A revision of Phan's theorem for type D_n

R. Gramlich,[*] C. Hoffman,[†] W. Nickel, S. Shpectorov[‡]

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1 Introduction

In 1977 Kok-Wee Phan [9] published a theorem on generation of the special unitary group $SU(n+1,q^2)$ by a system of its subgroups isomorphic to $SU(3,q^2)$. The proof of Phan's theorem given in his 1977 paper is somewhat incomplete. This motivated the paper [1] in which a new and complete proof of Phan's theorem was provided. The approach of [1] is based on the concepts of diagram geometries and amalgams of groups. It turns out that Phan's configuration arises as the amalgam of *stripped* rank two parabolics in the flag-transitive action of $SU(n+1,q^2)$ on the geometry of nondegenerate subspaces of the underlying unitary space (*stripped* in the sense that the torus of $SU(n+1,q^2)$ has been removed). This point of view leads to a twofold interpretation of Phan's theorem: its complete proof must include (1) a classification of related amalgams; and (2) a verification that—apart from some small exceptional cases—the above geometry is simply connected. These two parts are tied together by a lemma due to Tits, that implies that if a group G acts flag-transitively on a simply connected geometry then the corresponding amalgam of maximal parabolics provides a presentation for G. For an outline of the idea how to re-prove, extend and generalize Phan's theorems the reader is referred to [2].

Notice that this new approach has already yielded an unexpected new Phan-type theorem for the group $Sp(2n,q)$. See [5] for the simple connectedness of the corresponding geometry and [6] for a classification of related amalgams. In terms of the Dynkin diagrams, the original Phan's theorem corresponds to the diagram A_n, while the new theorem for $Sp(2n,q)$ corresponds to the diagram C_n. We conjecture that there is also a similar result, a Phan-type theorem, for every spherical diagram of rank at least three. In a later paper [10], Phan himself claimed such theorems for the groups $Spin^{\pm}(2n,q)$ (diagram D_n), ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$ (diagrams E_6, E_7, and E_8, respectively). So the proof of our conjecture requires new proofs for these results of Phan's, as well as new theorems for groups $SO(2n+1,q)$ (diagram B_n) and $F_4(q)$ (diagram F_4). The purpose of the present paper is to do the case D_n, that is, to reprove and extend Phan's theorem on $Spin^{\pm}(2n,q)$, using our new methods.

As defined in [1], subgroups U_1 and U_2 of $SU(3,q^2)$ form a *standard pair* whenever each U_i is the stabilizer in $SU(3,q^2)$ of a nonsingular vector v_i of the natural module U of $SU(3,q^2)$ and, furthermore, v_1 and v_2 are perpendicular. By Witt's theorem, standard pairs are exactly the conjugates of the pair

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formed by the two subgroups $SU(2,q^2)$ arising from the 2×2 blocks on the main diagonal with respect to an orthonormal basis of U. Standard pairs in quotients of $SU(3,q^2)$ over a subgroup of its center are defined as the images under the natural homomorphism of the standard pairs from $SU(3,q^2)$.

Similarly to [1], we say that a group G possesses a *weak Phan system of type D_n over $\mathbb{F}_{q^2}$* if G contains subgroups $U_i \cong SU(2,q^2)$, for $i = 1,\ldots,n$ and $U_{i,j}$, for distinct $i,j \in \{1,\ldots,n\}$, so that the following hold:

(wP1) If (i,j) is not an edge of the Dynkin diagram D_n then $U_{i,j}$ is a central product of U_i and U_j;

(wP2) If (i,j) is an edge of the Dynkin diagram D_n, then $U_{i,j}$ is isomorphic to a quotient of $SU(3,q^2)$ over a subgroup of its center; moreover, U_i and U_j form a standard pair in $U_{i,j}$; and

(wP3) the subgroups $U_{i,j}$, $1 \le i < j \le n$, generate G.

Note that we added (wP3) instead of just saying that the groups U_i generate G for the sake of the case $q = 2$. Indeed, the group $SU(3,2^2)$ is not generated by a standard pair of subgroups $SU(2,2^2)$, i.e., the geometry of nondegenerate subspaces of $\mathbb{F}_4^3$ with respect to a nondegenerate form, is not connected, cf. [1]. This fact influenced the wording of the entire definition: we did not introduce $U_{i,j}$ as $\langle U_i, U_j\rangle$ exactly in order to allow the case $q = 2$.

The main result of this paper is the following generalization of Phan's D_n theorem [10]. Notice that Phan only allowed odd prime powers $q \ge 5$. We start with the case of arbitrary prime power $q \ge 4$.

Main Theorem A
Let $q \ge 4$, $n \ge 3$, and let G be a group that contains a weak Phan system of type D_n over $\mathbb{F}_{q^2}$. Then G is isomorphic to a factor group of $Spin^+(2n,q)$ for n even and a factor group of $Spin^-(2n,q)$ for n odd.

Main Theorem A leaves us with two exceptional cases $q = 2, 3$. For these cases the following is true:

Main Theorem B
Let $q = 2$ or 3, and $n \ge 4$. Let G be a group that contains a weak Phan system of type D_n over $\mathbb{F}_{q^2}$. Suppose further that

(1) *for any triple i, j, k of nodes of the Dynkin diagram D_n that form a subdiagram $\underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{k}{\circ}$ of type A_3, the subgroup $\langle U_{i,j}, U_{j,k}\rangle$ is isomorphic to a factor group of $SU(4,q^2)$;*

(2) *additionally, if $q = 2$ then*

   (i) *for any triple i, j, k of nodes of D_n that form a subdiagram $\underset{i}{\circ} \quad \underset{j}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{k}{\circ}$ of type $A_1 \oplus A_2$ the groups U_i and $U_{j,k}$ commute elementwise; and*

   (ii) *for any quadruple i, j, k, l of nodes of D_n that form a subdiagram $\underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\circ} \quad \underset{k}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{l}{\circ}$ of type $A_2 \oplus A_2$ the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise.*

Then G is isomorphic to a factor group of $Spin^+(2n,q)$ for n even and isomorphic to a factor group of $Spin^-(2n,q)$ for n odd.

This paper is organized as follows. In Section 2 we state four important geometrical and group-theoretical results that form the cornerstones of our proof of Main Theorem A and B. In Section 3 we study unitary involutions on a nondegenerate quadratic space V over an arbitrary field of square order and in Section 4 we study the resulting geometry, some kind of folded building geometry. Section 5 provides some basic facts and methods from algebraic topology. Those methods are applied in Sections 6 and 7 to establish the simple connectedness of the folded building geometry from Section 4. Theorems 1 and 2 of Section 2 are proved in Section 7. In Section 8 we apply Tits' lemma and Theorems 1 and 2 to obtain a presentation of flag-transitive groups of automorphisms of our geometry, establishing Theorems 3 and 4 of Section 2.

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2 Relevant geometric results

Along the way to a proof of our Main Theorems we obtain a number of geometric and group-theoretic results. In this section we collect some of those. Let V be the natural module of the group $G \cong [SO^+(2n,q^2), SO^+(2n,q^2)]$ with the nondegenerate quadratic form f and let σ be an involutory semilinear transformation of V with $f(x^\sigma) = f(x)^q$ such that there exists a maximal f-singular subspace U of V with $U \cap U^\sigma = \{0\}$. We will show, see Proposition 3.10, that $G_\sigma = C_G(\sigma)$ is isomorphic to the commutator group of $SO^+(2n,q)$ if n is even and isomorphic to the commutator group of $SO^-(2n,q)$ if n is odd. The *flipflop geometry* Γ_σ consists of those f-singular subspaces of V (except the ones of dimension $n-1$) that trivially intersect the polar of their images under σ.

The geometry Γ_σ has the following properties. Refer to Section 4 for geometric terminology and to Section 5 for notions from algebraic topology.

Theorem 1
Let $q \geq 3$ and let $n \geq 2$. Then the following hold.

(1) *Γ_σ is a rank n geometry admitting a flag-transitive group of automorphisms $G_\sigma \cong S\Omega^+(2n,q)$ for n even and $G_\sigma \cong S\Omega^-(2n,q)$ for n odd.*

(2) *Γ_σ is residually connected.*

(3) *Γ_σ is simply connected unless $(n,q) = (3,3)$.*

In the case $(n,q) = (3,3)$, the geometry Γ_σ is not simply connected. It admits a triple cover which is universal. For $q = 2$ the geometry is connected (but not residually connected) and simply connected for $n \geq 4$.

Theorem 2
Let $q = 2$ and let $n \geq 2$. Then the following hold.

(1) *Γ_σ is a rank n geometry admitting a flag-transitive group of automorphisms $G_\sigma \cong S\Omega^+(2n,q)$ for n even and $G_\sigma \cong S\Omega^-(2n,q)$ for n odd.*

(2) *Γ_σ is connected.*

(3) *Γ_σ is simply connected if $n \geq 4$.*

The residual connectedness of Γ_σ fails because it admits a residue isomorphic to the geometry of nondegenerate subspaces of $\mathbb{F}_4^3$ with respect to a nondegenerate unitary form, which is not connected by [1]. Equivalently, a standard pair of $SU(3, 2^2)$ does not generate $SU(3, 2^2)$.

Theorem 1 and Theorem 2 have some group-theoretic implications via Tits' lemma. Let $F = U_1, \ldots, U_n$ be a maximal flag of Γ_σ. For $2 \leq s \leq n-1$, let $\mathcal{A}_{(s)}$ be the amalgam of all rank s parabolics, i.e., stabilizers in G_σ of subflags of F of corank s. Again, for geometric terminology see Section 4. Amalgams are defined in Section 8.

Theorem 3
Let n be an integer and let ϵ be the sign of $(-1)^n$. Then the following hold.

(1) *If $q \geq 4$ and $n \geq 3$ then $G_\sigma \cong S\Omega^\epsilon(2n, q)$ is the universal completion of $\mathcal{A}_{(2)}$.*

(2) *If $q = 2, 3$ and $n \geq 4$ then $G_\sigma \cong S\Omega^\epsilon(2n, q)$ is the universal completion of $\mathcal{A}_{(3)}$.*

The universal version $Spin^\epsilon(2n, q)$ of $S\Omega^\epsilon(2n, q)$ of course also acts flag-transtively on Γ_σ, so Theorem 3 also holds for this group. The maximal parabolics M_i of $Spin^\epsilon(2n, q)$ with respect to $F = U_1, \ldots, U_n$ are semisimple groups of the form $GU(i, q^2) \times Spin^\pm(2n-2i, q)$, $i = 1, \ldots, n-2$, and $GU(n, q^2)$, $i = n-1, n$. Each M_i, $1 \leq i \leq n-2$, stabilizes the direct decomposition $U_i \oplus U_i^\sigma \oplus \langle U_i, U_i^\sigma \rangle^\perp$ of the quadratic module U of $Spin^\epsilon(2n, q)$. For $1 \leq i \leq n-2$ they induce $GU(i, q^2)$ on U_i and $Spin^\pm(2n-2i, q)$ on $U_i^\perp$. The parabolics M_{n-1} and M_n induce $GU(n, q^2)$ on U_{n-1}, respectively U_n. The intersection of all M_i, i.e., the Borel subgroup arising from the action of $Spin^\epsilon(2n, q)$ on Γ_σ, is a maximal torus T of $Spin^\epsilon(2n, q)$ of order $(q+1)^n$. Let M_i^0 be the subgroup $SU(i, q^2) \times Spin^\pm(2n-2i, q)$, respectively $SU(n, q^2)$ of M_i. For an arbitrary parabolic $M_J = \cap_{i \in J} M_i$ define $M_J^0 = \cap_{i \in J} M_i^0$. Here J is a subset of the type set $I = \{1, \ldots, n\}$ of Γ_σ. It can be shown that $M_J = M_J^0 T$.

In case of a minimal parabolic $M_{I \setminus \{i\}}$, we have that $L_i := M^0_{I \setminus \{i\}} \cong SL(2, q)$. In fact, for all $1 \leq i \leq n$ the group L_i arises as $SU(2, q^2) \cong SL(2, q)$. Notice that $T_i = L_i \cap T$ is a torus in L_i of size $q+1$. Notice also that the subgroups T_i generate T as their direct product. If $q \neq 2$ we have $\langle L_i, L_j \rangle = M^0_{I \setminus \{i,j\}}$. In particular, if in that case Δ is the Dynkin diagram D_n then the subgroups L_i have the following properties:

(1) $L_i \cong SU(2, q)$ for $i = 1, \ldots, n$;

(2) $\langle L_i, L_j \rangle \cong \begin{cases} L_i \times L_j, & \text{if } (i,j) \text{ is not an edge of } \Delta; \\ SU(3, q), & \text{if } (i,j) \text{ is an edge of } \Delta. \end{cases}$

For (i, j) an edge of Δ the groups U_i, U_j form a standard pair of $U_{i,j}$. Moreover, the U_i, $1 \leq i \leq n$, form a weak Phan system inside $Spin^\epsilon(2n, q)$. In fact, $Spin^\epsilon(2n, q)$ is defined by this weak Phan system, as stated in Theorem 4.

For arbitrary q define $\mathcal{A}^0_{(s)}$ to be the amalgam formed by the subgroups M_J^0 for all parabolics M_J of rank s.

Theorem 4
Let n be an integer and let ϵ be the sign of $(-1)^n$. Then the following hold.

(1) *If $q \geq 4$ and $n \geq 3$ then $Spin^\epsilon(2n, q)$ is the universal completion of $\mathcal{A}^0_{(2)}$.*

(2) *If $q = 2, 3$ and $n \geq 4$ then $Spin^\epsilon(2n, q)$ is the universal completion of $\mathcal{A}^0_{(3)}$.*

3 Flips and forms

Let V be a $2n$-dimensional nondegenerate orthogonal space of plus type over $\mathbb{F}_{q^2}$. Let f be the quadratic form on V and let $(\cdot,\cdot)$ be the corresponding bilinear form, so that $(u,v)=f(u+v)-f(u)-f(v)$. When $n\geq 2$, the orthogonal space V gives rise to the building geometry $\mathcal{D}$ of type D_n. The elements of $\mathcal{D}$ of type $i=1,2,\dots,n-2$ are the f-singular subspaces of V of dimension i. The elements of $\mathcal{D}$ of the last two types, $n-1$ and n, are the maximal (i.e., n-dimensional) f-singular subspaces. Two such subspaces U and U' have the same type if and only if $U\cap U'$ has an even codimension in U. Incidence is symmetrized containment except for incidence between elements of type $n-1$ and n. Two elements of type $n-1$ resp. n are incident if they intersect each other in a hyperplane, i.e., a subspace of dimension $n-1$.

