Bounded Imaginary Powers and H_∞-Calculus of the Stokes Operator in Unbounded Domains

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Abstract

In the present contribution we study the Stokes operator $A_q = -P_q\Delta$ on $L^q_\sigma(\Omega)$, $1 < q < \infty$, where Ω is a suitable bounded or unbounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-boundary. We present some conditions on Ω and the related function space and basic equations which guarantee that $c + A_q$ for suitable $c \in \mathbb{R}$ is of positive type and admits a bounded H_∞-calculus. This implies the existence of bounded imaginary powers of $c + A_q$. Most domains studied in the theory of Navier-Stokes like e.g. bounded, exterior, and aperture domains as well as asymptotically flat layers satisfy the conditions. The proof is done by constructing an approximate resolvent based on the results of [3], which were obtained by applying the calculus of pseudodifferential boundary value problems. Finally, the result is used to proof the existence of a bounded H_∞-calculus of the Stokes operator A_q on an aperture domain.

Key words: Stokes equations, unbounded domains, bounded imaginary powers, H_∞-calculus, aperture domain
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1 Introduction

In this article we consider the Stokes operator $A_q = -P_q\Delta$ on $L^q_\sigma(\Omega)$ with domain

$$\mathcal{D}(A_q) = \{f \in W^2_q(\Omega)^n : f|_{\partial\Omega} = 0\} \cap L^q_\sigma(\Omega)$$

where $P_q\colon L^q(\Omega)^n \to L^q_\sigma(\Omega)$ denotes the Helmholtz projection, $L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $C^\infty_{0,\sigma}(\Omega) := \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}$, and $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a domain specified in Assumption 1.1 below. Properties of the Stokes operator are important for the

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associated instationary Stokes and Navier-Stokes equations. Since the latter equations arise in mathematical fluid mechanics, many different kinds of bounded and unbounded domains are of interest and have been studied.

The purpose of the present contribution is to present some conditions on Ω and the related function spaces which guarantee that $c + A_q$ for suitable $c \in \mathbb{R}$ is of positive type and admits a bounded H_∞-calculus w.r.t. $\delta \in (0,\pi)$. Here $c + A_q$ is of positive type w.r.t. δ if and only if $\Sigma_\delta \cup \{0\} \subseteq \rho(-c - A_q)$ and

$$\|(\lambda + c + A_q)^{-1}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq \frac{C_{q,\delta}}{|\lambda|}, \qquad \lambda \in \Sigma_\delta, \tag{1.1}$$

where $\Sigma_\delta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$. – Note that, if $\delta > \frac{\pi}{2}$, (1.1) implies that $-c - A_q$ generates a bounded, strongly continuous, analytic semi-group. – Moreover, $A := c + A_q$ is said to admit a *bounded H_∞-calculus* w.r.t. δ if and only if

$$h(A) := \frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + A)^{-1}\, d\lambda \tag{1.2}$$

is a bounded operator satisfying

$$\|h(A)\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq C_{q,\delta}\|h\|_\infty \qquad \text{for all } h \in H_\infty(\delta), \tag{1.3}$$

where $H_\infty(\delta)$ denotes the Banach algebra of all bounded holomorphic functions $h\colon \Sigma_{\pi-\delta} \to \mathbb{C}$, cf. McIntosh [25], and Γ is the negatively orientated boundary of Σ_δ. We note that in order to prove (1.3) for all $h \in H_\infty(\delta)$ it is sufficient to show the estimate for $h \in H(\delta)$, which consists of all $h \in H_\infty(\delta)$ such that

$$|h(z)| \leq C\frac{|z|^s}{1 + |z|^{2s}} \qquad \text{for all } z \in \Sigma_{\pi-\delta}$$

for some $s > 0$, cf. [8, Lemma 2.1]. For $h \in H(\delta)$ the integral (1.2) is well-defined as a Bochner integral.

The property of admitting a bounded H_∞-calculus is a generalization of possessing bounded imaginary powers since $h_y(z) = z^{iy} \in H_\infty(\delta)$ for all $\delta \in (0,\pi)$, which has many important consequences. In particular, (1.3) for $\delta > \frac{\pi}{2}$ yields the *maximal regularity* of $-c - A_q$ by the result of Dore and Venni [13].

The resolvent estimate (1.1) with arbitrary $\delta \in (0,\pi)$, $1 < q < \infty$ and $c = 0$, has been proved for various kinds of domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, cf. Giga [19] for bounded domains, Borchers and Sohr [9] and Borchers and Varnhorn [9] for exterior domains, Farwig and Sohr [16] for aperture domains, Abels and Wiegner [6] for an infinite layer $\Omega = \mathbb{R}^{n-1} \times (-1,1)$, and Abels [2] for asymptotically flat layers. Moreover, see Farwig and Sohr [15] for a general treatment and Desch, Hieber, and Prüss [12] for the case of a half-space $\mathbb{R}^n_+$ and $\mathbb{R}^n$, where also the borderline cases $q = 1, \infty$ have been studied.

The fact that A_q possesses bounded imaginary powers and admits a bounded H_∞-calculus was proved by Giga [20], Giga and Sohr [21], and Noll and Saal [27] for bounded domains and for exterior domains in $\mathbb{R}^n$, $n \geq 3$, by Giga and Sohr [22] for the half-space $\mathbb{R}^n_+$, see also [12], and by Abels [5, 4, 3] for two-dimensional exterior domains, an infinite layer, and asymptotically flat layers.

In the following we will present an approach, proving (1.1) for large λ and (1.3) for suitable c simultaneously for a class of domains, which includes all previously mentioned cases provided that some auxiliary results are known. More precisely, we make the following assumption.

