On material constants for micromorphic continua.

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Abstract

I investigate a geometrically exact generalized isotropic continua of micromorphic type in the sense of Eringen. The two-field problem for the macrodeformation φ and the "affine microdeformation" $\overline{P} \in \mathrm{GL}^+(3, \mathbb{R})$ in the quasistatic, elastic case is presented in a variational form. The relative elastic stress-strain relation is taken for simplicity as physically linear. The corresponding infinitesimal strain problem obtained by linearization is also presented. Focus of attention is shifted to the interpretation of the appearing material constants. I derive simple homogenization-like formulas which relate the Lamé constants of the substructure and the classical Lamé constants obtained for arbitrarily large samples with the effective parameters in the micromorphic model. The relation of the thus obtained model to the intrinsically linear representation of Mindlin and Eringen is also established. The results should be useful for finite-element simulations of micromorphic continua.

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1 Introduction

This article addresses the modelling and constitutive implications of **geometrically exact**[1] generalized continua of **micromorphic** type in the sense of Eringen in the elastic case. General continuum models involving **independent rotations** have already been introduced by the Cosserat brothers [9] at the beginning of the last century.

Their development has been largely forgotten for decades only to be rediscovered in the early sixties [42, 23, 1, 17, 15, 48, 49, 25, 37, 44, 50]. At that time theoretical investigations on non-classical continuum theories were the main motivation [34]. Since then, the Cosserat concept has been generalized in various directions, for an overview of these so called **microcontinuum** theories we refer to [16, 14, 4, 3, 5, 26, 35]. Recently, in [6, 7], the micromorphic balance equations derived by Eringen have been formally justified as a more realistic continuum model based on molecular dynamics and ensemble averaging.

The micromorphic model includes in a natural way **size effects**, i.e. small samples behave comparatively stiffer than large samples. These effects have recently received much attention in conjunction with nano-devices and foam-like structures.

The mathematical analysis of general micromorphic solids in the static case is at present restricted to the infinitesimal, linear elastic models, see e.g. [29, 12, 27, 21, 22] for linear micropolar models and [32, 30, 31] for linear microstretch models. The major difficulty of the mathematical treatment in the finite-strain static case is related to the **geometrically exact** formulation of the theory and the natural appearance of **nonlinear manifolds** necessary for the adequate description of the geometrical features of the microstructure. Both sources of

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[1] Fully frame-indifferent.

nonlinearity exclude the use of most techniques employed for the linear case. In addition, **coercivity** of the formulation w.r.t. deformations turns out to be a delicate problem related to the possible **fracture** of the material. This coercivity depends crucially on the level of smoothness provided by the microstructure. No general existence theorems for finite micromorphic models had been known until [39]. The simpler, geometrically exact nonlinear micropolar case has been dealt with in [40].

This contribution is organized as follows: first, we shortly review the basic concepts of the geometrically exact elastic micromorphic theories with affine microstructure in a variational context, i.e. we formulate the quasistatic case as a two-field minimization problem. For simplicity we restrict attention to a physically linear stress-(relative elastic) strain relation. We present the linearization of this model and and compare it to the intrinsically linear models of Mindlin and Eringen. In contrast to the latter models, positivity of the local strain-energy is automatically satisfied at the expense of having only five independent material parameters in contrast to seven parameters in the intrinsically linear models. Four of these five parameters can be directly related to simple experiments, while the remaining Cosserat couple modulus $\mu_c \geq 0$ must be viewed as a penalty-parameter without intrinsic physical significance. This couple modulus is strongly related to penalty formulations of variational problems when turning to numerical implementations. Usually, in finite-element simulations of micromorphic continua, values of material parameters are not discussed. Our result should prove useful in this case. The relevant notation is introduced in the appendix.

2 A finite-strain elastic micromorphic model with affine microstructure

Let us now motivate a finite-strain micromorphic approach.[2] For our development we choose a strictly Lagrangean description. We first introduce an independent kinematical field of **microdeformations** $P \in \mathrm{GL}^+(3,\mathbb{R})$ together with its right polar decomposition

$$P = \overline{R}_p \cdot U_p = \mathrm{polar}(P) \cdot U_p = \overline{R}_p \, e^{\frac{\overline{\alpha}_p}{3}} \, \overline{U}_p \,, \quad \det[P] = e^{\overline{\alpha}_p} \,,$$
$$\overline{U}_p = \frac{U_p}{\det[U_p]^{1/3}} \in \mathrm{SL}(3,\mathbb{R}) \,, \quad \overline{P} = \frac{P}{\det[P]^{1/3}} \in \mathrm{SL}(3,\mathbb{R}) \,, \tag{2.1}$$

with $\overline{R}_p \in \mathrm{SO}(3,\mathbb{R})$, $\overline{U_p} \in \mathrm{PSym}(3,\mathbb{R}) \cap \mathrm{SL}(3,\mathbb{R})$ and $\overline{\alpha}_p \in \mathbb{R}$. The microdeformations P are meant to describe the substructure of the material which can **rotate**, **stretch**, **shear** and **shrink**. We refer to $\overline{R}_p$ as **microrotations**.

The micromorphic theory we deal with can formally be obtained by introducing the **multiplicative decomposition** of the macroscopic deformation gradient F into **independent microdeformation** P and the **micromorphic, nonsymmetric right stretch tensor** $\overline{U}$ (first Cosserat deformation tensor) with

$$F = P \cdot \overline{U} \,, \quad \overline{U} = P^{-1} F \,, \quad \overline{U} \in \mathrm{GL}^+(3,\mathbb{R}), \tag{2.2}$$

leading altogether to a **micro-compressible, micromorphic formulation**.[3]

The notion **micro**morphic is prone to misunderstandings: the microdeformation P must be considered as a macroscopic (average) quantity as the deformation gradient and the resulting model is still phenomenological. However, geometrical features of the real substructure to be modelled determine the choice of geometric manifolds for P. Since the substructure of the material can in principle be crushed, the choice $P \in \mathrm{GL}^+(3,\mathbb{R})$ is mandatory.

In the **quasistatic** case, the micromorphic theory is now derived from a **two-field** variational principle by postulating the following **"action euclidienne"** [9, p.156] I for the finite macroscopic deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the independent microdeformation

[2]Following Eringen [14, p.13] we distinguish the **general micromorphic case**: $\overline{P} \in \mathrm{GL}^+(3,\mathbb{R}) = \mathbb{R}^+ \cdot \mathrm{SL}(3,\mathbb{R})$ with 9 additional **degrees of freedom** (dof), the **micro-incompressible micromorphic case**: $\overline{P} \in \mathrm{SL}(3,\mathbb{R})$ with 8 dof, the **microstretch case**: $\overline{P} \in \mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R})$ with 4 dof and the **micropolar case**: $\overline{P} \in \mathrm{SO}(3,\mathbb{R})$ with only 3 additional dof.

[3]The strain measure $\overline{U}$ which is induced by this definition corresponds to $\mathfrak{C}^T_{\mathrm{KL}}$ presented in $(1.5.11)_1$ of [14, p.15].

$P : [0,T] \times \overline{\Omega} \mapsto \mathrm{GL}^+(3,\mathbb{R})$:

$$I(\varphi, P) = \int_\Omega W(F, P, \mathrm{D_x} P) - \Pi_f(\varphi) - \Pi_M(P)\,\mathrm{dV} \\ - \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} - \int_{\Gamma_C} \Pi_{M_c}(P)\,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (\varphi, P), \\ P_{|_\Gamma} = P_\mathrm{d}\,, \quad \varphi_{|_\Gamma} = g_\mathrm{d}(t)\,. \tag{2.3}$$

The elastically stored energy density W depends on the macroscopic deformation gradient F as usual but in addition on the microdeformation P together with their first order space derivatives, represented through the third order tensor $\mathrm{D_x}P$. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions g, P_d for displacements and microdeformations, respectively, can be prescribed, while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. The potential of external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples.[4] In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary, where the potential of applied surface couples Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ corresponding natural boundary conditions for φ and P apply, which are obtained automatically in the variational process.