Recall that a semilinear transformation corresponding to an automorphism ϕ of $\mathbb{F}_{q^2}$ is a mapping $\sigma: V\longrightarrow V$ such that for all $u,v\in V$ and $a\in\mathbb{F}_{q^2}$ we have $(u+v)^\sigma=u^\sigma+v^\sigma$ and $(av)^\sigma=a^\phi v^\sigma$. We say that a semilinear transformation σ *weakly preserves* f if there is an $a\in\mathbb{F}_{q^2}$ such that for every $v\in V$ we have $f(v^\sigma)=af(v)^\phi$. Semilinear transformations weakly preserving f form the group $\Gamma O^+(2n,q^2)$. Every element of this group induces an automorphism of $\mathcal{D}$, possibly switching the types $n-1$ and n. The converse is also true. Every automorphism of $\mathcal{D}$, fixing the types $n-1$ and n, or switching them, is induced by a semilinear transformation weakly preserving f. Notice also that the only semilinear transformations acting trivially on $\mathcal{D}$ are the linear scalar transformations.

In this section we study involutory automorphisms of $\mathcal{D}$ induced by semilinear transformations σ with $\phi\neq\mathrm{Id}$. The map σ^2 has to be linear, as it acts trivially on $\mathcal{D}$, so ϕ must be of order two. We will use the bar to denote the action of this unique automorphism of $\mathbb{F}_{q^2}$. Such an automorphism σ of $\mathcal{D}$ is called a *unitary involution*. In other words a unitary involution σ satisfies $(\lambda v)^\sigma=\bar\lambda v^\sigma$ and $\sigma^2=a\mathrm{Id}$ for some $a\in\mathbb{F}_{q^2}$. We claim that $a\in\mathbb{F}_q$. Indeed, for $v\in V$, $v\neq 0$, we have $av^\sigma=(v^\sigma)^{\sigma^2}=v^{\sigma^3}=(v^{\sigma^2})^\sigma=(av)^\sigma=\bar a v^\sigma$. This shows that $\bar a=a$, hence $a\in\mathbb{F}_q$. Choose $\alpha\in\mathbb{F}_{q^2}$ such that $\alpha\bar\alpha=\frac{1}{a}$. This is possible by the surjectivity of the norm map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. Then, setting $\sigma'=\alpha\sigma$, we get $v^{\sigma'^2}=(v^{\sigma'})^{\sigma'}=\alpha(\alpha v^\sigma)^\sigma=\alpha\bar\alpha v^{\sigma^2}=\alpha\bar\alpha a v=v$. Thus, $(\sigma')^2=\mathrm{Id}$. Clearly, σ and σ' induce the same automorphism of $\mathcal{D}$ and so we may assume without loss of generality that $\sigma^2=\mathrm{Id}$. Since σ weakly preserves f, there is a $b\in\mathbb{F}_{q^2}$ such that $f(v^\sigma)=b\overline{f(v)}$ for all $v\in V$. Notice that $b\bar b=1$. Indeed, pick $v\in V$ so that $f(v)\neq 0$. Then $f(v)=f(v^{\sigma^2})=b\overline{f(v^\sigma)}=b\bar b f(v)$. Thus, $b\bar b=1$. Choose $\beta\in\mathbb{F}_{q^2}$ so that $b=\frac{\bar\beta}{\beta}$. Such a choice is possible as the subgroup of order $q+1$ of the multiplicative group of $\mathbb{F}_{q^2}$ (cyclic of order q^2-1) consists of precisely those elements which are of the form γ^{q-1} for some γ. Define $f'(v):=\beta f(v)$. Then $f'(v^\sigma)=\beta f(v^\sigma)=\beta b\overline{f(v)}=\frac{\beta b}{\bar\beta}\overline{f'(v)}=\overline{f'(v)}$. Clearly, f' is a quadratic form of plus type and, since the zeros of f coincide with the zeros of f', the quadratic form f' defines exactly the same building geometry $\mathcal{D}$. Consequently we can assume right from the beginning that f and σ have the property $f(v^\sigma)=\overline{f(v)}$, which by polarization also implies $(u^\sigma,v^\sigma)=\overline{(u,v)}$.

Hence studying unitary involutions of $\mathcal{D}$ means studying semilinear transformations σ of V satisfying

(F1) $(\lambda v)^\sigma=\bar\lambda v^\sigma$;

(F2) $f(v^\sigma)=\overline{f(v)}$; and,

(F3) $\sigma^2=\mathrm{Id}$.

From now on we will require any unitary involution to satisfy (F1) through (F3).

Let us now describe two examples of semilinear transformations σ inducing unitary involutions. Let $e_1,\dots,e_n,f_1,\dots,f_n$ be a *hyperbolic basis* in V. This means that the subspaces $\langle e_1,\dots,e_n\rangle$ and

$\langle f_1, \ldots, f_n \rangle$ are totally singular and that $(e_i, f_j) = \delta_{ij}$ for $1 \le i, j \le n$. Define σ_1 and σ_2 as follows:

$$(\sum_{i=1}^{n} x_i e_i + \sum_{i=1}^{n} y_i f_i)^{\sigma_1} = \sum_{i=1}^{n} \bar{y}_i e_i + \sum_{i=1}^{n} \bar{x}_i f_i$$

and

$$(\sum_{i=1}^{n} x_i e_i + \sum_{i=1}^{n} y_i f_i)^{\sigma_2} = \sum_{i=1}^{n-1} \bar{y}_i e_i + \sum_{i=1}^{n-1} \bar{x}_i f_i + \bar{x}_n e_n + \bar{y}_n f_n.$$

Then σ_1 and σ_2 satisfy (F1) through (F3). Therefore, both σ_1 and σ_2 induce unitary involutions. Observe that σ_1 sends $U = \langle e_1, \ldots, e_n \rangle$ to $\langle f_1, \ldots, f_n \rangle$, while σ_2 sends U to $\langle f_1, \ldots, f_{n-1}, e_n \rangle$. Thus, the codimension of $U \cap U^{\sigma_1}$ in U is n, while the codimension of $U \cap U^{\sigma_2}$ in U is $n-1$. Hence if n is odd then the unitary involution induced by σ_1 switches the types $n-1$ and n, while the one induced by σ_2 preserves them. If n is even then the opposite occurs: σ_1 preserves the types and σ_2 switches $n-1$ and n. In other words, if W is an arbitrary maximal f-singular subspace of V, the dimension of $W \cap W^{\sigma_1}$ is always even and the dimension of $W \cap W^{\sigma_2}$ is always odd. In particular, σ_1 and σ_2 induce nonconjugate unitary involutions.

We will eventually prove that every unitary involution is conjugate to either σ_1 or σ_2, but first we record some general facts. Define $((u, v)) := (u, v^\sigma) = f(u + v^\sigma) - f(u) - f(v^\sigma)$.

Lemma 3.1
The form $((\cdot,\cdot))$ is a nondegenerate Hermitian form. Furthermore, $((u^\sigma, v^\sigma)) = \overline{((u, v))}$ for $u, v \in V$.

Proof. Clearly, $((\cdot,\cdot))$ is a sesquilinear form. Also, $((v, u)) = (v, u^\sigma) = (u^\sigma, v) = \overline{(u^{\sigma^2}, v^\sigma)} = \overline{(u, v^\sigma)} = \overline{((u, v))}$. Thus, $((\cdot,\cdot))$ is Hermitian. If u is in the radical of $((\cdot,\cdot))$ then for any $v \in V$, we have $0 = ((u, v^\sigma)) = (u, v^{\sigma^2}) = (u, v)$. Therefore, $u = 0$, as $(\cdot,\cdot)$ is nondegenerate (recall that V has even dimension). Finally, $((u^\sigma, v^\sigma)) = (u^\sigma, v) = (v, u^\sigma) = ((v, u)) = \overline{((u, v))}$. $\square$

Let g be the unitary form related to $((\cdot,\cdot))$, i.e., $g(v) = ((v, v))$. Notice that g and $((\cdot,\cdot))$ have the same radical and rank on every subspace of V. This does not hold for f and $(\cdot,\cdot)$ when q is even. In this case the radical of f can be a hyperplane in the radical of $(\cdot,\cdot)$ and hence the rank of f can be one larger than the rank of $(\cdot,\cdot)$.

In what follows we will work with both $(\cdot,\cdot)$ and $((\cdot,\cdot))$. This calls for two different perpendicularity symbols. If U is a subspace of V then $U^\perp$ denotes its orthogonal complement with respect to $(\cdot,\cdot)$, while $U^{\perp\!\!\!\perp}$ will be used for $((\cdot,\cdot))$.

Lemma 3.2
For a subspace $U \subset V$, we have $U^{\perp\!\!\!\perp} = (U^\sigma)^\perp = (U^\perp)^\sigma$. Similarly, $U^\perp = (U^\sigma)^{\perp\!\!\!\perp} = (U^{\perp\!\!\!\perp})^\sigma$.

Proof. The first equality in the first claim immediately follows from the definition of $((\cdot,\cdot))$. If $u \in (U^\perp)^\sigma$ (say, $u = (u')^\sigma$ for $u' \in U^\perp$) and $v \in U$ then $((u, v)) = ((u')^\sigma, v^\sigma) = \overline{(u', v)} = 0$. The second claim follows by an application of σ to the equalities in the first claim. $\square$

Lemma 3.3
Let U be a subspace of V. Then f has the same rank on U and U^σ; likewise, it has the same rank on $U^\perp$ and $U^{\perp\!\!\!\perp} = (U^\perp)^\sigma$. The same statements hold also for $(\cdot,\cdot)$, g, and $((\cdot,\cdot))$.

Proof. The first claim follows from (F2) for f and $(\cdot,\cdot)$, and from Lemma 3.1 for g and $((\cdot,\cdot))$. The second claim follows from the first one and Lemma 3.2. $\square$

Now we focus on the case where U is σ-invariant. Let us start with the following general property of unitary involutions.

Lemma 3.4
Every σ-invariant subspace of V admits a σ-invariant complement. In particular if U and W are σ-invariant and $U \subseteq W$ then U has a σ-invariant complement in W.

Proof. We will just need the property (F1). It is clear that (F1) is inherited by the restrictions of σ to all σ-invariant subspaces and factor spaces. Let U be a σ-invariant proper subspace of V. We claim that there exists a one-dimensional σ-invariant subspace not contained in U. Once this is proved, we can factor out that invariant one-dimensional space and induction finishes the proof of the lemma. Let $v \in V \setminus U$. If $\langle v\rangle$ is σ-invariant then we are done. Otherwise consider $\langle v, v^\sigma\rangle$. This subspace contains $q+1$ one-dimensional σ-invariant subspaces $\langle v+\lambda v^\sigma\rangle$ where $\lambda\bar{\lambda}=1$. Clearly at most one of these lies in U.

For the second claim, if $U \subseteq W$ are σ-invariant and if T is a complement to U in V then $T \cap W$ is a σ-invariant complement to U in W. $\square$

Suppose U is σ-invariant. Clearly (F1), (F2), (F3) hold when you reduce σ and the forms to U. Also, it follows from Lemma 3.2 that $U^\perp = U^{\perp\!\!\perp}$. In other words, for a σ-invariant subspace U, the polar (and hence also the radical) of U is the same with respect to $(\cdot,\cdot)$ and $((\cdot,\cdot))$. Thus for a σ-invariant subspace we will speak simply of its radical, meaning the radical for $(\cdot,\cdot)$, $((\cdot,\cdot))$ and g. The radical for f will be referred to as the f-radical. Note that Lemma 3.4 implies that each of the radicals has an σ-invariant complement in U. Notice that the σ-invariant complement is automatically nondegenerate for the corresponding form.

Lemma 3.5
If U is an f-nondegenerate σ-invariant subspace of V of dimension at least three, then there exists a vector u of U that is f-singular and g-nonsingular.

Proof. Let W be a subspace of U which is maximal f-singular. If W and W^σ generate a subspace that is $(\cdot,\cdot)$-totally isotropic then W is the unique maximal f-singular subspace in $\langle W, W^\sigma\rangle$ which means that $W = W^\sigma$. So if $W \neq W^\sigma$ then $((\cdot,\cdot))$ is nontrivial on W, and so W contains the required vector. Therefore by way of contradiction we can assume that every W is σ-invariant. Since every f-singular one-dimensional subspace of U is the intersection of the maximal f-singular subspaces containing it, it follows that σ fixes all f-singular one-dimensional subspaces of U.

Let again W be a subspace of U which is maximal f-singular. Suppose W has dimension more than one. Since σ fixes each one-dimensional subspace of W, it will have to act on W as a scalar and this contradicts (F1).

If $\dim W = 1$ then there are two cases: $\dim U = 3$ or $\dim U = 4$ and f restricted to U is of minus type. First assume $\dim U = 3$. Let $\langle a\rangle$ be any one-dimensional subspace of U and let u_1 and u_2 be vectors of U with $f(u_1) = 0 = f(u_2)$ such that $\langle a, u_1, u_2\rangle = U$. Then $\langle a, u_1\rangle$ is either a tangent line or it contains another f-singular one-dimensional subspace besides $\langle u_1\rangle$. In either case σ leaves the subspace $\langle a, u_1\rangle$ invariant as it weakly preserves f. For the same reason also the subspace $\langle a, u_2\rangle$ is σ-invariant. Hence the intersection $\langle a\rangle = \langle a, u_1\rangle \cap \langle a, u_2\rangle$ is σ-invariant. Since a was chosen arbitrarily, σ leaves invariant all one-dimensional subspaces of U. This implies that σ acts as a linear scalar map on U, contradicting property (F1). Hence there exists a vector u of U with $f(u) = 0$ that is linearly independent from u^σ. This means $g(u) = ((u, u^\sigma)) \neq 0$.

Finally if $\dim U = 4$ then pick T to be an f-nondegenerate three-dimensional subspace of U. The space T is generated by f-singular vectors hence it is σ-invariant. Now the above argument applies. □

We have reached the stage where we can classify the unitary involutions.

Proposition 3.6
There are exactly two conjugacy classes of unitary involutions in $\Gamma O^+(2n, q^2)$.