Assumption 1.1 *Let $1 < q < \infty$ and $\delta \in (0, \pi)$ be fixed. Moreover, let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a domain satisfying the following conditions:*

(A1) *There is a finite covering of $\overline{\Omega}$ with relatively open sets U_j, $j = 1, \ldots, m$, such that U_j coincides (after rotation) with a relatively open set of $\overline{\mathbb{R}^n_{\gamma_j}}$, where $\mathbb{R}^n_{\gamma_j} := \{(x', x_n) \in \mathbb{R}^n : x_n > \gamma_j(x')\}$, $\gamma_j \in C^{1,1}(\mathbb{R}^{n-1})$. Moreover, suppose that there are cut-off functions $\varphi_j, \psi_j \in C^\infty_b(\overline{\Omega})$, $j = 1, \ldots, m$, such that φ_j, $j = 1, \ldots, m$, is a partition of unity, $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$, and $\operatorname{supp} \psi_j \subset U_j$, $j = 1, \ldots, m$.*

(A2) *The Helmholtz decomposition is valid for $L^r(\Omega)^n$ with $r = q, q'$, i.e., for every $f \in L^r(\Omega)^n$ there is a unique decomposition $f = f_0 + \nabla p$ with $f_0 \in L^r_\sigma(\Omega)$ and $p \in \dot{W}^1_r(\Omega)$. Moreover,*

$$L^q_\sigma(\Omega) = \{f \in L^q(\Omega)^n : \operatorname{div} f = 0, \gamma_\nu f = 0\}. \tag{1.4}$$

(A3) *For every $p \in \dot{W}^1_r(\Omega)$, $r = q, q'$, there is a decomposition $p = p_1 + p_2$ such that $p_1 \in W^1_r(\Omega)$, $p_2 \in L^r_{\mathrm{loc}}(\Omega)$ with $\nabla p_2 \in W^1_r(\Omega)$ and $\|(p_1, \nabla p_2)\|_{1,r} \leq C \|\nabla p\|_r$.*

We refer to Section 2 below for the definitions of the function spaces and the normal trace γ_ν.

Remark 1.2 First of all, we note that (A1) can be generalized to the case of a *locally finite* covering U_j, $j \in \mathbb{N}$, if uniform bounds on $\gamma_j, \varphi_j, \psi_j$ in $C^{1,1}$-norm are assumed and if for every $x \in \overline{\Omega}$ the number of sets U_j containing x is bounded by a constant independent of x. Moreover, it is easy to see that (A1) is fulfilled for all kinds of domains with $C^{1,1}$-boundary mentioned above. We refer to [28, 15, 16, 26, 14, 2] for the validity of the Helmholtz decomposition for these types of domains. The characterization (1.4) holds as well in these cases except for an aperture domain if $q > \frac{n}{n-1}$ for which the characterization is different, cf. [16, Lemma 3.1]. The identity (1.4) is used in Lemma 3.1 below. Moreover, (A3) is a technical condition needed in the proof of Lemma 4.2 below. It is satisfied if the following extension property is valid: For every $p \in \dot{W}^1_q(\Omega)$ there is an extension $\tilde{p} \in \dot{W}^1_q(\mathbb{R}^n)$ such that $\tilde{p}|_\Omega = p$ and $\|\nabla \tilde{p}\|_q \leq C \|\nabla p\|_q$. This is the case for every (ε, ∞)-domain, cf. [10], in particular, for exterior domains. This extension property does not hold for layer-like domains, cf. [2, Section 2.4]. Nevertheless (A3) is also valid in layer-like domains, cf. [2, Lemma 2.4].

The main result is the following:

THEOREM 1.3 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $\delta \in (0,\pi)$, and $1 < q < \infty$ satisfy Assumption 1.1. Then there is an $R > 0$ such that $(\lambda + A_q)^{-1}$ exists and*

$$\|(\lambda + A_q)^{-1}\|_q \leq \frac{C_{q,\delta}}{1+|\lambda|}$$

for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R$. Moreover,

$$\left\| \int_{\Gamma_R} h(-\lambda)(\lambda + A_q)^{-1}\, d\lambda \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq C_{q,\delta} \|h\|_\infty \tag{1.5}$$

for every $h \in H(\delta)$, where $\Gamma_R = \Gamma \setminus \overline{B_R(0)}$. In particular, for every $c \in \mathbb{R}$ and $0 < \delta' \leq \delta$ such that $c + \Sigma_{\delta'} \subset \rho(-A_q)$ the shifted Stokes operator $c + A_q$ admits a bounded H_∞-calculus with respect to δ'.

The proof of Theorem 1.3 is based on the construction of an approximative resolvent R_λ that coincides with $(\lambda + A_q)^{-1}$ modulo an operator which decays of order $O(|\lambda|^{-1-\varepsilon})$ as $|\lambda| \to \infty$ for some $\varepsilon > 0$. The construction is based on a localization procedure and a suitable result for the reduced Stokes operator on a curved half-space $\mathbb{R}^n_\gamma$, $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$, cf. Theorem 4.1 below. The latter result was basically obtained in [3] and is achieved using the calculus of pseudodifferential boundary value problems developed by Grubb [23] in a non-smooth version, cf. [3, Section 4] and [1].

In particular, if it is known that $\sigma(-A_q) \subseteq (-\infty, 0]$, then Theorem 1.3 implies that $c + A_q$ admits a bounded H_∞-calculus for every $c > 0$. In order to prove that this is also true for $c = 0$ it remains to analyze the resolvent near 0. This is done for the case of an aperture domain in $\mathbb{R}^n$, $n \geq 2$, in Section 5 below, where the following result is proved:

THEOREM 1.4 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an aperture domain with $C^{2,\mu}$-boundary, $\mu > 0$, as defined in Section 5 below. Then the Stokes operator A_q admits a bounded H_∞-calculus with respect to δ.*

In particular, this yields that the Stokes operator has maximal regularity on $L^q_\sigma(\Omega)$, which was also obtained by Fröhlich [18] in the context of weighted L^q-spaces.

2 Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a domain. Then $C_0^\infty(\Omega)$ denotes the set of all smooth $f\colon \Omega \to \mathbb{C}$ with compact support, and

$$C^\infty_{(0)}(\overline{\Omega}) := \{u = v|_{\overline{\Omega}} : v \in C_0^\infty(\mathbb{R}^n)\}.$$

Moreover, $C_b^\infty(\overline{\Omega})$ denotes the set of all smooth and bounded $f\colon \overline{\Omega} \to \mathbb{C}$ with bounded derivatives and $C^{1,1}(\mathbb{R}^n)$ is the set of all bounded functions $f\colon \mathbb{R}^n \to \mathbb{C}$ with bounded and globally Lipschitz continuous first order derivatives.