Variation of the action I with respect to φ yields the traditional equation for balance of linear momentum and variation of I with respect to P yields the additional balance of moment of momentum.

The standard conclusion from **frame-indifference** (here: invariance of the free energy under **s**uperposed **r**igid **b**ody **m**otions (**SRBM**) not merely **observer-invariance** of the model [47, 2, 38]: $\forall Q \in \mathrm{SO}(3,\mathbb{R}) : \ W(F, P, \mathrm{D_x}P) = W(QF, QP, \mathrm{D_x}[QP])$ leads to the reduced representation of the energy (specify $Q = \overline{R}_p^T$):

$$W(F, \overline{P}, \mathrm{D_x}P) = W(\overline{R}_p^T F, \overline{R}_p^T P, \overline{R}_p^T \mathrm{D_x}P) = W(U_p\overline{U}, U_p, \overline{R}_p^T \mathrm{D_x}P) = W^\sharp(\overline{U}, U_p, \mathfrak{K}_p, \nabla\overline{\alpha}_p)\,, \tag{2.4}$$

where for $\overline{P} = \overline{R}_p\,\overline{U}_p \in \mathrm{SL}(3,\mathbb{R})$ we set

$$\mathfrak{K}_p := \overline{R}_p^T \mathrm{D_x}\overline{P} = \left(\overline{R}_p^T \nabla(\overline{P}.e_1), \overline{R}_p^T \nabla(\overline{P}.e_2), \overline{R}_p^T \nabla(\overline{P}.e_3)\right) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3}\,, \tag{2.5}$$

which coincides with one specific representation[5] of the third order right **micropolar curvature tensor** (or torsion-curvature tensor, wryness tensor, second Cosserat deformation tensor, bending-twist tensor, etc.), if $\overline{P} \in \mathrm{SO}(3,\mathbb{R})$.

For a geometrically exact (macroscopically isotropic) theory we assume in the following an additive split of the total free energy density into micromorphic local stretch (macroscopic), stretch of the substructure (microscopic) and micromorphic curvature part according to

$$W^\sharp = \underbrace{W_{\mathrm{mp}}(\overline{U})}_{\text{macroscopic energy}} + \underbrace{W_{\mathrm{foam}}(\overline{U}_p, \overline{\alpha}_p)}_{\text{microscopic local energy}} + \underbrace{W_{\mathrm{curv}}(\mathfrak{K}_p, \nabla\overline{\alpha}_p)}_{\text{microscopic interaction energy}}\,, \tag{2.6}$$

since a possible coupling between $\overline{U}$ and $\mathfrak{K}_p$ for centrosymmetric bodies can be ruled out [41, p.14].

2.1 The elastic macroscopic micromorphic strain energy density

For a macroscopically **small elastic strain** theory[6] (physically linear), which should already cover many cases of physical interest, we require that $W_{\mathrm{mp}}(\overline{U})$ is a non-negative isotropic

[4] appearing in a non-mechanical context e.g. as influence of a magnetic field on the polarization of a substructure of the bulk.

[5] Note that $\mathfrak{K}_p^i = \overline{R_p}^T \nabla(\overline{P}.e_i) \notin \mathfrak{so}(3,\mathbb{R})$. Another representation of $\mathfrak{K}_p$ is given by $\overline{\mathfrak{K}}_p := \left(\overline{R}_p^T \partial_x \overline{P}, \overline{R}_p^T \partial_y \overline{P}, \overline{R}_p^T \partial_z \overline{P}\right) \in \mathfrak{T}(3)$. Since $\partial_x(\overline{R}_p^T \overline{P}) = 0$ for $\overline{P} = \overline{R}_p \in \mathrm{SO}(3,\mathbb{R})$, it holds that $\overline{\mathfrak{K}}_p \in \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})$ in this case. It is therefore possible to base all considerations of curvature in the **micropolar** case on a more compact expression $\widehat{\mathfrak{K}_p} := \left(\mathrm{axl}(\overline{R}_p^T \partial_x \overline{R}_p) | \,\mathrm{axl}(\overline{R}_p^T \partial_y \overline{R}_p) | \,\mathrm{axl}(\overline{R}_p^T \partial_z \overline{R}_p)\right) \in \mathbb{M}^{3\times3}$. This is the traditional micropolar approach, see e.g. [43, 18, 24]. For us it is, however, not possible to use $\widehat{\mathfrak{K}_p}$, since we allow $\overline{P} \in \mathrm{GL}^+(3,\mathbb{R})$.

[6] By this we mean that the part of the deformation which is superposed onto the substructure deformation has small elastic strains.

quadratic form. We assume moreover the macroscopic stretch energy density normalized to

$$W_{\mathrm{mp}}(1\!\!1) = 0, \qquad D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})_{|_{\overline{U}=1\!\!1}} = 0\,. \tag{2.7}$$

For the local energy contribution elastically stored in the substructure we assume the nonlinear expression

$$\begin{aligned} W_{\mathrm{foam}}(U_p) &= \underbrace{\mu^{\mathrm{m}}\, \| \frac{U_p}{\det[U_p]^{(1/3)}} - 1\!\!1\|^2}_{\text{isochoric substructure energy}} + \underbrace{\frac{\lambda^{\mathrm{m}}}{4} \left((\det[U_p]-1)^2 + (\frac{1}{\det[U_p]} - 1)^2 \right)}_{\text{volumetric energy}} \\ &= \mu^{\mathrm{m}}\, \|\overline{U}_p - 1\!\!1\|^2 + \frac{\lambda^{\mathrm{m}}}{4} \left((e^{\overline{\alpha}_p} - 1)^2 + (e^{-\overline{\alpha}_p} - 1)^2 \right) =: W_{\mathrm{foam}}(\overline{U}_p, \overline{\alpha}_p)\,, \end{aligned} \tag{2.8}$$

avoiding self-interpenetration in a variational setting, since $W_{\mathrm{foam}} \to \infty$ as $\det[P] = \det[U_p] \to 0$ if $\lambda^{\mathrm{m}} > 0$.[7] The most general form of W_{mp} consistent[8] with the requirement (2.7) is

$$W_{\mathrm{mp}}(\overline{U}) = \mu_e \, \| \operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \, \| \operatorname{skew}(\overline{U} - 1\!\!1)\|^2 + \frac{\lambda_e}{2} \operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2, \tag{2.9}$$

with material constants μ_e, μ_c, λ_e such that μ_e, $3\lambda_e + 2\mu_e, \mu_c \geq 0$ from non-negativity [14] of (2.9). **It is important to realize that μ_e, λ_e are effective elastic constants which in general do not coincide with the classical Lamé constants $\mu, \lambda > 0$.** Here, I take the classical Lamé constants to be obtained from standard experiments of sufficiently large samples of the materials, such that length scale effects do not interfere. The so-called **Cosserat couple modulus** μ_c (rotational couple modulus) remains for the moment unspecified, but we note that $\mu_c = 0$ is physically possible, since the **micromorphic reaction stress** $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T$ is not symmetric in general, i.e. the problem does not decouple. For comparison, in [14, p.111] for the infinitesimal micropolar case, the elastic moduli are taken to be $\mu_e = \mu + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$, $\lambda_e = \lambda$, but in this formula μ can neither be regarded as one of the Lamé constants.[9] [10] In [11, 45, 46, 19, 10, 13] the abbreviation μ_c is used while in [24] it is $\mu_c = \alpha$ and $\mu_c = G_c$ in [33] for the micropolar theory.

By formal similarity with the classical formulation we may call $\mu^{\mathrm{m}}, \lambda^{\mathrm{m}}$ the **microscopic Lamé moduli** of the affine substructure, which can be determined from classical experiments, e.g. dealing with a nickel-foam structure, they are the Lamé-constants of the smallest possible representative volume element in the foam, e.g. comprising 4 unit-cells. In the analytical section we will show, how to obtain consistent values for μ_e, λ_e if we know already $\mu^{\mathrm{m}}, \lambda^{\mathrm{m}}$ and μ, λ.

