Remarks on Nitsche's functional: The rotationally symmetric case

STEFFEN FRÖHLICH

Abstract

We investigate existence and stability of rotationally symmetric critical immersions of variational problems of higher order which were considered in [15] and [16].

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and two-fold connected domain. We consider two-dimensional immersions

$$X = X(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v)) \in C^{4+\alpha}(\Omega, \mathbb{R}^3) \cap C^0(\overline{\Omega}, \mathbb{R}^3), \quad \alpha \in (0,1), \tag{1.1}$$

with the property

$$W := |X_u \wedge X_v| > 0 \quad \text{in } \Omega \tag{1.2}$$

for the surface area element $W = W(u,v)$. Here, the indices u and v are the partial derivatives w.r.t. to the variables u resp. v, while $\wedge$ means the usual vector product in $\mathbb{R}^3$.

Let $\kappa_i = \kappa_i(u,v)$, $i = 1,2$, denote the principle curvatures of the surface. Then, by

$$H(u,v) := \frac{\kappa_1(u,v) + \kappa_2(u,v)}{2}, \quad K(u,v) := \kappa_1(u,v)\kappa_2(u,v) \tag{1.3}$$

we introduce its mean curvature and Gaussian curvature. We are concerned with rotationally symmetric critical immersions w.r.t. the variational problem

$$\mathcal{E}[X] := \iint\limits_{\Omega} (\alpha + \beta H^2 - \gamma K) W \, dudv \longrightarrow \text{extr!} \tag{1.4}$$

with positive constants $\alpha, \beta, \gamma \in \mathbb{R}$.

For $\alpha \neq 0$, $\beta, \gamma = 0$, the functional $\mathcal{E}[X]$ is proportional to the classical area functional. Critical points of the accessory variational problem are minimal surfaces, in this case the catenoid. In section 2 we investigate the stability of the catenoid solution w.r.t. small perturbations. The methods used here are presented in more general context in [14].

The case $\alpha, \gamma = 0$ and $\beta \neq 0$ leads to Willmore's functional for which we present selected numerical results. Further, we refer the reader to the textbook [27], in particular chapter 7.

Given two coaxial circular boundary curves $\Gamma_1, \Gamma_2 \in \mathbb{R}^3$ with common radius $R > 0$ of distance $d > 0$, we consider rotationally symmetric critical points of (1.4) encouraged by the J.C.C. Nitsche's treatises [15] and [16].

2 The catenoid

The catenary curve

Let $I := [x_\ell, x_r] \subset \mathbb{R}$, $|I| := |x_r - x_\ell| > 0$. A critical point $f \in C^{2+\alpha}(I, \mathbb{R})$, $\alpha \in (0,1)$, of the variational problem

$$\mathcal{A}[f] := 2\pi \int\limits_I f(x)\sqrt{1 + f'(x)^2} \, dx \longrightarrow \text{extr!} \tag{2.1}$$

is a solution of the non-linear Euler-Lagrange differential equation

$$f(x)f''(x) = 1 + f'(x)^2\,, \quad x \in (x_\ell, x_r), \tag{2.2}$$

with boundary conditions $y_\ell = f(x_\ell)$ and $y_r = f(x_r)$. Note that f and f'' has no zeros. With suitable integration constants $c_1, c_2 \in \mathbb{R}$ we get the catenary solution

$$f(x) = c_1 \cosh\left(\frac{x}{c_1} + c_2\right), \quad x \in I. \tag{2.3}$$

Detailed calculations can be found e.g. in [6], section 4, and in [1], chapter IV.

Stability of graphs

(a) Perturbation of the catenary curve

We investigate the stability of the catenary curve f w.r.t. small perturbations. Let $\psi \in C^{2+\alpha}(I,\mathbb{R})$, $\alpha \in (0,1)$, be given such that $f+\psi$ solves also the minimal surface equation (2.2), that is

$$(f+\psi)(f''+\psi'') - 1 - (f'+\psi')^2 = 0 \quad \text{in } I. \tag{2.4}$$

We arrive at the non-linear differential equation

$$\psi''(x) - \frac{2f'(x)}{f(x)}\,\psi'(x) + \frac{f''(x)}{f(x)}\,\psi(x) = \frac{1}{f}\,\psi'(x)^2 - \frac{1}{f}\,\psi(x)\psi''(x). \tag{2.5}$$

Due to $ff'' = 1 + f'^2 > 0$ the linear differential operator on the right hand side does not obey the maximum principle (see Proposition 6.2 of the appendix). Thus, we assume the existence of a positive stability function $\chi \in C^{2+\alpha}(I,\mathbb{R})$ such that

$$\chi''(x) - \frac{2f'(x)}{f(x)}\,\chi'(x) + \frac{f''(x)}{f(x)}\,\chi(x) \le 0 \quad \text{in } I, \quad \chi > 0 \quad \text{in } I. \tag{2.6}$$

With the product trick $\psi = \varphi\chi$ we calculate

$$\varphi''(x) + 2\left(\frac{\chi'}{\chi} - \frac{f'}{f}\right)\varphi'(x) + \left(\frac{\chi''}{\chi} - \frac{2f'}{f}\,\frac{\chi'}{\chi} + \frac{f''}{f}\right)\varphi(x) = \Phi(\varphi;\chi) \tag{2.7}$$

with the non-linear right hand side

$$\Phi(\varphi;\chi) := \frac{\chi}{f}\,\varphi'^2 - \frac{\chi}{f}\,\varphi\varphi'' + \left(\frac{\chi'^2}{f\chi} - \frac{\chi''}{f}\right)\varphi^2\,, \quad \Phi(0;\chi) = 0. \tag{2.8}$$

Now, the left hand side differential operator in (2.7) obeys the maximum principle. Corollary 6.3 ensures the uniqueness of the solution of the below boundary value problem (2.20).

(b) Schauder norms

Definition 2.1. We introduce the norms

$$\begin{aligned} \|u\|_{0,I} &:= \max_{x\in I}|u(x)|, \quad \|u\|_{1,I} := \|u\|_{0,I} + \max_{x\in I}|u'(x)|, \\ \|u\|_{2,I} &:= \|u\|_{1,I} + \max_{x\in I}|u''(x)|, \end{aligned} \tag{2.9}$$

furthermore the Hölder norms

$$\|u\|_{k+\alpha,I} := \|u\|_{k,I} + \max_{\substack{x_1,x_2\in I\\ x_1\neq x_2}} \frac{|u^{(k)}(x_1) - u^{(k)}(x_2)|}{|x_1-x_2|^\alpha}\,, \quad k = 0,1,2, \tag{2.10}$$

as well as the semi norms

$$[u]_{k,I} := \max_{x\in I}|u^{(k)}(x)|, \quad [u]_{k+\alpha,I} := \max_{\substack{x_1,x_2\in I\\ x_1\neq x_2}} \frac{|u^{(k)}(x_1) - u^{(k)}(x_2)|}{|x_1-x_2|^\alpha} \quad \text{for } k = 0,1,2. \tag{2.11}$$

(c) Contraction property of $\Phi(\eta;\chi)$

Lemma 2.1. *It holds*

$$\|\Phi(\eta_1;\chi)-\Phi(\eta_2;\chi)\|_{\alpha,I} \le \mathcal{C}_1(\|\eta_1\|_{2+\alpha,I}+\|\eta_2\|_{2+\alpha,I})\|\eta_1-\eta_2\|_{2+\alpha,I} \tag{2.12}$$

for all $\eta_1,\eta_2 \in C^{2+\alpha}(I,\mathbb{R})$ with the constant

$$\mathcal{C}_1 := \Big\{2\|\chi f^{-1}\|_{\alpha,I}+\|\chi'^2\chi^{-1}f^{-1}-\chi'' f^{-1}\|_{\alpha,I}(1+|I|^{1-\alpha})^2\Big\}(1+|I|^{1-\alpha})^2\,. \tag{2.13}$$

Proof. We calculate

$$\begin{aligned}
\Phi(\eta_1;\chi)-\Phi(\eta_2;\chi) &= \frac{\chi}{f}(\eta_1'^2-\eta_2'^2)-\frac{\chi}{f}(\eta_1\eta_1''-\eta_2\eta_2'')-\left(\frac{\chi'^2}{f\chi}-\frac{\chi''}{f}\right)(\eta_1^2-\eta_2^2)\\
&= \frac{\chi}{f}(\eta_1'+\eta_2')(\eta_1'-\eta_2')-\frac{\chi}{f}\eta_1(\eta_1''-\eta_2'')-\frac{\chi}{f}\eta_2''(\eta_1-\eta_2)\\
&\quad -\left(\frac{\chi'^2}{f\chi}-\frac{\chi''}{f}\right)(\eta_1+\eta_2)(\eta_1-\eta_2).
\end{aligned} \tag{2.14}$$

Now, we estimate as follows:

$$\begin{aligned}
&\|\Phi(\eta_1;\chi)-\Phi(\eta_2;\chi)\|_{\alpha,I}\\
&\quad\le \|\chi f^{-1}\|_{\alpha,I}\|\eta_1'+\eta_2'\|_{\alpha,I}\|\eta_1'-\eta_2'\|_{\alpha,I}\\
&\qquad +\|\chi f^{-1}\|_{\alpha,I}\|\eta_1\|_{\alpha,I}\|\eta_1''-\eta_2''\|_{\alpha,I}+\|\chi f^{-1}\|_{\alpha,I}\|\eta_2''\|_{\alpha,I}\|\eta_1-\eta_2\|_{\alpha,I}\\
&\qquad +\|\chi'^2\chi^{-1}f^{-1}-\chi''f^{-1}\|_{\alpha,I}\|\eta_1+\eta_2\|_{\alpha,I}\|\eta_1-\eta_2\|_{\alpha,I}\\
&\quad\le \|\chi f^{-1}\|_{\alpha,I}(\|\eta_1\|_{1+\alpha,I}+\|\eta_2\|_{1+\alpha,I})\|\eta_1-\eta_2\|_{1+\alpha,I}\\
&\qquad +\|\chi f^{-1}\|_{\alpha,I}\|\eta_1\|_{\alpha,I}\|\eta_1-\eta_2\|_{2+\alpha,I}+\|\chi f^{-1}\|_{\alpha,I}\|\eta_2\|_{2+\alpha,I}\|\eta_1-\eta_2\|_{\alpha,I}\\
&\qquad +\|\chi'^2\chi^{-1}f^{-1}-\chi''f^{-1}\|_{\alpha,I}(\|\eta_1\|_{\alpha,I}+\|\eta_2\|_{\alpha,I})\|\eta_1-\eta_2\|_{\alpha,I}\,.
\end{aligned} \tag{2.15}$$