The usual Lebesgue space will be denoted by $L^q(\Omega)$ and $L^q(\partial\Omega)$, $1 \le q \le \infty$, normed by $\|.\|_q \equiv \|.\|_{L^q(\Omega)}$ and $\|.\|_{q,\partial\Omega} \equiv \|.\|_{L^q(\partial\Omega)}$, respectively. Furthermore, $L^q_{\mathrm{loc}}(\overline{\Omega})$ consists of all $f\colon \Omega \to \mathbb{C}$ such that $f \in L^q(B\cap\Omega)$ for all balls B with $B\cap\Omega \neq \emptyset$. The scalar product on $L^2(M)$ is denoted by $(.,.)_M$ for $M = \Omega$ or $M = \partial\Omega$. Moreover, the usual Sobolev-Slobodeckij spaces based on L^q, $1 < q < \infty$, are denoted by $W^s_q(\Omega)$ and $W^s_q(\partial\Omega)$, $s \ge 0$, with norms $\|.\|_{s,q}$ and $\|.\|_{s,q,\partial\Omega}$, resp., cf. e.g. [7]. As usual $W^s_{q,0}(\Omega)$, $s \ge 0$ with $s - \frac{1}{q} \notin \mathbb{N}$, is defined as the closure of $C_0^\infty(\Omega)$ in $W^s_q(\Omega)$, and

$$W_q^{-s}(\Omega) := (W^s_{q',0}(\Omega))', \quad W_{q,0}^{-s}(\Omega) := (W^s_{q'}(\Omega))', \quad W_q^{-t}(\partial\Omega) := (W^t_{q'}(\partial\Omega))'$$

for $s,t > 0$ with $s - \frac{1}{q'} \notin \mathbb{N}$ where $\frac{1}{q} + \frac{1}{q'} = 1$. Finally, the homogeneous Sobolev space of order 1 is defined as

$$\dot{W}^1_q(\Omega) := \left\{ p \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \nabla p \in L^q(\Omega) \right\}$$

normed by $\|\nabla \cdot\|_q$.

In the following let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a domain satisfying (A1). Using the cut-off functions φ_j, ψ_j, $j = 1,\dots,m$, many properties of the Sobolev-Slobodeckij space on Ω and $\partial\Omega$ can be reduced to the case of a curved half-space $\mathbb{R}^n_\gamma$, $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$. Then the diffeomorphism

$$F\colon \mathbb{R}^n_+ \to \mathbb{R}^n_\gamma : x \mapsto (x', x_n + \gamma(x')) \tag{2.1}$$

can be used to reduce the statement on $\mathbb{R}^n_\gamma$ to the case of a half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. More precisely, if $(F^*f)(x) := f(F(x))$ and $(F^{*,-1}f)(x) := f(F^{-1}(x))$ denote the pull-back and push-forward, resp., of a scalar function f, then

$$\begin{aligned} &F^*\colon W^s_q(\mathbb{R}^n_+) \to W^s_q(\mathbb{R}^n_\gamma), \qquad && F^*\colon W^s_{q,0}(\mathbb{R}^n_+) \to W^s_{q,0}(\mathbb{R}^n_\gamma),\\ &F_0^*\colon W^s_q(\mathbb{R}^{n-1}) \to W^s_q(\partial\mathbb{R}^n_\gamma) \end{aligned}$$

are linear isomorphisms for $|s| \le 2$, where F_0 denotes the restriction of F on $\partial\mathbb{R}^n_+$ and $\partial\mathbb{R}^n_+$ is identified with $\mathbb{R}^{n-1}$. In particular, we note that, if $2 \ge s > j + \frac{1}{q}$, $j = 0,1$, with $s - \frac{1}{q} \notin \mathbb{N}$, there is a bounded and surjective linear operator

$$\gamma_j\colon W^s_q(\Omega) \to W_q^{s-\frac{1}{q}}(\partial\Omega) \tag{2.2}$$

such that $\gamma_j u = \partial^j_\nu u|_{\partial\Omega}$ for all $f \in C^\infty_{(0)}(\overline{\Omega})$, cf. e.g. [7], where ν denotes the exterior normal. Moreover, if $0 < s < \frac{1}{q}$, by [29, Theorem 2.9.3] $W^s_{q,0}(\Omega) = W^s_q(\Omega)$, where Ω is as in Assumption 1.1. Thus

$$(W^s_q(\Omega))' = W^{-s}_{q'}(\Omega) \qquad \text{for all } s \in \left(-\frac{1}{q'}, \frac{1}{q}\right). \tag{2.3}$$

Furthermore, recall that for $f \in L^q(\Omega)$ with $\operatorname{div} f = 0$ it is possible to define a weak trace of the normal component $\gamma_\nu f \in W_{q,\nu}^{-\frac{1}{q}}(\partial\Omega) := \left(\gamma_0 \dot{W}^1_{q'}(\Omega)\right)'$ by

$$\langle \gamma_\nu f, \gamma_0 v\rangle_{\partial\Omega} := (f, \nabla v)_\Omega \qquad \text{for all } v \in \dot{W}^1_{q'}(\Omega), \tag{2.4}$$

where $\gamma_0 \dot{W}^1_{q'}(\Omega) := \{a \in L^{q'}_{\mathrm{loc}}(\partial\Omega) : a = A|_{\partial\Omega}, A \in \dot{W}^1_{q'}(\Omega)\}$ is equipped with the quotient norm. Of course, if $f \in C^\infty_{(0)}(\overline{\Omega})$ with $\operatorname{div} f = 0$, the definition of $\gamma_\nu f$ by (2.4) coincides with the usual trace $\nu \cdot f|_{\partial\Omega}$, i.e., $\langle \gamma_\nu f, \gamma_0 v\rangle_{\partial\Omega} = (\nu \cdot f, v)_{\partial\Omega}$ for all $v \in \dot{W}^1_{q'}(\Omega)$.

Moreover, we note that, if $f = f_0 + \nabla p$, $f_0 \in L^q_\sigma(\Omega)$, $p \in \dot{W}^1_q(\Omega)$, is the Helmholtz decomposition of $f \in L^q(\Omega)^n$ and (1.4) is valid, then p is uniquely determined as solution of the weak Neumann problem

$$\Delta p = \operatorname{div} f \qquad \text{in } \Omega, \tag{2.5}$$

$$\partial_\nu p|_{\partial\Omega} = \nu \cdot f|_{\partial\Omega} \quad \text{on } \partial\Omega, \tag{2.6}$$

where (2.5) is understood in the sense of distributions and (2.6) is understood as $\gamma_\nu(f - \nabla p) = 0$, cf. [28]. Because of the definition of γ_ν, $p \in \dot{W}^1_q(\Omega)$ solves (2.5)-(2.6) if and only if

$$(\nabla p, \nabla v)_\Omega = (f, \nabla v)_\Omega \qquad \text{for all } v \in \dot{W}^1_{q'}(\Omega). \tag{2.7}$$