2.2 The nonlinear elastic curvature energy density

The curvature energy is responsible for the size-dependent resistance of the cell-structure against local twisting and inhomogeneous volume change. Thus inhomogeneous microstructural rearrangements are penalized. For the curvature term, to be specific, we assume the general form

$$\begin{aligned} W_{\mathrm{curv}}(\mathfrak{K}_p, \nabla\overline{\alpha}_p) &= \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 \, L_c^q \, \|\mathfrak{K}_p\|^q\right) \left(\alpha_5 \, \| \operatorname{sym} \mathfrak{K}_p\|^2 + \alpha_6 \, \| \operatorname{skew} \mathfrak{K}_p\|^2 + \alpha_7 \operatorname{tr}\left[\mathfrak{K}_p\right]^2 \right)^{\frac{1+p}{2}} \\ &\quad + \mu \, \frac{L_c^{1+p}}{12} \, \left(\alpha_8 \|\nabla\overline{\alpha}_p\|^{1+p} + \alpha_8 \, L_c \, \|\nabla\overline{\alpha}_p\|^{2+p} \right) \end{aligned} \tag{2.10}$$

where $L_c > 0$ is setting an internal length scale with units of length, $\alpha_4 \geq 0, p > 0, q \geq 0$ are additional material constants. The factor $\frac{1}{12}$ appears only for convenience and $\alpha_5 > 0, \alpha_6, \alpha_7 \geq 0, \alpha_8 > 0$ should be satisfied as a minimal requirement. We mean $\operatorname{tr}\left[\mathfrak{K}_p\right]^2 = \|\operatorname{tr}\left[\mathfrak{K}_p\right]\|^2$ by abuse of notation. This choice for W_{curv} does not presuppose any knowledge of the magnitude of the

[7] Note that $\left((\det[U_p]-1)^2 + (\frac{1}{\det[U_p]} - 1)^2 \right) = 2 \operatorname{tr}\left[U_p - 1\!\!1\right]^2 + O(\|U_p - 1\!\!1\|^3)$.

[8] Mixed products like $\langle \overline{U} - 1\!\!1, \overline{U}_p - 1\!\!1\rangle$ and $\operatorname{tr}\left[\overline{U} - 1\!\!1\right] \cdot \operatorname{tr}\left[\overline{U}_p - 1\!\!1\right]$ are excluded by non-negativity.

[9] A simple definition of the Lamé constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where E and ν are uniquely determined from uniform traction experiments for sufficiently large samples.

[10] Uniform traction and uniform compression do not activate rotations, hence the classical identification of the Lamé constants is achieved **independent** of μ_c. Uniform traction alone allows to determine the Young modulus E and the Poisson ratio ν [8, p.126]. Contrary to [20, p.411] we do not see the possibility to define a specific "micropolar Young modulus" or "micropolar Poisson ratio".

micromorphic curvature in the material and is non-degenerate in the origin $\|\mathfrak{K}_p\| = \|\nabla\overline{\alpha}_p\| = 0$. Some care has to be exerted in the finite-strain regime: W_{curv} should preferably be **coercive** in the sense that we impose pointwise

$$\exists\, c^+ > 0\; \exists\, r > 1 : \;\forall\, \mathfrak{K}_p \in \mathfrak{T}(3)\; \forall \xi \in \mathbb{R}^3 : \quad W_{\mathrm{curv}}(\mathfrak{K}_p, \xi) \geq c^+ \, \|(\mathfrak{K}_p, \xi)\|^r , \tag{2.11}$$

or less demanding

$$\exists\, r > 1 : \quad \frac{W_{\mathrm{curv}}(\mathfrak{K}_p, \xi)}{\|(\mathfrak{K}_p, \xi)\|^r} \to \infty \quad \text{as } \|(\mathfrak{K}_p, \xi)\| \to \infty \,, \tag{2.12}$$

which implies necessarily $\alpha_6,\, \alpha_8 > 0$ in (2.10). Observe that our formulation of the micromorphic curvature tensor is mathematically convenient in the sense that $\|\mathfrak{K}_p\| = \|\overline{R}_p^T \mathrm{D_x} \overline{P}\| = \|\mathrm{D_x}\overline{P}\|$ provides pointwise control of all first derivatives of $\overline{P}$ independent of the values of $\overline{P}$ itself.[11] Thus, coercivity of W_{curv} ensures a certain minimal level of smoothness of the microstucture without which coercivity w.r.t. deformations cannot be guaranteed. A lack of smoothness of the microstructure may therefore give rise to fracture on the macroscale.

Note that the presented formulation still includes a finite Cosserat micropolar model as a special case, if we set $\overline{P} = \overline{R} \in \mathrm{SO}(3,\mathbb{R})$. In this fashion, we have the following correspondence of limit problems:

$$\begin{aligned}
\lambda^{\mathrm{m}} \to \infty \quad &\Rightarrow \quad \textbf{micro-incompressible model: manifold } \mathrm{SL}(3,\mathbb{R}) \,, \\
\mu^{\mathrm{m}} \to \infty \quad &\Rightarrow \quad \textbf{microstretch model: manifold } \mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R}) \,, \\
\mu^{\mathrm{m}}, \lambda^{\mathrm{m}} \to \infty \quad &\Rightarrow \quad \textbf{micropolar model: manifold } \mathrm{SO}(3,\mathbb{R}) \,, \\
\mu_c \to \infty \quad &\Rightarrow \quad \textbf{higher gradient continua} \,.
\end{aligned} \tag{2.13}$$

Note also that $\mathrm{SO}(3,\mathbb{R})$, $\mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{R})$ are the only connected subgroups of $\mathrm{GL}^+(3,\mathbb{R})$ which contain $\mathrm{SO}(3,\mathbb{R})$.

3 The infinitesimal micromorphic elastic solid

3.1 The variational formulation

Starting from the proposed finite-strain formulation and not intrinsically linear, we may obtain a linear, infinitesimal micromorphic model by expanding all appearing variables to first order and keeping only quadratic terms in the energy expression. Thus we write $F = 1\!\!1 + \nabla u$, $P = 1\!\!1 + p$, and the model turns into the problem of finding a pair $(u, p) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathfrak{gl}^+(3,\mathbb{R})$ of macroscopic displacement u and **independent, infinitesimal microdeformation** p satisfying

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}, p) + W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla \mathrm{tr}\,[p]) \,\mathrm{dV} \mapsto \min .\ \text{w.r.t. } (u,p), \\
&\qquad \overline{\varepsilon} = \nabla u - p, \quad p_{|_\Gamma} = p_{\mathrm{d}} \in \mathfrak{gl}^+(3,\mathbb{R}) = \mathbb{M}^{3\times 3}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}} \,, \qquad (3.14) \\
&W_{\mathrm{mp}}(\overline{\varepsilon}, p) = \mu_e \,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c \,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\overline{\varepsilon}]^2 + \mu^{\mathrm{m}}\,\|\mathrm{sym}\,p\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\mathrm{tr}\,[\mathrm{sym}\,p]^2 \\
&\qquad = \mu_e \,\|\mathrm{sym}\,\nabla u - \mathrm{sym}\,p\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u - p)\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\,[\nabla u - p]^2 \\
&\qquad\quad + \mu^{\mathrm{m}}\,\|\mathrm{sym}\,p\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\mathrm{tr}\,[p]^2 \,, \\
&W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla \mathrm{tr}\,[p]) = \mu \frac{L_c^2}{12}\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{k}_p\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{k}_p\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{k}_p]^2 + \alpha_8\,\|\nabla \mathrm{tr}\,[p]\|^2\right) , \\
&\qquad \mathfrak{k}_p = \mathrm{D_x}[\mathrm{dev}\,p] = (\nabla(\mathrm{dev}\,p.e_1), \nabla(\mathrm{dev}\,p.e_2), \nabla(\mathrm{dev}\,p.e_3)) \,.
\end{aligned}$$

Here, $\mathfrak{k}_p$ is the third order infinitesimal curvature tensor, defined only on the purely distortional part of the infinitesimal microdeformation $\mathrm{dev}\,p$. If $\mu_e, \mu^{\mathrm{m}} > 0$ and $\mu_c, \lambda_e, \lambda^{\mathrm{m}} \geq 0$ it is an easy matter to show existence and uniqueness. For $\mu_c = 0$ we have to invoke the classical Korn's first inequality.