Note that

$$\begin{aligned}
\|\eta\|_{\alpha,I} &\le \|\eta\|_{0,I}+\max_{\substack{x_1,x_2\in I\\ x_1\neq x_2}}\frac{|\eta(x_1)-\eta(x_2)|}{|x_1-x_2|}|I|^{1-\alpha}\\
&\le \|\eta\|_{0,I}+|I|^{1-\alpha}\max_{\substack{x_1,x_2\in I\\ x_1\neq x_2}}\max_{\xi\in[x_1,x_2]}|\eta'(\xi)|\\
&\le \|\eta\|_{0,I}+|I|^{1-\alpha}\max_{\substack{x_1,x_2\in I\\ x_1\neq x_2}}\|\eta\|_{1,[x_1,x_2]}\\
&\le (1+|I|^{1-\alpha})\|\eta\|_{1+\alpha,I}\,.
\end{aligned} \tag{2.16}$$

In the same way we have

$$\|\eta\|_{1+\alpha,I}\le(1+|I|^{1-\alpha})\|\eta\|_{2+\alpha,I}\,,\quad \|\eta\|_{\alpha,I}\le(1+|I|^{1-\alpha})^2\|\eta\|_{2+\alpha,I}\,. \tag{2.17}$$

We insert these inequalities into (2.15) to get

$$\begin{aligned}
&\|\Phi(\eta_1;\chi)-\Phi(\eta_2;\chi)\|_{\alpha,I}\\
&\quad\le \|\chi f^{-1}\|_{\alpha,I}(1+|I|^{1-\alpha})^2(\|\eta_1\|_{2+\alpha,I}+\|\eta_2\|_{2+\alpha,I})\|\eta_1-\eta_2\|_{2+\alpha,I}\\
&\qquad +\|\chi f^{-1}\|_{\alpha,I}(1+|I|^{1-\alpha})^2\|\eta_1\|_{2+\alpha,I}\|\eta_1-\eta_2\|_{2+\alpha,I}\\
&\qquad +\|\chi f^{-1}\|_{\alpha,I}(1+|I|^{1-\alpha})^2\|\eta_2\|_{2+\alpha,I}\|\eta_1-\eta_2\|_{2+\alpha,I}\\
&\qquad +\|\chi'^2\chi^{-1}f^{-1}-\chi''f^{-1}\|_{\alpha,I}(1+|I|^{1-\alpha})^4(\|\eta_1\|_{2+\alpha,I}+\|\eta_2\|_{2+\alpha,I})\|\eta_1-\eta_2\|_{2+\alpha,I}\,.
\end{aligned} \tag{2.18}$$

This is the statement. ∎

For $\varphi \in C^{2+\alpha}(I,\mathbb{R})$ we define the linear differential operator

$$\mathcal{L}[\varphi] := \varphi''(x) + 2\left(\frac{\chi'}{\chi} - \frac{f'}{f}\right)\varphi'(x) + \left(\frac{\chi''}{\chi} - \frac{2f'\chi'}{f\chi} + \frac{f''}{f}\right)\varphi(x). \tag{2.19}$$

Due to (2.6) it obeys the maximum principle.
Successively we will solve the boundary value problem (let $\Phi(\varphi) := \Phi(\varphi;\chi)$)

$$\begin{aligned} &\mathcal{L}[\varphi] = \Phi(\varphi) \quad \text{in } I, \\ &\varphi(x_\ell) = \varphi_\ell\,,\ \varphi(x_r) = \varphi_r \end{aligned} \tag{2.20}$$

with the non-linear right hand side (2.8) and given boundary data $\varphi(x_\ell) = \varphi_\ell$ and $\varphi(x_r) = \varphi_r$. For this we start with the function $\varphi_0 \equiv 0$ and consider the linear problems

$$\begin{aligned} &\mathcal{L}[\varphi_k] = \Phi(\varphi_{k-1}) \quad \text{in } I, \\ &\varphi(x_\ell) = \varphi_\ell\,,\ \varphi(x_r) = \varphi_r \quad \text{for } k = 1,2,\ldots \end{aligned} \tag{2.21}$$

From $\mathcal{L}[\varphi_1] = \Phi(0) = 0$ and the Schauder estimate

$$\|\varphi_k\|_{2+\alpha,I} \le \mathcal{C}_2\|\Phi(\varphi_{k-1})\|_{\alpha,I} + \mathcal{C}_3\max\{|\varphi(x_\ell)|,|\varphi(x_r)|\} \tag{2.22}$$

(see the global $C^{2+\alpha}$-estimate (6.40) together with the C^0-estimate (6.7) and the a priori constants $\mathcal{C}_2, \mathcal{C}_3 \in (0,+\infty)$ following from it) we conclude

$$\|\varphi_1\|_{2+\alpha,I} \le \mathcal{C}_3\max\{|\varphi(x_\ell)|,|\varphi(x_r)|\} =: \mathcal{C}_3 a, \tag{2.23}$$

where $a := \max\{|\varphi(x_\ell)|,|\varphi(x_r)|\}$. Using (2.12) we get

$$\begin{aligned} \|\varphi_{k+1} - \varphi_k\|_{2+\alpha,I} &\le \mathcal{C}_2\|\Phi(\varphi_k) - \Phi(\varphi_{k-1})\|_{\alpha,I} \\ &\le \mathcal{C}_1\mathcal{C}_2(\|\varphi_k\|_{2+\alpha,I} + \|\varphi_{k-1}\|_{2+\alpha,I})\|\varphi_k - \varphi_{k-1}\|_{2+\alpha,I}\,. \end{aligned} \tag{2.24}$$

For given $\varepsilon \in (0,1)$ we choose a sufficiently small such that

$$2a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\left(1 + \frac{a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3}{1-\varepsilon}\right) \le \varepsilon. \tag{2.25}$$

Then, by induction we prove

$$\|\varphi_n\|_{2+\alpha,I} \le a\,\mathcal{C}_3\left(1 + \frac{a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3}{1-\varepsilon}\right) \quad \text{for } n = 1,2,\ldots \tag{2.26}$$

First, it holds $\|\varphi_1\|_{2+\alpha,I} \le a\mathcal{C}_3$. Furthermore, (2.24) gives

$$\begin{aligned} \|\varphi_2\|_{2+\alpha,I} &\le \|\varphi_2 - \varphi_1\|_{2+\alpha,I} + \|\varphi_1\|_{2+\alpha,I} \le \mathcal{C}_1\mathcal{C}_2\|\varphi_1\|^2_{2+\alpha,I} + \|\varphi_1\|_{2+\alpha,I} \\ &\le a\,\mathcal{C}_3(1 + a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3). \end{aligned} \tag{2.27}$$

Thus, the statement is true for $n = 1$ und $n = 2$. Let it be proved for $n = 1,2,\ldots,m$. For $k \le m$ we calculate (see (2.24))

$$\|\varphi_{k+1} - \varphi_k\|_{2+\alpha,I} \le 2a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\left(1 + \frac{a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3}{1-\varepsilon}\right)\|\varphi_k - \varphi_{k-1}\|_{2+\alpha,I} \le \varepsilon\|\varphi_k - \varphi_{k-1}\|_{2+\alpha,I}\,. \tag{2.28}$$

It follows that (see (2.27))

$$\begin{aligned} \|\varphi_{m+1}\|_{2+\alpha,I} &\le \|\varphi_1\|_{2+\alpha,I} + \sum_{k=1}^{m}\|\varphi_{k+1} - \varphi_k\|_{2+\alpha,I} \\ &\le \|\varphi_1\|_{2+\alpha,I} + (\varepsilon^{m-1} + \varepsilon^{m-2} + \ldots + \varepsilon + 1)\|\varphi_2 - \varphi_1\|_{2+\alpha,I} \\ &\le a\,\mathcal{C}_3 + \frac{1}{1-\varepsilon}\,a^2\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3^2 = a\,\mathcal{C}_3\left(1 + \frac{a\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3}{1-\varepsilon}\right). \end{aligned} \tag{2.29}$$

This proves (2.26) and we arrive at

$$\|\varphi_{k+1}-\varphi_k\|_{2+\alpha,I} \le \varepsilon\|\varphi_k-\varphi_{k-1}\|_{2+\alpha,I} \quad \text{for all } k=1,2,\ldots \tag{2.30}$$

for $\varepsilon\in(0,1)$. By Banach's fix point theorem the sequence $\{\varphi_k\}_{k=0,1,\ldots}$ converges uniformly in $C^{2+\alpha}$ to a function $\varphi\in C^{2+\alpha}(I,\mathbb{R})$ which solves the boundary value problem (2.20). Via $\psi=\varphi\chi$ we have found also a solution of (2.5) with sufficiently small boundary data.

