Pseudodifferential Boundary Value Problems with Non-Smooth Coefficients

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Abstract

In this contribution we establish a calculus of pseudodifferential boundary value problems with Hölder continuous coefficients. It is a generalization of the calculus of pseudodifferential boundary value problems introduced by Boutet de Monvel. We discuss their mapping properties in Bessel potential and certain Besov spaces. Although having non-smooth coefficients and the operator classes being not closed under composition, we will prove that the composition of Green operators $a_1(x, D_x)a_2(x, D_x)$ coincides with a Green operator $a(x, D_x)$ up to order $m_1 + m_2 - \theta$, where $\theta \in (0, \tau_2)$ is arbitrary, $a_j(x, \xi)$ is in $C^{\tau_j}(\mathbb{R}^n)$ w.r.t. x, and m_j is the order of $a_j(x, D_x)$, $j = 1, 2$. Moreover, $a(x, D_x)$ is obtained by the asymptotic expansion formula of the smooth coefficient case leaving out all terms of order less than $m_1 + m_2 - \theta$. This result is used to construct a parametrix of a uniformly elliptic Green operator $a(x, D_x)$.

Key words: Pseudodifferential boundary value problems, non-smooth pseudodifferential operators
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1 Introduction

In [5] L. Boutet de Monvel introduced an operator class modeling differential and pseudodifferential boundary problems, which is closed under composition and can be used to construct parametrices to elliptic operators. It gave great impact in many directions. This calculus and further developed calculi, cf. e.g. Grubb [12], have been used in index theory, cf. [5], Rempel and Schulze [22], in the theory of Navier-Stokes equations, cf. Grubb and Solonnikov [16, 13], in geometrical problems as trace expansions, cf. e.g. Grubb and Schrohe [15], and others, cf. [12].

Although the original calculus of Boutet de Monvel was generalized in many directions, it is usually assumed that the symbols of the operators are smooth in the space variable x. In order to treat boundary value problems in domains with non-smooth boundary or apply the theory to quasi-linear equations, it is necessary to allow symbols with limited smoothness in the space variable x.

In the present contribution we generalize the so-called Green operators in [5] to operators with symbols which are Hölder continuous in x – also called Green operator with "Hölder continuous coefficients". We discuss their mapping properties and behavior under composition. The present work extends and improves the results of [1, 2], where some partial results in this direction were proved and applied to show the existence of a bounded H^∞-calculus of the Stokes operator in so-called asymptotically flat layers with $C^{1,1}$-boundary.

A Green operator in the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,\infty)$ is of the form

$$a(x,D_x) = \begin{pmatrix} p(x,D_x)_+ + g(x,D_x) & k(x,D_x) \\ t(x,D_x) & s(x',D_{x'}) \end{pmatrix} : \begin{matrix} \mathcal{S}(\mathbb{R}^n_+)^N \\ \times \\ \mathcal{S}(\mathbb{R}^{n-1})^M \end{matrix} \to \begin{matrix} C^0(\mathbb{R}^n_+)^{N'} \\ \times \\ C^0(\mathbb{R}^{n-1})^{M'} \end{matrix} \tag{1.1}$$

Here $p(x,D_x)_+ = r^+ p(x,D_x) e^+$ is a truncated pseudodifferential operator, $k(x,D_x)$ is a Poisson operator (also called potential operator), $t(x',D_x)$ is a trace operator, $g(x,D_x)$ is a singular Green operator, and $s(x',D_{x'})$ is a pseudodifferential operator on $\mathbb{R}^{n-1}$, cf. [5], [22], or [12] for the definition in the smooth coefficient case. The precise definitions in the Hölder continuous case are given below. They are based on the definition of the class $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, cf. Kumano-Go and Nagase [19] or Taylor [25], i.e.,

$$p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \Leftrightarrow \|\partial_\xi^\alpha p(.,\xi)\|_{C^\tau(\mathbb{R}^n)} \le C_\alpha (1+|\xi|)^{m-|\alpha|} \text{ for all } \alpha \in \mathbb{N}_0^n,$$

where $C^\tau(\mathbb{R}^n)$ is the space of all $[\tau]$-times differentiable functions with bounded and Hölder continuous $[\tau]$-th derivatives of degree $\tau - [\tau]$.

Having non-smooth coefficients there are several new aspects: First of all, the mapping properties in Bessel potential and Besov spaces are of course limited by the smoothness of the coefficients. It is well-known that, if $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, the associated pseudodifferential operator $p(x,D_x)$ is a bounded operator

$$p(x,D_x)\colon H^{s+m}_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n) \qquad \text{if } |s| < \tau,$$

cf. e.g. [25, Proposition 2.1.D]. Using the latter mapping properties in a vector-valued variant, we will prove our first main result:

THEOREM 1.1 *Let $a(x,D_x)$ be a Green operator of order $m \in \mathbb{R}$, class $r \in \mathbb{Z}$, with C^τ-regularity in x. Then for every $s \in \mathbb{R}$*

$$a(x,D_x)\colon H^{s+m}_q(\mathbb{R}^n_+)^N \times B^{s+m-\frac{1}{q}}_q(\mathbb{R}^{n-1})^M \to H^s_q(\mathbb{R}^n_+)^{N'} \times B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1})^{M'}$$

provided that $|s| < \tau$ if $N' \neq 0$, $|s - \frac{1}{q}| < \tau$ if $M' \neq 0$, $s + m > r - \frac{1}{q'}$ if $N \neq 0$, and $m \in \mathbb{Z}$ if $p(x,D_x) \neq 0$.

Considering compositions of pseudodifferential or Green operators with non-smooth coefficients the situation is more complicated. The class of pseudodifferential operators with non-smooth coefficients is of course not closed under composition since e.g. $[\partial_{x_j}, p(x,D_x)] = (\partial_{x_j} p)(x,D_x)$. In particular the statement that $p_1(x,D_x)p_2(x,D_x) = (p_1\#p_2)(x,D_x)$ where $p_1\#p_2$ has the asymptotic expansion

$$p_1\#p_2(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) \tag{1.2}$$

cannot hold if p_2 is not smooth in x. However it will be shown under certain restrictions on m_1 that, if $p_j \in C^{\tau_j} S^{m_j}(\mathbb{R}^n\times\mathbb{R}^n)$, $j=1,2$, for any $\theta\in(0,\tau_2)$, $\theta\notin\mathbb{N}$,

$$p_1(x,D_x)p_2(x,D_x) = \sum_{|\alpha|<\theta} \frac{1}{\alpha!}\operatorname{OP}(\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi)) + R_\theta,$$

where R_θ is of order $m_1+m_2-\theta$ in the sense of the mapping properties in Bessel potential and Besov spaces, cf. Theorem 3.6 below. Hence in some sense the asymptotic expansion is valid as long as $D_x^\alpha p_2(x,\xi)$ exists and is Hölder continuous w.r.t x.

The corresponding statement for compositions of Green operators is as follows:

THEOREM 1.2 *Let $a_j(x,D_x)$, $j=1,2$, be Green operators of order $m_j\in\mathbb{R}$, class $r_j\in\mathbb{Z}$, and coefficients in C^{τ_j}, $\tau_j>0$, $j=1,2$, and let $p_j, g_j, k_j, t_j, s_j, N_j, M_j, N_j', M_j'$ denote the corresponding operators and parameters due to (1.1). Moreover, let $N_2' = N_1$, $M_2' = M_1$ and assume that the coefficients of $\tilde{g}_2$ and $\tilde{k}_2$ are independent of x_n and that $m_j\in\mathbb{Z}$ if $p_j\neq 0$. Then for every $\theta\in(0,\tau_2)$, $\theta\notin\mathbb{N}$, there is a Green operator $(a_1\#_{[\theta]}a_2)(x,D_x)$ of order m_1+m_2, class* $\max(r_1+m_2, r_2)$, *and with coefficients in C^τ, $\tau:=\min(\tau_1,\tau_2-[\theta])$, such that*

$$a_1(x,D_x)a_2(x,D_x) - (a_1\#_{[\theta]}a_2)(x,D_x) : \begin{array}{c} H_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+)^{N_2} \\ \times \\ B_q^{s+m_1+m_2-\frac{1}{q}-\theta}(\mathbb{R}^{n-1})^{M_2} \end{array} \to \begin{array}{c} H_q^{s}(\mathbb{R}^n_+)^{N_1'} \\ \times \\ B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1})^{M_1'} \end{array}$$

is a bounded linear mapping if the following conditions are satisfies:

1. $|s|<\tau$, $s-\theta>-\tau_2$ *if* $N_1'\neq 0$, $\left|s-\frac{1}{q}\right|<\tau$, $s-\frac{1}{q}-\theta>-\tau_2$ *if* $M_1'\neq 0$,
2. $-\tau_2+\theta<s+m_1<\tau_2$ *if* $N_1\neq 0$ *and* $-\tau_2+\theta<s+m_1-\frac{1}{q}<\tau_2$ *if* $M_1\neq 0$,
3. $s+m_1>r_1-\frac{1}{q'}$ *if* $N_1\neq 0$ *and* $s+m_1+m_2-\theta>r_2-\frac{1}{q'}$ *if* $N_2\neq 0$.

More precisely,

$$(a_1\#_{[\theta]}a_2)(x,D_x) = \begin{pmatrix} p_1\#_{[\theta]}p_2(x,D_x)_+ + g(x,D_x) & k(x,D_x) \\ t(x',D_x) & s(x',D_{x'}) \end{pmatrix},$$

where

1. $g(x, D_x) = (p_1 \#'_{[\theta]} g_2)(x, D_x) + (g_1 \#_{[\theta]} p_2)(x, D_x) + (g_1 \#'_{[\theta]} g_2)(x, D_x)$
   $+ (k_1 \#'_{[\theta]} t_2)(x, D_x) - l_\theta(p_1, p_2)(x, D_x)$,

2. $t(x', D_x) = (t_1 \#_{[\theta]} p_2)(x, D_x) + (t_1 \#'_{[\theta]} g_2)(x', D_x) + (s_1 \#'_{[\theta]} t_2)(x', D_x)$,

3. $k(x, D_x) = (p_1 \#_{[\theta]} k_2)(x, D_x) + (g_1 \#'_{[\theta]} k_2)(x, D_x) + (k_1 \#'_{[\theta]} s_2)(x, D_x)$,

4. $s(x', D_{x'}) = (t_1 \#'_{[\theta]} k_2)(x', D_{x'}) + (s_1 \#'_{[\theta]} s_2)(x', D_{x'})$,

and the terms are defined by (3.5), (4.8), (5.9)-(5.10), and (5.18) below.

Theorem 1.2 will be used to construct an inverse of a uniformly elliptic Green operator $a(x, D_x)$ up to order $-\theta$, where $0 < \theta < \tau$ and $\tau > 0$ is the regularity of the coefficients of $a(x, D_x)$.

The structure of the article is as follows: In Section 2 we summarize the necessary preliminaries on vector-valued and weighted function spaces. Then in Section 3 we consider the mapping properties and the compositions of operator-valued pseudodifferential operators with Hölder continuous coefficients, which will be the basis for the further discussion since Green operators can be considered as operator-valued pseudodifferential operators. The main results of this contributions are proved in Section 4 and Section 5. In Section 4, the Poisson, trace, and singular Green operators are defined and the corresponding mapping properties and statements on compositions are proved. Then truncated pseudodifferential operator enter the discussion in Section 5, where first of all a transmission condition for non-smooth pseudodifferential operators is given. Finally, Section 6 is devoted to the parametrix construction in the case of non-smooth coefficient.

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2 Preliminaries

2.1 Vector-Valued Besov and Bessel Potential Spaces

First of all, $\mathbb{N}$ denotes the set of natural numbers (without 0), $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ is the set of complex numbers.

We will keep close to the notation of the monograph [12]. In particular, $\partial_{x_j} f = \partial_j f$, $j = 1, \ldots, n$, denotes the partial derivatives of $f \colon \mathbb{R}^n \to \mathbb{C}$ and $D_{x_j} := -i\partial_{x_j}$. For $s \in \mathbb{R}$ we define $[s]$ to be the largest integer $\leq s$ and set $[s]_+ = \max\{s, 0\}$.

