DiPerna [7, Lemma II.1].

$$\begin{aligned}
&\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n))\cdot\mathbf{D}(\boldsymbol{v}^n)-(\eta^{\frac{1}{n}}*(\mathbf{T}\cdot\mathrm{Id}))(\mathbf{D}(\boldsymbol{v}^n))\\
&=\int_{\mathbb{R}^{d^2}}\Big[T(\mathbf{D}(\boldsymbol{v}^n)-\boldsymbol{\zeta})\eta^{\frac{1}{n}}(\boldsymbol{\zeta})\cdot\mathbf{D}(\boldsymbol{v}^n)-\eta^{\frac{1}{n}}(\boldsymbol{\zeta})\mathbf{T}(\mathbf{D}(\boldsymbol{v}^n)-\boldsymbol{\zeta})\cdot(\mathbf{D}(\boldsymbol{v}^n)-\boldsymbol{\zeta})\Big]\,d\boldsymbol{\zeta}\\
&=\int_{\mathbb{R}^{d^2}}\eta^{\frac{1}{n}}(\boldsymbol{\zeta})\mathbf{T}(\mathbf{D}(\boldsymbol{v}^n)-\boldsymbol{\zeta})\cdot\boldsymbol{\zeta}\,d\boldsymbol{\zeta}=g(\mathbf{D}(\boldsymbol{v}^n)).
\end{aligned}$$

Recall that $\operatorname{supp}\eta^{\frac{1}{n}}\subset B(0,\frac{1}{n})$, which together with growth conditions enables to estimate the last term as follows

$$\begin{aligned}
\Big|\int_{\mathbb{R}^{d^2}}\eta^{\frac{1}{n}}(\boldsymbol{\zeta})\mathbf{T}(\mathbf{D}(\boldsymbol{v}^n)-\boldsymbol{\zeta})\cdot\boldsymbol{\zeta}\,d\boldsymbol{\zeta}\Big| &\leq \varepsilon\widetilde{c}_3\int_{\mathbb{R}^{d^2}}\eta^{\frac{1}{n}}(\boldsymbol{\zeta})(1+|\mathbf{D}(\boldsymbol{v}^n)-\boldsymbol{\zeta}|)^{q-1}d\boldsymbol{\zeta}\\
&\leq \varepsilon\widetilde{c}_3(1+|\mathbf{D}(\boldsymbol{v}^n)|+\tfrac{1}{n})^{q-1}.
\end{aligned}$$

Since $(1+|\mathbf{D}(\boldsymbol{v}^n)|+\varepsilon)^{q-1}$ is bounded in $L^{q'}(\Omega)$, thus $\int_\Omega|g(\mathbf{D}(\boldsymbol{v}^n)|dx\to 0$ as $n\to\infty$ and (4.34) holds. Then

$$\begin{aligned}
&\liminf_{n\to\infty}\int_\Omega\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n))\cdot\mathbf{D}(\boldsymbol{v}^n)dx\\
&\geq\int_\Omega\left[\int_{\overline{U}_1}\mathbf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^1(\boldsymbol{\xi})+\int_{\overline{U}_2}\mathbf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu_x^2(\boldsymbol{\xi})\right]dx.
\end{aligned}\tag{4.35}$$

Recall (4.27), which together with (4.35) leads to (4.33). Thus (4.32) and (4.33) imply

$$\int_\Omega\left[\int_{\overline{U}_1}h^1(x,\boldsymbol{\xi})d\mu_x^1(\boldsymbol{\xi})+\int_{\overline{U}_2}h^2(x,\boldsymbol{\xi})d\mu_x^2(\boldsymbol{\xi})\right]dx=0.$$

