Knowledge acquisition under incomplete knowledge using methods from formal concept analysis Part II

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Abstract

Attribute exploration is an interactive computer algorithm which helps the expert to get informations about the attribute implications of a formal context. In the part I of this paper (see [H04]) an algorithm for attribute exploration with incomplete knowledge was presented. In this part we prove the main results of the algorithm: At the end of the attribute exploration the expert gets maximal information with respect to his knowledge about the unknown universe: He gets a list of implications which are certainly valid, a list of implications which are possibly valid, a list of counterexamples against the implications which are certainly not valid and a list of fictitious counterexamples against the implications which he answered by "unknown". He only has to check the implications which he answered by "unknown" and if he can decide for each of these implications whether it is valid or not, he gets complete knowledge about the implications of the context. For the definitions see part I.

1 Attribute exploration

Lemma 1 *Let* $j > 0$, $C \subseteq M, d \in M$ *and* $G^U \cap G_j \subseteq S \subseteq G_j$ *with* $C \to d \in Sat(\mathbb{K}_j|_S) \setminus Sat(\mathbb{K}_j)$. *Let* $\alpha = \bigwedge C \to d \vee \bigvee\{\bigwedge A \mid C \subseteq A \subseteq M,\ g_{A,d} \in G_j \setminus S\}$. *Then* $\alpha \in Th(Resp^{\mathcal{R}}(Sat(\mathbb{K}_j)))$.

Proof. Let $E \in Resp^{\mathcal{R}}(Sat(\mathbb{K}_j))$. Now we show that E is a model of α. Assume $C \subseteq E$. We have $E \to d \in Sat(\mathbb{K}_j|_S)$ because of rule (AU). If $d \in E$ then E is a model of α, now let $d \notin E$. We have $E \to d \notin Sat(\mathbb{K}_j)$ because of $E \in Resp^{\mathcal{R}}(Sat(\mathbb{K}_j))$. There exists $i > 0$ with $E \to d \in Sat(\mathbb{K}_i) \setminus Sat(\mathbb{K}_{i+1})$. In step i a fictitious counterexample $g_{A,b} \in G_{i+1} \setminus G_i$ against the implication $E \to d$ is added to the context, so $E \subseteq A$ and $b = d$. In $\mathbb{K}_i$ we have $E^{\Box\Diamond} \neq E \in Resp^{\mathcal{R}}(Sat(\mathbb{K}_j)) \subseteq Resp^{\mathcal{R}}(P_i)$. With Theorem 28 of part I the set A is minimal with $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_i)$, so we get $E = A$ and $g_{A,b} = g_{E,d} \in G_j \setminus S$. Therefore E is a model of α. ■

Corollary 2 *Let $j > 0$, $G^U \cap G_j \subseteq S \subseteq G_j$ and $Q = \{A \to d \mid A \subseteq M,\ d \in M$ with $g_{A,d} \in G_j \setminus S\}$. Then $Sat(\mathbb{K}_j|_S) \subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q)$.*

Proof. Let $C \to D \in Sat(\mathbb{K}_j|_S)$ and $d \in D$. If $C \to d \in Sat(\mathbb{K}_j)$, then we get $C \to d \in \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q)$. Now assume $C \to d \notin Sat(\mathbb{K}_j)$. We prove $C \to d \in Imp(Resp^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q))$: Let $E \in Resp^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q) \subseteq Resp^{\mathcal{R}}(Sat(\mathbb{K}_j))$ with $C \subseteq E$. With Lemma 1 we get $d \in E$ or $A \subseteq E$ for a set $A \subseteq M$ with $C \subseteq A$ and $g_{A,d} \in G_j \setminus S$. Because of $E \in Resp(Q)$ we get $d \in E$ in both cases. So we have $C \to d \in Imp(Resp^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q)) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q)$ and $C \to D \in \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q)$ with rule (AD), so we get $Sat(\mathbb{K}_j|_S) \subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_j) \cup Q)$. ∎

Lemma 3 *Let $j > 0$, $G^U \cap G_j \subseteq S \subseteq G_j$, $A \to B \in P_j$ and $b \in M \setminus B$ with $A \to b \in Sat(\mathbb{K}_j|_S)$. Then $g_{A,b} \in G_j \setminus S$.*

Proof. Let $i < j$ with $A \to B \in P_{i+1} \setminus P_i$. We have $A \to b \in Sat(\mathbb{K}_j|_S)$, so we get $A \to b \in Sat(\mathbb{K}_i)$, because with Corollary 29 of part I there does not exist a fictitious counterexample $g_{C,d}$ in $\mathbb{K}_i$ (with $A \subseteq C$ and $b = d$) against the implication $A \to b$. By Corollary 27 of part I we have $b \notin B = A^{\Box\Diamond}$ in $\mathbb{K}_{i+1}$, so we get $A \to b \notin Sat(\mathbb{K}_{i+1})$. Therefore $g_{A,b} \in G_{i+1} \subseteq G_j$, and we have $g_{A,b} \notin S$ because of $A \to b \in Sat(\mathbb{K}_j|_S)$. ∎

During the exploration it may happen that an implication $A \to b$ which was accepted as unknown is derivable from the implications which were accepted as valid. In this case the implication $A \to b$ must also be valid and the fictitious counterexample $g_{A,b}$ can be removed from the context.[1]

Lemma 4 *Let $j \geq i > 0$, $g_{A,b} \in G_i$ be a fictitious object such that A is a $Sat(\mathbb{K}_i)$-intent. Then $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_j)$.*

Proof. We have $A \in Resp^{\mathcal{R}}(Sat(\mathbb{K}_i)) \subseteq Resp^{\mathcal{R}}(Sat(\mathbb{K}_j)) \subseteq Resp^{\mathcal{R}}(P_j)$, so $A \to b \notin Imp(Resp^{\mathcal{R}}(P_j)) = \mathbf{Cons}^{\mathcal{R}}(P_j)$ because of $b \notin A$. ∎

[1]In Example 2 in the next section it will be shown that it may be better to keep such a fictitious object in the context until the end of the exploration (see also Theorem 9).

Lemma 5 *Let $j > 0$ and $g_{A,b} \in G_j$ be a fictitious object. The following conditions are equivalent:*

1. $Red^{\mathcal{R}}_{P_j}(g_{A,b})$ *exists*
2. $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_j)$
3. $\mathbf{Cons}^{\mathcal{R}}(P_j)$ *is satisfyable for* $g_{A,b}$

Proof.
1 ⇒ 3:
See Corollary 21 of part I.
3 ⇒ 2:
$A \to b$ is not satisfyable for $g_{A,b}$.
2 ⇒ 1:
We have $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_j) = Imp(Resp^{\mathcal{R}}(P_j))$, so there exists $T \in Resp^{\mathcal{R}}(P_j)$ with $A \subseteq T$ and $b \notin T$, so $g^{\Box}_{A,b} \subseteq T \subseteq g^{\Diamond}_{A,b}$, and $Red^{\mathcal{R}}_{P_j}(g_{A,b})$ exists because of Theorem 20 of part I. ■

Lemma 6 *Let n be the step in which the exploration ends. The following conditions are equivalent:*

1. $Red^{\mathcal{R}}_{P_n}(\mathbb{K}_n)$ *exists*
2. $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_n)$ *for all fictitious objects* $g_{A,b} \in G_n$
3. $\mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n)) = Sat(\mathbb{K}_n)$

Proof.
1 ⇒ 3:
If $Red^{\mathcal{R}}_{P_n}(\mathbb{K}_n)$ exists, then $\mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n)) = \mathbf{Cons}^{\mathcal{R}}(P_n) \subseteq Sat(\mathbb{K}_n)$ because of Corollary 31 of part I and Corollary 21 of part I.
3 ⇒ 2:
For $g_{A,b} \in G_n$ we have $A \to b \notin Sat(\mathbb{K}_n) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n)) = \mathbf{Cons}^{\mathcal{R}}(P_n)$ because of Corollary 31 of part I.
2 ⇒ 1:
Because of $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_n)$ for $g_{A,b} \in G_n$ we get the existence of $Red^{\mathcal{R}}_{P_n}(g_{A,b})$ with Lemma 5. The context rows of the normal objects $g \in G_n \cap G^U$ are P_n-reduced. ■

If there exists a fictitious object $g_{A,b} \in G_j$ and a normal object $g \in G_j \cap G^U$ in a step j of the exploration with $I_j(g,a) = \times$ for all $a \in A$ and $I_j(g,b) = o$, then the implication $A \to b$ which was accepted as unknown can not be valid in the universe $\mathbb{K}^U$ because of $g \in G^U$. Every implication $C \to D$ which is not valid for $g_{A,b}$ is not valid for g either, because the context row of $g_{A,b}$ is smaller or equal (in the information order) to the context row of g. The fictitious object $g_{A,b}$ can be removed from the context without changing the satisfyable implications:

Lemma 7 *Let $j > 0$ and $G_j \cap G^U \subseteq T \subseteq G_j$ and $S = T \setminus \{g_{A,b} \in T \mid I_j(g_{A,b}, \cdot) \leq I_j(g, \cdot)$ for some $g \in G_j \cap G^U\}$. Then $Sat(\mathbb{K}_j|_S) = Sat(\mathbb{K}_j|_T)$.*

Proof. We have $Sat(\mathbb{K}_j|_T) \subseteq Sat(\mathbb{K}_j|_S)$ because of $S \subseteq T$. Each implication $C \to D \in Imp_M$ for which a counterexample of $\{g_{A,b} \in T \mid I_j(g_{A,b}, \cdot) \leq I_j(g, \cdot)$ for some $g \in G_j \cap G^U\}$ exists, is not satisfyable for the corresponding object $g \in G_j \cap G^U \subseteq S$ with $I_j(g_{A,b}, \cdot) \leq I_j(g, \cdot)$. So we get $Sat(\mathbb{K}_j|_S) \subseteq Sat(\mathbb{K}_j|_T)$ and therefore $Sat(\mathbb{K}_j|_S) = Sat(\mathbb{K}_j|_T)$. ∎

Definition 8 *Let n be the step in which the exploration ends and $k \leq n$.*

$G_k^\sim := G_k \setminus \{g_{A,b} \in G_k \mid A \to b \in \mathbf{Cons}^{\mathcal{R}}(P_k)\}$

$G^\sim := G_n^\sim$

$G^* := G^\sim \setminus \{g_{A,b} \in G_n \mid I_n(g_{A,b}, \cdot) \leq I_n(g, \cdot)$ *for some* $g \in G_n \cap G^U\}$

$\mathbb{K}_k^\sim := \mathbb{K}_k|_{G_k^\sim} =: (G_k^\sim, M, \{\times, ?, o\}, I_k^\sim)$

$\mathbb{K}^* := \mathbb{K}_n|_{G^*} =: (G^*, M, \{\times, ?, o\}, I^*)$

$P^* := \{A \to B \cup \{d \in M \mid g_{A,d} \in G_n \setminus G^\sim\} \mid A \to B \in P_n\}$

Theorem 9 *Let $k > 0$ and $A \in \mathcal{R}$.*

1. $\mathbf{Cons}^{\mathcal{R}}(P_k) \subseteq Sat(\mathbb{K}_k^\sim)$.
2. *A is a $Sat(\mathbb{K}_k)$-intent iff A is a $Sat(\mathbb{K}_k^\sim)$-intent.*
3. *$A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_k)$ holds in $\mathbb{K}_k$ iff $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_k)$ holds in $\mathbb{K}_k^\sim$.*
4. *If $\mathcal{R}$ is a closure system and A is not a premise of P_k then A is $Sat(\mathbb{K}_k)$-pseudoclosed iff A is $Sat(\mathbb{K}_k^\sim)$-pseudoclosed.*

Proof.
Proof of 1:
For each normal object $g \in G^U \cap G_k$ the set $\mathbf{Cons}^{\mathcal{R}}(P_k)$ is satisfyable for g in $Sat(\mathbb{K}_k^\sim)$ because of $\mathbf{Cons}^{\mathcal{R}}(P_k) \subseteq Imp(\mathbb{K}^U)$. For each fictitious object $g_{C,d} \in G_k^\sim$ the set $\mathbf{Cons}^{\mathcal{R}}(P_k)$ is satisfyable for $g_{C,d}$ because of Lemma 5.
Proof of 2:

First assume that A is a $Sat(\mathbb{K}_k)$-intent. If A is not a $Sat(\mathbb{K}_k^{\sim})$-intent then with Corollary 16 of part I there exists an attribute $m \in M$ with $A \to m \in Sat(\mathbb{K}_k^{\sim})$ and $A \to m \notin Sat(\mathbb{K}_k)$, so there exists $j < k$ with $A \to m \in Sat(\mathbb{K}_j) \setminus Sat(\mathbb{K}_{j+1})$. There exists a fictitious object $g_{C,d} \in G_{j+1} \setminus G_j$ with $A \subseteq C$ and $d = m$. The set A is a $Sat(\mathbb{K}_k)$-intent and we have $C \to d \in \mathbf{Cons}^{\mathcal{R}}(P_k)$ because of $A \to m \in Sat(\mathbb{K}_k^{\sim})$, so with Lemma 4 we get $A \neq C$. With Theorem 28 of part I C is a minimal set with $C^{\Box\Diamond} \neq C \in Resp^{\mathcal{R}}(P_j)$ in $\mathbb{K}_j$, so we get $A \notin Resp^{\mathcal{R}}(P_j)$ which is a contradiction to $A \in Resp^{\mathcal{R}}(Sat(\mathbb{K}_k)) \subseteq Resp^{\mathcal{R}}(P_j)$. Therefore A is a $Sat(\mathbb{K}_k^{\sim})$-intent. The other direction follows from $Sat(\mathbb{K}_k) \subseteq Sat(\mathbb{K}_k^{\sim})$, so the set A is a $Sat(\mathbb{K}_k)$-intent iff A is a $Sat(\mathbb{K}_k^{\sim})$-intent.

Proof of 3:

Condition 3 follows from condition 2 with Corollary 16 of part I.

Proof of 4:

Let $\mathcal{R}$ be a closure system. Assume that condition 4 is not true and let $k > 0$ be minimal such that the assertion does not hold. Let $A \in \mathcal{R}$ be minimal such that A is not a premise of P_k and condition 4 does not hold for this set A. In the following let

$$B^{\Box\Diamond k} := \{m \in M \mid B \to m \in Sat(\mathbb{K}_k)\},$$
$$B^{\Box\Diamond\sim} := \{m \in M \mid B \to m \in Sat(\mathbb{K}_k^{\sim})\}$$

for $B \subseteq M$.

Case 1: A is $Sat(\mathbb{K}_k^{\sim})$-pseudoclosed (with respect to $\mathcal{R}$).

Then A is not $Sat(\mathbb{K}_k)$-pseudoclosed and A is no $Sat(\mathbb{K}_k^{\sim})$-intent, and therefore with $A \in \mathcal{R}$ and condition 2 the set A is no $Sat(\mathbb{K}_k)$-intent. There exists a proper subset

$$B \subset A \tag{1}$$

such that

$$B \text{ ist } Sat(\mathbb{K}_k)\text{-pseudoclosed,} \tag{2}$$
$$B^{\Box\Diamond k} \not\subseteq A. \tag{3}$$

If B is $Sat(\mathbb{K}_k^{\sim})$-pseudoclosed then we get $B^{\Box\Diamond k} \subseteq B^{\Box\Diamond\sim} \subseteq A$ because of the assumption that A is $Sat(\mathbb{K}_k^{\sim})$-pseudoclosed, but this is a contradiction to (3), so

$$B \text{ is not } Sat(\mathbb{K}_k^{\sim})\text{-pseudoclosed.} \tag{4}$$

By the minimality of A, the set B is a premise of P_k, and with Corollary 27 of part I we get

$$B \to B^{\Box\Diamond k} \in P_k. \tag{5}$$

With (2) the set B is not a $Sat(\mathbb{K}_k)$-intent, so with condition 2 it is not a $Sat(\mathbb{K}_k^{\sim})$-intent and with (4) there exists a proper subset

$$D \subset B \tag{6}$$

such that

$$D \text{ is } Sat(\mathbb{K}_k^{\sim})\text{-pseudoclosed}, \tag{7}$$
$$D^{\Box\Diamond\sim} \not\subseteq B. \tag{8}$$

