A Martingale Characterization of Pólya-Lundberg Processes

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Abstract

We study exponential families within the class of counting processes and show that a mixed Poisson process belongs to an exponential family if and only if it is either a Poisson process or it has a Gamma structure distribution. This property can be expressed via exponential martingales.

1 Introduction

Since mixed Poisson processes were introduced as a generalization of homogeneous Poisson processes they have been intensively studied. A detailed survey of the developed theory and obtained results is given by the monograph "Mixed Poisson Processes" by J. Grandell [Gra].

An important question is how mixed Poisson processes can be characterized within more general classes of processes. A well known result in this context is the characterization of mixed Poisson processes within the class of general point processes via the conditional uniformity of its occurence times by K. Nawrotzki [Naw]. Some recent articles were published by Y. Hayakawa [Hay], B. Grigelionis [Gri] and D. Pfeifer and U. Heller [PH]. While the first article proves a characterization within general point processes via normalised event occurence times, the latter two characterize mixed Poisson processes within the class of birth processes via martingales involving transistion intensities.

The present article, however, does not deal with characterizations of mixed Poisson processes within more general classes of processes but proves a characterization of Pólya-Lundberg processes within the class of mixed Poisson processes. Pólya-Lundberg processes, i.e., mixed Poisson processes whose structure distribution are Gamma distributions, were of special interest ever since mixed Poisson processes were studied. Not only that they seem to be appropriate to model the number of occurences of certain events in applications but they also are probably the easiest to treat analytically.

The characterization given in this article underlines the special role of these processes. The characterizing property is an exponential martingale property which will be deduced from studies of exponential families of stochastic processes. An overview of this topic is given in the monograph "Exponential Families of Stochastic Processes" by U. Küchler and M. Sørensen [KS].

The article is organized as follows: In section 2 we first concentrate on the definition of such exponential families. We follow a concept proposed by I. Küchler and U. Küchler [KK], where exponential families are introduced as equivalence classes with at least two elements of an equivalence relation which is defined on a set of probability measures on a filtered measurable space. Then, we study exponential families of mixed Poisson processes and we determine all existing exponential families. It turns out that these are only the family of Poisson processes and families of Pólya-Lundberg processes. Finally, in section 3 we deduce the main theorem, which states that Pólya-Lundberg processes can be characterized within the class of all mixed Poisson processes by exponential martingales.

2 Exponential Families of Mixed Poisson Processes

Consider the following canonical model. Let Ω be the space of all simple counting functions $\omega : T \to \mathbb{N}$ and for $\mathcal{F}$ the σ-algebra generated by all cylindric sets. Furthermore, we consider the canonical process $X_T, T = [0, \infty)$, with $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in T$ and the natural filtration $\{\mathcal{F}_t\}_{t \in T}$ generated by X_T. For $\mathcal{P}$ we choose the set of all probability measures on $(\Omega, \mathcal{F})$. For a measure $P \in \mathcal{P}$ we denote by P_t, $t \in T$, the restriction of P to $\mathcal{F}_t$.
Since we consider a canonical model, in the sequel, we will sometimes refer to a measure $P \in \mathcal{P}$ as the corresponding process.
In order to define exponential families we follow a concept proposed by I. Küchler and U. Küchler in [KK]. The definition is based on the following relation on $\mathcal{P}$:

Definition 1: *We say that two measures $P, Q \in \mathcal{P}$ are in relation, in symbols $P \sim Q$, if for every $t \in T$ the measure Q_t is absolutely continuous with respect to P_t and if there are functions $c, d : T \to [0, \infty)$ such that the Radon-Nikodym derivative dQ_t/dP_t satisfies*

$$dQ_t/dP_t = e^{c(t)X_t + d(t)} \qquad P_t - a.s. \tag{1}$$

This relation is an equivalence relation with the help of which we can now define exponential families by

Definition 2: *An equivalence class of $\sim$ with at least two elements is called an exponential family.*

This nonparametric approach to exponential families which are usually defined as parametric families of measures has overall two advantages: the independence of any parametrization and the more general mathematical structuring.

Consider now the set $\mathcal{M} \subset \mathcal{P}$ which consists of all mixed Poisson processes. Recall that under $P \in \mathcal{M}$ the process X_T is called a mixed Poisson process if its distribution P_{X_T} satisfies

$$P_{X_T}(A) = \int_0^\infty P^\lambda(A)\, dU(\lambda), \quad A \in \mathcal{F},$$

where P^λ describes the distribution of a Poisson process with intensity λ and where U is a distribution concentrated on $[0, \infty)$. The distribution U is called structure distribution

of the mixed Poisson process. If U is a Gamma distribution with scale parameter $\varphi > 0$ and shape parameter γ, $\Gamma(\varphi, \gamma)$, we call the corresponding mixed Poisson process a Pólya-Lundberg process.
By $\hat{u}$ we denote the Laplace transform of U, i.e. $\hat{u}(t) = \int_0^\infty e^{-\lambda t} dU(\lambda)$.
In order to determine exponential families of mixed Poisson processes, we will deduce equivalent descriptions of the equivalence $P \sim Q$ of two measures $P, Q \in \mathcal{M}$ via the structure distributions and their Laplace transforms U_P, $\hat{u}_P$ resp. U_Q, $\hat{u}_Q$ corresponding to P resp. Q. The connection between P and the Laplace transform $\hat{u}_P$ is described by

$$P(X_t = k) = \frac{(-t)^k}{k!} \hat{u}_P^{(k)}(t), \quad t > 0,\, k \in \mathbb{N}_0,$$

where $\hat{u}_P^{(k)}$ denotes the k-th derivative of $\hat{u}_P$.
Now, we consider the following issue: Can a measure $P \in \mathcal{M}$ be equivalent to a measure Q which does not correspond to a mixed Poisson process, $Q \in \mathcal{P} \backslash \mathcal{M}$, that is, can we restrict $\sim$ to $\mathcal{M}$ without reducing the equivalence classes? This question is answered by the following proposition:

Proposition 3: *Let $P, Q \in \mathcal{P}$ be two equivalent measures. Let additionally P be a mixed Poisson process, i. e., $P \in \mathcal{M}$. Then also Q is a mixed Poisson process, $Q \in \mathcal{M}$.*

Proof: We will apply the fact that a mixed Poisson process is characterized by the conditional uniformity of its event occurence times (see [Naw]), which can be expressed as follows:

$$P(X_{t_1} = k_1, \ldots, X_{t_{n-1}} = k_{n-1} | X_{t_n} = k_n) = \frac{k_n!\, t_1^{k_1}}{t_n^{k_n}\, k_1!} \prod_{l=2}^{n} \frac{(t_l - t_{l-1})^{k_l - k_{l-1}}}{(k_l - k_{l-1})!} \tag{2}$$

holds for $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ with $0 \le t_1 < \cdots < t_n$ and for $k_1, \ldots, k_n \in \mathbb{N}_0$ with $0 \le k_1 \le \cdots \le k_n$.

Consider two measures $P \in \mathcal{M}$ and $Q \in \mathcal{P}$ with $P \sim Q$. By condition (1) we have

$$Q\left(X_{t_1} = k_1, \ldots, X_{t_n} = k_n\right) = e^{c(t_n) X_{t_n} + d(t_n)}\, P\left(X_{t_1} = k_1, \ldots, X_{t_n} = k_n\right)$$

for $n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{N}$, $t_1, \ldots, t_n \in [0, t]$, $t_n \ge t_i$, $i = 1, \ldots, n$ and for some non-negative functions c and d. Since then

$$\begin{aligned}
Q(X_{t_1} = k_1, \ldots, X_{t_{n-1}} = k_{n-1} | X_{t_n} = k_n) &= \frac{Q(X_{t_1} = k_1, \ldots, X_{t_n} = k_n)}{Q(X_{t_n} = k_n)} \\
&= \frac{e^{c(t_n) k_n + d(t_n)}\, P(X_{t_1} = k_1, \ldots, X_{t_n} = k_n)}{e^{c(t_n) k_n + d(t_n)}\, P(X_{t_n} = k_n)} \\
&= \frac{P(X_{t_1} = k_1, \ldots, X_{t_n} = k_n)}{P(X_{t_n} = k_n)} = P(X_{t_1} = k_1, \ldots, X_{t_{n-1}} = k_{n-1} | X_{t_n} = k_n),
\end{aligned}$$

equation (2) holds under Q if and only if it holds under P. ■

Consequently, we can simply restrict $\sim$ onto $\mathcal{M}$. Then we are able to establish the following characterization for the equivalence of two measures:

Proposition 4: *Let P and Q be two mixed Poisson processes, $P, Q \in \mathcal{M}$, and $c, d : T \to \mathbb{R}$ real functions. Then the following statements are equivalent:*

(i) The measure P is equivalent to Q, $P \sim Q$ and the equivalence is determined by the functions c and d.

(ii) The relation $Q(X_t = k) = e^{c(t)k+d(t)}\, P(X_t = k)$ holds for all $t \in T$ and $k \in \mathbb{N}_0$.

(iii) For the Laplace transforms $\hat{u}_P$ and $\hat{u}_Q$ of the structure distributions of P and Q we have

$$\begin{aligned} \hat{u}_Q^{(k)}(t) &= e^{c(t)k+d(t)}\, \hat{u}_P^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N}_0, \\ d(0) &= 0. \end{aligned} \tag{3}$$

Proof: First, (i) holds if and only if

$$Q(X_{t_1} = k_1, \ldots, X_{t_n} = k_n) \;=\; e^{c(t_n)k_n+d(t_n)}\, P(X_{t_1} = k_1, \ldots, X_{t_n} = k_n) \tag{4}$$

holds for all $n \in \mathbb{N}$ and $t_1, \ldots t_n$ with $0 \le t_1 < \ldots < t_n \le t$ and $k_1, \ldots, k_n \in \mathbb{N}_0$. Without loss of generality, $k_1 \le k_2 \le \cdots \le k_n$.
Be now $n \in \mathbb{N}$, $t_1, \ldots t_n$ and $k_1, \ldots, k_n \in \mathbb{N}_0$, accordingly. By (2) we have

$$P(X_{t_1} = k_1, \ldots, X_{t_n} = k_n) \;=\; \frac{k_n!\, t_1^{k_1}}{t_n^{k_n}\, k_1!} \prod_{l=2}^{n} \frac{(t_l - t_{l-1})^{k_l - k_{l-1}}}{(k_l - k_{l-1})!} P(X_{t_n} = k_n).$$

Since this is similarly valid for Q, equation (4) reduces to

$$Q(X_{t_n} = k_n) \;=\; e^{c(t_n)k_n+d(t_n)}\, P(X_{t_n} = k_n). \tag{5}$$

If additionally $t_n > 0$, this is equivalent to

$$\hat{u}_Q^{(k_n)}(t_n) \;=\; e^{c(t_n)k_n+d(t_n)}\, \hat{u}_P^{(k_n)}(t_n).$$

For $t_n = 0$ the condition $d(0) = 0$ is necessary and sufficient for equation (5) to hold for all $k_n \ge 0$.
The equivalence of (i), (ii) und (iii) follows from these considerations. ■

As an additional consequence of this proposition we obtain that, since Laplace transforms of distributions are continuous and infinitely often differentiable on $(0, \infty)$, also the functions c and d are alike.
The following lemma is an essential contribution to find exponential families. It supplies necessary and sufficient conditions for a measure $P \in \mathcal{M}$ to belong to an exponential family depending only on the Laplace transform corresponding to P and not on any other Laplace transform corresponding to an equivalent measure distinct from P.

