Very Weak Solutions of the Navier-Stokes Equations in Exterior Domains with Nonhomogeneous Data

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Abstract

We investigate the nonstationary Navier-Stokes equations in an exterior domain $\Omega \subset \mathbb{R}^3$ in a solution class $L^s\big(0,T;L^q(\Omega)\big)$ of very low regularity in space and time, satisfying Serrin's condition $\frac{2}{s}+\frac{3}{q}=1$ but not necessarily any differentiability property. The nonhomogeneous boundary data $u|_{\partial\Omega} = g \in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big)$ are the weakest possible beyond the notion of usual trace theorems; moreover, we prescribe a divergence $k = \operatorname{div} u \in L^s\big(0,T;L^r(\Omega)\big)$, where $\frac{1}{3}+\frac{1}{q}=\frac{1}{r}$. In principle, we follow Amann's notion of very weak solutions, see [3], [4].

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1 Introduction and Main Theorems

Throughout this paper, $\Omega \subset \mathbb{R}^3$ is an exterior domain with nonempty, compact boundary $\partial\Omega$ of class $C^{2,1}$ and $[0,T)$, $0 < T \le \infty$, denotes the time interval. In $[0,T)\times\Omega$ we consider the instationary Navier-Stokes equations

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p &= f && \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= k && \text{in } (0,T)\times\Omega \\ u &= g && \text{on } (0,T)\times\partial\Omega \\ u &= u_0 && \text{at } t=0 \end{aligned} \tag{1.1}$$

with constant viscosity $\nu > 0$, nonhomogeneous external force $f = \operatorname{div} F = \big(\sum_{i=1}^3 \partial_i F_{ij}\big)_{j=1}^3$, divergence k, boundary data g, and initial value u_0 satisfying

$$\begin{aligned} F &= (F_{ij})_{i,j=1}^3 \in L^s\big(0,T;L^r(\Omega)\big) \\ k &\in L^s\big(0,T;L^r(\Omega)\big) \\ g &\in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big) \\ u_0 &\in \mathcal{J}_\nu^{q,s}(\Omega) \end{aligned} \tag{1.2}$$

where

$$\frac{2}{s}+\frac{3}{q}=1, \quad 2<s<\infty, \quad 3<q<\infty \quad \text{and} \quad \frac{1}{3}+\frac{1}{q}=\frac{1}{r}\,; \tag{1.3}$$

see Subsection 2.5 for the definition of the space $\mathcal{J}_\nu^{q,s}(\Omega)$ of initial values. Following Amann [3], [4] in principle, we define a very weak solution of (1.1):

Definition 1.1 Suppose that the data $f = \operatorname{div} F$, k, g and u_0 satisfy (1.2), (1.3). Then $u \in L^s\big(0,T;L^q(\Omega)\big)$ is called a *very weak solution* of the Navier-Stokes system (1.1) in the exterior domain $\Omega \subset \mathbb{R}^3$ if for all $w \in C_0^1\big([0,T);\, C^2_{0,\sigma}(\overline{\Omega})\big)$ where $C^2_{0,\sigma}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : \operatorname{div} v = 0,\ \operatorname{supp} v \text{ compact in } \overline{\Omega},\ v|_{\partial\Omega} = 0\}$ the relation

$$\begin{aligned} &\int_0^T \big(-\langle u, w_t\rangle_\Omega - \nu\langle u, \Delta w\rangle_\Omega + \nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega} - \langle u\otimes u, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega\big)\, dt \\ &\qquad = \langle u_0, w(0)\rangle_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega\, dt \quad \text{for all } w \in C_0^1\big([0,T);\, C^2_{0,\sigma}(\overline{\Omega})\big) \end{aligned} \tag{1.4}$$

and the conditions

$$\operatorname{div} u(t) = k(t) \text{ in } \Omega, \quad N\cdot u(t)|_{\partial\Omega} = N\cdot g(t) \quad \text{for a.a.} \quad t \in (0,T) \tag{1.5}$$

are satisfied.

Here, at $x = (x_1,x_2,x_3) \in \partial\Omega$ the outer normal is denoted by $N = N(x) \in \mathbb{R}^3$. The term $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual $L^q - L^{q'}$–pairing in Ω or the application of the functional $u_0 \in \mathcal{J}_\nu^{q,s}(\Omega)$ at $w(0) = w|_{t=0} \in C^2_{0,\sigma}(\overline{\Omega})$, cf. Subsection 2.5, and $\langle g(t),\ N\cdot\nabla w(t)\rangle_{\partial\Omega}$ is the value of the distribution $g(t) \in W^{-1/q,q}(\Omega)$ at the normal derivative $N\cdot\nabla w(t)$ of $w(t)$. Note that we used the elementary relation $u\cdot\nabla u = \operatorname{div}(u\otimes u) - ku$ where $u\otimes u = (u_i u_j)_{i,j=1}^3$.

An elementary calculation shows that for a solenoidal vector field w

$$N\cdot\nabla w(t) = \operatorname{curl} w(t) \times N \quad \text{on } \partial\Omega. \tag{1.6}$$

Therefore, (1.4) contains a condition only on the tangential component $N\times g$ of g on $\partial\Omega$, and we have to suppose the additional condition in (1.5) for the normal component $N\cdot g = N\cdot u|_{\partial\Omega}$. Note that the data (1.2) need not satisfy any compatibility condition as for bounded domains, see [10].

Then our main theorem reads as follows:

Theorem 1.2 *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{2,1}$. Suppose that the data $f = \operatorname{div} F$, k, g and u_0 satisfy* (1.2), (1.3). *Then there exists a $T' = T'(f,k,g,u_0,\nu) \in (0,T]$ and a unique very weak solution $u \in L^s\big(0,T';L^q(\Omega)\big)$ of the nonhomogeneous Navier-Stokes system* (1.1). *The interval of existence $[0,T')$ is determined by the condition* (5.12) *below and includes the case $T' = T = \infty$.*

There are not many references on the system (1.1) for the very general nonhomogeneous case $\operatorname{div} u = k \neq 0$ and $u|_{\partial\Omega} = g \neq 0$, but there are several results for $k = 0$, $g \neq 0$, see [3], [4], [8], [10], [14], [16], [19], and [20]. Amann's approach in Besov spaces [3], [4] seems to be the first one working in solution classes with $u|_{\partial\Omega} \neq 0$ beyond the usual trace theorems. Our purpose is to extend the solution class to the weakest possible class by keeping uniqueness, and to the case $\operatorname{div} u \neq 0$. Furthermore, we develop the corresponding theory also for the linear stationary and instationary Stokes equation with inhomogeneous data. For further references see [14].

We will see in Remark 5.2 that a very weak solution satisfies the first equation of (1.1) in the sense of distributions, together with some distribution p. Moreover, the boundary condition $u|_{\partial\Omega} = g$ is well defined in the sense of distributions on $\partial\Omega$, but not in the sense of usual trace theorems. Actually, the tangential condition $N \times u|_{\partial\Omega} = N \times g$ is implicitly defined as a distribution by the relation (1.4) via the boundary term $\nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$, see Remark 4.2 (2). Moreover, the trace $N \cdot u|_{\partial\Omega} = N \cdot g$ of the normal component is well defined in the usual weak sense, see (2.2). Finally, we see that the initial condition $u(0,\cdot) = u_0$ in (1.1) has a precise meaning "modulo gradients", see Subsection 2.5, since $w(0) \in C^2_{0,\sigma}(\overline{\Omega})$ in (1.4) is solenoidal.

It is remarkable that a very weak solution u of (1.1) *need not satisfy any energy inequality* like weak solutions in the sense of Leray and Hopf; in particular, u need not have finite energy $\frac{1}{2}\|u\|^2_{2,\infty} + \frac{1}{2}\|\nabla u\|^2_{2,2} < \infty$. This justifies the notion of a very weak solution. On the other hand, a very weak solution possesses the *uniqueness property* on its interval of existence $[0,T')$ because of the Serrin condition, cf. (1.3). Note that the uniqueness of weak solutions in the sense of Leray and Hopf is open.

The proof of Theorem 1.2 is based on the unique decomposition $u = \hat{u} + E$ where $E \in L^s\big(0,T;L^q(\Omega)\big)$ is the very weak solution of the linearized nonhomogeneous system

$$\begin{aligned} E_t - \nu\Delta E + \nabla h &= f, \qquad \operatorname{div} E = k \quad \text{in } (0,T)\times\Omega \\ E|_{\partial\Omega} &= g, \qquad E(0,\cdot) = u_0 \end{aligned}$$

and where $\hat{u} \in L^s\big(0,T;L^q(\Omega)\big)$ is the very weak solution of the "homogeneous" nonlinear system

$$\hat{u}_t - \nu\Delta\hat{u} + (\hat{u}+E)\cdot\nabla(\hat{u}+E) + \nabla\hat{p} = 0, \quad \operatorname{div}\hat{u} = 0 \quad \text{in} \quad (0,T)\times\Omega,$$

satisfying $\hat{u}_{|_{\partial\Omega}} = 0$, $u(0,\cdot) = 0$, cf. (5.1), (5.3) below.

The generalized nonstationary Stokes system we consider here has the form

$$\begin{aligned} u_t - \nu\Delta u + \nabla p &= f, \quad & \operatorname{div} u &= k \quad & \text{in } (0,T)\times\Omega, \\ u_{|_{\partial\Omega}} &= g, & u &= u_0 & \text{at } t = 0, \end{aligned} \tag{1.7}$$

where $f = \operatorname{div} F$, k, g and u_0 satisfy (1.2) and where $1 < s < \infty$, $3 < q < \infty$ and $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$ yielding $\frac{3}{2} < r < 3$. Note that Serrin's condition $\frac{2}{s} + \frac{3}{q} = 1$ is not needed for this linear problem. See Subsections 2.3 and 2.5 concerning the Stokes operator A_q and the generalized meaning of $A_q^{-1}P_q u_0$ of the distribution u_0. In this linear case the definition of a very weak solution reads as follows:

Definition 1.3 Suppose that the data $f = \operatorname{div} F, k, g$ and u_0 satisfy (1.2) with $1 < s < \infty$, $3 < q < \infty$ and $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Then $u \in L^s\big(0,T; L^q(\Omega)\big)$ is called a *very weak solution of the nonstationary Stokes system* (1.7) if for all $w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$

$$\begin{aligned} &\int_0^T \big(-\langle u, w_t\rangle_\Omega - \nu\langle u, \Delta w\rangle_\Omega + \nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega}\big)\,dt \\ &\qquad = \langle u_0, w(0)\rangle_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega\,dt \end{aligned} \tag{1.8}$$

and if

$$\operatorname{div} u(t) = k(t) \text{ in } \Omega, \quad N\cdot u(t)_{|_{\partial\Omega}} = N\cdot g(t) \quad \text{for a.a.} \quad t \in (0,T),$$

cf. (1.5), are satisfied.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^{2,1}$, let $f = \operatorname{div} F, k, g$ and u_0 satisfy* (1.2) *with $1 < s < \infty$, $3 < q < \infty$ and $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Then there exists a unique very weak solution $u \in L^s\big(0,T; L^q(\Omega)\big)$ of* (1.7) *satisfying*

$$\begin{aligned} &A_q^{-1}P_q u_t \in L^s\big(0,T; L^q_\sigma(\Omega)\big), \quad A_q^{-1}P_q u \in C\big([0,T); L^q_\sigma(\Omega)\big), \\ &A_q^{-1}P_q u_{|_{t=0}} = A_q^{-1}P_q u_0, \end{aligned} \tag{1.9}$$

and the a priori estimate

$$\begin{aligned} &\|A_q^{-1}P_q u_t\|_{q,s,\Omega,T} + \|\nu u\|_{q,s,\Omega,T} \\ &\qquad \le c\big(\|u_0\|_{\mathcal{J}_\nu^{q,s}(\Omega)} + \|F\|_{r,s,\Omega,T} + \|\nu k\|_{r,s,\Omega,T} + \|\nu g\|_{-1/q;q,s,\partial\Omega,T}\big) \end{aligned} \tag{1.10}$$

where $c = c(q,s,\Omega) > 0$. Moreover, the term $\|u_0\|_{\mathcal{J}_\nu^{q,s}(\Omega)}$ may be replaced by the smaller term $\big(\int_0^T \|\nu A_q e^{-\nu t A_q} A_q^{-1} P_q u_0\|^s_{q,\Omega}\,dt\big)^{1/s}$. The solution u possesses the explicit representation (4.5) *below.*

Finally we consider – indeed as a starting point of the proofs – the nonhomogeneous stationary Stokes system

$$-\nu\Delta u + \nabla p = f, \quad \operatorname{div} u = k \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g \tag{1.11}$$

with data $f = \operatorname{div} F$, k and g satisfying

$$F \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in W^{-1/q,q}(\partial\Omega), \quad 3 < q < \infty, \ \frac{1}{3} + \frac{1}{q} = \frac{1}{r} \tag{1.12}$$

yielding $\frac{3}{2} < r < 3$.

Definition 1.5 Given data $f = \operatorname{div} F$, k, g as in (1.12) a vector field $u \in L^q(\Omega)$ is called a *very weak solution of the stationary Stokes system* (1.11) if the relation

$$-\nu\langle u, \Delta w\rangle_\Omega + \nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega} = -\langle F, \nabla w\rangle_\Omega, \quad w \in C^2_{0,\sigma}(\overline{\Omega}), \tag{1.13}$$

and the conditions

$$\operatorname{div} u = k \quad \text{in } \Omega, \quad N\cdot u|_{\partial\Omega} = N\cdot g \tag{1.14}$$

are satisfied.

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary of class $C^{2,1}$, and let the data $f = \operatorname{div} F$, k, g satisfy* (1.12)*. Then there exists a unique very weak solution $u \in L^q(\Omega)$ of the stationary Stokes system* (1.11) *in the sense of* (1.13) – (1.14) *satisfying the a priori estimate*

$$\|\nu u\|_{q,\Omega} \le c\big(\|F\|_{r,\Omega} + \|\nu k\|_{r,\Omega} + \|\nu g\|_{-1/q;q,\partial\Omega}\big)$$

where $c = c(\Omega, q) > 0$. Moreover, u possesses the representation (3.14) *below.*

This paper is organized as follows. In Section 2 we introduce several function spaces and operators and recall important properties of them. The proof of the main Theorem 1.2 is based on Theorems 1.4, 1.6 and on a fixed point argument. Therefore, Section 3 deals with the proof of Theorem 1.6, Section 4 with the proof of Theorem 1.4, and the final Section 5 is devoted to the nonlinear case in Theorem 1.2. Note that the reals c, c_1, $c_2 > 0$ are generic constants depending on the exponents q, r, s etc., and on the exterior domain Ω, but not on the functions involved in subsequent estimates.

2 Notations and Preliminaries

2.1 Classical Function Spaces

Given $1 < q < \infty$ and $q' = \frac{q}{q-1}$ we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega)$, $W^{\alpha,q}(\Omega)$, where $\alpha \ge 0$, and $W^{\alpha,q}_0(\Omega) \subset W^{\alpha,q}(\Omega)$ with norms $\|\cdot\|_{L^q(\Omega)} =$

$\|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{W^{\alpha,q}(\Omega)} = \|\cdot\|_{\alpha;q,\Omega}$, resp. The space $W^{-\alpha,q}(\Omega) := W_0^{\alpha,q'}(\Omega)'$ denotes the dual space of $W_0^{\alpha,q'}(\Omega)$ with the natural pairing $\langle\cdot,\cdot\rangle_\Omega$ and the norm $\|\cdot\|_{W^{-\alpha,q}(\Omega)} = \|\cdot\|_{-\alpha;q,\Omega}$. If $\alpha = 0$, then $\langle f,h\rangle_\Omega = \int_\Omega f\cdot h\,dx$ for $f \in L^q(\Omega)$, $h \in L^{q'}(\Omega)$; here $f\cdot h$ denotes the scalar product of vector or matrix fields. Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector- or matrix-valued fields.

For the boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{R}^3$ let $L^q(\partial\Omega)$, $W^{\alpha,q}(\partial\Omega)$, $W^{-\alpha,q}(\partial\Omega) = W^{\alpha,q'}(\partial\Omega)'$, where $0 < \alpha \le 2$, denote the corresponding function spaces, using the norms $\|\cdot\|_{L^q(\partial\Omega)} = \|\cdot\|_{q,\partial\Omega}$, $\|\cdot\|_{W^{\alpha,q}(\partial\Omega)} = \|\cdot\|_{\alpha;q,\partial\Omega}$ and $\|\cdot\|_{W^{-\alpha,q}(\partial\Omega)} = \|\cdot\|_{-\alpha;q,\partial\Omega}$, resp., and the natural duality pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$. Note that the restriction $\alpha \le 2$ in this case is needed since $\partial\Omega \in C^{2,1}$. In particular, the pairing between $L^q(\partial\Omega)$ and its dual $L^{q'}(\partial\Omega)$ is given by

$$\langle f,g\rangle_{\partial\Omega} = \int_{\partial\Omega} f\cdot g\,dS$$

where $\int_{\partial\Omega}\ldots dS$ denotes the surface integral on $\partial\Omega$. For more details cf. [1], [12], [12] and [28].

Let $C^m(\Omega), C_0^m(\Omega)$ and $C^m(\overline{\Omega}), m \in \mathbb{N}\cup\{+\infty\}$, denote the usual spaces of smooth functions. An important function space is

$$C_0^m(\overline{\Omega}) := \{v \in C^m(\overline{\Omega}) : \operatorname{supp} v \text{ compact in } \overline{\Omega}, v = 0 \text{ on } \partial\Omega\}.$$

For the space $C_0^\infty(\Omega)'$ of distributions, the dual space of $C_0^\infty(\Omega)$, the duality pairing on Ω is denoted by $\langle\cdot,\cdot\rangle_\Omega$. Finally, we use the boundary distributions $C^1(\partial\Omega)'$ with test functions from $C^1(\partial\Omega)$ and with the pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$.

The subspaces of solenoidal vector fields are denoted by appending the subscript $'\sigma'$ leading to the spaces $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$ and $C_{0,\sigma}^m(\overline{\Omega}) = \{v \in C_0^m(\overline{\Omega}) : \operatorname{div} v = 0\}$ as well as to the dual space $C_{0,\sigma}^m(\Omega)'$ of $C_{0,\sigma}^m(\Omega)$ with pairing $\langle\cdot,\cdot\rangle_\Omega$. By a theorem of de Rham, [27], I, Prop. 1.1, a distribution $d \in C_0^\infty(\Omega)'$ vanishing at all $v \in C_{0,\sigma}^\infty(\Omega)$ may be written in the form $d = \nabla h$ with a scalar distribution h. Let $L^q_\sigma(\Omega)$ denote the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\cdot\|_{q,\Omega}$. It is well known that $L^q_\sigma(\Omega)' = L^{q'}_\sigma(\Omega)$ using the standard pairing $\langle\cdot,\cdot\rangle_\Omega$.

2.2 Traces and Extensions

Let $\alpha = 1,2$. Given an exterior domain $\Omega \subset \mathbb{R}^3$ with boundary of class $C^{2,1}$, let $B \subset \mathbb{R}^3$ be an open ball with $\partial\Omega \subset B$ and let $\Omega_0 := \Omega \cap B$. Then the *trace map* $f \mapsto f|_{\partial\Omega}$ is a well defined linear bounded operator from $W^{\alpha,q}(\Omega)$ onto $W^{\alpha-1/q,q}(\partial\Omega)$, and there exists a linear bounded *extension operator* E : $W^{\alpha-1/q,q}(\partial\Omega) \to W^{\alpha,q}(\Omega), h \mapsto E_h$, such that $E_h|_{\partial\Omega} = h$. The extension operator can be constructed in such a way that $\operatorname{supp} E_h \subset \overline{\Omega}_0$ for all $h \in W^{\alpha-1/q,q}(\partial\Omega)$.

Let $1 < r < 3$ and let $q > r$ be defined by $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Given $f \in L^q(\Omega)$ with $\operatorname{div} f \in L^r(\Omega)$ we use Green's identity in Ω_0 and the trace space $W^{1-1/q',q'}(\partial\Omega) = W^{1/q,q'}(\partial\Omega)$ to get that

$$\langle \operatorname{div} f, E_h \rangle_{\Omega_0} = \langle N \cdot f, h \rangle_{\partial\Omega} - \langle f, \nabla E_h \rangle_{\Omega_0}, \quad h \in W^{1/q,q'}(\partial\Omega). \tag{2.1}$$

Since $q > \frac{3}{2}$ and consequently $1 < q' < 3$, the embedding and extension estimate

$$\|E_h\|_{r',\Omega_0} \le c(\|E_h\|_{q',\Omega_0} + \|\nabla E_h\|_{q',\Omega_0}) \le c\|h\|_{1/q;q',\partial\Omega}$$

holds with $\frac{1}{3} + \frac{1}{r'} = \frac{1}{q'}$ and $c = c(\Omega, q) > 0$. Consequently,

$$\begin{aligned} |\langle N \cdot f, h \rangle_{\partial\Omega}| &\le c(\|f\|_{q,\Omega_0} + \|\operatorname{div} f\|_{r,\Omega_0})\|h\|_{1/q;q',\partial\Omega} \\ &\le c(\|f\|_{q,\Omega} + \|\operatorname{div} f\|_{r,\Omega})\|h\|_{1/q;q',\partial\Omega} \end{aligned} \tag{2.2}$$

for all $h \in W^{1/q,q'}(\partial\Omega)$. Hence the trace $N \cdot f|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ of the normal component of f on $\partial\Omega$ is well defined and satisfies the estimate

$$\|N \cdot f\|_{-1/q;q,\partial\Omega} \le c(\|f\|_{q,\Omega_0} + \|\operatorname{div} f\|_{r,\Omega_0}) \le c(\|f\|_{q,\Omega} + \|\operatorname{div} f\|_{r,\Omega}) \tag{2.3}$$

with the same $c > 0$ as in (2.2).

