Knowledge acquisition under incomplete knowledge using methods from formal concept analysis Part I

Richard Holzer

Abstract

Formal contexts with unknown entries can be represented by three-valued contexts $\mathbb{K} = (G, M, \{\times, o, ?\}, I)$, where a questionmark indicates that it is not known whether the object $g \in G$ has the attribute $m \in M$. To describe logical formulas between columns of such incomplete contexts the Kripke-semantics are used for propositional formulas over the set M of attributes. Attribute implications are considered as special propositional formulas. If a context is too large to be fully represented, an interactive computer algorithm may help the user to get maximal information (with respect to his knowledge) about the valid attribute implications of the unknown context. This computer algorithm is called "attribute exploration".

Introduction

If we have some knowledge and would like to derive more information from this knowledge, then it is useful to have an interactive computer algorithm, which helps the user to derive the information. Attribute exploration[1] is such a tool in formal concept analysis: An interactive computer algorithm helps the expert, to get knowledge about the validity of attribute implications of an unknown formal context, where an attribute implication $A \to B$ describes the dependencies between the attributes: If an object has all attributes of A then it must also have all attributes of B for some sets $A, B \subseteq M$ of attributes. The program asks some questions about the validity of implications, and the expert must find answers to these questions. If he can answer all questions, he gets a base of all valid implications and a complete list of counterexamples against the implications which are not valid. If the expert does not know the answers to all questions, then he only gets approximations for the valid implications: He gets a list of implications which are certainly valid, a list of implications which are possibly valid, a list of counterexamples against the implications which are certainly not valid and a list of fictitious counterexamples against the implications which he

[1] see section 2 in [H04] (part II of this paper) for some examples of attribute explorations

answered by "unknown". After the exploration he only has to check the implications which he answered by "unknown" and if he can decide for each of these implications whether it is valid or not, he gets complete knowledge about the implications of the context. Such an algorithm and the proof of its correctness is provided in these notes.

In literature already many different examples of attribute explorations can be found, for example the exploration of the dependencies between some properties of triangles (see [B91]), properties of natural numbers (see [B91], [BH00] or [H01]), properties of binary relations (see [GW99]), properties of rings (see [H01]), classification of two-dimensional crystallographic point groups (see [G99]), and there are also many nonmathematical examples of explorations: properties of music (see [W89]), properties of planets (see [H01]) ... Other examples can also be found in section 2 of part II (see [H04]).

An attribute exploration algorithm was implemented by Peter Burmeister in the program "ConImp" (see [B96b] and [B91b]), version 4.18. The algorithm described in this paper is very similar to the algorithm of ConImp. Only the use of frame contexts, which describe background knowledge in form of "possible object intents" for the attribute exploration, have not been implemented in ConImp yet. ConImp uses background knowledge in form of implications instead of frame contexts.

In older versions of ConImp incomplete knowledge about the validity of implications was treated differently: An implication which was accepted as unknown, was treated in the further process of the exploration like a valid implication, which led to the fact, that not all necessary implications were asked by the program. This problem was eliminated by using fictitious counterexamples in newer versions of ConImp (see also section 2 of this paper).

In [G99] another algorithm for attribute exploration is given: Background knowledge can be entered in form of universal sequents (= clauses) and existential sequents, where an existential sequent describes an (incomplete) context row. During the exploration algorithm the program asks by and by whether some attribute implications are valid and the expert can either answer by a universal sequent or by an existential sequent. The algorithm of [G99] was implemented in the program "Impex".

Another form of exploration in formal concept analysis is "concept exploration". It can be found in [S97]. Concept exploration helps the user to get informations about the relations "subconcept" and "superconcept" in a formal context.

These notes are a translation of the main results of the thesis [H01]. The thesis [H01] was written in German.

Since attribute exploration is highly connected with context implications, we have to start with the consideration of such implications, and since it may happen that not all questions can be decided, we need incomplete contexts. This will be handled in section 1. In section 2 the algorithm for attribute exploration with incomplete knowledge will be explained. The main results and some examples are given in part II.[2]

[2] see [H04]

1 Incomplete contexts

A *formal context* $\mathbb{K} := (G, M, I)$ consists of a set G of objects, a set M of attributes, and a relation $I \subseteq G \times M$ which indicates which object has which attribute. A formal context can be represented as a data table where each row of the table belongs to an object $g \in G$ and each column of the table belongs to an attribute $m \in M$. The entries in the table indicate whether an object has an attribute.

Example:

$\mathbb{K}$	even	odd	prime	square
1	o	$\times$	o	$\times$
2	$\times$	o	$\times$	o
3	o	$\times$	$\times$	o

In this example the set of objects consists of some natural numbers $G = \{1, 2, 3\}$ and the set of attributes consists of some properties of natural numbers: $M = \{even, odd, prime, square\}$. The entry $\times$ in the table means that the object has the attribute and the entry o means that the object does not have the attribute.[3] If we also would like to capture unknown knowledge then we need a third value: A questionmark indicates that it is not known (at this moment) if the object has the attribute or not.

For example if we want to create a table which contains some information about the weather on different days in summer, then we do not have complete information about the days in the future:

	sunshine	rain	snow
yesterday	o	o	o
today	$\times$	o	o
tomorrow	?	?	o

This is a representation of an *incomplete context* where an incomplete context $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ is a threevalued context which consists of a set G of objects, a set M of attributes, the set $\{\times, ?, o\}$ of values, and a function[4] $I : G \times M \to \{\times, ?, o\}$. The value $\times$ means that it is known that the object has the attribute, the value o means that it is known that the object does not have the attribute, and the questionmark means that it is not known whether the object has the attribute. If such a threevalued context is complete (what means that it has no questionmark as entry) then it can be identified with the corresponding formal context (G, M, J) with

[3] Sometimes a blank in the table is used instead of o, see [GW99].

[4] In [GW99] many-valued contexts have a relation I which is the graph of a partial function, so also missing values are allowed in [GW99].

$(g, m) \in J$ iff $I(g, m) = \times$. For two incomplete contexts $\mathbb{K}_1 := (G, M, \{\times, ?, o\}, I_1)$ and $\mathbb{K}_2 := (G, M, \{\times, ?, o\}, I_2)$ with the same set G of objects and the same set M of attributes we can compare the informations which are described by the context: If $\mathbb{K}_2$ can be derived from $\mathbb{K}_1$ by replacing some questionmarks by $\times$ or o then the context $\mathbb{K}_2$ contains more information than the context $\mathbb{K}_1$, and this will be denoted by $\mathbb{K}_1 \leq \mathbb{K}_2$. So $\leq$ is the *information order.* A *completion* of a context $\mathbb{K}$ is a complete context (=formal context) which can be derived from $\mathbb{K}$ by replacing all questionmarks by other values. So the completions of $\mathbb{K}$ are the maximal contexts (in the information order) above $\mathbb{K}$. The set of all completions of $\mathbb{K}$ is denoted by $Compl(\mathbb{K})$. The restriction of the context $\mathbb{K}$ to a subset $S \subseteq G$ is denoted by $\mathbb{K}|_S := (S, M, \{\times, ?, o\}, I|_{S\times M})$.

Let $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ be an incomplete context. For $B \subseteq M$ the *certain extent* $B^\Box$ of B is the set of all objects which certainly have all attributes of B, that means $B^\Box := \{g \in G \mid I(g, m) = \times \text{ for all } m \in B\}$. The *possible extent* $B^\Diamond$ of B is the set of all objects which possibly have all attributes of B: $B^\Diamond := \{g \in G \mid I(g, m) \neq o \text{ for all } m \in B\}$. The *certain intent* $S^\Box$ and the *possible intent* $S^\Diamond$ for $S \subseteq G$ are defined analogously:
$S^\Box := \{m \in M \mid I(g, m) = \times \text{ for all } g \in S\}$,
$S^\Diamond := \{m \in M \mid I(g, m) \neq o \text{ for all } g \in S\}$.
For $g \in G$ we use the abbreviation $g^\Box := \{g\}^\Box$ and $g^\Diamond := \{g\}^\Diamond$.
If $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ is complete (or a formal context $\mathbb{K} = (G, M, I)$) then we have $B^\Box = B^\Diamond =: B' =: B^I$ and $S^\Box = S^\Diamond =: B' =: B^I$, and they are just the extent and intent defined in [GW99]. The operators $' : \mathcal{P}(G) \to \mathcal{P}(M)$ and $' : \mathcal{P}(M) \to \mathcal{P}(G)$ (where $\mathcal{P}(G)$ is the powerset of G) are called *derivations.* A *(formal) concept* of a formal context $\mathbb{K} = (G, M, I)$ is a pair (S, B) with $S \subseteq G$ and $B \subseteq M$ such that $S' = B$ and $B' = S$. The set S is called *extent* of the concept, and the set B is called *intent* of the concept. Let $Ext(\mathbb{K})$ be the set of all extents of $\mathbb{K}$ and $Int(\mathbb{K})$ be the set of all intents of $\mathbb{K}$.