Lemma 5: *For a measure $P \in \mathcal{M}$ and functions $c, d : T \to \mathbb{R}$ the following statements are equivalent:*

(i) There is a measure $Q \in \mathcal{M}$ distinct from P, which is equivalent to P. The equivalence $P \sim Q$ is determined by c and d.

(ii) It is $d \not\equiv 0$ and $d(0) = 0$. Furthermore, c, d are continuous and differentiable on $(0, \infty)$ and verify for all $t \in (0, \infty)$ and all $k \in \mathbb{N}$ the equation

$$\left(c'(t)(k-1) + d'(t)\right) \hat{u}_P^{(k-1)}(t) = \left(e^{c(t)} - 1\right) \hat{u}_P^{(k)}(t). \tag{6}$$

Proof: "⇒" Suppose (i) to be valid. Then we have by proposition 4

$$\begin{aligned} \hat{u}_Q^{(k)}(t) &= e^{c(t)k+d(t)}\, \hat{u}_P^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N}_0, \\ d(0) &= 0. \end{aligned}$$

The functions c and d are continuous and differentiable on $(0, \infty)$. Considering the first equation for $k = 0$, we see that $d \not\equiv 0$ since $P \neq Q$ and consequently $\hat{u}_P \neq \hat{u}_Q$. So, we have

$$\begin{aligned} & e^{c(t)k+d(t)}\, \hat{u}_P^{(k)}(t) = \hat{u}_Q^{(k)}(t) \\ &= \frac{d}{dt}\, \hat{u}_Q^{(k-1)}(t) = \frac{d}{dt}\, e^{c(t)\cdot(k-1)+d(t)}\, \hat{u}_P^{(k-1)}(t) \\ &= \left(c'(t)(k-1) + d'(t)\right) e^{c(t)\cdot(k-1)+d(t)}\, \hat{u}_P^{(k-1)}(t) + e^{c(t)\cdot(k-1)+d(t)}\, \hat{u}_P^{(k)}(t) \\ &= \left[\left(c'(t)(k-1) + d'(t)\right) \hat{u}_P^{(k-1)}(t) + \hat{u}_P^{(k)}(t)\right] e^{c(t)\cdot(k-1)+d(t)}, \qquad t > 0,\ k \in \mathbb{N}. \end{aligned}$$

Finally, we obtain

$$\left(c'(t)(k-1) + d'(t)\right) \hat{u}_P^{(k-1)}(t) = \left(e^{c(t)} - 1\right) \hat{u}_P^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N}, \tag{7}$$

and (ii) is proven.

"⇐" Suppose (ii) to be valid. Define a function $\hat{u}_Q : T \to \mathbb{R}$ by

$$\hat{u}_Q(t) := e^{d(t)}\, \hat{u}_P(t), \qquad t \in T.$$

We will show that $\hat{u}_Q$ is a Laplace transform and that the measure $Q \in \mathcal{M}$ which corresponds to $\hat{u}_Q$, is equivalent to P. At first, we prove the following by induction on k:

$$\hat{u}_Q^{(k)}(t) = e^{c(t)k+d(t)}\, \hat{u}_P^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N}_0. \tag{8}$$

By definition, the equation holds for $k = 0$. Now suppose the upper equation to be valid for $k-1$, $k \in \mathbb{N}$. Then it also holds for k:

$$\begin{aligned}
\hat{u}_Q^{(k)}(t) = \frac{d}{dt}\,\hat{u}_Q^{(k-1)}(t) &= \frac{d}{dt}\left(e^{c(t)(k-1)+d(t)}\,\hat{u}_P^{k}(t)\right) \\
&= \left(c'(t)(k-1)+d'(t)\right)e^{c(t)(k-1)+d(t)}\,\hat{u}_P^{(k-1)}(t) + e^{c(t)(k-1)+d(t)}\,\hat{u}_P^{(k)}(t) \\
&= e^{c(t)(k-1)+d(t)}\left(\left(c'(t)(k-1)+d'(t)\right)\hat{u}_P^{(k-1)}(t) + \hat{u}_P^{(k)}(t)\right) \\
&\overset{(6)}{=} e^{c(t)(k-1)+d(t)}\left(\left(e^{c(t)}-1\right)\hat{u}_P^{(k)}(t) + \hat{u}_P^{(k)}(t)\right) \\
&= e^{c(t)k+d(t)}\,\hat{u}_P^{(k)}(t), \qquad t > 0.
\end{aligned}$$

Because $\hat{u}_Q$ is completely monotone on $(0,\infty)$, cp. (8), and $\hat{u}_Q(0) = 1$, $\hat{u}_Q$ actually is the Laplace transform of a distribution (see for instance [FEL]). Let U_Q be this distribution and $Q \in \mathcal{M}$ the corresponding mixed Poisson process. Because of $d \not\equiv 0$ we have $\hat{u}_Q \neq \hat{u}_P$ and hence $Q \neq P$. Moreover (8) implies that Q is equivalent to P. Thus, there is an exponential family which contains P. ■