Theorem 2.1. *Let $f\in C^{2+\alpha}(I,\mathbb{R})$ be a solution of the minimal surface equation (2.2) and let it be stable in the sense of (2.6). For sufficiently small $\varepsilon>0$ there exists $\psi\in C^{2+\alpha}(I,\mathbb{R})$ with boundary values $|\psi(x_\ell)|\le\varepsilon$ and $|\psi(x_r)|\le\varepsilon$ such that $f+\psi$ solves also (2.2).*

3 Variational problems of higher order

The first variation

Now we address the first variation of the functional $\mathcal{E}[X]$ from (1.4). We introduce conformal parameters $(u,v)\in\Omega$ with the properties

$$|X_u|^2=W=|X_v|^2\,,\quad X_u\cdot X_v^t=0 \quad \text{in } \Omega. \tag{3.1}$$

We consider the perturbation

$$\widetilde{X}(u,v):=X(u,v)+\varepsilon\{\varphi(u,v)X_u(u,v)+\psi(u,v)X_v(u,v)+\chi(u,v)N(u,v)\},\quad (u,v)\in\overline{\Omega}, \tag{3.2}$$

where $\varepsilon\in(-\varepsilon_0,+\varepsilon_0)$. Let

$$Z(u,v):=\varphi(u,v)X_u(u,v)+\psi(u,v)X_v(u,v)+\chi(u,v)N(u,v). \tag{3.3}$$

From [16] and [22] we have

Lemma 3.1. *For the first variation w.r.t. the vector field $Z=Z(u,v)$ there hold*

$$\begin{aligned}
\delta\iint\limits_{\Omega} W\,dudv &= -2\iint\limits_{\Omega} H(Z\cdot N^t)W\,dudv-\int\limits_{\partial\Omega}[Z,N,X']\,ds,\\
\delta\iint\limits_{\Omega} KW\,dudv &= -\int\limits_{\partial\Omega}K[Z,N,X']\,ds+\int\limits_{\partial\Omega}\left\{\Lambda^{(1)}\frac{\partial(Z\cdot N^t)}{\partial\nu}-\Lambda^{(2)}(Z\cdot N^t)\right\}ds,\\
\delta\iint\limits_{\Omega} H^2W\,dudv &= \iint\limits_{\Omega}\{\triangle H+2H(H^2-K)W\}(Z\cdot N^t)\,dudv\\
&\quad -\int\limits_{\partial\Omega}H^2[Z,N,X']\,ds+\int\limits_{\partial\Omega}\left\{H\frac{\partial(Z\cdot N^t)}{\partial\nu}-(Z\cdot N^t)\frac{\partial H}{\partial\nu}\right\}ds
\end{aligned} \tag{3.4}$$

with the settings

$$\begin{aligned}
\Lambda^{(1)} &= \kappa_n\,,\\
\Lambda^{(2)} &= \frac{\partial\tau}{\partial s}+\frac{\partial}{\partial s}\left(\frac{\kappa_n}{\kappa^2}\frac{\partial\kappa_g}{\partial s}\right)-\frac{\partial}{\partial s}\left(\frac{\kappa_g}{\kappa^2}\frac{\partial\kappa_n}{\partial s}\right).
\end{aligned} \tag{3.5}$$

Here, κ and τ are the curvature and the torsion of the boundary curves, while κ_n and κ_g mean their normal curvature and geodesic curvature, resp., w.r.t. the surface $X=X(u,v)$.

The Euler-Lagrange differential equation

$$\beta\{\triangle H+2H(H^2-K)W\}-2\alpha HW=0 \tag{3.6}$$

has to be coupled with the natural boundary conditions

$$0 = \int\limits_{\partial\Omega} \{\gamma K - \beta H^2 - \alpha\}[Z, N, X']\, ds + \int\limits_{\partial\Omega} \left\{\gamma \Lambda^{(2)} - \beta \frac{\partial H}{\partial \nu}\right\} (Z \cdot N^t)\, ds + \int\limits_{\partial\Omega} \{\beta H - \gamma \Lambda^{(1)}\} \frac{\partial (Z \cdot N^t)}{\partial \nu}\, ds. \tag{3.7}$$

Furthermore, we add the mean curvature differential system

$$\triangle X = 2H(X) X_u \wedge X_v\,. \tag{3.8}$$

In the case of fixed boundary conditions there hold $\chi|_{\partial\Omega} = 0$ and $(Z \wedge X')|_{\partial\Omega} = 0$, that is

$$\beta H - \gamma \Lambda^{(1)} = 0 \tag{3.9}$$

(for a detailed discussion we refer to [15] and [14].)

Differential equations for graph solutions

Let $\underline{\triangle}$ denote the Laplace-Beltrami operator w.r.t. the metric $ds^2 = |X_u|^2\, du^2 + 2X_u \cdot X_v^t\, dudv + |X_v|^2\, dv^2$. If $\beta \neq 0$ we can rewrite (3.6) and (3.8) to get

$$\underline{\triangle} H + 2H(H^2 - K) - \frac{2\alpha}{\beta} H = 0, \quad W \underline{\triangle} X = 2H(X_u \wedge X_v). \tag{3.10}$$

Now, assume the surface represents a graph $(x, y, z(x, y))$. We calculate

$$\begin{aligned} W \underline{\triangle}\, z &= \frac{\partial}{\partial x}\left(\frac{1 + z_y^2}{W}\, z_x\right) - \frac{\partial}{\partial x}\left(\frac{z_x z_y}{W}\, z_y\right) - \frac{\partial}{\partial y}\left(\frac{z_x z_y}{W}\, z_x\right) + \frac{\partial}{\partial y}\left(\frac{1 + z_x^2}{W}\, z_y\right) \\ &= \frac{1}{W^3}\left\{(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy}\right\}. \end{aligned} \tag{3.11}$$

Thus, it holds

$$(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy} = 2H (1 + z_x^2 + z_y^2)^{\frac{3}{2}}\,. \tag{3.12}$$

Analogously we have

$$\begin{aligned} W \underline{\triangle} H = \; & \frac{2 z_y z_{xy}}{W}\, H_x - \frac{(1 + z_y^2)(z_x z_{xx} + z_y z_{xy})}{W^3}\, H_x + \frac{1 + z_y^2}{W}\, H_{xx} \\ & - \frac{z_y z_{xx} + z_x z_{xy}}{W}\, H_y + \frac{z_x z_y (z_x z_{xx} + z_y z_{xy})}{W^3}\, H_y - \frac{z_x z_y}{W}\, H_{xy} \\ & - \frac{z_y z_{xy} + z_x z_{yy}}{W}\, H_x + \frac{z_x z_y (z_x z_{xy} + z_y z_{yy})}{W^3}\, H_x - \frac{z_x z_y}{W}\, H_{xy} \\ & + \frac{2 z_x z_{xy}}{W}\, H_y - \frac{(1 + z_x^2)(z_x z_{xy} + z_y z_{yy})}{W^3}\, H_y + \frac{1 + z_x^2}{W}\, H_{yy} \end{aligned} \tag{3.13}$$

for the mean curvature.

The rotationally symmetric case

We consider rotationally symmetric surfaces: The meridian function $f(x) := z(x, 0)$, where $f(x) > 0$ for all $x \in [x_\ell, x_r]$, rotates about the x-axis. We have

$$\begin{aligned} & z_x(x, 0) = f'(x), \quad z_y(x, 0) = 0, \\ & z_{xx}(x, 0) = f''(x), \quad z_{xy}(x, 0) = 0, \quad z_{yy}(x, 0) = -\frac{1}{f(x)} \end{aligned} \tag{3.14}$$

as well as

$$\begin{aligned} &H_x(x,0) = H'(x), \quad H_y(x,0) = 0, \\ &H_{xx}(x,0) = H''(x), \quad H_{xy}(x,0) = 0, \quad H_{yy}(x,0) = 0. \end{aligned} \tag{3.15}$$

The mean curvature equation reads as

$$f'' = 2H(1+f'^2)^{\frac{3}{2}} + \frac{1+f'^2}{f} \tag{3.16}$$

while from (3.13) we conclude

$$\underline{\triangle} H = \frac{f'(x)}{1+f'(x)^2}\left(\frac{1}{f(x)} - \frac{f''(x)}{1+f'(x)^2}\right) H'(x) + \frac{1}{1+f'(x)^2} H''(x). \tag{3.17}$$

Investing

$$K(x) = -\frac{f''(x)}{f(x)\{1+f'(x)^2\}} \tag{3.18}$$

along the meridian curve, from (3.10) and (3.17) we get

$$H'' + f'\left(\frac{1}{f} - \frac{f''}{1+f'^2}\right) H' + 2H^3(1+f'^2) + 2\frac{f''}{f} H - \frac{2\alpha}{\beta}(1+f'^2)H = 0. \tag{3.19}$$

4 Successive approximation

Perturbation of the mean curvature equation

We start with

$$H = \frac{f''}{2(1+f'^2)^{\frac{3}{2}}} - \frac{1}{2f\sqrt{1+f'^2}}\,. \tag{4.1}$$

Let $f \in C^{2+\alpha}(I,\mathbb{R})$ solve the minimal surface equation. For a perturbation $\psi \in C^{2+\alpha}(I,\mathbb{R})$ we consider the mean curvature

$$\widetilde{H} = \frac{f''+\psi''}{2(1+(f'+\psi')^2)^{\frac{3}{2}}} - \frac{1}{2(f+\psi)\sqrt{1+(f'+\psi')^2}} \tag{4.2}$$

along the meridian curve of the rotationally symmetric surface $f+\psi$. Using

$$\begin{aligned} \frac{1}{f+\psi} &= \frac{1}{f} - \frac{\psi}{f^2} + \ldots, \\ \frac{1}{\sqrt{1+f'^2+2f'\psi'+\psi'^2}} &= \frac{1}{\sqrt{1+f'^2}} - \frac{f'\psi'}{(1+f'^2)^{\frac{3}{2}}} + \ldots, \\ \frac{1}{(1+f'^2+2f'\psi'+\psi'^2)^{\frac{3}{2}}} &= \frac{1}{(1+f'^2)^{\frac{3}{2}}} - \frac{3f'\psi'}{(1+f'^2)^{\frac{5}{2}}} + \ldots \end{aligned} \tag{4.3}$$

we calculate

$$\begin{aligned} \widetilde{H} &= \frac{1}{2}\left[\frac{1}{(1+f'^2)^{\frac{3}{2}}} - \frac{3f'\psi'}{(1+f'^2)^{\frac{5}{2}}}\right](f''+\psi'') \\ &\quad -\frac{1}{2}\left[\frac{1}{f} - \frac{\psi}{f^2}\right]\left[\frac{1}{\sqrt{1+f'^2}} - \frac{f'\psi'}{(1+f'^2)^{\frac{3}{2}}}\right] + \ldots \\ &= \frac{1}{2\sqrt{1+f'^2}}\left[\frac{f''}{1+f'^2} - \frac{1}{f}\right] + \frac{1}{2(1+f'^2)^{\frac{3}{2}}}\psi'' + \frac{f'}{2(1+f'^2)^{\frac{3}{2}}}\left[\frac{1}{f} - \frac{3f''}{1+f'^2}\right]\psi' \\ &\quad + \frac{1}{2f^2\sqrt{1+f'^2}}\psi + \ldots \end{aligned} \tag{4.4}$$

for $\|\psi\|_{1,I} \le \varepsilon$ sufficiently small.