Since μ_x^i are nonnegative measures and μ_x is a probability measure, then at least one of μ^i has to be non-zero measure. Moreover monotonicity of $\mathcal{T}$ implies that $h^i(\boldsymbol{\xi})$ are strictly positive for all $\boldsymbol{\xi}\neq\nabla\boldsymbol{v}$, which follows that $\operatorname{supp}\mu_x=\{\nabla\boldsymbol{v}(x)\}$. Hence $\mu_x=\delta_{\{\nabla\boldsymbol{v}(x)\}}$ a.e. in Ω and for almost all $x\in\Omega$ holds

$$\begin{aligned}
\mathbf{S}=&\lambda\int_{\overline{U}_1}\mathbf{T}_1\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\delta_{\{\nabla\boldsymbol{v}(x)\}}(\boldsymbol{\xi})+(1-\lambda)\int_{\overline{U}_2}\mathbf{T}_2\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\delta_{\{\nabla\boldsymbol{v}(x)\}}(\boldsymbol{\xi})\\
=&\lambda\mathbf{T}_1(\mathbf{D}(\boldsymbol{v}))+(1-\lambda)\mathbf{T}_2(\mathbf{D}(\boldsymbol{v}))
\end{aligned}$$

where $\lambda\in[0,1]$. For all those $\mathbf{D}(v)$, for which $\mathbf{T}$ is single-valued $\lambda\in\{0,1\}$ and since $(\mathbf{D}(\boldsymbol{v}),\mathbf{T}_i(\mathbf{D}(\boldsymbol{v})))\in\mathcal{T}$ for $i=1,2$, obviously also $(\mathbf{D}(\boldsymbol{v}(x)),\mathbf{S}(x))$ belongs to $\mathcal{T}$. Whereas in other case since both the points $(\mathbf{D}(\boldsymbol{v}),\mathbf{T}_1(\mathbf{D}(\boldsymbol{v})))$ and $(\mathbf{D}(\boldsymbol{v}),\mathbf{T}_2(\mathbf{D}(\boldsymbol{v})))$ belong to the vertical part of the graph and that any interval is a convex set, then also $(\mathbf{D}(\boldsymbol{v}(x)),\mathbf{S}(x))\in\mathcal{T}$. ■

5 Measure-valued solutions

In the present section we skip the assumption on monotonicity of $\mathcal{T}$. The graph $\mathcal{T}$ is now only piecewise monotone.

Theorem 5.1 *Let $q \geq \frac{2d}{d+2}$ and let $\boldsymbol{v}_0 \in L^2_{div}(\Omega)$, $\boldsymbol{b} \in L^{q'}(I,(W^{1,q}_0(\Omega))')$. Then there exists a measure-valued solution $(\boldsymbol{v},\mu)$, i.e.,*

$$\begin{aligned}\boldsymbol{v} &\in L^\infty(I;L^2(\Omega)) \cap L^q(I;W^{1,q}_{0,div}(\Omega))\\ \mu &\in L^\infty(Q_T;\mathcal{M}(\mathbb{R}^{d^2}))\end{aligned}$$

and for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$

$$\begin{aligned}&\int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_{t,x}(\boldsymbol{\xi})\cdot\mathsf{D}(\boldsymbol{\varphi})\,dxdt\\ &=\int_{Q_T}[\boldsymbol{v}\cdot\boldsymbol{\varphi}_t-\boldsymbol{v}\otimes\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}-\boldsymbol{b}\cdot\boldsymbol{\varphi}]\,dxdt-\int_\Omega\boldsymbol{v}_0\cdot\boldsymbol{\varphi}(0)dx\end{aligned}\tag{5.36}$$

is satisfied. Moreover

$$\nabla\boldsymbol{v}=\int_{\mathbb{R}^{d^2}}\boldsymbol{\xi}d\mu_{t,x}(\boldsymbol{\xi})\quad a.e.\ in\ \Omega\tag{5.37}$$

and for all $t \in I$

$$\begin{aligned}&\|\boldsymbol{v}(t)\|_2^2+\int_{Q_t}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_{t,x}(\boldsymbol{\xi})\,dxd\tau\\ &\leq\|\boldsymbol{v}_0^n\|_2^2+\int_0^t\langle\boldsymbol{b},\boldsymbol{v}^n\rangle d\tau,\end{aligned}\tag{5.38}$$

where $Q_t=(0,t)\times\Omega$.