It should be observed that even if $\mu_c = 0$ there remains a coupling of the two fields (u,p) due to the remaining coupling in the symmetric terms.

[11] This is not true for other possible basic invariant curvature expressions like $\overline{P}^{-1}\mathrm{D_x}\overline{P}$ or $\overline{P}^T\mathrm{D_x}\overline{P}$ or $F^T\mathrm{D_x}\overline{P}$, see [14, 1.5.4,1.5.11].

3.2 The linear system of balance equations

The linearized macroscopic force balance equation is obtained by taking free variations with respect to the displacement u. Hence we obtain

$$\mathrm{Div}\, \sigma(\nabla u, p) = 0, \quad u_{|\Gamma}(x) = g_{\mathrm{d}}(x) - x \tag{3.15}$$

with

$$\sigma(\nabla u, p) = 2\mu_e\ (\mathrm{sym}\,\nabla u - \mathrm{sym}\, p) + 2\mu_c\ (\mathrm{skew}\,\nabla u - \mathrm{skew}\, p) + \lambda_e\, \mathrm{tr}\,[\nabla u - p] \cdot 1\!\!1\,. \tag{3.16}$$

The remaining system of nine balance equations for the nine additional components of $p \in \mathfrak{gl}^+(3,\mathbb{R}) = \mathbb{M}^{3\times 3}$ is obtained by taking free variations with respect to p which results in

$$\begin{aligned}
\mathrm{dev}\,\mathrm{Div}\, D_{\mathfrak{k}_p} W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla \mathrm{tr}\,[p]) &= \mathrm{dev}\,(-2\mu_e\ (\mathrm{sym}\,\nabla u - \mathrm{sym}\, p) \\
&\quad - 2\mu_c\ (\mathrm{skew}\,\nabla u - \mathrm{skew}\, p) - \lambda_e\, \mathrm{tr}\,[\nabla u - p]\, 1\!\!1 \\
&\quad + 2\mu^{\mathrm{m}}\ \mathrm{sym}\, p + \lambda^{\mathrm{m}}\, \mathrm{tr}\,[p] \cdot 1\!\!1)\ , \\
\mathrm{Div}\, D_{\nabla \mathrm{tr}[p]} W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla \mathrm{tr}\,[p]) &= \mathrm{tr}\,(-2\mu_e\ (\mathrm{sym}\,\nabla u - \mathrm{sym}\, p) \\
&\quad - 2\mu_c\ (\mathrm{skew}\,\nabla u - \mathrm{skew}\, p) - \lambda_e\, \mathrm{tr}\,[\nabla u - p]\, 1\!\!1 \\
&\quad + 2\mu^{\mathrm{m}}\ \mathrm{sym}\, p + \lambda^{\mathrm{m}}\, \mathrm{tr}\,[p] \cdot 1\!\!1)\ ,
\end{aligned} \tag{3.17}$$

which is equivalent to

$$\begin{aligned}
0 &= \mathrm{dev}\,\sigma(\nabla u, p) - 2\mu^{\mathrm{m}}\ \mathrm{dev}\,\mathrm{sym}\, p + \mathrm{dev}\,\mathrm{Div}\, D_{\mathfrak{k}_p} W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla \mathrm{tr}\,[p])\,, \\
0 &= \mathrm{tr}\,[\sigma(\nabla u, p)] - (2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})\, \mathrm{tr}\,[p] + \mathrm{Div}\, D_{\nabla \mathrm{tr}[p]} W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla \mathrm{tr}\,[p])\,.
\end{aligned} \tag{3.18}$$

3.3 Calculation of consistent effective elastic moduli

It is of prime importance to have values of μ_e, λ_e at hand which are consistent with the classical linear elastic model for long wave-length (large samples). Considering very large samples of the cellular structure amounts to letting L_c, the characteristic length, tend to zero. As a consequence of $L_c = 0$ equation (3.18) looses the curvature terms and turns into

$$\begin{aligned}
0 &= \mathrm{dev}\,\sigma(\nabla u, p) - 2\mu^{\mathrm{m}}\ \mathrm{dev}\,\mathrm{sym}\, p\,, \\
0 &= \mathrm{tr}\,[\sigma(\nabla u, p)] - (2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})\, \mathrm{tr}\,[p]\,,
\end{aligned} \tag{3.19}$$

expressing an algebraic side-condition. Inserting formula (3.16) for σ into (3.19) allows us to obtain after some lengthy but straightforward computations the following algebraic relations

$$\begin{aligned}
\mathrm{tr}\,[p] &= \frac{(2\mu_e + 3\lambda_e)}{2(\mu_e + \mu^{\mathrm{m}}) + 3(\lambda_e + \lambda^{\mathrm{m}})}\, \mathrm{tr}\,[\nabla u]\,, \\
\mathrm{dev}\,\mathrm{sym}\, p &= \frac{\mu_e}{(\mu_e + \mu^{\mathrm{m}})}\ \mathrm{dev}\,\mathrm{sym}\,\nabla u\ , \\
\mathrm{dev}\,\mathrm{skew}\, p &= \mathrm{dev}\,\mathrm{skew}\,\nabla u\ , \qquad (\text{without } \mu_c)\,,
\end{aligned} \tag{3.20}$$

where we used that dev is orthogonal to $\mathbb{R} \cdot 1\!\!1$ and sym is orthogonal to skew and $\mathrm{dev}\,\mathrm{skew} = \mathrm{skew}$. Moreover,

$$\begin{aligned}
\mathrm{tr}\,[\nabla u - p] &= \left(1 - \frac{(2\mu_e + 3\lambda_e)}{2(\mu_e + \mu^{\mathrm{m}}) + 3(\lambda_e + \lambda^{\mathrm{m}})}\right) \mathrm{tr}\,[\nabla u] \\
&= \frac{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})}{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}}) + (2\mu_e + 3\lambda_e)}\, \mathrm{tr}\,[\nabla u]\,.
\end{aligned} \tag{3.21}$$

Reinserting the results into (3.16) yields, after taking dev on both sides

$$\begin{aligned}
\mathrm{dev}\,\sigma(\nabla u, p) &= 2\mu_e\ (\mathrm{dev}\,\mathrm{sym}\,\nabla u - \mathrm{dev}\,\mathrm{sym}\, p) + 2\mu_c\ (\mathrm{skew}\,\nabla u - \mathrm{skew}\, p) \\
&= 2\mu_e\ \left(\mathrm{dev}\,\mathrm{sym}\,\nabla u - \frac{\mu_e}{(\mu_e + \mu^{\mathrm{m}})}\ \mathrm{dev}\,\mathrm{sym}\,\nabla u\right) + 2\mu_c\ (\mathrm{skew}\,\nabla u - 1 \cdot \mathrm{skew}\,\nabla u) \\
&= 2\mu_e\ \left(1 - \frac{\mu_e}{(\mu_e + \mu^{\mathrm{m}})}\right) \mathrm{dev}\,\mathrm{sym}\,\nabla u = 2\mu_e \frac{\mu^{\mathrm{m}}}{(\mu_e + \mu^{\mathrm{m}})}\ \mathrm{dev}\,\mathrm{sym}\,\nabla u\,.
\end{aligned} \tag{3.22}$$

