Resolvent Estimates for the Stokes Operator on an Infinite Layer

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October 4, 2004

Abstract

In this paper we prove unique solvability of the generalized Stokes resolvent equations in an infinite layer $\Omega = \mathbb{R}^{n-1} \times (-1,1)$, $n \geq 2$, in L^q-Sobolev spaces, $1 < q < \infty$, with non-slip boundary condition $u|_{\partial\Omega} = 0$. The unique solvability is proved for every $\lambda \in \mathbb{C} \setminus (-\infty, -\pi^2/4]$, where $-\frac{\pi^2}{4}$ is the least upper bound of the spectrum of Dirichlet realization of the Laplacian and the Stokes operator in Ω. Moreover, we provide uniform estimates of the solutions for large spectral parameter λ as well as λ close to $-\frac{\pi^2}{4}$. Because of the special geometry of the domain, partial Fourier transformation is used to calculate the solution explicitly. Then Fourier multiplier theorems are used to estimate the solution operator.

Key words: Stokes equations, Stokes operator, infinite layer, resolvent estimates
AMS-Classification: 35 Q 30, 76 D 07

1 Introduction

In the present contribution, we study the generalized Stokes resolvent equation

$$(\lambda - \Delta)u + \nabla q = f \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} u = g \qquad \text{in } \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega = \mathbb{R}^{n-1} \times (-1,1)$, $n \geq 2$, is an infinite layer. The Stokes resolvent equations are the starting point for the construction of strong solutions of the Navier-Stokes equations using semi-group theory, cf. e.g. [10, 11, 18].

Our main result is

THEOREM 1.1 *Let $1 < p < \infty$ and $\lambda \in \mathbb{C} \setminus (-\infty, -\frac{\pi^2}{4}]$. Then for every $f \in L_p(\Omega)^n$ and $g \in W^1_p(\Omega) \cap \hat{W}^{-1}_p(\Omega)$ there is a unique solution $(u,q) \in W^2_p(\Omega)^n \times \dot{W}^1_p(\Omega)$ of (1.1)-(1.3). Moreover, for every $\varepsilon > 0$ and $p \leq r < \infty$ with $\frac{n-1}{2}(\frac{1}{p} - \frac{1}{r}) \leq 1$ there is a constant C_ε such that*

$$\left|\lambda + \pi^2/4\right| \|u\|_p + \frac{|\lambda + \pi^2/4|}{(1+|\lambda|)} \|\nabla^2 u\|_p + \|\nabla q\|_p \leq C_\varepsilon \left(\|(f, \nabla g)\|_p + (1+|\lambda|)\|g\|_{\hat{W}^{-1}_p} \right)$$

$$\left|\lambda + \pi^2/4\right|^{1-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{r})} \|u\|_r \leq C_\varepsilon \left(\|(f, \nabla g)\|_p + (1+|\lambda|)\|g\|_{\hat{W}^{-1}_p} \right)$$

uniformly in $\lambda \in S_\varepsilon := \mathbb{C} \setminus V_\varepsilon$, where $V_\varepsilon := \{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq -\varepsilon(\operatorname{Re} \lambda + \pi^2/4)\}$.

Here $W^m_p(\Omega)$, $m \in \mathbb{N}$, denotes the usual Sobolev space of order m based on the standard Lebesgue space $L_p(\Omega)$, $1 \leq p \leq \infty$, and

$$\begin{aligned} \dot{W}^1_p(\Omega) &:= \left\{ q \in L_{p,\mathrm{loc}}(\overline{\Omega}) : \nabla q \in L_p(\Omega) \right\} \\ \hat{W}^{-1}_p(\Omega) &:= \left\{ g \in L_{p,\mathrm{loc}}(\overline{\Omega}) : \sup_{\varphi \in C^\infty_{(0)}(\overline{\Omega})} |\langle g, \varphi \rangle| \|\nabla \varphi\|^{-1}_{p'} < \infty \right\}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L_{p,\mathrm{loc}}(\overline{\Omega})$ if and only if $f \in L_p(\Omega \cap B)$ for all balls $B \subset \mathbb{R}^n$ with $B \cap \Omega \neq \emptyset$.

The result has been known since 1994 in an unpublished version [17] and was used by several author; e.g. by Abels [6] the unique solvability of the Stokes resolvent equations was used in order to prove the existence of bounded imaginary powers of the Stokes operator in an infinite layer.

An alternative proof of unique solvability of the Stokes resolvent equations, i.e., $g = 0$, was later given by Abe and Shibata [1, 2] for $f \in L_p(\Omega)$, $1 < p < \infty$, and $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. An approach to the generalized Stokes resolvent equations using a reduction to a pseudodifferential boundary value problem, which works for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, may be found in [7, Section 5]; see also [3, Remark 1.2]. Moreover, Nazarov and Pileckas [14, 15] considered the solvability of the Stokes resolvent equations in weighted L_p-Sobolev spaces. But the latter results do not cover the case $f \in L_p(\Omega)$.

Since the unique solvability of the *generalized* Stokes resolvent equations, i.e., $g \neq 0$, is important for perturbation argument, cf. [4, 5], the authors decided to give a rigorous proof of Theorem 1.1 which has also been simplified at some steps in comparison to its first version [17]. Moreover, we note that we give precise estimates of the solutions near $-\pi^2/4$, which is the largest value in the spectrum of the Laplace and the Stokes operator on Ω. In particular, Theorem 1.1 implies that the Stokes operator $-A_p$ generates an analytic semi-group e^{-tA_p}, $t \geq 0$, satisfying

$$\|e^{-tA_p}u_0\|_r \leq C_{p,r}t^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{r})}e^{-\frac{\pi^2}{4}t}\|u_0\|_p, \qquad t \geq 0, u_0 \in L_{p,\sigma}(\Omega), \tag{1.4}$$

where $1 < p \leq r < \infty$ and $L_{p,\sigma}(\Omega) = \overline{\{u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0\}}^{\|\cdot\|_p}$. For $r = p$, the latter estimate is a consequence of the well-known characterization of analytic semi-groups and their generators. For $r > p$, (1.4) is obtained by estimating e^{-tA_p} in a straight-forward manner and using the semi-group property.

The proof of Theorem 1.1 is based on (partial) Fourier transformation in the tangential coordinates $x' = (x_1, \dots, x_{n-1})$. By this (1.1)-(1.3) is transformed to an ordinary two-point boundary value problem in dependence of (λ, ξ'), $\xi' \in \mathbb{R}^{n-1}$, which can be solved explicitly. Then we use the Mikhlin and Lizorkin multiplier theorems to estimate the solution operators. More precisely, we first estimate ∇q and then use the corresponding estimates for the Laplace resolvent equation to get the estimates for u.

Besides the study of the Stokes and Laplace resolvent equations, we give an explicit formula of the Helmholtz decomposition of $L_q(\Omega)$, $1 < q < \infty$, which was proved by Miyakawa [13] and by Farwig [9] in more general context.

The structure of the article is as follows:

In Section 2 we study some Mikhlin multipliers, which will appear in the solution operator of the Stokes and Laplace resolvent equations, and estimate the corresponding operators. Then the explicit formula for the Helmholtz decomposition in an infinite layer is given in Section 3. In Section 4 we estimate the solutions of the Laplace resolvent equation, which can easily be calculated by the same technique of partial Fourier transformation. Finally, in Section 5 we derive an explicit formula for the pressure q in (1.1)-(1.3). Then the main part of the proof consists in a careful analysis of the solution operator for q. Once the pressure is estimated, the estimates of u are obtained using the estimates for the Laplace resolvent equation.

2 Partial Fourier transformation and multiplier estimates

First let us introduce some notations. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a domain. Then $C_0^\infty(\Omega)$ will denote the space of all smooth functions $f \colon \Omega \to \mathbb{C}$ with compact support $\operatorname{supp} f \subset \Omega$. Moreover,

$$C_{(0)}^\infty(\overline{\Omega}) := \{u = v|_{\overline{\Omega}} : v \in C_0^\infty(\mathbb{R}^n)\}.$$

Finally, we note that by [4, Lemma 2.4] $C_{(0)}^\infty(\overline{\Omega})$ is dense in $\dot{W}_p^1(\Omega)$ for every $1 < p < \infty$.

The special type of domain $\Omega = \mathbb{R}^{n-1} \times (-1,1)$, we are dealing with, suggests the use of a partial Fourier transformation. Denoting points of $\overline{\Omega}$ by $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, $x_n \in [-1,1]$, we define for $u\colon \Omega \to \mathbb{C}$

$$U(\xi', x_n) := \mathcal{F}_{x' \mapsto \xi'}[u](\xi', x_n) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n) dx'$$

and the inverse transformation

$$u(x', x_n) := \mathcal{F}^{-1}_{\xi' \mapsto x'}[U](x', x_n) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} U(\xi', x_n) d\xi'.$$

In the following we will use the abbreviation $\mathcal{F} \equiv \mathcal{F}_{x' \mapsto \xi'}$. The Fourier and inverse Fourier transformation of $f\colon \mathbb{R}^n \to \mathbb{C}$ with respect to $x \in \mathbb{R}^n$ will be denoted by $\hat{f}$ and $\check{f}$, respectively.

Throughout this section L will denote the smallest integer larger than $\frac{n}{2}$, where $n \in \mathbb{N}$. Basic for deriving L_p-estimates is the Mikhlin multiplier theorem, cf. e.g. [8]:

THEOREM 2.1 *Let $f \in L_p(\mathbb{R}^n)$ with $1 < p < \infty$ and let $m\colon \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ denote an L-times differentiable function with*

$$[m] := \sup_{\xi \neq 0, |\alpha| \leq L} \{|\xi|^{|\alpha|} |D_\xi^\alpha m(\xi)|\} < \infty.$$

Then $(m \cdot \hat{f})^\vee \in L_p(\mathbb{R}^n)$ with

$$\|(m \cdot \hat{f})^\vee\|_p \leq C_p [m] \|f\|_p.$$

It is well-known that $m(\xi) := \frac{\xi_j}{|\xi|}$, $j = 1, \dots, n$, satisfies the assumptions of the latter theorem. This is a consequence of the fact that $m(\xi)$ is homogeneous of degree 0.

Moreover, we will use the following variant of Theorem 2.1 due to Lizorkin [12].

THEOREM 2.2 *Let $m\colon \mathbb{R}^n \to \mathbb{C}$ be continuous with continuous derivatives $\partial_\xi^\alpha m(\xi)$ on $\{\xi \in \mathbb{R}^n : \xi_i \neq 0, i = 1, \dots, n\}$, $\alpha \in \{0,1\}^n$, and let $1 < p \leq q < \infty$. Moreover, suppose that*

$$\sup_{\xi_i \neq 0, i=1,\dots,n} \left| \xi_1^{\alpha_1 + \beta} \cdots \xi_n^{\alpha_n + \beta} \partial_\xi^\alpha m(\xi) \right| \leq M$$

for all $\alpha \in \{0,1\}^n$ and $\beta = \frac{1}{p} - \frac{1}{q}$. Then $\|(m \cdot \hat{f})^\vee\|_q \leq C_{p,q} M \|f\|_p$.

In the following, we have to calculate estimates for various multipliers. For convenience, let us define for $0 \leq \rho_1 < \rho_2 \leq \infty$

$$[m]_{\rho_1}^{\rho_2} := \sup_{\rho_1 < |\xi| \leq \rho_2, |\alpha| \leq L} \{|\xi|^{|\alpha|} |D_\xi^\alpha m(\xi)|\}$$

and $[m] := [m]_0^\infty$.

Then we have the following lemma, the proof of which is elementary.