In the following let X be a Banach space and $\Omega \subseteq \mathbb{R}^n$ be a domain. Then $L^p(\Omega; X)$, $1 \leq p < \infty$, is defined as the space of strongly measurable functions $f \colon \Omega \to X$ with

$$\|f\|_{L^p(\Omega;X)} := \left(\int_\Omega \|f(x)\|_X^p dx \right)^{\frac{1}{p}} < \infty$$

and $L^\infty(\Omega;X)$ is the space of all strongly measurable and essentially bounded functions. Moreover, $L^p(\Omega)$ denotes the standard Lebesgue space and $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$. Similarly, $\ell^p(\mathbb{N}_0;X)$, $1\leq p\leq\infty$, denotes the X-valued variant of $\ell^p(\mathbb{N}_0)$.

Furthermore, let $\mathcal{S}(\mathbb{R}^n;X)$ be the space of smooth rapidly decreasing function $f\colon\mathbb{R}^n\to X$ and let $\mathcal{S}(\mathbb{R}^n):=\mathcal{S}(\mathbb{R}^n,\mathbb{C})$. Moreover, $\mathcal{S}'(\mathbb{R}^n;X):=\mathcal{L}(\mathcal{S}(\mathbb{R}^n),X)$ denotes the space of tempered X-valued distributions, cf. e.g. Amann [3]. As in the scalar case the Fourier transformation is an isomorphism $\mathcal{F}\colon\mathcal{S}(\mathbb{R}^n;X)\to\mathcal{S}(\mathbb{R}^n;X)$ and $\mathcal{F}\colon\mathcal{S}'(\mathbb{R}^n;X)\to\mathcal{S}'(\mathbb{R}^n;X)$, cf [3]. Moreover, if $p\colon\mathbb{R}^n\to\mathbb{C}$ is a smooth function such that p and all its derivatives are of at most polynomial growth, then

$$p(D_x)f := \mathcal{F}^{-1}[p(\xi)\hat{f}], \qquad \hat{f} := \mathcal{F}[f],$$

is a bounded operator on $\mathcal{S}(\mathbb{R}^n;X)$ and $\mathcal{S}'(\mathbb{R}^n;X)$. In particular let $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$ and let $\varphi_j(\xi)$, $j\in\mathbb{N}_0$, be a partition of unity on $\mathbb{R}^n$ with $\operatorname{supp}\varphi_0\subseteq\{|\xi|\leq 2\}$ and $\operatorname{supp}\varphi_j\subseteq\{2^{j-1}\leq|\xi|\leq 2^{j+1}\}$ for $j\in\mathbb{N}$.

Then the X-valued variants of the Bessel potential and Besov spaces of order $s\in\mathbb{R}$ are defined as

$$\begin{aligned} H^s_p(\mathbb{R}^n;X) &:= \{f\in\mathcal{S}'(\mathbb{R}^n;X) : \langle D_x\rangle^s f\in L^p(\mathbb{R}^n;X)\} \quad \text{if } 1<p<\infty,\\ B^s_{p,q}(\mathbb{R}^n;X) &:= \{f\in\mathcal{S}'(\mathbb{R}^n;X) : (2^{sj}\varphi_j(D_x)f)_{j\in\mathbb{N}_0}\in\ell^q(\mathbb{N}_0;L^p(\mathbb{R}^n;X))\}, \end{aligned}$$

where $1\leq p,q\leq\infty$. Moreover, we will use the abbreviations $B^s_p(\mathbb{R}^n;X) := B^s_{p,p}(\mathbb{R}^n;X)$, $B^s_p(\mathbb{R}^n) := B^s_p(\mathbb{R}^n;\mathbb{C})$, and $H^s_p(\mathbb{R}^n) := H^s_p(\mathbb{R}^n;\mathbb{C})$.

As in the scalar case, the following properties are simple consequences of the definition and the fact that $\varphi_j(D_x)\langle\xi\rangle^s f = k_j * f$ with $\|k_j\|_{L^1(\mathbb{R}^n)}\leq C2^{sj}$, $j\in\mathbb{N}_0$, cf. Stein [24, Chapter VI, Section 5.3]:

$$\begin{aligned} B^s_{p,q_1}(\mathbb{R}^n;X) &\subseteq B^s_{p,q_2}(\mathbb{R}^n;X) && \text{for } 1\leq q_1\leq q_2\leq\infty, 1\leq p\leq\infty\\ B^s_{p,1}(\mathbb{R}^n;X) \subseteq H^s_p(\mathbb{R}^n;X) &\subseteq B^s_{p,\infty}(\mathbb{R}^n;X) && \text{for } 1<p<\infty, \\ B^{s+\varepsilon}_{p,\infty}(\mathbb{R}^n;X) &\subseteq B^s_{p,1}(\mathbb{R}^n;X) && \text{for } 1\leq p\leq\infty, \varepsilon>0, \end{aligned} \tag{2.1}$$

where $s\in\mathbb{R}$. In particular,

$$B^{s+\varepsilon}_p(\mathbb{R}^n;X)\subseteq H^s_p(\mathbb{R}^n;X)\subseteq B^{s-\varepsilon}_p(\mathbb{R}^n;X) \quad \text{for } 1<p<\infty, \varepsilon>0. \tag{2.2}$$

For interpolation properties of the scalar Besov and Bessel potential spaces we refer to Bergh and Löfström [4] and Triebel [26]. As in the latter monographs we will denote the complex and real interpolation functor by $(.,.)_{[\theta]}$, $(.,.)_{\theta,q}$, respectively.

Lemma 2.1 *Let $1\leq p,q_0,q_1,q\leq\infty$, $s_0,s_1\in\mathbb{R}$, $s_0\neq s_1$, $\theta\in(0,1)$, X be a Banach space, and let H be a Hilbert space. Then*

$$\begin{aligned} (B^{s_0}_{p,q_0}(\mathbb{R}^n;X), B^{s_1}_{p,q_1}(\mathbb{R}^n;X))_{\theta,q} &= B^s_{p,q}(\mathbb{R}^n;X),\\ (H^{s_0}_p(\mathbb{R}^n;X), H^{s_1}_p(\mathbb{R}^n;X))_{\theta,q} &= B^s_{p,q}(\mathbb{R}^n;X) && \text{if } 1<p<\infty,\\ (B^{s_0}_{p,q}(\mathbb{R}^n;X), B^{s_1}_{p,q}(\mathbb{R}^n;X))_{[\theta]} &= B^s_{p,q}(\mathbb{R}^n;X) && \text{if } 1\leq p<\infty,\\ (H^{s_0}_q(\mathbb{R}^n;H), H^{s_1}_q(\mathbb{R}^n;H))_{[\theta]} &= H^s_q(\mathbb{R}^n;H) && \text{if } 1<p<\infty, \end{aligned}$$

where $s=(1-\theta)s_0+\theta s_1$.

Proof: For the first and third interpolation spaces we refer to Amann [3, Section 5] and the references given there. The second statement is a consequence of the first and (2.1). For the last statement, we note that the Mikhlin multiplier theorem holds for $L^p(\mathbb{R}^n;H)$ if $1<p<\infty$. Hence the standard proof remains valid in the H-valued case, cf. [4]. ■

Furthermore, we note that

$$\begin{aligned} F(H^s_p(\mathbb{R}^n;X_0),H^s_p(\mathbb{R}^n;X_1)) &= H^s_p(\mathbb{R}^n;F(X_0,X_1)) \quad \text{for } 1<p<\infty \\ F(B^s_p(\mathbb{R}^n;X_0),B^s_p(\mathbb{R}^n;X_1)) &= B^s_p(\mathbb{R}^n;F(X_0,X_1)) \quad \text{for } 1\leq p<\infty \end{aligned}$$

if $F(.,.)=(.,.)_{[\theta]}$ or $F(.,.)=(.,.)_{\theta,q}$. Since $\langle D_x\rangle^s$ is by definition an isomorphism from $H^s_p(\mathbb{R}^n;X_j)$ onto $L^p(\mathbb{R}^n;X_j)$, the statement for the Bessel potential spaces is a consequence of the statement for the Lebesgue spaces, cf. Triebel [26, Section 1.18.4]. Moreover, as in the scalar case $f\mapsto(\varphi_j(D_x)f)_{j\in\mathbb{N}_0}$ is a retraction from $B^s_{p,q}(\mathbb{R}^n;X_k)$, $k=0,1$, into $\ell^q(\mathbb{N}_0;L^p(\mathbb{R}^n;X_k))$ with coretraction $(f_j)_{j\in\mathbb{N}_0}\mapsto\sum_{j=0}^\infty 2^{-sj}\psi_j(D_x)f_j$, where $\psi_j(D_x):=\sum_{k=-1}^1\varphi_{j+k}(D_x)$. Hence the statement for $B^s_p(\mathbb{R}^n;X)$ is a consequence of the interpolation properties of vector-valued $\ell^p(\mathbb{N}_0)$ and L^p-spaces, cf. [26, Section 1.18.1/4].

Finally, if X is reflexive, $s\in\mathbb{R}$, and $1<p,q<\infty$, then

$$(B^s_{p,q}(\mathbb{R}^n;X))'\cong B^{-s}_{p',q'}(\mathbb{R}^n;X'), \qquad (H^s_p(\mathbb{R}^n;X))'\cong H^{-s}_{p'}(\mathbb{R}^n;X'),$$

cf. [3] for the Besov spaces and Edwards [6, 8.20.5] for $(L^p(\mathbb{R}^n;X))'\cong L^{p'}(\mathbb{R}^n;X')$, which implies the statement for $H^s_p(\mathbb{R}^n;X)$.

2.2 Weighted Function Spaces

In the following we will use a measurable function $\omega\colon\mathbb{R}^n\to(0,\infty)$ to define weighted Lebesgue, Besov-, and Bessel potential spaces.

First of all, if $M\subseteq\mathbb{R}^n$ is a non-empty measurable set, $L^p(M,\omega)$, $1\leq p<\infty$, denotes the vector space of all measurable functions $f\colon M\to\mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n,\omega)}:=\left(\int_M|f(x)|^p\omega(x)dx\right)^{\frac{1}{p}}<\infty.$$

Since $f\mapsto f\omega^{\frac{1}{p}}$ is an isometric isomorphism from $L^p(M,\omega)$ onto $L^p(M)$,

$$(L^p(M,\omega))'=L^{p'}(M,\omega'), \qquad \omega'(x):=\omega(x)^{-\frac{p'}{p}}, \tag{2.3}$$

if $1<p<\infty$ by the usual identification of functions with functionals. Moreover, we note that

$$(L^p(\mathbb{R}^n,\omega_0),L^p(\mathbb{R}^n,\omega_1))_{\theta,p}=(L^p(\mathbb{R}^n,\omega_0),L^p(\mathbb{R}^n,\omega_1))_{[\theta]}=L^p(\mathbb{R}^n,\omega)$$

where $\omega(x) := \omega_0(x)^{1-\theta}\omega_1(x)^\theta$, $0 < \theta < 1$, and $1 \le p < \infty$, cf. [4, Theorem 5.4.1/5.5.3].

In order to get continuity of classical singular integral operators on $L^p(\mathbb{R}^n, \omega)$ for $1 < p < \infty$ a necessary and sufficient condition is that ω is in the Muckenhoupt class $\mathcal{A}_p$, i.e.,

$$\sup_Q \frac{1}{|Q|}\int_Q \omega(x)dx \left(\frac{1}{|Q|}\int_Q \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty,$$

where the supremum is taken with respect to all cubes $Q \subset \mathbb{R}^n$, cf. [24, Chapter V]. In the case that $\omega(x) = |x_n|^{\delta p}$, it is an elementary calculation that $|x_n|^{\delta p} \in \mathcal{A}_p$ if and only if $-\frac{1}{p} < \delta < \frac{1}{p'}$.