If $q \geq \frac{3d+2}{d+2}$ then the energy equality holds

$$\begin{aligned}&\|\boldsymbol{v}(t)\|_2^2+\int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu_{t,x}(\boldsymbol{\xi})\cdot\mathsf{D}(\boldsymbol{v})\,dxdt\\ &=\int_{Q_T}\boldsymbol{b}\cdot\boldsymbol{v}\,dxdt+\|\boldsymbol{v}_0\|_2^2.\end{aligned}\tag{5.39}$$

Proof:

Let $\{\boldsymbol{\omega}_r\}_{r=1}^\infty$ be an orthonormal basis of $L^2_{div}(\Omega)$. We define $\boldsymbol{v}^n(t)=\sum_{r=1}^n c_r^n(t)\boldsymbol{\omega}_r$, $\boldsymbol{v}^n \in V^n=\operatorname{span}\{\boldsymbol{\omega}_1,...,\boldsymbol{\omega}_n\}$ as a solution to

$$\begin{aligned}\int_\Omega\left(\frac{d}{dt}\boldsymbol{v}^n\cdot\boldsymbol{\omega}_r+\mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{\omega}_r)+\boldsymbol{v}^n\otimes\boldsymbol{v}^n\cdot\nabla\boldsymbol{\omega}_r\right)dx &= \langle\boldsymbol{b},\boldsymbol{\omega}_r\rangle,\\ \boldsymbol{v}^n(0) &= P^n\boldsymbol{v}_0,\end{aligned}\tag{5.40}$$

for all $1 \le r \le n$ and P^n is the continuous orthogonal projector of $L^2(\Omega)$ onto V^n. Multiplying equations (5.40) by $c_r^n(t)$ and summing over r we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n(t)\|_2^2 + \int_\Omega \mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}^n)\, dx = \langle \boldsymbol{b}, \boldsymbol{v}^n \rangle. \tag{5.41}$$

The coercivity conditions, Korn's and Young's inequality imply that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n(t)\|_2^2 + c\|\nabla \boldsymbol{v}^n\|_q^q \le c(\|\boldsymbol{b}\|_{-1,q'}^{q'} + |\Omega|).$$

Integrating over $(0,t)$, with $t \in I$ yields the uniform estimates

$$\begin{aligned} &\|\boldsymbol{v}^n\|_{L^\infty(I;L^2(\Omega))} \le c, \\ &\|\boldsymbol{v}^n\|_{L^q(I;W^{1,q}_{0,div}(\Omega))} \le c. \end{aligned} \tag{5.42}$$

The above imply that at least for a subsequence

$$\begin{aligned} &\boldsymbol{v}^n \overset{*}{\rightharpoonup} \boldsymbol{v} \quad \text{in } L^\infty(I;L^2(\Omega)), \\ &\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \quad \text{in } L^q(I;W^{1,q}_{0,div}(\Omega)). \end{aligned}$$

The existence of approximate solutions follows in a standard way, compare [13]. Also we recall the uniform estimate for time derivative

$$\|\boldsymbol{v}_t\|_{L^{q'}(I;(W^{s,2}_{0,div}(\Omega))^*)} \le c. \tag{5.43}$$

Since $W^{1,q}_{0,div}(\Omega) \hookrightarrow\hookrightarrow L^2_{div}(\Omega) \hookrightarrow (W^{s,2}_{0,div}(\Omega))^*$, thus the Aubin-Lions lemma [13, p. 36] yields