Moreover, the existence of the (unique) Helmholtz decomposition is equivalent to the existence of a unique solution $p \in \dot{W}^1_q(\Omega)$ of (2.7) for every $f \in L^q(\Omega)^n$. We note that every $F \in \dot{W}^{-1}_{q,0}(\Omega) := (\dot{W}^1_{q'}(\Omega))'$ can be represented as $\langle F, v\rangle_\Omega = (f, \nabla v)_\Omega$ for some $f \in L^q(\Omega)^n$ with $\|f\|_q \le C\|F\|_{\dot{W}^{-1}_{q,0}}$, which is a consequence of the Hahn-Banach theorem. Finally, let $K_N \colon W_{q,\nu}^{-\frac{1}{q}}(\partial\Omega) \to \dot{W}^1_q(\Omega)$ be defined by

$$(\nabla K_N a, \nabla v)_\Omega = \langle a, \gamma_0 v\rangle_{\partial\Omega} \qquad \text{for all } v \in \dot{W}^1_{q'}(\Omega). \tag{2.8}$$

By definition of $W_{q,\nu}^{-\frac{1}{q}}(\partial\Omega)$, $a = \gamma_\nu A$ for some $A \in L^q_\sigma(\Omega)$. Hence (2.8) is equivalent to $\Delta K_N a = 0$ and $\partial_\nu K_N a|_{\partial\Omega} = a$.

3 The Reduced Stokes Operator

In the following let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, $\delta \in (0, \pi)$, and let $1 < q < \infty$ be as in Assumption 1.1.

In order to apply the results from [3] for the construction of an approximate resolvent for large λ, we need the *reduced* Stokes operator $A_{0,q}$ defined as

$$A_{0,q} u \;:=\; (-\Delta + \nabla K_N T)u, \qquad Tu := \gamma_\nu(\Delta - \nabla \operatorname{div})u,$$

for $u \in \mathcal{D}(A_{0,q}) := W^2_q(\Omega)^n \cap W^1_{q,0}(\Omega)^n$, where K_N is defined by (2.8). Since $\operatorname{div}(\Delta - \nabla \operatorname{div})u = 0$, we conclude that $T \colon \mathcal{D}(A_{0,q}) \to W_{q,\nu}^{-\frac{1}{q}}(\partial\Omega)$. Hence $\nabla K_N T \colon \mathcal{D}(A_{0,q}) \to$

$L^q(\Omega)^n$ is well-defined. – Note that $A_{0,q}$ is a densely defined unbounded operator on $L^q(\Omega)^n$ in contrast to the Stokes operator, which acts on the subspace $L^q_\sigma(\Omega)$. We refer to [2, Section 3] for explanations of the relation between the Stokes and the reduced Stokes operator.

Introducing local coordinates $Tu = \operatorname{div}_\tau \gamma_1 u_\tau$ for every $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ where div_τ denotes the tangential divergence, cf. [24, Lemma A.1]. Hence by (2.2)

$$T\colon W^{2+s}_q(\Omega)^n \cap W^1_{q,0}(\Omega)^n \to W^{s-\frac{1}{q}}_q(\partial\Omega) \quad \text{for all } 0 \geq s > -1 + \frac{1}{q}. \tag{3.1}$$

We will use the following additional assumption:

(A4) *There is an $R > 0$ such that for every $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R$ there is no non-trivial solution $g \in W^1_q(\Omega)$ of*

$$\lambda(g, v)_\Omega + (\nabla g, \nabla v)_\Omega = 0 \qquad \textit{for all } v \in W^1_{q'}(\Omega). \tag{3.2}$$

This assumption is needed in the proof of Lemma 3.1 below. In the following we will show that (A4) is a consequence of Assumption 1.1, cf. proof of Theorem 1.3 below.

The construction of the approximate resolvent is based on the following lemma.

Lemma 3.1 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, $1 < q < \infty$, and $\delta \in (0, \pi)$ be as in Assumption 1.1. Moreover, assume that* (A4) *holds and that $(\lambda + A_{0,q})^{-1}$ exists for some $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R$. Then $(\lambda + A_q)^{-1}$ exists and*

$$A_{0,q}|_{L^q_\sigma(\Omega)} = A_q, \qquad (\lambda + A_{0,q})^{-1}|_{L^q_\sigma(\Omega)} = (\lambda + A_q)^{-1}. \tag{3.3}$$

Proof: The first statement can be seen as follows: If $u \in \mathcal{D}(A_{0,q}) \cap L^q_\sigma(\Omega)$, then $\operatorname{div}(-\Delta u + \nabla K_N T u) = 0$ in the sense of distributions and

$$\gamma_\nu(-\Delta u + \nabla K_N T u) = -\gamma_\nu \Delta u + \partial_\nu K_N T u|_{\partial\Omega} = 0$$

in the sense of (2.4). Hence $-\Delta u = (-\Delta + \nabla K_N T)u - \nabla K_N T u$ is the Helmholtz decomposition of $-\Delta u$ by (A2), i.e., $(-\Delta + \nabla K_N T)u = P_q(-\Delta)u = A_q u$.

In order to prove the second relation let $u = (\lambda + A_{0,q})^{-1} f$ with $f \in L^q_\sigma(\Omega)$. Then multiplying $(\lambda + A_{0,q})u = f$ by ∇v, $v \in W^1_{q'}(\Omega)$, and using (2.8) and (2.4) we obtain that $g = \operatorname{div} u$ solves (3.2), which implies $\operatorname{div} u = 0$ by (A4). Therefore, $u \in L^q_\sigma(\Omega)$ by (1.4) and $(\lambda + A_q)u = (\lambda + A_{0,q})u = f$. Since by the first statement $\lambda + A_q = (\lambda + A_{q,0})|_{L^q_\sigma(\Omega)}$ is injective, we finally conclude that $(\lambda + A_q)^{-1} f = u = (\lambda + A_{q,0})^{-1} f$ for every $f \in L^q_\sigma(\Omega)$. ■

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and $1 < q < \infty$ be as in Assumption 1.1. If $\lambda + A_{q'}$ is surjective for $\lambda \notin (-\infty, 0]$, then there is no non-trivial solution of (3.2).*

Proof: Let $f \in L^{q'}(\Omega)$ be arbitrary and let $u \in \mathcal{D}(A_{0,q'})$ such that $(\lambda + A_{0,q'})u = f$. Then multiplying f with ∇g we observe that $\operatorname{div} u \in W^1_{q'}(\Omega)$ solves

$$-\lambda(\operatorname{div} u, g) - (\nabla \operatorname{div} u, \nabla g) = (f, \nabla g), \qquad \text{for all } g \in W^1_q(\Omega).$$

Hence, if $g \in W^1_q(\Omega)$ solves (3.2), then $(f, \nabla g) = 0$ for all $f \in L^{q'}(\Omega)$ and therefore $\nabla g = 0$. Because of (3.2) and $\lambda \neq 0$, we conclude $g = 0$. ■

4 Construction of the Approximative Resolvent

The proof of Theorem 1.3 is based on the following result.