Because f solves the minimal surface equation (2.2), we get

$$\widetilde{H} = \frac{1}{2(1+f'^2)^{\frac{3}{2}}}\,\psi'' - \frac{f'}{f(1+f'^2)^{\frac{3}{2}}}\,\psi' + \frac{f''}{2f(1+f'^2)^{\frac{3}{2}}}\,\psi + \ldots \tag{4.5}$$

Lemma 4.1. *Let the perturbation $\psi \in C^{2+\alpha}(I,\mathbb{R})$ be sufficiently small w.r.t. the C^1-norm. Let $f \in C^{2+\alpha}(I,\mathbb{R})$ solve the minimal surface equation. Then it holds*

$$\psi'' - \frac{2f'}{f}\,\psi' + \frac{f''}{f}\,\psi = 2(1+f'^2)^{\frac{3}{2}}\,\widetilde{H} + \Psi_1(\psi) \tag{4.6}$$

with the mean curvature $\widetilde{H}$ along $f+\psi$ and the non-linear term $\Psi_1(\psi)$ which collects all super-linear terms for ψ and its first and second derivatives. Moreover, with a stability function χ from (2.6), the product $\varphi = \psi\chi^{-1}$ satisfies

$$\varphi'' + 2\left(\frac{\chi'}{\chi} - \frac{f'}{f}\right)\varphi' + \left(\frac{\chi''}{\chi} - \frac{2f'\chi'}{f\chi} + \frac{f''}{f}\right)\varphi = \Phi_1(\varphi,\widetilde{H}) \tag{4.7}$$

with the non-linear right hand side

$$\Phi_1(\varphi,\widetilde{H}) := \frac{2(1+f'^2)^{\frac{3}{2}}}{\chi}\,\widetilde{H} + \chi^{-1}\Psi_1(\varphi\chi). \tag{4.8}$$

Let us denote the linear differential operator on the left hand side of (4.7) with $\mathcal{L}_1$. Then we consider the boundary value problem

$$\begin{aligned} &\mathcal{L}_1[\varphi] = \Phi_1(\varphi,\widetilde{H}) \quad \text{in } I,\\ &\varphi(x_\ell) = 0,\ \varphi(x_r) = 0. \end{aligned} \tag{4.9}$$

We supplement the boundary value problem (4.13) for $\widetilde{H}$.

Perturbation of the Euler-Lagrange equation

Let the variation $f+\varphi$, where $\|\varphi\|_{1,I}$ is sufficiently small and f solves the minimal surface equation, satisfy (3.19), that is

$$\begin{aligned} 0 \;=\; & \widetilde{H}'' + (f'+\varphi')\left(\frac{1}{f+\varphi} - \frac{f''+\varphi''}{1+f'^2+2f'\varphi'+\varphi'^2}\right)\widetilde{H}' + 2(1+f'^2+2f'\varphi'+\varphi'^2)\widetilde{H}^3\\ & + 2\,\frac{f''+\varphi''}{f+\varphi}\,\widetilde{H} - \frac{2\alpha}{\beta}\,(1+f'^2+2f'\varphi'+\varphi'^2)\widetilde{H}\,. \end{aligned} \tag{4.10}$$

Lemma 4.2. *For the mean curvature along $f+\varphi$ it holds*

$$\widetilde{H}'' + \frac{2f''}{f}\,\widetilde{H} - \frac{2\alpha}{\beta}\,(1+f'^2)\widetilde{H} = \Phi_2(\varphi,\widetilde{H}) \tag{4.11}$$

with the non-linear right hand side

$$\begin{aligned} \Phi_2(\varphi,\widetilde{H}) \;=\; & -2(1+f'^2)\widetilde{H}^3 + (f'+\varphi')\left\{\frac{\varphi}{f^2} - \frac{2f'f''\varphi'}{(1+f'^2)^2} + \frac{\varphi''}{1+f'^2} - \frac{2f'\varphi'\varphi''}{(1+f'^2)^2}\right\}\widetilde{H}'\\ & - 2(f'+\varphi')\varphi'\widetilde{H}^3 + \left\{\frac{2f''\varphi}{f^2} - \frac{2\varphi''}{f} + \frac{2\varphi\varphi''}{f^2} + \frac{2\alpha}{\beta}\,(2f'\varphi'+\varphi'^2)\right\}\widetilde{H} + \ldots, \end{aligned} \tag{4.12}$$

where $\ldots$ means terms of higher order of ψ and its derivatives.

Let $\mathcal{L}_2$ denote the linear differential operator on the left hand side of (4.11). We consider the boundary value problem

$$\begin{aligned} \mathcal{L}_2[\widetilde{H}] &= \Phi_2(\varphi, \widetilde{H}) \quad \text{in } I, \\ \widetilde{H}(x_\ell) &= 0, \ \widetilde{H}(x_r) = 0. \end{aligned} \tag{4.13}$$

The homogeneous boundary conditions appear naturally in the case of rotationally symmetric Willmore surfaces (see [22], section 6.3).

Note that $\mathcal{L}_2$ obeys the maximum principle if

$$\frac{f''}{f} - \frac{\alpha}{\beta}(1+f'^2) < 0 \quad \text{resp..} \quad \frac{1}{f^2} < \frac{\alpha}{\beta}\,. \tag{4.14}$$

Let us work on this assumption.

We remark that for sufficiently small perturbations $\|\psi\|_{2+\alpha,I} < \varepsilon$ die residual sums in (4.3) and (4.12) converge. For $\varepsilon < 1$ and $\|\widetilde{H}\|_{2+\alpha,I} < 1$ we find constants $\mathcal{C}_4, \mathcal{C}_5 \in (0, +\infty)$ such that

$$\begin{aligned} \|\Phi_1(\varphi, \widetilde{H})\|_{\alpha,I} &\leq \mathcal{C}_4(\|\widetilde{H}\|_{2+\alpha,I} + \|\varphi\|^2_{2+\alpha,I}), \\ \|\Phi_2(\varphi, \widetilde{H})\|_{\alpha,I} &\leq \mathcal{C}_4(\|\widetilde{H}\|^3_{2+\alpha,I} + \|\varphi\|_{2+\alpha,I}\|\widetilde{H}\|_{2+\alpha,I}) \end{aligned} \tag{4.15}$$

as well as

$$\begin{aligned} &\|\Phi_1(\varphi_1, \widetilde{H}_1) - \Phi_1(\varphi_2, \widetilde{H}_2)\|_{\alpha,I} \\ &\qquad \leq \mathcal{C}_5\{\|\widetilde{H}_1 - \widetilde{H}_2\|_{2+\alpha,I} + (\|\varphi_1\|_{2+\alpha,I} + \|\varphi_2\|_{2+\alpha,I})\|\varphi_1 - \varphi_2\|_{2+\alpha,I}\}, \\ &\|\Phi_2(\varphi_1, \widetilde{H}_1) - \Phi_2(\varphi_2, \widetilde{H}_2)\|_{\alpha,I} \\ &\qquad \leq \mathcal{C}_5(\|\varphi_1\|_{2+\alpha,I} + \|\varphi_2\|_{2+\alpha,I} + \|\widetilde{H}_1\|_{2+\alpha,I} + \|\widetilde{H}_2\|_{2+\alpha,I}) \cdot \ldots \\ &\qquad\qquad \ldots \cdot (\|\varphi_1 - \varphi_2\|_{2+\alpha,I} + \|\widetilde{H}_1 - \widetilde{H}_2\|_{2+\alpha,I}). \end{aligned} \tag{4.16}$$

We will solve (4.9) and (4.13) successively: Let f be stable, and let (4.14) be fulfilled. We start with a pair $(\varphi_0, \widetilde{H}_0)$. First, with the right hand side $\Phi_2(\varphi_0, \widetilde{H}_0)$ we get an unique solution $\widetilde{H}_1$ from (4.13). Now, we solve (4.9) with $\Phi_1(\varphi_0, \widetilde{H}_1)$ to get an unique solution φ_1. In this way we proceed with $(\varphi_1, \widetilde{H}_1)$.