Lemma 2.3 *There holds uniformly in $0 \leq \rho_1 < \rho_2 \leq \infty$:*

$$[m_1 m_2]_{\rho_1}^{\rho_2} \leq c[m_1]_{\rho_1}^{\rho_2} \cdot [m_2]_{\rho_1}^{\rho_2} \tag{2.1}$$

If $|m| \geq c_1 > 0$, then

$$[1/m]_{\rho_1}^{\rho_2} \leq C(1 + [m]_{\rho_1}^{\rho_2})^L. \tag{2.2}$$

If $m(\xi) = M(|\xi|)$, then

$$[m]_{\rho_1}^{\rho_2} \leq c \sup_{a \in [\rho_1, \rho_2]} \left\{ \sum_{k=0}^{L} |a^k M^{(k)}(a)| \right\}. \tag{2.3}$$

As a typical multiplier, we come across functions of the following type:

Lemma 2.4 *Let $f, g \colon [0, \infty) \to \mathbb{C}$ be L-times differentiable with*

(i) $\mathrm{Re} f(a) \leq -c_1 A a$

(ii) $\sup_{1 \leq l \leq L} a^{l-1} |f^{(l)}(a)| \leq c_2 A$

(iii) $\sup_{0 \leq l \leq L} a^l |g^{(l)}(a)| \leq c_3 a^k$

on $0 \leq \rho_1 \leq a \leq \rho_2 \leq \infty$ with some $c_i > 0, A > 0, k \in \mathbb{N}_0$. Let $h(\xi) = g(|\xi|)e^{f(|\xi|)}$. Then

$$[h]_{\rho_1}^{\rho_2} \leq cA^{-k}. \tag{2.4}$$

Proof: As $h^{(j)}(a)$, $j = 1, \ldots, L$, is a sum of terms of the form

$$g^{(l_0)}(a) f^{(l_1)}(a) \ldots f^{(l_m)}(a) e^{f(a)},$$

where $l_0 + \ldots + l_m = j$ with $l_1, \ldots, l_m \neq 0$, and

$$\sup_{\rho_1 \leq a \leq \rho_2} a^j \left| g^{(l_0)}(a) f^{(l_1)}(a) \ldots f^{(l_m)}(a) e^{f(a)} \right| \leq C_1 a^k a^m A^m e^{-c_1 A a} \leq C_2 A^{-k}$$

the statement is a consequence of (2.3). ■

The next lemma provides L_p-estimates for a typical operator.

Lemma 2.5 *Let $\rho(t) = \chi_{[0,\infty)}(t)$ or $\rho(t) = 1 - \chi_{[0,\infty)}(t)$. Define the operator G by*

$$(Gf)(x', x_n) = \mathcal{F}^{-1} \left[\int_{-1}^{1} |\xi'| e^{-|\xi'||x_n - z|} \rho(x_n - z) \mathcal{F}[f](\xi', z) dz \right]$$

for $f \in C_0^\infty(\Omega)$. Then G extends to a bounded linear operator $G \colon L_p(\Omega) \to L_p(\Omega)$ for every $1 < p < \infty$.

Proof: Let f_0 denote the extension of f onto $\mathbb{R}^n$ by zero and let $g := Gf$. Then

$$\mathcal{F}[g](\xi', x_n) = \int_{\mathbb{R}} |\xi'| e^{-|\xi'||x_n - z|} \rho(x_n - z) \mathcal{F}[f_0](\xi', z) dz.$$

The right hand side makes sense also for $|x_n| > 1$. Hence the inverse transformation defines a function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}|_\Omega = g$,

$$\tilde{g}(x', x_n) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^n} e^{ix' \cdot \xi'} |\xi'| e^{-|\xi'||x_n - z|} \rho(x_n - z) \mathcal{F}[f_0](\xi', z) d(z, \xi').$$

The claim follows if we can prove $\|\tilde{g}\|_{L_p(\mathbb{R}^n)} \leq c \|f_0\|_{L_p(\mathbb{R}^n)}$. Inserting the definition of $\mathcal{F}[f_0]$, we may write $\tilde{g}$ as an $\mathbb{R}^n$-convolution $\tilde{g} = k * f_0$ or equivalently $\tilde{g} = (2\pi)^{n/2} (\hat{k} \cdot \hat{f}_0)^\vee$ with

$$k(x', x_n) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \rho(x_n) |\xi'| e^{-|\xi'||x_n|} d\xi'.$$

Now $k(x) = \mathcal{F}_{\xi' \mapsto x'}[h](x)$ with $h(\xi', x_n) = \rho(x_n) |\xi'| e^{-|\xi'||x_n|}$ hence

$$\hat{k}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-i\xi_n x_n} \rho(x_n) |\xi'| e^{-|\xi'||x_n|} dx_n$$

for $\xi = (\xi', \xi_n)$. As by simple integration

$$(2\pi)^{1/2} \hat{k}(\xi) = \begin{cases} |\xi'|(|\xi'| + i\xi_n)^{-1} & \text{for } \rho(t) = \chi_{[0,\infty)}(t) \\ |\xi'|(|\xi'| - i\xi_n)^{-1} & \text{for } \rho(t) = 1 - \chi_{[0,\infty)}(t) \end{cases}$$

we have $[\hat{k}(\eta)] \leq c$, and the claim follows by Theorem 2.1. ∎

Lemma 2.6 *Let*

$$h(t) = \int_{-1}^{1} \frac{s(x)}{(2 - t - x)} dx, \qquad s \in L_p(-1, 1),$$

for $t \in (-1, 1)$. *Then* $\|h\|_{L_p(-1,1)} \leq C_p \|s\|_{L_p(-1,1)}$ *for every* $1 < p < \infty$.

Proof: Extend s by zero to $\tilde{s}$ and let $\tilde{h}(t) := h(2 - t)$. Then $\tilde{h}(t) = \int_{-\infty}^{\infty} \frac{\tilde{s}(x)}{t - x} dx$ and

$$\int_{-1}^{1} |h(t)|^p dt \leq \int_{\mathbb{R}} |\tilde{h}(t)|^p dt \leq C_p \|\tilde{s}\|_p^p = C_p \|s\|_p^p$$

by the Calderon-Zygmond estimate. ∎

The following lemma is (partly) known; we include a proof, as it contains a surprising borderline case.

Lemma 2.7 *Suppose q is smooth, $\nabla q \in L_p(\mathbb{R}^n)$, $n \geq 1$, $1 < p < \infty$. Then there is a constant c_q such that for $R \geq 1$*

$$\|q - c_q\|_{L_p(B_R)} \leq c(R)\|\nabla q\|_p$$

with

$$c(R) = \begin{cases} c_p \cdot R & \text{if } p \neq n, \\ cR(\ln R + 1)^{1-1/n} & \text{if } p = n. \end{cases}$$

Remark: The example $q(x) = \ln(|x|^2 + 1)^\gamma$ with $0 < \gamma < 1 - \frac{1}{n}$ shows that the case $p = n$ is indeed exceptional – an estimate with some smaller exponent for the logarithmic term does not hold for all $R \geq 1$.

Proof: First we consider the case $n \geq 2$. Let q_ρ denote the mean-value of q over B_ρ. Then, as

$$q(rRw) - q(r\rho w) = \int_{r\rho}^{rR} w \cdot \nabla q(sw)ds,$$

one gets

$$q_R - q_\rho = c_n \int_0^1 r^{n-1} \left(\int_{|w|=1} \int_{r\rho}^{rR} w \cdot \nabla q(sw) ds dw \right) dr.$$

Hence

$$\begin{aligned} |q_R - q_\rho| &\leq c_n \int_0^1 r^{n-1} \int_{B_{Rr}\setminus B_{\rho r}} \frac{|\nabla q(x)|}{|x|^{n-1}} dx dr \\ &\leq c_n \int_0^1 r^{n-1} \|\nabla q\|_p \cdot \left(\int_{B_{Rr}\setminus B_{\rho r}} \frac{dx}{|x|^{(n-1)p'}} \right)^{1/p'} dr. \end{aligned}$$

As

$$\left(\int_{\rho r}^{Rr} s^{(n-1)(1-p')} ds \right)^{1/p'} \leq C \begin{cases} r^{1-n/p} \left| R^{\frac{p-n}{p-1}} - \rho^{\frac{p-n}{p-1}} \right|^{1-1/p} & \text{if } p \neq n, \\ \ln(\frac{R}{\rho})^{1-\frac{1}{p}} & \text{if } p = n \end{cases}$$

and $\int_0^1 r^{n-1} \cdot r^{1-\frac{n}{p}} dr = \int_0^1 r^{n/p'} dr \leq c$, one gets

$$|q_R - q_\rho| \leq c\|\nabla q\|_p \cdot \begin{cases} \left| R^{\frac{p-n}{p-1}} - \rho^{\frac{p-n}{p-1}} \right|^{1-\frac{1}{p}} & \text{if } p \neq n, \\ \ln(\frac{R}{\rho})^{1-\frac{1}{p}} & \text{if } p = n. \end{cases}$$

Thus, for $p > n, q_\rho \to q(0)$ for $\rho \to 0$, while for $p < n, q_\rho \to q_\infty$ for $\rho \to \infty$, and we may take $c_q = q(0)$, $c_q = q_\infty$, resp., to get $|q_R - c_q| \leq c\|\nabla q\|_p \cdot R^{1-\frac{n}{p}}$. For $p = n$, take $c_q = q_1$. Together with Poincare's inequality, the claim follows.

Finally, if $n = 1$, we have $|q(x) - q(0)| \leq |x|^{\frac{1}{p'}} \|\frac{d}{dx} q\|_p$, which implies the statement.

∎

Other multipliers will involve a certain square-root of $\lambda + |\xi'|^2$, where $\lambda \in S_\varepsilon$, $\varepsilon > 0$. Besides the sector S_ε, it is useful to consider

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$$

for $\theta \in (0,\pi)$. If $\lambda \in \Sigma_\theta$, $\theta \in (0,\pi)$, $b \geq 0$, and $\mu \in \Sigma_{\frac{\theta}{2}}$ is defined by $\mu^2 = \lambda + b$, we have

$$\operatorname{Re}\mu \geq c_\theta \max\{|\lambda|^{\frac{1}{2}}, b^{\frac{1}{2}}\} \qquad \text{for all } \lambda \in \Sigma_\theta. \tag{2.5}$$

The latter estimate can easily be proved by using the homogeneity of $f(\lambda, b) := (\lambda + b)^{\frac{1}{2}}$.

Finally, we recall that for every $\alpha, \beta \in \mathbb{N}_0^n$

$$\left| D_\xi^\beta \frac{\xi^\alpha}{\lambda + |\xi|^2} \right| \leq C_{\theta,\alpha,\beta} (|\lambda|^{\frac{1}{2}} + |\xi|)^{-2+|\alpha|-|\beta|} \tag{2.6}$$

for $\lambda \in \Sigma_\theta$ with $\theta \in (0,\pi)$. The latter statement can be proved by using the homogeneity of $f_\theta(\mu,\xi) := D_\xi^\beta \frac{\xi^\alpha}{e^{i\theta}\mu^2 + |\xi|^2}$, $\theta \in (0,\pi)$.

For $\lambda \in S_\varepsilon$, the following lemma provides some important estimates.

Lemma 2.8 *Let $\lambda \in S_\varepsilon$, $\varepsilon > 0$. Then for $a \in \mathbb{R}$ define $\mu = \mu(a)$ uniquely by*

$$\mu^2 := \lambda + a^2$$

with $\operatorname{Re}\mu \geq 0$ and $\operatorname{Im}\mu \geq 0$ if $\operatorname{Re}\mu = 0$. Then, if $\max\{|\lambda|^{1/2}, a\} \geq \kappa\frac{\pi}{2}$, $\kappa > 1$,

$$\operatorname{Re}\mu \geq 2\delta \max\{|\lambda|^{1/2}, a\} \geq \delta|\mu| \tag{2.7}$$

where $\delta = \delta(\varepsilon,\kappa) > 0$. Furthermore, for any $k \in \mathbb{N}$ we have $\sup_{a\geq 0} a^{k-1}|\mu^{(k)}(a)| \leq C_{\varepsilon,\kappa,k}$ uniformly in $\lambda \in S_\varepsilon$ with $|\lambda|^{\frac{1}{2}} \geq \kappa\frac{\pi}{2}$, $\kappa > 1$. Moreover, if $a = |\xi'|$, $\xi' \in \mathbb{R}^{n-1}$, then $\sup_{\xi'\in\mathbb{R}^{n-1},|\alpha|\leq k,\alpha\neq 0} |\xi'|^{|\alpha|-1}|D_{\xi'}^\alpha \mu| \leq C_{\varepsilon,\kappa,k}$ uniformly in $\lambda \in S_\varepsilon$ with $|\lambda|^{\frac{1}{2}} \geq \kappa\frac{\pi}{2}$, $\kappa > 1$.

Proof: First of all, the estimate $|\mu| \leq 2\max\{|\lambda|^{1/2}, a\}$ is trivial. Suppose first that $\max\{|\lambda|^{\frac{1}{2}}, a\} = |\lambda|^{\frac{1}{2}} \geq \kappa\frac{\pi}{2}$ with $\kappa > 1$. Then obviously $\lambda \in \Sigma_\theta$ for some $\theta = \theta(\varepsilon,\kappa)$. Hence (2.7) is a consequence of (2.5) with $b = a^2$. Similarly, if $\max\{|\lambda|^{\frac{1}{2}}, a\} = a \geq \kappa\frac{\pi}{2}$ with $\kappa > 1$, (2.5) implies

$$|\operatorname{Re}\mu| \geq c_\theta \max\left\{ \left|\lambda + \frac{\pi^2}{4}\right|^{\frac{1}{2}}, \left(a^2 - \frac{\pi^2}{4}\right)^{\frac{1}{2}} \right\} \geq c_\theta \left(1 - \frac{1}{\kappa^2}\right)^{\frac{1}{2}} a.$$

This proves (2.7).