In the following let the set M of attributes be finite. To descibe dependencies between the columns of a (formal or incomplete) context we use propositional formulas over the attribute set M. Let $F(M)$ be the set of all propositional formulas where M is the set of propositional variables. Let $\alpha \in F(M)$ and $B \subseteq M$. The set B is a *model* of α (or B *respects* α) if the interpretation of α is true for the valuation $v_B : M \to \{true, false\}$ with $v_B(m) = true$ iff $m \in B$. For $P \subseteq F(M)$ and $\mathcal{R} \subseteq \mathcal{P}(M)$ define
$Th(\mathcal{R}) = \{\alpha \in F(M) \mid \text{every } B \in \mathcal{R} \text{ is a model of } \alpha\}$,
$Resp(P) = \{B \subseteq M \mid B \text{ is a model of each } \alpha \in P\}$,
$Resp^{\mathcal{R}}(P) = Resp(P) \cap \mathcal{R}$.
For $A \subseteq M$ define $< A >_P = \bigcap\{B \in Resp(P) \mid A \subseteq B\}$.
For $A = \{a_1, a_2, \ldots a_m\} \subseteq M$ and $B = \{b_1, b_2, \ldots b_n\} \subseteq M$ we use the abbreviation $A \to B := \bigwedge A \to \bigwedge B := (a_1 \wedge a_2 \wedge \ldots \wedge a_m) \to (b_1 \wedge b_2 \wedge \ldots \wedge b_n)$. This formula is called *attribute implication.*

The formula $A \to \bigvee B := (a_1 \wedge a_2 \wedge \ldots \wedge a_m) \to (b_1 \vee b_2 \vee \ldots \vee b_n)$ is called *clause*.
Let $Imp_M := \{A \to B \mid A, B \subseteq M\}$.
For $\mathcal{R} \subseteq \mathcal{P}(M)$ define $Imp(\mathcal{R}) := Th(\mathcal{R}) \cap Imp_M$.
Note that a set $C \subseteq M$ respects the attribute implication $A \to B$ iff $A \subseteq C$ implies $B \subseteq C$. Let $\mathbb{K} = (G, M, I)$ be a formal context. A formula $\alpha \in F(M)$ is *valid* for an object $g \in G$ if g' is a model of α. The formula α is valid in $\mathbb{K}$ if it is valid for each object $g \in G$. In particular an attribute implication $A \to B$ is valid in $\mathbb{K}$ iff every object $g \in G$ which has all attributes of A also has all attributes of B. Analogously, a clause $A \to \bigvee B = (a_1 \wedge a_2 \wedge \ldots \wedge a_m) \to (b_1 \vee b_2 \vee \ldots \vee b_n)$ is valid in $\mathbb{K}$ iff every object which has all attributes of A has at least one attribute of B.

For incomplete contexts there exist many different logics to evaluate formulas, for example the Kleene-Logic or other many-valued logics. Here we use the *Kripke-semantics*: A formula is *Kripke-valid* (or *certainly valid*) in an incomplete context if it is valid in every completion. A formula α is *satisfyable* (or *possibly valid*) in an incomplete context if it is valid in at least one completion. Note that for complete contexts these two forms of validity are equal. The Kripke-valid attribute implications of an incomplete context $\mathbb{K}$ are denoted by $Imp(\mathbb{K}) := \{A \to B \in Imp_M \mid A \to B$ is Kripke-valid in $\mathbb{K}\}$ and, and the satisfyable attribute implications of $\mathbb{K}$ are denoted by $Sat(\mathbb{K}) := \{A \to B \in Imp_M \mid A \to B$ is satisfyable in $\mathbb{K}\}$.

Sets of formulas can also be described by frame contexts: A system $\mathcal{R} \subseteq \mathcal{P}(M)$ of sets is called *frame context* of a formal context $\mathbb{K} = (G, M, I)$ if $g' \in \mathcal{R}$ for all $g \in G$. So a frame context of a formal context $\mathbb{K}$ describes a set of subsets of M containing all object intents of $\mathbb{K}$.[5] If P is a set of formulas and $\mathcal{R}$ is the set of all models of P, then P is valid in $\mathbb{K}$ iff $\mathcal{R}$ is a frame context of $\mathbb{K}$. If $\mathbb{K}$ is an unknown formal context,[6] then frame contexts can be used as background knowledge: For a given system $\mathcal{R}$ of sets we are only interested in those contexts $\mathbb{K}$, such that $\mathcal{R}$ is a frame context of $\mathbb{K}$, that means only the sets in $\mathcal{R}$ are allowed for object intents. For example in the context of all natural numbers $\mathbb{K} = (\mathbb{N}, M, I)$ with some attributes (even, odd, prime, square, ...) we do not see directly which object intents occur, but some restrictions can be seen directly from the attributes: even numbers are not odd, prime numbers are not square numbers, etc. So to analyse the context $\mathbb{K}$ we can use a frame context like $\mathcal{R} = \{T \subseteq M \mid \text{ even} \in T \text{ iff odd} \notin T\}$. This frame context restricts the possible object intents, so computer algorithms (see next section) can be more efficient by using this background knowledge $\mathcal{R}$.

Lemma 1 *For a formal context $\mathbb{K} = (G, M, I)$ the mappings $' : \mathcal{P}(M) \to \mathcal{P}(G)$ and $' : \mathcal{P}(G) \to \mathcal{P}(M)$ form a Galois connection.*
For an incomplete context $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ the mappings $\Box : \mathcal{P}(M) \to \mathcal{P}(G)$ and $\Box : \mathcal{P}(G) \to \mathcal{P}(M)$ form a Galois connection.
For an incomplete context $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ the mappings $\Diamond : \mathcal{P}(M) \to \mathcal{P}(G)$

[5] One may call it "a set of possible object intents for $\mathbb{K}$".

[6] for example if the set of objects is very large, so that the expert does not know which object intents occur in the context

and $\diamond : \mathcal{P}(G) \to \mathcal{P}(M)$ *form a Galois connection.*
The mappings $Th : \mathcal{P}(\mathcal{P}(M)) \to \mathcal{P}(F(M))$ *and* $Resp : \mathcal{P}(F(M)) \to \mathcal{P}(\mathcal{P}(M))$ *form a Galois connection.*
The mappings $Imp : \mathcal{P}(\mathcal{P}(M)) \to \mathcal{P}(Imp_M)$ *and* $Resp|_{\mathcal{P}(Imp_M)} : \mathcal{P}(Imp_M) \to \mathcal{P}(\mathcal{P}(M))$ *form a Galois connection.*
For $\mathcal{R} \subseteq \mathcal{P}(M)$ *the mappings* $Imp|_{\mathcal{P}(\mathcal{R})} : \mathcal{P}(\mathcal{R}) \to \mathcal{P}(Imp_M)$ *and* $Resp^{\mathcal{R}}|_{\mathcal{P}(Imp_M)} : \mathcal{P}(Imp_M) \to \mathcal{P}(\mathcal{R})$ *form a Galois connection.*

Proof. For the first statement see [GW99]. The mappings $\Box : \mathcal{P}(M) \to \mathcal{P}(G)$ and $\Box : \mathcal{P}(G) \to \mathcal{P}(M)$ are the derivations in the formal context $(G, M, I_\Box)$ with $(g, m) \in I_\Box$ iff $I(g, m) = \times$.
The mappings $\diamond : \mathcal{P}(M) \to \mathcal{P}(G)$ and $\diamond : \mathcal{P}(G) \to \mathcal{P}(M)$ are the derivations in the formal context $(G, M, I_\diamond)$ with $(g, m) \in I_\diamond$ iff $I(g, m) \neq o$.
The mappings Th and $Resp$ are the derivations in the formal context $(\mathcal{P}(M), F(M), \models)$ with $A \models \alpha$ iff A is a model of α.
The mappings Imp and $Resp|_{\mathcal{P}(Imp_M)}$ are the derivations in the formal context $(\mathcal{P}(M), Imp_M, \models)$. The mappings $Imp|_{\mathcal{P}(\mathcal{R})}$ and $Resp^{\mathcal{R}}|_{\mathcal{P}(Imp_M)}$ are the derivations in the formal context $(\mathcal{R}, Imp_M, \models)$. ■

Lemma 2 [7] *For* $P \subseteq Imp_M$ *the set* $Resp(P)$ *is a closure system on* M *and the mapping* $< \cdot >_P : \mathcal{P}(M) \to \mathcal{P}(M)$ *is the corresponding closure operator.*

Lemma 3 [8] *If* $\mathcal{R}$ *is a closure system on* M *then* $\mathcal{R} = Resp(Imp(\mathcal{R}))$.