Let $\|\varphi_0\|_{2+\alpha,I}, \|\widetilde{H}_0\|_{2+\alpha,I} < \varepsilon$ with the above $\varepsilon \in (0,1)$. The Schauder estimates

$$\|\widetilde{H}_1\|_{2+\alpha,I} \leq \mathcal{C}_6\|\Phi_2(\varphi_0, \widetilde{H}_0)\|_{\alpha,I}\,, \quad \|\varphi_1\|_{2+\alpha,I} \leq \mathcal{C}_6\|\Phi_1(\varphi_0, \widetilde{H}_1)\|_{\alpha,I} \tag{4.17}$$

with a constant $\mathcal{C}_6 \in (0, +\infty)$ due to (6.40), (4.7) and (4.11) yield (see (4.15))

$$\begin{aligned} &\|\widetilde{H}_1\|_{2+\alpha,I} \leq \mathcal{C}_4\mathcal{C}_6(\varepsilon^3 + \varepsilon^2) \leq 2\mathcal{C}_4\mathcal{C}_6\varepsilon^2\,, \\ &\|\varphi_1\|_{2+\alpha,I} \leq \mathcal{C}_4\mathcal{C}_6(\|\widetilde{H}_1\|_{2+\alpha,I} + \varepsilon^2) \leq \mathcal{C}_4\mathcal{C}_6(1 + 2\mathcal{C}_4\mathcal{C}_6)\varepsilon^2\,. \end{aligned} \tag{4.18}$$

We choose

$$\varepsilon \leq \min\left\{1, \frac{1}{2\mathcal{C}_4\mathcal{C}_6}\,, \frac{1}{\mathcal{C}_4\mathcal{C}_6(1 + \mathcal{C}_4\mathcal{C}_6)}\right\}. \tag{4.19}$$

Then there follow $\|\widetilde{H}_1\|_{2+\alpha,I} \leq \varepsilon$ and $\|\varphi_1\|_{2+\alpha,I} \leq \varepsilon$. We continue this procedure to get

$$\|\widetilde{H}_k\|_{2+\alpha,I} \leq \varepsilon, \quad \|\varphi_k\|_{2+\alpha,I} \leq \varepsilon \quad \text{for } k = 0, 1, 2, \ldots \tag{4.20}$$

From (4.16) we deduce

$$\|\widetilde{H}_{k+1} - \widetilde{H}_k\|_{2+\alpha,I} \leq 4\varepsilon\,\mathcal{C}_5\mathcal{C}_6(\|\widetilde{H}_k - \widetilde{H}_{k-1}\|_{2+\alpha,I} + \|\varphi_k - \varphi_{k-1}\|_{2+\alpha,I}) \tag{4.21}$$

as well as (for $\mathcal{C}_5$ and $\mathcal{C}_6$ be sufficiently large)

$$\begin{aligned} \|\varphi_{k+1} - \varphi_k\|_{2+\alpha,I} &\leq \mathcal{C}_5\mathcal{C}_6\|\widetilde{H}_{k+1} - \widetilde{H}_k\|_{2+\alpha,I} + 2\varepsilon\,\mathcal{C}_5\mathcal{C}_6\|\varphi_k - \varphi_{k-1}\|_{2+\alpha,I} \\ &\leq 4\varepsilon\,\mathcal{C}_5^2\mathcal{C}_6^2\|\widetilde{H}_k - \widetilde{H}_{k-1}\|_{2+\alpha,I} + 6\varepsilon\,\mathcal{C}_5^2\mathcal{C}_6^2\|\varphi_k - \varphi_{k-1}\|_{2+\alpha,I}\,. \end{aligned} \tag{4.22}$$

From the last inequalities we conclude

$$\begin{aligned}\|\varphi_{k+1}-\varphi_k\|_{2+\alpha,I}+\|\widetilde{H}_{k+1}-\widetilde{H}_k\|_{2+\alpha,I}\\ \leq 10\varepsilon\,\mathcal{C}_5^2\mathcal{C}_6^2(\|\varphi_k-\varphi_{k-1}\|_{2+\alpha,I}+\|\widetilde{H}_k-\widetilde{H}_{k-1}\|_{2+\alpha,I})\end{aligned} \tag{4.23}$$

for $k=1,2,\dots$ Additionally we assume the smallness property

$$\varepsilon<\frac{1}{20\,\mathcal{C}_5^2\mathcal{C}_6^2} \tag{4.24}$$

such that the sequence $(\varphi_k,\widetilde{H}_k)_{k=1,2,\dots}$ is contractive. Then we find solutions $\varphi\in C^{2+\alpha}(I,\mathbb{R})$ and $\widetilde{H}\in C^{2+\alpha}(I,\mathbb{R})$, $\|\varphi\|_{2+\alpha,I}<\varepsilon$ and $\|\widetilde{H}\|_{2+\alpha,I}<\varepsilon$, of the non-linear coupled boundary value problems (4.9) and (4.13). As a meridian curve of the rotationally symmetric immersion with mean curvature $H\in C^{2+\alpha}(I,\mathbb{R})$, we have $f\in C^{4+\alpha}(I,\mathbb{R})$ due to well-known regularity results.

Theorem 4.1. *Let $f\in C^{4+\alpha}(I,\mathbb{R})$ be a stable solution of the minimal surface equation (2.2) for given boundary values f_ℓ and f_r. Let f satisfy (4.14). Then we find a solution pair (f,H) of (3.16) and (3.19) as a sufficiently small perturbation in the $C^{4+\alpha}$-norm of the minimal surface.*

5 Numerical results

The catenoid

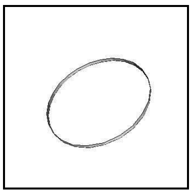
Two coaxial circular boundary curves of common radius $r>0$ span a minimal surface if the distance $h>0$ between the rings is sufficiently small.

We investigate the dependence between the ratio h/r and the surface area A numerically. For this, we increase the distance h of the rings starting from $h=0.1$ by adding successively $\triangle h=0.1$ while keeping the radius $r=1.5088795$ fixed (all calculations were done with Ken Brakke's Surface Evolver [2]).

In the case $h/r=1.3256$ there is no two-fold connected minimal surface. The given surface area is the total area of the two discs of radius r.

h/r	0.0663	0.1325	0.1988	0.2651	0.3314	0.3976	0.4639	0.5302	0.5965	0.6627
A	0.9053	1.8881	2.8292	3.7667	4.7001	5.6285	6.5506	7.4656	8.3723	9.2694

h/r	0.7290	0.7953	0.8616	0.9278	0.9941	1.0604	1.1267	1.1929	1.2592	1.3256
A	10.1558	11.0302	11.8810	12.7189	13.5347	14.3377	15.0976	15.8227	16.5026	14.3250



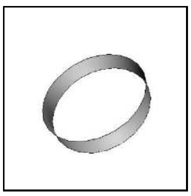
$h/r=0.0663$

$h/r=0.1325$

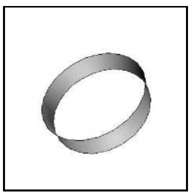
$h/r=0.1988$

$h/r=0.2651$

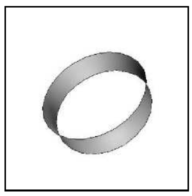
$h/r=0.3314$



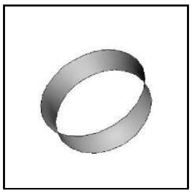
$h/r=0.3976$

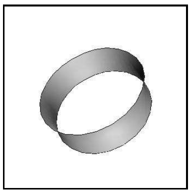


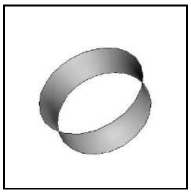
$h/r=0.4639$

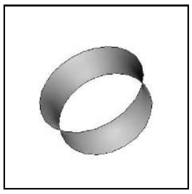


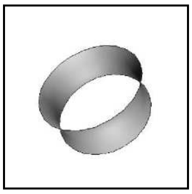
$h/r=0.5302$

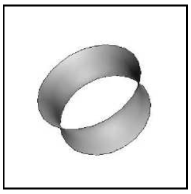

$^{h}/_{r} = 0.5965$

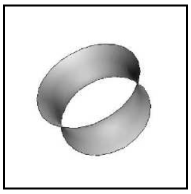

$^{h}/_{r} = 0.6627$

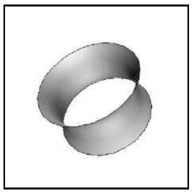

$^{h}/_{r} = 0.7290$

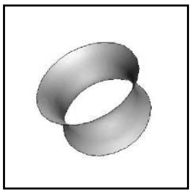

$^{h}/_{r} = 0.7953$

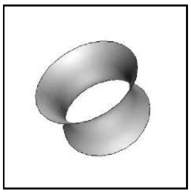

$^{h}/_{r} = 0.8616$

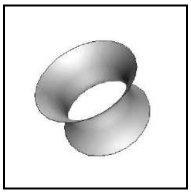

$^{h}/_{r} = 0.9278$

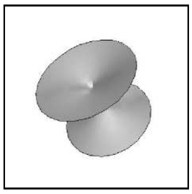

$^{h}/_{r} = 0.9941$


$^{h}/_{r} = 1.0604$


$^{h}/_{r} = 1.1267$


$^{h}/_{r} = 1.1929$


$^{h}/_{r} = 1.2592$


$^{h}/_{r} = 1.3256$

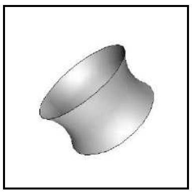
The Willmore catenoid

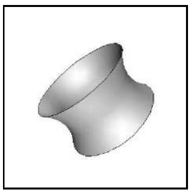
We investigate numerically critical points of Willmore's functional (1.4) with $\alpha = 0$, $\gamma = 0$, $\beta = 1$. The following tables compare the surface area A with the Willmore energy E for different distances h ($r = 1$).