Similarly, reinserting the results into (3.16) yields, after taking the trace on both sides

$$\begin{aligned}\operatorname{tr}[\sigma(\nabla u,p)] &= 2\mu_e \operatorname{tr}[\operatorname{sym}\nabla u - \operatorname{sym} p] + 2\mu_c \operatorname{tr}[\operatorname{skew}\nabla u - \operatorname{skew} p] + \lambda_e \operatorname{tr}[\nabla u - p]\cdot \operatorname{tr}[\mathbb{1}]\\ &= 2\mu_e \operatorname{tr}[\nabla u - p] + 3\lambda_e \operatorname{tr}[\nabla u - p] = (2\mu_e + 3\lambda_e)\,\operatorname{tr}[\nabla u - p]\\ &= (2\mu_e + 3\lambda_e)\,\frac{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})}{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}}) + (2\mu_e + 3\lambda_e)}\operatorname{tr}[\nabla u]\,. \end{aligned}\tag{3.23}$$

For a classical linear elastic isotropic solid, which represents the macroscopic stress-strain relation for large samples, one has the relation

$$\begin{aligned}\sigma &= 2\mu\,\operatorname{sym}\nabla u + \lambda \operatorname{tr}[\nabla u]\cdot \mathbb{1} \quad \Rightarrow\\ \operatorname{dev}\sigma &= 2\mu\,\operatorname{dev}\operatorname{sym}\nabla u \quad \text{and} \quad \operatorname{tr}[\sigma] = (2\mu+3\lambda)\,\operatorname{tr}[\nabla u]\,.\end{aligned}\tag{3.24}$$

Upon comparing coefficients of (3.24) with (3.22) and (3.23) we identify

$$\begin{aligned} 2\mu &= 2\mu_e \frac{\mu^{\mathrm{m}}}{(\mu_e + \mu^{\mathrm{m}})}\,,\\ (2\mu+3\lambda) &= (2\mu_e + 3\lambda_e)\,\frac{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})}{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}}) + (2\mu_e + 3\lambda_e)}\,.\end{aligned}\tag{3.25}$$

This shows that the **large scale shear modulus** μ is half the **harmonic mean**[12] of the **relative elastic shear modulus** μ_e and the **microstructural shear modulus** μ^{m}, while the **large scale bulk modulus** $\kappa = \frac{2\mu+3\lambda}{3}$ is half the **harmonic mean** of the **relative elastic bulk modulus** κ_e and the **microstructural bulk modulus** κ^{m}.

Hence, solving in a first step for the **relative elastic shear modulus** μ_e and the **relative elastic bulk modulus** $\kappa_e = \frac{2\mu_e+3\lambda_e}{3}$, yields

$$\mu_e = \frac{\mu^{\mathrm{m}}\,\mu}{(\mu^{\mathrm{m}} - \mu)}\,,\quad 3\kappa_e = (2\mu_e + 3\lambda_e) = \frac{(2\mu+3\lambda)\,(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})}{(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}}) - (2\mu+3\lambda)}\,.\tag{3.26}$$

Therefore

$$\mu_e = \frac{\mu^{\mathrm{m}}\,\mu}{(\mu^{\mathrm{m}} - \mu)}\,,\quad 3\lambda_e = \frac{(2\mu+3\lambda)\,(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})}{(2(\mu^{\mathrm{m}} - \mu) + 3(\lambda^{\mathrm{m}} - \lambda))} - 2\frac{\mu^{\mathrm{m}}\,\mu}{(\mu^{\mathrm{m}} - \mu)}\,.\tag{3.27}$$

This shows that the "macroscopic" Lamé moduli μ, λ must always be smaller than the microscopic moduli $\mu^{\mathrm{m}}, \lambda^{\mathrm{m}}$ related to the response of a representative volume element (REV) of the substructure. This is physically consistent: the large-scale sample cannot possibly be stiffer than the constitutive substructure.

Let us consider the interesting limit cases in (3.25):

$$\begin{aligned} \textbf{microincompressible:}\quad & \lambda^{\mathrm{m}} \to \infty\,,\quad \mu^{\mathrm{m}} < \infty \quad \Rightarrow \quad \lambda = \lambda_e + \frac{2\mu^2}{3(\mu^{\mathrm{m}} - \mu)}\,,\\ \textbf{microstretch:}\quad & \mu^{\mathrm{m}} \to \infty\,,\quad \lambda^{\mathrm{m}} < \infty \quad \Rightarrow \quad \lambda = \lambda_e\,,\quad \mu = \mu_e\,,\\ \textbf{micropolar:}\quad & \mu^{\mathrm{m}} \to \infty\,,\quad \lambda^{\mathrm{m}} \to \infty \quad \Rightarrow \quad \lambda = \lambda_e\,,\quad \mu = \mu_e\,.\end{aligned}\tag{3.28}$$

3.4 Identification with Mindlin's representation

Many papers on linearized micromorphic models start from a representation of the free-energy function based on Mindlin's work [36, 5.5], e.g. [28]. A major drawback of Mindlin's representation is, however, that now account has been taken, to ensure overall positivity of the quadratic energy. This has to be checked additionally and can be quite labourous because of many appearing coefficients. We consider only the local part (the part without curvature) of Mindlin's representation. Let us define

$$\varepsilon = \operatorname{sym}\nabla u\,,\qquad \overline{\varepsilon} := \nabla u - p\,.\tag{3.29}$$

Then Mindlin's local energy contribution with seven material constants $\widehat{\mu}, \widehat{\lambda}, b_1, b_2, b_3, g_1, g_2$ reads

$$\begin{aligned} W^{\mathrm{Mind}}(\nabla u, p) = W^{\mathrm{Mind}}(\varepsilon, \overline{\varepsilon}) &= \frac{\widehat{\lambda}}{2}\operatorname{tr}[\varepsilon]^2 + \widehat{\mu}\,\|\varepsilon\|^2 + \frac{b_1}{2}\operatorname{tr}[\overline{\varepsilon}]^2 + \frac{b_2}{2}\|\overline{\varepsilon}\|^2 + \frac{b_3}{2}\langle \overline{\varepsilon}, \overline{\varepsilon}^T\rangle\\ &\quad + g_1 \operatorname{tr}[\varepsilon]\operatorname{tr}[\overline{\varepsilon}] + g_2\,\langle \varepsilon, \overline{\varepsilon}\rangle\,.\end{aligned}\tag{3.30}$$

[12] $\mathcal{H}(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{2\alpha\beta}{\alpha+\beta}$ for $\alpha, \beta > 0$, compare with the **Reuss-bounds** in homogenization theory.

Note that this is a quadratic form, whose positiveness is not ensured by taking positive parameters! In comparison, I have proposed a five material constants representation, which automatically defines a positive quadratic form, if the coefficients are positive themselves.[13] The proposed quadratic representation in (3.14) reads