Proof. We have $\mathcal{R} \subseteq Resp(Imp(\mathcal{R}))$ because of Lemma 1. Let $h : \mathcal{P}(M) \to \mathcal{P}(M)$ be the closure operator corresponding to $\mathcal{R}$ and let $A \in Resp(Imp(\mathcal{R}))$. Then $A \to h(A) \in Imp(\mathcal{R})$, because for every $B \in \mathcal{R}$ with $A \subseteq B$ we get $h(A) \subseteq B$. Therefore we get $h(A) = A$ because of $A \in Resp(Imp(\mathcal{R}))$. So we get $A \in \mathcal{R}$ and $\mathcal{R} = Resp(Imp(\mathcal{R}))$. ■

The following lemma gives a characterisation of the valid implications of a formal context:

Lemma 4 [9] *Let* $\mathbb{K} = (G, M, I)$ *be a formal context and* $A, B \subseteq M$. *The following conditions are equivalent:*

1. $A \to B \in Imp(\mathbb{K})$

[7] see [G98] and [GW99]
[8] see [G98]
[9] see [GW99]

2. $A \to B \in Imp(\{g' \mid g \in G\})$
3. $A \to B \in Imp(Int(\mathbb{K}))$
4. $B \subseteq A''$
5. $A' \subseteq B'$
6. *For all* $g \in G$ *the following property holds: If* $(g, a) \in I$ *for all* $a \in A$ *then* $(g, b) \in I$ *for all* $b \in B$.

Now this Lemma will be generalized to the Kripke-validity and satisfyability of implications in incomplete contexts:

Lemma 5 *Let* $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ *be an incomplete context and* $A, B \subseteq M$. *The following conditions are equivalent:*

1. $A \to B \in Imp(\mathbb{K})$
2. $B \setminus A \subseteq A^{\Diamond\Box}$
3. $A^{\Diamond} \subseteq (B \setminus A)^{\Box}$
4. *For all* $g \in G$ *with* $A \subseteq g^{\Diamond}$ *we have* $B \setminus A \subseteq g^{\Box}$
5. *For all* $g \in G$ *the following property holds: If* $I(g, a) \neq o$ *for all* $a \in A$ *then* $I(g, b) = \times$ *for all* $b \in B \setminus A$.

Proof.
$3 \Leftrightarrow 5 \Leftrightarrow 4$: Trivial.
$1 \Rightarrow 5$:[10]
Let $A \to B \in Imp(\mathbb{K})$ and $g \in G$ with $I(g, a) \neq o$ for all $a \in A$. Let $\mathbb{K}' = (G, M, J)$ be the completion of $\mathbb{K}$ where the questionmarks in each column of $m \in A$ are replaced by $\times$, and the questionmarks in each column of $m \in M \setminus A$ are replaced by o. Then we get $(g, a) \in J$ for all $a \in A$. We have $A \to B \in Imp(\mathbb{K}')$, so $(g, b) \in J$ for all $b \in B$, and we get $I(g, m) = \times$ for all $m \in B \setminus A$ because of the definition of $\mathbb{K}'$.
$5 \Rightarrow 1$:[11]
Let $\mathbb{K}' = (G, M, J)$ be a completion of $\mathbb{K}$. We show condition 6 of Lemma 4: Let $g \in G$ with $(g, a) \in J$ for all $a \in A$. Then $I(g, a) \neq o$ for all $a \in A$, so by condition 5 we get $I(g, b) = \times$ for all $b \in B \setminus A$, so $(g, b) \in J$ for all $b \in B$. Therefore $A \to B \in Imp(\mathbb{K}')$, and we get $A \to B \in Imp(\mathbb{K})$.
$2 \Rightarrow 4$:
Let $g \in G$ with $A \subseteq g^{\Diamond}$, then we get $g \in A^{\Diamond}$ and with condition 2 and Lemma 1 we get $B \setminus A \subseteq A^{\Diamond\Box} \subseteq g^{\Box}$.
$3 \Rightarrow 2$:
With Lemma 1 and $A^{\Diamond} \subseteq (B \setminus A)^{\Box}$ we get $B \setminus A \subseteq (B \setminus A)^{\Box\Box} \subseteq A^{\Diamond\Box}$. ■

[10] See [B91a]
[11] See [B91a]

Lemma 6 *Let $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ be an incomplete context and $A, B \subseteq M$. The following conditions are equivalent:*

1. $A \to B \in Sat(\mathbb{K})$
2. $B \subseteq A^{\Box\Diamond}$
3. $A^{\Box} \subseteq B^{\Diamond}$
4. *For all $g \in G$ with $A \subseteq g^{\Box}$ we have $B \subseteq g^{\Diamond}$*
5. *For all $g \in G$ the following property holds: If $I(g, a) = \times$ for all $a \in A$ then $I(g, b) \neq o$ for all $b \in B$.*

Proof. The Proof works analogously to the proof of Lemma 5. ■

The attributes which are possibly implied by a set A of attributes can be computed with the derivation operators:

Corollary 7 *Let $\mathbb{K}$ be an incomplete context and $A \subseteq M$.*
Then $A^{\Box\Diamond} = \{m \in M \mid A \to m \in Sat(\mathbb{K})\}$.

If we have a set P of attribute implications which are valid in a formal context, then we can use a rule system to derive more implications from P, which are also valid in the context. Now some rules are defined to compute all consequences of a set of implications for complete and incomplete contexts:

Definition 8 *Let $\mathcal{R} \subseteq \mathcal{P}(M)$. The rules (AX), (PS), (AU), (PR), (AD), ($\mathcal{R}$-EX) are defined by*[12]

$$\frac{}{A \cup B \to A}\,(AX) \qquad \frac{A \to B \qquad B \cup C \to D}{A \cup C \to D}\,(PS)$$

$$\frac{A \to C}{A \cup B \to C}\,(AU) \qquad \frac{A \to B \cup C}{A \to B}\,(PR) \qquad \frac{A \to B \qquad A \to C}{A \to B \cup C}\,(AD)$$

$$\frac{(A \cup \{m\} \to B)_{m \in E}}{A \to B}\,(\mathcal{R}\text{-}EX) \quad \text{for } A \to \bigvee E \in Th(\mathcal{R})$$

for $A, B, C, D, E \subseteq M$. For $P \subseteq Imp_M$ let $\mathbf{Cons}(P)$ denote the smallest subset of Imp_M with $P \subseteq \mathbf{Cons}(P)$ which is closed with respect to the rules (AX) and (PS).

[12] see also [B91a], [M83], [G98]

Let $\mathbf{Cons}_{Sat}(P)$ denote the smallest subset of Imp_M with $P \subseteq \mathbf{Cons}_{Sat}(P)$ which is closed with respect to the rules (AX), (AU), (PR) and (AD). Let $\mathbf{Cons}^{\mathcal{R}}(P)$ be the smallest subset of Imp_M with $P \subseteq \mathbf{Cons}^{\mathcal{R}}(P)$ which is closed with respect to the rules (AX), (PS) and ($\mathcal{R}$-EX).

The exhaustion rule ($\mathcal{R}$-EX) can only be applied for clauses $A \to \bigvee E$ which are respected by te frame context $\mathcal{R}$.

Lemma 9 *For $\mathcal{R} \subseteq \mathcal{P}(M)$ and $P \subseteq Imp_M$ the sets $\mathbf{Cons}(P)$ and $\mathbf{Cons}^{\mathcal{R}}(P)$ are closed with respect to (AU), (PR) and (AD).*

Proof. The rules (AU), (PR) and (AD) are derivable from (AX) and (PS):

Proof of (AU):
$$\frac{\dfrac{}{A \cup B \to A}\,(AX) \qquad A \to C}{A \cup B \to C}\,(PS)$$

Proof of (PR):
$$\frac{A \to B \cup C \qquad \dfrac{}{B \cup C \to B}\,(AX)}{A \to B}\,(PS)$$

Proof of (AD):
$$\frac{A \to B \qquad \dfrac{A \to C \qquad \dfrac{}{C \cup B \to B \cup C}\,(AX)}{B \cup A \to B \cup C}\,(PS)}{A \cup A \to B \cup C}\,(PS)$$

■

The rules (AX) and (PS) are sound and complete for formal contexts, that means an implication $A \to B$ is derivable from a set of implications P iff for every formal context (with attribute set M) in which P is valid, the implication $A \to B$ is also valid:

Theorem 10 *Let $A, B \subseteq M$ and $P \subseteq Imp_M$. The following conditions are equivalent:*

1. $A \to B \in \mathbf{Cons}(P)$
2. $A \to B \in Imp(Resp(P))$
3. *For each formal context $\mathbb{K}$ with $P \subseteq Imp(\mathbb{K})$ we get $A \to B \in Imp(\mathbb{K})$.*
4. *For each incomplete context $\mathbb{K}$ with $P \subseteq Imp(\mathbb{K})$ we get $A \to B \in Imp(\mathbb{K})$.*
5. $B \subseteq < A >_P$

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 5$: See [G98] and [GW99].
$4 \Rightarrow 3$: Trivial.
$3 \Rightarrow 4$: Let $\mathbb{K}$ be an incomplete context with $P \subseteq Imp(\mathbb{K})$ and $\mathbb{K}' \in Compl(\mathbb{K})$, then with condition 3 we get $A \to B \in Imp(\mathbb{K}')$, so $A \to B \in Imp(\mathbb{K})$. ■