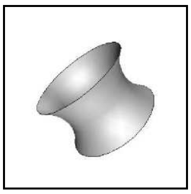
h	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
A	5.98	6.50	6.98	7.42	7.77	8.13	8.49	8.87	9.27	9.68	10.11
E	0.00	0.00	0.00	0.00	0.02	0.10	0.22	0.38	0.56	0.77	0.98

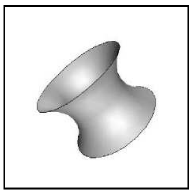
h	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
A	10.56	11.02	11.51	12.01	12.54	13.10	13.66	14.25	14.86
E	1.20	1.43	1.67	1.90	2.13	2.36	2.58	2.80	3.02

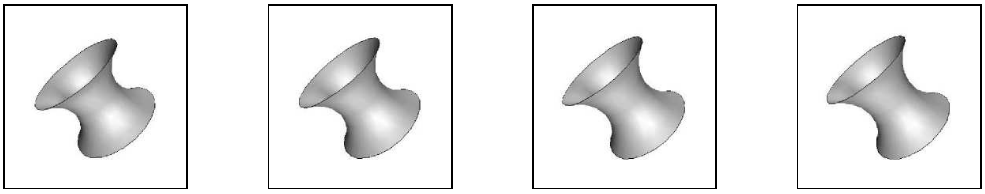
h	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
A	15.50	23.21	34.15	48.08	65.37	85.79	111.07	145.22
E	3.23	5.02	6.62	7.56	8.25	8.72	9.26	9.71


$h = 10$

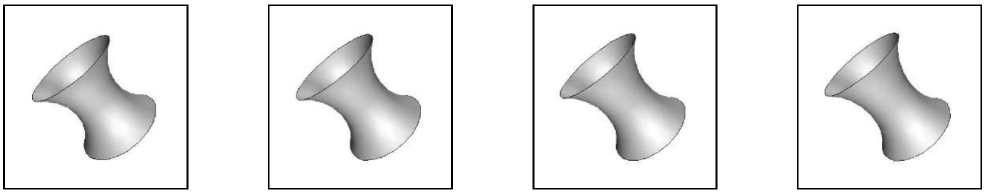

$h = 11$


$h = 12$

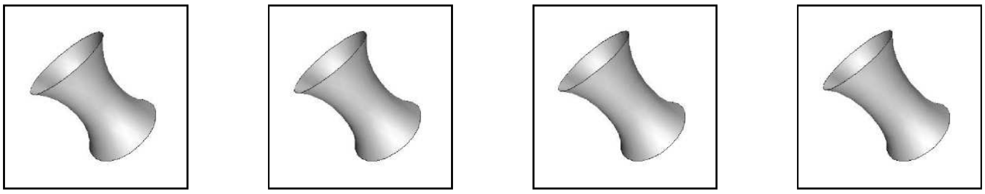

$h = 13$



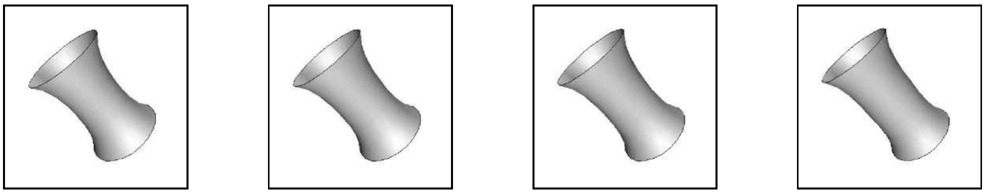
$h = 14$ $h = 15$ $h = 16$ $h = 17$



$h = 18$ $h = 19$ $h = 20$ $h = 21$

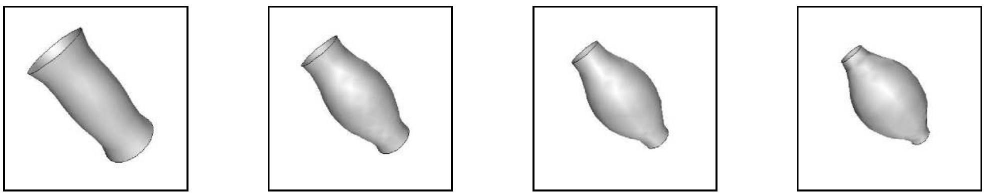


$h = 22$ $h = 23$ $h = 24$ $h = 25$



$h = 26$ $h = 27$ $h = 28$ $h = 29$

It is conjectured that such catenoid-type critical points of Willmore's functional exist for all distances $h > 0$. Furthermore, for h sufficiently large the critical immersions seem to tend to the sphere S^2 - except from a small neighborhood of the boundary curves where the mean curvature H vanishes. Therefore, we expect $\mathcal{W} \to 4\pi \approx 12.56$ for $r \to \infty$ due to the fact that 4π is exactly the Willmore energy for the round sphere.



$h = 40$ $h = 60$ $h = 80$ $h = 100$

6 Appendix: Linear ordinary differential equations

The maximum principle

Let $u \in C^{2+\alpha}(I, \mathbb{R})$, $\alpha \in (0,1)$, $I = [a,b]$, solve the boundary value problem

$$\widetilde{\mathcal{L}}[u](x) := u''(x) + \widetilde{p}(x)u'(x) + \widetilde{q}(x)\widetilde{u}(x) = \widetilde{f}(x) \quad \text{in } I, \quad u(a) = \eta_1\,,\ u(b) = \eta_2\,, \tag{6.1a}$$

where $\widetilde{p}, \widetilde{q} \in C^{\alpha}(I, \mathbb{R})$ and $f \in C^{\alpha}(I, \mathbb{R})$. Multiplication with the positive function

$$p(x) := \exp \int \widetilde{p}(x)\, dx \tag{6.2}$$

gives the Sturm-Liouville boundary value problem

$$\mathcal{L}[u] := (pu')' + qu = f \quad \text{in } I, \quad u(a) = \eta_1\,,\ u(b) = \eta_2\,, \tag{6.1b}$$

where $q := p\widetilde{q}$ and $f := p\widetilde{f}$. Note that $p \in C^1(I, \mathbb{R})$.

In [23], §26, Satz III we find the fundamental existence theorem:

Proposition 6.1. *(Existence and uniqueness)*
The inhomogeneous problem (6.1b) has a unique solution $u \in C^{2+\alpha}(I, \mathbb{R})$ if and only if the homogeneous problem

$$\mathcal{L}[u] = 0, \quad u(a) = 0, \; u(b) = 0, \tag{6.3}$$

has only the trivial solution $u \equiv 0$.

The Hölder regularity for u comes from the assumption $f \in C^\alpha(I, \mathbb{R})$. Furthermore, the uniqueness has its origin in the following maximum principle which can be found in [23], Ergänzung I for §26.

Proposition 6.2. *(Maximum principle)*
Let $u \in C^2(I, \mathbb{R})$ solve (6.1b) with $q \le 0$ (which is equivalent to $\widetilde{q} \le 0$).

(i) If $\mathcal{L}[u] \ge 0$ and u has an inner positive maximum, then $u \equiv$ const.

(ii) If $\mathcal{L}[u] \le 0$ and u has an inner negative minimum, then $u \equiv$ const.

Corollary 6.1. *(Uniqueness)*
Let $u, v \in C^2(I, \mathbb{R})$ solve

$$\begin{aligned} \mathcal{L}[u] = f, \quad u(a) = \eta_1\,, \; u(b) = \eta_2\,, \\ \mathcal{L}[v] = f, \quad v(a) = \eta_1\,, \; v(b) = \eta_2\,. \end{aligned} \tag{6.4}$$

Assume that $q \le 0$. Then it holds $u \equiv v$ in I.

Proof. Consider $w := u - v$ with $\mathcal{L}[w] = 0$ and $w(a) = 0$, $w(b) = 0$. The maximum principle yields $w \equiv 0$. □

For $q \le 0$ the homogeneous problem has only the trivial solution, that is (6.1b) is uniquely solvable.

Alternatively, if the homogeneous problem has non-trivial solutions, the inhomogeneous problem may have no solutions as the following example shows ([23], §26):

$$u''(x) + u(x) = 1, \quad u(0) = 0, \; u(\pi) = 1. \tag{6.5}$$

A priori estimates

Let $p_0, p_1 \in \mathbb{R}$ and $p_1' \in \mathbb{R}$ be constants such that

$$0 < p_0 \le p(x) \le p_1 < +\infty \quad \text{and} \quad |p'(x)| \le p_1' < +\infty \quad \text{for all } x \in I. \tag{6.6}$$

The maximum principle from Proposition 6.2 gives the following bounds on the C^0-norm.

Corollary 6.2. *(Estimate of the C^0-norm)*
Let $u \in C^2(I, \mathbb{R})$ solve (6.1b) with $q \le 0$. Then

$$\|u\|_{0,I} \le 3 \max_{x \in \partial I} |u(x)| + \frac{\|f\|_{0,I}}{\nu e^{\nu a}}\, e^{\nu(b-a)}\,, \quad \nu := \frac{1 + p_1'}{p_0}\,. \tag{6.7}$$

Proof. 1. Consider the positive functions

$$v(x) := \max_{x \in \partial I} |u(x)| + \frac{\|f\|_{0,I}}{\mu e^{\mu a}} \left(e^{b\mu} - e^{\mu x}\right), \quad x \in I, \tag{6.8}$$

where $\mu p_0 - p_1' = 1$ (note that $\mu > 0$), such that $p' + p\mu \ge 1$. It follows that

$$[p(e^{\mu x})']' = \mu[pe^{\mu x}]' = \mu[p' + p\mu]e^{\mu x} \ge \mu e^{\mu x} \ge \mu e^{\mu a}\,. \tag{6.9}$$

Because $qv \le 0$ we conclude

$$\mathcal{L}[v] = (pv')' + qv \le -\frac{\|f\|_{0,I}}{\mu e^{\mu a}}\, \mu e^{\mu a} + qv \le -\|f\|_{0,I}\,. \tag{6.10}$$

Thus, $\mathcal{L}[v-u] \le 0$. Due to the maximum principle, $v-u$ has no local negative minimum if it is not constant. Because $(v-u)|_{\partial I} \ge 0$ we have $v \ge u$ if $v - u \not\equiv$ const.

2. Let $\widetilde{v} = -v$, then $\mathcal{L}[\widetilde{v}] \geq \|f\|_{0,I}$. We have $\mathcal{L}[\widetilde{v} - u] \geq 0$. Due to the maximum principle, $\widetilde{v} - u$ has no local positive maximum if it is not constant. Because $(\widetilde{v} - u)|_{\partial I} \leq 0$ we have $\widetilde{v} \leq u$ if $\widetilde{v} - u \not\equiv \text{const}$.

3. If $v - u \not\equiv \text{const}$ and $\widetilde{v} - u \not\equiv \text{const}$ we have

$$-\max_{x\in\partial I}|u(x)| - \frac{\|f\|_{0,I}}{\mu e^{\mu a}}\left(e^{b\mu} - e^{\mu x}\right) \leq u(x) \leq \max_{x\in\partial I}|u(x)| + \frac{\|f\|_{0,I}}{\mu e^{\mu a}}\left(e^{b\mu} - e^{\mu x}\right) \quad \text{in } I. \tag{6.11}$$

The statement follows for this case.