If D is not a premise of P_k then D is also $Sat(\mathbb{K}_k)$-pseudoclosed by the minimality of A because in this case condition 4 is satisfied for D. But with (5) and (6) this is a contradiction to Corollary 29 of part I. Therefore

$$D \to D^{\Box\Diamond k} \in P_k, \tag{9}$$
$$D \text{ is } Sat(\mathbb{K}_k)\text{-pseudoclosed}, \tag{10}$$

and with (6) and (2) we get

$$D^{\Box\Diamond k} \subseteq B. \tag{11}$$

With (8) there exists an attribute

$$z \in M \setminus B \tag{12}$$

with

$$D \to z \in Sat(\mathbb{K}_k^{\sim}), \tag{13}$$

and with (9), (11), (12) and Lemma 3 we get $g_{D,z} \in G_k \setminus G_k^{\sim}$, so

$$D \to z \in \mathbf{Cons}^{\mathcal{R}}(P_k). \tag{14}$$

The system $\mathcal{R} \cap Resp(P_k)$ is a closure system, so with Lemma 2 of part I, Lemma 3 of part I and (5) we get

$$\begin{aligned} < D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)} &\in Resp(\mathbf{Cons}^{\mathcal{R}}(P_k)) & (15)\\ &= Resp(Imp(\mathcal{R} \cap Resp(P_k))) & (16)\\ &= \mathcal{R} \cap Resp(P_k) & (17)\\ &\subseteq Resp^{\mathcal{R}}(P_k \setminus \{B \to B^{\Box\Diamond k}\}). & (18) \end{aligned}$$

We have

$$B \in Resp^{\mathcal{R}}(P_k \setminus \{B \to B^{\Box\Diamond k}\}), \tag{19}$$

because for every $C \to E \in P_k \setminus \{B \to B^{\Box\Diamond k}\}$ with $C \subseteq B$ the set C is a $Sat(\mathbb{K}_k)$-pseudoclosed proper subset of B, so $E = C^{\Box\Diamond k} \subseteq B$ because of (2). We get

$$< D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)} \cap B \in Resp^{\mathcal{R}}(P_k \setminus \{B \to B^{\Box\Diamond k}\}), \tag{20}$$

because $\mathcal{R} \cap Resp(P_k \setminus \{B \to B^{\Box\Diamond k}\})$ is a closure system. If $< D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)} \cap B$ is a proper subset of B then we get

$$\begin{aligned} < D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)} \cap B &\in Resp^{\mathcal{R}}(P_k) &(21)\\ &= Resp^{\mathcal{R}}(Imp(Resp^{\mathcal{R}}(P_k))) &(22)\\ &= Resp^{\mathcal{R}}(\mathbf{Cons}^{\mathcal{R}}(P_k)), &(23)\end{aligned}$$

which is a contradiction to $D \subseteq < D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)} \cap B$ and (12) and (14). Therefore

$$< D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)} \cap B = B \tag{24}$$

and we get

$$\begin{aligned} B &\subseteq < D >_{\mathbf{Cons}^{\mathcal{R}}(P_k)}, &(25)\\ D \to B &\in \mathbf{Cons}^{\mathcal{R}}(P_k) &(26)\end{aligned}$$

because of Theorem 10 of part I. With (3) there exists an attribute

$$y \in B^{\Box\Diamond k} \tag{27}$$

with

$$y \notin A, \tag{28}$$

so because of (5), (26), (27) and condition 1 we get

$$D \to y \in \mathbf{Cons}^{\mathcal{R}}(P_k) \subseteq Sat(\mathbb{K}_k^{\sim}), \tag{29}$$

so $y \in D^{\Box\Diamond\sim}$, but this is a contradiction to (7) and (28) because D is a proper subset of the $Sat(\mathbb{K}_k^{\sim})$-pseudoclosed set A.
Case 2: A is $Sat(\mathbb{K}_k)$-pseudoclosed.
Then A is not $Sat(\mathbb{K}_k^{\sim})$-pseudoclosed. A is not a $Sat(\mathbb{K}_k)$-intent, so with condition 2 A is not a $Sat(\mathbb{K}_k^{\sim})$-intent. We get the existence of a proper subset

$$B \subset A \tag{30}$$

such that

$$\begin{aligned} &B \text{ is } Sat(\mathbb{K}_k^{\sim})\text{-pseudoclosed}, &(31)\\ &B^{\Box\Diamond\sim} \not\subseteq A. &(32)\end{aligned}$$

If B is a premise of P_k then B is $Sat(\mathbb{K}_k)$-pseudoclosed because of Corollary 27 of part I, and if B is not a premise of P_k then B is also $Sat(\mathbb{K}_k)$-pseudoclosed because of the minimality of A with respect to the violation of condition 4. So in both cases

$$\begin{aligned} &B \text{ is } Sat(\mathbb{K}_k)\text{-pseudoclosed}, &(33)\\ &B^{\Box\Diamond k} \subseteq A. &(34)\end{aligned}$$

With (32) and (34) there exists an attribute

$$m \in M \setminus A \tag{35}$$

with

$$B \to m \in Sat(\mathbb{K}_k^{\sim}), \tag{36}$$
$$B \to m \notin Sat(\mathbb{K}_k). \tag{37}$$

There exists a fictitious object

$$g_{C,d} \in G_k \setminus G_k^{\sim} \tag{38}$$

with

$$B \subseteq C, \tag{39}$$
$$m = d. \tag{40}$$

If $B = C$ then we have $g_{C,d} = g_{B,m} \in G_k \setminus G_k^{\sim}$, and if B is a proper subset of C then B is a premise of P_k because of (33), so we also get

$$g_{B,m} \in G_k \setminus G_k^{\sim} \tag{41}$$

with Lemma 3. So we have

$$B \to m \in \mathbf{Cons}^{\mathcal{R}}(P_k) \tag{42}$$

because of the definition of $G_k^{\sim}$. We have $A \in Resp^{\mathcal{R}}(P_k)$ because each premise of P_k is a $Sat(\mathbb{K}_k)$-pseudoclosed proper subset of A. Therefore

$$A \in Resp^{\mathcal{R}}(P_k) = Resp^{\mathcal{R}}(Imp(Resp^{\mathcal{R}}(P_k))) = Resp^{\mathcal{R}}(\mathbf{Cons}^{\mathcal{R}}(P_k)), \tag{43}$$

which is a contradiction to (35), (30) and (42). ■

Theorem 10 *If $\mathcal{R}$ is a closure system then during the exploration in a step k all fictitious counterexamples $g_{A,b} \in G_k$ which are recognized as superfluous*[2] *can be removed from the current context during the exploration, and the questions of the exploration program remain the same.*

Proof. If $I_k(g_{A,b}, \cdot) \leq I_k(g, \cdot)$ for some $g \in G_k \cap G^U$ then the satisfyable implications of the current context do not change by removing $g_{A,b}$ because of Lemma 7, so in this case the questions of the exploration program remain the same. From Theorem 9 it follows that after removing the fictitious objects $g_{A,b}$ with $A \to b \in \mathbf{Cons}^{\mathcal{R}}(P_j)$ the questions also remain the same. ■

[2] that means $g_{A,b} \in \mathbf{Cons}^{\mathcal{R}}(P_k)$ or $I_k(g_{A,b}, \cdot) \leq I_k(g, \cdot)$ for some $g \in G_k \cap G^U$

If $\mathcal{R}$ is no closure system, it is also possible to remove the superfluous objects during the exploration, but it might happen that the expert has to answer more questions then (see Example 2 in the next section), because the pseudoclosed sets would change: With such a modification of the algorithm there may exist a minimal $Sat(\mathbb{K}_j)$-pseudoclosed set A, which is not a premise of P_j, such that A is not minimal with respect to the property $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_j)$, so in this case the result of the algorithm depends on how it chooses the implications to be asked: If the premise is always chosen as a minimal set with $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_j)$, then in the modified algorithm the same questions like in the normal algorithm are asked. But if the premise is chosen as a minimal $Sat(\mathbb{K}_j)$-pseudoclosed set A, which is not a premise of P_j, then it might happen that in the modified algorithm the validitiy of an implication is asked which is already derivable from the accepted implications P_j,[3] so in this case it is better to keep the wrong fictitious counterexamples $g_{A,b}$ for which $A \to b$ is derivable from P_j until the end of the exploration. If the frame context $\mathcal{R}$ is a closure system, then we get the same results in both algorithms.

The satisfyable implications of $\mathbb{K}^*$ are exactly the implications which are derivable from P_n:

Theorem 11 *Let n be the step in which the exploration ends.*
$\mathbf{Cons}^{\mathcal{R}}(P^*) = \mathbf{Cons}^{\mathcal{R}}(P_n) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n)) = Sat(\mathbb{K}_n^{\sim}) = Sat(\mathbb{K}^*)$

Proof.
Proof of $\mathbf{Cons}^{\mathcal{R}}(P^*) = \mathbf{Cons}^{\mathcal{R}}(P_n)$:
For $A \to B \in P_n$ we have $A \to B \in \mathbf{Cons}^{\mathcal{R}}(P^*)$ because of rule (PR), so $\mathbf{Cons}^{\mathcal{R}}(P_n) \subseteq \mathbf{Cons}^{\mathcal{R}}(P^*)$.
For $A \to D \in P^*$ there exists a set $B \subseteq M$ with $A \to B \in P_n$ and $D = B \cup \{d \in M \mid g_{A,d} \in G_n \setminus G^{\sim}\} = B \cup \{d \in M \mid g_{A,d} \in G_n,\ A \to d \in \mathbf{Cons}^{\mathcal{R}}(P_n)\}$, so with rule (AD) we get $A \to D \in \mathbf{Cons}^{\mathcal{R}}(P_n)$ and therefore $\mathbf{Cons}^{\mathcal{R}}(P_n) = \mathbf{Cons}^{\mathcal{R}}(P^*)$.