THEOREM 4.1 *Let $\mathbb{R}^n_\gamma$, $n \geq 2$, $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$, be a curved half-space, $1 < q < \infty$, and let $\delta \in (0, \pi)$. Then there is a bounded operator $R_{\gamma,\lambda}\colon L^q(\mathbb{R}^n_\gamma)^n \to W^2_q(\mathbb{R}^n_\gamma)^n$ such that*

$$(\lambda - \Delta + \nabla \widetilde{K}_{\gamma,N} T) R_{\gamma,\lambda} f = f + S_{\gamma,\lambda} f \quad \text{in } \mathbb{R}^n_\gamma, \tag{4.1}$$

$$R_{\gamma,\lambda} f = 0 \quad \text{on } \partial\mathbb{R}^n_\gamma \tag{4.2}$$

for every $f \in L^q(\mathbb{R}^n_\gamma)^n$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, where $\|S_{\gamma,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \leq C_{q,\delta}(1 + |\lambda|)^{-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$ for some $\varepsilon > 0$. Here $\widetilde{K}_{\gamma,N}\colon W_q^{s-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma) \to W^{s+1}_q(\mathbb{R}^n_\gamma)$, $s \in (-\frac{1}{q'}, 1 + \frac{1}{q})$, is a bounded operator satisfying

$$\Delta \widetilde{K}_{\gamma,N} a = R'_\gamma a \quad \text{in } \mathbb{R}^n_\gamma, \tag{4.3}$$

$$\partial_\nu \widetilde{K}_{\gamma,N} a|_{\partial\Omega} = a + S'_\gamma a \quad \text{on } \partial\mathbb{R}^n_\gamma, \tag{4.4}$$

where $R'_\gamma\colon W_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to W^{-1}_{q,0}(\mathbb{R}^n_\gamma)$ and $S'_\gamma\colon W_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to W_q^{-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma)$ are bounded operators. Moreover, for every $R > 0$

$$(1 + |\lambda|)\|R_{\gamma,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} + \|\nabla^2 R_{\gamma,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \leq C_{q,\delta}, \qquad \lambda \in \Sigma_\delta, \tag{4.5}$$

$$\left\| \int_{\Gamma_R} h(-\lambda) R_{\gamma,\lambda}\, d\lambda \right\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \leq C_{q,\delta}\|h\|_\infty, \quad h \in H(\delta). \tag{4.6}$$

Proof: The theorem is proved in [5, Theorem 4.1] but only for $s = 0$ in the mapping properties of $\widetilde{K}_{\gamma,N}$, which remains to be extended.

The operator $\widetilde{K}_{\gamma,N} \equiv \widetilde{K}_1$ is defined in [3, Section 5.5] as

$$\widetilde{K}_{\gamma,N} \;=\; F^{*,-1} \underline{k}_1(D_x, x') F^*_0,$$

where $\underline{k}_1(D_x, x')$ is a Poisson operator of order -1 in R-form with $C^{0,1}$-coefficients in the sense of [3, Section 4] and $F^{*,-1}$, F^*_0 are as in Section 2. By duality

$$(\partial^\alpha_x \underline{k}_1(D_x, x') a, f)_\Omega = (-1)^{|\alpha|}(a, t(x', D_x)\partial^\alpha_x f)_{\partial\Omega} \tag{4.7}$$

for $a \in C_0^\infty(\mathbb{R}^{n-1})$, $f \in C_{(0)}^\infty(\overline{\mathbb{R}^n_+})$, and $\alpha \in \mathbb{N}_0^n$, where

$$t(x', D_x)f := \mathcal{F}^{-1}_{\xi' \mapsto x'} \left[\int_0^\infty \overline{\underline{\tilde{k}}_1(x', \xi', y_n)} \tilde{f}(\xi', y_n) dy_n \right],$$

$\tilde{f}(\xi', x_n) = \mathcal{F}_{x' \mapsto \xi'}[f(.,x_n)]$, and $\underline{\tilde{k}}_1(x', \xi', y_n)$ denotes the symbol-kernel of $\underline{k}_1(D_x, x')$, cf. [3, Section 4]. Using (4.7) for suitable α and [1, Theorem 4.8] for $t(x', D_x)$, the mapping properties of $\underline{k}_1(D_x, x')$ are obtained by duality and interpolation. ■

Now we define the approximate resolvent R_λ on a domain Ω satisfying (A1) as

$$R_\lambda f = \sum_{j=1}^m \psi_j R_{\gamma_j, \lambda}(\varphi_j f), \qquad f \in L^q(\Omega)^n,$$

where $R_{\gamma_j,\lambda}$, $j = 1, \ldots, m$, is the approximate resolvent on $\mathbb{R}^n_{\gamma_j}$ due to Theorem 4.1. Moreover, we define the approximate Poisson operator

$$\widetilde{K}_N a = \sum_{j=1}^m \psi_j \widetilde{K}_{\gamma_j, N}(\varphi_j a), \qquad a \in W_q^{-\frac{1}{q}}(\partial\Omega),$$

where $\widetilde{K}_{\gamma_j,N}$ is the operator due to Theorem 4.1 for $\mathbb{R}^n_{\gamma_j}$. Now we have

Lemma 4.2 *Let $1 < q < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be as in Assumption 1.1 and let $r = q$ or $r = q'$. Moreover, let K_N be the Poisson operator of the Neumann problem as defined in (2.8). Then there is some $\varepsilon > 0$ such that*

$$\|\nabla(K_N - \widetilde{K}_N)Tu\|_r \le C_r \|u\|_{2-\varepsilon,r} \qquad \text{for all } u \in W_r^{2-\varepsilon}(\Omega)^n \cap W_{r,0}^1(\Omega)^n.$$