So for the Kripke-validity in incomplete contexts the rules (AX) and (PS) are also sound and complete: An implication $A \to B$ is derivable from a set of implications P iff for every incomplete context in which P is Kripke-valid the implication $A \to B$ is also Kripke-valid. In literature there are many other adequate rule systems, for example the Armstrong rules.[13]

For the satisfyability however the implications need not to be transitive:

$\mathbb{K}$	a	b	c
1	$\times$	?	o

In this context $a \to b$ is satisfyable, and $b \to c$ is also satisfyable, but $a \to c$ is not satisfyable. So the rule (PS) is not sound for satisfyability. This example also shows that the operator $\Box\Diamond : \mathcal{P}(M) \to \mathcal{P}(M)$ is not idempotent:

$$a^{\Box\Diamond} = \{a, b\}$$
$$\{a, b\}^{\Box\Diamond} = \{a, b, c\}$$

There are also adequate rule systems for satisfyability. The rules (AX), (PR), (AU) and (AD) are sound and complete for the satisfyability of implications in incomplete contexts:

Theorem 11 [14] *Let $A, B \subseteq M$ and $P \subseteq Imp_M$. The following conditions are equivalent:*

1. $A \to B \in \mathbf{Cons}_{Sat}(P)$
2. *For each incomplete context $\mathbb{K}$ with $P \subseteq Sat(\mathbb{K})$ we get $A \to B \in Sat(\mathbb{K})$.*

If we have background knowledge in form of a frame context $\mathcal{R} \subseteq \mathcal{P}(M)$, and ask "Which implications follow from P, if we only consider contexts with frame context $\mathcal{R}$?" then we need the exhaustion rule ($\mathcal{R}$-EX):

Theorem 12 [15] *Let $\mathcal{R} \subseteq \mathcal{P}(M)$, $A, B \subseteq M$ and $P \subseteq Imp_M$. The following conditions are equivalent:*

1. $A \to B \in \mathbf{Cons}^{\mathcal{R}}(P)$
2. $A \to B \in Imp(Resp^{\mathcal{R}}(P))$
3. *For each formal context $\mathbb{K} = (G, M, I)$ with $P \subseteq Imp(\mathbb{K})$ and $g' \in \mathcal{R}$ for all $g \in G$ we get $A \to B \in Imp(\mathbb{K})$.*

[13] see [G98], [G99], [M83]
[14] see [L82] and [AM84]
[15] see [G98] and [G99]

Therefore we have a sound and complete system of rules: An implication $A \to B$ is derivable from a set P of implications with the rules (AX), (PS), ($\mathcal{R}$-EX) iff for every formal context $\mathbb{K}$ with frame context $\mathcal{R}$ in which P is valid the implication $A \to B$ is also valid.

Corollary 13
$\mathbf{Cons}(Imp(\mathbb{K})) = Imp(\mathbb{K})$ *and* $\mathbf{Cons}_{Sat}(Sat(\mathbb{K})) = Sat(\mathbb{K})$ *for every incomplete context* $\mathbb{K}$.
$\mathbf{Cons}^{\mathcal{R}}(Imp(\mathbb{K})) = Imp(\mathbb{K})$ *for every complete context* $\mathbb{K} = (G, M, I)$ *with* $g' \in \mathcal{R}$ *for all* $g \in G$.

Definition 14 *Let* $\mathcal{R} \subseteq \mathcal{P}(M)$ *and* $P \subseteq Imp_M$. *A set* $A \subseteq M$ *is called* P-intent *(with respect to* $\mathcal{R}$*) if* $A \in Resp(\mathbf{Cons}^{\mathcal{R}}(P))$. *A set* $A \subseteq M$ *is called* P-pseudoclosed *(with respect to* $\mathcal{R}$*) if the following three conditions are satisfied:*[16]

1. $A \in \mathcal{R}$
2. *A is not a P-intent with respect to $\mathcal{R}$*
3. $\{m \in M \mid B \to m \in \mathbf{Cons}_{Sat}(P)\} \subseteq A$ *for every proper subset* $B \subset A$ *which is pseudoclosed with respect to* $\mathcal{R}$

Let $P, Q \subseteq Imp_M$. *The set* P *is called* base *of* Q *with respect to* $\mathcal{R}$ *if* $\mathbf{Cons}^{\mathcal{R}}(P) = \mathbf{Cons}^{\mathcal{R}}(Q)$ *holds and* $\mathbf{Cons}^{\mathcal{R}}(T) \neq \mathbf{Cons}^{\mathcal{R}}(Q)$ *for every proper subset* $T \subset P$. *Define* $DGB^{\mathcal{R}}(P) := \{A \to \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\} \mid A$ *is P-pseudoclosed with respect to* $\mathcal{R}\}$.

In Theorem 18 it will be shown that $DGB^{\mathcal{R}}(P)$ is a base of P: It generates P and it is irredundant. First we need a characterisation of P-intents which are in the frame context $\mathcal{R}$:

Lemma 15 *Let* $\mathcal{R} \subseteq \mathcal{P}(M)$, $A \subseteq M$ *and* $P \subseteq Imp_M$. *If* $A \in \mathcal{R}$ *then the following conditions are equivalent:*

1. *A is a P-intent with respect to $\mathcal{R}$*
2. $A \in Resp(P)$
3. $A = \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}$

Proof.
$1 \Leftrightarrow 2$:

[16] Here we can use an inductive definition, because M ist finite.

With Lemma 1 and Theorem 12 we get
$\mathcal{R} \cap Resp(P) = Resp^{\mathcal{R}}(P) = Resp^{\mathcal{R}}(Imp(Resp^{\mathcal{R}}(P))) = Resp^{\mathcal{R}}(\mathbf{Cons}^{\mathcal{R}}(P))$.
$1 \Rightarrow 3$:
With rule (AX) we have $A \subseteq \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}$. With $A \in Resp(\mathbf{Cons}^{\mathcal{R}}(P))$ and Lemma 9 we get $A \in Resp(\mathbf{Cons}_{Sat}(P))$, so $A = \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}$.
$3 \Rightarrow 2$:
Let $C \to D \in P$ with $C \subseteq A$. Then we get $A \to D \in \mathbf{Cons}_{Sat}(P)$ with rule (AU), and $A \to m \in \mathbf{Cons}_{Sat}(P)$ for $m \in D$ with rule (PR). We get $D \subseteq \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}$, and with condition 3 we get $D \subseteq A$, so $A \in Resp(P)$. ■

For incomplete contexts the $Sat(\mathbb{K})$-intents which are in the frame context $\mathcal{R}$ and the $Sat(\mathbb{K})$-pseudoclosed sets can be expressed with the derivations $\Box$ and $\Diamond$:

Corollary 16 *Let $\mathcal{R} \subseteq \mathcal{P}(M)$, $A \in \mathcal{R}$ and $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$. The set A is a $Sat(\mathbb{K})$-intent with respect to $\mathcal{R}$ iff $A = A^{\Box\Diamond}$.*

Proof. With Lemma 7 and Corollary 13 we have
$A^{\Box\Diamond} = \{m \in M \mid A \to m \in Sat(\mathbb{K})\} = \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(Sat((\mathbb{K})))\}$. ■

Corollary 17 *Let $\mathcal{R} \subseteq \mathcal{P}(M)$ and $\mathbb{K}$ be an incomplete context. A set $A \subseteq M$ is $Sat(\mathbb{K})$-pseudoclosed with respect to $\mathcal{R}$ iff the following two conditions are satisifed:*

1. *$A^{\Box\Diamond} \neq A \in \mathcal{R}$*
2. *$B^{\Box\Diamond} \subseteq A$ for all proper $Sat(\mathbb{K})$-pseudoclosed proper subsets $B \subset A$.*

We have $DGB^{\mathcal{R}}(Sat(\mathbb{K})) = \{A \to A^{\Box\Diamond} \mid A$ is $Sat(\mathbb{K})$-pseudoclosed with respect to $\mathcal{R}\}$.

Proof. For $A \in \mathcal{R}$ we have $A \neq A^{\Box\Diamond}$ iff A is no $Sat(\mathbb{K})$-intent. For $B \subset A$ we have $B^{\Box\Diamond} \subseteq A$ iff $\{m \in M \mid B \to m \in \mathbf{Cons}_{Sat}(Sat(\mathbb{K}))\} \subseteq A$. ■

Theorem 18 *Let $\mathcal{R} \subseteq \mathcal{P}(M)$ and $P \subseteq Imp_M$. Then $DGB^{\mathcal{R}}(P)$ is a base of P with respect to $\mathcal{R}$. This base is called* Duquenne-Gigue-Base.[17]

[17] In [GW99] the Duquenne-Gigue-Base was defined only for complete contexts without frame contexts. In [G98] also frame contexts are considered, but the set of implications contructed in [G98] is different to the Duquenne-Gigue-Base of this paper: it is not a base but only a generating set of implications. In some cases the generating set of [G98] contains less implications than the Duquenne-Gigue-Base. If the frame context $\mathcal{R}$ is a closure system, both sets are the same.