Let us now consider some examples. With Lemma 5 we can easily verify that each Poisson process is in an exponential family:
Under P, let X_T be a Poisson process with intensity $\lambda > 0$, i.e., $U_P = \delta_\lambda$ and $\hat{u}_P(t) = e^{-\lambda t}$, $t \in T$. Choose $c_0 \in \mathbb{R}\backslash\{0\}$ and let c and d be

$$c(t) = c_0, \qquad d(t) = -\lambda\,(e^{c_0}-1)\,t, \qquad t \in T. \tag{9}$$

Then c and d are continuous and on $(0,\infty)$ differentiable and

$$\left(c'(t)(k-1)+d'(t)\right)\hat{u}_P^{(k-1)}(t) = \left(e^{c(t)}-1\right)\hat{u}_P^{(k)}(t).$$

holds for $t > 0$ and $k \in \mathbb{N}$. Moreover, we have $d \not\equiv 0$ and $d(0) = 0$. So due to lemma 5, the ordinary Poisson process belongs to an exponential family.
Equally, we can show that Pólya-Lundberg processes are contained in exponential families:
Under P, let X_T be a Pólya-Lundberg process with structure distribution $U_P = \Gamma(\varphi,\gamma)$, $\varphi,\gamma > 0$. Then

$$\hat{u}_P(t) = \left(1+\frac{t}{\varphi}\right)^{-\gamma}, \qquad t \in T,\ k \in \mathbb{N},$$

is the Laplace transform of U_P. Let c and d be

$$c(t) = \ln\left(\frac{t+\varphi}{t+\varphi\,\alpha}\right), \qquad d(t) = \gamma\left(\ln\left(\frac{t+\varphi}{t+\varphi\,\alpha}\right)+\ln\,\alpha\right), \qquad t \in T, \tag{10}$$

where $\alpha \in \mathbb{R}\backslash\{1\}$. Then c and d are continuous and differentiable on $(0,\infty)$ and equation (6) of Lemma 5 holds. As additionally, $d(0) = 0$ and $d \not\equiv 0$, P belongs to an exponential family.

However, the conditions of Lemma 5 for a Laplace transform $\hat{u}_P$ in order to correspond to a measure P of an exponential family are very restrictive. The following proposition, in fact, shows that the above examples are the only ones possible:

Proposition 6: *For a measure $P \in \mathcal{M}$ the following statements are equivalent:*

(i) The measure P belongs to an exponential family.

(ii) There are $\lambda > 0$ resp. $\varphi, \gamma > 0$ such that the structure distribution U_P of P is $U_P = \delta_\lambda$ resp. $U_P = \Gamma(\varphi, \gamma)$.
In other words, X_T is either a Poisson process or a Pólya-Lundberg process under P.

The functions c, d that determine those measures which are equivalent to P are as in (9) *resp.* (10) *for $U_P = \delta_\lambda$ resp. $U_P = \Gamma(\varphi, \gamma)$.*

Before we can prove this proposition we need the following technical lemma:

Lemma 7: *Condition (ii) of Lemma 5 implies*

$$\left(e^{-c(t)} - 1\right)^2 v(t_0) = c'(t)\, e^{-c(t)}, \qquad t > 0,$$

where $v(t_0) = \left(\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} - \frac{\hat{u}''_P(t_0)}{\hat{u}'_P(t_0)}\right) \cdot \frac{1}{e^{-c(t_0)}-1}$ *for an arbitrary* $t_0 > 0$.

Proof: Let $t_0 > 0$. By equation (6) of Lemma 5 we have

$$\frac{\hat{u}_P^{(k)}(t)}{\hat{u}_P^{(k-1)}(t)} = \frac{c'(t)(k-1) + d'(t)}{e^{c(t)} - 1}, \qquad t > 0,\ k \in \mathbb{N}.$$

The solution $\hat{u}_P^{(k-1)}$ of this ordinary differential equation satisfies

$$\hat{u}_P^{(k-1)}(t) = \hat{u}_P^{(k-1)}(t_0) \left(\frac{e^{-c(t)} - 1}{e^{-c(t_0)} - 1}\right)^{k-1} e^{I(t)}, \qquad t > 0,\ k \in \mathbb{N}, \tag{11}$$

with $I(t) := \int_{t_0}^t \frac{d'(s)}{e^{c(s)}-1}\, ds$.
The following procedure is to evaluate equation (11) for $k = 1, 2, 3$ which first leads to a system of equations and at the end to a differential equation for the function c. For $k = 1$ we have by (11)

$$\hat{u}_P(t) = \hat{u}_P(t_0)\, e^{I(t)}, \qquad t > 0. \tag{12}$$

and consequently

$$\hat{u}'_P(t) \quad = \quad \hat{u}_P(t_0)\, e^{I(t)}\, \frac{d'(t)}{e^{c(t)} - 1}, \qquad t > 0. \tag{13}$$

Evaluating (11) for $k = 2$, we obtain

$$\hat{u}'_P(t) \;=\; \hat{u}'_P(t_0)\,\frac{e^{-c(t)}-1}{e^{-c(t_0)}-1}\,e^{I(t)}, \qquad t>0, \tag{14}$$

and combining (13) and (14) leads to

$$d'(t) \;=\; \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}\cdot\frac{(e^{-c(t)}-1)\,(e^{c(t)}-1)}{e^{-c(t_0)}-1}, \qquad t>0. \tag{15}$$