Proof of $\mathbf{Cons}^{\mathcal{R}}(P_n) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n))$:
See Corollary 31 of part I.

Proof of $\mathbf{Cons}^{\mathcal{R}}(P_n) \subseteq Sat(\mathbb{K}^*)$:
$\mathbf{Cons}^{\mathcal{R}}(P_n) \subseteq Sat(\mathbb{K}_n^{\sim}) \subseteq Sat(\mathbb{K}^*)$ follows from Theorem 9.(1) and $G^* \subseteq G^{\sim}$.

Proof of $Sat(\mathbb{K}^*) = Sat(\mathbb{K}_n^{\sim})$:
See Lemma 7.

Proof of $Sat(\mathbb{K}_n^{\sim}) \subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n))$:
For $Q = \{A \to b \in Imp_M \mid g_{A,b} \in G_n \setminus G^{\sim}\} \subseteq \mathbf{Cons}^{\mathcal{R}}(P_n) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n))$ we

[3] see Example 2

have $Sat(\mathbb{K}_n^{\sim}) \subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n) \cup Q) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n))$ because of Corollary 2.
■

Remark 12 *After the end of the exploration the questionmark reduction can also be done for the fictitious objects. The satisfyable implications of $\mathbb{K}^*$ do not change by this questionmark reduction at the end of the exploration.*[4] *But one loses the information about the involved implications. Before the questionmark reduction the corresponding implictions can be reconstructed by the context rows of the fictitious counterexamples: The attributes with the value "$\times$" are the premise and the attribute with the value "o" is the conclusion. After the questionmark reduction the values of the context row usually do not contain this information anymore.*

Corollary 13 *Let n be the step in which the exploration ends. Then $Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)$ exists and $Sat(\mathbb{K}^*) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*))$.*

Proof. For $g_{A,b} \in G^*$ we have $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_n)$, so $Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)$ exists because of Lemma 6. We have $Sat(\mathbb{K}^*) = \mathbf{Cons}^{\mathcal{R}}(P_n) \subseteq Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*))$ because of Corollary 21 of part I, so we get $Sat(\mathbb{K}^*) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*))$ because of $Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*) \geq \mathbb{K}^*$. ■

Theorem 14 *For each $C \to D \in P^*$ we get $D = C^{\Box\Diamond}$ in $\mathbb{K}^*$. For each $C \subseteq M$ we get $< C >_{\mathbf{Cons}^{\mathcal{R}}(P^*)} = < C >_{Sat(\mathbb{K}^*)} = C^{\Box\Diamond}$ in $\mathbb{K}^*$.*

Proof.
Proof of $D = C^{\Box\Diamond}$:
For $C \to D \in P^*$ there exists $C \to B \in P_n$ with $D = B \cup \{b \in M \mid g_{C,b} \in G_n \setminus G^{\sim}\}$. With Theorem 11 we get $C \to D \in Sat(\mathbb{K}^*)$, so with rule (PR) and Corollary 7 of part I we get $D \subseteq \{m \in M \mid C \to m \in Sat(\mathbb{K}^*)\} = C^{\Box\Diamond}$. Now let $d \in C^{\Box\Diamond}$. Then we get $C \to d \in Sat(\mathbb{K}^*) = Sat(\mathbb{K}_n|_{G^{\sim}})$ by Theorem 11. If $C \to d \in Sat(\mathbb{K}_n)$ then $d \in \{m \in M \mid C \to m \in Sat(\mathbb{K}_n)\} = B \subseteq D$ by Corollary 27 of part I, and if $C \to d \notin Sat(\mathbb{K}_n)$ then $g_{C,d} \in G_n \setminus G^{\sim}$ because of Lemma 3, so $d \in D$. Therefore $C^{\Box\Diamond} = D$.
Proof of $< C >_{\mathbf{Cons}^{\mathcal{R}}(P^*)} = < C >_{Sat(\mathbb{K}^*)} = C^{\Box\Diamond}$:
With Theorem 11 we have $C^{\Box\Diamond} = \{m \in M \mid C \to m \in Sat(\mathbb{K}^*)\} = \{m \in M \mid C \to m \in \mathbf{Cons}^{\mathcal{R}}(P^*)\} = < C >_{\mathbf{Cons}^{\mathcal{R}}(P^*)}$ because of Theorem 10 of part I. We have $< C >_{\mathbf{Cons}^{\mathcal{R}}(P^*)} = < C >_{Sat(\mathbb{K}^*)}$ because of Theorem 11. ■

[4]see the following corollary

In the context $\mathbb{K}^*$ (and also in the context $Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)$ after the questionmark reduction) the operator ${}^{\Box\Diamond} : \mathcal{P}(M) \to \mathcal{P}(M)$ is a closure operator: A set $C \subseteq M$ is closed with respect to this operator iff it respects all implications derivable from P^*, so $C^{\Box\Diamond}$ is the generated P^*-intent (which is also the generated P_n-intent[5]). During the exploration the operator ${}^{\Box\Diamond} : \mathcal{P}(M) \to \mathcal{P}(M)$ is only extensive and monotone, but in general not idempotent.

Lemma 15 *Let $G^U \cap G^* \subseteq S \subseteq G^*$ and $Q = \{A \to d \mid A \subseteq M,\ d \in M$ with $g_{A,d} \in G^* \setminus S\}$. Then $Sat(\mathbb{K}^*|_S) \subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}^*) \cup Q) = \mathbf{Cons}^{\mathcal{R}}(P^* \cup Q)$.*[6]

Proof. Let n be the step in which the exploration ends. Let $T := S \cup \{g_{A,b} \in G_n \mid I_n(g_{A,b}, \cdot) \leq I_n(g, \cdot)$ for some $g \in G_n \cap G^U\}$. Now we prove the following inclusions:

$$\begin{aligned} Sat(\mathbb{K}^*|_S) &\subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n) \cup \{A \to b \mid g_{A,b} \in G_n \setminus T\}) && (1) \\ &\subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}^*) \cup Q) && (2) \end{aligned}$$

With Lemma 7 we have $Sat(\mathbb{K}^*|_S) = Sat(\mathbb{K}_n|_T)$, so (1) follows from Corollary 2. Now let $g_{A,b} \in G_n \setminus T$. Then $g_{A,b} \notin S$.
Case 1: $g_{A,b} \in G^\sim$
Then we get $g_{A,b} \in G^*$ because of the definition of T, therefore $A \to b \in Q$.
Case 2: $g_{A,b} \notin G^\sim$
Then $A \to b \in \mathbf{Cons}^{\mathcal{R}}(P_n)$ because of the definition of $G^\sim$.
In both cases we have $A \to b \in \mathbf{Cons}^{\mathcal{R}}(P_n) \cup Q$, so

$$\begin{aligned} \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n) \cup \{A \to b \mid g_{A,b} \in G_n \setminus T\}) &\subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n) \cup \mathbf{Cons}^{\mathcal{R}}(P_n) \cup Q) \\ &= \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}^*) \cup Q) \end{aligned}$$

because of Theorem 11, and we get (2). Finally $\mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}^*) \cup Q) = \mathbf{Cons}^{\mathcal{R}}(P^* \cup Q)$ follows from Theorem 11. ■

Definition 16
Let $P^u := \{A \to b \in Imp_M \mid g_{A,b} \in G^*\}$ be the set of all implications accepted as unknown (after removing the superfluous unknown implications[7]).
Let $P^u_U := P^u \cap Imp(\mathbb{K}^U)$ be the set of all implications accepted as unknown which are valid in the universe $\mathbb{K}^U$. Let $G^*_U := G^* \setminus \{g_{A,b} \in G^* \mid A \to b \in Imp(\mathbb{K}^U)\}$ be the set of all objects of G^* which are either in the universe $\mathbb{K}^U$ or fictitious objects $g_{A,b}$ for which $A \to b$ is not valid in the universe $\mathbb{K}^U$.[8]

[5] see Theorem 11
[6] See also Corollary 2.
[7] see Definition 8
[8] At this stage it is still unknown to the expert whether $A \to b$ is valid in the universe.