Proof. First we prove $\mathbf{Cons}^{\mathcal{R}}(DGB^{\mathcal{R}}(P)) = \mathbf{Cons}^{\mathcal{R}}(P)$:
Each implication $A \to \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\} \in DGB^{\mathcal{R}}(P)$ is an element of $\mathbf{Cons}^{\mathcal{R}}(P)$ because of Lemma 9 (with rule (AD)), so we get $\mathbf{Cons}^{\mathcal{R}}(DGB^{\mathcal{R}}(P)) \subseteq \mathbf{Cons}^{\mathcal{R}}(P)$. Now we show $Resp^{\mathcal{R}}(DGB^{\mathcal{R}}(P)) \subseteq Resp^{\mathcal{R}}(P)$.
Let $A \in Resp^{\mathcal{R}}(DGB^{\mathcal{R}}(P))$, then we have $A \in \mathcal{R}$. For each P-pseudoclosed proper subset B of A we get $\{m \in M \mid B \to m \in \mathbf{Cons}_{Sat}(P)\} \subseteq A$ because A is a model of the implication $B \to \{m \in M \mid B \to m \in \mathbf{Cons}_{Sat}(P)\} \in DGB^{\mathcal{R}}(P)$. If A is not a P-intent, then A is P-pseudoclosed and $A \neq \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}$ which is a contradiction because then A must be a model of $A \to \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\} \in DGB^{\mathcal{R}}(P)$. So A must be a P-intent: $A \in Resp^{\mathcal{R}}(P)$. Therefore $Resp^{\mathcal{R}}(DGB^{\mathcal{R}}(P)) \subseteq Resp^{\mathcal{R}}(P)$ and with Theorem 12 we get
$\mathbf{Cons}^{\mathcal{R}}(P) = Imp(Resp^{\mathcal{R}}(P)) \subseteq Imp(Resp^{\mathcal{R}}(DGB^{\mathcal{R}}(P))) = \mathbf{Cons}^{\mathcal{R}}(DGB^{\mathcal{R}}(P))$.
Now let $A \subseteq M$ be a P-pseudoclosed set and $Q := DGB^{\mathcal{R}}(P) \setminus \{A \to \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}\}$. For each P-pseudoclosed proper subset $B \subset A$ we have $\{m \in M \mid B \to m \in \mathbf{Cons}_{Sat}(P)\} \subseteq A$, so we get $A \in Resp^{\mathcal{R}}(Q)$. With Lemma 15 we have $A \neq \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\}$, therefore we get $A \to \{m \in M \mid A \to m \in \mathbf{Cons}_{Sat}(P)\} \notin Imp(Resp^{\mathcal{R}}(Q))$ because A is no model of this implication. We get $\mathbf{Cons}^{\mathcal{R}}(DGB(P)) \neq Imp(Resp^{\mathcal{R}}(Q)) = \mathbf{Cons}^{\mathcal{R}}(Q)$. Therefore $DGB^{\mathcal{R}}(P)$ is a base. ■

If we have some background knowledge about an incomplete context $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$, then we can replace a questionmark $I(g, m) =?$ by $\times$ if it follows from the background knowledge that the object $g \in G$ must have the attribute $m \in M$ and we can replace a questionmark $I(g, m) =?$ by o if it follows from the background knowledge that the object $g \in G$ does not have the attribute $m \in M$. With this method we can reduce the questionmarks in $\mathbb{K}$. This questionmark reduction is described in the following definition:

Definition 19 *For any $\mathcal{R} \subseteq \mathcal{P}(M)$, $P \subseteq Imp_M$ and an incomplete context $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ define the rules (Red1) and (Red2) for $g \in G$ and $m \in M$ as follows:*

(Red1) If $m \in T$ for all $T \in Resp^{\mathcal{R}}(P)$ with $g^{\Box} \subseteq T \subseteq g^{\Diamond}$ then the value $I(g, m)$ is replaced by $\times$.

(Red2) If $m \notin T$ for all $T \in Resp^{\mathcal{R}}(P)$ with $g^{\Box} \subseteq T \subseteq g^{\Diamond}$ then the value $I(g, m)$ is replaced by o.

The context $\mathbb{K}$ is called P-reduced *(with respect to $\mathcal{R}$) if for all $g \in G$ and $m \in M$ the applications of (Red1) and (Red2) do not change $\mathbb{K}$.*

Theorem 20 *Let $\mathcal{R} \subseteq \mathcal{P}(M)$, let $\mathbb{K} = (G, M, \{\times, ?, o\}, I)$ be an incomplete context with $M \neq \emptyset$ and let $P \subseteq Imp_M$. The following conditions are equivalent:*

1. *There exists a smallest context (in the information order) $\mathbb{K}' \geq \mathbb{K}$ which is P-reduced with respect to $\mathcal{R}$.*

2. *There exists a context $\mathbb{K}' \geq \mathbb{K}$ which is P-reduced with respect to $\mathcal{R}$.*

3. *For every $g \in G$ there exists a set $T_g \in Resp^{\mathcal{R}}(P)$ with $g^{\Box} \subseteq T_g \subseteq g^{\Diamond}$, where the operators $\Box$ and $\Diamond$ refer to $\mathbb{K}$.*

4. *There exists a completion $\mathbb{K}' = (G, M, J) \in Compl(\mathbb{K})$ with $P \subseteq Imp(\mathbb{K}')$ and $g^J \in \mathcal{R}$ for all $g \in G$.*

5. *By applying the rules (Red1) and (Red2) on $\mathbb{K}$ finitely many times, the values $\times$ and o are not replaced by other values.*

Proof.
$1 \Rightarrow 2$: Trivial.
$2 \Rightarrow 3$:
Assume condition 2. Let $g \in G$ and $m \in M$. The context $\mathbb{K}' = (G, M, \{\times, ?, o\}, J)$ of condition 2 is P-reduced, so there exists a set $T_g \in Resp^{\mathcal{R}}(P)$ such that $g^{\Box} \subseteq T_g \subseteq g^{\Diamond}$ holds in $\mathbb{K}'$ because otherwise we would get $\times = J(g, m) = o$ with the rules (Red1) and (Red2). Because of $\mathbb{K} \leq \mathbb{K}'$ the inclusions $g^{\Box} \subseteq T_g \subseteq g^{\Diamond}$ also hold in $\mathbb{K}$.
$3 \Rightarrow 4$:
For $g \in G$ let $T_g \in Resp^{\mathcal{R}}(P)$ with $g^{\Box} \subseteq T_g \subseteq g^{\Diamond}$ in $\mathbb{K}$. Then we can define a completion $\mathbb{K}' = (G, M, J) \in Compl(\mathbb{K})$ by $g^J := T_g$ for $g \in G$. We have $P \subseteq Imp(\mathbb{K}')$ and $g^J \in \mathcal{R}$ for $g \in G$ because of $T_g \in Resp(P) \cap \mathcal{R}$.
$4 \Rightarrow 5$:
For $g \in G$ we have $g^J \in Resp^{\mathcal{R}}(P)$ and $g^{\Box} \subseteq g^J \subseteq g^{\Diamond}$ in $\mathbb{K}$. After each application of the rules (Red1) and (Red2) the condition $g^{\Box} \subseteq g^J \subseteq g^{\Diamond}$ is still satisfied because $I(g, m)$ can not be replaced by o for $m \in g^J$, and $I(g, m)$ can not be replaced by $\times$ for $m \in M \setminus g^J$. Therefore the values $\times$ and o are not replaced by other values.
$5 \Rightarrow 1$:
For each object $g \in G$ the rules (Red1) and (Red2) are applied until no more questionmarkes can be replaced by other values. Because of condition 5 there are no circles of replacements. Because of the finiteness of M this process terminates for each fixed object $g \in G$ after finitely many steps. So we get a P-reduced context $\mathbb{K}' \geq \mathbb{K}$. Let $\mathbb{K}''$ be another P-reduced context with $\mathbb{K}'' \geq \mathbb{K}$. Then each context after applying the rules (Red1) and (Red2) on $\mathbb{K}$ finitely many times is bounded by $\mathbb{K}''$, so we get $\mathbb{K}' \leq \mathbb{K}''$. ■

The smallest P-reduced context $\mathbb{K}' \geq \mathbb{K}$ is denoted by $Red_P^{\mathcal{R}}(\mathbb{K})$ if it exists. For each object $g \in G$ the smallest P-reduced context row, which is greater than or equal to $I(g, \cdot)$ is denoted by $Red_P^{\mathcal{R}}(g)$.

Corollary 21 *If* $Red_P^{\mathcal{R}}(\mathbb{K})$ *exists then* $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$.