The estimate for $\mu^{(k)}(a)$ follows from the fact that $\mu^{(k)}(a) = p_k\left(\frac{a}{\mu}\right) a^{1-k}, k \geq 1$, where p_k is a polynomial. Then the last statement is a consequence of the chain rule and the fact that $\left|D_{\xi'}^\alpha |\xi'|\right| \leq C_\alpha |\xi'|^{1-|\alpha|}$, which is a consequence of the homogeneity of the mapping $\xi' \mapsto |\xi'|$. ∎

Lemma 2.9 *For $\lambda \in S_\varepsilon$, $\varepsilon > 0$, $\xi' \in \mathbb{R}^{n-1}$, and $a := |\xi|$ let μ be as in Lemma 2.8. Moreover, let $k \in \mathbb{N}_0$, $z, w \geq 0$ with $z + w > 0$, and let $M = \max\{\frac{|\lambda|^{\frac{1}{2}}}{4}, \pi\}$. Then*

$$\left[\frac{\mu}{a}\right]_M^\infty \leq C_\varepsilon, \quad \left[\frac{\mu}{\mu + a}\right]_0^\infty \leq C_\varepsilon \quad \text{if } |\lambda|^{\frac{1}{2}} \geq \pi, \quad \left[\frac{\mu}{\mu - a}\right]_0^{\frac{1}{2}|\lambda|^{\frac{1}{2}}} \leq C_\varepsilon,$$
$$\left[a^k e^{-az-\mu w}\right]_0^\infty \leq \frac{C_\varepsilon}{(z+w)^k} \quad \text{if } |\lambda|^{\frac{1}{2}} \geq \pi, \quad \left[a^k e^{-az-\mu w}\right]_\pi^\infty \leq \frac{C_\varepsilon}{(z+w)^k},$$

uniformly in $\lambda \in S_\varepsilon$.

Proof: If $a \geq \max\{\frac{|\lambda|^{\frac{1}{2}}}{4}, \pi\}$, then $|\mu| \leq 5a$ and Lemma 2.8 yields $\left[\frac{\mu}{a}\right]_M^\infty \leq C_\varepsilon$. Moreover, because of Lemma 2.8 and $\frac{a}{\mu} = \frac{d}{da}\mu(a)$, $\left[\frac{a}{\mu}\right]_0^\infty \leq C_\varepsilon$ uniformly in $\lambda \in S_\varepsilon$ with $|\lambda|^{\frac{1}{2}} \geq \pi$. Hence $\frac{\mu}{\mu+a} = (1 + \frac{a}{\mu})^{-1}$, $\mathrm{Re}(1 + \frac{a}{\mu}) \geq 1$, and (2.2) imply the statement for $\frac{\mu}{\mu+a}$. Furthermore, if $a \leq \frac{1}{2}|\lambda|^{\frac{1}{2}}$, $|\mu| \geq \frac{\sqrt{3}}{2}|\lambda|^{\frac{1}{2}}$ and therefore $|1 - \frac{a}{\mu}| \geq 1 - \frac{1}{\sqrt{3}}$, which implies $\left[\frac{\mu}{\mu-a}\right]_0^{\frac{1}{2}|\lambda|^{\frac{1}{2}}} \leq C_\varepsilon$.

In order to prove $\left[a^k e^{-az-\mu w}\right]_0^\infty \leq \frac{C_\varepsilon}{(z+w)^k}$ if $|\lambda| \geq \pi$ and $\left[a^k e^{-az-\mu w}\right]_\pi^\infty \leq \frac{C_\varepsilon}{(z+w)^k}$, we apply Lemma 2.4 with $g(a) = a^k$, $f(a) = -az - \mu w$, and $A = z + w$. Here the assumptions of Lemma 2.4 are consequences of Lemma 2.8. ∎

Lemma 2.10 *Let $\delta_\pm := (e^{\mu \pm a} - e^{-(\mu \pm a)})/(\mu \pm a)$ and let $m_\pm := (\delta_+ \pm \delta_-)^{-1} e^{a+\mu}$, where μ and a are the same as in Lemma 2.8. Then there is a constant $a_\varepsilon \geq \pi$ such that*

$$\left[a^{-1} m_\pm\right]_M^\infty \leq C_\varepsilon \tag{2.8}$$

uniformly in $\lambda \in S_\varepsilon$, $\varepsilon > 0$, where $M := \max\{a_\varepsilon, \frac{1}{4}|\lambda|^{\frac{1}{2}}\}$, and

$$\left[\mu^{-1} m_+\right]_0^{2M} \leq C_\varepsilon, \qquad \left[\mu^{-1}\frac{a}{1+a} m_-\right]_0^{2M} \leq C_\varepsilon$$

uniformly in $\lambda \in S_\varepsilon$, $\varepsilon > 0$, with $\frac{1}{4}|\lambda|^{\frac{1}{2}} \geq a_\varepsilon$.

Proof: First we consider (2.8). We will use (2.2), where we note that

$$m_\pm^{-1} = \frac{1}{\mu + a}(1 - e^{-2a-2\mu}) \pm \delta_- e^{-(a+\mu)}, \qquad \delta_\pm = \int_{-1}^1 e^{t(\mu \pm a)} dt.$$

By Lemma 2.4 with $k = 1$ and $A = \min_{|t| \leq 1}\{(1-t)2\delta + (1+t)\} = \min\{4\delta, 2\}$, we have $[a\delta_- e^{-a-\mu}]_M^\infty \leq C_\varepsilon$. Because of Lemma 2.9, this implies $[a m_\pm^{-1}]_M^\infty \leq C_\varepsilon$.

As $|\mu| \leq 5a$ if $a \geq M$, we have $\left|\frac{a}{a+\mu}\right| \geq \frac{a}{a+|\mu|} \geq \frac{1}{6}$ and

$$|\delta_-| = \left|\int_{-1}^{1} e^{t(\mu-a)}dt\right| \leq 2\max\{e^{\mathrm{Re}\mu-a}, e^{a-\mathrm{Re}\mu}\}.$$

Hence using Lemma 2.8 again

$$|am_{\pm}^{-1}| \;\geq\; \frac{1}{6}(1-e^{-2a-2\mathrm{Re}\mu}) - 2a\max\{e^{-2a}, e^{-2\mathrm{Re}\mu}\} \geq \frac{1}{8}$$

for $a \geq a_\varepsilon$ and a_ε large enough (this is one of the conditions for a_ε).

Concerning the other estimates, we use (2.2) and prove for $\tilde{m}_\pm = \mu(\delta_+ \pm \delta_-)e^{-(a+\mu)}$ that $|\tilde{m}_+| \geq c$ on $[0, 2M]$, $|\tilde{m}_-| \geq c$ on $[1, 2M]$, $|a^{-1}\tilde{m}_-| \geq c$ on $[0, 1]$ and that $[\tilde{m}_+]_0^{2M}$, $[\tilde{m}_-]_1^{2M}$, $[a^{-1}\tilde{m}_-]_0^1$ are uniformly bounded in $\lambda \in S_\varepsilon$, $|\lambda| \geq a_\varepsilon$. As $\tilde{m}_\pm = \frac{\mu}{\mu+a}(1-e^{-2a-2\mu}) \pm \frac{\mu}{\mu-a}(e^{-2a}-e^{-2\mu})$, the estimates for $[\tilde{m}_+]_0^{2M}$ and $[\tilde{m}_-]_1^{2M}$ follow by Lemma 2.9. Moreover, using

$$\tilde{m}_- = \frac{\mu}{\lambda}\left(\mu(1-e^{-2a})(1+e^{-2\mu}) - a(1-e^{-2\mu})(1+e^{-2a})\right),$$

we conclude $[a^{-1}\tilde{m}_-]_0^1 \leq C_\varepsilon$ uniformly in $|\lambda| \geq a_\varepsilon$, $\lambda \in S_\varepsilon$.

Last we need estimates from below. We remember that $\mathrm{Re}\mu \geq 2\delta|\lambda|^{1/2} \geq 8\delta a_\varepsilon$, $c_1|\mu| \geq |\mu \pm a| \geq c_2|\mu|$, and $\mathrm{Re}\mu \geq \delta|\mu|$. Hence

$$\begin{aligned} |\tilde{m}_+| &= \left|\frac{\mu}{\mu+a}\right| \left|1 - e^{-2a-2\mu} + \frac{\mu+a}{\mu-a}(e^{-2a}-e^{-2\mu})\right| \\ &\geq c_1^{-1}\left\{(1+e^{-2a}) - e^{-2\mathrm{Re}\mu} - \frac{2a}{|\mu-a|}(e^{-2a}+e^{-2\mathrm{Re}\mu}) - e^{-2\mathrm{Re}\mu}\right\} \\ &\geq c_1^{-1}\left((1+e^{-2a}) - \frac{c}{|\mu|} - 2e^{-2\mathrm{Re}\mu}\right) \geq \frac{1}{2}c_1^{-1} \end{aligned}$$

for a_ε large enough (the second condition on a_ε).

Similarly,

$$|\tilde{m}_-| \geq \left|\frac{\mu}{\mu+a}\right| \left(\left|1 - e^{-2a} - \frac{2a}{\mu-a}e^{-2a}\right| - e^{-2\mathrm{Re}\mu}\left|e^{-2a} - 1 - \frac{2a}{\mu-a}\right|\right)$$

and for $a \geq 1$, we have

$$|\tilde{m}_-| \geq c_1^{-1}\left\{\left(1 - e^{-2} - \frac{c}{|\mu|}\right) - ce^{-2\mathrm{Re}\mu}\right\} \geq c_1^{-1}\left(\frac{1}{2} - e^{-2}\right)$$

for a_ε suitably large (the third condition on a_ε), while for $a \leq 1$

$$\begin{aligned} |a^{-1}\tilde{m}_-| &\geq c_1^{-1}\left\{\left(\frac{1-e^{-2a}}{a}\right) - \frac{2e^{-2a}}{|\mu-a|} - e^{-2\mathrm{Re}\mu}\left(\frac{1-e^{-2a}}{a} + \frac{2}{|\mu-a|}\right)\right\} \\ &\geq c_1^{-1}\left(2e^{-2} - \frac{c}{|\mu|} - ce^{-2\mathrm{Re}\mu}\right) \geq c_1^{-1}e^{-2} \end{aligned}$$

for a_ε large (the last condition on a_ε). This finishes the proof. ∎

Lemma 2.11 *Let $f(a,\mu) := D/(a\mu)$. Then f is a smooth function in $a \in \mathbb{R}, \mu \in \mathbb{C}$ and $f(a,\mu) \neq 0$ if $a \geq 0$ and $\operatorname{Re}\mu \geq 0$ with $\mu^2 \in \overline{S_\varepsilon}$, $\varepsilon > 0$. In particular, if $a = a(\xi')$ and $\mu = \mu(\xi', \lambda)$ are defined as in Lemma 2.8, then for every $A, B > 0$*

$$[a\mu/D]_0^A \leq c(A,B,\varepsilon)$$

uniformly in $\lambda \in S_\varepsilon$ with $|\lambda| \leq B$.

Proof: First of all, since $\delta_\pm = \int_{-1}^1 e^{t(\mu\pm a)}dt$,

$$\begin{aligned} \delta_+ + \delta_- &= 4\int_0^1 \cosh(t\mu)\cosh(ta)\;dt = 4P_1(a,\mu) \\ \frac{\delta_+ - \delta_-}{a\mu} &= 4\int_0^1 \frac{\sinh(t\mu)}{\mu}\frac{\sinh(ta)}{a}\;dt = 4P_2(a,\mu) \end{aligned}$$

Hence $f(a,\mu) = D/(a\mu) = 16P_1(a,\mu)P_2(a,\mu)$ is a smooth function for all $a \in \mathbb{R}$, $\mu \in \mathbb{C}$. Therefore it remains to prove that $f(a,\mu) \neq 0$ if $a \geq 0$, $\operatorname{Re}\mu \geq 0$, and $\mu^2 \in \overline{S_\varepsilon}$.