Proof: For simplicity let $r = q$. Let $f \in L^{q'}(\Omega)^n$ be arbitrary and let $f = f_0 + \nabla p$, $f_0 \in L^{q'}_\sigma(\Omega)$, $p \in \dot{W}^1_{q'}(\Omega)$ be its Helmholtz decomposition. By (A3) we have $p = p_1 + p_2$, where $p_1 \in W^1_{q'}(\Omega)$ and $p_2 \in L^{q'}_{\text{loc}}(\overline{\Omega})$ with $\nabla p_2 \in W^1_{q'}(\Omega)$ and $\|(p_1, \nabla p_2)\|_{1,q'} \le C_{q'} \|\nabla p\|_{1,q'}$. Then by (4.3)-(4.4)

$$\begin{aligned}(\nabla(K_N - \widetilde{K}_N)Tu, f)_\Omega &= (\nabla(K_N - \widetilde{K}_N)Tu, \nabla p)_\Omega \\ &= (\nabla(K_N - \widetilde{K}_N)Tu, \nabla p_2)_\Omega + \langle (I - \partial_\nu \widetilde{K}_N)Tu, p_1\rangle_{\partial\Omega} - (\Delta \widetilde{K}_N Tu, p_1)_\Omega.\end{aligned}$$

The term $\langle (I - \partial_\nu \widetilde{K}_N)Tu, p_1\rangle_{\partial\Omega}$ can be estimated by $\|u\|_{2-\varepsilon,q}\|f\|_{q'}$ for some $\varepsilon > 0$ in a straight-forward manner using Theorem 4.1 and (3.1). Moreover,

$$\begin{aligned}|(\Delta \widetilde{K}_N Tu, p_1)_\Omega| &\le \sum_{j=1}^m |(\psi_j R'_{\gamma_j}(\varphi_j Tu), p_1)_\Omega| \\ &\quad + \sum_{j=1}^m |(2(\nabla\psi_j)\cdot \nabla \widetilde{K}_{\gamma_j,N}(\varphi_j Tu) + (\Delta\psi_j)\widetilde{K}_{\gamma_j,N}(\varphi_j Tu), p_1)_\Omega|,\end{aligned}$$

where

$$|(\psi_j R'_{\gamma_j}(\varphi_j Tu), p_1)_\Omega| \leq C\|R'_{\gamma_j}\varphi_j Tu\|_{W^{-1}_{q,0}(\mathbb{R}^n_{\gamma_j})}\|p_1\|_{1,q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'}$$

for some $\varepsilon > 0$. Since $p_1 \in W^1_{q'}(\Omega) \hookrightarrow W^\varepsilon_{q'}(\Omega)$ with $0 < \varepsilon < \frac{1}{q'}$, by the mapping property $\widetilde{K}_{\gamma_j,N}\colon W_q^{-\varepsilon-\frac{1}{q}}(\partial\mathbb{R}^n_{\gamma_j}) \to W_q^{1-\varepsilon}(\mathbb{R}^n_\gamma)$, and by (2.3)

$$|(2(\nabla\psi_j)\cdot\nabla\widetilde{K}_{\gamma_j,N}(\varphi_j Tu) + (\Delta\psi_j)\widetilde{K}_{\gamma_j,N}(\varphi_j Tu), p_1)_\Omega| \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'}$$

for $0 < \varepsilon < \frac{1}{q'}$. The term $(\nabla\widetilde{K}_N Tu, \nabla p_2)_\Omega$ is estimated in the same way using $\nabla p_2 \in W^1_q(\Omega)$. Finally, by the definitions of K_N and T

$$\begin{aligned} |(\nabla K_N Tu, \nabla p_2)_\Omega| &= |((\Delta - \nabla\operatorname{div})u, \nabla p_2)_\Omega| \\ &\leq |(\nabla u, \nabla^2 p_2)_\Omega| + |(\partial_\nu u, \nabla p_2)_{\partial\Omega}| + |(\operatorname{div} u, \Delta p_2)_\Omega| + |(\operatorname{div} u, \partial_\nu p_2)_{\partial\Omega}| \\ &\leq C\|u\|_{2-\varepsilon,q}\|\nabla p_2\|_{1,q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'} \end{aligned}$$

for some $\varepsilon > 0$. – The proof for $r = q'$ is done in the same way. ■

Proof of Theorem 1.3: The proof is the same as in [5, Theorem 4.4] with minor modifications. We include it for the convenience of the reader.

First of all, by (4.5) and interpolation

$$\|R_{j,\lambda}(\varphi_j f)\|_{s,q} \leq C_{q,\delta,R}(1+|\lambda|)^{-1+\frac{s}{2}}\|f\|_q, \qquad \lambda \in \Sigma_\delta, \tag{4.8}$$

for all $s \in [0,2]$ and $f \in L^q(\Omega)^n$, $j = 1,\ldots,m$. Moreover, by (4.1)

$$\begin{aligned} &(\lambda - \Delta + \nabla K_N T)R_\lambda f \\ &= f + \sum_{j=1}^m \psi_j S_{\gamma_j,\lambda}(\varphi_j f) - \sum_{j=1}^m (2(\nabla\psi_j)\cdot\nabla R_{j,\lambda}(\varphi_j f) + (\Delta\psi_j)R_{j,\lambda}(\varphi_j f)) \\ &\quad + (\nabla K_N T - \nabla\widetilde{K}_N T)R_\lambda f. \end{aligned}$$

Hence (4.8), Theorem 4.1, and Lemma 4.2 imply

$$(\lambda - \Delta + \nabla K_N T)R_\lambda = I + S'_\lambda,$$

where $\|S'_\lambda\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta}(1+|\lambda|)^{-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$ for some $\varepsilon > 0$. Therefore $(\lambda + A_{0,q})^{-1}$ exists for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R$ for some $R > 0$ and

$$(\lambda + A_{0,q})^{-1} = R_\lambda + S_\lambda,$$

where $\|S_\lambda\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta}(1+|\lambda|)^{-1-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$, $|\lambda| \geq R$. Furthermore, for $B_\lambda = R_\lambda, S_\lambda$ we have

$$\left\|\int_{\Gamma_R} h(-\lambda)B_\lambda\, d\lambda\right\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta}\|h\|_\infty, \qquad h \in H(\delta),$$

because of (4.6) and $\|S_\lambda\| \le C_{q,\delta}(1+|\lambda|)^{-1-\varepsilon}$.