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{\varepsilon}, p) &= \mu_e \,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 + \mu^{\mathrm{m}}\,\|\operatorname{sym} p\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\operatorname{sym} p]^2 \\
&= \mu_e \,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 + \mu^{\mathrm{m}}\,\|\operatorname{sym} p - \varepsilon + \varepsilon\|^2 \\
&\quad + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\operatorname{sym} p - \varepsilon + \varepsilon]^2 \\
&= \mu_e \,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\left(\|\operatorname{sym} p - \varepsilon\|^2 + 2\langle \operatorname{sym} p - \varepsilon, \varepsilon\rangle + \|\varepsilon\|^2\right) \\
&\quad + \frac{\lambda^{\mathrm{m}}}{2}\left(\operatorname{tr}[\operatorname{sym} p - \varepsilon]^2 + 2\operatorname{tr}[\operatorname{sym} p - \varepsilon]\operatorname{tr}[\varepsilon] + \operatorname{tr}[\varepsilon]^2\right) \qquad (3.31)\\
&= (\mu_e + \mu^{\mathrm{m}})\,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{(\lambda_e + \lambda^{\mathrm{m}})}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\,\|\varepsilon\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\varepsilon]^2 \\
&\quad - 2\mu^{\mathrm{m}}\langle \varepsilon - \operatorname{sym} p, \varepsilon\rangle - \lambda^{\mathrm{m}}\operatorname{tr}[\varepsilon - \operatorname{sym} p]\operatorname{tr}[\varepsilon] \\
&= (\mu_e + \mu^{\mathrm{m}})\,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{(\lambda_e + \lambda^{\mathrm{m}})}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\,\|\varepsilon\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\varepsilon]^2 - 2\mu^{\mathrm{m}}\langle \overline{\varepsilon}, \varepsilon\rangle - \lambda^{\mathrm{m}}\operatorname{tr}[\overline{\varepsilon}]\operatorname{tr}[\varepsilon] \\
&= (\mu_e + \mu^{\mathrm{m}})\,\|\frac{1}{2}(\overline{\varepsilon} + \overline{\varepsilon}^T)\|^2 + \mu_c\,\|\frac{1}{2}(\overline{\varepsilon} - \overline{\varepsilon}^T)\|^2 + \frac{(\lambda_e + \lambda^{\mathrm{m}})}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\,\|\varepsilon\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\varepsilon]^2 - 2\mu^{\mathrm{m}}\langle \overline{\varepsilon}, \varepsilon\rangle - \lambda^{\mathrm{m}}\operatorname{tr}[\overline{\varepsilon}]\operatorname{tr}[\varepsilon] \\
&= \frac{(\mu_e + \mu^{\mathrm{m}})}{4}\,\|\overline{\varepsilon} + \overline{\varepsilon}^T\|^2 + \frac{\mu_c}{4}\,\|\overline{\varepsilon} - \overline{\varepsilon}^T\|^2 + \frac{(\lambda_e + \lambda^{\mathrm{m}})}{2}\operatorname{tr}[\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\,\|\varepsilon\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\varepsilon]^2 - 2\mu^{\mathrm{m}}\langle \overline{\varepsilon}, \varepsilon\rangle - \lambda^{\mathrm{m}}\operatorname{tr}[\overline{\varepsilon}]\operatorname{tr}[\varepsilon] \\
&= \frac{(\mu_e + \mu^{\mathrm{m}})}{2}\left(\|\overline{\varepsilon}\|^2 + \langle \overline{\varepsilon}, \overline{\varepsilon}^T\rangle\right) + \frac{\mu_c}{2}\left(\|\overline{\varepsilon}\|^2 - \langle \overline{\varepsilon}, \overline{\varepsilon}^T\rangle\right) + \frac{(\lambda_e + \lambda^{\mathrm{m}})}{2}\operatorname{tr}[\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\,\|\varepsilon\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\varepsilon]^2 - 2\mu^{\mathrm{m}}\langle \overline{\varepsilon}, \varepsilon\rangle - \lambda^{\mathrm{m}}\operatorname{tr}[\overline{\varepsilon}]\operatorname{tr}[\varepsilon] \\
&= \frac{(\mu_e + \mu^{\mathrm{m}} + \mu_c)}{2}\,\|\overline{\varepsilon}\|^2 + \frac{(\mu_e + \mu^{\mathrm{m}} - \mu_c)}{2}\langle \overline{\varepsilon}, \overline{\varepsilon}^T\rangle + \frac{(\lambda_e + \lambda^{\mathrm{m}})}{2}\operatorname{tr}[\overline{\varepsilon}]^2 \\
&\quad + \mu^{\mathrm{m}}\,\|\varepsilon\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\operatorname{tr}[\varepsilon]^2 - 2\mu^{\mathrm{m}}\langle \overline{\varepsilon}, \varepsilon\rangle - \lambda^{\mathrm{m}}\operatorname{tr}[\overline{\varepsilon}]\operatorname{tr}[\varepsilon]\,.m
\end{aligned}$$

Hence, comparing with Mindlin's representation (3.30) we can identify

$$\begin{aligned}
&\widehat{\mu} = \mu^{\mathrm{m}}, \qquad \widehat{\lambda} = \lambda^{\mathrm{m}}, \qquad b_1 = \lambda_e + \lambda^{\mathrm{m}}\,, \\
&b_2 = \mu_e + \mu^{\mathrm{m}} + \mu_c, \qquad b_3 = \mu_e + \mu^{\mathrm{m}} - \mu_c\,, \\
&g_1 = -\lambda^{\mathrm{m}}, \qquad g_2 = -2\mu^{\mathrm{m}}\,.
\end{aligned} \qquad (3.32)$$

Mindlin proposes [36, p.60]

$$\begin{aligned}
&3b_1 + b_2 + b_3 \geq 0\,, \quad b_2 + b_3 \geq 0\,, \quad b_2 - b_3 \geq 0 \quad \Rightarrow \\
&\kappa_e + \kappa^{\mathrm{m}} \geq 0\,, \quad \mu_e + \mu_m \geq 0\,, \quad \mu_c \geq 0\,,
\end{aligned} \qquad (3.33)$$

as necessary conditions for a positive definite energy function which is verified for (3.14).

Remark 3.1
It is not clear to the author, whether Mindlin's seven parameter representation of the local strain-energy could be obtained by consistently linearizing a finite-strain micromorphic model.

[13] This can be slightly weakened: $2\mu_e + 3\lambda_e \geq 0, 2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}} \geq 0, \mu_e, \mu^{\mathrm{m}}, \mu_c \geq 0$ is sufficient.

3.5 Identification with Eringen's formulation

In [14, 7.1.15] the following local energy representation with seven independent parameters $\tilde{\lambda}, \tilde{\nu}, \tilde{\tau}, \tilde{\mu}, \tilde{\sigma}, \tilde{\eta}, \tilde{\kappa}$ has been taken:

$$\begin{aligned} W^{\mathrm{Er}}(\nabla u, p) = W^{\mathrm{Er}}(\overline{\varepsilon}, \operatorname{sym} p) &= \frac{\tilde{\lambda}}{2} \operatorname{tr}[\overline{\varepsilon}]^2 + \frac{(\tilde{\lambda} + \tilde{\tau} + 2\tilde{\nu})}{2} \operatorname{tr}[\operatorname{sym} p]^2 + (\tilde{\lambda} + \tilde{\tau}) \operatorname{tr}[\overline{\varepsilon}] \operatorname{tr}[\operatorname{sym} p] \\ &+ \frac{\tilde{\mu}}{2} \langle \overline{\varepsilon}, \overline{\varepsilon}^T \rangle + (\tilde{\mu} + \tilde{\sigma} + \tilde{\sigma} + \tilde{\eta}) \| \operatorname{sym} p \|^2 \\ &+ 2(\tilde{\mu} + \tilde{\sigma}) \langle \overline{\varepsilon}, \operatorname{sym} p \rangle + \frac{(\tilde{\mu} + \tilde{\kappa})}{2} \| \overline{\varepsilon} \|^2 . \end{aligned} \tag{3.34}$$

Again, positivity of this quadratic form has to be ensured a posteriori. Consider the five parameter representation in (3.14)

$$\begin{aligned} W_{\mathrm{mp}}(\overline{\varepsilon}, p) &= \mu_e \| \operatorname{sym} \overline{\varepsilon} \|^2 + \mu_c \| \operatorname{skew} \overline{\varepsilon} \|^2 + \frac{\lambda_e}{2} \operatorname{tr}[\operatorname{sym} \overline{\varepsilon}]^2 + \mu^{\mathrm{m}} \| \operatorname{sym} p \|^2 + \frac{\lambda^{\mathrm{m}}}{2} \operatorname{tr}[\operatorname{sym} p]^2 \\ &= \mu_e \| \overline{\varepsilon} \|^2 + (\mu_c - \mu_e) \| \operatorname{skew} \overline{\varepsilon} \|^2 + \frac{\lambda_e}{2} \operatorname{tr}[\operatorname{sym} \overline{\varepsilon}]^2 + \mu^{\mathrm{m}} \| \operatorname{sym} p \|^2 + \frac{\lambda^{\mathrm{m}}}{2} \operatorname{tr}[\operatorname{sym} p]^2 \\ &= \frac{\mu_e + \mu_c}{2} \| \overline{\varepsilon} \|^2 + \frac{\mu_e - \mu_c}{2} \langle \overline{\varepsilon}, \overline{\varepsilon}^T \rangle + \frac{\lambda_e}{2} \operatorname{tr}[\operatorname{sym} \overline{\varepsilon}]^2 + \mu^{\mathrm{m}} \| \operatorname{sym} p \|^2 + \frac{\lambda^{\mathrm{m}}}{2} \operatorname{tr}[\operatorname{sym} p]^2 . \end{aligned} \tag{3.35}$$