Firstly, let $\mu = x \in \mathbb{R}$. Then obviously $P_1(a,\mu), P_2(a,\mu) > 0$ and therefore $f(a,\mu) > 0$. Moreover, if $\mu = iy$, $y \in \mathbb{R}$, then

$$P_1(a,iy) \;=\; \int_0^1 \cos ty \cosh ta\;dt > 0, \qquad P_2(a,iy) = \int_0^1 \frac{\sin ty}{ty}\frac{\sinh ta}{ta}t^2 dt > 0,$$

where we have used that $\mu^2 = -y^2 \in \overline{S_\varepsilon}$ implies $|y| \leq \frac{\pi}{2}$.

Finally, let $\mu = x + iy$, $y \neq 0$, $x > 0$. Using

$$\begin{aligned} \lambda\frac{\delta_+ + \delta_-}{4} &= \mu\sinh\mu\cosh a - a\cosh\mu\sinh a \\ &= \left(\mu\frac{\sinh\mu}{\cosh\mu} - a\frac{\sinh a}{\cosh a}\right)\cosh\mu\cosh a, \\ \lambda\frac{\delta_+ - \delta_-}{4} &= \mu\cosh\mu\sinh a - a\sinh\mu\cosh a \\ &= a\mu\left(\frac{\sinh a}{a\cosh a} - \frac{\sinh\mu}{\mu\cosh\mu}\right)\cosh\mu\cosh a, \end{aligned}$$

where $\lambda = \mu^2 - a^2 \neq 0$ if $\operatorname{Im}\mu = y \neq 0$ and $\cosh\mu \neq 0$ if $\operatorname{Re}\mu > 0$, it is sufficient to show that

$$\operatorname{Im}\left(\mu\frac{\sinh\mu}{\cosh\mu}\right) \neq 0 \quad \text{and} \quad \operatorname{Im}\left(\frac{\sinh\mu}{\mu\cosh\mu}\right) \neq 0.$$

By elementary calculations

$$\begin{aligned} \operatorname{Im}\left(\mu\frac{\sinh\mu}{\cosh\mu}\right) &= \frac{1}{2}\frac{x\sin 2y + y\sinh 2x}{\cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y} \\ \operatorname{Im}\left(\frac{\sinh\mu}{\mu\cosh\mu}\right) &= \frac{1}{2|\mu|^2}\frac{x\sin 2y - y\sinh 2x}{\cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y}, \end{aligned}$$

where $|y \sinh 2x \pm x \sin 2y| \geq |y|(\sinh 2x - 2x) > 0$ if $y \neq 0$, $x > 0$.

The last statement is a trivial consequence of the first part and (2.3). ■

Lemma 2.12 *Let $k(\xi', w, x_n)$ be measurable and sufficiently smooth in $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. Moreover, suppose that*

$$[\overline{k}(\xi', w, x_n)] + \left[\frac{\partial}{\partial w}\overline{k}(\xi', w, x_n)|\xi'|^{-1}\right] \leq c(2 \pm w - x_n)^{-1}.$$

and define

$$h(\xi', x_n) = \mathcal{F}^{-1}\left[\int_{-1}^{1} k(\xi', w, x_n)|\xi'|^{-1}\mathcal{F}[g](\xi', w)dw\right]$$

for $g \in C^\infty_{(0)}(\overline{\Omega}) \cap \hat{W}^{-1}_p(\Omega)$. Then for every $1 < p < \infty$

$$\|h\|_{L_p(\Omega)} \leq C_p \|g\|_{\hat{W}^{-1}_p(\Omega)} \qquad \textit{for all } g \in C^\infty_{(0)}(\overline{\Omega}) \cap \hat{W}^{-1}_p(\Omega).$$

Proof: Let $\varphi \in C^\infty_0(\Omega)$ and define ψ by

$$\mathcal{F}[\psi](\xi', w) := \int_{-1}^{1} \overline{k}(\xi', w, z)|\xi'|^{-1}\mathcal{F}[\varphi](\xi', z)dz.$$

Then

$$\begin{aligned}\langle g, \psi\rangle &= \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{R}^{n-1}} k(\xi', w, z)|\xi'|^{-1}\mathcal{F}[g](\xi', w)\overline{\mathcal{F}[\varphi]}(\xi', z)d\xi' dz dw \\ &= \int_{-1}^{1}\int_{\mathbb{R}^{n-1}} \mathcal{F}[h](\xi', z)\overline{\mathcal{F}[\varphi]}(\xi', z)d\xi' dz = \langle h, \varphi\rangle\end{aligned}$$

and therefore $|\langle h, \varphi\rangle| \leq \|g\|_{\hat{W}^{-1}_p(\Omega)} \cdot \|\nabla\psi\|_{p'}$.

Now $\mathcal{F}[\frac{\partial\psi}{\partial x_j}](\xi', w) = \int_{-1}^{1} \overline{k}(\xi', w, z) i\xi_j |\xi'|^{-1}\mathcal{F}[\varphi](\xi', z)dz$, for $j = 1, \ldots, n-1$, which implies

$$\left\|\frac{\partial\psi}{\partial x_j}(\cdot, w)\right\|_{L_p(\mathbb{R}^{n-1})} \leq C_p \int_{-1}^{1} \|\varphi(\cdot, z)\|_{p'}(2 \pm w - z)^{-1}dz$$

by Theorem 2.1 and the same estimate follows for $j = n$. By Lemma 2.6, $\|\nabla\psi\|_{p'} \leq C_p\|\varphi\|_{p'}$. Hence the claim is proved. ■

3 The Helmholtz Projection

Let $L_{p,\sigma}(\Omega) = \overline{\{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}}^{\|\cdot\|_p}$. We want to give an explicit construction for the Helmholtz projection $P\colon L_p(\Omega)^n \to L_{p,\sigma}(\Omega)$.

THEOREM 3.1 *For every $f \in L_p(\Omega)^n$ there is a unique decomposition $f = f^0 + \nabla q$ with $f^0 \in L_{p,\sigma}(\Omega), q \in \dot{W}_p^1 = \{q \in L_{p,loc}(\overline{\Omega}) | \nabla q \in L_p(\Omega)^n\}$. Moreover,*

$$\|f^0\|_p + \|\nabla q\|_p \leq C_p \|f\|_p \tag{3.1}$$

and $\|q\|_{L_p(\Omega \cap B_R)} \leq C_p \|f\|_p + C_p R \|(f_1, \ldots, f_{n-1})\|_p$ for $p \neq n-1$ if we choose an appropriate constant for q.

Remark 3.2 The theorem (except for the $L_p(\Omega \cap B_R)$-estimate) was proved by Miyakawa [13] and Farwig [9] in a more general context. Since we will need the explicit formula for $f_0 = Pf$, we include a proof.

Proof: We may assume $f \in C_0^\infty(\Omega)^n$, so that all integrals appearing are well defined.

Define $F_j(\xi', x_n) := \mathcal{F}[f_j](\xi', x_n), j = 1, \ldots, n$ and $F_{n+1}(\xi', x_n) := \frac{i\xi'}{|\xi'|} \cdot F'(\xi', x_n)$, where $F' = (F_1, \ldots, F_{n-1})$. Note that

$$\|\mathcal{F}^{-1}[F_{n+1}](\xi', x_n)\|_{L_p(\mathbb{R}^{n-1})} \leq C_p \|f'(\cdot, x_n)\|_{L_p(\mathbb{R}^{n-1})}.$$

Now define a pressure q by

$$\begin{aligned} Q(\xi', x_n) &:= \mathcal{F}[q](\xi', x_n) \\ &= -\frac{1}{2} \int_{-1}^{1} e^{-a|x_n - z|} F_{n+1}(\xi', z) dz - \frac{1}{2} \int_{-1}^{1} e^{-a|x_n - z|} \mathrm{sign}(z - x_n) F_n(\xi', z) dz \\ &\quad + \alpha e^{a x_n} + \beta e^{-a x_n} \end{aligned}$$

where $a = |\xi'|$ and α, β are chosen such that $\partial_n Q - F_n|_{w=\pm 1} \equiv 0$.

Since $(\partial_n - a^2) Q = i\xi' \cdot F' + \partial_n F_n$, the Helmholtz projection $f^0 = Pf$ is then given by

$$\begin{aligned} F_j^0 &= \mathcal{F}[f_j^0] := F_j - i\xi_j Q, \quad j = 1, \ldots, n-1, \\ F_n^0 &= \mathcal{F}[f_n^0] := F_n - \partial_n Q \end{aligned} \tag{3.2}$$

But it remains to determine α, β. By differentiation

$$\begin{aligned} \partial_n Q - F_n &= -\frac{a}{2} \int_{-1}^{1} e^{-a|x_n - z|} \mathrm{sign}(z - w) F_{n+1}(\xi', z) dz - \frac{a}{2} \int_{-1}^{1} e^{-a|x_n - z|} F_n(\xi', z) dz \\ &\quad + \alpha a e^{a x_n} - \beta a e^{-a x_n} \end{aligned}$$

and $\partial_n(\partial_n Q - F_n) = a^2 Q + a F_{n+1}$. From the boundary conditions we conclude

$$\begin{aligned} \alpha a e^{a x_n} &= \frac{a}{2(1 - e^{-4a})} \Bigg(\int_{-1}^{1} e^{-a(2 - x_n - z)} (F_n(\xi', z) - F_{n+1}(\xi', z)) dz \\ &\quad - \int_{-1}^{1} e^{-a(4 + z - x_n)} (F_n(\xi', z) + F_{n+1}(\xi', z)) dz \Bigg) \end{aligned}$$

and

$$\begin{aligned}\beta a e^{-ax_n} &= \frac{a}{2(1-e^{-4a})}\left(\int_{-1}^{1} e^{-a(4-z+x_n)}(F_n(\xi',z)-F_{n+1}(\xi',z))dz\right.\\&\left.-\int_{-1}^{1} e^{-a(2+x_n+z)}(F_n(\xi',z)+F_{n+1}(\xi',z))dz\right)\end{aligned}$$

These terms are of the type

$$R(\xi',x_n)=\int_{-1}^{1} k(\xi',z,x_n)\mathcal{F}[g](\xi',z)dz.$$

Hence P is a linear operation, and

$$\|\mathcal{F}^{-1}[R](\cdot,x_n)\|_{L_p(\mathbb{R}^{n-1})}\le C_p\int_{-1}^{1}[k(\cdot,z,x_n)]\|g(\cdot,z)\|_{L_p(\mathbb{R}^{n-1})}dz.$$

As $|z|,|x_n|\le 1$ and the kernels are real analytic in a, we have $[k(\cdot,z,x_n)]_0^1\le c$. Furthermore, for $b(a)=(1-e^{-4a})^{-1}$, we have $[b]_1^\infty\le c$. Together with Lemma 2.9, we obtain $[k(\cdot,z,x_n)]\le c((2+z+x_n)^{-1}+(2-z-x_n)^{-1})$, and by Lemma 2.6 $\|\mathcal{F}^{-1}[R]\|_p\le C_p\|f\|_p$. For the first two terms of $\partial_n Q-F_n$, we use Lemma 2.5 to conclude finally

$$\left\|\frac{\partial\pi}{\partial x_n}-f_n\right\|_p\le C_p\|f\|_p.$$

For later purposes, let us note that

$$2a^{-1}F_n^0(\xi',x_n)=\int_{-1}^{1}k_n(a,x_n,z)F_n(\xi',z)dz+\int_{-1}^{1}k_{n+1}(a,x_n,z)F_{n+1}(\xi',z)dz \quad (3.3)$$

with

$$\begin{aligned}&k_n(a,w,z)=\\&\quad e^{-a|w-z|}+(1-e^{4a})^{-1}\left(e^{-a(4-z+w)}-e^{-a(2+w+z)}-e^{-a(2-w-z)}+e^{-a(4+z-w)}\right)\end{aligned}$$

and

$$\begin{aligned}k_{n+1}(a,w,z) &= e^{-a|w-z|}\mathrm{sign}(z-w)-(1-e^{-4a})^{-1}\left(e^{-a(4-z+w)}+e^{-a(2+w+z)}\right.\\&\left.-e^{-a(2-w-z)}-e^{-a(4+z-w)}\right).\end{aligned}$$

Note that these kernels are analytic in a (uniformly for $|z|,|w|\le 1$).

Now for $j=1,\dots,n-1$,

$$\begin{aligned}-i\xi_j Q(\xi',x_n) &= \frac{a}{2}\int_{-1}^{1} e^{-a|x_n-z|}\frac{i\xi_j}{|\xi'|}F_{n+1}(\xi',z)dz\\&+\frac{a}{2}\int_{-1}^{1} e^{-a|x_n-z|}\mathrm{sign}(z-x_n)\frac{i\xi_j}{|\xi'|}F_n(\xi',z)dz\\&-\alpha a e^{ax_n}\frac{i\xi_j}{|\xi'|}-\beta a e^{-ax_n}\frac{i\xi_j}{|\xi'|}.\end{aligned}$$

This is obviously a sum of terms of the same type as in $\partial_n Q - F_n$ with F_n, F_{n+1} substituted by $\frac{i\xi_j}{|\xi'|}F_n$, $-\frac{\xi_j}{|\xi'|}F_{n+1}$, respectively. Hence also $\left\|\frac{\partial q}{\partial x_j}\right\|_p \leq C_p\|f\|_p, j = 1,\ldots,n-1$. Thus $P\colon L_p(\Omega)^n \to L_p(\Omega)^n$ is continuous. For the uniqueness of the decomposition, we refer to [13].