Proof. We first construct f-singular vectors $e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}$ such that $e_i^\sigma = f_i$, $(e_i, e_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. There is nothing to prove for $n = 1$ so assume that n is at least two. By Lemma 3.5 there is a vector v which is singular with respect to f and nonsingular with respect to g. Let $c = g(v) = ((v, v))$. Since g is unitary, we have $c \in \mathbb{F}_q$. By surjectivity of the norm map, there is a $\gamma \in \mathbb{F}_{q^2}$ such that $c = \gamma\bar{\gamma}$. Set $e_1 = \frac{1}{\gamma} v$ and $f_1 = e_1^\sigma$. Then e_1 and f_1 are singular for f and $(e_1, f_1) = ((e_1, e_1)) = \frac{1}{\gamma\bar{\gamma}}((v, v)) = 1$. Let $U = \langle e_1, f_1 \rangle$ and $V_0 = U^\perp$. Since U is σ-invariant, Lemma 3.2 shows that $V_0 = U^{\perp\!\perp}$ is also σ-invariant. Furthermore, f is nondegenerate on V_0 of plus type. If $\dim V_0 \geq 4$ then σ induces an unitary involution on the building geometry of V_0. This means that, working inductively and applying Lemma 3.5 in each step, we can complement e_1 and f_1 by further f-singular vectors $e_2, f_2, \dots, e_{n-1}, f_{n-1}$, such that for $1 \leq i, j \leq n-1$ we have $(e_i, e_j) = 0 = (f_i, f_j)$, $(e_i, f_j) = \delta_{ij}$ and $e_i^\sigma = f_i$.

For arbitrary n let now $U = \langle e_1, f_1 \dots, e_{n-1}, f_{n-1} \rangle^\perp$ and $V_0 = U^\perp$. Then both U and V_0 are σ-invariant. Since f is nondegenerate on U of plus type, it is so on V_0 as well. This means that V_0 contains exactly two singular one-dimensional spaces, say $\langle e \rangle$ and $\langle f \rangle$. There are two possibilities: either σ interchanges these two subspaces, or it stabilizes both of them. Consider the first possibility. Setting $c = g(e) = (e, e^\sigma)$, we see that $c \in \mathbb{F}_q$. Choosing $\gamma \in \mathbb{F}_{q^2}$ such that $c = \gamma\bar{\gamma}$ and setting $e_n = \frac{1}{\gamma} e$ and $f_n = e_n^\sigma$ we obtain a complete hyperbolic basis $e_1, \dots, e_n, f_1, \dots, f_n$ on which σ acts the way σ_1 does. Hence σ is conjugate to σ_1 in this first case. Consider now the second possibility. Suppose $e^\sigma = ce$. Since $\sigma^2 = 1$, we obtain $e = e^{\sigma^2} = (ce)^\sigma = \bar{c}ce$, which shows that $c\bar{c} = 1$. This means that there is a $\gamma \in \mathbb{F}_{q^2}$ such that $c = \frac{\gamma}{\bar{\gamma}}$. Indeed, the subgroup of order $q+1$ of the multiplicative group of $\mathbb{F}_{q^2}$ (cyclic of order $q^2 - 1$) consists of precisely those elements which are of the form γ^{q-1} for some γ. Taking $e_n = \gamma e$, we compute: $(e_n)^\sigma = (\gamma e)^\sigma = \bar{\gamma}(ce) = \frac{\bar{\gamma} c}{\gamma}(\gamma e) = e_n$. Let f_n be the unique vector in $\langle f \rangle$ such that $(e_n, f_n) = 1$. Since $(e_n, f_n^\sigma) = (e_n^\sigma, f_n^\sigma) = \overline{(e_n, f_n)} = 1$, we must also have that $f_n^\sigma = f_n$. Thus, in the second case σ acts on the hyperbolic basis $e_1, \dots, e_n, f_1, \dots, f_n$ the way σ_2 does. Hence σ is conjugate to σ_2. □

In view of [2] studying flipflop geometries related to σ only makes sense when one actually has a chamber of the building mapped to an opposite chamber, as otherwise the flipflop geometry would be empty. The following result shows that such maps σ are the ones that are conjugate to σ_1.

Corollary 3.7
Suppose σ is a unitary involution. The following are equivalent:

(F4a) *σ is conjugate to σ_1.*

(F4b) *V contains a maximal f-singular subspace U such that $\langle U, U^\sigma \rangle = V$.*

(F4c) *V contains a maximal f-singular subspace U such that $U \cap U^\sigma = \{0\}$.*

(F4d) *V contains a maximal f-singular subspace U such that $\dim U \cap U^\sigma$ is even.*

(F4e) *For every maximal f-singular subspace U of V we have that* $\dim U \cap U^\sigma$ *is even.*

Proof. By analyzing the action of σ_1 and σ_2 on the two types of maximal f-singular subspaces of V one immediately deduces that (F4a) and (F4e) are equivalent and that (F4d) implies (F4a). Also it is clear that (F4b) and (F4c) are equivalent. The implication (F4a) $\Rightarrow$ (F4b) follows from the fact that $\langle e_1, \ldots, e_n \rangle^{\sigma_1} = \langle f_1, \ldots, f_n \rangle$. Clearly, (F4c) implies (F4d). $\square$

Any unitary involution satisfying the equivalent conditions above is called a *flip*. From now on we assume that σ is a flip on V.

Next, let us study the "eigenspaces" of σ in V. For $\lambda \in \mathbb{F}_{q^2}$, define $V_\lambda = \{u \in V \,|\, u^\sigma = \lambda u\}$. Note that V_λ is not a true eigenspace, because σ is not linear. We will see that every non-empty V_λ is a $2n$-dimensional $\mathbb{F}_q$-vector space.

Lemma 3.8
The following hold.

(1) *For* $0 \neq \mu \in \mathbb{F}_{q^2}$, *we have* $\mu V_\lambda = V_{\lambda'}$, *where* $\lambda' = \frac{\bar\mu}{\mu}\lambda$; *in particular,* V_λ *is an* $\mathbb{F}_q$*-subspace of* V.

(2) $V_\lambda \neq 0$ *if and only if* $\lambda\bar\lambda = 1$; *furthermore, if* $V_\lambda \neq 0$, *then* V_λ *contains a basis of* V.

Proof. Suppose $u \in V_\lambda$. Then $(\mu u)^\sigma = \bar\mu u^\sigma = \bar\mu \lambda u = \frac{\bar\mu}{\mu}\lambda(\mu u)$. This proves (1). Also, $u = u^{\sigma^2} = \bar\lambda\lambda u$. Thus, if $u \neq 0$ then $\lambda\bar\lambda = 1$. This proves the 'only if' part of (2). To prove the 'if' part, choose a canonical basis $\{e_1, \ldots, f_n\}$ of V for σ. Fix a $\lambda \in \mathbb{F}_{q^2}$ such that $\lambda\bar\lambda = 1$. Define $u_i = e_i + \bar\lambda f_i$ and $v_i = \bar\lambda e_i + f_i$ for $1 \leq i \leq n$. A simple check shows that u_i and v_i are in V_λ. This shows that $V_\lambda \neq 0$. Furthermore, u_i and v_i are not proportional unless $\bar\lambda = \lambda$, that is, $\lambda \in \mathbb{F}_q$. Thus, if $\lambda \notin \mathbb{F}_q$ then $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a basis of V. If $\lambda \in \mathbb{F}_q$ then consider $\lambda' = \frac{\bar\mu}{\mu}\lambda$, where μ is chosen so that $\frac{\bar\mu}{\mu} \notin \mathbb{F}_q$. By (1), $V_{\lambda'} = \mu V_\lambda$. Also, since $\lambda' \notin \mathbb{F}_q$, the space $V_{\lambda'}$ contains a basis of V, and hence so does V_λ. $\square$

Now fix a $\lambda \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ such that $\lambda\bar\lambda = 1$.

Lemma 3.9
The restriction of λf *to* V_λ *is a nondegenerate* $\mathbb{F}_q$*-quadratic form. It is of plus type if* n *is even and of minus type if* n *is odd.*

Proof. Clearly, the form λf is $\mathbb{F}_q$-quadratic. Since V_λ contains a basis of V by Lemma 3.8 (2), the form is nondegenerate. It remains to see that it takes values in $\mathbb{F}_q$. If $u \in V_\lambda$, then $\overline{\lambda f(u)} = \bar\lambda f(u^\sigma) = \bar\lambda\lambda^2 f(u) = \lambda f(u)$.

To determine the type of λf we compute the form λf on the $\mathbb{F}_q$-vector space $\langle u_i, v_i \rangle$ with respect to the basis u_i, v_i for $1 \leq i \leq n$ where, as above, $u_i = e_i + \bar\lambda f_i$ and $v_i = \bar\lambda e_i + f_i$. We have

$$\lambda f(u_i) = \lambda f(e_i + \bar\lambda f_i) = \lambda f(e_i) + \lambda f(\bar\lambda f_i) + \lambda(e_i, \bar\lambda f_i) = \lambda\bar\lambda = 1;$$
$$\lambda f(v_i) = \lambda f(\bar\lambda e_i + f_i) = \lambda f(\bar\lambda e_i) + \lambda f(f_i) + \lambda(\bar\lambda e_i, f_i) = \lambda\bar\lambda = 1; \text{ and}$$
$$\lambda(u_i, v_i) = \lambda(e_i + \bar\lambda f_i, \bar\lambda e_i + f_i) = \lambda + \bar\lambda.$$

So the form λf on $\langle u_i, v_i \rangle$ with respect to the basis u_i, v_i equals $\alpha^2 + \beta^2 + \alpha\beta(\lambda + \bar\lambda) = (\beta + \alpha\lambda)(\beta + \alpha\bar\lambda)$. We are looking for solutions in $\mathbb{F}_q$ of the equation $0 = (\beta + \alpha\lambda)(\beta + \alpha\bar\lambda)$. However, since $\lambda \notin \mathbb{F}_q$, this equation does not have any solutions in $\mathbb{F}_q$. Therefore the restriction of λf to the $\mathbb{F}_q$-vector space $\langle u_i, v_i \rangle$ is elliptic. The claim about the type of λf follows. $\square$

Observe that the conjugation by σ is an automorphism of $G = S\Omega^+(2n, q^2)$. Let G_σ be the centralizer of σ in G. The above setup gives us means to identify G_σ. Let H be the commutator group of the group of linear transformations of V_λ of determinant 1 preserving the (restriction of the) form λf. By Lemma 3.9 it is isomorphic to $S\Omega^+(2n, q)$ in case n even and isomorphic to $S\Omega^-(2n, q)$ in case n odd. Since V_λ contains a basis of V, we can use $\mathbb{F}_{q^2}$-linearity to extend the action of the elements of H to the entire V. This allows us to identify H with a subgroup of G. Clearly, since $h \in H$ preserves λf, it must also preserve f.

Proposition 3.10
$G_\sigma = H$.

Proof. Choose a basis $\{w_1, \ldots, w_{2n}\}$ in V_λ. Then this set is also a basis of V. Let $h \in H$. If $u = \sum_{i=1}^{2n} x_i w_i \in V$ then $u^{\sigma h} = (\sum_{i=1}^{2n} \bar{x}_i \lambda w_i)^h = \sum_{i=1}^{2n} \bar{x}_i \lambda w_i^h$. On the other hand, $u^{h\sigma} = (\sum_{i=1}^{2n} x_i w_i^h)^\sigma = \sum_{i=1}^{2n} \bar{x}_i \lambda w_i^h$. Therefore, $H \leq G_\sigma$. Now take $h \in G_\sigma$. If $u \in V_\lambda$ then $(u^h)^\sigma = (u^\sigma)^h = (\lambda u)^h = \lambda u^h$. This proves that h leaves V_λ invariant. It remains to see that h preserves λf. However, this is clear, because h is $\mathbb{F}_{q^2}$-linear and it preserves f. □

4 The flipflop geometry Γ

Before studying the geometry we are interested in, let us recall some definitions. Let I be a finite set, called the *set of types*. Its elements as well as its subsets are called *types*. Let $\Gamma = (X, *, \mathrm{typ})$ be a triple where X is a set, $* \subseteq X \times X$ is a symmetric and reflexive relation and $\mathrm{typ} : X \to I$ is a map, such that, for $x, y \in X$ we have $x = y$ if and only if $x * y$ and $\mathrm{typ}(x) = \mathrm{typ}(y)$. Then Γ is called a *pregeometry over* I. The elements of X are called the *elements of* Γ, the relation $*$ is called the *incidence relation of* Γ, the map typ is called the *type function of* Γ.

Let $\Gamma = (X, *, \mathrm{typ})$ be a pregeometry over I. If $A \subseteq X$, then A is of *type* $\mathrm{typ}(A) \subseteq I$, of *cotype* $I \backslash \mathrm{typ}(A)$, of *rank* $|\mathrm{typ}(A)|$, and of *corank* $|I \backslash \mathrm{typ}(A)|$. The rank of A is also denoted by $\mathrm{rk}\,(A)$. The cardinality $|I|$ of I is called the *rank of* Γ.

A *flag* F of a pregeometry Γ is a set of mutually incident elements of Γ. Notice that $\mathrm{typ}_{|F} : F \to I$ is a injection. A *maximal flag* of Γ is a flag that is maximal with respect to inclusion. Flags of type I are called *chambers*. A *geometry over* I is a pregeometry Γ over I in which every maximal flag is a chamber.

Let F be a flag of Γ, say of type $J \subseteq I$. Then the *residue* Γ_F *of* F is the geometry $(X', *_{|X' \times X'}, \mathrm{typ}_{|I \backslash J})$ over $I \backslash J$, with $X' := \{x \in X \mid F \cup \{x\}$ is a flag of Γ and $\mathrm{typ}(x) \notin \mathrm{typ}(F)\}$.

The geometry Γ is *connected* if the graph $(X, *)$ is connected. The geometry Γ is *residually connected* if for any flag F of corank at least two the residue Γ_F is connected.

Finally, if $\Gamma = (X, *, \mathrm{typ})$ and $\Gamma' = (X', *', \mathrm{typ}')$ are two geometries, over I and I', respectively, with $I \cap I' = \emptyset$, then the *direct sum* $\Gamma \oplus \Gamma'$ is the geometry $(X \cup X', *'', \mathrm{typ} \cup \mathrm{typ}')$ over $I \sqcup I'$, with $*''_{|X} = *$, $*''_{|X'} = *'$ and $(X \times X') \subseteq *''$.

We will use the notation from the previous section. In particular, V is a nondegenerate orthogonal $\mathbb{F}_{q^2}$-space of dimension $2n$ with a quadratic form f of plus type and associated symmetric bilinear form $(\cdot, \cdot)$, the map σ is a flip and $((\cdot, \cdot))$ the corresponding Hermitian form. Also, G is isomorphic to $S\Omega^+(2n, q^2)$. Furthermore, G_σ is the centralizer $C_G(\sigma)$ of σ in G. The group G_σ is isomorphic to $S\Omega^+(2n, q)$ if n is even and isomorphic to $S\Omega^-(2n, q)$ if n is odd.