If $\omega \in \mathcal{A}_p$, $1 < p < \infty$, and $s \in \mathbb{R}$, then we define the weighted Bessel potential space as

$$H^s_p(\mathbb{R}^n, \omega) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x\rangle^s u \in L^p(\mathbb{R}^n, \omega)\}$$

normed by $\|\langle D_x\rangle^s \cdot \|_{L^p(\mathbb{R}^n,\omega)}$. Using the variant of the Mikhlin multiplier theorem for weighted L^p-spaces when $\omega \in \mathcal{A}_p$, cf. Garcia-Cuerva and Rubio de Francia [8, Chapter IV, Theorem 3.9], one can prove in the same way as for the standard Bessel potential spaces that

$$H^m_p(\mathbb{R}^n, \omega) = W^m_p(\mathbb{R}^n, \omega) := \{u \in L^p(\mathbb{R}^n, \omega) : D^\alpha_x u \in L^p(\mathbb{R}^n, \omega) \text{ for } |\alpha| \le m\} \quad (2.4)$$

for $m \in \mathbb{N}_0$ and that

$$(H^{s_0}_p(\mathbb{R}^n, \omega), H^{s_1}_p(\mathbb{R}^n, \omega))_{[\theta]} = H^s_p(\mathbb{R}^n, \omega), \qquad s = (1-\theta)s_0 + \theta s_1, \qquad (2.5)$$

for all $s_0, s_1 \in \mathbb{R}$, $1 < p < \infty$, and $\omega \in \mathcal{A}_p$, cf. Fröhlich [7, Lemma 8.1/Satz 8.3].

Moreover, since $\langle D_x\rangle^s$ is an isomorphism from $H^s_p(\mathbb{R}^n, \omega)$ onto $L^p(\mathbb{R}^n, \omega)$ and because of (2.3),

$$(H^s_p(\mathbb{R}^n, \omega))' = H^{-s}_{p'}(\mathbb{R}^n, \omega'), \qquad \omega'(x) = \omega(x)^{-\frac{p'}{p}}. \qquad (2.6)$$

If $\omega \in \mathcal{A}_p$, the weighted Bessel potential spaces on $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty)$ are defined as

$$\begin{aligned} H^s_p(\mathbb{R}^n_+, \omega) &:= r^+ H^s_p(\mathbb{R}^n; \omega), \\ H^s_{p;0}(\mathbb{R}^n_+, \omega) &:= \{u \in H^s_p(\mathbb{R}^n, \omega) : \operatorname{supp} u \subseteq \overline{\mathbb{R}}^n_+\}, \end{aligned}$$

where $r^+ f$ denotes the restriction of a distribution f to $\mathbb{R}^n_+$. As usual

$$H^m_p(\mathbb{R}^n_+, \omega) = W^m_p(\mathbb{R}^n_+, \omega) := \{u \in L^p(\mathbb{R}^n_+, \omega) : D^\alpha_x u \in L^p(\mathbb{R}^n_+, \omega) \text{ for } |\alpha| \le m\},$$

cf. [7, Section 8.2.2]. Moreover, because of (2.6) and the definitions,

$$H^s_p(\mathbb{R}^n_+, \omega)' = H^{-s}_{p';0}(\mathbb{R}^n_+, \omega'), \qquad \omega'(x) = \omega(x)^{-\frac{p'}{p}}.$$

In particular we will use $H^s_p(\mathbb{R}_+, |x_n|^{p\delta})$ and $H^s_{p;0}(\mathbb{R}_+, |x_n|^{p\delta})$ also denoted by $H^s_p(\mathbb{R}_+, x_n^{p\delta})$, $H^s_{p;0}(\mathbb{R}_+, x_n^{p\delta})$, resp., with $-\frac{1}{p} < \delta < \frac{1}{p'}$. Since $(1+i\xi_n)^t \in S^t_{1,0}(\mathbb{R}\times\mathbb{R})$ is a classical pseudodifferential symbol and because of the continuity of pseudodifferential operators in weighted Bessel potential spaces, cf. Marschall [20],

$$(1+iD_n)^t \colon H^{s+t}_p(\mathbb{R}, |x_n|^{\delta p}) \to H^s_p(\mathbb{R}, |x_n|^{\delta p}).$$

By the Paley-Wiener theorem, $\operatorname{supp}(1+iD_n)^t f \subseteq \overline{\mathbb{R}}_+$ if $\operatorname{supp} f \subseteq \overline{\mathbb{R}}_+$. Hence

$$(1+iD_n)^t \colon H^{s+t}_{p;0}(\mathbb{R}_+, x_n^{\delta p}) \to H^s_{p;0}(\mathbb{R}_+, x_n^{\delta p}) \tag{2.7}$$
$$(1-iD_n)^t \colon H^{s+t}_{p}(\mathbb{R}_+, x_n^{\delta p}) \to H^s_{p}(\mathbb{R}_+, x_n^{\delta p}) \tag{2.8}$$

are isomorphisms by (2.6) and $((1+iD_n)^t)' = (1-iD_n)^{-t}$.

As a consequence we obtain the following generalization of Grubb and Kokholm [14, Theorem 1.8]:

Lemma 2.2 *Let $s \in \mathbb{R}$. If $1 < q \le 2$, $-\frac{1}{2} < \delta_1 < \frac{1}{q} - \frac{1}{2} < \delta_2 < \frac{1}{2}$, and $\theta = (\frac{1}{q} - \frac{1}{2} - \delta_1)/(\delta_2 - \delta_1)$, then*

$$(H^s_2(\mathbb{R}_+, x_n^{2\delta_1}), H^s_2(\mathbb{R}_+, x_n^{2\delta_2}))_{\theta,q} \subseteq H^s_q(\mathbb{R}_+), \tag{2.9}$$
$$(H^{s-\delta_1}_{2;0}(\mathbb{R}_+), H^{s-\delta_2}_{2;0}(\mathbb{R}_+))_{\theta,q} \supseteq H^s_{q;0}(\mathbb{R}_+). \tag{2.10}$$

Moreover, if $2 \le q < \infty$, $s \in \mathbb{R}$, $-\frac{1}{2} < \delta_1 < \frac{1}{2} - \frac{1}{q} < \delta_2 < \frac{1}{2}$, and $\theta = (\frac{1}{2} - \frac{1}{q} - \delta_1)/(\delta_2 - \delta_1)$, then

$$(H^s_{2;0}(\mathbb{R}_+, x_n^{-2\delta_1}), H^s_{2;0}(\mathbb{R}_+, x_n^{-2\delta_2}))_{\theta,q} \supseteq H^s_{q;0}(\mathbb{R}_+), \tag{2.11}$$
$$(H^{s+\delta_1}_{2}(\mathbb{R}_+), H^{s+\delta_2}_{2}(\mathbb{R}_+))_{\theta,q} \subseteq H^s_{q}(\mathbb{R}_+). \tag{2.12}$$

Proof: The lemma was proved by Grubb and Kokholm [14, Theorem 1.8] for the case $s = 0$, where we note that $H^{\pm\delta_j}_{2,0}(\mathbb{R}_+) = H^{\pm\delta_j}_2(\mathbb{R}_+)$ since $|\delta_j| < \frac{1}{2}$. Then the general case is a consequence of (2.7)-(2.8). ■

Finally, we note that

$$X^s_p(\mathbb{R}^n_+) = X^s_p(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) \cap L^p(\mathbb{R}^{n-1}; X^s_p(\mathbb{R}_+)) \tag{2.13}$$
$$X^{-s}_p(\mathbb{R}^n_+) = X^{-s}_p(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) + L^p(\mathbb{R}^{n-1}; X^{-s}_p(\mathbb{R}_+)) \tag{2.14}$$

where $X = H$ or $X = B$ and $s > 0$, cf. e.g. Grubb [11, (A.23)] and the references given there.

If $\delta \ge 0$, $s > 0$, we define the weighted Besov space

$$\begin{aligned} B^s_q(\mathbb{R}^n_+, x_n^{\delta q}) &:= \{f \in W^{[s]}_q(\mathbb{R}^n_+, x_n^{\delta q}) : \|f\|_{B^s_q(\mathbb{R}^n_+, x_n^{\delta q})} < \infty\} \\ \|f\|^q_{B^s_q(\mathbb{R}^n_+, x_n^{\delta q})} &:= \|f\|^q_{W^{[s]}_q(\mathbb{R}^n_+, x_n^{\delta q})} + \sum_{|\alpha|\le k} \int_{\mathbb{R}^n_+} |h|^{-(s-k)q} \|\Delta^l_h D^\alpha f\|^q_{L^q(\mathbb{R}^n_+, x_n^{\delta q})} \frac{dh}{|h|^n}, \end{aligned}$$

where $k, l \in \mathbb{N}_0$ such that $k < s$ and $l > s - k$. Then

$$B^s_q(\mathbb{R}^n_+, x_n^{\delta q}) = (L^q(\mathbb{R}^n_+, x_n^{\delta q}), W^m_q(\mathbb{R}^n_+, x_n^{\delta q}))_{\theta,q}, \tag{2.15}$$

where $s = \theta m$, $0 < \theta < 1$, $m \in \mathbb{N}$, cf. [26, Theorem 3.3.1]. Finally, we define

$$B^{-s}_{q;0}(\mathbb{R}^n_+, x_n^{-\delta q}) := (B^s_{q'}(\mathbb{R}^n_+, x_n^{\delta q'}))'.$$

Note that the definition of this weighted Besov space for $\delta = 0$ is consistent with the Besov spaces defined by

$$B^s_q(\mathbb{R}^n_+) := r^+ B^s_q(\mathbb{R}^n), \qquad B^s_{q;0}(\mathbb{R}^n_+) = \{f \in B^s_q(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\mathbb{R}^n_+}\}, \qquad s \in \mathbb{R}$$

and that $B^s_q(\mathbb{R}^n_+) = B^s_{q;0}(\mathbb{R}^n_+)$ if and only if $-\frac{1}{q'} < s < \frac{1}{q}$.

3 Operator-Valued Pseudodifferential Operators with Non-Smooth Coefficients

In the following we will use operator-valued pseudodifferential operators with coefficients in the Hölder space $C^\tau(\mathbb{R}^n)$ of all functions $f\colon \mathbb{R}^n \to \mathbb{C}$ with Hölder continuous derivatives $\partial_x^\alpha f$ of degree $\tau - [\tau]$ for all $|\alpha| \leq [\tau]$ normed by

$$\|f\|_{C^\tau(\mathbb{R}^n)} := \sum_{|\alpha|\leq[\tau]} \|\partial_x^\alpha f\|_\infty + \sum_{|\alpha|=[\tau]} \sup_{x\neq y} \frac{|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)|}{|x-y|^{\tau-[\tau]}}.$$

Here $[\tau]$ denotes the largest integer not larger than τ. The vector-valued variant $C^\tau(\mathbb{R}^n; X)$, where X is a Banach space, is defined in an obvious way.

In the following we will often use that

$$C^\tau(\mathbb{R}^n) \hookrightarrow C^{\tau-\tau'}(\mathbb{R}^{n-1}; C^{\tau'}(\mathbb{R})) \qquad \text{for } 0 < \tau' < \tau. \tag{3.1}$$

Definition 3.1 *Let X be a Banach space. The symbol space $C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$, $\tau > 0$, $\delta \in [0,1]$, $m \in \mathbb{R}$, is the set of all functions $p\colon \mathbb{R}^n \times \mathbb{R}^n \to X$ that are smooth with respect to ξ and are in C^τ with respect to x satisfying the estimates*

$$\|D_\xi^\alpha D_x^\beta p(.,\xi)\|_{L^\infty(\mathbb{R}^n;X)} \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|+\delta|\beta|}, \quad \|D_\xi^\alpha p(.,\xi)\|_{C^\tau(\mathbb{R}^n;X)} \leq C_\alpha\langle\xi\rangle^{m-|\alpha|+\delta\tau}$$

for all $\alpha \in \mathbb{N}_0^n$ and $|\beta| \leq [\tau]$.

For short we also write $C^\tau S^m_{1,\delta}(X)$ or even $C^\tau S^m_{1,\delta}$ if X is known from the context. Obviously, $\bigcap_{\tau>0} C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$ coincides with the usual Hörmander class $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$ in the vector-valued variant.

Remark 3.2 Note that if $p \in C^\tau S^{m_0}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X_0) \cap C^\tau S^{m_1}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X_1)$ and (X_0, X_1) is an interpolation couple, then $p \in C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$ with $X = (X_0, X_1)_{\theta,q}$ or $X = (X_0, X_1)_{[\theta]}$, $\theta \in (0,1)$, $1 \leq q \leq \infty$, and $m = (1-\theta)m_0 + \theta m_1$.