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{in } L^q(I;L^2(\Omega)), \tag{5.44}$$

which provides the limit passage

$$\int_{Q_T} \boldsymbol{v}^n \otimes \boldsymbol{v}^n \cdot \boldsymbol{\varphi}\, dxdt \to \int_{Q_T} \boldsymbol{v} \otimes \boldsymbol{v} \cdot \boldsymbol{\varphi}\, dxdt.$$

The growth conditions (cf. Lemma 3.1) provide that $(\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)))$ is bounded in $L^{q'}(Q_T)$, thus there exists a subsequence of $(\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)))$ and some $\mathbf{S} \in L^{q'}(Q_T)$ such that

$$\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n)) \rightharpoonup \mathbf{S} \quad \text{in} \quad L^{q'}(Q_T). \tag{5.45}$$

According to Theorem 3.2 there exists a family of measures μ^l with $\operatorname{supp} \mu^l_{t,x} \subset \overline{U}_l$ such that $\mu_{t,x} = \sum_l \mu^l_{t,x}$ and $\mu \in L^\infty(\Omega; \mathcal{M}(\mathbb{R}^{d^2}))$. Theorem 3.2 (iii) allows to conclude that

$$\mathbf{S} = \sum_{l \in J} \int_{\overline{U}_l} \mathbf{T}_l \left(\frac{\boldsymbol{\xi} + \boldsymbol{\xi}^T}{2} \right) d\mu^l_{t,x}(\boldsymbol{\xi}) \tag{5.46}$$

for almost all $(t,x)\in Q_T$ and also

$$\nabla \boldsymbol{v} = \int_{\mathbb{R}^{d^2}} \boldsymbol{\xi}\, d\mu_{t,x}(\boldsymbol{\xi}),$$

which proves (5.36) and (5.37).
To prove (5.38) we integrate (5.41) over $(0,t)$ with $t\in I$ to obtain

$$\|\boldsymbol{v}^n(t)\|_2^2 + \int_{Q_t} \mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot \mathsf{D}(\boldsymbol{v}^n) dx d\tau = \|\boldsymbol{v}_0^n\|_2^2 + \int_0^t \langle \boldsymbol{b}, \boldsymbol{v}^n\rangle d\tau. \tag{5.47}$$

Before we let $n\to\infty$ let us observe some properties of solutions. Notice that the strong convergence (5.44) implies

$$\boldsymbol{v}^n(t) \to \boldsymbol{v}(t) \quad \text{in } L^2(\Omega) \text{ for a.a. } t\in I.$$

Considering an arbitrarty $t\in I$ and a sequence (t_k) with $t=\lim_{k\to\infty} t_k$, see [18, pp. 67-68] for details, one can show that

$$\boldsymbol{v}^n(t) \rightharpoonup \boldsymbol{v}(t) \quad \text{in } L^2(\Omega) \text{ for all } t\in I, \tag{5.48}$$

hence

$$\liminf_{n\to\infty} \|\boldsymbol{v}^n(t)\|_2 \ge \|\boldsymbol{v}(t)\|_2 \quad \text{for all} \quad t\in I. \tag{5.49}$$

However T is not monotone any more, but still the term $\mathsf{T}(\mathsf{D}(\boldsymbol{v}^n))\cdot \mathsf{T}(\mathsf{D}(\boldsymbol{v}^n))$ is nonnegative[5]. Next, repeating the argumentation used in Section 4 (cf. (4.34)) and Lemma 3.3 we claim

$$\liminf_{n\to\infty} \int_{Q_t} \mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot \mathsf{D}(\boldsymbol{v}^n) dx d\tau \ge \int_{Q_t} \sum_{l\in J} \int_{\overline{U}_l} \mathsf{T}_l(\boldsymbol{\xi})\cdot \boldsymbol{\xi} d\mu^l_{t,x}(\boldsymbol{\xi})\, dx d\tau. \tag{5.50}$$