At the same time we have

$$\hat{u}''_P(t) \;=\; \hat{u}'_P(t_0)\left(\frac{-c'(t)e^{-c(t)}}{e^{-c(t_0)}-1}+\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}\cdot\left(\frac{e^{-c(t)}-1}{e^{-c(t_0)}-1}\right)^2\right)e^{I(t)}, \qquad t>0, \tag{16}$$

by deriving equation (14) and substituting $d'(t)$ by (15). Equation (11) for $k = 3$ is

$$\hat{u}''_P(t) \;=\; \hat{u}''_P(t_0)\,\left(\frac{e^{-c(t)}-1}{e^{-c(t_0)}-1}\right)^2 e^{I(t)}, \qquad t>0. \tag{17}$$

Joining (16) and (17) finally leads to the following differential equation for c

$$\left(e^{-c(t)}-1\right)^2\underbrace{\left(\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}-\frac{\hat{u}''_P(t_0)}{\hat{u}'_P(t_0)}\right)\cdot\frac{1}{e^{-c(t_0)}-1}}_{=:v(t_0)} \;=\; c'(t)\,e^{-c(t)}, \qquad t>0.$$

Notice that the continuous function v, as a function of t_0, does not change its sign on $(0,\infty)$, because $v(t_0) = 0$ for a $t_0 > 0$ implies that U_P is a Dirac distribution and therefore we had $v \equiv 0$. ■

Proof of Proposition 6: It remains to show that (i) implies (ii). Assume (i) to be valid, i.e., assume $P \in \mathcal{M}$ to belong to an exponential family. Let $t_0 > 0$. Then Lemma 7 implies the following differential equation:

$$\left(e^{-c(t)}-1\right)^2 v(t_0) = c'(t)\,e^{-c(t)}, \qquad t>0. \tag{18}$$

In the sequel, we will solve this equation for c and deduce $\hat{u}_P$.
We distinguish the cases $v \equiv 0$ and $v(t_0) \neq 0$ for all $t_0 > 0$. Assume $v \equiv 0$, that is, U_P is a Dirac distribution. We are interested in the points in which U_P can be concentrated and in what the corresponding functions c and d are. Equation (18) now looks like $0 = c'(t)\,e^{-c(t)}$ and implies $c(t) = c_0,\ t > 0$. Thus, by (15) we have

$$d'(t) = \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}\,(e^{c_0}-1)$$

Consequently, taking into account that $d(0)=0$, we obtain

$$d(t) \;=\; \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}\,(e^{c_0}-1)\,t, \qquad t>0.$$

Since

$$I(t) \;=\; \int\limits_{t_0}^{t} \frac{d'(s)}{e^{c(s)}-1}ds \;=\; \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}\,(t-t_0), \qquad t>0,$$

equation (12) leads to

$$\hat{u}_P(t) \;=\; \hat{u}_P(t_0)\,e^{\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}\,(t-t_0)}, \qquad t>0. \tag{19}$$

Taking the logarithmic derivative with respect to t, we have

$$\frac{\hat{u}'_P(t)}{\hat{u}_P(t)} \;=\; \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}, \qquad t>0. \tag{20}$$

That means, for $t>0$ the quotient $\hat{u}'_P(t)/\hat{u}_P(t)$ is independent of t. Additionally this quotient is ≤ 0 and $=0$ if and only if $U=\delta_0$. The latter case has no longer to be considered since $U=\delta_0$ implies $d\equiv 0$ which contradicts condition (ii) from Lemma 5. Thus with $\lambda := -\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)}$, we have the following representation for $\hat{u}_P$

$$\hat{u}_P(t) = Ce^{-\lambda t}, \qquad t>0. \tag{21}$$

Since a Laplace transform $\hat{u}_P$ is continuous and satisfies $\hat{u}_P(0)=1$, the constant C has to be equal to 1. Altogether, we have the following representations for $\hat{u}_P$ and c,d

$$\hat{u}_P(t) = e^{-\lambda t}, \qquad c(t) = c_0, \qquad d(t) = -\lambda\,(e^{c_0}-1)\,t, \qquad t>0.$$

Because these functions are continuous, they can be extended onto the whole interval $T=[0,\infty)$. Apparently, $\hat{u}_P$ is the Laplace transform of a Dirac distribution in $\lambda>0$ and the measure $P\in\mathcal{M}$ corresponds to an ordinary Poisson process with intensity $\lambda>0$.

Consider now the case $v(t_0)\neq 0$. With $v := v(t_0)$ and $g(t) := e^{-c(t)}-1$ the differential equation

$$\left(e^{-c(t)}-1\right)^2\, v(t_0) = c'(t)\,e^{-c(t)},$$

can be transformed into

$$g(t)^2 v = -g'(t) \qquad \text{resp.} \qquad v = -\frac{g'(t)}{g(t)^2}, \qquad t>0,$$

and thus,

$$g(t) = \frac{1}{v\,t+a}, \qquad t>0.$$

Since $g(t) \in (-1, \infty)$ for $t > 0$, the integration constant a is restricted by

$$\begin{aligned} a \geq \ & 0 \qquad \text{for } v > 0, \\ a \leq & -1 \qquad \text{for } v < 0. \end{aligned}$$

From g we obtain a representation for c

$$c(t) \ = \ -\ln\left(1 + \frac{1}{v \cdot t + a}\right), \qquad t > 0.$$

By (15) we have

$$d'(t) \ = \ \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \frac{v\,t_0 + a}{v\,t + a} \cdot \frac{-1}{v\,t + a + 1}, \qquad t > 0,$$

and consequently, taking $d(0) = 0$ into account,

$$d(t) \ = \ \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \frac{vt_0 + a}{v} \cdot \left(\ln\left(\frac{vt + a + 1}{vt + a}\right) - \ln\left(\frac{a+1}{a}\right)\right), \qquad t > 0.$$