In particular we are interested in the case $\delta = 0$. But we need the classes $C^\tau S^m_{1,\delta}$ with $\delta > 0$ when working with the technique called *symbol smoothing*: If $p \in C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$, $\delta \in [0,1)$, then for every $\gamma \in (\delta, 1)$ there is a decomposition $p = p^\# + p^b$ with

$$p^\# \in S^m_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; X), \qquad p^b \in C^\tau S^{m-(\gamma-\delta)\tau}_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; X), \tag{3.2}$$

cf. [25, Equation (1.3.21)]. Moreover, if $\delta = 0$, we have

$$\partial_x^\beta p^\# \in S^m_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; X) \qquad \text{for } |\beta| \le \tau, \tag{3.3}$$

$$\partial_x^\beta p^\# \in S^{m+\gamma(|\beta|-\tau)}_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; X) \quad \text{for } |\beta| > \tau, \tag{3.4}$$

cf. Taylor [25, Proposition 1.3.D]. Note that the proofs in [25] are formulated for scalar symbols only, but they still hold in the X-valued setting since they are based on elementary estimates.

In the case $X = \mathcal{L}(X_0, X_1)$ is the space of all bounded linear operators $A\colon X_0 \to X_1$ for some Banach spaces X_0 and X_1 we define the pseudodifferential operator of a symbol $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$ as

$$p(x, D_x)u = \mathrm{OP}(p)u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\!\!\bar{}\xi \quad \text{for } u \in \mathcal{S}(\mathbb{R}^n; X_0),$$

where $d\!\!\bar{}\xi := (2\pi)^{-n} d\xi$. Moreover, $\mathrm{OP}'(p)$ and $\mathrm{OP}_n(p)$ will denote the pseudodifferential operator of a symbol depending on $x', \xi' \in \mathbb{R}^{n-1}$, $x_n, \xi_n \in \mathbb{R}$, respectively, where we use the decomposition $x = (x', x_n)$, $\xi = (\xi', \xi_n)$ for $x, \xi \in \mathbb{R}^n$.

Note that, if $p \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$, $\delta \in [0,1)$, the well-known statements on composition, adjoints, and asymptotic expansion of pseudodifferential operators with scalar symbols directly carry over to the present operator valued setting, cf. e.g. Kumano-Go [18].

The proofs of the mapping properties of Green operators with non-smooth coefficients are based on the following two theorems.

THEOREM 3.3 *Let $\tau > 0$, $1 < q < \infty$, $m \in \mathbb{R}$, and let H_0, H_1 be Hilbert spaces. If $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$, then $p(x, D_x)$ extends to a bounded linear operator*

$$p(x, D_x)\colon H^{s+m}_q(\mathbb{R}^n; H_0) \to H^s_q(\mathbb{R}^n; H_1) \quad \textit{for all } |s| < \tau.$$

Proof: Theorem 3.3 is an operator-valued variant of [25, Proposition 2.1.D]. As indicated in [1, Appendix] the proof given in [25] directly carries over to the present setting by using the Mikhlin multiplier theorem in the $\mathcal{L}(H)$-valued version, where it is essential that H is a Hilbert space. ■

It is known that in general $p(x, D_x)$ does not have to be a bounded operator from $H^{s+m}_q(\mathbb{R}^n; X_0)$ to $H^s_q(\mathbb{R}^n; X_1)$ if X_0 and X_1 are merely Banach spaces, see [14, Remark 1.7] for a counterexample. But in the case of vector-valued Besov spaces the situation is easier:

THEOREM 3.4 *Let $\tau > 0$, $0 \le \rho \le 1$, $1 \le q, r \le \infty$, $m \in \mathbb{R}$, and let X_0, X_1 be Banach spaces. If $p \in C^\tau S^m_{1,\rho}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$, then $p(x, D_x)$ extends to a bounded linear operator*

$$p(x, D_x)\colon B^{s+m}_{q,r}(\mathbb{R}^n; X_0) \to B^s_{q,r}(\mathbb{R}^n; X_1) \quad \text{for all } -\tau(1-\rho) < s < \tau.$$

Proof: Once the case $\rho = 1$ is proved, the case $\rho \in [0, 1)$ is easily obtained by the same arguments as in [25, Section 2.1].

The case $X_0 = X_1 = \mathbb{C}$ and $q = r = \infty$ is proved in [25, Theorem 2.1.A]. The proof can be adapted to our situation as follows: For the case $q, r \in [1, \infty]$ we just have to replace [25, Lemma 2.1.H] with [17, Theorem 2.4] and have to use [17, Lemma 2.5] instead of [25, Equation (2.1.23)]. Then the proof in the present vector-valued case is literally the same as in the scalar case since all inequalities are obtained by direct (and in principle elementary) estimates. In particular the Mikhlin multiplier theorem is not needed in contrary to the proof for the Bessel potential spaces. ■

The following variant of the latter theorem will be useful in order to analyze some remainder terms in the composition of Green operators.

Lemma 3.5 *Let $\tau, \tau' > 0$, $1 \le q, r \le \infty$, and let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$, $m \in \mathbb{R}$. If additionally $p \in C^{\tau'} S^{m-\theta}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$ for some $0 < \theta < \tau$, then*

$$p(x, D_x)\colon B^{s+m-\theta}_{q,r}(\mathbb{R}^n; X_0) \to B^s_{q,r}(\mathbb{R}^n; X_1)$$

is a continuous mapping for all $-\tau + \theta < s < \tau$.

Proof: Since $0 < \theta < \tau$, there is a $\delta \in (0, 1)$ such that $\theta = \tau\delta$. Let $p = p^\# + p^b$ be the decomposition as described above with $\gamma = \delta$. Since $p \in C^{\tau'} S^{m-\theta}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$, $p^\# \in S^{m-\theta}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$. Moreover, because of $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$ and $\theta = \delta\tau$, $p^b \in C^\tau S^{m-\theta}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$. Hence the lemma is a consequence of Theorem 3.4. ■

We denote for $k \in \mathbb{N}_0$

$$(p_1 \#_k p_2)(x, \xi) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \partial^\alpha_\xi p_1(x, \xi) D^\alpha_x p_2(x, \xi). \tag{3.5}$$

Moreover, if $p_j(x', \xi')$ are the symbols of operator-valued pseudodifferential operators on $\mathbb{R}^{n-1}$, $p_1 \#'_k p_2$ is defined as above with (x, ξ, α) replaced by $(x', \xi', \alpha') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{N}_0^{n-1}$. In the following we will often use the abbreviation

$$R_\theta(p_1, p_2) := p_1(x, D_x) p_2(x, D_x) - (p_1 \#_{[\theta]} p_2)(x, D_x)$$

and $R'_\theta(p_1, p_2)$ for operators on $\mathbb{R}^{n-1}$ where $\theta \in (0, \tau_2)$.

The following theorem shows that R_θ is of order $m_1+m_2-\theta$ in the sense of mapping properties in Besov and Bessel potential spaces, where $\theta \in (0,\tau_2)$ is arbitrary. This theorem is the basis for all statements on compositions of Green operators with non-smooth coefficients.

THEOREM 3.6 *Let $1 \leq p,q \leq \infty$, $m_1,m_2 \in \mathbb{R}$, $\tau_1,\tau_2 > 0$, $\theta \in (0,\tau_2)$, $\tau := \min(\tau_1,\tau_2-[\theta])$, and let $p_1 \in C^{\tau_1}S^{m_1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n;\mathcal{L}(X_1,X_2))$ and $p_2 \in C^{\tau_2}S^{m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n;\mathcal{L}(X_0,X_1))$, where X_0,X_1,X_2 are Banach spaces. Then for every $s \in \mathbb{R}$ such that $|s|<\tau$, $s-\theta > -\tau_2$, and $-\tau_2+\theta < s+m_1 < \tau_2$*

$$p_1(x,D_x)p_2(x,D_x) - p_1\#_{[\theta]}p_2(x,D_x)\colon B^{s+m_1+m_2-\theta}_{p,q}(\mathbb{R}^n;X_0) \to B^s_{p,q}(\mathbb{R}^n;X_2).$$

are bounded operators (defined by extension from $\mathcal{S}(\mathbb{R}^n;X_0)$). The analogous statement holds for Bessel potential spaces instead of Besov spaces if $1<p<\infty$ and $\theta \notin \mathbb{N}$.

Proof: First of all, if p_1 is chosen according to the assumptions of the theorem and $p_2 \in S^{m_2}_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;\mathcal{L}(X_0,X_1))$, $0 \leq \delta < 1$, is a smooth symbol, then there is a symbol $p_1\#p_2 \in C^{\tau_1}S^{m_1+m_2}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n;\mathcal{L}(X_0,X_1))$ such that

$$p_1(x,D_x)p_2(x,D_x)f = p_1\#p_2(x,D_x)f \tag{3.6}$$

for all $f \in \mathcal{S}(\mathbb{R}^n;X_0)$ and the asymptotic expansion (1.2) holds. The latter statement can be proved by a simple modification of the standard proof for compositions of smooth symbols, cf. e.g. [18, Chapter 2, Theorem 1.7]. The crucial fact is that only smoothness of p_1 in $\xi \in \mathbb{R}^n$ and smoothness of p_2 in x are needed in order to make the proof using oscillatory integrals work.

Let $\delta := \frac{\theta}{\tau_2}$. Then by (3.2)-(3.4) $p_2(x,\xi) = p_2^\#(x,\xi)+p_2^b(x,\xi)$ with $p_2^b \in C^{\tau_2}S^{m_2-\theta}_{1,\delta}$ and

$$\begin{aligned} \partial_x^\alpha p_2^\# &\in S^{m_2}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n;\mathcal{L}(X_0,X_1)) && \text{if } |\alpha| \leq [\tau_2],\\ \partial_x^\alpha p_2^\# &\in S^{m_2-\delta(\tau_2-|\alpha|)}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n;\mathcal{L}(X_0,X_1)) && \text{if } |\alpha| > [\tau_2]. \end{aligned}$$

Hence we get

$$p_1(x,D_x)p_2(x,D_x) = p_1(x,D_x)p_2^\#(x,D_x) + p_1(x,D_x)p_2^b(x,D_x),$$

where $p_1(x,D_x)p_2^b(x,D_x)\colon B^{s+m_1+m_2-\theta}_{p,q}(\mathbb{R}^n;X_0) \to B^s_{p,q}(\mathbb{R}^n;X_1)$ is a bounded operator since $-\tau_2(1-\delta) = -\tau_2+\theta < s+m_1 < \tau_2$ and $|s|<\tau_1$.

Moreover, $p_1(x,\xi) = p_1^\#(x,\xi)+p_1^b(x,\xi)$ with $p_1^\# \in S^{m_1}_{1,\delta}$ and $p_1^b \in C^{\tau_1}S^{m_1-\delta\tau_1}_{1,\delta}$. Because of (3.6), $p_1^\#(x,D_x)p_2^\#(x,D_x) = p^\#(x,D_x)$ and $p_1^b(x,D_x)p_2^\#(x,D_x) = p^b(x,D_x)$ with

$$p^\#(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha p_1^\#(x,\xi)D_x^\alpha p_2^\#(x,\xi), \quad p^b(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha p_1^b(x,\xi)D_x^\alpha p_2^\#(x,\xi),$$

where

$$\partial_\xi^\alpha p_1^\#(x,\xi) D_x^\alpha p_2^\#(x,\xi) \in C^{\tau_1} S_{1,\delta}^{m_1+m_2-(1-\delta)|\alpha|-\theta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0,X_2))$$
$$\partial_\xi^\alpha p_1^b(x,\xi) D_x^\alpha p_2^\#(x,\xi) \in C^{\tau_1} S_{1,\delta}^{m_1+m_2-(1-\delta)|\alpha|-\theta-\delta\tau_1}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0,X_2))$$

if $|\alpha| > [\theta]$. Thus, since $(1-\delta)|\alpha| \geq 0$,

$$p_1(x,\xi) p_2^\#(x,\xi) = \sum_{|\alpha|\leq[\theta]} \frac{1}{\alpha!} \partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2^\#(x,\xi) + r^\#(x,\xi) + r^b(x,\xi)$$

with $r^\# \in S_{1,\delta}^{m_1+m_2-\theta}$ and $r^b \in C^{\tau_1} S_{1,\delta}^{m_1+m_2-\theta-\delta\tau_1}$. Hence

$$r^b(x,D_x)\colon B_{p,q}^{s+m_1+m_2-\theta}(\mathbb{R}^n;X_0) \to B_{p,q}^{s+\delta\tau_1}(\mathbb{R}^n;X_2) \quad \text{if } -\tau_1+\delta\tau_1 < s+\delta\tau_1 < \tau_1$$

and therefore

$$r^\#(x,D_x) + r^b(x,D_x)\colon B_{p,q}^{s+m_1+m_2-\theta}(\mathbb{R}^n;X_0) \to B_{p,q}^{s}(\mathbb{R}^n;X_2) \quad \text{if } |s| < \tau_1.$$