Throughout this section, we assume $n \geq 2$. Let $\mathcal{D}$ be the building geometry associated with G. The elements of $\mathcal{D}$ of type $i = 1, 2, \ldots, n-2$ are the singular subspaces of V of dimension i. The elements of $\mathcal{D}$ of the last two types, $n-1$ and n, are the maximal (n-dimensional) singular subspaces. Two such subspaces U and U' have the same type if and only if $U \cap U'$ has an even codimension in U. Incidence is given by symmetrized containment except incidence between elements of type $n-1$ and n. Two such elements are incident if their intersection is a hyperplane of either element. We will use the customary geometric terminology. In particular, *points*, *lines*, and *planes* are elements of vector space dimension 1, 2, and 3, respectively.

Let $\Gamma = \Gamma_\sigma$ be the pregeometry consisting of those f-singular proper subspaces of V that do not intersect the polar of their image under σ. (See [2] for an explanation why this is a natural object to consider.) The pregeometry Γ is called the *flipflop geometry* of $\mathcal{D}$ associated with σ. Alternatively, we can describe the flipflop geometry Γ as follows.

Proposition 4.1
The elements of Γ are all proper subspaces $U \subset V$ of dimension other than $n-1$, which are singular with respect to f and nondegenerate with respect to $((\cdot,\cdot))$.

Proof. By Lemma 3.2, $U^{\perp\!\!\perp} = (U^\perp)^\sigma$. Hence, if X is the $((\cdot,\cdot))$-radical of U, we have $X = U \cap U^{\perp\!\!\perp} = U \cap (U^\perp)^\sigma$. Therefore $X = \{0\}$ if and only if $U \cap (U^\perp)^\sigma = \{0\}$. □

Recall that a group G of automorphisms of some pregeometry Δ is called *flag-transitive* if for each pair F_1, F_2 of flags with $\mathrm{typ}(F_1) = \mathrm{typ}(F_2)$ there exists a $g \in G$ with $F_1^g = F_2$. Notice that for a geometry Δ this condition is equivalent to the condition that for each pair F_1, F_2 of chambers there exists a $g \in G$ with $F_1^g = F_2$.

Proposition 4.2
The pregeometry Γ is a geometry of rank n. Moreover, G_σ acts flag-transitively on Γ.

Proof. For the first claim we need to show that a maximal flag F in Γ contains elements of all types. If F contains an element of type i less than n then clearly it also contains elements of all types less than i. Suppose m is the highest type present in F and let U be the element of type m in F. Suppose first that $m < n-1$. Let $W = \langle U, U^\sigma \rangle$ and $T = W^\perp$. Since σ is a flip of W, it also is a flip of T. Therefore, by Corollary 3.7 there exists a maximal f-singular subspace X in T, such that $X^\sigma \cap X = \{0\}$. The space X has dimension $n-m$ and thus $\langle U, X \rangle$ is an element of Γ of type $n-1$ or n incident to each element of F. So $m = n-1$ or n. By symmetry it suffices to consider only one of these cases. Suppose that $m = n$. Then the only type possibly missing in F is $n-1$. Let X be a hyperplane in U such that X contains the element of type $n-2$ from F and X is nondegenerate with respect to $((\cdot,\cdot))$. Then X is contained in exactly two f-singular subspaces of dimension n. One of them is U, let Y be the other one. Since $X \cap X^{\perp\!\!\perp} = \{0\}$, the space $Y \cap Y^{\perp\!\!\perp}$ has dimension at most one. Since σ is a flip, Corollary 3.7 implies that $Y \cap Y^{\perp\!\!\perp} = \{0\}$, so Y is nondegenerate for $((\cdot,\cdot))$ and it can be added to F as the missing element of type $n-1$. This shows that Γ is a geometry.

For the second claim, let $V_1, V_2, \ldots, V_n$ and $V_1', V_2', \ldots, V_n'$ be two chambers ordered by types. Choose bases $\mathcal{B} = \{e_1, \ldots, e_n\}$, $\mathcal{B}' = \{e_1', \ldots, e_n'\}$ for V_n and V_n' that are orthonormal with respect to $((\cdot,\cdot))$ and such that $V_i = \langle e_1, \ldots, e_i \rangle$, $V_i' = \langle e_1', \ldots, e_i' \rangle$ for $1 \leq i \leq n-2$ and also $V_{n-1} \cap V_n = \langle e_1, \ldots, e_{n-1} \rangle$ and $V_{n-1}' \cap V_n' = \langle e_1', \ldots, e_{n-1}' \rangle$. Choose some $h \in G$ such that $e_i^h = e_i'$, $(e_i^\sigma)^h = (e_i')^\sigma$. Such an h exists, since G acts transitively on the set of hyperbolic bases of V. Notice that $\sigma \circ h = h \circ \sigma$ on the basis $\mathcal{B} \cup \mathcal{B}^\sigma$ of V. Therefore $h \in G_\sigma$. □

Let us first discuss the cases n equal to two and three. In case $n = 3$, our flipflop geometry has already been studied in [1] in guise of the geometry of nondegenerate subspaces of a four-dimensional $\mathbb{F}_{q^2}$-vector space with respect to a nondegenerate unitary form. Indeed, for $n = 3$ our flipflop geometry is obtained as the geometry Γ_σ of the twin building geometry of type D_3 over the field $\mathbb{F}_{q^2}$. (See Proposition 1 of [14] for a characterization of spherical twin buildings.) Building-theoretically σ interchanges the positive and the negative part of the twin building, interchanges the distances and preserves the codistance, cf. [2]. The twin buildings of type D_3 and of type A_3 over $\mathbb{F}_{q^2}$ are isomorphic, so the image of σ under this isomorphism will be a flip of the twin building geometry of type A_3 over $\mathbb{F}_{q^2}$. It remains to see which flip this image is. It is clear that a flip of the twin building of type $A_3(q^2)$ is induced by a nondegenerate polarity on the projective space $\mathbb{P}(\mathbb{F}_{q^2}^4)$, and since the flip admits a chamber that is mapped to its opposite, this polarity cannot be a symplectic one. So it is orthogonal or unitary. A nondegenerate two-dimensional subspace of a orthogonal space has at least $q^2 - 1$ nondegenerate points, while a nondegenerate two-dimensional subspace of a unitary space has $q^2 - q$ nondegenerate points. Hence, indeed, our flipflop geometry in case $n = 3$ coincides with the flipflop geometry for $n = 3$ from [1]. Therefore all properties of our geometry Γ for $n = 3$ follow from [1].

Theorem 4.3
Let $n = 3$. The geometry Γ is isomorphic to the geometry of nondegenerate subspaces of a nondegenerate unitary space of dimension four over $\mathbb{F}_{q^2}$. In particular, it is connected for all q and simply connected for $q \geq 4$.

See Section 5 for a definition of simple connectedness.

Proof. The first claim follows from the above discussion. The second claim follows from [1]. $\square$

In case $n = 2$, by the above paragraph our Γ_σ is isomorphic to the residue of a line of the geometry of nondegenerate subspaces of a four-dimensional $\mathbb{F}_{q^2}$-vector space with respect to a nondegenerate unitary form. Hence Γ_σ is a generalized digon, which certainly is connected.

This discussion shows that the desired properties of Γ_σ hold true for n equal to two and three. This means that in the remainder of the paper we can assume $n \geq 4$, which we will do unless it is specified otherwise.

The following lemma will be very useful throughout the article. Recall that the points and the lines are the elements of Γ of types one and two respectively.

Lemma 4.4
Let p be a point of Γ and $W \supset p$ be a three-dimensional f-singular subspace of V of $((\cdot,\cdot))$-rank at least two. Let U be a two-dimensional subspace of W that contains at least one point and does not contain p. Then U contains at least $q^2 - 2q - 1$ (respectively, $q^2 - q - 1$) points that are collinear with p if it is (respectively, is not) a line.

Proof. Since W is f-singular, we only need to consider $((\cdot,\cdot))$. Notice that, if U is a two-dimensional subspace of W that is not totally isotropic with respect to $((\cdot,\cdot))$, then U contains $q^2 - q$ points, if U is a line, and it contains q^2 points if it is not a line.

Consider $U_1 = p^{\perp\!\perp} \cap W$. Then, by the above, among the $q^2 + 1$ two-dimensional subspaces on p in W, at least $q^2 - q$ meet U_1 in a point and hence they are lines. If U is itself a line, then at most $q + 1$ of those lines do not meet U in a point of Γ. This leaves at least $q^2 - 2q - 1$ lines on p meeting U in a point. If U is not a line then at most one of the $q^2 - q - 1$ lines on p does not meet U in a point. Hence the lemma follows. $\square$

We need to prove that the geometry Γ is connected. This follows from the connectedness of the *collinearity graph* of Γ, i.e., the graph on the points of Γ in which two points are adjacent if and only if they are collinear.

Lemma 4.5
The collinearity graph of the geometry Γ has diameter two. In particular, Γ is connected.

Proof. Suppose $n \geq 5$. Let p_1, p_2 be distinct points of the geometry. Consider $W := \langle p_1, p_2 \rangle^\perp \cap \langle p_1, p_2 \rangle^{\perp\!\!\perp}$. Then $\dim W \geq 2n - 4$. Moreover, the space W is σ-invariant and has rank at least $2n-6$, which is at least four. Indeed, $W^\perp = \langle p_1, p_2, p_1^\sigma, p_2^\sigma \rangle$ has rank at least two and hence its radical is at most two-dimensional. Therefore Lemma 3.5 yields a point of Γ inside that complement. This point is collinear in Γ to both p_1 and p_2.

So now suppose $n = 4$. Take p_1, p_2, and W as above. If the rank of W is at least three, then again Lemma 3.5 yields a common neighbor of p_1 and p_2. The only case that W does not have rank at least three occurs in case of $\dim W = 4$ and $\mathrm{rk}\, W = 2$. If that happens, let W_1 be equal to $p_1^\perp \cap p_1^{\perp\!\!\perp}$, which is six-dimensional and nondegenerate. Moreover, σ is a flip of W_1, since it is a flip of $\langle p_1, p_1^\sigma \rangle$. Take U to be a maximal f-singular subspace of W. Notice that U necessarily has to be three-dimensional, as W is the direct sum of its radical and a hyperbolic line. Since σ is a flip of W_1, the intersection $U \cap U^\sigma$ has to be even-dimensional by Corollary 3.7. Hence W contains a point of Γ, which is collinear to p_1 and p_2.

Connectedness of the collinearity graph and hence the geometry follow from the finiteness of its diameter. □

We summarize Lemma 4.5 and the results of [1] on connectedness in the following theorem and corollary.

Theorem 4.6
Let $n \geq 2$. Then Γ is connected. □

Corollary 4.7
The geometry Γ is residually connected unless $q = 2$.

Proof. The residues of Γ are either direct sums (and as such connected) or isomorphic to our geometry Γ in some smaller dimension (and as such connected by Theorem 4.6) or isomorphic to geometry of nondegenerate subspaces of some $\mathbb{F}_{q^2}$-vector space with respect to a nondegenerate unitary form. The latter one however is not connected in case of a three-dimensional $\mathbb{F}_{2^2}$-vector space, see [1]. □

5 Fundamental group and simple connectedness

Let Γ be a connected geometry. A path of length k in the geometry is a sequence of elements $(x_0, \ldots, x_k)$ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. A *cycle* based at an element x is a path in which $x_0 = x_k = x$. Two paths are *homotopically equivalent* if one can be obtained from the other via the following operations (called *elementary homotopies*): inserting or deleting a repetition (i.e., replacing x by xx or vice versa), a return (i.e., replacing x by xyx or vice versa), or a triangle (i.e., replacing x by $xyzx$ or vice versa). The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the *fundamental group* of Γ and denoted by $\pi_1(\Gamma, x)$. A cycle based at x that is homotopically equivalent to the trivial cycle (x) is called *null-homotopic*. Every cycle of length 2 or 3 is null-homotopic.

Suppose Γ and $\hat{\Gamma}$ are geometries over the same type set and suppose $\phi : \hat{\Gamma} \to \Gamma$ is a *homomorphism* of geometries, i.e., ϕ preserves the types and sends incident elements to incident elements. A surjective homomorphism ϕ between connected geometries $\hat{\Gamma}$ and Γ is called a *covering* if and only if for every non-empty flag $\hat{F}$ in $\hat{\Gamma}$ the mapping ϕ induces an isomorphism between the residue of $\hat{F}$ in $\hat{\Gamma}$ and the residue of $F = \phi(\hat{F})$ in Γ. Coverings of a geometry correspond to the usual topological coverings of the flag complex. It is well-known that a surjective homomorphism ϕ between connected geometries $\hat{\Gamma}$ and Γ is a covering if and only if for every element $\hat{x}$ in $\hat{\Gamma}$ the map ϕ induces an isomorphism between the residue of $\hat{x}$ in $\hat{\Gamma}$ and the residue of $x = \phi(\hat{x})$ in Γ. If ϕ is an isomorphism, then the covering is said to be *trivial*.

Recall the following result.

Theorem 5.1
Let Γ be a connected geometry and let x be an element of Γ. Then every covering of the geometry Γ is trivial if and only if $\pi_1(\Gamma, x)$ is trivial.

Proof. See [11]. $\square$

A geometry satisfying the equivalent conditions in the previous theorem is called *simply connected*.

A *geometric cycle* in the geometry $\mathcal{G}$ is a cycle each element of which is incident with a common element x.

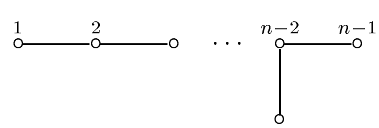
Proposition 5.2
Every geometric cycle is null-homotopic.

Proof. Suppose $\gamma = x_1x_2 \ldots x_kx_1$ is a cycle without returns all of its elements are incident with some element x. If $k \leq 3$ then γ is null-homotopic by definition. So we assume that $k > 3$. If $x_1 = x$ or $x_3 = x$ then x_1 is incident to x_3 and so γ is homotopic to a shorter geometric cycle, namely $x_1x_3 \ldots x_1$. Similarly, if $x_2 = x$ or $x_4 = x$ then γ is homotopic to $x_1x_2x_4 \ldots x_1$. Finally, if $x \neq x_i$, $i \leq 4$, then γ is homotopic to $x_1xx_4 \ldots x_1$, by inserting the triangle (x_4, x_2, x, x_4). Thus, in all cases γ is homotopic to a shorter geometric cycle, and the claim follows by induction. $\square$

Corollary 5.3
If two cycles are obtained from one another by inserting or erasing a geometric cycle then they are homotopic. $\square$

Let $\mathcal{G}$ be a geometry over the set I. Let $i, j \in I$, then we define $i \sim j$ if there exists a flag F of cotype $\{i, j\}$ such that the residue of F is a geometry containing two elements that are not incident. Then the graph $(I, \sim)$ is called the *digon diagram of $\mathcal{G}$*.