Now, take $\tilde{\lambda} + \tilde{\tau} = 0$, $\tilde{\mu} + \tilde{\sigma} = 0$. Then (3.34) reduces to

$$W^{\mathrm{Er}}(\overline{\varepsilon}, \operatorname{sym} p) = \frac{(\tilde{\mu} + \tilde{\kappa})}{2} \| \overline{\varepsilon} \|^2 + \frac{\tilde{\lambda}}{2} \operatorname{tr}[\overline{\varepsilon}]^2 + \tilde{\nu} \operatorname{tr}[\operatorname{sym} p]^2 + \frac{\tilde{\mu}}{2} \langle \overline{\varepsilon}, \overline{\varepsilon}^T \rangle + (\tilde{\eta} - \tilde{\mu}) \| \operatorname{sym} p \|^2 . \tag{3.36}$$

Identification is now obtained by setting

$$(\tilde{\mu} + \tilde{\kappa}) = \mu_e + \mu_c , \quad \tilde{\lambda} = \lambda_e , \quad \tilde{\mu} = \mu_e - \mu_c , \quad (\tilde{\eta} - \tilde{\mu}) = \mu^{\mathrm{m}} , \quad \tilde{\nu} = \frac{\lambda^{\mathrm{m}}}{2} . \tag{3.37}$$

This implies notably

$$\tilde{\kappa} = 2\mu_c , \quad \tilde{\eta} = \mu_e + \mu^{\mathrm{m}} - \mu_c = b_3 . \tag{3.38}$$

It should be noted again, that despite notation, Eringen's $\tilde{\mu}$ is neither the corresponding Lamé constant related to the representative volume element of the substructure nor of the bulk or the relative shear modulus, while $\tilde{\lambda}$ is the corresponding relative modulus. Unfortunately, this has led to some confusion in the literature.

3.6 Formulation as a three-field problem

In a constitutive context it is useful to clearly separate influences due to volumetric changes and due to distortional effects. This can be most easily done if we incorporate an additional curvature term due to only volumetric changes of the substructure. Introducing the additive split

$$p = \underbrace{\operatorname{dev} p}_{\mathfrak{sl}(3,\mathbb{R})} + \frac{1}{3} \operatorname{tr}[p] \cdot 1\!\!1 = \overline{p} + \frac{1}{3} \overline{\alpha}_p \cdot 1\!\!1 , \quad \overline{p} := \operatorname{dev} p , \quad \overline{\alpha}_p := \operatorname{tr}[p] , \tag{3.39}$$

we may write the linearized micromorphic problem (3.14) equivalently as a **three-field problem**

$$\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon},\overline{p},\overline{\alpha}_p) + W_{\mathrm{curv}}(\mathfrak{k}_p,\nabla\overline{\alpha}_p)\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } (u,\overline{p},\overline{\alpha}_p), \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}\,, \tag{3.40}$$

$$\overline{\varepsilon} = \nabla u - \overline{p} - \frac{1}{3}\overline{\alpha}_p\,1\!\!1, \quad p_{|_\Gamma} = p_{\mathrm{d}} \in \mathfrak{gl}^+(3,\mathbb{R}) = \mathbb{M}^{3\times 3}, \quad \overline{p}\in\mathfrak{sl}(3,\mathbb{R}),\ \overline{\alpha}_p\in\mathbb{R},$$

$$W_{\mathrm{mp}}(\overline{\varepsilon},p) = \mu_e\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\left[\mathrm{sym}\,\overline{\varepsilon}\right]^2 + \mu^{\mathrm{m}}\,\|\mathrm{sym}\,p\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\mathrm{tr}\left[\mathrm{sym}\,p\right]^2$$

$$= \mu_e\,\|\mathrm{sym}\,\nabla u - \mathrm{sym}\,\overline{p} - \frac{1}{3}\overline{\alpha}_p\cdot 1\!\!1\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u - \overline{p})\|^2$$

$$+ \frac{\lambda_e}{2}\,\mathrm{tr}\left[\nabla u - \frac{1}{3}\overline{\alpha}_p\cdot 1\!\!1\right]^2 + \mu^{\mathrm{m}}\,\|\mathrm{sym}\,\overline{p}\|^2 + \left(\frac{2\mu^{\mathrm{m}}+3\lambda^{\mathrm{m}}}{6}\right)\overline{\alpha}_p^2\,,$$

$$W_{\mathrm{curv}}(\mathfrak{k}_p,\nabla\overline{\alpha}_p) = \mu\frac{L_c^2}{12}\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{k}_p\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{k}_p\|^2 + \alpha_7\,\mathrm{tr}\left[\mathfrak{k}_p\right]^2 + \alpha_8\,\|\nabla\overline{\alpha}_p\|^2\right)\,,$$

$$\mathfrak{k}_p = \mathrm{D_x}\overline{p} = (\nabla(\overline{p}.e_1),\nabla(\overline{p}.e_2),\nabla(\overline{p}.e_3))\,, \quad \text{infinitesimal } \textbf{curvature tensor}\,.$$

3.7 The infinitesimal micro-incompressible micromorphic elastic solid

Starting with the finite-strain formulation we may obtain a linear, infinitesimal microincompressible, micromorphic model by expanding all appearing variables to first order and keeping quadratic terms in the energy expression. Thus we write $F = 1\!\!1 + \nabla u$, $\overline{P} = 1\!\!1 + \overline{p}$, $\mathrm{tr}\left[\overline{p}\right] = 0$ and the model turns into the problem of finding a pair $(u,\overline{p}) : \Omega\subset\mathbb{R}^3 \mapsto \mathbb{R}^3\times\mathfrak{sl}(3,\mathbb{R})$ of displacement u and **independent, infinitesimal microdeformation** $\overline{p}$ satisfying

$$\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon},\overline{p}) + W_{\mathrm{curv}}(\mathfrak{k}_p)\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } (u,\overline{p}),$$

$$\overline{\varepsilon} = \nabla u - \overline{p}, \quad \overline{p}_{|_\Gamma} = \overline{p}_{\mathrm{d}}\in\mathfrak{sl}(3,\mathbb{R}), \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}\,, \tag{3.41}$$

$$W_{\mathrm{mp}}(\overline{\varepsilon},\overline{p}) = \mu_e\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\left[\mathrm{sym}\,\overline{\varepsilon}\right]^2 + \mu^{\mathrm{m}}\,\|\mathrm{sym}\,\overline{p}\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\mathrm{tr}\left[\mathrm{sym}\,\overline{p}\right]^2$$

$$= \mu_e\,\|\mathrm{sym}\,\nabla u - \mathrm{sym}\,\overline{p}\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u - \overline{p})\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\left[\mathrm{sym}\,\nabla u\right]^2 + \mu^{\mathrm{m}}\,\|\mathrm{sym}\,\overline{p}\|^2\,,$$

$$W_{\mathrm{curv}}(\mathfrak{k}_p) = \mu\frac{L_c^2}{12}\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{k}_p\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{k}_p\|^2 + \alpha_7\,\mathrm{tr}\left[\mathfrak{k}_p\right]^2\right)\,,$$

$$\mathfrak{k}_p = \mathrm{D_x}\overline{p} = (\nabla(\overline{p}.e_1),\nabla(\overline{p}.e_2),\nabla(\overline{p}.e_3))\,, \quad \text{third order, infinitesimal } \textbf{curvature tensor}\,.$$

If $\mu_e,\mu^{\mathrm{m}} > 0$ and $\mu_c,\lambda_e,\lambda^{\mathrm{m}}\geq 0$ it is an easy matter to show existence and uniqueness.