Moreover, if $|\alpha| \leq [\theta]$,

$$\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2^\#(x,\xi) =$$
$$\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) - \partial_\xi^\alpha p_1^\#(x,\xi) D_x^\alpha p_2^b(x,\xi) - \partial_\xi^\alpha p_1^b(x,\xi) D_x^\alpha p_2^b(x,\xi),$$

where $\partial_\xi^\alpha p_1^\# D_x^\alpha p_2^b \in C^{\tau_2-|\alpha|} S_{1,\delta}^{m_1+m_2-(1-\delta)|\alpha|-\theta}$ and $\partial_\xi^\alpha p_1^b D_x^\alpha p_2^b \in C^\tau S_{1,\delta}^{m_1+m_2-\theta-\delta\tau_1}$. Hence

$$\mathrm{OP}(\partial_\xi^\alpha p_1^\# D_x^\alpha p_2^b)\colon B_{p,q}^{s+m_1+m_2-\theta}(\mathbb{R}^n;X_0) \to B_{p,q}^{s+(1-\delta)|\alpha|}(\mathbb{R}^n;X_2)$$

if $-(\tau_2-|\alpha|)(1-\delta) < s+(1-\delta)|\alpha| < \tau_2-|\alpha|$. Thus

$$\mathrm{OP}(\partial_\xi^\alpha p_1^\# D_x^\alpha p_2^b)\colon B_{p,q}^{s+m_1+m_2-\theta}(\mathbb{R}^n;X_0) \to B_{p,q}^{s}(\mathbb{R}^n;X_2)$$

for all $|s| < \tau$. Moreover,

$$\mathrm{OP}(\partial_\xi^\alpha p_1^b D_x^\alpha p_2^b)\colon B_{p,q}^{s+m_1+m_2-\theta}(\mathbb{R}^n;X_0) \to B_{p,q}^{s+\delta\tau_1}(\mathbb{R}^n;X_2)$$

if $-\tau+\tau\delta < s+\delta\tau_1 < \tau$ and therefore

$$\mathrm{OP}(\partial_\xi^\alpha p_1^b D_x^\alpha p_2^b)\colon B_{p,q}^{s+m_1+m_2-\theta}(\mathbb{R}^n;X_0) \to B_{p,q}^{s}(\mathbb{R}^n;X_2)$$

for $|s| < \tau$. Combining all terms, we have proved the theorem for the case of Besov spaces. Because of (2.2) and since $\theta \in (0,\tau_2)$ is arbitrary, the statement for Bessel potential spaces is a consequence of the one for Besov spaces. ■

Remark 3.7 In the case of scalar Bessel potential spaces Marschall proved a similar theorem in the context of non-smooth symbols of the class $S_{\rho,\delta}^m$, cf. [21, Theorem 3.5]. It covers the case $X_0 = X_1 = \mathbb{C}$, $\theta \leq 1$, and $\tau_1 = \tau_2$ of the latter theorem.

4 Poisson Operators, Trace Operators, and Singular Green Operators

We assume that the reader is familiar with the basic definitions of the Boutet de Monvel calculus, cf. [5], [12], [22], or [23]. Recall that $\mathcal{S}(\overline{\mathbb{R}}_+)$ is the space of smooth rapidly decreasing functions on $\overline{\mathbb{R}}_+$. Moreover, since $\mathcal{S}(\overline{\mathbb{R}}_+)$ is a nuclear space, $\mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\overline{\mathbb{R}}^2_{++})$, where $\overline{\mathbb{R}}^2_{++} := \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$.

We start with the definition of the symbol-kernels of the non-smooth Poisson, trace, and singular Green operators.

Definition 4.1 *The space $C^\tau S^d_{1,0}(\mathbb{R}^N \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $d \in \mathbb{R}$, $n, N \in \mathbb{N}$, consists of all functions $\tilde{f}(x,\xi',y_n)$, which are smooth in $(\xi', y_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+$, are in $C^\tau(\mathbb{R}^N)$ with respect to x, and satisfy*

$$\|y_n^l \partial_{y_n}^{l'} D_{\xi'}^\alpha \tilde{f}(.,\xi',.)\|_{C^\tau(\mathbb{R}^N;L^2_{y_n}(\mathbb{R}_+))} \leq C_{\alpha,l,l'} \langle \xi' \rangle^{d+\frac{1}{2}-l+l'-|\alpha|} \tag{4.1}$$

for all $\alpha \in \mathbb{N}_0^{n-1}, l, l' \in \mathbb{N}_0$.

Similarly, the space $C^\tau S^d_{1,0}(\mathbb{R}^N \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $d \in \mathbb{R}$, $n, N \in \mathbb{N}$, is the space of all $\tilde{f}(x,\xi',y_n,z_n)$, which are smooth in $(\xi', y_n, z_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{R}}^2_{++}$ and which are in $C^\tau(\mathbb{R}^N)$ with respect to x such that

$$\|y_n^k \partial_{y_n}^{k'} z_n^l \partial_{z_n}^{l'} D_{\xi'}^\alpha \tilde{f}(.,\xi',.)\|_{C^\tau(\mathbb{R}^N;L^2_{y_n,z_n}(\mathbb{R}^2_{++}))} \leq C_{\alpha,k,k',l,l'} \langle \xi' \rangle^{d+1-k+k'-l+l'-|\alpha|} \tag{4.2}$$

for all $\alpha \in \mathbb{N}_0^{n-1}, k, k', l, l' \in \mathbb{N}_0$.

Now the Poisson operators with non-smooth coefficients are defined in almost the same way as in the smooth case:

Definition 4.2 *Let $\tilde{k} = \tilde{k}(x,\xi',y_n) \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $d \in \mathbb{R}$. Then we define the* Poisson operator *of order d by*

$$k(x,D_x)a = \mathcal{F}^{-1}_{\xi' \mapsto x'} \left[\tilde{k}(x,\xi',x_n) \acute{a}(\xi') \right], \qquad a \in \mathcal{S}(\mathbb{R}^{n-1}),$$

where $\acute{a}(\xi') := \mathcal{F}_{x' \mapsto \xi'}[a]$ denotes the partial Fourier transform *applied to a.*

Remarks 4.3 1. In the following many symbol-kernels $\tilde{k}(x,\xi',y_n)$ will depend only on $x' \in \mathbb{R}^{n-1}$ as in the standard calculus with smooth coefficient. This fact will be denoted by $\tilde{k} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$. But we will also need these more general symbol-kernels since they occur naturally when considering $k(x,D_x)a := r^+ p(x,D_x)\delta_0 \otimes a$, where p is a pseudodifferential operator satisfying the transmission condition defined below and δ_0 denotes the delta distribution w.r.t. x_n.

2. If $\tilde{k} \in S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+)) = \bigcap_{\tau>0} C^\tau S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ is a smooth symbol, then $\tilde{k}(x, \xi', x_n) \in \mathcal{S}(\overline{\mathbb{R}}_+)$ w.r.t x_n and one can prove that $k(x, D_x) = k'(x', D_x)$ with $\tilde{k}'(x', \xi', x_n) \in S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, cf. [12, Remark 2.4.9]. Moreover,
$$\tilde{k}'(x', \xi', x_n) \sim \sum_{k \in \mathbb{N}_0} \frac{1}{k!} x_n^k \partial_{x_n}^k k(x', 0, \xi', y_n)|_{y_n = x_n} \tag{4.3}$$
Of course the latter statement no longer holds if $k(x, \xi', y_n)$ is not smooth in x_n. Nevertheless $k(x, D_x)$ can be approximated by an operator $k'(x', D_x)$ with symbol-kernel derived from (4.3) with $k < \tau$, cf. Theorem 4.11 below.

3. For each fixed $x \in \mathbb{R}^n$ the symbol-kernel $\tilde{k}_x(\xi', y_n) := \tilde{k}(x, \xi', y_n)$ belongs to $S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ w.r.t (ξ', y_n). Moreover, let
$$|\tilde{f}|_m^{(d-1)} := \sup_{x', \xi' \in \mathbb{R}^{n-1}, l + l' + |\alpha| + |\beta| \le m} \|y_n^l \partial_{y_n}^{l'} D_{x'}^\beta D_{\xi'}^\alpha \tilde{f}(x', \xi', .))\|_{L^2(\mathbb{R}_+)} \langle \xi' \rangle^{-d + \frac{1}{2} + l - l' + |\alpha|}$$
for $f \in S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$. Then $\tilde{k} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ if and only if $\tilde{k}_x \in S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ for each fixed $x \in \mathbb{R}^n$ and
$$\begin{aligned} |\partial_x^\beta \tilde{k}(x, .)|_m^{(d-1)} &\le C_{\beta,m} && \text{for } |\beta| \le [\tau], \\ |\partial_x^\beta \tilde{k}(x, .) - \partial_x^\beta \tilde{k}(y, .)|_m^{(d-1)} &\le C_{\beta,m} |x - y|^{\tau - [\tau]} && \text{for } |\beta| = [\tau] \end{aligned}$$
uniformly in $x, y \in \mathbb{R}^n$ and for all $m \in \mathbb{N}_0$.

Finally, we note that the *boundary symbol operator* $k(x, \xi', D_n)$ is defined as a one-dimensional operator with symbol-kernel $\tilde{k}(x, \xi', y_n)$ for fixed (x', ξ').

The trace and singular Green operators are defined as follows:

Definition 4.4 *Let $d \in \mathbb{R}$ and let $r \in \mathbb{N}_0$.*

1. *If $\tilde{t} \in C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $s_j \in C^\tau S^{d-j}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $j = 0, \ldots, r-1$, then the associated trace operator of order d and class r is defined as*
$$\begin{aligned} t(x', D_x) f &= \sum_{j=0}^{r-1} s_j(x', D_{x'}) \gamma_j f + t_0(x', D_x) f \\ t_0(x', D_x) f &= \mathcal{F}^{-1}_{\xi' \mapsto x'} \left[\int_0^\infty \tilde{t}_0(x', \xi', y_n) \acute{f}(\xi', y_n) dy_n \right], \end{aligned}$$
where $\acute{f}(\xi', x_n) = \mathcal{F}_{x' \mapsto \xi'}[f(., x_n)]$ and $\gamma_j f = D_n^j f|_{x_n = 0}$.

2. *If $\tilde{g} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $\tilde{k}_j \in C^\tau S^{d-j-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ for $j = 0, \ldots, r-1$, then the associated singular Green operator of order d and*

class r is defined as

$$\begin{aligned} g(x,D_x)f &= \sum_{j=0}^{r-1} k_j(x,D_x)\gamma_j f + g_0(x,D_x)f, \\ g_0(x,D_x)f &= \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\int_0^\infty \tilde{g}_0(x,\xi',x_n,y_n)\acute{f}(\xi',y_n)dy_n\right], \end{aligned}$$

where $\acute{f}$ and $\gamma_j f$ are as above.

Finally, the boundary symbol operators $t(x',\xi',D_n)$ and $g(x,\xi',D_n)$ are defined in the same way as for the Poisson operator.

Remark 4.5 Let $a_j(x,\xi',D_n)$, $j=1,2$, be the boundary symbol operator of a Poisson, trace, or singular Green operator of order d_j, class r_j, with coefficients in C^{τ_j}. Using the observation of Remark 4.3.2 it follows from the standard calculus that the composition $a_1(x,\xi',D_n)a_2(x',\xi',D_n) = a(x,\xi',D_n)$ of boundary symbol operators is again a boundary symbol operator if the composition is well-defined and the coefficients of a_2 are independent of x_n. The new boundary symbol operator is of order d_1+d_2, class r_2, and has coefficients in $C^{\min(\tau_1,\tau_2)}$.