For $I(t)$ we get

$$I(t) \ = \ \int_{t_0}^{t} \frac{d'(s)}{e^{c(s)} - 1}\, ds \ = \ \frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \frac{v\,t_0 + a}{v} \ \ln\left(\frac{v\,t + a}{v\,t_0 + a}\right), \qquad t > 0.$$

Equation (12) leads to

$$\hat{u}_P(t) = \hat{u}_P(t_0)\, e^{I(t)} \ = \ \hat{u}_P(t_0)\left(\frac{v\,t + a}{v\,t_0 + a}\right)^{\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \frac{v\,t_0 + a}{v}}, \qquad t > 0. \tag{22}$$

As additionally

$$\frac{v\,t_0 + a}{v} \ = \ \frac{g(t_0)^{-1}}{v} \ = \ \left(v\left(e^{-c(t_0)} - 1\right)\right)^{-1} \tag{23}$$

is valid for $t > 0$, we obtain the following representation of $\hat{u}_P$ from (22):

$$\hat{u}_P = \hat{u}_P(t_0)\left(\frac{v\,t + a}{v\,t_0 + a}\right)^{\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \left(v\left(e^{-c(t_0)} - 1\right)\right)^{-1}}, \qquad t > 0. \tag{24}$$

Taking twice the logarithmic derivative one can show that neither $\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \left(v\left(e^{-c(t_0)} - 1\right)\right)^{-1}$ nor $\frac{a}{v}$ depend on the choice of t_0. Thus, setting

$$\gamma := -\frac{\hat{u}'_P(t_0)}{\hat{u}_P(t_0)} \cdot \frac{1}{v(t_0)\left(e^{-c(t_0)} - 1\right)} > 0 \qquad \text{and} \qquad \varphi := a/v(t_0) > 0,$$

we can express $\hat{u}_P$ as

$$\hat{u}_P(t) \ = \ C\left(1 + \frac{t}{\varphi}\right)^{-\gamma}, \qquad t > 0.$$

Since $\hat{u}_P$ is a Laplace transform and thus continuous in 0 with $\hat{u}_P(0) = 1$, C has to be equal to 1 . Apparently, $\hat{u}_P$ is the Laplace transform of a Γ-distribution with parameters $\varphi, \gamma > 0$. With $\alpha := \frac{a+1}{a}$, where $a > 0$ or $a < -1$ imply $\alpha \in \mathbb{R}\backslash\{1\}$, we obtain

$$c(t) \;=\; \ln\left(\frac{t+\varphi}{t+\varphi\,\alpha}\right), \quad d(t) \;=\; \gamma\left(\ln\left(\frac{t+\varphi}{t+\varphi\,\alpha}\right) + \ln\,\alpha\right), \qquad t > 0.$$

The above representations for $\hat{u}_P$, c and d can be continuously extended to the point 0. ■

We will now indicate an alternative way to prove the last proposition: R. S. Liptser and A. N. Shiryayev (cp. [LS], theorem 19.7.) characterize absolute continuity of two measures $P, Q \in \mathcal{P}$ by their compensators $\{A_t^P\}_{t\geq 0}$, $\{A_t^Q\}_{t\geq 0}$. The measure Q is absolute continuous with respect to P if and only if there exists a nonnegative process $\left\{\lambda_t^{P,Q}\right\}_{t\geq 0}$ wich is predictable with respect to $\{\mathcal{F}_t\}_{t\in T}$, such that

$$A_t^Q(\omega) = \int_0^\infty \lambda_s^{P,Q}(\omega) dA_s^P(\omega), \qquad t < \infty,$$

and

$$\int_0^\infty \left(1 - \sqrt{\lambda_s^{P,Q}(\omega)}\right)^2 dA_s^P(\omega) < \infty$$

are verified for P-almost all $\omega \in \Omega$.
Now for $P \in \mathcal{M}$ and analogously for $Q \in \mathcal{M}$ we have (cp. [Gra])

$$A_t^P(\omega) = \int_0^t \kappa^P_{X_s(\omega)}(s) ds\,, \qquad P\text{-a.e.},$$

where $\kappa_n^P(t)$, $n \in \mathbb{N}$, $t > 0$, are the transition intensities of the mixed Poisson process (and hence birth process) P. So, for the case $P, Q \in \mathcal{M}$ we obtain:

$$\lambda_t^{P,Q}(\omega) = \frac{\kappa^Q_{X_t(\omega)}(t)}{\kappa^P_{X_t(\omega)}(t)}\,, \qquad P - a.e.,$$

which is a predictable process if and only if the quotient on the right side does not depend on ω, i.e., $\lambda_t^{P,Q}(\omega) \equiv \lambda_t^{P,Q}$ for $\omega \in \Omega$. Additionally, we have $0 < \lambda_t^{P,Q} < 1$ for $P \neq Q$.
A second part of theorem 19.7. in [LS] says that the Radon-Nikodym derivatives dQ_t/dP_t for $t \geq 0$ can be represented as

$$\frac{dQ_t}{dP_t}(\omega) \;=\; \exp\left\{\int_0^t \ln\lambda_s^{P,Q}(\omega)\, dX_s(\omega) - \left(A_t^Q(\omega) - A_t^P(\omega)\right)\right\}, \qquad P_t - a.e. \qquad (25)$$