The same arguments apply to $A_{0,q'}$ instead of $A_{0,q}$. Hence we can assume that also $(\lambda + A_{0,q'})^{-1}$ exists for all $\lambda \in \Sigma_\delta$ with $|\lambda| \ge R$. In particular, this implies that (A4) is valid because of Lemma 3.2. Now $(\lambda + A_{0,q})^{-1}|_{L^q_\sigma(\Omega)} = (\lambda + A_q)^{-1}$ by Lemma 3.1, which proves (1.5). The rest of the theorem follows from (1.5) and the fact that $f(\lambda) = (\lambda + A_q)^{-1}$ is uniformly bounded on compact subsets of $\rho(-A_q)$. ■

5 Bounded H_∞-Calculus for an Aperture Domain

Roughly speaking an aperture domain is a domain separated by a wall with a hole (aperture) inside. More precisely, an aperture domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a domain such that

$$\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B$$

for some ball B, where $\mathbb{R}^n_- := \{x \in \mathbb{R}^n : x_n < -d\}$ for some $d > 0$. Concerning the Stokes equations the aperture domain is of particular interest since under certain circumstances depending on the dimension n and the integral exponent q an additional condition has to be posed to get unique solutions, cf. [16]. This may be done by prescribing the *flux* $\phi(u)$ of the velocity field u through the hole, i.e.,

$$\phi(u) := \int_{M'} \nu \cdot u d\sigma,$$

where M' is an $(n-1)$-dimensional compact manifold dividing Ω into an upper and lower part Ω_+, Ω_-, resp., such that $\Omega_\pm \cup B = \mathbb{R}^n_\pm \cup B$. In the following let $\varphi_\pm, \psi_\pm \in C^\infty_b(\overline{\mathbb{R}^n_\pm})$ be cut-off functions such that $\varphi_\pm(x) = 1$ for $x \in \mathbb{R}^n_\pm$ with $|x| \ge R$ for some suitable $R > 0$, $\psi_\pm \equiv 1$ on $\operatorname{supp} \varphi_\pm$, and $\operatorname{supp} \psi_\pm \subset \overline{\mathbb{R}^n_\pm} \setminus B$.

In order to apply Theorem 1.3, it remains to verify Assumption 1.1.

Proposition 5.1 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be an aperture domain with $C^{1,1}$-boundary. Then Assumption 1.1 is valid for Ω and every $1 < q \le \frac{n}{n-1}$ and $\delta \in (0,\pi)$.*

Proof: Obviously, (A1) is true. The first part of (A2) is true for every $1 < q < \infty$ but (1.4) only holds if $1 < q \le \frac{n}{n-1}$, cf. [16, Theorem 2.6]. The condition (A3) can be easily verified by using the cut-off functions $\varphi_\pm$ and the corresponding statement for the half-spaces $\mathbb{R}^n_\pm$. ■

We will analyze the resolvent $(\lambda + A_q)^{-1}$ near the origin by comparing it with the resolvents of the Stokes operator on $\mathbb{R}^n_\pm$. Hence let $R^\pm_\lambda = (\lambda + A_{\mathbb{R}^n_\pm,q})^{-1}$, $1 < q < \infty$, denote the resolvent of the Stokes operator on $\mathbb{R}^n_\pm$. Then

$$|\lambda|\|u\|_q + \|\nabla^2 u\|_q \le C_{q,\delta}\|f\|_q, \qquad \lambda \in \Sigma_\delta, f \in L^q(\mathbb{R}^n_\pm), \tag{5.1}$$

cf. e.g. [15, Theorem 1.3]. Moreover,

Lemma 5.2 *Let $1 < q < n$, $\delta \in (0, \pi)$, and let $B^\pm_M = B_M(0) \cap \mathbb{R}^n_\pm$. Then for every $M > 0$*

$$\|(R^\pm_\lambda f, \nabla R^\pm_\lambda f)\|_{L^q(B^\pm_M)} \leq C_{q,\delta,M}\|f\|_q$$

uniformly in $\lambda \in \Sigma_\delta$ and $f \in L^q(\mathbb{R}^n_\pm)^n$.

Proof: By Poincaré's inequality, $\|R^\pm_\lambda f\|_{L^q(B^\pm_M)} \leq C_{q,M}\|\nabla R^\pm_\lambda f\|_{L^q(B^\pm_M)}$. Hence it remains to estimate $\nabla R^\pm_\lambda$. Then, if $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$,

$$\|\nabla R^\pm_\lambda f\|_{L^q(B^\pm_M)} \leq C_M\|\nabla R^\pm_\lambda f\|_{L^{q^*}(\mathbb{R}^n_\pm)} \leq C_{q,M}\|\nabla^2 R_\lambda f\|_q \leq C_{q,\delta,M}\|f\|_q$$

uniformly in $\lambda \in \Sigma_\delta$ and $f \in L^q(\mathbb{R}^n_\pm)^n$ by (5.1). ■

In order to analyze (1.2) for $A = A_q$ with Γ replaced by $\Gamma'_R := \Gamma \cap B_R(0)$, $R > 0$, we consider

$$u_\lambda := (\lambda + A_q)^{-1} f - \psi^+ R^+_\lambda \varphi^+ f - \psi^- R^-_\lambda \varphi^- f, \qquad f \in L^q_\sigma(\Omega). \tag{5.2}$$

Moreover, let $p_\lambda = q_\lambda - \psi^+ p^+_\lambda - \psi^- p^-_\lambda$, where $\nabla q_\lambda = (I - P_q)\Delta(\lambda + A_q)^{-1} f$ and $\nabla p^\pm_\lambda = (I - P^\pm_q)\Delta R^\pm_\lambda \varphi^\pm f$ is chosen such that $\int_{B^\pm_M} p^\pm_\lambda = 0$ for $M > 0$ so large that $\psi^\pm \equiv 1$ on $\mathbb{R}^n_\pm \setminus B_M(0)$. Then (u_λ, p_λ) solves

$$\begin{aligned}
(\lambda - \Delta)u_\lambda + \nabla p_\lambda = (1 - \varphi^+ - \varphi^-)f + f^+_\lambda + f^-_\lambda =: \tilde{f}_\lambda \qquad & \text{in } \Omega,\\
\operatorname{div} u_\lambda = g^+_\lambda + g^-_\lambda =: \tilde{g}_\lambda \qquad & \text{in } \Omega,\\
u_\lambda|_{\partial\Omega} = 0 \qquad & \text{on } \partial\Omega,
\end{aligned}$$

where $g^\pm_\lambda = -\nabla\psi^\pm \cdot R^\pm_\lambda \varphi^\pm f$ and

$$f^\pm_\lambda = 2(\nabla\psi^\pm) \cdot \nabla R^\pm_\lambda \varphi^\pm f + (\Delta\psi^\pm) R^\pm_\lambda \varphi^\pm f - (\nabla\psi^\pm) p^\pm_\lambda.$$

Moreover, $\phi(u_\lambda) = \phi((\lambda + A_q)^{-1} f) = 0$.