Proof. Let $\mathbb{K}'$ be the context of condition 4 of Theorem 20. Then we have $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Imp(\mathbb{K}') \subseteq Sat(\mathbb{K})$. ■

In section 2 of part II it will be shown that $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$ does not imply the existence of $Red_P^{\mathcal{R}}(\mathbb{K})$.[18] But for closure systems we have the following result:

Lemma 22 *If* $\mathcal{R}$ *is a closure system then* $Red_P^{\mathcal{R}}(\mathbb{K})$ *exists iff* $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$.

Proof. If $Red_P^{\mathcal{R}}(\mathbb{K})$ exists then $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$ by Corollary 21. Now assume $\mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$ and let $g \in G$. For $H := Imp(\mathcal{R})$ we have $\mathcal{R} = Resp(H)$ by Lemma 3. For $T_g :=< g^{\square} >_{P\cup H}$ we get $T_g \in Resp(P \cup H) = Resp(P) \cap Resp(H) = Resp^{\mathcal{R}}(P)$ by Lemma 2, and with Theorem 10 and Theorem 12 we get $g^{\square} \to < g^{\square} >_{P\cup H} \in \mathbf{Cons}(P \cup H) = Imp(Resp(P \cup H)) = Imp(Resp^{\mathcal{R}}(P)) = \mathbf{Cons}^{\mathcal{R}}(P) \subseteq Sat(\mathbb{K})$. Therefore $g^{\square} \subseteq T_g \subseteq g^{\diamond}$ follows from Lemma 6. So $Red_P^{\mathcal{R}}(\mathbb{K})$ exists because of Theorem 20. ■

2 Attribute exploration

Let $\mathbb{K}^U = (G^U, M, I^U)$ be a formal context (with a finite set M of attributes), called universe, which is too large to be fully represented; in what follows we also call such contexts "unknown contexts". The attribute exploration is an interactive (computer) algorithm which helps the expert to get maximal information about the attribute implications valid in the universe $\mathbb{K}^U$. Moreover a list of counterexamples against non-valid attribute implications is produced.[19] The exploration program asks for the validity of some implications $A \to B$ and the expert must search for an answer to these questions. The following algorithm describes the exploration:[20]

Attribute exploration algorithm:

(E1) At the beginning of the exploration algorithm the user can enter some background knowledge about the universe: He enters the (finite) attribute set M, a frame context $\mathcal{R} \subseteq \mathcal{P}(M)$ of $\mathbb{K}^U$ (with $g' \in \mathcal{R}$ for all $g \in G^U$) and an incomplete context $\mathbb{K}_0 = (G_0, M, \{\times, ?, o\}, I_0)$ containing some objects $G_0 \subseteq G^U$ of the universe $\mathbb{K}^U$, such that a completion of $\mathbb{K}_0$ is a subcontext of $\mathbb{K}^U$. So $\mathbb{K}_0$ may contain questionmarks if the user does not know, which attributes the objects have.

[18] See Example 5.
[19] See Remark 20 in part II.
[20] see section 2 of part II for some examples

(E2) The set of accepted implications is initialised with the empty set: $P_1 = \emptyset$. Then $\mathbb{K}_1 := (G_1, M, \{\times, ?, o\}, I_1) := Red^{\mathcal{R}}_{P_1}(\mathbb{K}_0)$ exists because of Theorem 20 (equivalence 1. $\Leftrightarrow$ 4.). Let $j := 1$.

(E3) In the $j-th$ step of the algorithm the program chooses a *minimal set* $A \subseteq M$ *which is* $Sat(\mathbb{K}_j)$*-pseudoclosed with respect to* $\mathcal{R}$ *such that* A *is no premise of* P_j.[21] Let $B = A^{\Box\Diamond} = \{m \in M \mid A \to m \in Sat(\mathbb{K}_j)\}$. The program asks the expert whether $A \to B$ is valid in the universe $\mathbb{K}^U$. The set P_j contains the implications accepted as valid so far.

(a) If the expert gives the answer "no" then he must enter a counterexample $g \in G^U$ against the implication $A \to B$. The expert enters the context row of g which may contain questionmarks. Let $\mathbb{K}' = (G_j \cup \{g\}, M, \{\times, ?, o\}, J)$ be the context which consists of $\mathbb{K}_j$ and the context row of g which is entered by the expert. Then $J(g, a) = \times$ for all $a \in A$ and $J(g, b) = o$ for at least one $b \in B$ must hold. The context row of g in $\mathbb{K}^U$ has to be a completion of the context row of g in $\mathbb{K}'$, so $Red^{\mathcal{R}}_{P_j}(g)$ exists in $\mathbb{K}'$ because of Theorem 20. Let $P_{j+1} := P_j$ and $\mathbb{K}_{j+1} = (G_{j+1}, M, \{\times, ?, o\}, I_{j+1})$ be the context $\mathbb{K}'$ after replacing the context row of g by $Red^{\mathcal{R}}_{P_j}(g)$.

(b) If the expert gives the answer "unknown" then the program asks the expert for which attributes $b \in B$ the implication $A \to b$ is unknown. Let $Z = \{b \in B \mid A \to b \text{ is unknown}\}$. For $b \in B \setminus Z$ the implication $A \to b$ is valid in the universe $\mathbb{K}^U$, because every counterexample against $A \to b$ would also be a counterexample against $A \to B$. For $b \in Z$ the computer program checks, whether $A \to b \in \mathbf{Cons}^{\mathcal{R}}(P_j \cup \{A \to B \setminus Z\})$ holds, in this case the attribute b can be removed from Z because $A \to b$ follows from the implications which are known to be valid in the universe $\mathbb{K}^U$, so $A \to b$ must also be valid. In the following we assume $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_j \cup \{A \to B \setminus Z\})$ for $b \in Z$. Now fictitious objects are added to $\mathbb{K}_j$: Let $\mathbb{K}' := (G_j \cup \{g_{A,b} \mid b \in Z\}, M, \{\times, ?, o\}, J)$ with $J(g, m) = I_j(g, m)$ for $g \in G_j, m \in M$ and for $b \in Z$ let $J(g_{A,b}, a) = \times$ for $a \in A$, $J(g_{A,b}, b) = o$ and $J(g_{A,b}, m) =?$ for $m \in M \setminus (A \cup \{b\})$. Here we assume that $g_{A,b}$ is a new object: $g_{A,b} \notin G_j$ and $g_{A,b} \notin G^U$. Let $P_{j+1} := P_j \cup \{A \to B \setminus Z\}$ if $B \setminus Z \neq A$ and $P_{j+1} := P_j$ if $B \setminus Z = A$. Then $Red^{\mathcal{R}}_{P_{j+1}}(g)$ exists for all objects $g \in G^U \cap G_j$ which are not fictitious, because of $P_j \cup \{A \to B \setminus Z\} \subseteq Imp(\mathbb{K}^U)$. Let $\mathbb{K}_{j+1} = (G_{j+1}, M, \{\times, ?, o\}, I_{j+1})$ be the context $\mathbb{K}'$ after replacing the context row of each $g \in G^U \cap G_j$ by $Red^{\mathcal{R}}_{P_{j+1}}(g)$. The

[21] If the background knowledge is given by a set of implications ($\mathcal{R} = Resp(H)$ with $H \subseteq Imp(\mathbb{K}^U)$), then the "next concept"-algorithm of Ganter can be used to find such a set A (see [GW99]). If the user wants to get a generating system (or a base) of the valid implications in $\mathbb{K}^U$, such that he only has to use the rules (AX) and (PS) to compute all consequences (so the rule ($\mathcal{R}$-EX) is not needed), then the exploration algorithm can be modified (see Remark 21 in part II).

context rows of the fictitious objects remain unchanged.[22]

(c) If the expert gives the answer "yes" then the implication is added to the set of all accepted implications: $P_{j+1} := P_j \cup \{A \to B\}$. Then $Red^{\mathcal{R}}_{P_{j+1}}(g)$ exists for all objects $g \in G^U \cap G_j$ which are not fictitious, because of $P_{j+1} \subseteq Imp(\mathbb{K}^U)$. Let $\mathbb{K}_{j+1} = (G_{j+1}, M, \{\times, ?, o\}, I_{j+1})$ be the context $\mathbb{K}_j$ after replacing the context row of each $g \in G^U \cap G_j$ by $Red^{\mathcal{R}}_{P_{j+1}}(g)$. The context rows of the fictitious objects remain unchanged.

(E4) Let $j := j + 1$. Step (E3) is repeated until every $Sat(\mathbb{K}_j)$-pseudoclosed set $A \subseteq M$ is a premise of P_j. Then the algorithm ends.

Note that the context row of a fictitious object $g_{A,b}$ is the smallest context row (with respect to the information order) which is a counterexample against $A \to b$.