Thus finally letting $n\to\infty$ in (5.47) we finish the proof of (5.38). To prove (5.39) let us rewrite (5.36) as follows

$$\int_{Q_t} \boldsymbol{v}_t \cdot \boldsymbol{\varphi}\, dx d\tau = \int_{Q_t} (\boldsymbol{v}\otimes\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi} - \mathsf{S}\cdot\mathsf{D}(\boldsymbol{\varphi}) - \boldsymbol{b}\cdot\boldsymbol{\varphi}) dx d\tau \tag{5.51}$$

for all $\boldsymbol{\varphi}\in D(-\infty,T;\mathcal{V})$. Notice that for $q\ge \frac{3d+2}{d+2}$ the r.h.s. is a linear bounded functional on $L^q(I;W^{1,q}_{0,div}(\Omega))$, cf. Málek et al. [13, Lemma 2.44 p. 220] for detailed estimates. Thus $\boldsymbol{v}_t$ is an element of $L^{q'}(I;(W^{1,q}_{0,div}(\Omega))^*)$, which provides that (5.51) holds for all $\boldsymbol{\varphi}\in L^q(I;W^{1,q}_{0,div}(\Omega))$. This conclusion was necessary to be allowed to test (5.51) with a solution $\boldsymbol{v}$. Since $\boldsymbol{v}\in L^q(I;W^{1,q}_{0,div}(\Omega))$ and $\boldsymbol{v}_t\in L^{q'}(I;(W^{1,q}_{0,div}(\Omega))^*)$, then for all $0\le s\le t\le T$ it holds (cf. [19, Prop. 1.5.8.])

$$\int_s^t \langle \boldsymbol{v}_t, \boldsymbol{v}\rangle d\tau = \frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 - \frac{1}{2}\|\boldsymbol{v}(s)\|_2^2.$$

[5] Notice that $\mathsf{T}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n) = \nu_l(|\mathsf{D}(\boldsymbol{v}^n)|^2)\mathsf{D}(\boldsymbol{v}^n)\cdot\mathsf{D}(\boldsymbol{v}^n) = \nu_l(|\mathsf{D}(\boldsymbol{v}^n)|^2)|\mathsf{D}(\boldsymbol{v}^n)|^2 \ge 0$.

Hence in particular

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2+\int_{Q_t}\mathbf{S}\cdot\mathbf{D}(\boldsymbol{v})\,dxd\tau=\frac{1}{2}\|\boldsymbol{v}_0\|_2^2+\int_{Q_t}\langle\boldsymbol{b},\boldsymbol{v}\rangle dxd\tau, \tag{5.52}$$

which together with (5.46) completes the proof. ■

6 Unsteady flows. Weak solutions

Theorem 6.1 *Let $q\geq\frac{3d+2}{d+2}$. Given $\boldsymbol{b}\in L^{q'}(I;(W^{1,q}_{0,div}(\Omega))^*)$ and $\boldsymbol{v}_0\in L^2_{div}(\Omega)$ there exists a function $\boldsymbol{v}\in L^\infty(I;L^2(\Omega))\cap L^q(I;W^{1,q}_{0,div}(\Omega))$ and a selection $\mathbf{S}\in L^{q'}(I;L^{q'}(\Omega))$ such that*

1. $(\mathbf{D}(\boldsymbol{v}(t,x)),\mathbf{S}(t,x))\in\mathcal{T}$ *a.e. in* Q_T.
2. *For all* $\boldsymbol{\varphi}\in\mathcal{D}(-\infty,T;\mathcal{V})$:

$$\int_{Q_T}\mathbf{S}\cdot\mathbf{D}(\boldsymbol{\varphi})dxdt=\int_{Q_T}(\boldsymbol{v}\cdot\boldsymbol{\varphi}_t+\boldsymbol{v}\otimes\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}+\boldsymbol{b}\cdot\boldsymbol{\varphi})\,dxdt-\int_\Omega\boldsymbol{v}_0\cdot\boldsymbol{\varphi}(0)dx. \tag{6.53}$$

Proof of Theorem 6.1 To show that measure-valued solutions are weak solutions we will prove that the Young measure $\mu_{t,x}$ is a dirac. Let there $\mathbf{S}$ be the limit of the Cauchy stress tensor as in (5.45).