Lemma 5.4
Let Γ be a geometry of rank $n \geq 4$ with digon diagram



and assume that for each element x of type $n-1$ or n the collinearity graph of Γ_x is connected. Furthermore, suppose that if the residue Γ_y of some element y has a disconnected diagram falling into connected components Δ_1, Δ_2, Δ_3 (one of those may be empty), then Γ_x is equal to the direct sum of the three truncations of Γ_y with respect to $\mathrm{typ}(\Delta_1)$, $\mathrm{typ}(\Delta_2)$ *and* $\mathrm{typ}(\Delta_3)$. *Then every cycle of Γ based at some element of type 1 or 2 is homotopically equivalent to a cycle passing exclusively through elements of type 1 or 2.*

Proof. We will induct on the number of elements of the path that are not of type 1 or 2. If this number is zero there is nothing to prove. Take an arbitrary cycle $\gamma := x = x_0x_1 \dots x_{t-1}x_t = x$. Let x_s be the first element that is not of type 1 or 2. Clearly $s \notin \{0, t\}$. There are the following cases to consider:

If the type of x_s is less or equal $n-2$ and if the type of x_{s+1} is bigger than the type of x_s then x_{s-1} and x_{s+1} are incident as they belong to two different direct summands of Γ_{x_s}. Thus γ is homotopically equivalent to the cycle $xx_1 \dots x_{s-1}x_{s+1} \dots x$.

Suppose the type of x_{s+1} is smaller than the type of x_s. Let y be an element of type $n-1$ or n which is incident to x_s (in particular, take x_s, if the type of x_s is $n-1$ or n), then y is incident to both x_{s-1} and x_{s+1}. Indeed, either $y = x_s$ and there is nothing to prove or y is contained in another direct summand of Γ_{x_s} than x_{s-1}, x_{s+1}. Therefore, by Proposition 5.2, γ is homotopically equivalent to the path $xx_1 \dots x_{s-1}yx_{s+1} \dots x$. Now pick two elements z, w of type 1 such that z is incident to x_{s+1} and w is x_{s-1}, if x_{i-1} is a point, or a point incident to x_{i-1}, otherwise. Using the hypothesis we can connect w and z with a path $ww_1 \dots w_kz$ passing exclusively through elements of type 1 and 2, all of which are incident with y. Again by Proposition 5.2 γ is homotopically equivalent to $xx_1 \dots x_{s-1}w_1 \dots w_kzx_{s+2} \dots x$ which contains fewer elements that are not of type 1 or 2.

Notice that the above paragraph includes the case $\mathrm{typ}(x_{s-1}) \in \{1, 2\}$, $\mathrm{typ}(x_{s+1}) = n-1$, $\mathrm{typ}(x_s) = n$. The only case missing altogether is $\mathrm{typ}(x_{s-1}) \in \{1, 2\}$, $\mathrm{typ}(x_{s+1}) = n$, $\mathrm{typ}(x_s) = n-1$, which holds by interchanging the labels $n-1$ and n. □

Lemma 5.5
Assume that $\Gamma = \Gamma_1 \oplus \Gamma_2$ with Γ_1 connected of rank at least two. Then Γ is simply connected.

Proof. See Lemma 7.2 of [5]. □

Our strategy of proof for the simple connectedness of the geometry Γ is to establish that its fundamental group is trivial. We want to apply Lemma 5.4 to Γ so that it will suffice to prove that every cycle passing through only points and lines is homotopically trivial. The residue of an element of type $n-1$ or n of Γ is isomorphic to the geometry of nondegenerate subspaces with respect to some unitary form studied in [1]. There it is proved that Γ satisfies the hypothesis of Lemma 5.4 on the collinearity graph of the residues for $n \geq 4$. The direct sum property required in the hypothesis of Lemma 5.4 follows by the definition of Γ. Hence we can restrict ourselves to cycles passing through points and lines only. Since any pair of distinct collinear points of Γ uniquely determines the line incident to both points it actually suffices to study cycles in the collinearity graph of Γ. Finally, by Lemma 4.5, any cycle of length six or more automatically decomposes into smaller cycles. Therefore all we need to establish is that arbitrary triangles, quadrangles, and pentagons in the collinearity graph of Γ are homotopically trivial. The next two sections deal with that problem.

6 Simple connectedness, Part 1

In this section we only deal with the case $n \geq 4$ and $q \geq 3$. First we show that every triangle can be decomposed into geometric triangles.

Lemma 6.1
All triangles are decomposable.

Proof. Consider a triangle with vertices $\langle u \rangle$, $\langle v \rangle$ and $\langle w \rangle$. Let $U = \langle u, v, w \rangle$. Then U is totally singular with respect to $(\cdot, \cdot)$. If it is nondegenerate with respect to $((\cdot, \cdot))$ then the triangle is geometric. So we can assume that U is degenerate. Since U contains a line it cannot have rank less than two. This means it has rank exactly two and its radical R is one-dimensional, hence $R = \langle r \rangle$ for some $r \in U$.

First let us suppose that $R^\sigma \neq R$. Consider the line $L = \langle u, v \rangle$. Let $W = L^\perp \cap L^{\perp\!\!\perp}$. Then W is a nondegenerate σ-invariant subspace of codimension four and, hence, of dimension at least four. Furthermore, with respect to $(\cdot, \cdot)$, the space W is of plus type, since V and $\langle L, L^\sigma \rangle = W^\perp$ are of plus type. Notice that $R, R^\sigma \subset W$. Let $X = R^\perp \cap W$. Consider W with respect to just $(\cdot, \cdot)$. Since W is nondegenerate of plus type, X is generated by singular vectors. In particular, there is a singular one-dimensional space $\langle t \rangle$ which is contained in X but not in X^σ. Then the space $\langle u, v, w, t \rangle$ is totally singular with respect to $(\cdot, \cdot)$ and nondegenerate with respect to $((\cdot, \cdot))$. Thus, $\langle u, v, w, t \rangle$ is an element of Γ, and so our triangle is geometric.

Now suppose $R = R^\sigma$ and choose $r \in R$ such that $r = r^\sigma$. We claim that we can choose a four-dimensional subspace W such that $U \subset W$, W is totally singular with respect to $(\cdot, \cdot)$ and its rank with respect to $((\cdot, \cdot))$ is exactly two. Indeed consider first a maximal totally singular subspace X containing U. Since σ is a flip, the radical Y of X with respect to $((\cdot, \cdot))$ has even dimension (since $Y = X \cap X^\sigma$). Notice that $R \subset Y$. Let S be any other one-dimensional space in Y. Then $W = \langle U, S \rangle$ is as required. Let now Y be the two-dimensional radical of W.

Let x be a nonzero vector in $\langle u, v \rangle \cap \langle w, r \rangle$. After a suitable scaling we can assume that $x = u + v$ and $w = r + x$. Notice that x is a nonsingular vector, since w is nonsingular. Pick $a \in \mathbb{F}_{q^2}$, $a \neq 0, 1$, so that $u + av$ is nonsingular. Let us consider vectors t of the form $u + av + y$, where $y \in Y \setminus R$. Then t is nonsingular and the point $\langle t \rangle$ is collinear with $\langle u \rangle$, $\langle v \rangle$, and $\langle w \rangle$ (since $\langle u, t \rangle$, $\langle v, t \rangle$, and $\langle w, t \rangle$ are complements to Y in W). This allows us to decompose the triangle $\langle u \rangle$, $\langle v \rangle$, $\langle w \rangle$ as a product of three triangles.

Let us compute the radicals of the three-dimensional spaces that these triangles generate. Those radicals are the intersections of the respective three-dimensional spaces with Y. It is easy to compute that $\langle u, v, t \rangle \cap Y = \langle y \rangle$, $\langle u, w, t \rangle \cap Y = \langle y - ar \rangle$, and $\langle v, w, t \rangle \cap Y = \langle y - r \rangle$.

We first assume that Y is not σ-invariant. In that case $Y \cap Y^\sigma = R$ and so none of the above three radicals can be σ-invariant. Hence the three new triangles are geometric, and hence the triangle $\langle u \rangle$, $\langle v \rangle$, $\langle w \rangle$ is decomposable. Finally, we deal with the case $Y = Y^\sigma$. If we can choose y so that none of the one-dimensional spaces $\langle y \rangle$, $\langle y - r \rangle$, and $\langle y - ar \rangle$ is σ-invariant then again the three new triangles are geometric and our initial triangle is decomposable. Note that Y contains exactly q one-dimensional σ-invariant subspaces aside from r. Indeed, assuming that s and r are σ-invariant vectors in Y, the one-dimensional space $\langle s + \alpha r \rangle$ is σ invariant if and only if $\bar{\alpha} = \alpha$ and so there are q choices for α.

Now each of the $(q^2 - 1)q$ vectors in the invariant spaces besides R can occur in at most three triples $\langle y \rangle$, $\langle y - r \rangle$, and $\langle y - ar \rangle$. The total number of triples is $q^4 - q^2$ (once we pick y, the triple is determined) and there are at most $3(q^2 - 1)q$ triples that contain at least one bad one-dimensional space. Note however that if we pick y to be an σ-invariant vector, then both $\langle y \rangle$, $\langle y - r \rangle$ are σ-invariant one-dimensional spaces hence the number of bad triples is strictly less than $3(q^3 - q)$. If $q \geq 3$ this assures the existence of a good triple. $\square$

We will consider 4-gons next. When studying them, the following lemma will prove useful.

Lemma 6.2
Let U be a four-dimensional $(\cdot,\cdot)$-nondegenerate subspace of V of Witt index two and of $((\cdot,\cdot))$-rank at least one. Then V contains a point of Γ.

Proof. If U has $((\cdot,\cdot))$-rank one, any two-dimensional $(\cdot,\cdot)$-totally singular subspace of U not inside the $((\cdot,\cdot))$-radical of U contains points of Γ.

If U has $((\cdot,\cdot))$-rank two, then it has a two-dimensional $((\cdot,\cdot))$-radical X. Any $(\cdot,\cdot)$-totally singular two-dimensional subspace of U that does not intersect X necessarily is $((\cdot,\cdot))$-nondegenerate, so it contains points of Γ.

If the $((\cdot,\cdot))$-rank of U equals three, then any $(\cdot,\cdot)$-totally singular two-dimensional subspace of U not containing the $((\cdot,\cdot))$-radical of U has $((\cdot,\cdot))$-rank at least one. Indeed, U does not contain three-dimensional $((\cdot,\cdot))$-totally isotropic subspaces. Hence U contains points of Γ.

If U is $((\cdot,\cdot))$-nondegenerate, then the claim follows from the fact that the unitary quadrangle $H(3,q^2)$ does not contain a subquadrangle isomorphic to $Q^+(3,q^2)$, see [8]. $\square$

Lemma 6.3
Let $q \geq 3$. Then any quadrangle inside a $(\cdot,\cdot)$-totally isotropic subspace of V is null-homotopic.

Proof. Let a, b, c, d be a quadrangle such that $(\cdot,\cdot)$ vanishes on $\langle a,b,c,d\rangle$.

If $\langle a,b,c,d\rangle$ is three-dimensional then it can have $((\cdot,\cdot))$-rank two or three. If its $((\cdot,\cdot))$-rank is three, then a, b, c, d is a geometric cycle and, thus, null-homotopic. So we can assume that its $((\cdot,\cdot))$-rank is two. But then any complement of its radical X is a line of Γ. Therefore c and d have a common neighbor on the line $\langle a,b\rangle$, since that contains at least six points.

If $\langle a,b,c,d\rangle$ is four-dimensional then it can have $((\cdot,\cdot))$-rank two, three or four. In case of $((\cdot,\cdot))$-rank four the cycle a, b, c, d again is geometric, whence null-homotopic. If its $((\cdot,\cdot))$-rank is two, then the span $\langle a,b,c\rangle$ intersects the two-dimensional radical X in a one-dimensional space X_1. Any two-dimensional subspace of $\langle a,b,c\rangle$ missing X_1 is a complement of X and, thus, a line of Γ. Hence exactly one of the at least six points of Γ on the line $\langle a,b\rangle$ is not collinear to c, leaving at least five points that are collinear to c. By symmetry, d is not collinear to a unique point of $\langle a,b\rangle$, whence there are at least four points on $\langle a,b\rangle$ collinear to both c and d, decomposing the quadrangle.

Finally, assume the $((\cdot,\cdot))$-rank of $\langle a,b,c,d\rangle$ is three. Let X be its one-dimensional $((\cdot,\cdot))$-radical. If X is contained in $\langle a,b,c\rangle$, then $\langle a,b,c\rangle$ has $((\cdot,\cdot))$-rank two and any two-dimensional subspace of it missing X is a line of Γ. Hence, in this case c is collinear to all points of $\langle a,b\rangle$ except one. On the other hand, by Lemma 4.4 the point d is collinear to q^2-2q-1 points of $\langle a,b\rangle$. Removing the point that c is not collinear to if necessary, there remain q^2-2q-2 points of $\langle a,b\rangle$ collinear to both c and d. Since $q \geq 3$, this is a positive number.

So we can assume that X is not contained in $\langle a,b,c\rangle$. Consider the two-dimensional subspace $\langle d,X\rangle$. It intersects $\langle a,b,c\rangle$ in some one-dimensional space e distinct from X. Therefore, as the $((\cdot,\cdot))$-rank of $\langle d,X\rangle$ is one, e is a point of Γ. Since the $((\cdot,\cdot))$-rank of $\langle a,d,X\rangle = \langle a,d,e\rangle$ is two (indeed, it contains the $((\cdot,\cdot))$-radical X, but also the line $\langle a,d\rangle$), any two-dimensional subspace of $\langle a,d,X\rangle$ missing X is a line of Γ. In particular, $\langle a,e\rangle$ is a line of Γ. For the same reason, the space $\langle c,e\rangle$ is a line of Γ. We have decomposed the quadrangle a, b, c, d into the quadrangle a, b, c, e (which lies inside the three-dimensional space $\langle a,b,c\rangle$ and by the above is null-homotopic) and the quadrangle a, e, c, d (which has the property that the radical X lies inside $\langle a,d,e\rangle$ and hence is null-homotopic by the preceding paragraph). $\square$

Remark 6.4 There exists a much shorter proof for $q \geq 4$. Indeed, by Lemma 4.4 there exist q^2-3q-2 points on $\langle a,b\rangle$ collinear to both c and d, decomposing the quadrangle. This example illustrates that studying the flipflip geometries over small fields may be quite difficult. And, indeed, we did not succeed to decompose pentagons in case $(n,q)=(4,3)$, but we have to rely on a computer based computation instead.