Lemma 23 *Let $j > 0$ and $g_{A,b} \in G_j$ be a fictitious object. Let $C \to D \in Imp_M$. The implication $C \to D$ is not satisfyable for $g_{A,b}$ iff $C \subseteq A$ and $b \in D$.*

Proof. $C \to D$ is not satisfyable for $g_{A,b}$ iff
$I_j(g_{A,b}, c) = \times$ for all $c \in C$ and $I_j(g_{A,b}, d) = o$ for some $d \in D$ iff
$C \subseteq A$ and $b \in D$. ■

Corollary 24 *Let $j > 0$ and $G_j \cap G^U \subseteq S \subseteq T \subseteq G_j$. Then*
$Sat(\mathbb{K}_j|_T) = Sat(\mathbb{K}_j|_S) \setminus \{E_1 \to E_2 \mid \textit{there exists } g_{A,b} \in T \setminus S \textit{ with } E_1 \subseteq A, b \in E_2\}$.

Proof.
$\subseteq$:
For $C \to D \in Sat(\mathbb{K}_j|_T)$ we have $C \to D \in Sat(\mathbb{K}_j|_S)$ because of $S \subseteq T$, and with Lemma 23 we get $C \to D \notin \{E_1 \to E_2 \mid \text{there exists } g_{A,b} \in T \setminus S \text{ with } E_1 \subseteq A, b \in E_2\}$.
$\supseteq$:
Let $C \to D \in Sat(\mathbb{K}_j|_S) \setminus \{E_1 \to E_2 \mid \text{there exists } g_{A,b} \in T \setminus S \text{ with } E_1 \subseteq A, b \in E_2\}$. With Lemma 23 $C \to D$ is satisfyable for all $g_{A,b} \in T \setminus S$, so $C \to D \in Sat(\mathbb{K}_j|_T)$. ■

The implications accepted as valid are satisfyable in the current context during the exploration:

[22] In section 2 of part II it will be shown that the expert gets a worse result of the exploration if the algorithm reduces also the questionmarks in context rows of the fictitious objects before the algorithm has ended (see Example 3).

Theorem 25 *$P_j \subseteq Sat(\mathbb{K}_j)$ for all $j > 0$.*

Proof. For $S = G_j \cap G^U$ the set P_j is satisfyable in $\mathbb{K}_j|_S$ because of $P_j \subseteq Imp(\mathbb{K}^U)$. Let $C \to D \in P_j \subseteq Sat(\mathbb{K}_j|_S)$. Assume $C \to D \notin Sat(\mathbb{K}_j)$. With Corollary 24 there exists an object $g_{A,b} \in G_j \setminus S$ with $C \subseteq A$ and $b \in D$. Let $i < j$ with $C \to D \in P_{i+1} \setminus P_i$ and $k < j$ with $g_{A,b} \in G_{k+1} \setminus G_k$. In $\mathbb{K}_i$ we have $b \in C^{\Box\Diamond} = \{m \in M \mid C \to m \in Sat(\mathbb{K}_i)\}$, so $g_{A,b} \notin G_i$ because $C \to b$ is not satisfyable for $g_{A,b}$. So we get $i \leq k$. The validity of $A \to b$ was unknown in step k, so $A \to b \notin \mathbf{Cons}^{\mathcal{R}}(P_{k+1})$ and we get $C \to D \notin P_{k+1}$ because of the rules (AU) and (PR). Therefore $k < i$ which is a contradiction. Therefore $C \to D \in Sat(\mathbb{K}_j)$ and we get $P_j \subseteq Sat(\mathbb{K}_j)$. ∎

Theorem 26 *Let $k > j > 0$ and $A \to B$ be the implication, such that the program asks in step j for the validity of $A \to B$ and the expert gives either the answer "yes" or "unknown". Let $Z = \{b \in B \mid A \to b$ is unknown$\}$ or $Z = \emptyset$ if the implication is accepted as valid.*

1. *$B \setminus Z = A^{\Box\Diamond}$ holds in $\mathbb{K}_k$*
2. *Each $Sat(\mathbb{K}_k)$-pseudoclosed proper subset $C \subset A$ is $Sat(\mathbb{K}_j)$-pseudoclosed.*
3. *If $B \setminus Z \neq A$ then A is $Sat(\mathbb{K}_k)$-pseudoclosed.*
4. *Each $Sat(\mathbb{K}_j)$-pseudoclosed proper subset $C \subset A$ is $Sat(\mathbb{K}_k)$-pseudoclosed.*

Proof.
Proof of 1:
For $B \setminus Z \neq A$ we have $A \to B \setminus Z \in P_k \subseteq Sat(\mathbb{K}_k)$ and for $B \setminus Z = A$ we also have $A \to B \setminus Z \in Sat(\mathbb{K}_k)$. Because of $k \geq j+1$ we have $Sat(\mathbb{K}_k) \subseteq Sat(\mathbb{K}_{j+1})$, so $B \setminus Z \subseteq \{m \in M \mid A \to m \in Sat(\mathbb{K}_k)\} \subseteq \{m \in M \mid A \to m \in Sat(\mathbb{K}_{j+1})\}$. We have $\{m \in M \mid A \to m \in Sat(\mathbb{K}_j)\} = B$ because in step j the algorithm asks for the validity of the implication $A \to \{m \in M \mid A \to m \in Sat(\mathbb{K}_j)\}$. For the attributes of Z the context $\mathbb{K}_{j+1}$ contains fictitious counterexamples $\{g_{A,b} \mid b \in Z\}$, so we get $\{m \in M \mid A \to m \in Sat(\mathbb{K}_{j+1})\} \subseteq B \setminus Z$ and $\{m \in M \mid A \to m \in Sat(\mathbb{K}_k)\} = B \setminus Z$.
Proof of 2 and 3 by induction over j:
Let the conditions 2 and 3 be true for all $i < j$.
Proof of 2:
Let C be a $Sat(\mathbb{K}_k)$-pseudoclosed proper subset of A. Then we have $C \in \mathcal{R}$ (see definition of pseudoclosed sets), but C is not a $Sat(\mathbb{K}_k)$-intent, so C is not a $Sat(\mathbb{K}_j)$-intent because of $Sat(\mathbb{K}_k) \subseteq Sat(\mathbb{K}_j)$. Let D be a $Sat(\mathbb{K}_j)$-pseudoclosed proper subset of C. Then $D \to E \in P_j$ for a set $E \subseteq M$ because otherwise the exploration

program must ask in step j for the validity of $D \to E$ because D is a proper subset of A. So there exists an $i < j$ with $D \to E$ in $P_{i+1} \setminus P_i$. By inducion hypothesis condition 3 holds fo the step i, so D is $Sat(\mathbb{K}_k)$-pseudoclosed and it is a proper subset of the $Sat(\mathbb{K}_k)$-pseudoclosed set C, so with condition 1 (for step i) we get $\{m \in M \mid D \to m \in Sat(\mathbb{K}_j)\} = E = \{m \in M \mid D \to m \in Sat(\mathbb{K}_k)\} \subseteq C$. Therefore C is $Sat(\mathbb{K}_j)$-pseudoclosed.
Proof of 3:
Assume $B \setminus Z \neq A$. We have $A \in \mathcal{R}$ because A is $Sat(\mathbb{K}_j)$-pseudoclosed. Because of condition 1 and $B \setminus Z \neq A$ and Corollary 16 the set A is no $Sat(\mathbb{K}_k)$-intent. Let C be a $Sat(\mathbb{K}_k)$-pseudoclosed proper subset of A. Because of condition 2 the set C is $Sat(\mathbb{K}_j)$-pseudoclosed, so we get $\{m \in M \mid C \to m \in Sat(\mathbb{K}_k)\} \subseteq \{m \in M \mid C \to m \in Sat(\mathbb{K}_j)\} \subseteq A$ and A is $Sat(\mathbb{K}_k)$-pseudoclosed.
Proof of 4:
Let $C \subset A$ be a $Sat(\mathbb{K}_j)$-pseudoclosed proper subset of A. Then C is a premise of P_j because otherwise the exploration program must ask in step j for the validity of $C \to C^{\Box\Diamond}$. Let $i < j$ and $D \subseteq M$ with $C \to D \in P_{i+1} \setminus P_i$. With condition 3 (for step i) the set C is $Sat(\mathbb{K}_k)$-pseudoclosed. ■

Corollary 27 *Let $j > 0$.*

1. *For each implication $A \to B \in P_j$ we get $B = A^{\Box\Diamond}$ in $\mathbb{K}_j$ and A is $Sat(\mathbb{K}_j)$-pseudoclosed.*
2. *Let $g_{A,b} \in G_j$ be a fictitious object. $A = A^{\Box\Diamond}$ holds in $\mathbb{K}_j$ iff A is no premise of P_j.*