Analogously, by the identity

$$\langle \operatorname{curl} f, E_h \rangle_{\Omega_0} = \langle N \times f, h \rangle_{\partial\Omega} + \langle f, \operatorname{curl} E_h \rangle_{\Omega_0}, \tag{2.4}$$

we obtain the following trace property: Given $f \in L^q(\Omega)$ with $\operatorname{curl} f \in L^r(\Omega)$ where $1 < r < 3, \frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, the trace $N \times f|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ of the tangential component of f on $\partial\Omega$ is well defined, and it holds the estimate

$$\|N \times f\|_{-1/q;q,\partial\Omega} \le c\big(\|f\|_{q,\Omega_0} + \|\operatorname{curl} f\|_{r,\Omega_0}\big) \le c\big(\|f\|_{q,\Omega} + \|\operatorname{curl} f\|_{r,\Omega}\big). \tag{2.5}$$

Consider the divergence problem

$$\operatorname{div} b = f \quad \text{in } \Omega_0, \quad b = 0 \quad \text{on } \partial\Omega_0 \tag{2.6}$$

for given right-hand side f. If $1 < q < \infty$ and $f \in L^q(\Omega_0)$ satisfying $\int_{\Omega_0} f(x)dx = 0$, then there exists some $b = b^f \in W_0^{1,q}(\Omega_0)$ solving (2.6) such that

$$\|b^f\|_{1;q,\Omega_0} \le c(\Omega_0, q)\, \|f\|_{q,\Omega_0}. \tag{2.7}$$

Moreover, if additionally $f \in W_0^{1,q}(\Omega_0)$, then $b^f \in W_0^{2,q}(\Omega)$ and

$$\|b^f\|_{2;q,\Omega_0} \le c(\Omega_0, q)\, \|\nabla f\|_{q,\Omega_0}, \tag{2.8}$$

cf. [11], III, Theorem 3.2.

In the case of an exterior domain $\Omega \subset \mathbb{R}^3$ let $1 < r < 3$ and $f \in L^r(\Omega)$. Then by [11], III, Theorem 3.4 and II, Remark 5.2, there exists $b \in L^q(\Omega)$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, with $\nabla b \in L^r(\Omega)$, $b|_{\partial\Omega} = 0$ satisfying $\operatorname{div} b = f$ and the estimate

$$\|b\|_{q,\Omega} \le c\|\nabla b\|_{r,\Omega} \le c'\|f\|_{r,\Omega} \tag{2.9}$$

with constants $c, c' > 0$ depending only on Ω and on r. Note that in each case $b = b^f$ can be chosen to depend linearly on f.

Using properties of the weak Neumann problem [23] we find for each $h \in W^{-1/q,q}(\partial\Omega)$ a vector field $E^h \in L^q(\Omega)$ depending linearly on h such that $\operatorname{div} E^h \in L^r(\Omega)$, $N \cdot E^h|_{\partial\Omega} = h$, $\operatorname{supp} E^h \subset \overline{\Omega}_0$ satisfying the estimate

$$\|E^h\|_{q,\Omega} + \|\operatorname{div} E^h\|_{r,\Omega} \le c\|h\|_{-1/q;q,\partial\Omega}\,. \tag{2.10}$$

By an extension theorem for the bounded domain Ω_0, cf. [22], Theorem 5.8 or [28], Subsection 5.4.4 we obtain the following result: For every $h \in W^{1-1/q,q}(\partial\Omega)$ there exists an extension $w^h \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ depending linearly on h such that $N \cdot \nabla w^h|_{\partial\Omega} = h$, $\operatorname{supp} w^h \subset \overline{\Omega}_0$ and

$$\|w^h\|_{2,q,\Omega} = \|w^h\|_{2,q,\Omega_0} \le c\|h\|_{1-1/q;q,\partial\Omega}. \tag{2.11}$$

If additionally $N \cdot h = 0$ on $\partial\Omega$, then a calculation shows that

$$\operatorname{div} w^h|_{\partial\Omega_0} = 0, \quad N \cdot \nabla w^h|_{\partial\Omega} = \operatorname{curl} w^h|_{\partial\Omega} \times N = h.$$

Moreover, since $\int_{\Omega_0} \operatorname{div} w^h\, dx = 0$, $\operatorname{div} w^h \in W_0^{1,q}(\Omega_0)$, we may use (2.6) – (2.8) to find $\hat{w}^h = w^h - b^f \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $f = \operatorname{div} w^h$, depending linearly on h such that $\operatorname{supp} \hat{w}^h \subset \overline{\Omega}_0$,

$$\begin{aligned} &\operatorname{div} \hat{w}^h = 0 \quad \text{in } \Omega, \quad N \cdot \nabla \hat{w}^h = \operatorname{curl} \hat{w}^h|_{\partial\Omega} \times N = h, \\ &\|\hat{w}^h\|_{2;q,\Omega_0} = \|\hat{w}^h\|_{2;q,\Omega} \le c\|h\|_{1-1/q;q,\partial\Omega} \end{aligned} \tag{2.12}$$

with $c = c(\Omega, \Omega_0, q) > 0$ in (2.10) – (2.12). Note that the extensions E^h, w^h, $\hat{w}^h$ are first of all constructed for Ω_0 by setting $N \cdot E^h|_{\partial B} = 0$, and $w^h|_{\partial\Omega_0} = 0$, $N \cdot \nabla w^h|_{\partial B} = 0$. Then $\operatorname{div} E^h \in L^r(\Omega)$ and $w^h, \hat{w}^h \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$.

2.3 Helmholtz Projection and Stokes Operator

Given a vector field $f \in L^q(\Omega)$, $1 < q < \infty$, on the exterior domain $\Omega \subset \mathbb{R}^3$, the weak Neumann problem

$$\Delta p = \operatorname{div} f, \quad N \cdot (\nabla p - f)|_{\partial\Omega} = 0$$

has a unique solution $\nabla p \in L^q(\Omega)$ satisfying the estimate $\|\nabla p\|_{q,\Omega} \le c\|f\|_{q,\Omega}$ with $c = c(\Omega, q) > 0$. Then the Helmholtz projection P_q defined by$P_q f = f - \nabla p$ is a bounded linear operator from $L^q(\Omega)$ onto the solenoidal subspace $L^q_\sigma(\Omega)$ satisfying $P_q^2 = P_q$ and $P_q' = P_{q'}$, i.e., $\langle P_q f, g\rangle_\Omega = \langle f, P_{q'} g\rangle_\Omega$ for all $f \in L^q(\Omega)$, $g \in L^{q'}(\Omega)$. Note that $P_q f = P_\varrho f$ if $f \in L^q(\Omega) \cap L^\varrho(\Omega)$ and $1 < q, \varrho > \infty$, see [23].

The Stokes operator $A_q = \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$ with dense domain

$$\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \ \subset \ L^q_\sigma(\Omega)$$

is defined by $A_q u = -P_q \Delta u$, $u \in \mathcal{D}(A_q)$; its range $\{A_q u : u \in \mathcal{D}(A_q)\}$ will be denoted by $\mathcal{R}(A_q)$. Note that for two exponents $1 < q, r < \infty$ and for $u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_r)$ we get $A_q u = A_r u$. As usual, $\mathcal{D}(A_q)$ will be equipped with the graph norm $\|u\|_{q,\Omega} + \|A_q u\|_{q,\Omega}$ for $u \in \mathcal{D}(A_q)$. Concerning more details on the Stokes operator see [6], [8] – [19], [25] – [27].

For $\alpha \in [-1, 1]$ the fractional power $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L^q_\sigma(\Omega)$ with dense domain $\mathcal{D}(A_q^\alpha) \subset L^q_\sigma(\Omega)$ is a well defined, injective operator such that

$$(A_q^\alpha)^{-1} = A_q^{-\alpha}, \quad \mathcal{R}(A_q^\alpha) = \mathcal{D}(A_q^{-\alpha}), \quad (A_q^\alpha)' = A_{q'}^\alpha\,.$$

We mention several important embedding estimates for the sequel:

$$\begin{aligned} \|A_q^{1/2} u\|_{q,\Omega} &\leq c\|\nabla u\|_{q,\Omega}, \qquad 1 < q < \infty,\ u \in \mathcal{D}(A_q^{1/2}), \\ \|A_q u\|_{q,\Omega} &\leq c\|\nabla^2 u\|_{q,\Omega}, \qquad 1 < q < \infty,\ u \in \mathcal{D}(A_q), \end{aligned} \tag{2.13}$$

and, by [5], Theorem 4.4 and [18], Theorem 3.1, resp.,

$$\begin{aligned} \|\nabla u\|_{q,\Omega} &\leq c\|A_q^{1/2} u\|_{q,\Omega}, \qquad 1 < q < 3,\ u \in \mathcal{D}(A_q^{1/2}), \\ \|\nabla^2 u\|_{q,\Omega} &\leq c\|A_q u\|_{q,\Omega}, \qquad 1 < q < \tfrac{3}{2},\ u \in \mathcal{D}(A_q); \end{aligned} \tag{2.14}$$

in each case $c = c(\Omega, q) > 0$. In particular, $\mathcal{D}(A_q^{1/2} u) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ when $1 < q < 3$. Concerning further fractional powers of A_q let $1 < q \leq \gamma < \infty$, $0 \leq \alpha \leq 1$ and $u \in \mathcal{D}(A_q^\alpha)$. Then, by [6], Corollary 4.6 and [17], Corollary 6.7, resp.,

$$\begin{aligned} \|u\|_{\gamma,\Omega} &\leq c\|A_q^\alpha u\|_{q,\Omega}, \qquad 0 \leq \alpha \leq \frac{1}{2}\,, \quad 1 < q < 3\,, \quad 2\alpha + \frac{3}{\gamma} = \frac{3}{q}\,, \\ \|u\|_{\gamma,\Omega} &\leq c\|A_q^\alpha u\|_{q,\Omega}, \qquad 0 \leq \alpha \leq 1\,, \quad 1 < q < \frac{3}{2}\,, \quad 2\alpha + \frac{3}{\gamma} = \frac{3}{q}\,, \end{aligned} \tag{2.15}$$

where $c = c(\Omega, \alpha, q, \gamma) > 0$.