Now let us return to our initial question: Which measures $P, Q \in \mathcal{M}$ can be in relation $\sim$ to each other?
The definition of $P \sim Q$ requires for $t \in [0, \infty)$ that

$$\frac{dQ_t}{dP_t}(\omega) \;=\; e^{c(t)X_t(\omega)+d(t)}, \qquad P_t - a.e.,$$

which together with equation (25) leads to

$$\int_0^t \ln \lambda_s^{P,Q}\, dX_s(\omega) - \Big(A_t^Q(\omega) - A_t^P(\omega)\Big) \;=\; c(t)X_t(\omega) + d(t), \qquad P_t - a.e.$$

Partial integration yields

$$\ln \lambda_t^{P,Q} X_t(\omega) - \int_0^t \frac{\frac{d}{ds}\lambda_s^{P,Q}}{\lambda_s^{P,Q}} X_s(\omega)\, ds - \lambda_t^{P,Q} A_t^P(\omega) + \int_0^t \frac{d}{ds}\lambda_s^{P,Q} A_s^P(\omega)\, ds + A_t^P(\omega)$$
$$= \;\; c(t)X_t(\omega) + d(t)$$

for P_t-almost all ω. If for a fixed $\omega \in \Omega$ this equation is valid and if $t > 0$ is a continuity point of the path $X_T(\omega)$ we can derive with respect to t and obtain

$$\Big(1 - \lambda_t^{P,Q}\Big) \frac{d}{dt} A_t^P(\omega) \;=\; c'(t)X_t(\omega) + d'(t)$$

which implies

$$\kappa^P_{X_t(\omega)}(t) \;=\; \frac{d}{dt} A_t^P(\omega) \;=\; \frac{c'(t)}{1 - \lambda_t^{P,Q}} X_t(\omega) + \frac{d'(t)}{1 - \lambda_t^{P,Q}}.$$

Furthermore, for $t > 0$ and $n \in \mathbb{N}$, the set $\{\omega \in \Omega \,:\, X_t(\omega) = n,\, \lim_{s\to t-0} X_s(\omega) = n\}$, i.e. the set of counting functions which at time t do not jump and are in state n, has positive P_t-measure. So, the transition intensities must verify

$$\kappa_n^P(t) \;=\; \frac{c'(t)}{1 - \lambda_t^{P,Q}} n + \frac{d'(t)}{1 - \lambda_t^{P,Q}}, \qquad n \in \mathbb{N},\; t > 0.$$

But the only processes $P \in \mathcal{M}$ with transition intensities $\kappa_n^P(t)$ that are linear in n for fixed t are Poisson processes and Pólya-Lundberg processes (cp. [Gra]).
To find those measures $Q \in \mathcal{M}$ which can be equivalent to P consider the quotient $\frac{\kappa^Q_{X_t(\omega)}(t)}{\kappa^P_{X_t(\omega)}(t)}$. Since it must not depend on ω if the two measures P and Q shall be in relation to each other, we find that a Poisson process can only be equivalent to a Poisson process and a Pólya-Lundberg process with structure distribution $\Gamma(\varphi,\gamma)$, $\varphi,\gamma > 0$, and transition intensities $\kappa_n(t) = \frac{\gamma+n}{\varphi+t}$ can only be equivalent to Pólya-Lundberg processes with the same parameter γ.

An immediate consequence of proposition 6 is the following corollary which specifies exponential families in $\mathcal{M}$.

Corollary 8: *The only existing exponential families in $\mathcal{M}$ are the exponential family of homogeneous Poisson processes $\{P \in \mathcal{M} \,:\, U_P = \delta_\lambda,\, \lambda > 0\}$ and exponential families of Pólya-Lundberg processes $\{P \in \mathcal{M} \,:\, U_P = \Gamma(\varphi,\gamma),\, \varphi > 0\}$, where the shape parameter $\gamma > 0$ of the corresponding Gamma structure distributions remains constant within such an exponential family.*

Proof: Proposition 6 determines the only measures $P \in \mathcal{M}$ belonging to exponential families and the only functions c and d leading to equivalent measures. Combining these and therefore calculating for such a $P \in \mathcal{M}$ and all possible appropriate functions c and d the equivalent measures $\tilde{P} \in \mathcal{M}$ via $\hat{u}_{\tilde{P}}(t) = e^{d(t)}\hat{u}_P(t)$ we obtain the above stated exponential families. ■

3 A Martingale Characterization of Pólya-Lundberg Processes

In the preceeding section we emphasized the special position of the Pólya-Lundberg process within mixed Poisson processes. In the sequel, we will deduce a martingale characterization. First, consider the succeeding proposition characterizing the process of densities $\{dQ_t/dP_t\}_{t\geq 0} = \left\{e^{c(t)X_t+d(t)}\right\}_{t\geq 0}$ of two equivalent measures P, $Q \in \mathcal{M}$ as a martingale:

Proposition 9: *Given a measure $P \in \mathcal{M}$ and functions $c, d : T \to \mathbb{R}$ the following statements are equivalent:*

(i) There is a measure $Q \in \mathcal{M}$, so that for every $t \in T$ the measure Q_t is absolutely continuous with respect to P_t and the corresponding Radon-Nikodym derivative verifies $dQ_t/dP_t = e^{c(t)X_t+d(t)}$ P_t-a.e.

(ii) Under P, the process $\left\{ e^{c(t)X_t+d(t)} \right\}_{t\in T}$ is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\in T}$ of X_T and the expectation of $e^{c(t)X_t+d(t)}$ is equal to 1 for all $t \in T$.