Because of the construction, $\operatorname{div} f^0 = 0$, $f_n^0|_{\partial\Omega} = 0$, and $P^2 f = Pf$. Therefore P extends to a continuous projection $P\colon L_p(\Omega)^n \to L_{p,\sigma}(\Omega)$ such that $Pf = f - \nabla q$.

It remains to prove a local estimate for q. Take a cutoff function φ with $\varphi(t) = 1$ for $t \leq 1$ and $\varphi \equiv 0$ for $t \geq 2$. Let

$$Q = \varphi(a)Q + \left(\frac{1-\varphi(a)}{a}\right) Qa = Q_1 + Q_2.$$

Now $(1-\varphi(a))/a$ is a multiplier on $\mathbb{R}^{n-1}$ with $[(1-\varphi(a))/a] \leq c$, hence

$$\|\mathcal{F}^{-1}[Q_2]\|_p \leq C_p\|\mathcal{F}^{-1}[Qa]\|_p \leq C_p\|f\|_p,$$

as shown above. Next Q_1 is of the type

$$Q_1(\xi', x_n) \;=\; \sum_{j=1}^{n}\int_{-1}^{1}\varphi(a)k_j(a,x_n,z)F_j(z)dz - \frac{\varphi(a)}{2a}\int_{-1}^{1}F_{n+1}(z)dz$$

with $[\varphi(\cdot)k_j(\cdot,x_n,z)] \leq c$ uniformly. Inverting,

$$q = q_0 + \int_{-1}^{1}\sum_{j=1}^{n-1}\mathcal{F}^{-1}\left[\frac{\varphi(a)}{a^2}\xi_j\right] * f_j(\cdot,z)dz$$

with $\|q_0\|_p \leq c\|f\|_p$, $\|\nabla q_0\|_p \leq c\|f\|_p$. As $\mathcal{F}^{-1}[\frac{\varphi(a)}{a^2}\xi_j]$ is smooth and bounded if $n \geq 3$, the convolution is well defined for $f_j \in C_0^\infty$ in that case. If $n = 2$, $q(x)$ is well defined since it is determined up to an constant by $\partial_2 q(x) \in L^q(\Omega)$. Hence $q = q_0 + q_1(x')$ with $\|\nabla_{x'}q_1\|_{L^p(\mathbb{R}^{n-1})} \leq C\|f'\|_p$. The local estimates follow now by Lemma 2.7. ∎

It is well-known that the existence of the Helmholtz decomposition of $L_p(\Omega)$, $1 < p < \infty$, is equivalent to the unique solvability of the weak Neumann problem for the Laplace equation, i.e., for every $f \in (\dot{W}^1_{p'}(\Omega))'$ there is a unique $u \in \dot{W}^1_p(\Omega)$ such that

$$\langle \nabla u, \nabla v\rangle = \langle f, v\rangle \quad \text{for all } v \in \dot{W}^1_{p'}(\Omega), \tag{3.4}$$

cf. e.g. [16]. Moreover, $\|\nabla u\|_p \leq C_p\|f\|_{(\dot{W}^1_{p'}(\Omega))'}$. This will be used in the proof of the following lemma.

Lemma 3.3 *Let $1 < p < \infty$. Then $C^\infty_{(0)}(\overline{\Omega}) \cap \hat{W}^{-1}_p(\Omega)$ is dense in $W^1_p(\Omega) \cap \hat{W}^{-1}_p(\Omega)$.*

Proof: Let $g \in W^1_p(\Omega) \cap \hat{W}^{-1}_p(\Omega)$ be arbitrary. Since $g \in (\dot{W}^1_{p'}(\Omega))'$ and by (3.4), there is a $u \in L_p(\Omega)$ such that

$$\langle g, v\rangle = \langle \nabla u, \nabla v\rangle, \quad \text{for all } v \in \dot{W}^1_{p'}(\Omega). \tag{3.5}$$

Moreover, since even $g \in W^1_p(\Omega)$, approximation of $\partial_j u$, $\partial_i\partial_j u$, $i,j = 1,\dots,n-1$, by difference quotients yields that $\partial_j u, \partial_i\partial_j u \in L_p(\Omega)$. Using (3.5) we obtain also $\partial_n u, \partial_n \nabla u \in L_p(\Omega)$ and therefore $u \in W^2_p(\Omega)$. Furthermore, (3.5) yields $\Delta u = g$ almost everywhere and $\partial_n u|_{\partial\Omega} = 0$.

Now let $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ with $\psi(0) = 1$ and set $g_R(x) = \operatorname{div}(\psi(Rx')\nabla u)$. Then $g_R \in W^1_p(\Omega) \cap \hat{W}^{-1}_p(\Omega)$ and $\lim_{R\to\infty} g_R = g$ in $W^1_p(\Omega) \cap \hat{W}^{-1}_p(\Omega)$. Since g_R is compactly supported, [4, Lemma 2.8] implies that

$$\int g_R dx = 0 \qquad \text{if } 1 < p \leq \frac{n-1}{n-2}.$$

Hence, if $1 < p \leq \frac{n-1}{n-2}$, we can find $g_{k,R} \in C^\infty_{(0)}(\overline{\Omega})$ with $\int g_{k,R} dx = 0$ such that $\lim_{k\to\infty} g_{k,R} = g_R$ in $W^1_p(\Omega)$. Since $\int g_R dx = \int g_{k,R} dx = 0$, this implies that $\lim_{k\to\infty} g_{k,R} = g_R$ in $\hat{W}^{-1}_p(\Omega)$ by Poincaré's inequality. In the case $p > \frac{n-1}{n-2}$, by [4, Lemma 2.8] every $h \in L_p(\Omega)$ with support in $B_R(0)$ is in $\hat{W}^{-1}_p(\Omega)$ and $\|h\|_{\hat{W}^{-1}_p(\Omega)} \leq C_R \|h\|_p$. Hence, if $g_{k,R} \in C^\infty_{(0)}(\overline{\Omega})$ such that $\lim_{k\to\infty} g_{k,R} = g_R$ in $W^1_p(\Omega)$, also $\lim_{k\to\infty} g_{k,R} = g_R$ in $\hat{W}^{-1}_p(\Omega)$.

In particular, we have proved that $C^\infty_{(0)}(\overline{\Omega}) \cap \{g : \int g dx = 0\}$ if $1 < p \leq \frac{n-1}{n-2}$ and $C^\infty_{(0)}(\overline{\Omega})$ if $p > \frac{n-1}{n-2}$ are dense in $W^1_p(\Omega) \cap \hat{W}^{-1}_p(\Omega)$. ■

4 Laplace Resolvent Equation

We consider

$$(\lambda - \Delta)u = f \quad \text{in } \Omega, \tag{4.1}$$
$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

In order to prove the estimates of Theorem 1.1, we will use the corresponding statements for the latter system.

THEOREM 4.1 *Let $1 < q \leq r < \infty$, $n \geq 2$, and let $\varepsilon > 0$. Then for every $\lambda \in \mathbb{C} \setminus (-\infty, \frac{\pi^2}{4}]$ and $f \in L_q(\Omega)$ there is a unique solution $u \in W^2_q(\Omega)$ of (4.1)-(4.2). Moreover,*

$$\left|\lambda + \frac{\pi^2}{4}\right| \left(\|u\|_q + (1+|\lambda|)^{-\frac{1}{2}}\|\nabla u\|_q + (1+|\lambda|)^{-1}\|\nabla^2 u\|_q\right) \leq C_\varepsilon \|f\|_q,$$
$$\left|\lambda + \frac{\pi^2}{4}\right|^{1-\frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} (1+|\lambda|)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|u\|_r \leq C_\varepsilon \|f\|_q$$

uniformly in $\lambda \in S_\varepsilon$ provided that $\frac{n-1}{2}\left(\frac{1}{q} - \frac{1}{r}\right) \leq 1$.

First we calculate the solution u for $f \in C_0^\infty(\Omega)$ and estimate the solution operator. Then the general case is obtained by continuous extension to $L_p(\Omega)$. Using partial Fourier transformation, (4.1)-(4.2) reduces to

$$(\lambda + |\xi'|^2 - \partial_n^2)U = F \quad \text{in } (-1,1), \tag{4.3}$$

$$U|_{x_n=\pm 1} = 0 \tag{4.4}$$

for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in S_\varepsilon$, where $U(\xi', x_n) = \mathcal{F}_{x' \mapsto \xi'}[u(.,x_n)]$ and $F(\xi', x_n) = \mathcal{F}_{x' \mapsto \xi'}[f(.,x_n)]$. The solution of this boundary value problem is given by

$$U(\xi', x_n) = \int_{-1}^{1} k(\mu, x_n, z_n) F(\xi', z_n) dz_n,$$

where

$$k(\mu, x_n, z_n) = \frac{1}{2\mu(1 - e^{-4\mu})} \left(e^{-\mu(2-x_n-z_n)} + e^{-\mu(2+x_n+z_n)} - e^{-\mu|x_n-z_n|} - e^{-\mu(4-|x_n-z_n|)} \right)$$

with μ as in Lemma 2.8. Since $\mu^2 = \lambda + |\xi'|^2 = (-\mu)^2$, we obtain $k(-\mu, x_n, z_n) = k(\mu, x_n, z_n)$. Moreover, it is easy to observe that $k(\mu, x_n, z_n)$ is holomorphic in $\mathbb{C} \setminus \{\frac{il\pi}{2} : l \in \mathbb{Z}, l \neq 0\}$ and depends smoothly on $x_n, z_n \in \mathbb{R}$. Since $k(\mu, x_n, z_n)$ has a pole of first order for $\mu = \pm i\frac{\pi}{2}$ and $k(-\mu, x_n, z_n) = k(\mu, x_n, z_n)$, we have

$$k(\mu, x_n, z_n) = \frac{k'(\mu^2, x_n, z_n)}{\mu^2 + \frac{\pi^2}{4}} \qquad \text{for all } \mu \text{ such that } \mu^2 \in B_{\pi^2}(0)$$

where $k'(z, x_n, z_n)$ is holomorphic in $z \in B_{\pi^2}(0)$.

Lemma 4.2 *Let $\mu = \mu(\lambda, \xi')$ be as in Lemma 2.8 with $|\mu| < \pi$. Moreover, let $\varepsilon > 0$, $\theta \in [0,1]$, and let $\kappa \in (0,1)$. Then*

$$|\partial_{\xi'}^\alpha k(\mu, x_n, z_n)| \leq C_{\alpha,\kappa,\varepsilon} \left| \lambda + \frac{\pi^2}{4} \right|^{-1+\theta} |\xi'|^{-2\theta-|\alpha|}$$

for all $\lambda \in S_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ such that $|\mu| \leq \kappa\pi$ and uniformly in $x_n, z_n \in [-1,1]$.