4. Let $v - u \equiv C$, $C \in \mathbb{R}$. With $C = v(b) - u(b) = v(b) - \eta_2$ we get

$$|C| \leq v(b) + |\eta_2| \leq 2\max_{x\in\partial I}|u(x)|, \tag{6.12}$$

and the statement follows from $|u| \leq |v| + |C|$. Analogously, for $\widetilde{v} - u \equiv \widetilde{C}$ we get $|\widetilde{C}| \leq 2\max_{x\in\partial I}|u(x)|$. This makes the proof complete.

□

In the following calculations we can replace $\|u\|_{0,I}$ by the above estimate if $q \leq 0$.

Lemma 6.1. *(Estimate of the C^1-norm)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.1b). Then, for all real $0 < \mu < \frac{1}{2}$ it holds

$$\|u\|_{1,I} \leq \frac{b-a}{2p_0}\|f\|_{0,I} + \left(1 + \frac{2p_1}{\mu p_0(b-a)} + \frac{q_1(b-a)}{2p_0}\right)\|u\|_{0,I} + \frac{\mu p_1(b-a)}{2p_0}\|u\|_{2+\alpha,I}\,. \tag{6.13}$$

Proof. 1. First, we estimate $|u'(c)|$ with $c := \frac{a+b}{2}$. Let $d := \mu\frac{b-a}{2}$ with $\mu < \frac{1}{2}$ sufficiently small. Set $x_1 := c - d$ and $x_2 := c + d$. By the mean value theorem there exists $\widetilde{x} \in [x_1, x_2]$ such that

$$|u'(\widetilde{x})| = \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|} = \frac{|u(x_1) - u(x_2)|}{2d} \leq \frac{1}{d}\|u\|_{0,I}\,. \tag{6.14}$$

Because $|c - \widetilde{x}| \leq d$ we get

$$\begin{aligned} |u'(c)| &= |u'(\widetilde{x}) + \int_{\widetilde{x}}^{c} u''(\xi)\,d\xi\,| \;\leq\; \frac{1}{d}\|u\|_{0,I} + d\sup_{y\in[x_1,x_2]}|u''(y)| \\ &\leq \frac{2}{\mu(b-a)}\|u\|_{0,I} + \frac{\mu(b-a)}{2}[u]_{2,I}\,. \end{aligned} \tag{6.15}$$

2. Integrating $(pu')' = f - qu$ from $\xi = c$ to $\xi = x$ yields

$$p(x)u'(x) = \int_c^x f(\xi)\,d\xi - \int_c^x q(\xi)u(\xi)\,d\xi + p(c)u'(c), \tag{6.16}$$

that is,

$$[u]_{1,I} \leq \frac{b-a}{2p_0}\left(\|f\|_{0,I} + q_1\|u\|_{0,I}\right) + \frac{2p_1}{\mu p_0(b-a)}\|u\|_{0,I} + \frac{\mu p_1(b-a)}{2p_0}\|u\|_{2+\alpha,I}\,. \tag{6.17}$$

The statement follows.

□

Lemma 6.2. *(Estimate of the C^α-norm)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.1b). Then

$$\|u\|_{\alpha,I} \leq \|u\|_{0,I} + (b-a)^{1-\alpha}\|u\|_{1,I} \tag{6.18}$$

with the above bounds on $\|u\|_{0,I}$ and $\|u\|_{1,I}$.

Proof. For arbitrary $x_1, x_2 \in [a,b]$, $x_1 \neq x_2$, we have

$$|u(x_1) - u(x_2)| = |u'(\xi)||x_1 - x_2| \tag{6.19}$$

with $\xi \in [x_1, x_2]$. We conclude

$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} = |u'(\xi)||x_1 - x_2|^{1-\alpha} \leq (b-a)^{1-\alpha}\|u\|_{1,I}\,, \tag{6.20}$$

therefore

$$\|u\|_{\alpha,I} \leq \|u\|_{0,I} + (b-a)^{1-\alpha}\|u\|_{1,I}\,. \tag{6.21}$$

□

Lemma 6.3. *(Estimate of the $C^{1+\alpha}$-norm)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.1b). Then, for all real $0 < \mu < \frac{1}{2}$ it holds

$$\|u\|_{1+\alpha,I} \leq \|u\|_{1,I} + c_1\|u\|_{0,I} + c_2\|f\|_{0,I} + \mu c_3\|u\|_{2+\alpha,I} \tag{6.22}$$

with the above bounds on $\|u\|_{1,I}$ and the constants

$$\begin{aligned} c_1 &:= (b-a)^{1-\alpha}\left(\frac{p_1'}{p_0} + \frac{2p_1p_1'}{\mu p_0^2(b-a)} + \frac{p_1'q_1(b-a)}{2p_0^2} + \frac{q_1}{p_0}\right), \\ c_2 &:= (b-a)^{1-\alpha}\left(\frac{p_1'(b-a)}{2p_0^2} + \frac{1}{p_0}\right), \quad c_3 := \frac{p_1p_1'(b-a)^{2-\alpha}}{2p_0^2}\,. \end{aligned} \tag{6.23}$$

Proof. Let $x_1, x_2 \in I$. We find $\xi \in [x_1, x_2]$ with the property

$$\frac{|u'(x_1) - u'(x_2)|}{|x_1 - x_2|^\alpha} = |u''(\xi)||x_1 - x_2|^{1-\alpha} \leq (b-a)^{1-\alpha}[u]_{2,I}\,. \tag{6.24}$$

From the equation

$$u'' + \frac{p'}{p}\,u' + \frac{q}{p}\,u = \frac{1}{p}\,f \tag{6.25}$$

we conclude

$$[u]_{2,I} \leq \frac{p_1'}{p_0}\,\|u\|_{1,I} + \frac{q_1}{p_0}\,\|u\|_{0,I} + \frac{1}{p_0}\,\|f\|_{0,I}\,. \tag{6.26}$$

Using the above C^1-estimate we get

$$[u]_{2,I} \leq c_1\|u\|_{0,I} + c_2\|f\|_{0,I} + c_3\|u\|_{2+\alpha,I} \tag{6.27}$$

with the given constants c_1, c_2, c_3. □

Let us define

$$c_4 := c_2 + \frac{b-a}{2p_0}\,, \quad c_5 := 1 + c_1 + \frac{2p_1}{\mu p_0(b-a)} + \frac{q_1(b-a)}{2p_0}\,, \quad c_6 := c_3 + \frac{p_1(b-a)}{2p_0}\,. \tag{6.28}$$

From (6.22) it follows that

$$\|u\|_{1+\alpha,I} \leq c_4\|f\|_{0,I} + c_5\|u\|_{0,I} + \mu c_6\|u\|_{2+\alpha,I}\,. \tag{6.29}$$

To prove global estimates of the $C^{2+\alpha}$-norm we note that

$$u''(x) = \frac{f(x)}{p(x)} - \frac{p'(x)}{p(x)}\,u'(x) - \frac{q(x)}{p(x)}\,u(x) \equiv F(x), \tag{6.30}$$

that is, we have to investigate the Poisson-type equation $u'' = F$.

Lemma 6.4. *(Estimate of the C^0-norm for Poisson's equation)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.30). Then

$$\|u\|_{0,I} \leq 3\max_{x\in\partial I}|u(x)| + \|F\|_{0,I}e^{b-2a}\,. \tag{6.31}$$

This follows from the C^0-estimate (6.7) with $p \equiv 1$, $q \equiv 0$.

Lemma 6.5. *(Estimate of the C^1-norm for Poisson's equation)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.30). For all real $0 < \mu < \frac{1}{2}$ it holds

$$\|u\|_{1,I} \le \frac{(1+\mu)(b-a)}{2}\,\|F\|_{0,I} + \frac{2+\mu(b-a)}{\mu(b-a)}\,\|u\|_{0,I}\,. \tag{6.32}$$

Proof. As in (6.15) we have

$$|u'(c)| \le \frac{2}{\mu(b-a)}\,\|u\|_{0,I} + \frac{\mu(b-a)}{2}\,[u]_{2,I} = \frac{2}{\mu(b-a)}\,\|u\|_{0,I} + \frac{\mu(b-a)}{2}\,\|F\|_{0,I} \tag{6.33}$$

because $[u]_{2,I} = \|F\|_{0,I}$. Integrating $u''(x) = F(x)$ from $\xi = c$ to $\xi = x$ gives

$$[u]_{1,I} \le \frac{b-a}{2}\,\|F\|_{0,I} + \frac{2}{\mu(b-a)}\,\|u\|_{0,I} + \frac{\mu(b-a)}{2}\,\|F\|_{0,I}\,. \tag{6.34}$$

This proves the statement. □

Furthermore, $[u'']_{2,I} = \|F\|_{0,I}$ yields immediately

Lemma 6.6. *(Estimate of the C^2-norm for Poisson's equation)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.30). For all real $0 < \mu < \frac{1}{2}$ it holds

$$\|u\|_{2,I} \le \frac{2+(1+\mu)(b-a)}{2}\,\|F\|_{0,I} + \frac{2+\mu(b-a)}{\mu(b-a)}\,\|u\|_{0,I}\,. \tag{6.35}$$

As in (6.18) and (6.22) we prove

Lemma 6.7. *(Estimate of the C^α-norm for Poisson's equation)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.30). Then it holds

$$\|u\|_{\alpha,I} \le \|u\|_{0,I} + (b-a)^{1-\alpha}[u]_{1,I} \tag{6.36}$$

with the above estimates of $\|u\|_{0,I}$ and $[u]_{1,I}$.

Lemma 6.8. *(Estimate of the $C^{1+\alpha}$-norm for Poisson's equation)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.30). Then it holds

$$\|u\|_{1+\alpha,I} \le \|u\|_{1,I} + (b-a)^{1-\alpha}[u]_{2,I} \tag{6.37}$$

with the above bounds on $\|u\|_{1,I}$ and $[u]_{2,I}$.