Lemma 17 *$P_U^u = \{A \to b \in Imp_M \mid g_{A,b} \in G^* \setminus G_U^*\}$ and $G_U^* = G^* \setminus \{g_{A,b} \in G^* \mid A \to b \in P_U^u\}$*

Proof.
$A \to b \in P_U^u$ iff $A \to b \in P^u$ and $A \to b \in Imp(\mathbb{K}^U)$ iff $g_{A,b} \in G^*$ and $g_{A,b} \notin G_U^*$.
$g \in G_U^*$ iff $g \in G^*$ and $g \neq g_{A,b}$ for $A \to b \in Imp(\mathbb{K}^U)$ iff $g \in G^*$ and $g \neq g_{A,b}$ for $A \to b \in P_U^u$ iff $g \in G^* \setminus \{g_{A,b} \in G^* \mid A \to b \in P_U^u\}$. ■

Lemma 18
$G_U^* = \{g \in G^* \mid$ there exists a context row in $\mathbb{K}^U$, which is a completion of the context row of g in $\mathbb{K}^*\}$

Proof.
$\subseteq$:
Each context row of an object $g \in G_U^* \cap G^U$ has a completion in $\mathbb{K}^U$. For each fictitious object $g_{A,b} \in G_U^*$ there exists a counterexample $g \in G^U$ in $\mathbb{K}^U$ against the implication $A \to b$ because of $A \to b \notin Imp(\mathbb{K}^U)$. The context row of g in $\mathbb{K}^U$ is a completion of the context row of $g_{A,b}$ in $\mathbb{K}^*$. So for each context row of $\mathbb{K}^*|_{G_U^*}$ there exists a completion in $\mathbb{K}^U$.
$\supseteq$:
Let $g \in G^*$ be an object such that there exists a completion in $\mathbb{K}^U$ of the context row of g in $\mathbb{K}^*$. For $g_{A,b} \in G^*$ with $A \to b \in Imp(\mathbb{K}^U)$ we have $g \neq g_{A,b}$ because $A \to b$ is not valid in a completion of the context row of $g_{A,b}$. So we get $g \in G_U^*$. ■

The exploration helps to get knowledge about the implications valid in the universe $\mathbb{K}^U$. If the expert gives the answer "unknown" to some questions of the program, then he may not get complete knowledge of the universe, but the following theorem shows that all incomplete knowledge about the implications of $\mathbb{K}^U$ is coded in the fictitious counterexamples of $\mathbb{K}^*$.

Theorem 19 *Let n be the step in which the exploration ends. Then $\mathbf{Cons}^{\mathcal{R}}(P_n \cup P_U^u) = \mathbf{Cons}^{\mathcal{R}}(P^* \cup P_U^u) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}^*) \cup P_U^u) = Sat(\mathbb{K}^*|_{G_U^*}) = Imp(\mathbb{K}^U) = Sat(\mathbb{K}^U)$.*

Proof. $\mathbf{Cons}^{\mathcal{R}}(P_n \cup P_U^u) = \mathbf{Cons}^{\mathcal{R}}(P^* \cup P_U^u)$ follows from Theorem 11.
The context $\mathbb{K}^U$ does not contain questionmarks, so we get $Imp(\mathbb{K}^U) = Sat(\mathbb{K}^U)$.
Proof of $\mathbf{Cons}^{\mathcal{R}}(P_n \cup P_U^u) \subseteq Imp(\mathbb{K}^U)$:
$P_U^u \subseteq Imp(\mathbb{K}^U)$ follows from the definition of P_U^u. $P_n \subseteq Imp(\mathbb{K}^U)$ holds because the implications of P_n are accepted as valid. Therefore $\mathbf{Cons}^{\mathcal{R}}(P_n \cup P_U^u) \subseteq Imp(\mathbb{K}^U)$.

Proof of $Imp(\mathbb{K}^U) \subseteq Sat(\mathbb{K}^*|_{G^*_U})$:
For $A \to B \in Imp(\mathbb{K}^U)$ we get $A \to B \in Sat(\mathbb{K}^*|_{G^*_U})$ by Lemma 18.
Proof of $Sat(\mathbb{K}^*|_{G^*_U}) \subseteq \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}^*) \cup P^u_U) = \mathbf{Cons}^{\mathcal{R}}(P^* \cup P^u_U)$:
See Lemma 15 and Lemma 17. ■

Remark 20 *An implication $A \to B$ is valid in the universe $\mathbb{K}^U$ iff it is derivable from $P_n \cup P^u_U$ iff it is satisfyable in the subcontext $Sat(\mathbb{K}^*|_{G^*_U})$ of $\mathbb{K}^*$. The sets G^*_U and P^u_U are unknown to the expert but the expert knows that there exists a subset $P^u_U \subseteq P^u$ of the unknown implications such that this subset together with the accepted implications P_n is a generating set of all valid implications. So after the exploration the expert only has to check the implications $A \to b$ for $g_{A,b} \in G^*$, and as soon as he can decide for each such implication whether it is valid in $\mathbb{K}^U$ or not, then he has complete knowledge about the valid implications of $\mathbb{K}^U$: An implication is valid in $\mathbb{K}^U$ iff it is derivable from the implications accepted as valid and the implications accepted as unknown which are valid in $\mathbb{K}^U$. Moreover, then the subcontext $Sat(\mathbb{K}^*|_{G^*_U})$ contains a complete list of counterexamples against the implications which are not valid in $\mathbb{K}^U$.*

Remark 21 *Note that the set $P_n \cup P^u_U$ is only a generating system with respect to the frame context, that means if we want to compute all consequences, we also need the exhaustion rule ($\mathcal{R}$-EX). Sometimes the user wants to have a generating system (or a base) of the valid implications, such that he only has to use the rules (AX) and (PS) to compute all consequences. In this case the set P_n must contain the informations of the frame context $\mathcal{R}$. It is possible to modify the exploration algorithm, such that the expert gets this result at the end of the algorithm: At the beginning of the exploration algorithm the expert still can enter a frame context $\mathcal{R}$, but in each step j the program does not search for a minimal set A satisfying $A^{\Box\Diamond} \neq A \in \mathcal{R} \cap Resp(P_j)$ but it searches for a minimal set A satisfying $A^{\Box\Diamond} \neq A \in Resp(P_j)$.[9] Before the program asks for the validity of $A \to A^{\Box\Diamond}$ it checks whether this implication is derivable from the accepted implications P_j with the rules (AX), (PS) and ($\mathcal{R}$-EX), in this case it can be accepted automatically as valid, otherwise the expert is asked. With this modified algorithm the expert gets a base P_n such that the frame context is not needed after the exploration anymore: $Sat(\mathbb{K}^*)$ is derivable from P_n using only the rules (AX) and (PS), and $Imp(\mathbb{K}^U)$ is derivable from $P_n \cup P^u_U$ using only the rules (AX) and (PS).*

After reducing the questionmarks of $\mathbb{K}^*$ each context row which still contains some questionmarks is redundant: Every object $g \in G^*$ which has a questionmark in the context row of the reduced context can be removed and the satisfyable implications do not change. So we get a complete context:

[9] Or equivalently: for a minimal set A which is $Sat(\mathbb{K}_j)$-pseudoclosed with respect to $\mathcal{P}(M)$ but not a premise of P_j.