Proof: First of all,

$$\partial_{\xi'}^\alpha \left[\frac{1}{\mu^2 + \frac{\pi^2}{4}} \right] = \sum_{k=\lceil |\alpha|/2 \rceil}^{|\alpha|} \frac{p_k(\xi')}{\left(\lambda + |\xi'|^2 + \frac{\pi^2}{4}\right)^{1+k}}$$

where $p_k(\xi')$ is a homogeneous polynomial of degree $2k - |\alpha|$. Hence

$$\begin{aligned} \left| \partial_{\xi'}^\alpha \left[\frac{1}{\mu^2 + \frac{\pi^2}{4}} \right] \right| &\leq \sum_{k=\lceil |\alpha|/2 \rceil}^{|\alpha|} \frac{C_k |\xi'|^{2k-|\alpha|}}{|\lambda + \pi^2/4 + |\xi'|^2|^{1+k}} \leq \sum_{k=\lceil |\alpha|/2 \rceil}^{|\alpha|} \frac{C_{k,\varepsilon} |\xi|^{2k-|\alpha|}}{(|\lambda + \pi^2/4| + |\xi'|^2)^{1+k}} \\ &\leq \frac{C_{\alpha,\varepsilon}}{|\lambda + \pi^2/4|^{(1-\theta)}} |\xi'|^{-2\theta-|\alpha|} \end{aligned}$$

where we have used $|\tilde{\lambda}+|\xi'|^2| \geq c_\theta(|\tilde{\lambda}|+|\xi'|^2)$ for all $\tilde{\lambda} \in \Sigma_\theta$ and some $\theta = \theta(\varepsilon) \in (0,\pi)$, which is a consequence of (2.6). Moreover, since $k'(z, x_n, z_n)$ is holomorphic in $B_{\pi^2}(0)$ and smooth in $x_n, z_n \in \mathbb{R}$, $|\partial_{\xi'}^\alpha k'(\mu^2, x_n, z_n)| \leq C_\alpha$ for all $\lambda \in S_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ such that $|\mu| \leq \kappa\pi$. Hence the statement of the lemma is a consequence of the product and chain rule. ■

Proof of Theorem 4.1: Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(s) = 1$ for $|s| \leq \frac{1}{2}\pi^2$ and $\operatorname{supp}\varphi \subseteq B_{\frac{3}{4}\pi^2}(0)$. Then $U = U_1 + U_2$, where

$$U_j(\xi', x_n) \;=\; \int_{-1}^{1} k_j(\mu, x_n, z_n) F(\xi', z_n)\, dz_n, \qquad j = 1, 2,$$

$k_1(\mu, x_n, y_n) = \varphi(|\mu^2|)k(\mu, x_n, y_n)$ and $k_2(\mu, x_n, y_n) = (1-\varphi(|\mu^2|))k(\mu, x_n, y_n)$. Since $|\partial_{\xi'}^\alpha \varphi(|\mu^2|)| \leq C_\alpha$, we have for $\theta \in [0,1]$ by Lemma 4.2

$$|\partial_{\xi'}^\alpha k_1(\mu, x_n, y_n)| \leq \frac{C_{\alpha,\varepsilon}}{|\lambda + \pi^2/4|^{(1-\theta)}} |\xi'|^{-2\theta-|\alpha|}$$

uniformly in $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in S_\varepsilon$, and $x_n, z_n \in [-1,1]$. Hence

$$\begin{aligned}
\left\|\mathcal{F}^{-1}_{\xi'\mapsto x'}[U_1(\xi', x_n)]\right\|_r &\leq C \sup_{x_n\in[-1,1]} \|\mathcal{F}^{-1}_{\xi'\mapsto x'}[k_1(\mu, ., z_n)F(\xi', .)]\|_{r,\Omega} \\
&\leq C_{\varepsilon,q,r} \left|\lambda + \frac{\pi^2}{4}\right|^{-1+\frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{q,\Omega}.
\end{aligned}$$

by Theorem 2.2, where we have chosen $\theta = \frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)$. Moreover, if $\beta \in \mathbb{N}_0^{n-1}$, $|\beta| \leq 2$, we get by choosing $\theta = |\beta|/2$

$$\begin{aligned}
\left\|\partial_{x'}^\beta \mathcal{F}^{-1}_{\xi'\mapsto x'}[U_1(\xi', x_n)]\right\|_q &\leq C \sup_{x_n\in[-1,1]} \|\mathcal{F}^{-1}_{\xi'\mapsto x'}[(\xi')^\beta k_1(\mu, x_n, .)F(\xi', .)]\|_q \\
&\leq C_{\varepsilon,q} \left|\lambda + \frac{\pi^2}{4}\right|^{-1+\theta} \|f\|_q \leq C_{\varepsilon,q} \left|\lambda + \frac{\pi^2}{4}\right|^{-1} (1+|\lambda|)^{\frac{|\beta|}{2}} \|f\|_q
\end{aligned}$$

since $k_1(\mu, x_n, z_n) = 0$ for $|\lambda| \geq \pi^2$. Because of (4.3),

$$\left\|\partial_{x_n}^2 \mathcal{F}^{-1}_{\xi'\mapsto x'}[U_1(\xi', x_n)]\right\|_q \leq C_\varepsilon (1+|\lambda|) \left|\lambda + \frac{\pi^2}{4}\right|^{-1} \|f\|_q.$$

Using $\|\partial_n v\|_q \leq C\|v\|_q^{\frac{1}{2}} \left(\|v\|_q + \|\partial_n^2 v\|_q\right)^{\frac{1}{2}}$ for $v \in W_q^2(\Omega)$, we obtain the corresponding estimate for $\partial_{x_n}\mathcal{F}^{-1}_{\xi'\mapsto x'}[U_1(\xi', x_n)]$.

In order to estimate $U_2(\xi', x_n)$, we use

$$\begin{aligned}
&\mathcal{F}^{-1}_{\xi'\mapsto x'}\left[(1-\varphi(|\mu^2|)) \int_{-1}^{1} \frac{e^{-\mu(2\mp x_n\mp z_n)}}{2\mu(1-e^{-4\mu})} F(\xi', z_n)\, dz_n\right] \\
&\quad= \tau_{\mp 1}\mathcal{F}^{-1}_{\xi\mapsto x}\left[\frac{(1-\varphi(|\mu^2|))}{(1-e^{-4\mu^2})(\lambda+|\xi|^2)} \mathcal{F}_{x\mapsto\xi}[\tau_{\pm 1} e f]\right],
\end{aligned}$$

where $(\tau_y f)(x) = f(x+y)$ and ef denotes the extension by 0 of f to $\mathbb{R}^n$. Because of (2.6) and $1-\varphi(|\mu^2|) = 0$ if $|\mu|^2 \le \frac{1}{2}\pi^2$,

$$\left[\frac{\xi^\alpha(1-\varphi(|\mu^2|))}{(1-e^{-4\mu})(\lambda+|\xi|^2)}\right] \le C_\varepsilon(1+|\lambda|)^{-1+\frac{|\alpha|}{2}}$$

for every $|\alpha| \le 2$ uniformly in $\lambda \in S_\varepsilon$, $\varepsilon > 0$. Hence we obtain by Theorem 2.1

$$\left\| D_x^\alpha \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[(1-\varphi(|\mu^2|))\int_{-1}^{1}\frac{e^{-\mu(2\mp x_n\mp z_n)}}{2\mu(1-e^{-4\mu})}F(\xi',z_n)dz_n\right]\right\|_q \le C_{\varepsilon,q}(1+|\lambda|)^{-1+\frac{|\alpha|}{2}}\|f\|_q$$

uniformly in $\lambda \in S_\varepsilon$ for all $|\alpha| \le 2$ and $1 < q < \infty$. In the same way, we obtain by Theorem 2.2

$$\left\| \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[(1-\varphi(|\mu^2|))\int_{-1}^{1}\frac{e^{-\mu(2\mp x_n\mp z_n)}}{2\mu(1-e^{-4\mu})}F(\xi',z_n)dz_n\right]\right\|_r \le C_\varepsilon(1+|\lambda|)^{-1+\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|f\|_q$$

uniformly in $\lambda \in S_\varepsilon$ for all $1 < q \le r < \infty$. Moreover,

$$\frac{e^{-\mu|x_n-z_n|}+e^{-\mu(4-|x_n-z_n|)}}{2\mu(1-e^{-4\mu})} = \frac{e^{-\mu|x_n-z_n|}}{2\mu} + \frac{e^{-\mu(4-x_n+z_n)}+e^{-\mu(4+x_n-z_n)}}{2\mu(1-e^{-4\mu})},$$

where

$$\mathcal{F}^{-1}_{\xi'\mapsto x'}\left[(1-\varphi(|\mu^2|))\int_{-1}^{1}\frac{e^{-\mu|x_n-z_n|}}{2\mu}F(\xi',z_n)dz_n\right] = \mathcal{F}^{-1}_{\xi\mapsto x}\left[\frac{1-\varphi(|\mu^2|)}{\lambda+|\xi|^2}\mathcal{F}_{x\mapsto\xi}[ef]\right],$$

and

$$\begin{aligned}&\mathcal{F}^{-1}_{\xi'\mapsto x'}\left[(1-\varphi(|\mu^2|))\int_{-1}^{1}\frac{e^{-\mu(4\mp x_n\pm z_n)}}{2\mu(1-e^{-4\mu})}F(\xi',z_n)dz_n\right]\\ &= \tau_{\mp2}\mathcal{F}^{-1}_{\xi\mapsto x}\left[\frac{(1-\varphi(|\mu^2|))}{(1-e^{-4\mu})(\lambda+|\xi|^2)}\mathcal{F}_{x\mapsto\xi}[\tau_{\mp2}ef]\right].\end{aligned}$$

Hence the last terms can be estimated as before.

It remains to prove the uniqueness of the solution. Let $\Delta_q\colon \mathcal{D}(\Delta_q) \to L_q(\Omega)$, $1 < q < \infty$, with $\mathcal{D}(\Delta_q) = \{u \in W_q^2(\Omega)\colon u|_{\partial\Omega} = 0\}$. Then $\lambda - \Delta_q$ is surjective for every $\lambda \in \mathbb{C}\setminus(-\infty,\pi^2/4]$ by the first part of the proof. Moreover, since $(\lambda-\Delta_q)' \supseteq \overline{\lambda}-\Delta_{q'}$, the adjoint of $\lambda-\Delta_q$ is surjective. Hence $\lambda-\Delta_q$ is injective for every $\lambda \in \mathbb{C}\setminus(-\infty,\pi^2/4]$, which finishes the proof. ■

5 Proof of Theorem 1.1

We may start with two simplifications. First we may restrict ourselves to $f \in C_0^\infty(\Omega)$ and $g \in C^\infty_{(0)}(\overline{\Omega})\cap\hat{W}_p^{-1}(\Omega)$ by Lemma 3.3. Next we may use the Helmholtz decomposition $f = f^0+\nabla q$, where f^0 is defined explicitly via its partial Fourier transformation

F^0, being well defined and smooth. The pressure part may be absorbed on the left side. Hence we may assume for simplicity that $f = f^0$. Thus applying the partial Fourier transformation we have to solve (with $a = |\xi'|$)

$$\begin{aligned} (\lambda + a^2)U' - \partial_n^2 U' + i\xi' Q &= F', && (5.1)\\ (\lambda + a^2)U_n - \partial_n^2 U_n + \partial_n Q &= F_n, && (5.2)\\ \partial_n U_n + i\xi' \cdot U' &= G, && (5.3)\\ U|_{x_n=\pm 1} &= 0, && (5.4) \end{aligned}$$

where $U' = (U_1, \dots, U_{n-1})$ and $F' = (F_1, \dots, F_{n-1})$. Differentiating (5.2) with respect to x_n, multiplying (5.1) by $i\xi'$ and adding, we obtain

$$(\lambda + a^2)G - \partial_n^2 G - a^2 Q + \partial_n^2 Q = 0.$$

Hence

$$\partial_n^2 (Q - G) - a^2 (Q - G) = -\lambda G. \tag{5.5}$$

Thus we define

$$Q(\xi', x_n) = G(\xi', x_n) + \frac{\lambda}{2a} \int_{-1}^{1} e^{-a|x_n - w|} G(\xi', w) dw + \frac{A}{a} e^{a x_n} - \frac{B}{a} e^{-a x_n} \tag{5.6}$$

with parameters A, B to be chosen below. Setting

$$L_j := i\xi_j Q - F_j, \; j = 1, \dots, n-1, \quad L_n := \partial_n Q - F_n \tag{5.7}$$

we may solve

$$\partial_n^2 U_j - (\lambda + a^2) U_j = L_j, \quad U_j = 0 \text{ on } z = \pm 1 \quad \text{for } j = 1, \dots, n. \tag{5.8}$$

It remains to satisfy (5.3). Let $R := i\xi' \cdot U' + \partial_n U_n - G$. We have to show that $R \equiv 0$. Now

$$\begin{aligned} \partial_n R &= i\xi' \cdot \partial_n U' + (\partial_n^2 U_n - \partial_n Q) + \partial_n (Q - G)\\ &= i\xi' \cdot \partial_n U' + (\lambda + a^2) U_n - F_n + \partial_n (Q - G) \qquad \text{and}\\ \partial_n^2 R &= i\xi' \cdot \partial_n^2 U' + (\lambda + a^2) \partial_n U_n - \partial_n F_n + \partial_n^2 (Q - G)\\ &= (i\xi' \cdot U' + \partial_n U_n)(\lambda + a^2) - a^2 Q - (i\xi' \cdot F' + \partial_n F_n) + \partial_n^2 (Q - G)\\ &= (\lambda + a^2) R \end{aligned}$$

by (5.5) and div $f = 0$. Therefore, with μ as defined in Lemma 2.8,

$$\begin{aligned} R = R(\xi', x_n) &= a(\xi') e^{\mu x_n} + b(\xi') e^{-\mu x_n} \quad \text{for } \mu \neq 0\\ \text{resp. } &= a(\xi') x_n + b(\xi') \qquad\qquad \text{for } \mu = 0 \end{aligned}$$

Suppose we can choose A, B such that $R(\xi', \pm 1) = 0$. Then $R \equiv 0$, as $e^{4\mu} - 1 \neq 0$ due to $\lambda + |\xi'|^2 \neq \frac{k^2}{4}\pi^2$ (the case $\mu = 0$ is trivial), hence the claim follows.