Lemma 6.5
Let $q \geq 3$. Any quadrangle with a $(\cdot,\cdot)$-perpendicular pair of opposite points is null-homotopic.

Proof. Let a, b, c, d be a quadrangle with $a \perp c$. In view of the preceding lemma we can assume that $b \not\perp d$. In that case $\langle a,c\rangle$ is the radical of $\langle a,b,c,d\rangle$ and also of $W=\langle a,c\rangle^\perp$. It follows that $\langle a,c\rangle$ is $((\cdot,\cdot))$-degenerate or that a and c are collinear and the quadrangle decomposes in two triangles. Hence assume $\langle a,c\rangle$ is $((\cdot,\cdot))$-degenerate. Then it has a one-dimensional $((\cdot,\cdot))$-radical X. For each $v \in W$ we will denote by v' its image in $W'=W/\langle a,c\rangle$. We will identify W' with some complement of $\langle a,c\rangle$ in W containing b and d. (If no such complement exists, then $\langle a,c\rangle$ and $\langle b,d\rangle$ have a nontrivial intersection, whence $\langle a,b,c,d\rangle$ is $(\cdot,\cdot)$-totally isotropic, so we are in the case of the preceding lemma.) Note that the pre-image of a vector of W' is an affine two-dimensional subspace of W and the pre-image of a one-dimensional subspace of W' is a three-dimensional subspace of W.

Choose a $(\cdot,\cdot)$-totally singular two-dimensional subspace l of W' through b and an opposite $(\cdot,\cdot)$-totally singular two-dimensional subspace m of W' through d. Notice that the pre-images $\langle a,b,c\rangle$ of b and $\langle a,c,d\rangle$ of d in W have rank two or three with respect to $((\cdot,\cdot))$ as they contain lines of Γ. Therefore both l and m each contain at most $q+1$ one-dimensional subspaces whose pre-images in W have $((\cdot,\cdot))$-rank one. Consequently, we can find a one-dimensional subspace z_1 of l and a $(\cdot,\cdot)$-perpendicular one-dimensional subspace z_2 of m such that the pre-images of both z_1 and z_2 in W have $((\cdot,\cdot))$-rank two or three.

It is possible to find a common neighbor p of a and c in $\langle a,c,z_1\rangle$ and a common neighbor q of a and c in $\langle a,c,z_2\rangle$. Since $z_1 \perp z_2$ and $\langle a,c\rangle$ is the $(\cdot,\cdot)$-radical of W we also have $p \perp q$. Similarly, $b \perp z_1$ and $d \perp z_2$ implies $b \perp p$ and $d \perp q$. Consequently we have decomposed the quadrangle a, b, c, d into the quadrangles a, b, c, p and a, p, c, q and a, q, c, d, all three span a $(\cdot,\cdot)$-totally isotropic subspace and by Lemma 6.3 are null-homotopic. $\square$

Lemma 6.6
Let $q \geq 3$. Any quadrangle is null-homotopic.

Proof. In view of Lemmas 6.3 and 6.5 we may assume that the quadrangle a, b, c, d has the property $a \not\perp c$ and $b \not\perp d$. Therefore the span $\langle a,b,c,d\rangle$ must be four-dimensional and the $(\cdot,\cdot)$-rank of $\langle a,b,c,d\rangle$ must be four. Therefore the space $\langle a,b,c,d\rangle^\perp$ contains a point e of Γ by Lemma 6.2. By Lemma 4.4 there exist at least q^2-2q-1 points of each line of the quadrangle a, b, c, d collinear to e. We have decomposed the quadrangle a, b, c, d into quadrangles satisfying the hypothesis of Lemma 6.5, whence a, b, c, d is null-homotopic. $\square$

Lemma 6.7
Let $q \geq 3$. Any pentagon a, b, c, d, e with $a \perp c$ and $a \perp d$ is null-homotopic.

Proof. By Lemma 4.4 the line $\langle c, d\rangle$ contains $q^2 - 2q - 1$ points of Γ collinear to a, decomposing the pentagon. $\square$

Lemma 6.8
Let $q \geq 3$ and $n \geq 5$. Then any pentagon is null-homotopic.

Proof. Let a, b, c, d, e be a pentagon. In view of the preceding lemma we can assume $a \not\perp d$ and $c \not\perp e$. Consequently, $\langle a, c, d\rangle$ has $(\cdot,\cdot)$-rank two, and its radical X, which is distinct from d, is contained in $\langle c, d\rangle$. Since $c \not\perp e$ and $d \perp e$ we have $e \not\perp X$. Therefore $\langle a, c, d, e\rangle$ has to be four-dimensional and its $(\cdot,\cdot)$-rank is four. Hence the $(\cdot,\cdot)$-rank of $\langle a, b, c, d, e\rangle$ is at least four, and so is the $(\cdot,\cdot)$-rank of $\langle a, b, c, d, e\rangle^\perp$, which has at least dimension five. Moreover, the $((\cdot,\cdot))$-rank of $\langle a, b, c, d, e\rangle$ is at least two, as it contains points, so the $((\cdot,\cdot))$-rank of $\langle a, b, c, d, e\rangle^\perp$ also is at least two. Hence we can choose a $(\cdot,\cdot)$-nondegenerate four-dimensional subspace of $\langle a, b, c, d, e\rangle^\perp$ that has $((\cdot,\cdot))$-rank at least one, so by Lemma 6.2 the space $\langle a, b, c, d, e\rangle$ contains a point f of Γ. By Lemma 4.4 there exists points on $\langle a, b\rangle$, $\langle b, c\rangle$, $\langle c, d\rangle$, $\langle d, e\rangle$, $\langle e, a\rangle$ collinear to f, decomposing a, b, c, d, e into quadrangles. $\square$

Lemma 6.9
Let $q \geq 4$ and let $n = 4$. Then any pentagon is null-homotopic.

Proof. Let a, b, c, d, e be a pentagon. In view of Lemma 6.7 we can assume that $a \not\perp d$. Hence the $(\cdot,\cdot)$-rank of $\langle a, c, d\rangle$ is two, and the $(\cdot,\cdot)$-radical X of $\langle a, c, d\rangle$ is contained in $\langle c, d\rangle$. Therefore the $(\cdot,\cdot)$-rank of $\langle a, c, d\rangle^\perp$ is four. Note that X cannot be the $((\cdot,\cdot))$-radical of $\langle a, c, d\rangle^\perp$, as then it also would be the $((\cdot,\cdot))$-radical of $\langle a, c, d\rangle^\sigma$. However, this is impossible as $\langle c, d\rangle$ and $\langle c, d\rangle^\sigma$ are $((\cdot,\cdot))$-nondegenerate.

If X is a point of Γ, then consider $\langle a, c, d\rangle^\perp \cap X^{\perp\!\!\perp}$. This space is a complement of X inside $\langle a, c, d\rangle^\perp$ and as such it is $(\cdot,\cdot)$-nondegenerate. Moreover, the $((\cdot,\cdot))$-rank of $\langle a, c, d\rangle$ is at least two, hence so is the $((\cdot,\cdot))$-rank of $\langle a, c, d\rangle^\perp$, whence $\langle a, c, d\rangle^\perp \cap X^{\perp\!\!\perp}$ is not $((\cdot,\cdot))$-totally isotropic and by Lemma 6.2 we can find a point p of Γ in that space. Then $\langle X, p\rangle$ is a line of Γ.

If X is not a point of Γ, then we chose any $(\cdot,\cdot)$-singular one-dimensional subspace p of $\langle a, c, d\rangle^\perp \backslash X^{\perp\!\!\perp}$ and $\langle X, p\rangle$ is also a line of Γ.

That line $\langle X, p\rangle$ contains at least $q^2 - 3q - 2$ points of Γ collinear to a and d, which is at least one since $q \geq 4$, say f. Therefore we have decomposed the pentagon a, b, c, d, e into a quadrangle a, f, d, e and a pentagon a, b, c, d, f, in which $f \perp c$.

If also $f \perp b$, then we are done by Lemma 6.7. If $f \not\perp b$, then we can repeat the whole argument of the present proof for the pentagon f, d, c, b, a instead of a, b, c, d, e. We will then obtain another quadrangle and a pentagon f, d, c, b, g with $c \perp f$ and $c \perp g$, which is null-homotopic by Lemma 6.7. $\square$

7 Simple connectedness, Part 2

In this section we deal with the case $q = 2$ and $n \geq 5$. First of all note that in this case two points p and q of Γ are collinear if and only if $p \perp q$ and $p \perp\!\!\perp q$. Therefore we do not have to worry about triangles.

Lemma 7.1
Any triangle of Γ is geometric.

Proof. Let a, b, c be a triangle. Then $a \perp\!\!\perp b$, $b \perp\!\!\perp c$ and $c \perp\!\!\perp a$ implies that the $((\cdot,\cdot))$-rank of $\langle a,b,c\rangle$ is three, whence $\langle a,b,c\rangle$ is an element of Γ, so that the triangle a, b, c is geometric. $\square$

Lemma 7.2
Any quadrangle of Γ is null-homotopic.

Proof. Let a, b, c, d be a quadrangle. Assume first that $\langle a,c,a^\sigma,c^\sigma\rangle$ is of dimension four. Let $W := \langle a,c\rangle^\perp \cap \langle a,c\rangle^{\perp\!\!\perp}$. It is σ-invariant, has dimension $2n-4$, (which is at least six) and has a zero- or two-dimensional $(\cdot,\cdot)$-radical X (as $(\cdot,\cdot)$ is alternating), which at the same time is the $((\cdot,\cdot))$-radical. By Lemma 3.4 there exists a σ-invariant $(\cdot,\cdot)$- and $((\cdot,\cdot))$-nondegenerate complement W' to that radical in W, and W' has dimension at least four.

On the space W' the map σ acts as a flip. Indeed, consider a maximal $(\cdot,\cdot)$-totally singular subspace of V that is generated by X, a and a maximal $(\cdot,\cdot)$-totally isotropic subspace U of W'. Since a is a point of Γ, it is moved by σ, whence the intersection $U \oplus \langle X,a\rangle \cap U^\sigma \oplus \langle X,a\rangle^\sigma$ equals the intersection $U \oplus X \cap U^\sigma \oplus X^\sigma$, which in turn is equal to $(U \cap U^\sigma) \oplus X$. Since X has even dimension and $U \oplus \langle X,a\rangle \cap U^\sigma \oplus \langle X,a\rangle^\sigma$ has even dimension (it is the intersection of a maximal $(\cdot,\cdot)$-totally singular subspace with its image under the flip σ) also the intersection $U \cap U^\sigma$ has even dimension. Therefore, σ has to be a flip of W', as it is conjugate to either σ_1 or σ_2 by Proposition 3.6 and only σ_1 has the property that a maximal $(\cdot,\cdot)$-totally isotropic subspace intersects its image in a subspace of even dimension.

Notice that b and d live in W and that their projections onto W' (with respect to the decomposition $W = W' \oplus X$) are points and they can be connected in W' by Theorem 4.6, hence so can b and d and, thus, the quadrangle can be decomposed into triangles, since all these points are collinear with a and c.

It remains to consider the case where both $\langle a,c,a^\sigma,c^\sigma\rangle$ and $\langle b,d,b^\sigma,d^\sigma\rangle$ are three-dimensional. Notice that each of these two spaces is of $(\cdot,\cdot)$- and $((\cdot,\cdot))$-rank two and that they are $(\cdot,\cdot)$- and $((\cdot,\cdot))$-perpendicular. Let $U = \langle a,b,c,d\rangle + \langle a,b,c,d\rangle^\sigma$. If U is of dimension less than six, then it has $(\cdot,\cdot)$-radical of dimension at most one, which means that $U^\perp$ has $(\cdot,\cdot)$-rank at least four. Now consider a σ-invariant complement to the radical in $U^\perp$ and using Lemma 3.5 we see that $U^\perp$ contains a point of Γ, which is collinear to a, b, c, d.

So we can assume that U has dimension six and that its $(\cdot,\cdot)$-radical X has dimension two. Let W be σ invariant complement to X in $U^\perp$. Then W is at least two-dimensional and $(\cdot,\cdot)$-nondegenerate. If X contains a σ-invariant f-nonsingular one-dimensional subspace, its span with W is σ-invariant and f-nondegenerate of dimension at least three. So it contains a point by Lemma 3.5. Thus we can assume that X is f-singular. We claim that W is of plus type and that σ is a flip of W. Indeed, the span of U and W equals to $X^\perp$. Consider the quotient $X^\perp/X$. Since X is f-singular, this is a space of plus type. The image of U is of plus type, so the image of W (which is isometric to W) is of plus type, too. Since X is two-dimensional, σ induces a flip on $X^\perp/X$. It also induces a flip on the image of U, thus it induces a flip on the image of W, whence on W. Therefore W contains points. $\square$

Lemma 7.3
Any pentagon of Γ is null-homotopic.

Proof. Let a, b, c, d, e be a pentagon. Let W be $\langle c,d\rangle^\perp \cap \langle c,d\rangle^{\perp\!\!\perp}$ and let U be $\langle a,c,d\rangle^\perp \cap \langle a,c,d\rangle^{\perp\!\!\perp}$. The space W is nondegenerate σ-invariant of dimension $2n-4$. Moreover, σ is a flip on W, because σ is a flip on $\langle c,d,c^\sigma,d^\sigma\rangle$, as that space contains the line $\langle c,d\rangle$. The space U is σ-invariant of dimension at least $2n-6$ and has rank at least $2n-8$. By Lemma 3.5 the space U contains a point of Γ unless it has rank exactly two, in which case $n=5$, and σ is not a flip on the complement of the radical. If the

$(\cdot,\cdot)$-radical X of U is not f-singular, then we can choose a σ-invariant f-nonsingular vector in X. Taking the span of this vector together with a σ-invariant complement Y of X in U produces a three-dimensional f-nondegenerate σ-invariant space, in which we can find a point of Γ by Lemma 3.5. So we can assume that X is f-singular. In this last case, as in the previous proof, show that Y is of plus type and σ is a flip on Y. Indeed, the image of $U^\perp$ in $X^\perp/X$ is of plus type and σ induces a flip on it. □

Proof of Theorem 1 and Theorem 2. The claims (1) follow by Proposition 3.10 and Proposition 4.2. The claims (2) follow by Theorem 4.6 and Corollary 4.7. Claims (3) follow by Theorem 4.3 and Sections 6 and 7 and Appendix A. □

8 Consequences of simple connectedness

In the present paper an *amalgam* $\mathcal{A}$ of groups is a set with a partial operation of multiplication and a collection of subsets $\{H_i\}_{i\in I}$, for some index set I, such that the following hold:

(1) $\mathcal{A} = \cup_{i\in I} H_i$;

(2) the product ab is defined if and only if $a, b \in H_i$ for some $i \in I$;

(3) the restriction of the multiplication to each H_i turns H_i into a group; and

(4) $H_i \cap H_j$ is a subgroup in both H_i and H_j for all $i, j \in I$.