In order to apply Theorem 3.3 and Theorem 3.6 it is an important fact that we can consider the Poisson, trace, and singular Green operators as operator-valued pseudodifferential operators as follows:

Lemma 4.6 *Let $1<q<\infty$, $d\in\mathbb{R}$, $\tau>0$. Moreover, let $\tilde{k}\in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{t}\in C^\tau S^{d}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{S}(\overline{\mathbb{R}}_+))$, and let $\tilde{g}\in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{S}(\overline{\mathbb{R}}^2_{++}))$. Then*

$$\begin{aligned} &k(x',\xi',D_n)\in C^\tau S^{d-\frac{1}{q}+s-\delta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},H^s_q(\mathbb{R}_+,x_n^{q\delta}))),\\ &t(x',\xi',D_n)\in C^\tau S^{d+\frac{1}{q}+s-\delta'}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(H^{-s}_{q;0}(\mathbb{R}_+,x_n^{-q\delta'}),\mathbb{C})),\\ &g(x',\xi',D_n)\in C^\tau S^{d+s+s'-\delta-\delta'}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(H^{-s}_{q;0}(\mathbb{R}_+,x_n^{-q\delta'}),H^{s'}_q(\mathbb{R}_+,x_n^{q\delta}))) \end{aligned}$$

for all $s,s'\geq 0$, $0\leq\delta<\frac{1}{q'}$, and $0\leq\delta'<\frac{1}{q}$.

Proof: First of all, if $\tilde{f}\in C^\tau S^d_{1,0}(\mathbb{R}^N\times\mathbb{R}^{n-1},\mathcal{S}(\overline{\mathbb{R}}_+))$, $N\in\mathbb{N}_0$, $d\in\mathbb{R}$, then (4.1) implies

$$y_n^l\partial_{y_n}^{l'}\tilde{f}(x,\xi',y_n)\in C^\tau S^{d+\frac{1}{2}-l+l'}_{1,0}(\mathbb{R}^N\times\mathbb{R}^{n-1};L^2(\mathbb{R}_+)),$$

where the $L^2(\mathbb{R}_+)$-norm is taken with respect to y_n. Moreover, we will use the elementary interpolation inequalities

$$\|f\|_p\leq C_{p,q}\|f\|_q^{1+\frac{1}{q}-\frac{1}{p}}\|x_nf\|_q^{\frac{1}{p}-\frac{1}{q}}\qquad \|f\|_q\leq C_{p,q}\|f\|_p^{1+\frac{1}{q}-\frac{1}{p}}\|f'\|_p^{\frac{1}{p}-\frac{1}{q}}\tag{4.4}$$

for all $f \in \mathcal{S}(\overline{\mathbb{R}}_+)$ and $1 \le p \le q \le \infty$ such that $\frac{1}{p} - \frac{1}{q} < 1$. The first one is proved by using $1 = 1/(1+\varepsilon|x_n|) + \varepsilon|x_n|/(1+\varepsilon|x_n|)$, applying Hölder's inequality, and choosing a suitable $\varepsilon > 0$. The second inequality is a consequence of Sobolev's embedding theorem applied to $f(\varepsilon x)$ for $\varepsilon = \|f\|_p/\|f'\|_p$.

Therefore we conclude

$$y_n^l \partial_{y_n}^{l'} \tilde{f}(x,\xi',y_n) \in C^\tau S_{1,0}^{d+1-\frac{1}{q}-l+l'}(\mathbb{R}^N \times \mathbb{R}^{n-1}; L^q(\mathbb{R}_+)).$$

Hence

$$\partial_{y_n}^{l'} \tilde{f}(x,\xi',y_n) \in C^\tau S_{1,0}^{d+1-\frac{1}{q}-\delta+l'}(\mathbb{R}^N \times \mathbb{R}^{n-1}; L^q(\mathbb{R}_+, y_n^{q\delta}))$$

for $\delta \ge 0$ because of $(L^q(\mathbb{R}_+), L^q(\mathbb{R}_+, y_n^{lq}))_{\theta,q} = L^q(\mathbb{R}_+, y_n^{q\theta l})$, $\theta \in (0,1)$. Thus

$$\tilde{f}(x,\xi',y_n) \in C^\tau S_{1,0}^{d+1-\frac{1}{q}-\delta+s}(\mathbb{R}^N \times \mathbb{R}^{n-1}; H_q^s(\mathbb{R}_+, y_n^{q\delta}))$$

for all $s \ge 0$ and $\delta \in [0, \frac{1}{q'})$ by (2.4)-(2.5). This implies the statements for the Poisson and trace boundary symbol operators since $k(x',\xi',D_n)a = \tilde{k}(x',\xi',x_n)a$, $a \in \mathbb{C}$, and $t(x',\xi',D_n)f = \int_0^\infty \tilde{t}(x',\xi',y_n)f(y_n)dy_n$, $f \in \mathcal{S}(\mathbb{R}_+)$.

In the case of singular Green symbol-kernels, the symbol-kernel estimates imply in the same way as before that

$$x_n^\delta y_n^{\delta'} \partial_{x_n}^m \partial_{y_n}^{m'} \tilde{g}(x',\xi',x_n,y_n) \in C^\tau S_{1,0}^{d-\delta-\delta'+m+m'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; L^q(\mathbb{R}_+; L^{q'}(\mathbb{R}_+)))$$

for all $m, m' \in \mathbb{N}_0$ and $\delta, \delta' \ge 0$. Hence

$$\|\partial_{\xi'}^\alpha g(x',\xi',D_n)f\|_{H_q^m(\mathbb{R}_+, x_n^{\delta q})} \le C_{\alpha,m,m'} \langle \xi' \rangle^{d-|\alpha|+m+m'-\delta-\delta'} \|f\|_{H_{q;0}^{-m}(\mathbb{R}_+, x_n^{-q\delta'})}$$

for all $m, m' \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$, $\delta \in [0, \frac{1}{q'})$, and $\delta' \in [0, \frac{1}{q})$, i.e.,

$$g(x',\xi',D_n) \in C^\tau S_{1,0}^{d+m+m'-\delta-\delta'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H_{q;0}^{-m'}(\mathbb{R}_+, x_n^{-q\delta'}), H_q^m(\mathbb{R}_+, x_n^{q\delta}))).$$

for all $m, m' \in \mathbb{N}_0$, $\delta \in [0, \frac{1}{q'})$, and $\delta' \in [0, \frac{1}{q})$. Then interpolation finishes the proof. ■

Remark 4.7 Let $X_q^s(\mathbb{R}_+, x_n^{q\delta'}) := B_q^s(\mathbb{R}_+, x_n^{q\delta'}) \cap H_q^s(\mathbb{R}_+, x_n^{q\delta'})$ and $X_{q;0}^{-s}(\mathbb{R}_+, x_n^{-q\delta}) := B_{q;0}^{-s}(\mathbb{R}_+, x_n^{-q\delta}) + H_{q;0}^{-s}(\mathbb{R}_+, x_n^{-q\delta})$ for $s > 0$ and $X_q^0(\mathbb{R}_+, x_n^{q\delta}) := X_{q;0}^0(\mathbb{R}_+, x_n^{q\delta}) := L^q(\mathbb{R}_+, x_n^{q\delta})$. Then by (2.15) the latter lemma implies

$$\begin{aligned}
&k(x',\xi',D_n) \in C^\tau S_{1,0}^{d-\frac{1}{q}+s-\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, X_q^s(\mathbb{R}_+, x_n^{q\delta}))),\\
&t(x',\xi',D_n) \in C^\tau S_{1,0}^{d+\frac{1}{q}+s-\delta'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(X_{q;0}^{-s}(\mathbb{R}_+, x_n^{-q\delta'}), \mathbb{C})),\\
&g(x',\xi',D_n) \in C^\tau S_{1,0}^{d+s+s'-\delta-\delta'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(X_{q;0}^{-s}(\mathbb{R}_+, x_n^{-q\delta'}), X_q^{s'}(\mathbb{R}_+, x_n^{q\delta'})))
\end{aligned}$$

for all $s, s' \ge 0$, $\delta \in [0, \frac{1}{q'})$, and $\delta' \in [0, \frac{1}{q})$.

THEOREM 4.8 *Let* $\tilde{k} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{t} \in C^\tau S^{d}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, *and let* $\tilde{g} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $d\in\mathbb{R}$, $\tau>0$. *Then for every* $1<q<\infty$

$$\begin{aligned} k(x',D_x)\colon B^{d+s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) &\to H^s_q(\mathbb{R}^n_+) && \text{if } |s|<\tau,\\ t(x',D_x)\colon H^{d+s}_q(\mathbb{R}^n_+) &\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) && \text{if } \left|s-\frac{1}{q}\right|<\tau \text{ and } d+s>-\frac{1}{q'},\\ g(x',D_x)\colon H^{d+s}_q(\mathbb{R}^n_+) &\to H^s_q(\mathbb{R}^n_+) && \text{if } |s|<\tau \text{ and } d+s>-\frac{1}{q'} \end{aligned}$$

are continuous operators.

Proof: By Lemma 4.6 and Theorem 3.3,

$$\begin{aligned} k(x',D_x)\colon & H^{d+s+s'-\frac{1}{2}-\delta'}_q(\mathbb{R}^{n-1}) \to H^s_q(\mathbb{R}^{n-1}; H^{s'}_2(\mathbb{R}_+, x_n^{2\delta'}))\\ g(x',D_x)\colon & H^{d+s+s'+s''-\delta-\delta'}_q(\mathbb{R}^{n-1}; H^{-s''}_{2;0}(\mathbb{R}_+, x_n^{-2\delta})) \to H^s_q(\mathbb{R}^{n-1}; H^{s'}_2(\mathbb{R}_+, x_n^{2\delta'})) \end{aligned}$$

if $|s|<\tau$ and

$$t(x',D_x)\colon H^{d+s}_q(\mathbb{R}^{n-1}; H^{-s''}_{2,0}(\mathbb{R}_+, x_n^{-2\delta})) \to H^{s-s''-\frac{1}{2}-\delta}_q(\mathbb{R}^{n-1})$$

if $\left|s-s''-\frac{1}{2}-\delta\right|<\tau$ for all $s',s''\geq 0$, and $0\leq\delta,\delta'<\frac{1}{2}$. Hence using (2.9) if $1<q\leq 2$ and (2.12) if $2\leq q<\infty$,

$$k(x',D_x)\colon B^{d+s+s'-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \to H^s_q(\mathbb{R}^{n-1}; H^{s'}_q(\mathbb{R}_+))$$

for all $|s|<\tau$ and $s'\geq 0$ which implies the statement for $k(x',D_x)$ due to (2.13)-(2.14). Similarly, using (2.10) if $1<q\leq 2$ and (2.11) if $2\leq q<\infty$,

$$t(x',D_x)\colon H^{d+s}_q(\mathbb{R}^{n-1}; H^{-s''}_{q;0}(\mathbb{R}_+)) \to B^{s-s''-\frac{1}{q}}_q(\mathbb{R}^{n-1})$$

for all $s\in\mathbb{R}$, $s''\geq 0$ with $|s-s''-\frac{1}{q}|<\tau$. Because of $H^{-s''}_{q;0}(\mathbb{R}_+) = H^{-s''}_q(\mathbb{R}_+)$ if $0\leq s''<\frac{1}{q'}$ and (2.13)-(2.14), the statement for $t(x',D_x)$ is proved. Finally, if $1<q\leq 2$,

$$g(x',D_x)\colon H^{d+s+s'+s''}_q(\mathbb{R}^{n-1}; H^{-s''-\delta}_{2,0}(\mathbb{R}_+)) \to H^s_q(\mathbb{R}^{n-1}; H^{s'}_2(\mathbb{R}_+, x_n^{2\delta}))$$

for $s',s''\geq 0$, $0\leq\delta<\frac{1}{2}$, and $|s|<\tau$ and therefore using (2.9)-(2.10) yields

$$g(x',D_x)\colon H^{d+s+s'+s''}_q(\mathbb{R}^{n-1}; H^{-s''}_{q;0}(\mathbb{R}_+)) \to H^s_q(\mathbb{R}^{n-1}; H^{s'}_q(\mathbb{R}_+)).$$

If $2\leq q<\infty$, the latter mapping property is proved with the aid of (2.11)-(2.12) in the same way. Because of (2.13)-(2.14), also the last statement is proved. ■

The following lemma is the fundamental result on x_n-dependent Poisson and singular Green symbol-kernels.