Lemma 6.9. *(Estimate of the $C^{2+\alpha}$-norm for Poisson's equation)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.30). Then it holds

$$\|u\|_{2+\alpha,I} = \|u\|_{2,I} + [u]_{2+\alpha,I} = \|u\|_{2,I} + \|F\|_{\alpha,I} \tag{6.38}$$

with the above bound on $\|u\|_{2,I}$.

Our main result is a global estimate of the $C^{2+\alpha}$-norm for regular solutions of (6.1b).

Proposition 6.3. *(Global estimate of the $C^{2+\alpha}$-norm)*
Let $u \in C^{2+\alpha}(I,\mathbb{R})$ solve (6.1b). Let

$$\widehat{p}_1 := \|p\|_{\alpha,I}\,, \quad \widehat{q}_1 := \|q\|_{\alpha,I}\,, \quad \widehat{p}_1{}' := \|p\|_{1+\alpha,I}\,. \tag{6.39}$$

Then there exist real constants $C_1, C_2 = C_1, C_2(p_0, \widehat{p}_1, \widehat{q}_1, \widehat{p}_1{}', b-a, \alpha) \in (0,+\infty)$ such that

$$\|u\|_{2+\alpha,I} \le C_1\|f\|_{\alpha,I} + C_2\|u\|_{0,I}\,. \tag{6.40}$$

Proof. Note that

$$u''(x) = -\frac{p'(x)}{p(x)}\,u'(x) - \frac{q(x)}{p(x)}\,u(x) + \frac{f(x)}{p(x)} \equiv F(x). \tag{6.41}$$

From (6.35) it follows that

$$\begin{aligned} \|u\|_{2+\alpha,I} &= \|u\|_{2,I} + [u]_{2+\alpha,I} \le \|u\|_{2,I} + \|F\|_{\alpha,I} \\ &\le \frac{2+(1+\mu)(b-a)}{2}\,\|F\|_{0,I} + \|F\|_{\alpha,I} + \frac{2+\mu(b-a)}{\mu(b-a)}\,\|u\|_{0,I}\,. \end{aligned} \tag{6.42}$$

1. We estimate $\|F\|_{0,I}$. From (6.29) we infer

$$\begin{aligned} \|F\|_{0,I} &\leq \frac{1}{p_0}\,\|f\|_{0,I} + \frac{p_1'}{p_0}\,\|u\|_{1,I} + \frac{q_1}{p_0}\,\|u\|_{0,I} \\ &\leq \frac{1}{p_0}\,(1+p_1'c_4)\|f\|_{0,I} + \frac{1}{p_0}\,(p_1'c_5+q_1)\|u\|_{0,I} + \mu\frac{p_1'}{p_0}\,c_6\|u\|_{2+\alpha,I}\,. \end{aligned} \tag{6.43}$$

We set

$$c_7 := \frac{1}{p_0}\,(1+p_1'c_4), \quad c_8 := \frac{1}{p_0}\,(p_1'c_5+q_1), \quad c_9 := \frac{p_1'}{p_0}\,c_6\,, \tag{6.44}$$

such that

$$\|F\|_{0,I} \leq c_7\|f\|_{0,I} + c_8\|u\|_{0,I} + \mu c_9\|u\|_{2+\alpha,I}\,. \tag{6.45}$$

2. Analogously we estimate

$$\|F\|_{\alpha,I} \leq \frac{\|f\|_{\alpha,I}}{p_0} + \frac{\widehat{p_1}'}{p_0}\,\|u\|_{1+\alpha,I} + \frac{\widehat{q}_1}{p_0}\,\|u\|_{\alpha,I}\,. \tag{6.46}$$

We define

$$\begin{aligned} c_{10} &:= \frac{1}{p_0}\big(1+\widehat{p}_1{}'c_4+\widehat{q}_1(b-a)^{1-\alpha}c_4\big), \\ c_{11} &:= \frac{1}{p_0}\big(\widehat{p}_1{}'c_5+\widehat{q}_1+\widehat{q}_1(b-a)^{1-\alpha}c_5\big), \\ c_{12} &:= \frac{1}{p_0}\big(\widehat{p}_1{}'c_6+\widehat{q}_1(b-a)^{1-\alpha}c_6\big). \end{aligned} \tag{6.47}$$

Together with (6.18) and (6.22) we have

$$\|F\|_{\alpha,I} \leq c_{10}\|f\|_{\alpha,I} + c_{11}\|u\|_{0,I} + \mu c_{12}\|u\|_{2+\alpha,I}\,. \tag{6.48}$$

Now, we set

$$\begin{aligned} c_{13} &:= \frac{2+(1+\mu)(b-a)}{2}\,c_7 + c_{10}\,, \\ c_{14} &:= \frac{2+(1+\mu)(b-a)}{2}\,c_8 + \frac{2+\mu(b-a)}{\mu(b-a)} + c_{11}\,, \\ c_{15} &:= \frac{2+(1+\mu)(b-a)}{2}\,c_9 + c_{12}\,. \end{aligned} \tag{6.49}$$

Summarizing (6.42), (6.45) and (6.48) we arrive at the estimate

$$\|u\|_{2+\alpha,I} \leq c_{13}\|f\|_{\alpha,I} + c_{14}\|u\|_{0,I} + \mu c_{15}\|u\|_{2+\alpha,I}\,. \tag{6.50}$$

Finally, we choose $0<\mu<\frac{1}{2}$ such that $1-\mu c_{15}>0$. It follows that

$$\|u\|_{2+\alpha,I} \leq \frac{c_{13}}{1-\mu c_{15}}\,\|f\|_{\alpha,I} + \frac{c_{14}}{1-\mu c_{15}}\,\|u\|_{0,I}\,. \tag{6.51}$$

Setting $C_1 := c_{13}(1-\mu c_{15})^{-1}$ and $C_2 := c_{14}(1-\mu c_{15})^{-1}$ proves the statement. □

The presented a priori estimates are not sharp. In the foregoing application we needed only a qualitative form of these constants.

Literatur

[1] Bliss, G.A.: *Variationsrechnung.* B.G. Teubner, 1932.

[2] Brakke, K.A.: *The surface evolver.* Experimental Mathematics **1,** 141–165, 1992.

[3] Bryant, R.; Griffiths, P.: *Reduction for constrained variational problems and $\int \kappa^2/2\,ds$.* Amer. J. Math. **108,** 525–570, 1986.

[4] Courant, R.: *Dirichlet's principle, conformal mappings, and minimal surfaces.* Interscience publ., Inc., 1950.

[5] Eells, J.: *The surfaces of Delaunay.* Math. Int. **9,** 53–57, 1987.

[6] Fomin, S.V.; Gelfand, I.M.: *Calculus of variations.* Prentice Hall, Inc., 1994.

[7] Gilbarg, D.; Trudinger, N.S.: *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer, 2001.

[8] Helfrich, W.: *Elastic properties of lipid bilayers: Theory and possible experiments.* Z. Naturforsch. **28c,** 693–703, 1973.

[9] Hildebrandt, S.; Tromba, A.: *The parsimonious universe.* Springer-Verlag, 1996.

[10] Hille, E.: *Lectures on ordinary differential equations.* Addison-Wesley Publishing Company, Inc., 1969.

[11] Kühnel, W.: *Differentialgeometrie.* vieweg studium. vieweg, 1999.

[12] Langer, J.; Singer, D.A.: *Curves in the hyperbolic plane and mean curvature of tori in 3-space.* Bull. London Math. Soc. **16,** 531–534, 1984.

[13] Langer, J.; Singer, D.A.: *The total squared curvature of closed curves.* J. Differential Geometry **20,** 1–22, 1984.

[14] Nitsche, J.C.C.: *Vorlesungen über Minimalflächen.* Grundlehren der mathematischen Wissenschaften **199,** Springer, 1973.

[15] Nitsche, J.C.C.: *Periodical surfaces that are extremal for energy functionals containing curvature functions.* In: Statistical thermodynamics and differential geometry of microstructured materials, H.T. Davis, J.C.C. Nitsche (Ed.), IMA Volumes in Mathematics and its Applications **51,** Springer, 69–98, 1993.

[16] Nitsche, J.C.C.: *Boundary value problems for variational integrals involving surface curvatures.* Quart. Appl. Math. **LI,** No. 2, 353–387, 1993.

[17] Oprea, J.: *Differential geometry and its applications.* Prentice Hall, Inc., 1997.

[18] Sauvigny, F.: *Introduction of isothermal parameters into a Riemannian metric by the continuity method.* Analysis **19,** No.3, 235–242, 1999.

[19] Schmidt, M.U.: *A proof of the Willmore conjecture.* Preprint, SFB 288 "Differentialgeometrie und Quantenmechanik", FU Berlin, 2002.

[20] Simon, L.: *Existence of surfaces minimizing the Willmore functional.* Commun. Anal. Geom. **1,** No.2, 281–326, 1993.

[21] Thompson, D'A.W.: *On growth and form.* Cambridge University Press, 1961.

[22] von der Mosel, H.: *Geometrische Variationsprobleme höherer Ordnung.* Bonner math. Schriften **293,** 1996.

[23] Walter, W.: *Gewöhnliche Differentialgleichungen.* Springer, 2000.

[24] Wente, H.C.: *Constant mean curvature surfaces of annular type.* Preprint, 2000.

[25] Wente, H.C.: *Explicit solutions to the H-surface equation on tori.* Michigan Math. J. **49,** 501–517, 2001.

[26] White, J.H.: *A global invariant of conformal mappings in space.* Proc. Amer. Math. Soc. **38,** 162–164, 1973.

[27] Willmore, T.J.: *Riemannian geometry.* Oxford Science Publications, Clarendon Press, 1993.

Steffen Fröhlich
Technische Universität Darmstadt
Fachbereich Mathematik, AG 4
Differentialgeometrie und Geometrische Datenverarbeitung
Schloßgartenstraße 7
D-64289 Darmstadt
Germany

e-mail: sfroehlich@mathematik.tu-darmstadt.de