It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{e^{-tA_q} : t \geq 0\}$ on $L^q_\sigma(\Omega)$ satisfying the decay estimate

$$\|A_q^\alpha e^{-tA_q} u\|_{q,\Omega} \leq c t^{-\alpha} \|u\|_{q,\Omega}, \quad t > 0, \tag{2.16}$$

where $\alpha \geq 0$, $1 < q < \infty$ and $c = c(\Omega, q, \alpha) > 0$; see [6], (3.3) or [18], (3.16).

Let $0 < \alpha \leq 1$, $1 < q < \infty$ and consider suitable distributions $d = (d_1, d_2, d_3) \in C_0^\infty(\Omega)'$ for which the term $A_q^{-\alpha} P_q d \in L^q_\sigma(\Omega)$ will be well defined by applying the operations $A_q^{-\alpha}$ and P_q in the corresponding orders to the "test function side". To be more precise, suppose that $\langle d, v\rangle$ is well defined for all $v \in \mathcal{D}(A^\alpha_{q'})$ and satisfies the estimate

$$|\langle d, v\rangle_\Omega| \leq c\|A^\alpha_{q'} v\|_{q',\Omega}. \tag{2.17}$$

Hence there exists $d^* \in L^q_\sigma(\Omega)$ such that

$$\langle d, v\rangle_\Omega = \langle d^*, A^\alpha_{q'} v\rangle_\Omega \quad \text{for all } v \in \mathcal{D}(A^\alpha_{q'}); \tag{2.18}$$

note that d^* is unique, since $\mathcal{R}(A^\alpha_{q'})$ is dense in $L^{q'}_\sigma(\Omega)$. For simplicity we write $d^* = A_q^{-\alpha} P_q d$, since then formally

$$\langle d^*, A^\alpha_{q'} v\rangle_\Omega = \langle A_q^{-\alpha} P_q d, A^\alpha_{q'} v\rangle_\Omega = \langle P_q d, v\rangle_\Omega = \langle d, P_{q'} v\rangle_\Omega = \langle d, v\rangle_\Omega \tag{2.19}$$

giving $A_q^{-\alpha} P_q d$ a generalized meaning. If $d \in C_0^\infty(\Omega)'$ satisfies (2.17), we say that $A_q^{-\alpha} P_q d \in L^q_\sigma(\Omega)$, well defined by d^* in (2.18). For similar operations see [26], III, Lemma 2.6.1.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $\frac{3}{2} < r < 3$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, and let $F = (F_{ij})^3_{i,j=1} \in L^r(\Omega)$. Then $A_q^{-1} P_q \operatorname{div} F \in L^q_\sigma(\Omega)$ and*

$$\|A_q^{-1} P_q \operatorname{div} F\|_{q,\Omega} \leq c\|F\|_{r,\Omega}, \tag{2.20}$$

where $c = c(\Omega, r) > 0$. Hence, $A_q^{-1} P_q \operatorname{div} : L^r(\Omega) \to L^q_\sigma(\Omega)$ is a bounded linear operator.

Proof. Considering (2.17), (2.18) with $d = \operatorname{div} F$, $d^* = A_q^{-1} P_q \operatorname{div} F$ and $\alpha = 1$ we have to estimate the term $\langle \operatorname{div} F, v\rangle_\Omega =: \langle A_q^{-1} P_q \operatorname{div} F, A_{q'} v\rangle_\Omega$ using $\|A_{q'} v\|_{q',\Omega}$ only. Since $\frac{1}{3} + \frac{1}{r'} = \frac{1}{q'}$ where $1 < q' < \frac{3}{2}$, we know that $\mathcal{D}(A_{q'}) \subset \mathcal{D}(A^{1/2}_{r'})$, cf. (2.14), and $A^{1/2}_{r'} v = A^{1/2}_{q'} v \in \mathcal{D}(A^{1/2}_{q'})$ for all $v \in \mathcal{D}(A_{q'})$. Hence $(2.14)_1$ (with r' instead of q) implies for $v \in \mathcal{D}(A_{q'})$

$$|\langle \operatorname{div} F, v\rangle_\Omega| = |-\langle F, \nabla v\rangle_\Omega| \leq c\|F\|_{r,\Omega}\, \|A^{1/2}_{r'} v\|_{r',\Omega}\,.$$

Moreover, by $(2.15)_1$ $\big($with $\alpha = \frac{1}{2}$, $1 + \frac{3}{r'} = \frac{3}{q'}$ and $u = A^{1/2}_{r'} v \in \mathcal{D}(A^{1/2}_{q'})\big)$

$$\|A^{1/2}_{r'} v\|_{r',\Omega} \leq c\|A^{1/2}_{q'} A^{1/2}_{q'} v\|_{q',\Omega} = c\|A_{q'} v\|_{q',\Omega}\,.$$

Now, (2.20) is proved. ■

2.4 The Spaces $L^s(0,T;X)$

Given a Banach space $(X, \|\cdot\|_X)$ and $1 < s < \infty$, let $L^s(0,T;X)$ denote the usual Bochner space with norm $\|\cdot\|_{L^s(0,T;X)} = \big(\int_0^T \|\cdot\|_X^s\,dt)^{1/s}$. If $X = W^{\alpha,q}(\Omega)$ or $X = W^{\alpha,q}(\partial\Omega)$, $1 < q < \infty$, $\alpha \in [-1,1]$, we set $\|\cdot\|_{L^s(0,T;W^{\alpha,q}(\Omega))} = \|\cdot\|_{\alpha;q,s,\Omega,T}$ and $\|\cdot\|_{L^s(0,T;W^{\alpha,q}(\partial\Omega))} = \|\cdot\|_{\alpha;q,s,\partial\Omega,T}$, resp. If $\alpha = 0$, i.e. $X = L^q(\Omega)$ or $L^q(\partial\Omega)$, we simply write $\|\cdot\|_{q,s,\Omega,T}$ or $\|\cdot\|_{q,s,\partial\Omega,T}$, resp. As duality pairing we define

$$\langle f, g\rangle_{\Omega,T} = \int_0^T \langle f, g\rangle_\Omega\,dt, \quad f \in L^s\big(0,T;L^q(\Omega)\big),\ g \in L^{s'}\big(0,T;L^{q'}(\Omega)\big),$$

and analogously $\langle f, g\rangle_{\partial\Omega,T}$ for all $f \in L^s\big(0,T;L^q(\partial\Omega)\big)$, $g \in L^{s'}\big(0,T;L^{q'}(\partial\Omega)\big)$.

We will also need the classical spaces $C^m([0,T);X)$, $m = 0,1,2,\dots$, of X-valued functions $v(t)$ such that $(\frac{d}{dt})^j v(t)$, $0 \le j \le m$, is continuous on $[0,T)$ in X. The space $C_0^1([0,T);X)$ is the subspace of $C^1([0,T);X)$ of function v with compact support in $[0,T)$, and $C_0^1\big((0,T);X\big)$ is that subspace where $\operatorname{supp} v$ is compact in $(0,T)$.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{2,1}$, let $f \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$, $1 < s,\ q < \infty$, and let $v_0 \in L^q_\sigma(\Omega)$ such that $\int_0^\infty \|\nu A_q e^{-\nu t A_q} v_0\|_{q,\Omega}^s\,dt < \infty$. Then the Stokes evolution system*

$$v_t + \nu A_q v = f \quad \text{in } (0,T), \quad v(0) = v_0,$$

has a unique solution $v \in L^s\big(0,T;\mathcal{D}(A_q)\big)$ such that $v_t \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ and $v \in C^0\big([0,T);\ L^q_\sigma(\Omega)\big)$. Moreover, v admits the maximal regularity estimate

$$\|v_t\|_{q,s,\Omega,T} + \|\nu A_q v\|_{q,s,\Omega,T} \le c\Big(\big(\int_0^T \|\nu A_q e^{-\nu t A_q} v_0\|_{q,\Omega}^s\,dt\big)^{1/s} + \|f\|_{q,s,\Omega,T}\Big) \quad (2.21)$$

with $c = c(\Omega, q, s) > 0$ not depending on T, ν, and

$$v(t) = e^{-\nu t A_q} v_0 + \int_0^t e^{-\nu(t-\tau)A_q} f(\tau)\,d\tau, \quad 0 \le t \le T. \quad (2.22)$$

Proof. See [18] (3.15) or [25]. The case $v_0 \neq 0$ is easily reduced to the case $v_0 = 0$ by considering $\hat{v}(t) = v(t) - e^{-\nu t A_q} v_0$. ■

2.5 The Space of Initial Values

Let $1 < q,\ s < \infty$. Then the space of initial values, $\mathcal{J}_\nu^{q,s}(\Omega)$, is defined as a space of distributions on Ω as follows:

$$\mathcal{J}_\nu^{q,s}(\Omega) := \Big\{u_0 \in C_0^\infty(\Omega)' :\ A_q^{-1} P_q u_0 \in L^q_\sigma(\Omega),\ \int_0^\infty \|\nu A_q e^{-\nu t A_q} A_q^{-1} P_q u_0\|_{q,\Omega}^s\,dt < \infty\Big\}$$

equipped with the seminorm

$$\|u_0\|_{\mathcal{J}_\nu^{q,s}(\Omega)} := \nu^{1-1/s}\|A_q^{-1}P_q u_0\|_{q,\Omega} + \Big(\int_0^\infty \|\nu A_q e^{-\nu t A_q} A_q^{-1}P_q u_0\|_{q,\Omega}^s\,dt\Big)^{1/s};$$

here, $A_q^{-1}P_q u_0$ is defined as in (2.17) – (2.19). Obviously $\|\cdot\|_{\mathcal{J}_\nu^{q,s}(\Omega)}$ becomes a norm if we identify $u_0, \hat{u}_0 \in \mathcal{J}_\nu^{q,s}(\Omega)$ when $\|A_q^{-1}P_q(u_0 - \hat{u}_0)\|_{q,\Omega} = 0$, i.e., when $u_0 - \hat{u}_0$ is a gradient field, see (2.18) with $d^* = 0$. Note that $\mathcal{J}_\nu^{q,s}(\Omega)$ can be considered as a real interpolation space, cf. [18], (2.5).

Consider a function $u \in L^s\big(0,T; L^q(\Omega)\big)$ such that $A_q^{-1}P_q u \in L^s\big(0,T; L^q(\Omega)\big)$ is well defined and $(A_q^{-1}P_q u)_t = A_q^{-1}P_q u_t \in L^s\big(0,T; L^q(\Omega)\big)$ holds for its time derivative in the sense of distributions. Then – redefining u on a null set of $[0,T)$ if necessary – we obtain that

$$A_q^{-1}P_q u \in C\big([0,T);\, L^q_\sigma(\Omega)\big), \quad A_q^{-1}P_q u(t) \in L^q_\sigma(\Omega) \quad \text{for all } t \in [0,T); \tag{2.23}$$

in particular, the initial condition $A_q^{-1}P_q u\big|_{t=0} = A_q^{-1}P_q u_0$ in (1.7) is well defined.