Proof: (i) ⇒ (ii): Since for $t \geq 0$, $s \in [0,t]$ and $A \in \mathcal{F}_s$ we have $\displaystyle\int_A \frac{dQ_t}{dP_t}\,dP = \int_A \frac{dQ_s}{dP_s}\,dP$ the process of Radon-Nikodym derivatives $\left\{ e^{c(t)X_t+d(t)} \right\}_{t\in T}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in T}$.

(ii) ⇒ (i): It is evident, that if we define a projective family of measures $\{Q_t\}_{t\geq 0}$ by $dQ_t/dP_t := e^{c(t)X_t+d(t)}$, then there exists a measure $Q \in \mathcal{P}$ such that the measures Q_t are the restrictions of Q to $\mathcal{F}_t$. What we have to show is, that Q necessarily lies in $\mathcal{M}$.

For $d \equiv 0$ the measure P itself verifies (i). Notice that $d \equiv 0$ and hence $1 = E_P\left(e^{c(t)X_t}\right)$ imply $c \equiv 0$ or $U_P = \delta_0$.

Let $d \not\equiv 0$. We will apply lemma 5, that is to show that $d(0) = 0$ and that c and d are continuous and differentiable on $(0, \infty)$ such that

$$\left(c'(t)(k-1) + d'(t)\right)\hat{u}_P^{(k-1)}(t) \;=\; \left(e^{c(t)} - 1\right)\hat{u}_P^{(k)}(t) \tag{26}$$

holds for $t > 0$ and $k \in \mathbb{N}$.

At first we have $d(0) = -\ln E_P\left(e^{c(0)X_0}\right) = 0$.

Continuity and differentiability can be deduced from the martingale property:

Let $t > 0$ fixed. The martingale property corresponds to

$$E_P\left(e^{c(t)X_t+d(t)} \,\middle|\, X_s\right) = e^{c(s)X_s+d(s)}, \qquad s \in [0,t],$$

because X_T is a markovian process under P. We can equivalently write:

$$\sum_{k=0}^{\infty} e^{c(t)k+d(t)}\, P(X_t = k \,|\, X_s = k_s) \;=\; e^{c(s)k_s+d(s)}, \qquad s \in [0,t], k_s \in \mathbb{N}_0. \tag{27}$$

Since the conditional probabilities $P(X_t = k \,|\, X_s = k_s)$ are

$$P(X_t = k \,|\, X_s = k_s) \;=\; \begin{cases} (-1)^{k-k_s} \dfrac{(t-s)^{k-k_s}}{(k-k_s)!} \cdot \dfrac{\hat{u}_P^{(k)}(t)}{\hat{u}_P^{(k_s)}(s)} & \text{for } s \in [0,t],\ k \geq k_s, \\ 0 & \text{else,} \end{cases}$$

we have

$$e^{c(s)k_s+d(s)} \hat{u}_P^{(k_s)}(s) \;=\; \sum_{k=k_s}^{\infty} e^{c(t)k+d(t)}\, (-1)^{k-k_s} \frac{(t-s)^{k-k_s}}{(k-k_s)!} \hat{u}_P^{(k)}(t), \qquad s \in [0,t],\ k_s \in \mathbb{N}_0.$$

Since for $k_s \in \{0,1\}$ the right side of the equation is a continuous and differentiable function of s for $s \in [0,t]$ and since t can be chosen arbitrarily big, the functions c and d must be continuous and differentiable on $(0,\infty)$.

To prove (26) we differentiate the last equation with respect to s and obtain:

$$\begin{aligned} &\big(c'(s)k_s + d'(s)\big)\, e^{c(s)k_s+d(s)}\, \hat{u}_P^{(k_s)}(s) + e^{c(s)k_s+d(s)}\, \hat{u}_P^{(k_s+1)}(s) \\ &\quad = \sum_{k=k_s+1}^{\infty} e^{c(t)k+d(t)} (-1)^{k-k_s} \frac{(t-s)^{k-k_s-1}}{(k-k_s-1)!}\, \hat{u}_P^{(k)}(t), \qquad s \in (0,t),\ k_s \in \mathbb{N}_0. \end{aligned}$$

Since power series are continuous in their convergence interval, taking the limit $s \uparrow t$ finally leads to

$$\big(c'(t)k_s + d'(t)\big)\, e^{c(t)k_s+d(t)}\, \hat{u}_P^{(k_s)}(t) + e^{c(t)k_s+d(t)}\, \hat{u}_P^{(k_s+1)}(t) \;=\; e^{c(t)(k_s+1)+d(t)}\, \hat{u}_P^{(k_s+1)}(t)$$

resp.

$$\big(c'(t)k_s + d'(t)\big)\, \hat{u}_P^{(k_s)}(t) = \left(e^{c(t)} - 1\right)\, \hat{u}_P^{(k_s+1)}(t), \qquad k_s \in \mathbb{N}_0.$$

Such a representation exists for every $t > 0$. Hence, all the conditions of lemma 5 (ii) are verified and consequently there exists a measure $Q \in \mathcal{M}$, that verifies the required (cp. lemma 5 (i)). ∎

An immediate consequence of proposition 9 is the following martingale characterization of Pólya-Lundberg processes within the class of mixed Poisson processes:

Main theorem: *The following statements are equivalent for a measure $P \in \mathcal{M}$:*

(i) There are functions $c, d : T \to \mathbb{R}$ with d non-constant, such that $\left\{ e^{c(t)X_t + d(t)} \right\}_{t \in T}$ under P is a martingale with respect to $\{\mathcal{F}_t\}_{t \in T}$.

(ii) The measure P corresponds either to a Poisson or a Pólya-Lundberg process.

Proof: Use propositions 6 and 9. ■

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