Lemma 5.3 *For every $1 < q < \infty$ there is some $0 < a < 1$ such that $|\lambda|^a\|u_\lambda\|_q \leq C\|\tilde{f}_\lambda\|_q$ uniformly in $\lambda \in \Sigma_\delta$, $|\lambda| \leq 1$.*

Proof: By [16, Theorem 2.1] for every $1 < q < n$

$$|\lambda|\|u_\lambda\|_q + \|\nabla^2 u_\lambda\|_q \leq C_{q,\delta}\left(\|\tilde{f}_\lambda\|_q + \|\nabla\tilde{g}_\lambda\|_q + (1 + |\lambda|)\|\tilde{g}_\lambda\|_{\dot{W}^{-1}_{q,0}}\right) \tag{5.3}$$

uniformly in $\lambda \in \Sigma_\delta$. Moreover,

$$\int_{B^\pm_M} g^\pm_\lambda dx = \int_{B^\pm_M} \operatorname{div}\left((1 - \psi^\pm) R^\pm_\lambda \varphi^\pm f\right) dx = 0.$$

Hence $\|g_\lambda^\pm\|_{\dot{W}^{-1}_{q,0}(\mathbb{R}^n_\pm)} \leq C_{R,q}\|g_\lambda^\pm\|_q$ by Poincaré's inequality. Furthermore, if $n < r < \infty$ and $1 < q < \frac{n}{n-1}$ is defined by $\frac{1}{q} = \frac{1}{r} + \frac{1}{n}$, then by [16, Corollary 2.4]

$$\begin{aligned}\|\nabla u_\lambda\|_r &\leq C_{r,\delta}\left(|\lambda|\|u_\lambda\|_q + \|\tilde{f}_\lambda\|_q + \|\nabla\tilde{g}_\lambda\|_q + \|\tilde{g}_\lambda\|_{\dot{W}^{-1}_{q,0}}\right)\\ &\leq C_{r,\delta}\left(\|\tilde{f}_\lambda\|_r + \|\tilde{g}_\lambda\|_{1,r}\right)\end{aligned}$$

for $\lambda \in \Sigma_\delta$, $|\lambda| \leq 1$, since $\operatorname{supp}\tilde{f}_\lambda, \operatorname{supp}\tilde{g}_\lambda \subset B_M(0)$. Moreover, since $\|\nabla u_\lambda\|_q \leq C_q\|(u_\lambda, \nabla^2 u_\lambda)\|_q$, interpolation of the latter inequality with (5.3) yields that for every $1 < q < \infty$ there is some $0 < a < 1$ such that

$$|\lambda|^a\|\nabla u_\lambda\|_q \leq C_q\left(\|\tilde{f}_\lambda\|_q + \|\tilde{g}_\lambda\|_{1,q}\right), \qquad \lambda \in \Sigma_\delta, |\lambda| \leq 1.$$

This implies that for $n < r < \infty$ and $1 < q < \frac{n}{n-1}$ defined by $\frac{1}{q} = \frac{1}{r} + \frac{1}{n}$ there is some $0 < a < 1$ such that

$$|\lambda|^a\|u_\lambda\|_r \leq C_r|\lambda|^a\|\nabla u_\lambda\|_q \leq C_{r,\delta}\left(\|\tilde{f}_\lambda\|_q + \|\tilde{g}_\lambda\|_{1,q}\right) \leq C_{r,\delta}\left(\|\tilde{f}_\lambda\|_r + \|\tilde{g}_\lambda\|_{1,r}\right)$$

for $\lambda \in \Sigma_\delta$ with $|\lambda| \leq 1$ using Sobolev's inequality, cf. [16, Lemma 3.1]. Interpolating again finishes the proof. ■

Combining Lemma 5.2 and Lemma 5.3 we obtain:

Corollary 5.4 *Let u_λ be defined as in (5.2). Then for every $1 < q < n$ there is some $0 < a < 1$ such that $\|u_\lambda\|_q \leq C|\lambda|^{-a}\|f\|_q$ for all $f \in L^q_\sigma(\Omega)$ uniformly in $\lambda \in \Sigma_\delta$ with $|\lambda| \leq 1$.*

Proof of Theorem 1.4: First of all, $(\lambda + A_q)^{-1}$ exists for every $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ and $(A_q)' = A_{q'}$, cf. [16, Theorem 2.5]. Hence by duality and interpolation it is sufficient to prove the statement for $1 < q < \frac{n}{n-1}$. Moreover, $\mathcal{R}(A_q)$ is dense, cf. Franzke [17, Theorem 6]. Therefore A_q is sectorial and it is sufficient to prove (1.3) for $h \in H(\delta)$, cf. Denk, Hieber, and Prüss [11, Section 2.4]. Moreover, because of Theorem 1.3 it remains to prove

$$\left\|\int_{\Gamma'_R} h(-\lambda)(\lambda + A_q)^{-1}d\lambda\right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq C_{\delta,q}\|h\|_\infty \qquad \text{for all } h \in H(\delta). \tag{5.4}$$

Since $(\lambda + A_q)^{-1}$ is bounded on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$, it is sufficient to consider $R = 1$. Because of (5.2) and Corollary 5.4, we can replace A_q by $A_{q,\mathbb{R}^n_\pm}$. Moreover, using

$$\frac{1}{2\pi i}\int_{\Gamma'_1} h(-\lambda)(\lambda + A_{\mathbb{R}^n_\pm,q})^{-1}d\lambda = h(A_{\mathbb{R}^n_\pm,q}) - \frac{1}{2\pi i}\int_{\Gamma_1} h(-\lambda)(\lambda + A_{\mathbb{R}^n_\pm,q})^{-1}d\lambda$$

with $\Gamma_1 = \Gamma \setminus \overline{B_1(0)}$ the estimate (5.4) is a consequence of the bounded H_∞-calculus for the Stokes operator on $\mathbb{R}^n_\pm$, cf. Desch, Hieber, and Prüss [12], and of Theorem 1.3 for $\Omega = \mathbb{R}^n_\pm$. ■

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