3 The Stationary Stokes System, Proof of Theorem 1.6

Given data $f = \operatorname{div} F$, k and g, see (1.12), consider a very weak solution $u \in L^q(\Omega)$ of the stationary Stokes system (1.11), i.e., of (1.13) – (1.14). First we assume $\nu = 1$; the case $\nu \neq 1$ will be an easy consequence when considering $-\Delta u + \nabla(\frac{p}{\nu}) = \frac{1}{\nu} f$. We will prove the unique representation formula

$$u = \hat{F} + \hat{G} + \nabla H \tag{3.1}$$

where $\hat{F}, \hat{G}$ and $\nabla H \in L^q(\Omega)$ solve suitable auxiliary problems and satisfy the estimates

$$\begin{aligned} \|\hat{F}\|_{q,\Omega} &\le c\,\|F\|_{r,\Omega}\,, & (3.2)\\ \|\nabla H\|_{q,\Omega} &\le c\big(\|k\|_{r,\Omega} + \|g\cdot N\|_{-1/q;q,\partial\Omega}\big), & (3.3)\\ \|\hat{G}\|_{q,\Omega} &\le c\big(\|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega}\big). & (3.4) \end{aligned}$$

The first term $\hat{F} := A_q^{-1}P_q \operatorname{div} F$ is well defined by Lemma 2.1 and satisfies (3.2), cf. (2.20). Obviously, cf. Definition 1.5 and (2.18), (2.20), $u_1 = \hat{F}$ is the unique very weak solution of the system

$$\begin{aligned} -\langle u_1, \Delta w\rangle_\Omega &= -\langle F, \nabla w\rangle_\Omega \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}),\\ \operatorname{div} u_1 &= 0 \quad \text{in } \Omega, \quad u_1\big|_{\partial\Omega} = 0. \end{aligned} \tag{3.5}$$

Next we solve the system

$$\begin{aligned} -\langle u_2, \Delta w\rangle_\Omega + \langle g, N\cdot\nabla w\rangle_{\partial\Omega} &= 0 \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}),\\ \operatorname{div} u_2 = 0 \quad \text{in } \Omega, \quad N\cdot u_2\big|_{\partial\Omega} &= 0 \end{aligned} \tag{3.6}$$

matching only the tangential part of g on $\partial\Omega$. To find u_2 we estimate $\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$ as follows: Since $1 < q' < \frac{3}{2}$, $q' < r' < 3$, $1 + \frac{3}{r'} = \frac{3}{q'}$, Poincaré's inequality on the bounded subdomain $\Omega_0 = \Omega \cap B$ and the properties (2.13), $(2.14)_1$, $(2.15)_1$ yield the estimates

$$\|w\|_{q',\Omega_0} \leq c\|\nabla w\|_{q',\Omega_0} \leq c\|\nabla w\|_{r',\Omega} \leq c\|\nabla^2 w\|_{q',\Omega} \leq c\|A_{q'}w\|_{q',\Omega}.$$

Moreover, it holds the inequality

$$\begin{aligned} |\langle g, N\cdot\nabla w\rangle_{\partial\Omega}| &\leq c\|g\|_{-1/q;q,\partial\Omega}\|\nabla w\|_{1/q;q',\partial\Omega} \\ &\leq c\|g\|_{-1/q;q,\partial\Omega}\|w\|_{2;q',\Omega_0} \\ &\leq c\|g\|_{-1/q;q,\partial\Omega}\|A_{q'}w\|_{q',\Omega}\,, \end{aligned}$$

which immediately extends to all $w \in \mathcal{D}(A_{q'})$. Hence for all $v \in \mathcal{R}(A_{q'})$, $v = A_{q'}w$, we get

$$|\langle g, N\cdot\nabla A_{q'}^{-1}v\rangle_{\partial\Omega}| \leq c\|g\|_{-1/q;q,\partial\Omega}\|v\|_{q',\Omega}$$

which extends to all $v \in L^{q'}_\sigma(\Omega)$ since $\mathcal{R}(A_{q'})$ is dense in $L^{q'}_\sigma(\Omega)$. Since $L^{q'}_\sigma(\Omega)' = L^q_\sigma(\Omega)$, there exists a unique $G \in L^q_\sigma(\Omega)$ such that

$$-\langle G, v\rangle_\Omega + \langle g, N\cdot\nabla A_{q'}^{-1}v\rangle_{\partial\Omega} = 0 \quad \text{for all } v \in \mathcal{R}(A_{q'}), \tag{3.7}$$

and $\|G\|_{q,\Omega} \leq c\|g\|_{-1/q;q,\partial\Omega}$. Using the identity $g = (g\cdot N)N + N\times(g\times N)$ and (1.6) we see that g in (3.7) may be replaced by $g - N(g\cdot N)$, and we get that even

$$\|G\|_{q,\Omega} \leq c\|g - N(g\cdot N)\|_{-1/q;q,\partial\Omega} \leq c\|N\times g\|_{-1/q;q,\partial\Omega}. \tag{3.8}$$

Due to (3.7) we conclude with $v = A_{q'}w = -P_{q'}\Delta w$ that $u_2 = G$ is the unique solution of (3.6) in $L^q_\sigma(\Omega)$.

However, G will be modified in the third step in which we look for a very weak solution $u_3 \in L^q(\Omega)$ of the system

$$-\langle u_3, \Delta w\rangle_\Omega = 0 \quad \forall w \in C^2_{0,\sigma}(\overline{\Omega}),\ \operatorname{div} u_3 = k, \quad u_3\cdot N|_{\partial\Omega} = g\cdot N. \tag{3.9}$$

To find the unique solution u_3 of (3.9) we first consider the weak solution ∇H of the Neumann problem

$$\Delta H = k, \quad N\cdot\nabla H|_{\partial\Omega} = N\cdot g. \tag{3.10}$$

To construct ∇H we use the extension $E^h \in L^q(\Omega)$ of $h = N\cdot g$ with $\operatorname{div} E^h \in L^r(\Omega), N\cdot E^h|_{\partial\Omega} = h$ and with compact support in $\overline{\Omega}$, see (2.10). Moreover, cf. (2.9), there exists $b \in L^q(\Omega)$ satisfying $\operatorname{div} b = \operatorname{div} E^h - k$, $b|_{\partial\Omega} = 0$ and $\nabla b \in L^r(\Omega)$. Hence (3.10) may be written in the form

$$\Delta H = \operatorname{div}(E^h - b), \quad N\cdot\big(\nabla H - (E^h - b)\big)|_{\partial\Omega} = 0$$

which, cf. [23], has a unique solution $\nabla H \in L^q(\Omega)$ satisfying

$$\begin{aligned}\|\nabla H\|_{q,\Omega} &\leq c\|E^h - b\|_{q,\Omega}\\ &\leq c\big(\|E^h\|_{q,\Omega} + \|\operatorname{div} E^h - k\|_{r,\Omega}\big)\\ &\leq c\big(\|N\cdot g\|_{-1/q;q,\partial\Omega} + \|k\|_{r,\Omega}\big)\end{aligned}$$

by (2.9), (2.10). This estimate proves (3.3).

To solve (3.9) for u_3 we use the relation

$$\langle \nabla H, \Delta w\rangle_\Omega = \langle \nabla H, N\cdot\nabla w\rangle_{\partial\Omega} \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}) \tag{3.11}$$

which will be proved below. Further, we observe, see (2.3), (2.5), that $\|\nabla H\|_{-1/q;q,\partial\Omega} < \infty$ is well defined, and that $\nabla H_{|_{\partial\Omega}}$ satisfies the same estimates as g in (3.7), (3.8). Therefore, we get, instead of G in (3.7), a unique vector field $G' \in L^q_\sigma(\Omega)$ satisfying

$$-\langle G', \Delta w\rangle_\Omega + \langle \nabla H,\ N\cdot\nabla w\rangle_{\partial\Omega} = 0 \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}) \tag{3.12}$$

and, using (3.8), (2.5),

$$\|G'\|_{q,\Omega} \leq c\|N\times\nabla H\|_{-1/q;q,\partial\Omega} \leq c\|\nabla H\|_{q,\Omega}. \tag{3.13}$$

Now, looking at (3.10) – (3.12), we conclude that $u_3 := \nabla H - G'$ is the unique solution of (3.9).

Summarizing the previous steps, we see that $u = u_1 + u_2 + u_3$ satisfies (1.13), (1.14), u is a very weak solution of system (1.11), and it holds the representation (3.1) with $\hat{G} = G - G'$ satisfying (3.4). Moreover, u depends only on the data F, k, g and satisfies the estimate

$$\|u\|_{q,\Omega} \leq c\big(\|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega}\big)$$

due to (3.2) – (3.4). It is unique, since (3.5) with right-hand side $F = 0$ admits only the trivial solution.

Finally, we prove (3.11). For this purpose, we approximate H by smooth functions (H_j) such that $\|\nabla H - \nabla H_j\|_{q,\Omega} \to 0$ and $\|\nabla H - \nabla H_j\|_{-1/q;q,\partial\Omega} \to 0$ as $j \to \infty$. To find H_j, $j \in \mathbb{N}$, we approximate k and g in (3.10) by smooth functions k_j, g_j, let ∇H_j be the corresponding solutions, and use the estimate (3.3) with $\nabla H, k, g$ replaced by $\nabla H - \nabla H_j, k - k_j, g - g_j$. Then an integration by parts yields for every $w \in C^2_{0,\sigma}(\overline{\Omega})$

$$\begin{aligned}\langle \nabla H_j, \Delta w\rangle_\Omega &= \langle \nabla H_j, N\cdot\nabla w\rangle_{\partial\Omega} - \langle \nabla(\nabla H_j), \nabla w\rangle_\Omega\\ &= \langle \nabla H_j, N\cdot\nabla w\rangle_{\partial\Omega} + \langle \Delta(\nabla H_j), w\rangle_\Omega\\ &= \langle \nabla H_j, N\cdot\nabla w\rangle_{\partial\Omega},\end{aligned}$$

since $\operatorname{div} w = 0$ and $w_{|_{\partial\Omega}} = 0$. As $j \to \infty$ we get (3.11). The general case $\nu \neq 1$ is reduced to $\nu = 1$ by considering $-\Delta u + \nabla\big(\frac{p}{\nu}\big) = \frac{f}{\nu}$ and replacing F, $A_q^{-1}P_q\operatorname{div} F$ by $\frac{F}{\nu}$, $(\nu A_q^{-1})P_q\operatorname{div} F$. This proves Theorem 1.6. ∎