3.8 The infinitesimal microstretch elastic solid

Such a model is obtained by assuming $P = e^{\frac{\overline{\alpha}_p}{3}}\,\overline{R}_p$, $\overline{\alpha}_p\in\mathbb{R}$, $\overline{R}_p\in\mathrm{SO}(3,\mathbb{R})$ with independent variables $\overline{\alpha}_p,\overline{R}_p$ and independent curvature parts $\mathfrak{K}_p = \overline{R}_p^T\mathrm{D_x}\overline{R}_p$ and $\nabla\overline{\alpha}_p(x,y,z)$. Inserting this assumption into the finite-strain model and expanding $e^{\frac{\overline{\alpha}_p}{3}} = 1 + \frac{\overline{\alpha}_p}{3} + \ldots$, $\overline{R}_p = 1\!\!1 + \overline{A} + \ldots$ for small $(\overline{A},\overline{\alpha}_p)$ yields to first order the **three-field** problem

$$\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon},\overline{\alpha}_p) + W_{\mathrm{curv}}(\mathfrak{k}_p,\nabla\overline{\alpha}_p)\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } (u,\overline{A},\overline{\alpha}_p), \tag{3.42}$$

$$\overline{\varepsilon} = \nabla u - \overline{A} - \frac{\overline{\alpha}_p}{3}\cdot 1\!\!1, \quad \overline{A}_{|_\Gamma} = \overline{A}_{\mathrm{d}}\in\mathfrak{so}(3,\mathbb{R}), \quad \overline{\alpha}_{p_{|_\Gamma}} = \overline{\alpha}_{p,\mathrm{d}_{|_\Gamma}}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}\,,$$

$$\overline{U}_p = \sqrt{\overline{P}^T\overline{P}} = \sqrt{e^{2\overline{\alpha}_p/3}\overline{R}_p^T\overline{R}_p} = e^{\frac{\overline{\alpha}_p}{3}}\cdot 1\!\!1 = (1 + \frac{\overline{\alpha}_p}{3} + \ldots)\cdot 1\!\!1 = 1\!\!1 + \frac{\overline{\alpha}_p}{3}\cdot 1\!\!1 + \ldots\,,$$

$$W_{\mathrm{mp}}(\overline{\varepsilon},\overline{\alpha}_p) = \mu_e\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\left[\mathrm{sym}\,\overline{\varepsilon}\right]^2 + \mu^{\mathrm{m}}\,\|\frac{\overline{\alpha}_p}{3}\cdot 1\!\!1\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\mathrm{tr}\left[\frac{\overline{\alpha}_p}{3}\cdot 1\!\!1\right]^2$$

$$= \mu_e\,\|\mathrm{sym}\,\nabla u - \frac{\overline{\alpha}_p}{3}\cdot 1\!\!1\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda_e}{2}\,\mathrm{tr}\left[\mathrm{sym}\,\nabla u - \frac{\overline{\alpha}_p}{3}\cdot 1\!\!1\right]^2$$

$$+ \left(\frac{2\mu^{\mathrm{m}}+3\lambda^{\mathrm{m}}}{6}\right)\overline{\alpha}_p^2\,,$$

$$W_{\mathrm{curv}}(\mathfrak{k}_p, \nabla\overline{\alpha}_p) = \mu \frac{L_c^2}{12}\left(\alpha_5 \,\|\operatorname{sym}\mathfrak{k}_p\|^2 + \alpha_6 \,\|\operatorname{skew}\mathfrak{k}_p\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{k}_p]^2 + \alpha_8 \|\nabla\overline{\alpha}_p\|^2\right),$$
$$\mathfrak{k}_p = \mathrm{D_x}\overline{A} = \left(\nabla(\overline{A}.e_1), \nabla(\overline{A}.e_2), \nabla(\overline{A}.e_3)\right), \quad \text{infinitesimal } \textbf{curvature tensor}.$$

3.9 The infinitesimal micropolar elastic solid

Such a model is obtained by setting $\overline{\alpha}_p \equiv 0$ in (3.42). We are left with the **two-field** problem

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\mathfrak{k}_p)\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } (u, \overline{A}),\\
&\overline{\varepsilon} = \nabla u - \overline{A}, \quad \overline{A}_{|\Gamma} = \overline{A}_{\mathrm{d}} \in \mathfrak{so}(3,\mathbb{R}), \quad \varphi_{|\Gamma} = g_{\mathrm{d}}, \qquad (3.43)\\
&W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu_e \,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda_e}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2\\
&\qquad = \mu_e \,\|\operatorname{sym}\nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda_e}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2,\\
&W_{\mathrm{curv}}(\mathfrak{k}_p) = \mu\frac{L_c^2}{12}\left(\alpha_5 \,\|\operatorname{sym}\mathfrak{k}_p\|^2 + \alpha_6 \,\|\operatorname{skew}\mathfrak{k}_p\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{k}_p]^2\right),\\
&\mathfrak{k}_p = \mathrm{D_x}\overline{A} = \left(\nabla(\overline{A}.e_1), \nabla(\overline{A}.e_2), \nabla(\overline{A}.e_3)\right), \quad \text{micropolar } \textbf{curvature tensor}.
\end{aligned}$$

Note that for $\mu_c = 0$ the two fields completely decouple which must be seen as a deficiency of the infinitesimal micropolar model. This allows us to appreciate the exceptional role played by a coupling only through (infinitesimal) rotations.

3.10 The infinitesimal, non-polar classical linear elastic solid

Only for completeness we note the classical **one-field** linear elasticity formulation

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\varepsilon)\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } u, \quad \varepsilon = \operatorname{sym}\nabla u, \quad \varphi_{|\Gamma} = g_{\mathrm{d}}, \qquad (3.44)\\
&W_{\mathrm{mp}}(\varepsilon) = \mu\,\|\varepsilon\|^2 + \frac{\lambda}{2}\operatorname{tr}[\varepsilon]^2 = \mu\,\|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2.
\end{aligned}$$

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5 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and by $\mathfrak{T}(3)$ the set of all third order tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \;|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|X^TX = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \;|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \;|\operatorname{tr}[X] = 0\}$ of traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\, \eta_j$. We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. For a second order tensor X we define the third order tensor $\mathfrak{h} = \mathrm{D}_{\mathrm{x}}X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3}$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}\left[\mathfrak{h}^1\right], \operatorname{tr}\left[\mathfrak{h}^2\right], \operatorname{tr}\left[\mathfrak{h}^3\right]) \in \mathbb{R}^3$. Moreover, for any second order tensor X we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$ correspondingly. Quantities with a bar, e.g. the micropolar rotation $\overline{R}_p$, represent the micropolar replacement of the corresponding classical continuum rotation R. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$ and we use ∇ in general only for column-vectors in $\mathbb{R}^3$. The first differential of a scalar valued function $W(F)$ is written $D_F W(F).H$. Sometimes we use also $\partial_X W(X)$ to denote the first derivative of W with respect to X. For $X \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we define $\operatorname{Div} X(x)$ as the operation Div applied row wise. For $\mathfrak{h} \in \mathfrak{T}(3)$ we define $\operatorname{Div}\mathfrak{h} = \left(\operatorname{Div}\mathfrak{h}^1 | \operatorname{Div}\mathfrak{h}^2 | \operatorname{Div}\mathfrak{h}^3\right)^T \in \mathbb{M}^{3\times 3}$. Finally, w.r.t. abbreviates with respect to.