A and B will of course depend on ξ' and λ, and we have to calculate them in order to estimate the solution. Skipping the ξ'-dependence, we get from (5.2) and (5.4)

$$\begin{aligned} \partial_n U_n(z) e^{\pm\mu z}|_{-1}^{1} &= \int_{-1}^{1} (\partial_n^2 U_n(z) - \mu^2 U_n(z)) e^{\pm\mu z} dz = \int_{-1}^{1} (\partial_n Q(z) - F_n(z)) e^{\pm\mu z} dz \\ &= Q(z) e^{\pm\mu z}\Big|_{-1}^{1} - \int_{-1}^{1} (\pm\mu Q(z) + F_n(z)) e^{\pm\mu z} dz \end{aligned}$$

hence

$$R(z) e^{\pm\mu z}|_{-1}^{1} = (Q(z) - G(z)) e^{\pm\mu z}|_{-1}^{1} - \int_{-1}^{1} (\pm\mu Q(z) + F_n(z)) e^{\pm\mu z} dz$$

Again due to $e^{4\mu} - 1 \neq 0$, the right-hand sides should vanish for $+\mu$ as well as for $-\mu$. For the "+"-sign, this means

$$\begin{aligned} 0 &= \frac{\lambda}{2a} \left(\int_{-1}^{1} e^{-a(1-x)+\mu} G(x) dx - \int_{-1}^{1} e^{-a(1+x)-\mu} G(x) dx \right) \\ &+ \frac{A}{a} \left(e^{a+\mu} - e^{-(a+\mu)} \right) - \frac{B}{a} \left(e^{\mu-a} - e^{-(\mu-a)} \right) - \int_{-1}^{1} F_n(z) e^{\mu z} dz - \mu \int_{-1}^{1} Q(z) e^{\mu z} dz \end{aligned}$$

Calculating the last term:

$$\begin{aligned} \int_{-1}^{1} Q(z) e^{\mu z} dz &= \int_{-1}^{1} G(z) e^{\mu z} dz + \frac{\lambda}{2a} \int_{-1}^{1} \int_{-1}^{1} e^{-a|w-z|} G(w) e^{\mu z} dw dz \\ &+ \frac{A}{a} \int_{-1}^{1} e^{(a+\mu)z} dz - \frac{B}{a} \int_{-1}^{1} e^{(-a+\mu)z} dz \end{aligned}$$

and noting that

$$\int_{-1}^{1} e^{-a|w-z|} e^{\mu z} dz = -\frac{2a}{\lambda} e^{\mu w} + e^{aw} \frac{e^{\mu-a}}{\mu-a} - e^{-aw} \frac{e^{-(\mu+a)}}{\mu+a}$$

we get

$$\begin{aligned} \int_{-1}^{1} Q(z) e^{\mu z} dz &= \frac{\lambda}{2a} \int_{-1}^{1} \left(e^{aw} \frac{e^{\mu-a}}{\mu-a} - e^{-aw} \frac{e^{-(\mu+a)}}{\mu+a} \right) G(w)\, dw \\ &+ \frac{A}{a} \left(\frac{e^{a+\mu}}{a+\mu} - \frac{e^{-(a+\mu)}}{a+\mu} \right) - \frac{B}{a} \left(\frac{e^{\mu-a}}{\mu-a} - \frac{e^{-(\mu-a)}}{\mu-a} \right). \end{aligned}$$

Inserting, we end with

$$A\delta_+ + B\delta_- = \int_{-1}^{1} F_n(\xi', x) e^{\mu x} dx + \frac{1}{2} \int_{-1}^{1} \lambda G(\xi', x) k(\xi', x) dx,$$

where $k(\xi',x) = e^{ax}\frac{e^{\mu-a}}{\mu-a} + e^{-ax}\frac{e^{-(\mu+a)}}{\mu+a}$ and $\delta_\pm := (e^{\mu\pm a} - e^{-(\mu\pm a)})/(\mu\pm a)$.

Changing from μ to $-\mu$, δ_+ goes over to δ_- and vice versa and the second condition is

$$A\delta_- + B\delta_+ = \int_{-1}^{1} F_n(\xi',x)e^{-\mu x}dx - \frac{1}{2}\int_{-1}^{1} \lambda G(\xi',x)k(\xi',-x)dx.$$

Finally,

$$A = \int_{-1}^{1} F_n(\xi',x)H(\xi,x)dx + \frac{1}{2}\int_{-1}^{1} \lambda G(\xi',x)K(\xi',x)dx \tag{5.9}$$

$$B = \int_{-1}^{1} F_n(\xi',x)H(\xi',-x)dx - \frac{1}{2}\int_{-1}^{1} \lambda G(\xi',x)K(\xi',-x)dx \tag{5.10}$$

with

$$H(\xi',x) = (\delta_+ e^{\mu x} - \delta_- e^{-\mu x})/D \qquad \text{and}$$
$$K(\xi',x) = (\delta_+ k(\xi',x) + \delta_- k(\xi',-x))/D$$

where $D = \delta_+^2 - \delta_-^2$. Thus (5.6) - (5.10) gives the explicit solution of our problem.

Now we may start to estimate the quantities of (5.7) after inverting the partial Fourier transformation. Obviously,

$$\|\mathcal{F}^{-1}[\partial_n G]\|_p + \|\mathcal{F}^{-1}[i\xi' G]\|_p \le c\|\nabla g\|_p.$$

Next, multiplying by $i\xi_j$ and differentiating with respect to x_n, resp., we have to treat the following three types of integrals

$$h_1(x',x_n) = \mathcal{F}^{-1}\left[\int_{-1}^{1} e^{-|\xi'||w-x_n|}\rho(w-x_n)\lambda G(\xi',w)dw\right] \tag{5.11}$$

$$h_2(x',x_n) = \mathcal{F}^{-1}\left[\int_{-1}^{1} e^{-|\xi'|x_n} F_n(\xi',w)H(\xi',w)dw\right] \tag{5.12}$$

$$h_3(x',x_n) = \mathcal{F}^{-1}\left[\int_{-1}^{1} e^{-|\xi'|x_n} \lambda G(\xi',w)K(\xi',w)dw\right] \tag{5.13}$$

with $\rho(t) = \operatorname{sign} t$ or $\rho(t) = 1$. We start with

Lemma 5.1 *Let h_1 be defined as in (5.11) and let $1 < p < \infty$. Then*

$$\|h_1\|_{L_p(\Omega)} \le C_p\|\lambda g\|_{\hat{W}_p^{-1}} \qquad \text{for all } g \in C_{(0)}^\infty(\overline{\Omega}) \cap \hat{W}_p^{-1}(\Omega).$$

Proof: Let $\varphi \in C_0^\infty(\Omega)$. Define ψ by

$$\mathcal{F}[\psi](\xi',x_n) = \int_{-1}^{1} e^{-|\xi'||x_n-z|}\rho(x_n-z)\mathcal{F}[\varphi](\xi',z)dz.$$

Then

$$\begin{aligned}
&\int_\Omega \lambda g(x',x_n)\overline{\psi}(x',x_n)d(x',x_n)\\
&= \int_{\mathbb{R}^{n-1}}\int_{-1}^{1}\int_{-1}^{1}\lambda G(\xi',x_n)e^{-|\xi'||x_n-z|}\rho(x_n-z)\overline{\mathcal{F}[\varphi](\xi',z)}dx_n dz d\xi'\\
&= \int_{\mathbb{R}^{n-1}}\int_{-1}^{1}\mathcal{F}[h_1](\xi',z)\overline{\mathcal{F}[\varphi]}(\xi',z)dzd\xi' = \int_\Omega h_1(x',x_n)\overline{\varphi}(x',x_n)dx
\end{aligned}$$

As for $j=1,\dots,n-1$

$$\begin{aligned}
\mathcal{F}\left[\frac{\partial\psi}{\partial x_j}\right](\xi',x_n) &= \int_{-1}^{1} e^{-|\xi'||x_n-z|}|\xi'|\rho(x_n-z)\frac{i\xi_j}{|\xi'|}\mathcal{F}[\varphi](\xi',z)dz,\\
\mathcal{F}\left[\frac{\partial\psi}{\partial x_n}\right](\xi',x_n) &= \int_{-1}^{1} e^{-|\xi'||x_n-z|}|\xi'|\,\mathrm{sign}(z-x_n)\rho(w-x_n)\mathcal{F}[\varphi](\xi',z)dz\\
&\qquad (+2\mathcal{F}[\varphi](\xi',x_n) \quad \text{in the case of } \rho(t)=\mathrm{sign}\,t)\,,
\end{aligned}$$

Lemma 2.5 implies that $\|\nabla\psi\|_{p'}\le C_p\|\varphi\|_{p'}$. Therefore $\left|\int_\Omega h_1\cdot\overline{\varphi}dx\right|\le C_p\|\lambda g\|_{\hat{W}_p^{-1}}\cdot\|\varphi\|_{p'}$, which implies the claim. ■

The most important estimates are contained in the following lemma.

Lemma 5.2 *Let h_2 be defined as in (5.12) and let $1<p<\infty$. Then*

$$\|h_2\|_p\le C_p\|f\|_p \qquad \textit{for all } f\in C_0^\infty(\Omega).$$

Proof: Here we need a little trick in splitting the phase space into two parts, depending on the size of $|\lambda|$. Therefore let $M:=\max\{\frac{1}{4}|\lambda|^{1/2},a_\varepsilon\}$ with $a_\varepsilon\ge\pi$ as in Lemma 2.10. Moreover, we choose a cut-off function $\chi_M(a)$, which vanishes outside $[0,2M]$ and equals 1 on $[0,M]$. We may assume that $0\le\chi_M(a)\le 1$ and $[\chi_M]\le C$, independent of M. Now we split $h_2=h_2^1+h_2^2$ with

$$h_2^j(x',x_n)=\mathcal{F}^{-1}\left[\int_{-1}^{1}e^{-ax_n}F_n(\xi',w)H^j(\xi,w)dw\right],\quad j=1,2,$$

where $H^1(\xi',w)=(1-\chi_M(a))H(\xi',w)$ and $H^2(\xi',w)=\chi_M(a)H(\xi',w)$. If we now set

$$H^3(\xi',w)=\chi_M(a)a(\mu D)^{-1}(\delta_+e^{\mu w}+\delta_-e^{-\mu w}),$$

then

$$\frac{\partial H^3}{\partial w}(\xi',w)=H^2(\xi',w)a.$$

Since $F_n(\xi',\pm1)=0$ and $-\partial_nF_n=i\xi'\cdot F'$, one gets

$$h_2^2(x',x_n)=\mathcal{F}^{-1}\left[\int_{-1}^{1}e^{-ax_n}\frac{i\xi'}{|\xi'|}\cdot F'(\xi',w)H^3(\xi',w)dw\right].$$

Thus by Theorem 2.1

$$\begin{aligned}\|h_2(\cdot,x_n)\|_{L_p(\mathbb{R}^{n-1})} &\leq C_p\int_{-1}^{1}[e^{-ax_n}H^1(\xi',w)]\|f_n(\cdot,w)\|_{L_p(\mathbb{R}^{n-1})}dw\\ &\quad+C_p\int_{-1}^{1}[e^{-ax_n}H^3(\xi',w)]\|f'(\cdot,w)\|_{L_p(\mathbb{R}^{n-1})}dw\end{aligned}$$

By (2.1) and the definition of χ

$$[e^{az}H^1(\xi',w)] \leq C\sum_{+,-}[D^{-1}e^{az}\delta_\pm e^{\pm\mu w}]_M^\infty \qquad \text{and} \tag{5.14}$$

$$[e^{az}H^3(\xi',w)] \leq C\sum_{+,-}[D^{-1}(a\mu^{-1})e^{az}\delta_\pm e^{\pm\mu w}]_0^{2M}. \tag{5.15}$$

We will show that the right-hand sides are bounded by $C(2-z-w)^{-1}+C(2-z+w)^{-1}$. Then the statements of the lemma is a consequence of Lemma 2.6.