It follows that the groups H_i share the same identity element, which is then the only identity element in $\mathcal{A}$, and that $a^{-1} \in \mathcal{A}$ is well-defined for every $a \in \mathcal{A}$. We will call the groups H_i the *members* of the amalgam $\mathcal{A}$. Notice that our definition is a special case of the general definition of an amalgam of groups as found, say, in [12].

A group H is called a *completion* of an amalgam $\mathcal{A}$ if there exists a map $\pi : \mathcal{A} \to H$ such that

(1) for all $i \in I$ the restriction of π to H_i is a homomorphism of H_i to H; and

(2) $\pi(\mathcal{A})$ generates H.

Among all completions of $\mathcal{A}$ there is one "largest" which can be defined as the group having the following presentation:

$$U(\mathcal{A}) = \langle t_h \mid h \in \mathcal{A},\ t_x t_y = t_z,\ \text{whenever } xy = z \text{ in } \mathcal{A}\rangle.$$

Obviously, $U(\mathcal{A})$ is a completion of $\mathcal{A}$ since one can take π to be the mapping $h \mapsto t_h$. Every completion of $\mathcal{A}$ is isomorphic to a quotient of $U(\mathcal{A})$, and because of that $U(\mathcal{A})$ is called the *universal completion*.

Suppose a group $H \le \operatorname{Aut}\Gamma$ acts flag-transitively on a geometry Γ. A *rank k parabolic* is the stabilizer in H of a flag of corank k from Γ. Parabolics of rank $n-1$ (where n is the rank of Γ) are called *maximal parabolics*. They are exactly the stabilizers in H of elements of Γ.

Let F be a maximal flag in Γ, and let H_x denote the stabilizer in H of $x \in \Gamma$. The amalgam $\mathcal{A} = \mathcal{A}(F) = \cup_{x\in F} H_x$ is called the amalgam of maximal parabolics in H. Since the action of H is flag-transitive, this amalgam is defined uniquely up to conjugation in H. For a fixed flag F we can also use the notation M_i for the maximal parabolic H_x, where $x \in F$ is of type i. (We defined this notation in the introduction.) For a subset $J \subset I = \{0, 1, \ldots, n-1\}$, define M_J to be $\cap_{j\in J} M_j$, including $M_\emptyset = H$. Notice that M_J is a parabolic of rank $|I \setminus J|$; indeed, it is the stabilizer of the subflag of F of type J.

Similarly to $\mathcal{A}$, we can define the amalgam $\mathcal{A}_{(s)}$ as the union of all rank s parabolics. With this notation we can write $\mathcal{A} = \mathcal{A}_{(n-1)}$. Moreover, according to our definition, $\mathcal{A}_{(n)} = H$.

Proposition 8.1 (Tits' Lemma)
Suppose a group H acts flag-transitively on a geometry Γ and let $\mathcal{A}$ be the amalgam of maximal parabolics associated with some maximal flag F. Then H is the universal completion of the amalgam $\mathcal{A}$ if and only if Γ is simply connected.

Proof. Follows from [13], *Corollaire* 1, applied to the flag complex of Γ. □

Theorem 8.2
Let Γ be a geometry over some finite set I with a flag-transitive group of automorphisms G, let $k \leq |I|$, let $\mathcal{A}$ and $\mathcal{A}_{k-1}$ be the amalgam of parabolics resp. rank-k-parabolics with respect to some maximal flag F, and assume that all residues of rank greater or equal k with respect to subsets of F are simply connected. Then $G = \mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{A}_{(k-1)})$.

Proof. We will proceed by induction and show that the universal completion of $\mathcal{A}_{(k-1)}$ coincides with the universal completion of $\mathcal{A}_{(k)}$. Denote by H_k the universal completion of $\mathcal{A}_{(k)}$.

The universal completion H_k of $\mathcal{A}_{(k)}$ is also a completion of $\mathcal{A}_{(k-1)}$. Indeed, if $n = k$, then $H_n = G$, which certainly is a completion of $\mathcal{A}_{(n-1)}$. In case $n > k$, the amalgam $\mathcal{A}_{(k)}$ is the union of all G_J with J of corank k and we have a map $\pi : \mathcal{A}_{(k)} \to H_k$ such that $\pi_{|G_J} : G_J \to H_k$ is a homomorphism. Consequently, also $\pi_{|G_J \cap G_{J'}} : G_J \cap G_{J'} \to H_k$ is a homomorphism. It remains to show that the set of all images $\pi(G_J \cap G_{J'})$ with $|I \backslash (J \cup J')| = k-1$ actually generate H_k. But since Γ_J is connected (simple connectedness assumes connectedness), the group $\pi(G_J) \leq H_k$ is generated by all those images for a fixed J (because the $G_J \cap G_{J'}$ are maximal parabolics in G_J). Thus, H_k is a completion of $\mathcal{A}_{(k-1)}$, as it is generated by the $\pi(G_J)$.

Therefore there is a canonical homomorphism ϕ from H_{k-1} onto H_k whose restriction to $\mathcal{A}_{(k-1)}$ is the identity. Let ψ be the inverse of the restriction of ϕ to $\mathcal{A}_{(k-1)}$. Let $J \subset I$ be such that $|I \setminus J| = k$ and let $\hat{G}_J$ be defined as $\langle \psi(G_J \cap \mathcal{A}_{(k-1)}) \rangle$. By simple connectedness of Γ_J and by Proposition 8.1 (Tits' Lemma), ϕ induces an isomorphism of $\hat{G}_J$ onto G_J. Therefore, ψ extends to an isomorphism of $\mathcal{A}_{(k)} \subset H_k$ onto

$$\hat{\mathcal{A}}_{(k)} = \bigcup_{J \subset I, |I \backslash J| = k} \hat{G}_J \qquad \subset H_{k-1}.$$

Hence the universal completion of $\mathcal{A}_{(k-1)}$ coincides with the universal completion of $\mathcal{A}_{(k)}$. The fact $H_n = G$ finishes the proof. □

Proof of Theorem 3. This follows immediately by Theorems 1, 2, and 8.2. □

Proof of Theorem 4. Let $s = 2$ if $q \geq 4$ and $s = 3$ if $q = 2, 3$, and suppose that $n \geq s+1$. Let $\hat{H}$ be the universal completion of the amalgam $\mathcal{A}^0_{(s)}$. Let ϕ be the canonical homomorphism of $\hat{H}$ onto H, that exists due to the fact that H is a completion of $\mathcal{A}^0_{(s)}$. Denote by $\hat{\mathcal{A}}^0_{(s)}$ the copy of $\mathcal{A}^0_{(s)}$ in $\hat{H}$, so that ϕ induces an isomorphism of $\hat{\mathcal{A}}^0_{(s)}$ onto $\mathcal{A}^0_{(s)}$. As in the proof of Theorem 2, let $\psi : \mathcal{A}^0_{(s)} \to \hat{\mathcal{A}}^0_{(s)}$ be the inverse of $\phi_{|\hat{\mathcal{A}}^0_{(s)}}$. Additionally, define $\hat{T}_i = \psi(T_i)$ and $\hat{T} = \langle \hat{T}_1, \ldots, \hat{T}_n \rangle$. Observe that $T_i, T_j \leq M^0_{I \backslash \{i,j\}} = \langle L_i, L_j \rangle \subset \mathcal{A}^0_{(s)}$. Since ψ restricted to the latter group is an isomorphism to $\psi(M^0_{I \backslash \{i,j\}})$, the groups $\hat{T}_i$ and $\hat{T}_j$ commute elementwise. Because T is the direct product of the groups T_i, the map ϕ establishes an isomorphism between $\hat{T}$ and T.

Let J be a subset of I with $|I \setminus J| = s$. Observe that $M_J = M_J^0 T$. Accordingly, we would like to define $\hat{M}_J$ as $\hat{M}_J^0 \hat{T}$, where $\hat{M}_J^0 = \psi(M_J^0)$. For this definition to make sense, we need to show that $\hat{T}$ normalizes $\hat{M}_J^0$. Assume first that $q > 2$. Since M_i^0 is normal in M_i and since $T \leq M_i$, we have that T normalizes all M_i and therefore T normalizes every $L_i = \cap_{j \in I \setminus \{i\}} M_j^0$. Observe that $T_j \leq L_j$ and $L_i, L_j \leq M^0_{I \setminus \{i,j\}} = \langle L_i, L_j \rangle$. Since ψ is an isomorphism from $\mathcal{A}^0_{(s)}$ to $\hat{\mathcal{A}}^0_{(s)}$, the group $\hat{T}_j$ normalizes $\hat{L}_i$ for all i and j. It is clear that M_J^0 is generated by L_i, $i \in I \setminus J$. The same must be true for $\hat{M}_J^0$ and $\hat{L}_i$'s. Therefore every $\hat{T}_j$ will normalize every $\hat{M}_J^0$ which means that also $\hat{T}$ normalizes $\hat{M}_J^0$. If $q = 2$ the same result can be achieved by using $M^0_{I \setminus \{i,j\}}$'s in place of L_i's; recall that in this case we assume $s = 3$.

Since $\hat{T}$ normalizes M_J^0 and since $\hat{T} \cap \hat{M}_J^0 = \langle \hat{T}_j \mid j \in I \setminus J \rangle$ is isomorphic (via ϕ) to $T \cap M_J^0$, the map ϕ establishes an isomorphism between $\hat{M}_J$ and M_J, and, thus, ϕ extends to an isomorphism

$$\hat{\mathcal{A}}_{(s)} = \bigcup_{J \subset I, |I \setminus J| = s} \hat{M}_J \qquad \longrightarrow \qquad \mathcal{A}_{(s)} \quad .$$

Therefore, the universal completions of $\mathcal{A}_{(s)}$ and $\mathcal{A}^0_{(s)}$ are isomorphic, and the claim follows from Theorem 3. ☐

The Main Theorems A and B can be proved using Theorem 4 in exactly the same fashion as the Phan-type theorems of [1] and of [6] are proved. The exact details are left to the reader.

A Computations in GAP

In this section we report on a computation done in the computer algebra system GAP in order to prove the following proposition.

Proposition A.1
Let Γ be the flipflop geometry for $n = 4$ and $q = 2$ or $q = 3$. Then Γ is simply connected.

Proof. We prove both cases of the statement by using Proposition 8.1 (Tits' Lemma). For this we consider the maximal parabolics in $SO^+(8, q)$ of a maximal flag F in Γ. Let $e_1, \ldots, e_4, f_1, \ldots, f_4$ be a hyperbolic basis of the underlying vector space (cf. Section 3). Since $SO^+(8, q)$ acts flag-transitively on Γ we may choose F to be $\langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3, e_4 \rangle, \langle e_1, e_2, e_3, f_4 \rangle$.

The main part of the proof is computer based. We determine a generating set for each maximal parabolic corresponding to F. For each generating set we compute a set of defining relators. The generating sets constructed have the property that the intersection of two maximal parabolics is generated be the intersection of their generating sets. Then the universal completion of the amalgam is defined by the presentation given by the union of the generating sets and the union of the sets of defining relators.

Finally, we determine the index of a preimage of one of the maximal parabolics in the universal completion. The group $SO^+(8, q)$ is the universal completion of the amalgam if and only if the index of the preimage in the universal completion is equal to the index of the maximal parabolic in $SO^+(8, q)$. This is checked by performing a coset enumeration for the presentation of the universal completion over the preimage of one of the maximal parabolics. ☐

A matrix $M \in SO^+(8,q^2)$ lies in (our copy of) $SO^+(8,q)$ if and only of $(M^{tr})^{-1} = \overline{M}$. The stabilizer of F is isomorphic to C_{q+1}^4. It is the set of all diagonal matrices in $SO^+(8,q)$ with elements of multiplicative order dividing $q+1$ on the diagonal.

Below we provide enough information for each of the two cases to make it possible to check the claims in any group theory computer system that provides the standard algorithms for working with matrix groups over finite fields, permutation groups and finitely presented groups. We used GAP [4] for our computations.

For each maximal parabolic we list a set of matrices and a finite presentation. The matrices together with generators for the flag stabilizer form a generating set for the parabolic subgroup. Upper case letters denoting matrices correspond to lower case letters in the finite presentations. It is routine to verify that the given matrices satisfy the given relators. Unfortunately it is a bit more complicated to check that the given presentations define each maximal parabolic. By direct inspection one checks that each matrix is in $SO^+(8,3)$ and fixes the required elements in F. In order to show that the specified sets of matrices generate the proposed parabolics it suffices to determine the order of the subgroup each set generates. These are routine computations in GAP. Again, the generating sets for the maximal parabolics are arranged such that the intersection of two parabolics is generated by the intersection of their generating sets.

A.1 The case $q = 3$

In this case, we will show that the universal completion of the amalgam of the maximal parabolics corresponding to $\langle e_1 \rangle$, $\langle e_1, e_2 \rangle$, $\langle e_1, e_2, e_3, e_4 \rangle$ is already $SO^+(8,3)$. From this it follows that the universal completion of the amalgam of all four maximal parabolics is $SO^+(8,3)$.