Theorem 22 *Let n be the step in which the exploration ends. For $g \in G^*$ let* $I^*(g, M) = \{I^*(g, m) \mid m \in M\}$*. Let*
$Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*) = (G^*, M, \{\times, ?, o\}, J)$,
$S = G^* \setminus \{g \in G^* \cap G^U \mid ? \in I^*(g, M)\}$,
$T = G^* \setminus \{g \in G^* \mid ? \in J(g, M)\} = \{g \in G^* \mid ? \notin J(g, M)\}$.
Then the following conditions hold:

1. $Sat(\mathbb{K}^*) = Sat(\mathbb{K}^*|_S) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T)$
2. $S \cap G^U = T \cap G^U$
3. $Imp(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T) \subseteq Imp(\mathbb{K}^U) \subseteq Imp(\mathbb{K}^*|_{T \cap G^U})$
4. $Int(\mathbb{K}^*|_{T \cap G^U}) \subseteq Int(\mathbb{K}^U) \subseteq Int(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T)$

Proof.
Proof of $Sat(\mathbb{K}^*) = Sat(\mathbb{K}^*|_S)$:
$\mathbb{K}^*|_S$ is a subcontext of $\mathbb{K}^*$, so we have $Sat(\mathbb{K}^*) \subseteq Sat(\mathbb{K}^*|_S)$. Assume that there exists an implication $A \to B \in Sat(\mathbb{K}^*|_S)$ with $A \to B \notin Sat(\mathbb{K}^*)$. Let A be maximal with these properties. Then in $\mathbb{K}^*$ there exists a counterexample $g \in G^* \cap G^U$ against $A \to B$ such that the context row of g contains a questionmark. Let $m \in M$ with $I^*(g, m) =?$. We have $I^*(g, a) = \times$ for all $a \in A$ and $I^*(g, b) = o$ for some $b \in B$. With rule (AU) we get $A \cup \{m\} \to B \in Sat(\mathbb{K}^*|_S)$. The set $A \cup \{m\}$ is a proper superset of A, so with the maximality of A we get $A \cup \{m\} \to B \in Sat(\mathbb{K}^*) = \mathbf{Cons}^{\mathcal{R}}(P_n) = Imp(Resp^{\mathcal{R}}(P_n))$. For each $E \in Resp^{\mathcal{R}}(P_n)$ with $g^{\Box} \subseteq E \subseteq g^{\Diamond}$ (in $\mathbb{K}^*$) we have $A \subseteq E$ and $B \not\subseteq E$, so $m \notin E$, because E respects the implication $A \cup \{m\} \to B$. By rule (Red2) we get $I^*(g, m) = o$ which is a contradiction to the assumption $I^*(g, m) =?$. Therefore $Sat(\mathbb{K}^*) = Sat(\mathbb{K}^*|_S)$.
Proof of $Sat(\mathbb{K}^*) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T)$:
We have $Sat(\mathbb{K}^*) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*))$ by Corollary 13, and the proof of $Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T)$ works analogously to the proof of $Sat(\mathbb{K}^*) = Sat(\mathbb{K}^*|_S)$.
Proof of 2:
The context rows of the objects of $G^* \cap G^U$ are P_n-reduced, so for $g \in G^*$ we have $g \in S \cap G^U$ iff $g \in G^U$ and $? \notin I^*(g, M)$ iff $g \in T \cap G^U$.
Proof of 3:
$Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T$ and $\mathbb{K}^*|_{T \cap G^U}$ are complete, so $Imp(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T) = Sat(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T) = Sat(\mathbb{K}^*) \subseteq Imp(\mathbb{K}^U) \subseteq Sat(\mathbb{K}^*|_{T \cap G^U}) = Imp(\mathbb{K}^*|_{T \cap G^U})$.
Proof of 4:
With [GW99] and condition 3 we have $Int(\mathbb{K}^*|_{T \cap G^U}) = Resp(Imp(\mathbb{K}^*|_{T \cap G^U})) \subseteq Resp(Imp(\mathbb{K}^U)) = Int(\mathbb{K}^U) \subseteq Resp(Imp(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T)) = Int(Red^{\mathcal{R}}_{P_n}(\mathbb{K}^*)|_T)$. ∎

This Theorem shows that we always can get a complete context at the end of the exploration: After reducing the questionmarks with the rules (Red1) and (Red2) all context rows which still contain questionmarks can be removed from the context without changing the satisfyable implications. But one should remember that these context rows may be needed for the equality $Sat(\mathbb{K}^*|_{G_U^*}) = Imp(\mathbb{K}^U)$ in Theorem 19; this equality may not hold anymore after removing the objects with questionmarks in the context rows.[10]

2 Some examples

Example 1:
This example contains an attribute exploration for properties of natural numbers. Let $\mathbb{K}^U = (G^U, M, I^U)$ where $G^U = \mathbb{N}^+$ are the positive integers and
$M = \{even, odd, prime, s2e, s2p\}$, where $(n, s2e) \in I^U$ iff the positive integer n is the sum of two even positive integers, and $(n, s2p) \in I^U$ iff the positive integer n is the sum of two primes. For the exploration we use the framecontext $\mathcal{R} = \mathcal{P}(M)$ and we start with the context

$\mathbb{K}_1$	even	odd	prime	s2e	s2p
1	o	×	o	o	o
2	×	o	×	o	o
3	o	×	×	o	o

Question 1:[11] $s2p \to \{even, odd, prime, s2e\}$
Answer: no
Counterexample: 4
The context row of 4 is added to the context:

$\mathbb{K}_2$	even	odd	prime	s2e	s2p
1	o	×	o	o	o
2	×	o	×	o	o
3	o	×	×	o	o
4	×	o	o	×	×

Question 2: $s2p \to \{even, s2e\}$
Answer: no
Counterexample: 5
The context row of 5 is added to the context.
Question 3: $s2e \to \{even, s2p\}$
Answer: unknown

[10] See Example 4 in the next section.
[11] In the examples we use the implications $A \to A^{\Box\Diamond} \setminus A$ instead of $A \to A^{\Box\Diamond}$ because it is clear that A implies A.

This question is equivalent to the Goldbach conjecture: It is not known whether every even number $n \geq 4$ is the sum of two primes.
The first part of the implication is accepted as valid: $P_4 = \{s2e \rightarrow even\}$
A fictitious counterexample is added to the context:

$\mathbb{K}_4$	even	odd	prime	s2e	s2p
1	o	$\times$	o	o	o
2	$\times$	o	$\times$	o	o
3	o	$\times$	$\times$	o	o
4	$\times$	o	o	$\times$	$\times$
5	o	$\times$	$\times$	o	$\times$
$g_{\{s2e\},s2p}$	?	?	?	$\times$	o

Question 4: $\{prime, s2p\} \rightarrow odd$
Answer: yes
Question 5: $\{odd, s2p\} \rightarrow prime$
Answer: no
Counterexample: 9
Question 6: $\{even, s2p\} \rightarrow s2e$
Answer: yes
Question 7: $\{even, s2e\} \rightarrow s2p$
Answer: unknown
Fictitious counterexample: $g_{\{even,s2e\},s2p}$
Question 8: $\{even, prime, s2e\} \rightarrow \{odd, s2p\}$
Answer: yes
Question 9: $\{even, odd\} \rightarrow \{prime, s2e, s2p\}$
Answer: yes
The algorithm ends.

At the end of the algorithm we get the following context $\mathbb{K}_{10} = \mathbb{K}^*$:

$\mathbb{K}^*$	even	odd	prime	s2e	s2p
1	o	$\times$	o	o	o
2	$\times$	o	$\times$	o	o
3	o	$\times$	$\times$	o	o
4	$\times$	o	o	$\times$	$\times$
5	o	$\times$	$\times$	o	$\times$
9	o	$\times$	o	o	$\times$
$g_{\{s2e\},s2p}$	?	?	?	$\times$	o
$g_{\{even,s2e\},s2p}$	$\times$	?	?	$\times$	o

We have a list of implications accepted as valid:
$P_{10} = \{$

$s2e \to even,$
$\{prime, s2p\} \to odd,$
$\{even, s2p\} \to s2e,$
$\{even, prime, s2e\} \to \{odd, s2p\},$
$\{even, odd\} \to \{prime, s2e, s2p\}$
$\}$

We reduce the questionmarks in $\mathbb{K}^*$ by using the valid implications in P_{10} and the background knowledge $\mathcal{R}$ (which does not contain any information in this case because of $\mathcal{R} = \mathcal{P}(M)$). This leads to a context $Red^{\mathcal{R}}_{P_{10}}(\mathbb{K}^*)$:

$Red^{\mathcal{R}}_{P_{10}}(\mathbb{K}^*)$	even	odd	prime	s2e	s2p
1	o	$\times$	o	o	o
2	$\times$	o	$\times$	o	o
3	o	$\times$	$\times$	o	o
4	$\times$	o	o	$\times$	$\times$
5	o	$\times$	$\times$	o	$\times$
9	o	$\times$	o	o	$\times$
$g_{\{s2e\},s2p}$	$\times$	o	o	$\times$	o
$g_{\{even,s2e\},s2p}$	$\times$	o	o	$\times$	o

Now we have the following two cases:

1. If the Goldbach conjecture is true then the context rows of the fictitious counterexamples do not occur in the universe $\mathbb{K}^U$, so the following subcontext $\mathbb{K}^*|_{G^* \cap G^U}$ already contains a complete list of counterexamples, that means an implication is valid in the universe iff it is satisfyable (or valid) in the subcontext $\mathbb{K}^*|_{G^* \cap G^U}$ iff it is derivable from $P_{10} \cup P^u_U$, where $P^u_U = \{s2e \to s2p, \{even, s2e\} \to s2p\}$.