Remark 3.1 (1) The proof of Theorem 1.6 shows that the very weak solution $u \in L^q(\Omega)$ of (1.11) possesses the representation

$$u = (\nu A_q)^{-1} P_q \operatorname{div} F + \hat{G} + \nabla H \tag{3.14}$$

where ∇H is defined by (3.10) and $\hat{G} = G - G'$ satisfies

$$\langle \hat{G}, v \rangle_\Omega = \langle g - \nabla H|_{\partial\Omega},\ N \cdot A_{q'}^{-1} v \rangle_{\partial\Omega} \quad \text{for all } v \in \mathcal{R}(A_{q'}).$$

(2) Let $u \in L^q(\Omega)$ be a very weak solution of (1.11). For $h \in W^{1/q,q'}(\partial\Omega)$ with $N \cdot h = 0$ let $\hat{w}^h \in \mathcal{D}(A_{q'})$ with $N \cdot \nabla \hat{w}^h|_{\partial\Omega} = h$ be the extension of h considered in (2.12). Using $\hat{w}^h$ as test function in (1.13) we get that

$$\nu \langle g, h \rangle_{\partial\Omega} = \nu \langle u, \Delta \hat{w}^h \rangle_\Omega - \langle F, \nabla \hat{w}^h \rangle_\Omega,$$

where $\langle g, h \rangle_\Omega$ equals $\langle N \times g, N \times h \rangle_{\partial\Omega}$, since $g = (g \cdot N)N + N \times (g \times N)$. Hence, in the sense of a boundary distribution, the tangential component $N \times u|_{\partial\Omega}$ is well-defined by

$$\nu \langle N \times u, N \times h \rangle_{\partial\Omega} = \nu \langle u, \Delta \hat{w}^h \rangle_\Omega - \langle F, \nabla \hat{w}^h \rangle_\Omega. \tag{3.15}$$

On the other hand, using the extension $E^h \in W^{1,q'}(\Omega)$ with compact support of an arbitrary function $h \in W^{1,q,q'}(\partial\Omega)$, (2.1) yields the identity

$$\langle N \cdot u|_{\partial\Omega}, h \rangle_{\partial\Omega} = \langle k, E^h \rangle_\Omega + \langle u, \nabla E^h \rangle_\Omega. \tag{3.16}$$

Therefore, (3.15) - (3.16) yield an explicit expression of the trace $u|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$. Thus we can define a well defined trace $u|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ – beyond the usual trace theorems – for each $u \in L^q(\Omega)$ satisfying the relations (1.13), (1.14).

(3) Using test functions $w \in C_{0,\sigma}^\infty(\Omega)$ in (1.13) de Rham's argument yields a distribution p such that

$$-\nu \Delta u + \nabla p = f$$

in the sense of distributions.

4 Nonstationary Stokes Systems, Proof of Theorem 1.4

Given data $f = \operatorname{div} F, k, g$ and u_0 as in (1.2) with $1 < s < \infty$, $3 < q < \infty$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, and let $u \in L^s\big(0, T; L^q(\Omega)\big)$ be a very weak solution of the instationary Stokes system (1.7), see Definition 1.3. First we assume that $\nu = 1$, the general case $\nu \neq 1$ will be reduced to $\nu = 1$ by a scaling transformation concerning t.

Let $E(t) = E^{k(t),g(t)}$ be the very weak solution of the stationary Stokes system

$$-\Delta E(t) + \nabla p(t) = 0, \quad \operatorname{div} E(t) = k(t), \quad E(t)\big|_{\partial\Omega} = g(t) \tag{4.1}$$

for a.a. $t \in (0,T)$. By Theorem 1.6, $E \in L^s\big(0,T;L^q(\Omega)\big)$ and

$$\|E\|_{q,s,\Omega,T} \le c\big(\|k\|_{r,s,\Omega,T} + \|g\|_{-1/q;q,s,\partial\Omega,T}\big). \tag{4.2}$$

Moreover, let $\nabla H \in L^s\big(0,T;L^q(\Omega)\big)$ be defined by $\nabla H(t) = u(t) - P_q u(t)$ for a.a. $t \in (0,T)$, i.e., $\nabla H(t)$ is the weak solution of the Neumann problem

$$\Delta H(t) = k(t), \quad N \cdot \nabla H(t)\big|_{\partial\Omega} = N \cdot g(t). \tag{4.3}$$

Note that, cf. (3.3), (3.10),

$$\|\nabla H\|_{q,s,\Omega,T} \le c\big(\|k\|_{r,s,\Omega,T} + \|g\|_{-1/q;q,s,\partial\Omega,T}\big). \tag{4.4}$$

Lemma 4.1 *Consider $f = \operatorname{div} F, k, g, u_0$ as in (1.2) with $1 < s < \infty$, $3 < q < \infty$ and $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, E as in (4.1), (4.2), and a very weak solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ of (1.7). Then the well defined representation formula*

$$u(t) = \nabla H(t) + A_q e^{-tA_q} A_q^{-1} P_q u_0 + \int_0^t A_q e^{-(t-\tau)A_q}\big(A_q^{-1} P_q \operatorname{div} F + P_q E\big)\, d\tau \tag{4.5}$$

holds for a.a. $t \in (0,T)$.

Proof. Consider the test function $w \in C_0^1\big([0,T);C^2_{0,\sigma}(\overline{\Omega})\big)$ and let $v = \tilde{A}_{q'} w$ where $\tilde{A}_q = A_{q'} + I$. It is well-known, see [17], [18], [25], that $\tilde{A}_{q'}^{-1}$ and $A_{q'}\tilde{A}_{q'}^{-1}$ are bounded operators on $L^q_\sigma(\Omega)$. By the weak formulation (1.13) of (4.1) we get that $\langle E(t), \Delta w(t)\rangle_\Omega = \langle g(t),\, N \cdot \nabla w(t)\rangle_{\partial\Omega}$ for a.a. $t \in (0,T)$ yielding

$$\langle g,\, N \cdot \nabla w\rangle_{\partial\Omega,T} = \langle E, \Delta w\rangle_{\Omega,T}.$$

Then the weak formulation (1.8), using $w \in C_0^1\big([0,T);C^2_{0,\sigma}(\overline{\Omega})\big)$, $v = \tilde{A}_{q'} w$, implies the identity

$$-\langle \tilde{A}_q^{-1} P_q u, v_t\rangle_{\Omega,T} - \langle u - E,\, \Delta \tilde{A}_{q'}^{-1} v\rangle_{\Omega,T} = \langle u_0, \tilde{A}_{q'}^{-1} v(0)\rangle_\Omega - \langle F, \nabla \tilde{A}_{q'}^{-1} v\rangle_{\Omega,T}\,. \tag{4.6}$$

Since $u(t) - E(t) \in L^q_\sigma(\Omega)$ for a.a. $t \in (0,T)$, the second term on the left hand side will be rewritten as

$$\langle u - E,\, (-A_{q'})\tilde{A}_{q'}^{-1} v\rangle_{\Omega,T} = -\langle A_q \tilde{A}_q^{-1} P_q(u - E), v\rangle_{\Omega,T}\,.$$

Moreover, the terms on the right-hand side equal

$$\langle A_q \tilde{A}_q^{-1}(A_q^{-1} P_q u_0), v(0)\rangle_\Omega \quad \text{and} \quad \langle A_q \tilde{A}_q^{-1}(A_q^{-1} P_q \operatorname{div} F), v\rangle_{\Omega,T}\,, \text{ resp.,}$$

where $A_q^{-1}P_q \operatorname{div} F \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$, cf. Lemma 2.1, and $A_q^{-1}P_q u_0 \in L^q_\sigma(\Omega)$, see Subsection 2.5, are well defined. Hence we get from (4.6) the relation

$$\begin{aligned}
&-\langle \tilde A_q^{-1}P_q u, v_t\rangle_{\Omega,T} + \langle A_q(\tilde A_q^{-1}P_q u), v\rangle_{\Omega,T} \qquad (4.7)\\
&\quad = \langle A_q\tilde A_q^{-1}(A_q^{-1}P_q u_0), v(0)\rangle_\Omega + \langle A_q\tilde A_q^{-1}P_q E, v\rangle_{\Omega,T} + \langle A_q\tilde A_q^{-1}A_q^{-1}P_q \operatorname{div} F, v\rangle_{\Omega,T}.
\end{aligned}$$

Then a standard argument, see [27], III 1.1 or [26], IV 1.3, shows that $U(t) = \tilde A_q^{-1}P_q u(t)$ is a strong solution of the instationary Stokes system

$$\begin{aligned}
U_t + A_q U &= A_q\tilde A_q^{-1}(A_q^{-1}P_q \operatorname{div} F + P_q E)\\
U(0) &= A_q\tilde A_q^{-1}(A_q^{-1}P_q u_0).
\end{aligned}$$

Since the right-hand side is contained in $L^s\big(0,T;L^q_\sigma(\Omega)\big)$ and in $L^q_\sigma(\Omega)$, resp., Lemma 2.2 yields $U_t, A_q U \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ and the representation

$$U(t) = A_q\tilde A_q^{-1}e^{-tA_q}(A_q^{-1}P_q u_0) + \int_0^t e^{-(t-\tau)A_q}A_q\tilde A_q^{-1}(A_q^{-1}P_q \operatorname{div} F + P_q E)\,d\tau\,.$$

We may apply $\tilde A_q$ to both sides of this identity to obtain that

$$P_q u(t) = A_q e^{-tA_q}A_q^{-1}P_q u_0 + \int_0^t A_q e^{-(t-\tau)A_q}(A_q^{-1}P_q \operatorname{div} F + P_q E)\,d\tau \qquad (4.8)$$

for a.a. $t \in (0,T)$. Since $u(t) = \nabla H(t) + P_q u(t)$, (4.5) is proved. ■

Given data $f = \operatorname{div} F, k, g$ and u_0, let u be defined by (4.5). Proceeding as in the proof of Lemma 4.1 we get that u is a very weak solution of (1.7).