Let z be a primitive element in $\mathbb{F}_9$ with minimal polynomial $x^2 - x - 1$. We define the following matrices:

$$U := \left(\begin{array}{cccc|cccc} z^5 & z^7 & & & & & & \\ z^5 & z^3 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & z^7 & z^5 & & \\ & & & & z^7 & z & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right) \qquad V := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & z^5 & z^3 & & & & & \\ & z^5 & z^7 & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & z^7 & z & \\ & & & & & z^7 & z^5 & \\ & & & & & & & 1 \end{array}\right)$$

$$W := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & z & z^5 & & & & \\ & & z^5 & z^5 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & z^3 & z^7 \\ & & & & & & z^7 & z^7 \end{array}\right) \qquad Y := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & & z^3 & & & z^3 & \\ & & z & & & & & z^5 \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & z & & & & & z \\ & & & z^7 & & & z^3 & \end{array}\right)$$

Each maximal parabolic in $SO^+(8,3)$ is generated by the matrices specified in the following table together with generators of the flag stabilizer.

stabilizer	element	generators
S_1	$\langle e_1\rangle$	V, W, Y
S_2	$\langle e_1, e_2\rangle$	U, W, Y
S_3	$\langle e_1, e_2, e_3, e_4\rangle$	U, V, W

In addition to these elements we use diagonal matrices D_i, $1 \le i \le 4$, that generate the stabilizer of the flag F, isomorphic to C_4^4. The following presentations define each maximal parabolic. To each presentation the relators d_i^4 for $1 \le i \le 4$ and $[d_i, d_j]$ for $1 \le i, j \le 4$ need to be added.

Generators for S_1: $d_1, d_2, d_3, d_4, v, w, y$.
Relators for S_1: $[v, d_1]$, $[v, d_4]$, $[w, d_1]$, $[w, d_2]$, $[y, d_2]$, $[y, d_1]$, y^2, v^3, $w^2 d_3 d_4$, $v d_3 v d_2 d_3^2$, $(y d_3^{-1} d_4)^2$, $(y w^{-1} d_3)^2$, $w^{-1} d_3^{-1} w d_4 w^{-1} d_4$, $(w^{-1} y)^3 d_3^2 d_4^2$, $v w^{-1} d_3 v^{-1} w^{-1} v^{-1} d_3 d_2^{-1} w$, $v w y v d_3 w^{-1} v^{-1} y w^{-1} v^{-1} y w^{-1} y d_3^{-1} w^{-1} d_4^{-1}$, $v w y v d_3 y d_3^{-1} v^{-1} y w^{-1} v^{-1} y d_3^{-1} w^{-1} y d_4^{-2} d_3$, $d_2 v y v^{-1} y w^{-1} v d_3^{-1} y v w^{-1} v^{-1} y w d_4 y w d_4^{-2}$.

Generators for S_2: $d_1, d_2, d_3, d_4, u, w, y$.
Relators for S_2: $[u, w]$, $[u, y]$, $[u, d_3]$, $[u, d_4]$, $[w, d_1]$, $[w, d_2]$, $[y, d_1]$, $[y, d_2]$, y^2, $w^2 d_3 d_4$, $d_1 u^{-3} d_2$, $(y d_3 d_4^{-1})^2$, $u d_1 u d_2^{-1}$, $w y w^{-1} d_4 y d_4^{-1} w^{-1} d_4$, $y d_3^{-1} w d_3^{-1} y d_4^{-1} d_3^{-2} w d_4 d_3^{-1} w y w^{-1}$.

Generators for S_3: $d_1, d_2, d_3, d_4, u, v, w$.
Relators for S_3: v^3, $[u, w]$, $[u, d_4]$, $[u, d_3]$, $[v, d_1]$, $[v, d_4]$, $[w, d_2]$, $[w, d_1]$, $w^2 d_3 d_4$, $u^{-1} d_1^{-1} u^{-1} d_2$, $u^{-2} d_1 u d_1$, $d_2^{-1} v^{-1} d_3^{-1} v^{-1} d_3^2$, $w d_3 w d_3 w d_4^{-1}$, $w v w^{-1} d_3 v^{-1} w^{-1} v^{-1} d_2^{-1} d_3$, $u v u^{-1} d_1^{-1} d_2^{-1} v^{-1} u^{-1} v^{-1} d_2^{-1} d_3$, $u^{-1} v w^{-1} v u^{-1} v^{-1} w v^{-1} u^{-1} v w^{-1} v d_4 w d_4$.

The union of the relators above together with the generators $d_1, d_2, d_3, d_4, u, v, w, y$ give a presentation for the universal completion of the amalgam of the maximal parabolics. Coset enumeration over the subgroup generated by u, v, w gives an index of 379040 which is the index of the maximal parabolic stabilizing $\langle e_1, e_2, e_3, e_4\rangle$ in $SO^+(8, 3)$. This shows that $SO^+(8, 3)$ is the universal completion of the amalgam of maximal parabolics.

A.2 The case $q = 2$

In the following, let z be a primitive element in $\mathbb{F}_4$ with minimal polynomial $x^2 + x + 1$. We define the following matrices:

$$U := \left(\begin{array}{cccc|cccc} & z^2 & & & & & & \\ z & & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & & z & & \\ & & & & z^2 & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right) \quad V := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & & z & & & & & \\ & z^2 & & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & & z^2 & \\ & & & & & z & & \\ & & & & & & & 1 \end{array}\right)$$

$$W := \left(\begin{array}{cccc|cccc} z^2 & & & & & & & \\ & 1 & & & & & & \\ & & & z & & & & \\ & & 1 & & & & & \\ \hline & & & & z & & & \\ & & & & & 1 & & \\ & & & & & & & z^2 \\ & & & & & & 1 & \end{array}\right) \quad Y := \left(\begin{array}{cccc|cccc} z & & & & & & & \\ & z^2 & & & & & & \\ & & z & & & & & \\ & & & & & & & 1 \\ \hline & & & & z^2 & & & \\ & & & & & z & & \\ & & & & & & z^2 & \\ & & & 1 & & & & \end{array}\right)$$

$$S := \left(\begin{array}{cccc|cccc} z^2 & z^2 & z & & & & & \\ 1 & z & z & & & & & \\ 1 & z^2 & 1 & & & & & \\ & & & z & & & & \\ \hline & & & & z & z & z^2 & \\ & & & & 1 & z^2 & z^2 & \\ & & & & 1 & z & 1 & \\ & & & & & & & z^2 \end{array}\right) \quad T := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & z^2 & 1 & z & & & & \\ & z^2 & z^2 & z^2 & & & & \\ & z & 1 & z^2 & & & & \\ \hline & & & & 1 & & & \\ & & & & & z & 1 & z^2 \\ & & & & & z & z & z \\ & & & & & z^2 & 1 & z \end{array}\right)$$

$$R := \left(\begin{array}{cccc|cccc} z & & & & & & & \\ & z^2 & & & & & & \\ & & & & & & & 1 \\ & & & & & & z & \\ \hline & & & & z^2 & & & \\ & & & & & z & & \\ & & & 1 & & & & \\ & & z^2 & & & & & \end{array}\right) \quad Q := \left(\begin{array}{cccc|cccc} z^2 & & & & & & & \\ & z & 1 & & & & & 1 \\ & z^2 & 1 & & & & & z^2 \\ & & & 1 & & z & z & \\ \hline & & & & z & & & \\ & & & 1 & & z^2 & 1 & \\ & & & z & & z & 1 & \\ & z^2 & z^2 & & & & & 1 \end{array}\right)$$

Each maximal parabolic in $SO^+(8,2)$ is generated by the matrices specified in the following table together with generators of the flag stabilizer.

stabilizer	element	generators
S_1	$\langle e_1 \rangle$	V, W, T, R, Q, Y
S_2	$\langle e_1, e_2 \rangle$	U, W, Y
S_3	$\langle e_1, e_2, e_3, e_4 \rangle$	U, V, S, W, T
S_4	$\langle e_1, e_2, e_3, f_4 \rangle$	U, V, S, R, Q

In addition to these elements we use diagonal matrices D_i, $1 \leq i \leq 4$, that generate the stabilizer of the flag F, isomorphic to C_3^4. The following presentations define each maximal parabolic. To each presentation the relators d_i^3 for $1 \leq i \leq 4$ and $[d_i, d_j]$ for $1 \leq i, j \leq 4$ need to be added.

Generators for S_1: $d_1, d_2, d_3, d_4, v, w, t, r, q, y$.
Relators for S_1: v^2, y^2, $d_1qd_1^{-1}q^{-1}$, $d_1rd_1^{-1}r^{-1}$, $d_3yd_3^{-1}y$, $d_2yd_2^{-1}y$, $d_1yd_1^{-1}y$, $w^{-1}yr^{-1}y$, $d_1vd_1^{-1}v$, $d_3r^2d_4^{-1}$, $vyvy$, $w^{-1}d_4^{-1}wd_3$, yd_4yd_4, $w^{-1}d_3wd_4^{-1}$, $d_1t^{-1}d_1^{-1}t$, $d_2vd_3^{-1}v$, $d_4^{-1}vd_4v$, $wd_4d_3^{-1}d_4w$, $rq^{-2}d_3^{-1}d_2^{-1}$, $d_3^{-1}d_4w^{-1}d_4^{-1}w^{-1}$, $q^{-1}vqd_3vd_3^{-1}$, $rw^{-1}r^{-1}d_3^{-1}wd_4^{-1}$, $d_2t^{-1}d_2^{-1}td_2^{-1}t^{-1}$, $vd_2^{-1}td_4td_4d_2$, $d_2d_4^2wtd_2t$, $r^2vwd_2d_3^{-1}w^{-1}v$, $d_3yqyvtd_4d_2^{-1}$, $d_2wt^{-1}wd_4t^{-1}d_3^{-1}d_4^{-1}w^{-1}$, $d_2^{-1}w^{-1}td_4^{-1}w^{-1}td_3d_4w$, $tqyd_4q^{-1}t^{-1}qd_4^{-1}yq^{-1}$, $d_4ytwqd_3yd_4^{-1}twqd_3$, $d_2^{-1}yrvtwqd_3tqd_4^{-1}yq^{-1}$, $w^{-1}d_3^{-1}vd_1wvd_2^{-1}d_1^{-1}d_3^{-1}d_2^{-1}w^{-1}vd_3^{-1}$, $tyd_4^{-1}twqd_3t^{-1}d_3^{-1}q^{-1}w^{-1}t^{-1}yd_4^{-1}$, $twd_4q^{-1}w^{-1}t^{-1}d_4yqyd_4q^{-1}wd_4d_3$, $rt^{-1}wqd_4ytwqd_4^{-1}w^{-1}qyd_4q^{-1}$, $qyd_4q^{-1}d_3^{-1}w^{-1}d_3^{-1}qd_4^{-1}yq^{-1}d_3^{-1}w^{-1}d_3^{-1}qd_4^{-1}yq^{-1}d_3^{-1}w^{-1}d_3^{-1}qd_4^{-1}yq^{-1}d_3^{-1}w^{-1}d_3^{-1}$.

Generators for S_2: $d_1, d_2, d_3, d_4, u, w, y$.
Relators for S_2: u^2, y^2, $d_1wd_1^{-1}w^{-1}$, $d_1yd_1^{-1}y$, $d_2wd_2^{-1}w^{-1}$, $d_3ud_3^{-1}u$, $d_3w^{-1}d_3w^{-1}$, $d_3yd_3^{-1}y$, $d_4^{-1}yd_4^{-1}y$, $d_4ud_4^{-1}u$, $ud_2^{-1}ud_1$, $uwuw^{-1}$, $uyuy$, $w^{-1}d_3d_4w^{-1}$, $w^{-1}d_4d_3w^{-1}$, $w^{-1}yw^{-1}yw^{-1}yw^{-1}y$.

Generators for S_3: $d_1, d_2, d_3, d_4, u, v, s, w, t$.
Relators for S_3: u^2, v^2, t^3, $d_1vd_1^{-1}v$, $d_1wd_1^{-1}w^{-1}$, $d_1td_1^{-1}t^{-1}$, $d_4ud_4^{-1}u$, $d_4vd_4^{-1}v$, $d_4wd_3^{-1}w^{-1}$, $d_4sd_4^{-1}s^{-1}$, $ud_1^{-1}ud_2$, $uwuw^{-1}$, $wd_2^{-1}w^{-1}d_2$, $wd_3^{-1}wd_3^{-1}$, $w^2d_4^{-1}d_3^{-1}$, $s^{-1}ud_2s^{-1}d_1$, $sd_1^{-1}vsd_2^{-1}$, $s^{-1}d_3d_2d_1s^{-2}$, $t^{-1}d_2^{-1}t^{-1}d_3w^{-1}d_2^{-1}$, $td_3st^{-1}s^{-1}d_3^{-1}$, $d_2vsd_3^{-1}d_2d_1^{-1}sd_3$.

Generators for S_4: $d_1, d_2, d_3, d_4, u, v, s, r, q$.
Relators for S_4: u^2, v^2, $d_3^{-1}d_4r^{-2}$, $r^{-1}d_4^{-1}rd_3^{-1}$, $r^2d_3d_4^{-1}$, $rur^{-1}u$, $qd_1^{-1}q^{-1}d_1$, $d_4vd_4^{-1}v$, $rd_1^{-1}r^{-1}d_1$, $rd_2^{-1}r^{-1}d_2$, $d_4s^{-1}d_4^{-1}s$, $vd_1^{-1}vd_1$, $d_4ud_4^{-1}u$, $d_3ud_3^{-1}u$, $d_2ud_1^{-1}u$, $d_3vd_2^{-1}v$, $s^{-1}d_1s^{-1}ud_2$, $d_4^{-1}q^{-2}r^{-1}d_2^{-1}d_3^{-1}$, $sud_2d_1^{-1}s^{-1}v$, $qd_4r^{-1}q^{-1}vd_3d_2^{-1}$, $sd_3d_1^{-1}sd_1^{-1}d_3sd_1^{-1}d_3$, $q^{-1}s^{-1}rs^{-1}d_1r^{-1}sd_1d_3d_2^{-1}d_1^{-1}$, $qd_3^{-1}s^{-1}d_1q^{-1}d_2d_1sd_1d_3d_2^{-1}$.

The union of the relators above together with the generators $d_1, d_2, d_3, d_4, u, v, w, y, s, t, r, q$ give a presentation for the universal completion of the amalgam of the maximal parabolics. Coset enumeration over the subgroup generated by v, w, y, t, r, q gives an index of 2240 which is the index of the maximal parabolic stabilizing $\langle e_1 \rangle$ in $SO^+(8,2)$. This shows that $SO^+(8,2)$ is the universal completion of the amalgam of maximal parabolics.

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Authors' addresses:

R. Gramlich, W. Nickel:

TU Darmstadt
FB Mathematik
Schloßgartenstraße 7
64289 Darmstadt
Germany

gramlich@mathematik.tu-darmstadt.de
nickel@mathematik.tu-darmstadt.de

C. Hoffman, S. Shpectorov:

Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403
USA

hoffman@bgnet.bgsu.edu
sergey@bgnet.bgsu.edu