Lemma 4.9 *Let* $\tilde{k}(x,\xi',y_n) \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $m \in \mathbb{R}$, *with* $\tilde{k}|_{x_n=0} = 0$ *and let* $\tilde{g}(x,\xi',y_n,z_n) \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$ *with* $\tilde{g}|_{x_n=0} = 0$. *Then*

$$\begin{aligned} k(x',x_n,\xi',D_n) &\in C^{\tau-\tau'} S^{m-\frac{1}{q}+s-\theta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},H^s_q(\mathbb{R}_+)))\\ &\quad\cap C^\tau S^{m-\frac{1}{q}}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},L^q(\mathbb{R}_+))),\\ g(x',x_n,\xi',D_n) &\in C^{\tau-\tau'} S^{m-\frac{1}{q}+s+s'-\theta-\delta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+,x_n^{-\delta q}),H^s_q(\mathbb{R}_+)))\\ &\quad\cap C^\tau S^{m-\frac{1}{q}+s'-\delta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+,x_n^{-\delta q}),L^q(\mathbb{R}_+))) \end{aligned}$$

for all $\theta \in [0,1)$ *with* $\theta < \tau'$, $0 < s < \tau' < \tau$, $s' > 0$, $\delta \in [0,\frac{1}{q})$.

Proof: We can assume that $s \in (0,1)$ since the general case can be reduced by differentiation and interpolation to this case. First of all, for $\tilde{f} = \tilde{k}$ and $d = m-1$ the symbol-kernel estimates (4.1) are equivalent to the estimates

$$\|y_n^l \partial_{y_n}^{l'} D^\alpha_{\xi'} \tilde{k}(x,\xi',y_n)\|_{C^\tau(\mathbb{R}^n_x;L^\infty_{y_n})} \leq C_{\alpha,l,l'} \langle\xi'\rangle^{m-l+l'-|\alpha|} \tag{4.5}$$

for all $\alpha \in \mathbb{N}_0^{n-1}, l, l' \in \mathbb{N}_0$ because of (4.4). The latter estimates imply

$$\left|\partial^{l'}_{z_n} D^\alpha_{\xi'}\left[\tilde{k}(x',x_n,\xi',z_n) - \tilde{k}(x',y_n,\xi',z_n)\right]\right| \leq C_{\alpha,s,l'}|x_n-y_n|^{\tau'} z_n^{-s}\langle\xi'\rangle^{m-s+l'-|\alpha|} \tag{4.6}$$

uniformly in $x,y \in \mathbb{R}^n, \xi' \in \mathbb{R}^{n-1}$ and for all $\alpha \in \mathbb{N}_0^{n-1}$, $l' \in \mathbb{N}_0$, $s \geq 0$, and $0 < \tau' \leq \min(1,\tau)$.

Claim: *Let* $f^{(\alpha)}(x',x_n,\xi') := D^\alpha_{\xi'}\tilde{k}(x',x_n,\xi',x_n)$. *Then*

$$x_n^{s'}\left|f^{(\alpha)}(x',x_n,\xi') - f^{(\alpha)}(x',y_n,\xi')\right| \leq C_{\alpha,s',\tau'}|x_n-y_n|^{\tau''}\langle\xi'\rangle^{m-|\alpha|-s'-\tau'+\tau''}$$

uniformly in $x',\xi' \in \mathbb{R}^{n-1}$, $x_n, y_n \geq 0$ *with* $|x_n-y_n| \leq 1$ *and for all* $\alpha \in \mathbb{N}_0^{n-1}$, $s' \geq 0$, *where* $0 \leq \tau'' \leq \tau' \leq \min(1,\tau)$.

Proof of the claim: It suffices to consider the case $0 \leq x_n \leq y_n$. Then

$$x_n^{s'}\left|D^\alpha_{\xi'}\tilde{k}(x,\xi',y_n) - D^\alpha_{\xi'}\tilde{k}(x',y_n,\xi',y_n)\right| \leq C_{\alpha,s'}|x_n-y_n|^{\tau''}\langle\xi'\rangle^{m-|\alpha|-s'+\tau'-\tau''} \tag{4.7}$$

by (4.6) with $s = s'+\tau'-\tau''$. Moreover,

$$\begin{aligned} & x_n^{s'}\left|D^\alpha_{\xi'}\tilde{k}(x,\xi',x_n) - D^\alpha_{\xi'}\tilde{k}(x,\xi',y_n)\right|\\ \leq\ & x_n^{s'}\left(|D^\alpha_{\xi'}\tilde{k}(x,\xi',x_n)| + |D^\alpha_{\xi'}\tilde{k}(x,\xi',y_n)|\right) \leq C_{\alpha,s',\tau'}\langle\xi'\rangle^{m-|\alpha|-s'-\tau'} \end{aligned}$$

by (4.6) with $s = s'+\tau'$ and $l' = 0$ since $\tilde{k}(x',0,\xi',y_n) = 0$. Furthermore,

$$\begin{aligned} & x_n^{s'}|D^\alpha_{\xi'}\tilde{k}(x,\xi',x_n) - D^\alpha_{\xi'}\tilde{k}(x,\xi',y_n)|\\ \leq\ & x_n^{s'}\int_{x_n}^{y_n}|\partial_{y_n}\tilde{k}(x,\xi',t)|dt \leq C_{\alpha,s',\tau'}|x_n-y_n|\langle\xi'\rangle^{m-|\alpha|-s'-\tau'+1} \end{aligned}$$

by (4.6) with $s = s' + \tau'$, $l' = 1$, and $y_n = 0$. Interpolation of the last two inequalities yields

$$x_n^{s'} |D_{\xi'}^\alpha \tilde{k}(x, \xi', x_n) - D_{\xi'}^\alpha \tilde{k}(x, \xi', y_n)| \leq C_{\alpha,s'} |x_n - y_n|^{\tau''} \langle \xi' \rangle^{m-|\alpha|-s'-\tau'+\tau''}.$$

Combining the latter inequality and (4.7) proves the claim.

Because of (4.4),

$$\begin{aligned} \|f\|_{B^{\tau''}_{q,\infty}(\mathbb{R}_+)} &\leq C \left(\|f\|_q + \sup_{h>0} h^{-\tau''} \|\Delta_h f\|_q \right) \\ &\leq C \left(\|f\|_\infty^{\frac{1}{q'}} \|x_n f\|_\infty^{\frac{1}{q}} + \sup_{h>0} h^{-\tau''} \|\Delta_h f\|_\infty^{\frac{1}{q'}} \|x_n \Delta_h f\|_\infty^{\frac{1}{q}} \right), \end{aligned}$$

where $\Delta_h f(x) = f(x+h) - f(x)$ and we have used an equivalent norm on $B^{\tau''}_{q,\infty}(\mathbb{R}_+)$ due to [26, Theorem 4.4.1]. Hence we conclude

$$\|D_{\xi'}^\alpha \tilde{k}(x, \xi', x_n)\|_{B^{\tau''}_{q,\infty}(\mathbb{R}_{+,x_n})} \leq C_{\alpha,q,\tau'} \langle \xi' \rangle^{m-\frac{1}{q}-|\alpha|-\tau'+\tau''}.$$

Moreover, using (3.1) it can be proved in the same way that

$$\|\partial_{\xi'}^\alpha \tilde{k}(x', x_n, \xi', x_n)\|_{C^{\tau-\tau'}(\mathbb{R}^{n-1}_{x'}; B^{\tau''}_{q,\infty}(\mathbb{R}_{+,x_n}))} \leq C_{\alpha,q,\tau'} \langle \xi' \rangle^{m-\frac{1}{q}-|\alpha|-\tau'+\tau''}.$$

Finally, let $0 < s < \theta < \tau'$ and set $\tau'' := s - \theta + \tau'$. Then $s < \tau'' < \tau'$ and therefore $B^{\tau''}_{q,\infty}(\mathbb{R}_+) \hookrightarrow H^s_q(\mathbb{R}_+)$ which together with the latter estimate proves

$$k(x', x_n, \xi', D_n) \in C^{\tau-\tau'} S_{1,0}^{m-\frac{1}{q}+s-\theta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, H^s_q(\mathbb{R}_+))).$$

Moreover, by (4.5),

$$\|x_n^l D_{\xi'}^\alpha \tilde{k}(x, \xi', x_n))\|_{C^\tau(\mathbb{R}^{n-1}_{x'}; L^\infty_{x_n})} \leq C_{\alpha,l} \langle \xi' \rangle^{m-l-|\alpha|}$$

for $\alpha \in \mathbb{N}_0^{n-1}$, $l \in \mathbb{N}_0$, which implies $k(x, \xi', D_n) \in C^\tau S_{1,0}^{m-\frac{1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, L^q(\mathbb{R}_+)))$ due to (4.4).

Using the arguments of Lemma 4.6, the proof above can be easily modified to prove the statement for $g(x, \xi', D_n)$. ■

Because of $\tilde{k}(x, \xi', y_n) = \tilde{k}(x', 0, \xi', y_n) + \tilde{k}_r(x, \xi', y_n)$, where $\tilde{k}_r(x, \xi', y_n)|_{x_n=0} = 0$, the latter lemma, Lemma 4.6, and real interpolation imply:

Corollary 4.10 *Let* $\tilde{k} \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ *and let* $\tilde{g} \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $m \in \mathbb{R}$. *Then*

$$\begin{aligned} &k(x, \xi', D_n) \in C^{\tau-\tau'} S_{1,0}^{m-\frac{1}{q}+s}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, H^s_q(\mathbb{R}_+) \cap B^s_q(\mathbb{R}_+))) \\ &g(x, \xi', D_n) \in C^{\tau-\tau'} S_{1,0}^{m-\frac{1}{q}+s+s'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+), H^s_q(\mathbb{R}_+) \cap B^s_q(\mathbb{R}_+))) \end{aligned}$$

for all $0 \leq s < \tau' < \tau$, $s' \geq 0$.

Moreover, Lemma 4.9 yields:

THEOREM 4.11 *Let $\tilde{k} \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ and let $\tilde{g} \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $m \in \mathbb{R}$. Then for every $0 < \theta < \tau$, $\theta \notin \mathbb{N}$, and $-\tau + \theta < s < \tau - [\theta]$*

$$k(x, D_x) - \sum_{j=0}^{[\theta]} \frac{x_n^j}{j!} (\partial_{x_n}^j k)(x', 0, D_x) \colon B_q^{s+m-\frac{1}{q}-\theta}(\mathbb{R}^{n-1}) \to B_q^s(\mathbb{R}^n_+),$$
$$g(x, D_x) - \sum_{j=0}^{[\theta]} \frac{x_n^j}{j!} (\partial_{x_n}^j g)(x', 0, D_x) \colon B_q^{s+m-\theta}(\mathbb{R}^n_+) \to B_q^s(\mathbb{R}^n_+) \quad \text{if } s + m - \theta > -\frac{1}{q'}.$$

Proof: First of all,

$$f(x, \xi', D_n) \;=\; \sum_{j=0}^{[\theta]} \frac{x_n^j}{j!} (\partial_{x_n}^j f)(x', 0, \xi', D_n) + x_n^{[\theta]} f_r(x, \xi', D_n)$$

for $f = k, g$, where

$$y_n^{[\theta]} \tilde{k}_r \in C^{\tau-[\theta]} S^{m-[\theta]-1}_{1,0}(\cdot, \mathcal{S}(\overline{\mathbb{R}}_+)), \quad y_n^{[\theta]} \tilde{g}_r \in C^{\tau-[\theta]} S^{m-[\theta]-1}_{1,0}(\cdot, \mathcal{S}(\overline{\mathbb{R}}^2_{++})),$$

and $\tilde{k}_r|_{x_n=0} = \tilde{g}_r|_{x_n=0} = 0$. Hence it suffices to consider the case $[\theta] = 0$.