Let $n\to\infty$ in (5.47), then with use of the lower semicontinuity of the norm (5.49) we conclude

$$\limsup_{n\to\infty}\int_{Q_T}\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n))\cdot\mathbf{D}(\boldsymbol{v}^n)dxdt\leq\int_I\langle\boldsymbol{b},\boldsymbol{v}\rangle dt-\frac{1}{2}\|\boldsymbol{v}(T)\|_2^2+\frac{1}{2}\|\boldsymbol{v}_0\|_2^2.$$

Applying energy equality (5.52) leads to

$$\limsup_{n\to\infty}\int_{Q_T}\mathbf{T}^{\frac{1}{n}}(\mathbf{D}(\boldsymbol{v}^n))\cdot\mathbf{D}(\boldsymbol{v}^n)dxdt\leq\int_{Q_T}\mathbf{S}\cdot\mathbf{D}(\boldsymbol{v})dxdt. \tag{6.54}$$

We will proceed similarly to the proof of steady case. Set again for $\boldsymbol{\xi},\mathbf{D}(\boldsymbol{v})\in\mathbb{R}^{d^2}$

$$h^l(\boldsymbol{\xi})=\left[\mathbf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)-\mathbf{T}(\mathbf{D}(\boldsymbol{v}))\right]\cdot\left[\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}-\mathbf{D}(\boldsymbol{v})\right].$$

Thus

$$\int_{Q_T}\sum_{l\in J}\int_{\overline{U_l}}h^l(\boldsymbol{\xi})d\mu^l_{t,x}(\boldsymbol{\xi})dxdt\geq 0. \tag{6.55}$$

The sequence $(\nabla \boldsymbol{v}^n)$ is bounded in $L^q(Q_T)$, thus the tightness condition is satisfied, which provides that $\mu_{t,x}$ is a probability measure. This allows to compute the above integral

$$\begin{aligned}\int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l} h^l(t,x,\boldsymbol{\xi})d\mu^l_{t,x}(\boldsymbol{\xi})dxdt \\ &= \int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu^l_{t,x}(\boldsymbol{\xi})dxdt \\ &- \int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\mu^l_{t,x}(\boldsymbol{\xi})\cdot\mathsf{D}(\boldsymbol{v})dxdt.\end{aligned} \tag{6.56}$$

Combining (6.55), (6.56) and (5.46) yields

$$\int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu^l_{t,x}(\boldsymbol{\xi})\,dxdt \geq \int_{Q_T}\mathsf{S}\cdot\mathsf{D}(\boldsymbol{v})\,dxdt. \tag{6.57}$$

Analogously to (4.35) we can claim that

$$\liminf_{n\to\infty}\int_{Q_T}\mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n)dxdt \geq \int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu^l_{t,x}(\boldsymbol{\xi})\,dxdt. \tag{6.58}$$

Thus (6.54) and (6.58) lead to

$$\int_{Q_T}\sum_{l\in J}\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}d\mu^l_{t,x}(\boldsymbol{\xi})\,dxdt \leq \int_{Q_T}\mathsf{S}\cdot\mathsf{D}(\boldsymbol{v})\,dxdt. \tag{6.59}$$

Hence (6.57) and (6.59) imply

$$\sum_{l\in J}\int_{\overline{U}_l} h^l(t,x,\boldsymbol{\xi})d\mu^l_{t,x}(\boldsymbol{\xi}) = 0.$$