Proof.
Proof of 1:
For $A \to B \in P_j$ there exists an $i < j$ with $A \to B \in P_{i+1} \setminus P_i$, so there exists a set $E \subseteq M$ such that exploration program asked in step i for the validity of $A \to E$. With condition 1 of Theorem 26 (for step i) we get $B = A^{\Box\Diamond}$ in $\mathbb{K}_j$ and A is $Sat(\mathbb{K}_j)$-pseudoclosed because of condition 3 of Theorem 26.
Proof of 2:
There exists $i < j$ with $g_{A,b} \in G_{i+1} \setminus G_i$, so there exists a set $B \subseteq M$ such that exploration program asked in step i for the validity of $A \to B$. For $b \in Z := \{m \in B \mid A \to m \text{ is unknown}\}$ a fictitious object $g_{A,b}$ is added, so with Corollary 16 and condition 1 of Theorem 26 the set A is a $Sat(\mathbb{K}_j)$-intent iff $B \setminus Z = A$ iff A is no premise of P_j. ■

Theorem 28 *Let $A \subseteq M$ and $j > 0$. The condition, that A is a minimal $Sat(\mathbb{K}_j)$-pseudoclosed set (with respect to $\mathcal{R}$) which is no premise of P_j is equivalent to the property that A is a minimal set with $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_j)$.*

Proof.
$\Rightarrow$:
Let A be a minimal $Sat(\mathbb{K}_j)$-pseudoclosed set, which is no premise of P_j. Then we get $A^{\Box\Diamond} \neq A$ because of Corollary 17. Let $B \to C \in P_j$ with $B \subseteq A$. Then B is a proper subset of A because A is no premise of P_j. With Corollary 27 the set B is $Sat(\mathbb{K}_j)$-pseudoclosed and $C = B^{\Box\Diamond} \subseteq A$ because A is $Sat(\mathbb{K}_j)$-pseudoclosed. We have $A \in \mathcal{R}$, so we get $A \in Resp(P_j) \cap \mathcal{R} = Resp^{\mathcal{R}}(P_j)$. Before we prove the minimality we prove the other direction.
$\Leftarrow$:
Let A be minimal with $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_j)$, then we have $A \in \mathcal{R}$, and A is no $Sat(\mathbb{K}_j)$-intent because of Corollary 16. The set A is no premise of P_j because A does not respect the implication $A \to A^{\Box\Diamond}$. Let B be a $Sat(\mathbb{K}_j)$-pseudoclosed proper subset of A. Now we show $B^{\Box\Diamond} \subseteq A$. Assume that B is no premise of P_j. For $C \to D \in P_j$ with $C \subseteq B$ we have $B \neq C$ because B is no premise of P_j, so with Corollary 27 we get $D = C^{\Box\Diamond} \subseteq B$ because C is a $Sat(\mathbb{K}_j)$-pseudoclosed proper subset of B. So we get $B \in Resp(P_j)$ and $B^{\Box\Diamond} \neq B \in Resp^{\mathcal{R}}(P_j)$, which is a contradiction to the minimality of A. Therefore B is a premise of P_j. With Corollary 27 we get $B \to B^{\Box\Diamond} \in P_j$ and $B^{\Box\Diamond} \subseteq A$ because of $A \in Resp(P_j)$. Therefore A is $Sat(\mathbb{K}_j)$-pseudoclosed. If A is not a minimal $Sat(\mathbb{K}_j)$-pseudoclosed set which is not a premise of P_j then (because of the finiteness of M) there exists a minimal proper subset C of A which is a $Sat(\mathbb{K}_j)$-pseudoclosed set and which is no premise of P_j, so with the first part of the proof of this theorem we get $C^{\Box\Diamond} \neq C \in Resp^{\mathcal{R}}(P_j)$ which is a contradiction to the minimality of A.
Analogously we get the minimality of A in the first part of the proof: If A is not minimal with $A^{\Box\Diamond} \neq A \in Resp^{\mathcal{R}}(P_j)$ then there exists a minimal proper subset C of A with $C^{\Box\Diamond} \neq C \in Resp^{\mathcal{R}}(P_j)$, so with the second part of the proof C is a $Sat(\mathbb{K}_j)$-pseudoclosed set which is no premise of P_j, but this is a contradiction to the minimality of A. ■

Corollary 29 *Let $k > j > 0$. If the program asks in step j for the validity of $A \to B$ and in step k for the validity of $C \to D$ then C is not a proper subset of A.*

Proof. $C^{\Box\Diamond} \neq C \in Resp^{\mathcal{R}}(P_k)$ holds in $\mathbb{K}_k$, so $C^{\Box\Diamond} \neq C \in Resp^{\mathcal{R}}(P_j)$ holds in $\mathbb{K}_j$ because of $j < k$, so C is not a proper subset of A because of the minimality of A. ■

Lemma 30 *Let $j > 0$ and let $A \to B \in Imp_M$ be the implication such that the program asks in step j for the validity of $A \to B$ or $A = M$ if the exploration ends in step j. Let $D \subseteq M$ and C be a proper subset of A. Then $C \to D \in P_j$ iff $C \to D \in DGB^{\mathcal{R}}(Sat(\mathbb{K}_j))$.*

Proof.
$\Rightarrow$:
Let $C \to D \in P_j$, then C is $Sat(\mathbb{K}_j)$-pseudoclosed because of Corollary 27. We have $D = \{m \in M \mid C \to m \in Sat(\mathbb{K}_j)\}$ and $C \to D \in DGB^{\mathcal{R}}(Sat(\mathbb{K}_j))$.
$\Leftarrow$:
Let $C \to D \in DGB^{\mathcal{R}}(Sat(\mathbb{K}_j))$. Then C is $Sat(\mathbb{K}_j)$-pseudoclosed, so C is a premise of P_j because the program asks in step j for the validity of an implication with minimal $Sat(\mathbb{K}_j)$-pseudoclosed premise which is no premise of P_j. We have $D = \{m \in M \mid C \to m \in Sat(\mathbb{K}_j)\} = C^{\Box\Diamond}$ and $C \to D \in P_j$ with condition 1 of Corollary 27.
■

So at the end of the exploration the set of all accepted implications is the Duquenne-Gigue-Base:

Corollary 31 *Let n be the step in which the exploration ends.*
Then $P_n = DGB^{\mathcal{R}}(Sat(\mathbb{K}_n))$ and $\mathbf{Cons}^{\mathcal{R}}(P_n) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n))$. If there was no unknown implication during the exploration then $Imp(\mathbb{K}^U) = Sat(\mathbb{K}_n)$ and $P_n = DGB^{\mathcal{R}}(Imp(\mathbb{K}^U))$ and $\mathbf{Cons}^{\mathcal{R}}(P_n) = Imp(\mathbb{K}^U)$.

Proof. $P_n = DGB^{\mathcal{R}}(Sat(\mathbb{K}_n))$ follows from Lemma 30.
$\mathbf{Cons}^{\mathcal{R}}(P_n) = \mathbf{Cons}^{\mathcal{R}}(Sat(\mathbb{K}_n))$ follows from Theorem 18.
If there are no fictitious objects, then we have
$Sat(\mathbb{K}_n) \subseteq \mathbf{Cons}^{\mathcal{R}}(P_n) \subseteq Imp(\mathbb{K}^U) \subseteq Sat(\mathbb{K}_n)$ and
$P_n = DGB^{\mathcal{R}}(Sat(\mathbb{K}_n)) = DGB^{\mathcal{R}}(Imp(\mathbb{K}^U))$ and $\mathbf{Cons}^{\mathcal{R}}(P_n) = Imp(\mathbb{K}^U)$. ■

Now let n be the step in which the exploration ends. The set P_n of accepted implications is the Duquenne-Gigue-Base of the satisfyable implications $Sat(\mathbb{K}_n)$ of the last context. This set is a generating set (with respect to the frame context) of $Sat(\mathbb{K}_n)$: $Sat(\mathbb{K}_n)$ is derivable from P_n with the rules (AX), (PS), ($\mathcal{R}$-EX). The set P_n is nonredundant in the sense that no implication of P_n follows from other implications of P_n. If there have been no unknown implication during the exploration then the last context contains a complete system of counterexamples and the set P_n is also the Duquenne-Gigue-Base of the valid implications $Imp(\mathbb{K}^U)$ of the universe $\mathbb{K}^U$, so in this case the expert has complete knowledge about the implications of the universe. Then for each attribute implication $A \to B$ the following three conditions are equivalent:

- $A \to B$ is valid in $\mathbb{K}^U$,
- $A \to B$ is derivable from P_n,
- $A \to B$ is satisfyable in the last context $\mathbb{K}_n$.

If there have been some unknown implications during the exploration then the set of all satisfyable implications of the last context $\mathbb{K}_n$ is only a subset of the valid implications of the universe because $\mathbb{K}_n$ may contain some fictitious counterexample which do not correspond to objects in the universe $\mathbb{K}^U$. In this case P_n does not generate all valid implications of the universe but only the implications which are certainly valid with respect to the knowledge of the expert: For every implication which is not derivable from P_n there exists a (fictitious or normal) counterexample in $\mathbb{K}_n$.

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Address of the author:
Richard Holzer
Department of Mathematics, AG1
Darmstadt University of Technology
Schloßgartenstr. 7
64289 Darmstadt
Germany