In order to prove (5.14), we write the multipliers as a product $m_1\cdot m_2$ with

$$m_1 = ae^{a(z-1)+\mu(\pm w-1)} \quad \text{and} \quad m_2=(e^{a+\mu}\delta_\pm)(aD)^{-1}.$$

Since $a\geq M=\max\{\frac{1}{4}|\lambda|^{1/2},a_\varepsilon\}\geq\pi$, we have $[m_1]_M^\infty\leq C_\varepsilon(2-z\mp w)^{-1}$ by Lemma 2.9. Instead of estimation m_2, we may as well estimate

$$m_2^\pm = e^{a+\mu}(\delta_+\pm\delta_-)(aD)^{-1}=a^{-1}(\delta_+\mp\delta_-)^{-1}e^{a+\mu}$$

as $D=\delta_+^2-\delta_-^2$, which was done in Lemma 2.10.

For the proof of (5.15), suppose first that $\frac{1}{4}|\lambda|^{1/2}\geq a_\varepsilon$, which implies $a\leq\frac{1}{2}|\lambda|^{1/2}$. Similarly as before, the necessary estimates follow from the estimates of $\left[\frac{a}{\mu}\frac{e^{az+\mu w}}{(\delta_+\pm\delta_-)}\right]_0^{2M}$ (and by symmetry also for $-w$), as $D=\delta_+^2-\delta_-^2$. For the case of the "+"-sign, we factor into

$$ae^{a(z-1)+\mu(w-1)} \text{ and } \mu^{-1}(\delta_++\delta_-)^{-1}e^{a+\mu}.$$

For the "-"-sign, we take the factors

$$(1+a)e^{a(z-1)+\mu(w-1)} \text{ and } \mu^{-1}\frac{a}{1+a}(\delta_+-\delta_-)^{-1}e^{a+\mu}$$

on $[0,2M]$. All these terms were estimated in Lemma 2.9 and Lemma 2.10.

For the second part we have to consider the other possibility $|\lambda|^{1/2}/4\leq a_\varepsilon=M$, where now a_ε is fixed. Then we do *not* change the original form of h_2^2, but still consider

$$h_2^2(x',x_n)=\mathcal{F}^{-1}\left[\int_{-1}^{1}e^{-ax_n}F_n(\xi',w)\chi_M(a)H(\xi',w)dw\right]$$

The crucial observation now is that, because of (3.3) and $F_n=F_n^0$, $F_n(\xi',w)$ contains factor $a=|\xi'|$! Hence we estimate

$$\|h_2^2(\cdot,x_n)\|_p\leq C_p\int_{-1}^{1}[aH(\xi',w)e^{-ax_n}]_0^{2M}\left\|\mathcal{F}^{-1}\left[F_n(\xi',w)a^{-1}\chi_M(a)\right]\right\|_p dw$$

For the second factor, (3.3) and $M = a_\varepsilon$ imply for all $|w| \le 1$ that

$$\left\| \mathcal{F}^{-1} \left[F_n(\xi', w) a^{-1} \chi_M(a) \right] \right\|_p \le C_p(M) \|f\|_p$$

As $M = a_\varepsilon$ is fixed, we only need to show, that $[aH(\xi', w)e^{-ax_n}]_0^{2M} \le C$ (uniformly for $|x_n|, |w| \le 1$) which follows from $[aH(\xi', w)]_0^{2M} \le c$.

Now $aH(\xi', w) = a(\delta_+ e^{\mu w} - \delta_- e^{-\mu w})/D$. As $\delta_+ e^{\mu w} - \delta_- e^{-\mu w}$ is odd in μ and analytic in a and μ, one gets

$$\delta_+ e^{\mu w} - \delta_- e^{-\mu w} = \mu Z_w(a, \mu^2)$$

with $Z_w(a, b)$ analytic in a, b. Thus

$$[aH(\xi', w)]_0^{2M} \le c[Z_w(a, \lambda + a^2)]_0^{2M} \cdot [a\mu/D]_0^{2M}$$

and the conclusion follows with the help of Lemma 2.11. ■

Lemma 5.3 *Let h_3 be defined as in (5.13) and let $1 < p < \infty$. Then*

$$\|h_3\|_p \le C \left(\|\nabla g\|_p + (1 + |\lambda|) \|g\|_{\hat{W}_p^{-1}} \right) \qquad \text{for all } g \in C_{(0)}^\infty(\overline{\Omega}) \cap \hat{W}_p^{-1}(\Omega).$$

Proof: Remember that

$$h_3(x', x_n) = \mathcal{F}^{-1} \left[\int_{-1}^{1} \lambda e^{-ax_n} K(\xi', w) G(\xi', w) dw \right]$$

with $K(\xi', w) = (\delta_+ k(\xi', w) + \delta_- k(\xi', -w))/D$ and

$$k(\xi', w) = e^{aw} \frac{e^{\mu - a}}{\mu - a} + e^{-aw} \frac{e^{-(\mu + a)}}{\mu + a}.$$

Due to $\lambda = (\mu - a)(\mu + a)$, there is no singularity for $\mu = a$.

We use the same cut-off function $\chi_M(a)$ as in the proof of the previous lemma to split h_3 into $h_3^1 + h_3^2$, where the support of the multiplier in h_3^2 is contained in $[0, 2M]$, and $M = \max\{|\lambda|^{1/2}/4, a_\varepsilon\}$.

Let us estimate h_3^1 first. Using (2.1)

$$\begin{aligned} &[\lambda a^{-1} e^{az} K(\xi, w)]_M^\infty \\ &\le \quad c[e^{a+\mu} \delta_\pm a^{-1} D^{-1}]_M^\infty [a e^{a(z \pm w - 2)}]_M^\infty \left(\left[\lambda a^{-1} e^{a-\mu} \frac{e^{\mu - a}}{\mu - a} \right]_M^\infty + \left[\lambda a^{-1} e^{a-\mu} \frac{e^{-\mu - a}}{\mu + a} \right]_M^\infty \right) \end{aligned}$$

The first factor is bounded by a constant because of Lemma 2.10, the second by $C(2 - z \mp w)^{-1}$ because of Lemma 2.9. The last factor may be simplified to

$$\left[\frac{\mu + a}{a} \right]_M^\infty + \left[\frac{\mu - a}{a} e^{-2\mu} \right]_M^\infty,$$

which is bounded due to Lemma 2.9. Thus

$$\|h_3^1(\cdot,x_n)\|_{L_p(\mathbb{R}^{n-1})} \le C_p \int_{-1}^{1} (2-x_n \mp w)^{-1} \|\mathcal{F}^{-1}[aG](\cdot,w)\|_{L_p(\mathbb{R}^{n-1})} dw.$$

As $\|\mathcal{F}^{-1}[aG](\cdot,w)\|_{L_p(\mathbb{R}^{n-1})} \le C_p \|\nabla g(\cdot,w)\|_{L_p(\mathbb{R}^{n-1})}$, the estimate for h_3^1 in $L_p(\Omega)$ follows by Lemma 2.6.

Turning to h_3^2, we have to estimate multipliers on $[0,2M]$. Suppose first that $\frac{1}{4}|\lambda|^{1/2} \ge a_\varepsilon$ and therefore $|\mu| \ge \sqrt{3}a$.

We calculate

$$\begin{aligned}\lambda K(\xi',w)e^{az} &= e^{a(w+z)}\left(e^{2\mu}-e^{-2\mu}\right)D^{-1}\\ &\quad + e^{-a(w-z)}\left(\frac{\mu-a}{\mu+a}(1-e^{-2\mu-2a}) - \frac{\mu+a}{\mu-a}(1-e^{2\mu-2a})\right)D^{-1}\end{aligned}$$

In order to apply Lemma 2.12, we estimate the multiplier of $\lambda G|\xi'|^{-1}$, which is

$$\begin{aligned}h(\xi',w,z) &= \left(e^{2\mu+2a}-e^{2a-2\mu}\right)\frac{ae^{a(w+z-2)}}{D\lambda}\\ &\quad + \left(\frac{\mu-a}{\mu+a}(e^{2a}-e^{-2\mu}) - \frac{\mu+a}{\mu-a}(e^{2a}-e^{2\mu})\right)\frac{ae^{a(z-w-2)}}{D\lambda}\end{aligned}$$

With $\lambda = (\mu-a)(\mu+a)$, we estimate on $[0,2M]$ using Lemma 2.9

$$\begin{aligned}[h(\xi',w,z)]_0^{2M} &\le \frac{C}{2-w-z}\left[e^{2\mu+2a}D^{-1}(\mu+a)^{-2}\frac{a}{1+a}\right]_0^{2M}\left[\frac{\mu+a}{\mu-a}(1-e^{-4\mu})\right]_0^{2M}\\ &\quad + \frac{C}{2-z+w}\left[e^{2\mu+2a}D^{-1}(\mu+a)^{-2}\frac{a}{1+a}\right]_0^{2M}\\ &\quad \cdot\left[\left(e^{-2\mu}-e^{-4\mu-2a}\right) - \left(\frac{\mu+a}{\mu-a}\right)^2\left(e^{-2\mu}-e^{-2a}\right)\right]_0^{2M}.\end{aligned}$$

Because of Lemma 2.9, Lemma 2.10, and

$$e^{2\mu+2a}D^{-1}(\mu+a)^{-2}\frac{a}{1+a} = \left(\frac{\mu}{\mu+a}\right)^2 \mu^{-1}m_+\mu^{-1}\frac{a}{1+a}m_-,$$

we conclude $\left[e^{2\mu+2a}D^{-1}(\mu+a)^{-2}\frac{a}{1+a}\right]_0^{2M} \le C_\varepsilon$. The remaining terms can also be estimated by Lemma 2.9. Hence we get

$$[h(\xi',w,z)]_0^{2M} \le c(2-z\pm w)^{-1}.$$

Now $|\xi'|^{-1}\frac{\partial h}{\partial w}(\xi',w,z)$ is of precisely the same structure, apart from a sign, which does not matter. Hence by Lemma 2.12, we get $\|h_3^2\|_p \le c\|\lambda g\|_{\hat{W}_p^{-1}}$.

It remains to consider the case $|\lambda|^{1/2} \leq 4a_\varepsilon, a \leq 2a_\varepsilon$. We calculate

$$\lambda K(\xi', w)e^{az} = a^{-1}e^{az}\frac{a\mu}{D} \cdot \left(e^{aw}\frac{e^{2\mu}-e^{-2\mu}}{\mu} + e^{-aw}\int_0^2 e^{-ta}(e^{t\mu}+e^{-t\mu})dt + ae^{-aw}\int_0^2 e^{-ta}\frac{e^{t\mu}-e^{-t\mu}}{\mu}dt\right),$$

where by Lemma 2.11 $[\frac{a\mu}{D}]_0^{2a_\varepsilon} \leq c$, and the terms in brackets are analytic in a and μ^2. Thus $\lambda K(\xi', w)e^{az} = a^{-1}R(\xi', w, z)$, with $\left[\frac{\partial}{\partial w}R(\xi', w, z)|\xi'|^{-1}\right]_0^{2a_\varepsilon} + [R(\xi', w, z)]_0^{2a_\varepsilon} \leq c$. By Lemma 2.12, we get in this case $\|h_3^2\|_p \leq c\|g\|_{\hat{W}_p^{-1}}$. This finishes the proof. ■

Combining Lemma 5.1-Lemma 5.3, we have proved that

$$\|\nabla q\|_p \leq C_{\varepsilon,p}\left(\|f\|_p + \|\nabla g\|_p + (1+|\lambda|)\|g\|_{\hat{W}_p^{-1}}\right)$$

uniformly in $\lambda \in S_\varepsilon$, $\varepsilon > 0$. Since $(\lambda - \Delta)u = f - \nabla q$, $u|_{\partial\Omega} = 0$, Theorem 4.1 implies the estimates for u stated in Theorem 1.1. Hence extending the solution operator by continuity we have proved the solvability of (1.1)-(1.3) for every $\lambda \in \mathbb{C} \setminus (-\infty, \frac{\pi^2}{4}]$.

Finally, it remains to prove uniqueness of the solution. Let $A_p = -P_p\Delta_p$ with $\mathcal{D}(A_p) = \mathcal{D}(\Delta_p) \cap L_{p,\sigma}(\Omega)$, $1 < p < \infty$, be the Stokes operator. Then, by the solvability of (1.1)-(1.3), $\lambda + A_p$ is surjective for every $\lambda \in \mathbb{C}\setminus(-\infty, \frac{\pi^2}{4}]$ and $1 < p < \infty$. Because of $(\lambda + A_p)' \supseteq \overline{\lambda} + A_{p'}$, $(\lambda + A_p)'$ is surjective. Hence $\lambda + A_p$ is injective and the solution of (1.1)-(1.3) is unique.

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