Since $\mu^l_{t,x}$ are nonnegative measures and $\mathcal{T}$ is strictly monotone then $\operatorname{supp}\mu_{t,x} = \{\nabla\boldsymbol{v}(t,x)\}$ which implies $\mu_{t,x} = \delta_{\{\nabla\boldsymbol{v}(t,x)\}}$ a.e. in Q_T. Thus for almost all $(t,x)\in Q_T$

$$\mathsf{S} = \sum_{l\in J}\lambda_l\int_{\overline{U}_l}\mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)d\delta_{\{\nabla\boldsymbol{v}(t,x)\}}(\boldsymbol{\xi}), \tag{6.60}$$

where $\sum_l\lambda_l = 1$, $\lambda_l \geq 0$. For all those $\mathsf{D}(v)$, for which T is single-valued $\lambda_l \in \{0,1\}$ and since $(\mathsf{D}(\boldsymbol{v}),\mathsf{T}_l(\mathsf{D}(\boldsymbol{v}))) \in \mathcal{T}$ for $l\in J$, obviously also $(\mathsf{D}(\boldsymbol{v}(x)),\mathsf{S}(x))$ belongs to $\mathcal{T}$. Whereas in other case since all the points $(\mathsf{D}(\boldsymbol{v}),\mathsf{T}_l(\mathsf{D}(\boldsymbol{v})))$ belong to the vertical part of the graph and that any interval is a convex set, then also $(\mathsf{D}(\boldsymbol{v}(x)),\mathsf{S}(x)) \in \mathcal{T}$, which completes the proof. ■

7 Uniform integrability

We will show the following property of solutions

Lemma 7.1 *Let $\boldsymbol{v}^n$ be a sequence of solutions to approximate problem (5.40) and $\boldsymbol{v}$ the solution to (2.11). Then*

$$\mathsf{D}(\boldsymbol{v}^n) \to \mathsf{D}(\boldsymbol{v}) \quad \text{in } L^q(Q_T).$$

7.1 Biting convergence

We recall the definition of the biting convergence and well known Biting Lemma. Then we formulate Lemma 7.3 with a simple proof due to [6] and will be the tool for the proof of Lemma 7.1.

Definition 7.1 *A bounded sequence $(\boldsymbol{z}^\varepsilon)$ in $L^1(\Omega)$ converges weakly in biting sense to a function $\boldsymbol{z} \in L^1(\Omega)$, written $\boldsymbol{z}^\varepsilon \xrightarrow{b} \boldsymbol{z}$ in Ω, provided there exists a sequence (E_k) of measurable subsets of Ω, satisfying $\lim_{k\to\infty} |E_k| = 0$, such that for each k*

$$\boldsymbol{z}^\varepsilon \rightharpoonup \boldsymbol{z} \quad \text{in} \quad L^1(\Omega \setminus E_k).$$

Lemma 7.2 (Biting lemma) *Let $\Omega \subset \mathbb{R}^n$ be bounded measurable, and let $(\boldsymbol{z}^\varepsilon)$ be a bounded sequence in $L^1(\Omega)$. Then there exists a function $\boldsymbol{z} \in L^1(\Omega)$ such that at least for a subsequence*

$$\boldsymbol{z}^\varepsilon \xrightarrow{b} \boldsymbol{z} \quad \text{in} \quad \Omega.$$

Lemma 7.3 *Let $g^n \xrightarrow{b} g$, $\quad g^n, g \in L^1(\Omega)$, $\quad g^n \geq 0$, $\quad \lim\limits_{n\to\infty} \int_\Omega g^n dx = \int_\Omega g$. Then $g^n \rightharpoonup g$ in $L^1(\Omega)$.*

Proof:
Let E_k be the family of sets described by Definition 7.1. By assumption

$$g^n \rightharpoonup g \quad \text{in} \quad L^1(\Omega \setminus E_k) \quad \forall k \in \mathbb{N},$$

which follows

$$\lim_{n\to\infty} \int_{\Omega\setminus E_k} g^n \, dx = \int_{\Omega\setminus E_k} g \, dx.$$