The right-hand side of (4.8) is contained in $\mathcal{R}(A_q)$ for a.a. $t \in (0,T)$. Therefore $A_q^{-1}P_q u(t)$ is well-defined and it holds

$$A_q^{-1}P_q u(t) = e^{-tA_q}A_q^{-1}P_q u_0 + \int_0^t e^{-(t-\tau)A_q}(A_q^{-1}P_q \operatorname{div} F + P_q E)\,d\tau\,. \qquad (4.9)$$

This identity has the form (2.22) with $v_0 = A_q^{-1}P_q u_0 \in L^q_\sigma(\Omega)$ and with f replaced $A_q^{-1}P_q \operatorname{div} F + P_q E \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$. By the maximal regularity estimate (2.21) we get using (2.20) and (4.2) that

$$\begin{aligned}
&\|A_q^{-1}P_q u_t\|_{q,s,\Omega,T} + \|P_q u\|_{q,s,\Omega,T}\\
&\quad\le c\Big(\Big(\int_0^T \|A_q e^{-tA_q}A_q^{-1}P_q u_0\|_q^s dt\Big)^{1/s} + \|A_q^{-1}P_q \operatorname{div} F\|_{q,s,\Omega,T} + \|E\|_{q,s,\Omega,T}\Big)\\
&\quad\le c\big(\|u_0\|_{\mathcal{J}^{q,s}_\nu(\Omega)} + \|F\|_{r,s,\Omega,T} + \|k\|_{r,s,\Omega,T} + \|g\|_{-1/q;q,s,\partial\Omega,T}\big).
\end{aligned}$$

Thus (1.10) is proved when $\nu = 1$. A scaling argument replacing (1.7) by the system

$$\tilde u_\tau - \Delta\tilde u + \nabla\tilde p = \tilde f, \quad \operatorname{div}\tilde u = \tilde k, \quad \tilde u_{|\partial\Omega} = \tilde g, \quad \tilde u(0) = \tilde u_0\,,$$

with $\tilde{u}(\tau) = u(t)$, $\tilde{k}(\tau) = k(t)$, $\tilde{g}(\tau) = g(t)$, $\tilde{p}(\tau) = \frac{1}{\nu}p(t)$, $\tilde{f}(\tau) = \frac{1}{\nu}f(t)$ and $\tau = \nu t$ will yield (1.10) when $\nu \neq 1$. Moreover, $A_q^{-1}P_q u \in C\big([0,T); L^q_\sigma(\Omega)\big)$ and $A_q^{-1}P_q u(0) = A_q^{-1}P_q u_0$, cf. (4.9). This completes the proof. ■

Remark 4.2 (1) Let $u \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$ be a very weak solution as in Theorem 1.4. Then, using test functions $w \in C_0^\infty\big((0,T); C^\infty_{0,\sigma}(\Omega)\big)$ in Definition 1.3, we get the existence of a distribution p such that

$$u_t - \nu\Delta u + \nabla p = f \quad \text{in } (0,T)\times\Omega$$

in the sense of distributions, cf. [26], [27].

(2) Let $u \in L^s\big(0,T; L^q(\Omega)\big)$ be a very weak solution as in Theorem 1.4. Given $h \in C_0^1\big((0,T);\, W^{1/q,q'}(\partial\Omega)\big)$ with $N\cdot h = 0$ we find an extension $\hat{w}^h(t) := \hat{w}^{h(t)} \in C_0^1\big((0,T);\, \mathcal{D}(A_{q'})\big)$ with $N\cdot\nabla\hat{w}^h(t)_{|_{\partial\Omega}} = h$, cf. (2.12). Then $h \mapsto \hat{w}^h$ is a linear mapping with $(\hat{w}^h)_t = \hat{w}^{h_t}$. Using $\hat{w}^h$ as test function in (1.8) a calculation as in Remark 3.1(2) yields the formula

$$\nu\langle N\times g,\, N\times h\rangle_{\partial\Omega,T} = \langle u, \hat{w}^{h_t}\rangle_{\Omega,T} + \nu\langle u, \Delta\hat{w}^h\rangle_{\Omega,T} - \langle F, \nabla\hat{w}^h\rangle_{\Omega,T}\,. \quad (4.10)$$

Since $\langle N\times g,\, N\times h\rangle_{\partial\Omega,T} = \langle N\times u_{|_{\partial\Omega}},\, N\times h\rangle_{\partial\Omega,T}$ for smooth u, the right hand side of (4.10) yields a definition of the boundary distribution $N\times u(t)_{|_{\partial\Omega}}$, the tangential part of $u_{|_{\partial\Omega}}$. Analogously to Remark 3.1(2) also the normal component $N\cdot u(t)_{|_{\partial\Omega}}$ is well-defined, cf. (3.16) and (1.5). Therefore, the general trace property $u_{|_{\partial\Omega}} = g$ in (1.7) is well defined beyond the usual trace theorem.

(3) As already mentioned, the property $A_q^{-1}P_q u_t \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$ yields $A_q^{-1}P_q u \in C^0\big([0,T); L^q_\sigma(\Omega)\big)$ and implies that the initial value u_0 of u is well-defined in the sense $A_q^{-1}P_q u_{|_{t=0}} = A_q^{-1}P_q u_0$. According to the definition of $\mathcal{J}^{q,s}_\nu(\Omega)$ the initial condition implies that $u(0)$ coincides with u_0 only up to a gradient. This is obvious from the variational formulation (1.8) since $w(0) \in C^2_{0,\sigma}(\overline{\Omega})$ is solenoidal.

5 The Navier-Stokes System, Proof of Theorem 1.2

Given data $f = \operatorname{div} F, k, g$ and u_0 as in (1.2), (1.3) let $u \in L^s\big(0,T; L^q(\Omega)\big)$ be a very weak solution of the nonstationary Navier-Stokes system (1.1). Further let $E = E^{f,k,g,u_0}$ be the very weak solution of the instationary nonhomogeneous Stokes system

$$\begin{aligned} E_t - \nu\Delta E + \nabla h &= f, & \operatorname{div} E &= k \quad \text{in } (0,T)\times\Omega \\ E_{|_{\partial\Omega}} &= g, & E_{|_{t=0}} &= u_0 \end{aligned} \quad (5.1)$$

such that $A_q^{-1}P_qE_t \in L^s\big(0,T;L^q(\Omega)\big)$ and

$$\|\nu E\|_{q,s,\Omega,T} \;\leq\; c\Bigg(\Big(\int_0^T \|\nu A_q e^{-\nu t A_q} A_q^{-1}P_q u_0\|_q^s\,dt\Big)^{1/s} + \|F\|_{r,s,\Omega,T} + \|\nu k\|_{r,s,\Omega,T} + \|\nu g\|_{-1/q;q,s,\partial\Omega,T}\Bigg), \tag{5.2}$$

see Theorem 1.4. Then the variational formulations (1.4) for u and (1.8) for E imply that $\hat u = u - E$ satisfies $\operatorname{div}\hat u = 0$, $N\cdot \hat u_{|_{\partial\Omega}} = 0$ and

$$-\langle \hat u, w_t\rangle_{\Omega,T} - \nu\langle \hat u, \Delta w\rangle_{\Omega,T} = \langle u\otimes u, \nabla w\rangle_{\Omega,T} + \langle ku, w\rangle_{\Omega,T} \tag{5.3}$$

for all $w \in C^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$. This nonlinear problem will be rewritten as a nonlinear integral equation in $\hat u$ which is the starting point to find a solution $\hat u$ by Banach's Fixed Theorem. For this purpose, we analyze the term $A_q^{-\alpha}P_q(u\cdot\nabla u)$ where $\alpha = \frac{3}{2q}+\frac12 < 1$.

Lemma 5.1 (1) *Let $u \in L^q(\Omega)$ such that $k = \operatorname{div} u \in L^r(\Omega)$ where $\frac13 + \frac1q = \frac1r$ and $3 < q < \infty$. Then for $\alpha = \frac{3}{2q}+\frac12 < 1$*

$$\|A^{-\alpha}P_q u\cdot\nabla u\|_{q,\Omega} \leq c\big(\|u\|^2_{q,\Omega} + \|k\|_{r,\Omega}\,\|u\|_{q,\Omega}\big) \tag{5.4}$$

with $c = c(\Omega,q) > 0$.

(2) *Let $u \in L^s\big(0,T,L^q(\Omega)\big)$ such that $k = \operatorname{div} u \in L^s\big(0,T;L^r(\Omega)\big)$ where r,s,q satisfy* (1.3). *Then for $\alpha = \frac{3}{2q}+\frac12 < 1$*

$$\|A^{-\alpha}P_q u\cdot\nabla u\|_{q,s/2,\Omega,T} \leq c\big(\|u\|^2_{q,s,\Omega,T} + \|k\|_{r,s,\Omega,T}\,\|u\|_{q,s,\Omega,T}\big) \tag{5.5}$$

with $c = c(\Omega,q) > 0$.

Proof.

(1) For an arbitrary test function $v \in \mathcal{D}(A^\alpha_{q'})$ we have to estimate $\langle u\cdot\nabla u, v\rangle = -\langle u\otimes u,\nabla v\rangle - \langle ku, v\rangle$ by $\|A^\alpha_{q'}v\|_{q',\Omega}$. By Hölder's inequality, $(2.14)_1$ and $(2.15)_1$ (with $\alpha' = \frac{3}{2q}$ instead of α, $2\alpha' + \frac{3}{(q/2)'} = \frac{3}{q'}$, applied to $A^{1/2}_{(q/2)'}v$) we get that

$$\begin{aligned} |\langle u\otimes u,\nabla v\rangle_\Omega| &\leq \|u\otimes u\|_{q/2,\Omega}\,\|\nabla v\|_{(q/2)',\Omega} \\ &\leq c\|u\|^2_{q,\Omega}\,\|A^{1/2}_{(q/2)'}v\|_{(q/2)',\Omega} \\ &\leq c\|u\|^2_{q,\Omega}\,\|A^\alpha_{q'}v\|_{q',\Omega}. \end{aligned}$$

Moreover, by $(2.15)_2$ (with $2\alpha + \frac{3}{\gamma'} = \frac{3}{q'}$ where $\gamma = \big(\frac1r + \frac1q\big)^{-1}$),

$$\begin{aligned} |\langle ku, v\rangle_\Omega| &\leq \|k\|_{r,\Omega}\,\|u\|_{q,\Omega}\,\|v\|_{\gamma',\Omega} \\ &\leq c\|k\|_{r,\Omega}\,\|u\|_{q,\Omega}\,\|A^\alpha_{q'}v\|_{q',\Omega}. \end{aligned}$$

Combining the previous inequalities we get (5.4).