$\mathbb{K}^*|_{G^* \cap G^U}$	even	odd	prime	s2e	s2p
1	o	$\times$	o	o	o
2	$\times$	o	$\times$	o	o
3	o	$\times$	$\times$	o	o
4	$\times$	o	o	$\times$	$\times$
5	o	$\times$	$\times$	o	$\times$
9	o	$\times$	o	o	$\times$

2. If the Goldbach conjecture is false then the context rows of the fictitious counterexamples occur in the universe $\mathbb{K}^U$, so the context $Red^{\mathcal{R}}_{P_{10}}(\mathbb{K}^*)$ contains a complete list of counterexamples, that means an implication is valid in the universe iff it is satisfyable (or valid) in $Red^{\mathcal{R}}_{P_{10}}(\mathbb{K}^*)$ iff it is derivable from P_{10}.

The concept lattice of $\mathbb{K}^U$ is either isomorphic to the concept lattice of $\mathbb{K}^*|_{G^* \cap G^U}$ or to the concept lattice of $Red^{\mathcal{R}}_{P_{10}}(\mathbb{K}^*)$. The following two figures show the concept lattices for both cases.

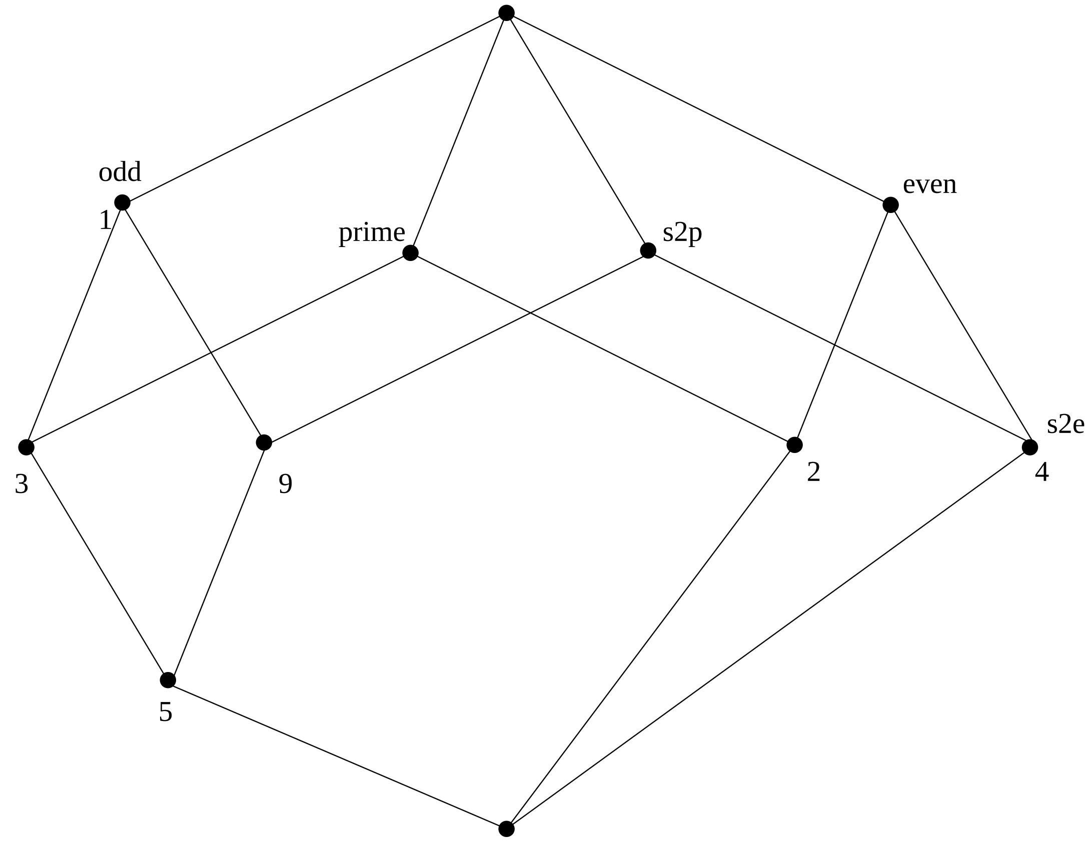

Figure 1: Concept lattice of $\mathbb{K}^*|_{G^* \cap G^U}$ if the Goldbach conjecture is true.

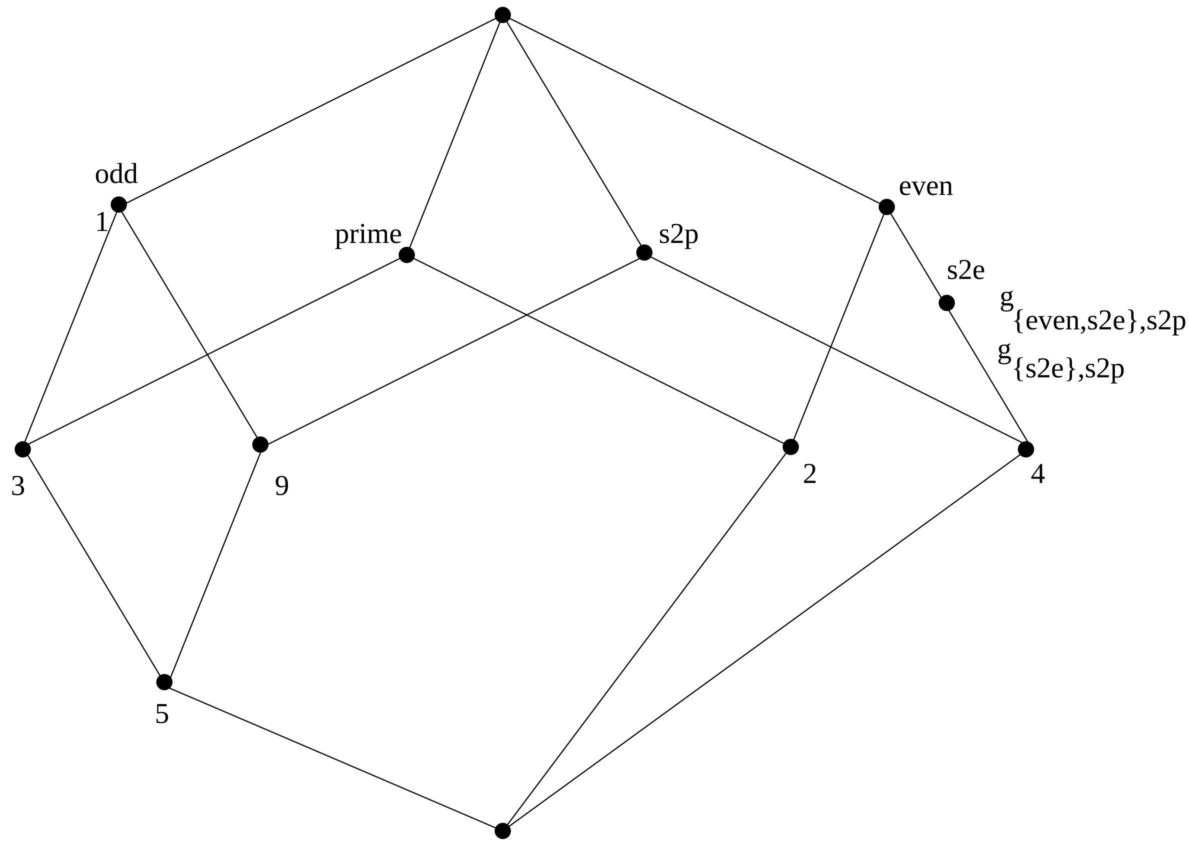

Figure 2: Concept lattice of $Red^{\mathcal{R}}_{P_{10}}(\mathbb{K}^*)$ if the Goldbach conjecture is false.

Example 2:
This example shows that a fictitious object should not be removed before the end of the exploration even if the corresponding unknown implication is recognized to be valid in the universe. Let $M = \{a, b, c, d, e\}$ and $\mathcal{R}$ be the set of all models of the clause $a \to b \vee c$. Let $\mathbb{K}_0 = \mathbb{K}_1$ be the following context:

$\mathbb{K}_1$	a	b	c	d	e
1	×	*o*	×	×	*o*
2	×	×	*o*	×	×

The empty set is $Sat(\mathbb{K}_0)$-pseudoclosed with respect to $\mathcal{R}$, so in step 1 the exploration program asks for the validity of $\emptyset \to \{a, d\}$. We give the answer unknown, and the program asks, for which attribute of the conclusion the implication is unknown. We give the answer that $\emptyset \to d$ is unknown and $\emptyset \to a$ is valid. A fictitious object is added to the context:

$\mathbb{K}_2$	a	b	c	d	e
1	×	o	×	×	o
2	×	×	o	×	×
$g_{\emptyset,d}$	?	?	?	o	?