Fix k. Then

$$\lim_{n\to\infty} \int_{E_k} g^n \, dx = \lim_{n\to\infty} \int_\Omega g^n \, dx - \lim_{n\to\infty} \int_{\Omega\setminus E_k} g^n \, dx$$
$$= \int_\Omega g \, dx - \int_{\Omega\setminus E_k} g \, dx = \int_{E_k} g \, dx.$$

Take an arbitrary $\varphi \in L^\infty(\Omega)$.

$$\lim_{n\to\infty}\int_\Omega (g^n - g)\varphi\, dx = \lim_{n\to\infty}\int_{\Omega\setminus E_k} (g^n - g)\varphi\, dx + \lim_{n\to\infty}\int_{E_k} (g^n - g)\varphi\, dx.$$

The first term on the r.h.s. converges to zero by assumption. To show the convergence of the second term observe that obviously

$$\lim_{k\to\infty}\int_{E_K} g\varphi\, dx = 0$$

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{E_K} g^n\varphi\, dx \le \lim_{k\to\infty}\lim_{n\to\infty}\|g^n\|_{L^1(E_k)}\|\varphi\|_{L^\infty(E_k)} = \lim_{k\to\infty}\|g\|_{L^1(E_k)}\|\varphi\|_{L^\infty(E_k)} = 0$$

7.2 Proof of Lemma 7.1

Theorem 3.2 (v) implies

$$\mathsf{D}(\boldsymbol{v}^n) \to \mathsf{D}(\boldsymbol{v}) \qquad \text{in measure.}$$

Combinig Lemma 3.3 and (6.54) we conclude

$$\lim_{n\to\infty}\int_{Q_T} \mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n)\, dxdt = \int_{Q_T} \mathsf{S}\cdot\mathsf{D}(\boldsymbol{v})\, dxdt.$$

The sequence $(|\mathsf{D}(\boldsymbol{v}^n)|^q)$ is bounded in $L^1(Q_T)$. Thus Lemma 7.2 implies that it is weakly relatively compact on the set $Q_T \setminus E_k$. Theorem 3.2 (iv) applied to the function $\mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n)$ in analogous way to (4.34) implies

$$\mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n))\cdot\mathsf{D}(\boldsymbol{v}^n) \xrightarrow{b} \sum_{l\in J}\int_{\overline{U}_l} \mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2} d\mu_{t,x}(\boldsymbol{\xi}).$$

Using that $\mu_{t,x}$ is a dirac measure and (6.60) provides

$$\begin{aligned}&\sum_{l\in J}\int_{\overline{U}_l} \mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2} d\mu_{t,x}(\boldsymbol{\xi})\\ &\qquad = \sum_{l\in J}\lambda_l(x)\int_{\overline{U}_l} \mathsf{T}_l\left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2}\right)\cdot\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^T}{2} d\delta_{\{\nabla\boldsymbol{v}(t,x)\}}(\boldsymbol{\xi}) = \mathsf{S}\cdot\mathsf{D}(\boldsymbol{v})\end{aligned}$$

with $\lambda_l(x) \ge 0$ and $\sum_l \lambda_l(x) = 1$. Finally Lemma 7.3 implies that the sequence $(\mathsf{T}^{\frac{1}{n}}(\mathsf{D}(\boldsymbol{v}^n)\cdot\mathsf{D}(\boldsymbol{v}^n))$ is weakly precompact in $L^1(Q_T)$, thus by Dunford-Pettis theorem it is uniformly integrable. Due to the coercivity condition also the sequence $(|\mathsf{D}(\boldsymbol{v}^n)|^q)$ is uniformly integrable. Using Vitali's Theorem yields that $\mathsf{D}(\boldsymbol{v}^n) \to \mathsf{D}(\boldsymbol{v})$ in $L^q(Q_T)$, which completes the proof. ■

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