(2) Using (5.4) for a.a. $t \in (0,T)$ and integrating its $\frac{s}{2}$-power on $(0,T)$ we prove (5.5). This proves Lemma 5.1 ■

To prove Theorem 1.2 we consider $w \in C^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$ in (5.3) and let $v = \tilde{A}_{q'} w$ where $\tilde{A}_q = A_q + I$. Then the calculation which led from (4.6) to (4.9) (with $u_0 = 0$, $E = 0$ and $A_q^{-1} P_q \operatorname{div} F \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$ replaced by $-A_q^{-\alpha} P_q(u \cdot \nabla u)$) yields the identity

$$-\langle \tilde{A}_q^{-1} P_q \hat{u}, v_t\rangle_{\Omega,T} + \nu \langle A_q \tilde{A}_q^{-1} P_q \hat{u}, v\rangle_{\Omega,T} = -\langle A_q^\alpha \tilde{A}_q^{-1} A_q^{-\alpha} P_q(u\cdot\nabla u), v\rangle_{\Omega,T}$$

and the representation formula

$$\tilde{A}_q^{-1} P_q \hat{u}(t) = -\int_0^t e^{-\nu(t-\tau)A_q}\, A_q^\alpha \tilde{A}_q^{-1} A_q^{-\alpha} P_q(u\cdot\nabla u)\, d\tau\,.$$

Since $\hat{u}(t) \in L^q_\sigma(\Omega)$ and $A_q^{-\alpha} P_q(u\cdot\nabla u) \in L^{s/2}\big(0,T; L^q_\sigma(\Omega)\big)$ we may apply $\tilde{A}_q$ to get that

$$\hat{u}(t) = -A_q^\alpha \int_0^t e^{-\nu(t-\tau)A_q}\, A_q^{-\alpha} P_q(u\cdot\nabla u)\, d\tau, \quad 0 \le t < T. \tag{5.6}$$

Moreover, we conclude from Lemma 2.2 that

$$\begin{aligned} &\hat{u}(t) \in \mathcal{R}(A_q^\alpha), \quad A_q^{-\alpha}\hat{u}(t) \in L^q_\sigma(\Omega) \quad \text{for all } t \in [0,T),\\ &A_q^{-\alpha}\hat{u}_t \in L^{s/2}\big(0,T; L^q_\sigma(\Omega)\big), \quad A_q^{-\alpha}\hat{u} \in C\big([0,T); L^q_\sigma(\Omega)\big),\\ &A_q^{1-\alpha}\hat{u} \in L^{s/2}\big(0,T; L^q_\sigma(\Omega)\big). \end{aligned} \tag{5.7}$$

Finally, the initial value $A_q^{-\alpha}\hat{u}(0) = 0$ is well-defined.

To construct a very weak solution $u = \hat{u} + E$ on some interval $[0,T')$, $0 < T' \le T$, we write (5.6) as the fixed point equation

$$\hat{u} = \mathcal{F}(\hat{u})$$

where

$$\mathcal{F}(\hat{u}) = -\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q}\, A_q^{-\alpha} P_q\big((\hat{u}+E)\cdot\nabla(\hat{u}+E)\big)\, d\tau\,. \tag{5.8}$$

For the application of Banach's Fixed Point Theorem we have to estimate $\mathcal{F}(\hat{u})$ in $L^s\big(0,T; L^q(\Omega)\big)$.

By (2.16)

$$\|\mathcal{F}(\hat{u})(t)\|_{q,\Omega} \le c\nu^{-\alpha} \int_0^t (t-\tau)^{-\alpha}\, \|A_q^{-\alpha} P_q\big((\hat{u}+E)\cdot\nabla(\hat{u}+E)\big)\|_{q,\Omega}\, d\tau.$$

Then the Hardy-Littlewood inequality ([26], [28], with $(1-\alpha) + \frac{1}{s} = \frac{1}{s/2}$, i.e., $\alpha = \frac{1}{s'}$) and Lemma 5.1 imply that

$$\begin{aligned} \|\mathcal{F}(\hat{u})\|_{q,s,\Omega,T} &\le c\nu^{-\alpha} \|A_q^{-\alpha} P_q\big((\hat{u}+E)\cdot\nabla(\hat{u}+E)\big)\|_{q,s/2,\Omega,T} \\ &\le c_1 \nu^{-\alpha}\big[\big(\|\hat{u}\|_{q,s,\Omega,T} + \|E\|_{q,s,\Omega,T}\big)^2 + \|k\|_{r,s,\Omega,T}\,\big(\|\hat{u}\|_{q,s,\Omega,T} + \|E\|_{q,s,\Omega,T}\big)\big]. \end{aligned} \tag{5.9}$$

To control the interval of existence $[0,T')$ let

$$A = c_1\nu^{-\alpha}, \quad B = B(T') = \|E\|_{q,s,\Omega,T'}, \quad C = C(T') = c_1\nu^{-\alpha}\|k\|_{r,s,\Omega,T'}$$

for $T' \in (0,T]$ to be chosen below. Hence, replacing T by T' in (5.9), we get that

$$\|\mathcal{F}(\hat{u})\|_{q,s,\Omega,T'} + B \le A\big(\|\hat{u}\|_{q,s,\Omega,T'} + B\big)^2 + C\big(\|\hat{u}\|_{q,s,\Omega,T'} + B\big) + B. \tag{5.10}$$

Consider the closed ball $\mathcal{B} = \{\hat{u} \in L^s\big(0,T'; L^q_\sigma(\Omega)\big) : \|\hat{u}\|_{q,s,\Omega,T'} + B \le y_1\}$ in $L^s\big(0,T'; L^q_\sigma(\Omega)\big)$ where $y_1 > B$ is the smallest positive root of the quadratic equation $y = Ay^2 + Cy + B$. Assuming

$$4AB + 2C < 1 \tag{5.11}$$

we get $y_1 = 2B\big(1 - C + \sqrt{1 + C^2 - (4AB + 2C)}\big)^{-1}$. The smallness condition (5.11) is satisfied if

$$\|E\|_{q,s,\Omega,T'} + \|k\|_{r,s,\Omega,T'} < \frac{1}{4c_1}\,\nu^\alpha,$$

or, due to (5.2), if

$$\begin{aligned} &\Big(\int_0^{T'} \|\nu A_q e^{-\nu t A_q}(A_q^{-1}P_q u_0)\|_q^s \, d\tau\Big)^{1/s} \\ &\quad + \|F\|_{r,s,\Omega,T'} + \|\nu k\|_{r,s,\Omega,T'} + \|\nu g\|_{-1/q;q,s,\partial\Omega,T'} < c\nu^{1+\alpha} \end{aligned} \tag{5.12}$$

where $c = c(\Omega, q) > 0$ is independent of the data and of T', ν. Obviously (5.12) is satisfied for a sufficiently small $T' = T'(f,k,g,u_0,\nu) \in (0,T]$. In particular, the interval of existence $(0,T')$ may be infinite.

The conditions (5.10), (5.11) or (5.12) imply that $\mathcal{F}$ maps the closed ball $\mathcal{B}$ into itself. For $\hat{u}, \hat{v} \in \mathcal{B}$ we similarly obtain that

$$\begin{aligned} \|\mathcal{F}(\hat{u}) - \mathcal{F}(\hat{v})\|_{q,s,\Omega,T'} &\le A\big(\|\hat{u}\|_{q,s,\Omega,T'} + \|\hat{v}\|_{q,s,\Omega,T'} + 2B\big)\,\|\hat{u} - \hat{v}\|_{q,s,\Omega,T'} \\ &\le 2Ay_1\|\hat{u} - \hat{v}\|_{q,s,\Omega,T'}\,. \end{aligned}$$

Since by (5.11) y_1 is shown to be less than $2B$, (5.11) proves that $\mathcal{F}$ is a strict contraction in $\mathcal{B}$. Then Banach's Fixed Point Theorem yields the existence of a fixed point $\hat{u}$ of $\mathcal{F}$ unique in $\mathcal{B}$. Finally we obtain that $u = \hat{u} + E$ is a very weak solution of (1.1).

It remains to prove the uniqueness within the class of *all* very weak solutions of (1.1) on $(0,T')$. In addition to u let $v \in L^s\big(0,T; L^q(\Omega)\big)$ be a very weak solution of (1.1). Then $\hat{v} = v - E$ has the representation (5.6) with $u, \hat{u}$ replaced by $v, \hat{v}$. Therefore, for $U = \hat{u} - \hat{v}$,

$$U(t) = -\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q\big(U \cdot \nabla(\hat{u} + E) + (\hat{v} + E)\cdot\nabla U\big)\,d\tau, \quad 0 \le t < T'.$$

The same estimate as for $\mathcal{F}(\hat{u})$ in (5.9) leads to the inequality

$$\|U\|_{q,s,\Omega,T''} \le c\big(\|u\|_{q,s,\Omega,T''} + \|v\|_{q,s,\Omega,T''} + \|k\|_{r,s,\Omega,T''}\big)\,\|U\|_{q,s,\Omega,T''} \tag{5.13}$$

where $c = c(\Omega,\nu,q) > 0$ is independent of $T'' \in (0,T']$. Hence we may choose $T'' \in (0,T']$ such that the term in front of $\|U\|_{q,s,\Omega,T''}$ on the right-hand side of (5.13) is less than 1. This choice of T'' yields $U = 0$ and consequently $u = v$ on $[0,T'')$. If $T'' < T'$, we may repeat this procedure finitely many times to get $u = v$ on $[0,T')$ if $T' < \infty$. For $T' = \infty$ we get $u = v$ on $[0,T')$ for every $T' < \infty$, i.e., $u = v$ on $[0,\infty)$. This completes the proof of Theorem 1.2. ■

Remark 5.2 (1) Let $u \in L^s\big(0,T;L^q(\Omega)\big)$ be a very weak solution of the Navier-Stokes system (1.1). Then as in the linear case, cf. Remark 4.2(1), there exists a distribution p on $(0,T)\times\Omega$ such that

$$u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = k$$

in the sense of distributions.

(2) For each very weak solution u of (1.1) there exists an explicit trace formula for $u|_{\partial\Omega}$ analogously to Remark 4.2(2). Thus $u|_{\partial\Omega} = g$ is well-defined in the sense of distributions on $\partial\Omega$.

(3) Each very weak solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ of (1.1) has the unique decomposition

$$u = \hat{u} + E \quad \text{with} \quad \hat{u}, E \in L^s\big(0,T;L^q(\Omega)\big),$$

where $E = E^{f,k,g,u_0}$ is defined by (5.1) and the "perturbation" $\hat{u}$ is a very weak solution of the "homogeneous" system

$$\begin{aligned}
\hat{u}_t - \nu\Delta\hat{u} + (\hat{u}+E)\cdot\nabla(\hat{u}+E) + \nabla\hat{h} &= 0, \quad & \operatorname{div}\hat{u} &= 0 \quad \text{in } (0,T)\times\Omega,\\
\hat{u}|_{t=0} &= 0, & \hat{u}|_{\partial\Omega} &= 0,
\end{aligned}$$

leading to the variational formulation

$$-\langle\hat{u}, w_t\rangle_{\Omega,T} - \nu\langle\hat{u},\Delta w\rangle_{\Omega,T} = \langle(\hat{u}+E)\otimes(\hat{u}+E),\nabla w\rangle_{\Omega,T} + \langle k(\hat{u}+E), w\rangle_{\Omega,T}$$

for all $w \in C_0^1\big([0,T);C^2_{0,\sigma}(\overline{\Omega})\big)$. The unique solution $\hat{u}$ has the regularity properties (5.7). Finally, since $A_q^{-1}P_qE_t \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$,

$$A_q^{-1}P_qE|_{t=0} = A_q^{-1}P_q u_0 \quad \text{and} \quad A_q^{-\alpha}P_q\hat{u}|_{t=0} = 0$$

yielding a precise formulation for $u(0) = u_0$ in (1.1).

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