The set $\{a\}$ is not an element of the frame context $\mathcal{R}$, so $\{a,b\}$ is a minimal $Sat(\mathbb{K}_2)$-pseudoclosed set which is not a premise of $P_2 = \{\emptyset \to a\}$. In step 2 the program asks for the validity of $\{a,b\} \to \{d,e\}$ and we give the answer "yes". In step 3 the program asks for the validity of $\{a,c\} \to d$ and we again give the answer "yes". We have $P_4 = \{\emptyset \to a, \{a,b\} \to \{d,e\}, \{a,c\} \to d\}$, therefore $\emptyset \to d$ is derivable from P_4, because $a \to b \vee c$ is valid in the frame context $\mathcal{R}$. So the unknown implication $\emptyset \to d$ must be valid in the universe $\mathbb{K}^U$. The set $\{a,b,d\}$ is not $Sat(\mathbb{K}_4)$-pseudoclosed because $\{a,b\}$ is a $Sat(\mathbb{K}_4)$-pseudoclosed proper subset with $\{a,b\}^{\Box\Diamond} \not\subseteq \{a,b,d\}$. But if we remove the "wrong" fictitious object $g_{\emptyset,d}$ from the current context $\mathbb{K}_4$ then we get $\emptyset^{\Box\Diamond} = \{a,d\}$, so the set $\{a,b\}$ is no longer $Sat(\mathbb{K}_4)$-pseudoclosed, and $\{a,b,d\}$ becomes a minimal $Sat(\mathbb{K}_4)$-pseudoclosed set which is not a premise of P_4, so the program would ask in step 4 for the validity of $\{a,b,d\} \to e$ which is already derivable from P_4. So in this case it is better to leave the wrong fictitious object in the context until the end of the exploration. If the program does not use pseudoclosed sets, but searches for minimal sets A with $A^{\Box\Diamond} \neq A \in Resp(P_j) \cap \mathcal{R}$ then it does not matter whether the wrong fictitious objects are removed during or after the exploration, because the questions remain the same.[12] In the example above after removing the wrong fictitious example $g_{\emptyset,d}$ in step 4 the set $\{a,b,d\}$ is a minimal $Sat(\mathbb{K}_4)$-pseudoclosed set which is no premise of P_4, but it does not belong to $Resp(P_4)$ because of $\{a,b\} \to \{d,e\} \in P_4$. If the universe $\mathbb{K}^U$ conists only of the objects 1 and 2, then a base (with respect to $\mathcal{R}$) of $Imp(\mathbb{K}^U)$ is $\{\emptyset \to a, \{a,b\} \to \{d,e\}, \{a,c\} \to d, \{a,c,d,e\} \to b\}$.

Example 3:
This example shows that questionmark reduction should not be done for fictitious objects before the exploration ends, because otherwise the expert may loose some information about the universe. Let $M = \{a,b,c\}$, $\mathcal{R} = \mathcal{P}(M)$ and $\mathbb{K}_0 = \mathbb{K}_1$ be the context

$\mathbb{K}_1$	a	b	c
1	o	o	o
2	o	×	o
3	o	×	×
4	o	?	×

In step 1 the exploration program asks for the validity of $a \to \{b,c\}$. We give the answer unknown, and the program asks, for which attribute of the conclusion the implication is unknown. We give the answer that $a \to b$ is unknown and $a \to c$ is

[12]See Theorem 9.

valid. A fictitious object is added to the context:

$\mathbb{K}_2$	a	b	c
1	o	o	o
2	o	×	o
3	o	×	×
4	o	?	×
$g_{\{a\},b}$	×	o	?

In step 2 the program asks for the validity of $c \to b$. We give the answer "unknown" and another fictitious counterexample is added to the context:

$\mathbb{K}_3$	a	b	c
1	o	o	o
2	o	×	o
3	o	×	×
4	o	?	×
$g_{\{a\},b}$	×	o	?
$g_{\{c\},b}$	?	o	×

In step 3 the program asks for the validity of $\{a, c\} \to b$. We give the answer "unknown" and another fictitious counter example is added to the context:

$\mathbb{K}_4$	a	b	c
1	o	o	o
2	o	×	o
3	o	×	×
4	o	?	×
$g_{\{a\},b}$	×	o	?
$g_{\{c\},b}$	?	o	×
$g_{\{a,c\},b}$	×	o	×

The algorithm ends in step 4. We have only one implication accepted as valid: $P_4 = \{a \to c\}$. If we would have done a questionmark reduction before step 2 for the fictitious counterexample $g_{\{a\},b}$, then the exploration would have ended in step 2 because the implications $c \to b$ and $\{a, c\} \to b$ are no longer satisfyable in $\mathbb{K}_2$:

$\mathbb{K}_2$	a	b	c
1	o	o	o
2	o	×	o
3	o	×	×
4	o	?	×
$g_{\{a\},b}$	×	o	×

If $\mathbb{K}^U$ is the context

$\mathbb{K}^U$	a	b	c
1	o	o	o
2	o	$\times$	o
3	o	$\times$	$\times$
4	o	$\times$	$\times$

then we have $\mathbb{K}^* = \mathbb{K}_2$, $P_U^u = \{a \to b\}$, $G_U^* = G^U = \{1,2,3,4\}$, $\mathbb{K}^*|_{G_U^*} = \mathbb{K}_1$ and $P_2 = \{a \to c\}$, so $c \to b$ is not derivable from $P_2 \cup P_U^u$, but $c \to b$ is valid in the universe, so in this case Theorem 19 would not hold. If $\mathbb{K}^U$ is the context

$\mathbb{K}^U$	a	b	c
1	o	o	o
2	o	$\times$	o
3	o	$\times$	$\times$
4	o	o	$\times$

then we have again $\mathbb{K}^* = \mathbb{K}_2$, $P_U^u = \{a \to b\}$, $G_U^* = G^U = \{1,2,3,4\}$, $\mathbb{K}^*|_{G_U^*} = \mathbb{K}_1$ and $P_2 = \{a \to c\}$, so $c \to b$ is satisfyable in $\mathbb{K}^*|_{G_U^*}$ but not valid in $\mathbb{K}^U$, so in this case Theorem 19 would not hold either.

Example 4:
This example shows that Theorem 19 may not hold after applying Theorem 22 to get a complete context at the end of the exploration. Let $\mathbb{K}^*$ be the following context after the exploration:

$\mathbb{K}^*$	a	b
1	o	?
2	$\times$	o
$g_{\{b\},a}$	o	$\times$

After removing the object 1 the satisfyable implications do not change, but if $\mathbb{K}^U$ is the context

$\mathbb{K}^U$	a	b
1	o	o
2	$\times$	o

then the object 1 is needed for the equality $Sat(\mathbb{K}^*|_{G_U^*}) = Imp(\mathbb{K}^U)$ because the implication $\emptyset \to a$ is not valid in $\mathbb{K}^U$. So if the expert removes the object 1 from $\mathbb{K}^*$ (for example to draw the line diagram of a concept lattice), then he should remember, that this object is only redundant for the context $\mathbb{K}^*$ but it may be irredundant in $\mathbb{K}^U$.

Example 5:
This example shows that $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$ *does not imply the existence of* $Red_P^{\mathcal{R}}(\mathbb{K})$.

Let $\mathbb{K}$ be the context

$\mathbb{K}$	a	b
g	$\times$	?

Let $P = \{a \to b\}$ and $\mathcal{R} = \{\{a\}, \{b\}\}$. Then we have $Sat(\mathbb{K}) = Imp_M$, but there is no $T_g \in Resp^{\mathcal{R}}(P)$ with $g^{\Box} \subseteq T_g \subseteq g^{\Diamond}$ in $\mathbb{K}$. So $Red_P^{\mathcal{R}}(\mathbb{K})$ does not exist because of Theorem 20.3 of part I.

3 Conclusion

At the end of the attribute exploration the expert gets maximal information (with respect to his knowledge) about the unknown universe $\mathbb{K}^U$: He gets a list of implications which are certainly valid, a list of implications which are possibly valid, a list of counterexamples against the implications which are certainly not valid and a list of fictitious counterexamples against the implications which he answered by “unknown”. He only has to check the implications which he answered by “unknown” and if he can decide for each of these implications whether it is valid or not, he gets complete knowledge about the implications of the context: An implication is valid in $\mathbb{K}^U$ iff it is derivable from the implications accepted as valid and the implications accepted as unknown which are valid in $\mathbb{K}^U$.

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