Now let $\theta' \in (\theta, \min(1, \tau))$ and let $-\tau + \theta < s < \tau$. Then Lemma 4.9 and Lemma 3.5 yield for $s' \geq 0$

$$k_r(x, D_x) \colon B_q^{m+s-\frac{1}{q}-\theta'+\varepsilon}(\mathbb{R}^{n-1}) \to B_q^s(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)),$$
$$g_r(x, D_x) \colon B_q^{m+s+s'-\theta'}(\mathbb{R}^{n-1}; H^{-s'}_{q;0}(\mathbb{R}_+)) \to B_q^s(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)),$$

and if $s > 0$

$$k_r(x, D_x) \colon B_q^{m+s-\frac{1}{q}-\theta'+\varepsilon}(\mathbb{R}^{n-1}) \to B_q^\varepsilon(\mathbb{R}^{n-1}; B_q^s(\mathbb{R}_+)),$$
$$g_r(x, D_x) \colon B_q^{m+s+s'-\theta'+\varepsilon}(\mathbb{R}^{n-1}; H^{-s'}_{q;0}(\mathbb{R}_+)) \to B_q^\varepsilon(\mathbb{R}^{n-1}; B_q^s(\mathbb{R}_+)),$$

where $\varepsilon > 0$ is arbitrary and $B_q^\varepsilon(\mathbb{R}^{n-1}; B_q^s(\mathbb{R}_+)) \hookrightarrow L^q(\mathbb{R}^{n-1}; B_q^s(\mathbb{R}_+))$. This implies the statement for $k(x, D_x)$ by (2.13)-(2.14) and

$$g_r(x, D_x) \colon B_q^{m+s+s'-\theta'+\varepsilon}(\mathbb{R}^{n-1}; B^{-s'}_{q;0}(\mathbb{R}_+)) \to B_q^s(\mathbb{R}^n_+)$$

for $s' \geq 0$. Hence, if $m + s - \theta \geq 0$, the statement for $g_r(x, D_x)$ is also proved. If $m + s - \theta < 0$, we use (2.15) and (2.2) to obtain

$$g_r(x, D_x) \colon L^q(\mathbb{R}^{n-1}; B^{m+s-\theta'+2\varepsilon}_{q;0}(\mathbb{R}_+)) \to B_q^s(\mathbb{R}^n_+),$$

for $\varepsilon > 0$ sufficiently small. Since $B^{m+s-\theta}_{q;0}(\mathbb{R}_+) = B^{m+s-\theta}_q(\mathbb{R}_+)$ if $-\frac{1}{q'} < m+s-\theta < \frac{1}{q}$, the theorem is proved. ■

Remark 4.12 The argument in the last part of the latter proof will be used many times: In order to show that $A\colon B^{s-\theta}_q(\mathbb{R}^n_+) \to X$ is a bounded operator into a Banach space X for an $s-\theta > -\frac{1}{q'}$, it is sufficient to prove

$$A\colon B^{s+s'-\theta'}_q(\mathbb{R}^{n-1}; H^{-s'}_{q;0}(\mathbb{R}_+)) \to X \quad \text{or} \quad B^{s+s'-\theta'}_q(\mathbb{R}^{n-1}; B^{-s'}_{q;0}(\mathbb{R}_+)) \to X$$

for all $s' \in (0, \frac{1}{q'})$ and some $\theta' > \theta$.

In the following let

$$(a_1 \#'_k a_2)(x,\xi',D_n) := \sum_{|\alpha'|\le k} \frac{1}{\alpha'!} \partial^{\alpha'}_{\xi'} a_1(x,\xi',D_n) D^{\alpha'}_{x'} a_2(x',\xi',D_n), \tag{4.8}$$

where $k \in \mathbb{N}_0$ and $a_j(x,\xi',D_n)$ denotes the boundary symbol operator of a Poisson, trace, singular Green operator, or a symbol of a pseudodifferential operator on $\mathbb{R}^{n-1}$. Moreover, it is assumed that a_2 is independent of x_n. – If a_2 does depend on x_n, Theorem 4.11 can be used to reduce the composition to the latter case. – Finally, let

$$R'_\theta(a_1,a_2) := a_1(x,D_x)a_2(x',D_x) - (a_1 \#'_k a_2)(x,D_x).$$

The following theorem treats compositions of Poisson, trace, singular Green, and $n-1$-dimensional pseudodifferential operators. It is a main step in the proof of Theorem 1.2 and the fundamental result of this section.

THEOREM 4.13 *Let* $\tilde{k}_j \in C^{\tau_j} S^{m_j-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{t}_j \in C^{\tau_j} S^{m_j}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+)))$, $\tilde{g}_j \in C^{\tau_j} S^{m_j-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $s_j \in C^{\tau_j} S^{m_j-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, *and* $c \in C^{\tau_2}(\mathbb{R}^{n-1})$, $\tau_j > 0$, $m_j \in \mathbb{R}$ *such that* $\tilde{k}_2(x,\xi',y_n)$ *and* $\tilde{g}_2(x,\xi',y_n,z_n)$ *are independent of* x_n. *Moreover, let* $\theta \in (0,\tau_2)$, $\theta \notin \mathbb{N}$, *and set* $\tau := \min(\tau_1, \tau_2 - [\theta])$, $m := m_1 + m_2$.

1. *Assume that* $|s| < \tau$, $s - \theta > -\tau_2$, *and* $-\tau_2 + \theta < s + m_1 < \tau_2$. *Then*

$$R'_\theta(g_1,k_2), R'_\theta(s_1,k_2)\colon B^{s+m-\frac{1}{q}-\theta}_q(\mathbb{R}^{n-1}) \to B^s_q(\mathbb{R}^n_+),$$
$$R'_\theta(g_1,g_2), R'_\theta(s_1,g_2)\colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \qquad \text{if } s+m-\theta > -\frac{1}{q'},$$
$$R'_\theta(g_1,c)\colon B^{s+m_1-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \qquad \text{if } s+m_1-\theta > -\frac{1}{q'}.$$

2. *Assume that* $|s| < \tau$, $s - \theta > -\tau_2$, *and* $-\tau_2 + \theta < s + m_1 - \frac{1}{q} < \tau_2$. *Then*

$$R'_\theta(k_1,t_2)\colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \qquad \text{if } s+m-\theta > -\frac{1}{q'},$$
$$R'_\theta(k_1,s_2)\colon B^{s+m-\frac{1}{q}-\theta}_q(\mathbb{R}^{n-1}) \to B^s_q(\mathbb{R}^n_+).$$

3. *Assume that* $\left|s-\frac{1}{q}\right|<\tau$, $s-\frac{1}{q}-\theta>-\tau_2$, *and* $-\tau_2+\theta<s+m_1<\tau_2$. *Then*

$$R'_\theta(t_1,g_2)\colon B^{s+m-\theta}_q(\mathbb{R}^n_+)\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \qquad \text{if } s+m-\theta>-\frac{1}{q'},$$
$$R'_\theta(t_1,k_2)\colon B^{s+m-\frac{1}{q}-\theta}_q(\mathbb{R}^{n-1})\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}),$$
$$R'_\theta(t_1,c)\colon B^{s+m_1-\theta}_q(\mathbb{R}^n_+)\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \qquad \text{if } s+m_1-\theta>-\frac{1}{q'}.$$

4. *Assume that* $\left|s-\frac{1}{q}\right|<\tau$, $s-\frac{1}{q}-\theta>-\tau_2$, *and* $-\tau_2+\theta<s+m_1-\frac{1}{q}<\tau_2$. *Then*

$$R'_\theta(s_1,t_2)\colon B^{s+m-\theta}_q(\mathbb{R}^n_+)\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \qquad \text{if } s+m-\theta>-\frac{1}{q'},$$
$$R'_\theta(s_1,s_2)\colon B^{s+m-\frac{1}{q}-\theta}_q(\mathbb{R}^{n-1})\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}).$$

Proof: First of all let $\theta'\in(\theta,\tau_2)$ with $[\theta']=[\theta]$ sufficiently close to θ such that all conditions still hold if θ is replaced by θ'.

We first consider the composition $g_1(x,D_x)g_2(x',D_x)$. The statement concerning this composition is a consequence of the fact that

$$R'_\theta(g_1,g_2)\colon B^{s+s'+m-\theta'}_q(\mathbb{R}^{n-1};H^{-s'}_{q;0}(\mathbb{R}_+))\to B^s_q(\mathbb{R}^{n-1};L^q(\mathbb{R}_+)) \qquad \text{and} \tag{4.9}$$
$$R'_\theta(g_1,g_2)\colon B^{s+s'+m-\theta'+\varepsilon}_q(\mathbb{R}^{n-1};H^{-s'}_{q;0}(\mathbb{R}_+))\to B^\varepsilon_q(\mathbb{R}^{n-1};B^s_q(\mathbb{R}_+)) \quad \text{if } s>0 \tag{4.10}$$

for some $\varepsilon>0$ and arbitrary $s'>0$, cf. Remark 4.12. Because of Remark 4.7 and Corollary 4.10,

$$g_1(x,\xi',D_n)\in C^{\tau_1-\tau'}S^{m_1+s}_{1,0}(\mathcal{L}(L^q(\mathbb{R}_+),B^s_q(\mathbb{R}_+)))\cap C^{\tau_1}S^{m_1}_{1,0}(\mathcal{L}(L^q(\mathbb{R}_+))),$$
$$g_2(x',\xi',D_n)\in C^{\tau_2}S^{m_2+s'}_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+),L^q(\mathbb{R}_+))),$$

where $s'\geq 0$, $0<s<\tau'<\tau_1$. Hence (4.9) and (4.10) are consequences of Theorem 3.6.

All other compositions of the operators $k_j(x,D_x),t_j(x',D_x),g_j(x,D_x)$, and $s_j(x',D_{x'})$ except $s_1(x',D_{x'})k_2(x',D_x)$, $s_1(x',D_{x'})g_2(x',D_x)$ and the compositions with $c(x')$ are treated in the same way.

In order to estimate $R_\theta(s_1,k_2)$ and $R_\theta(s_1,g_2)$, we use that

$$k_2(.,D_n)\in C^{\tau_2}S^{m_2+s-\frac{1}{q}}_{1,0}(\mathcal{L}(\mathbb{C},B^s_q(\mathbb{R}_+)))\cap C^{\tau_2}S^{m_2-\frac{1}{q}}_{1,0}(\mathcal{L}(\mathbb{C},L^q(\mathbb{R}_+))),$$
$$g_2(.,D_n)\in C^{\tau_2}S^{m_2+s+s'}_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+),B^s_q(\mathbb{R}_+)))\cap C^{\tau_2}S^{m_2+s'}_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+),L^q(\mathbb{R}_+))),$$
$$s_1\in C^{\tau_1}S^{m_1}_{1,0}(\mathcal{L}(B^s_q(\mathbb{R}_+))\cap\mathcal{L}(L^q(\mathbb{R}_+))),$$

where $s>0,s'\geq 0$ and apply Theorem 3.6 as before.

Finally, the statements for $g_1(x, D_x)c(x')$ and $t_1(x, D_x)c(x')$ are proved using

$$g_1(., D_n) \in C^{\tau_1 - \tau'} S^{m_1+s+s'}_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+), B^s_q(\mathbb{R}_+))) \cap C^{\tau_1} S^{m_1+s'}_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+), L^q(\mathbb{R}_+))),$$
$$t_1(., D_n) \in C^{\tau_1} S^{m_1+s'}_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+), \mathbb{C})), \quad c(x') \in C^{\tau_2} S^0_{1,0}(\mathcal{L}(H^{-s'}_{q;0}(\mathbb{R}_+))),$$

where $s' \geq 0$ and $0 < s < \tau' < \tau_1$. ■

Remark 4.14 Note that all singular Green and trace operators in the latter theorem are of class 0. The statements in the general case can be easily obtained from the latter one using that $\gamma_j k_2(x', D_x)$ and $\gamma_j g_2(x', D_x)$ are pseudodifferential operators, Poisson operators of order $m_2 + j$, $j \in \mathbb{N}_0$, respectively.

The following lemma treats some remainder terms, which will be needed when discussing compositions with differential operators.

Lemma 4.15 *Let $\tilde{t} \in C^{\tau_1} S^m_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{g} \in C^{\tau_1} S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $m \in \mathbb{R}$, $\tau_1, \tau_2 > 0$, and let $c \in C^{\tau_2}(\mathbb{R}^n)$ with $c(x', 0) = 0$. Then for every $\theta \in [0, 1)$ with $\theta < \tau_2$ and $s \in \mathbb{R}$ with $-\tau_2 + \theta < s + m < \tau_2$ and $s + m - \theta > -\frac{1}{q'}$*

$$t(x', D_x)c(x) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \quad \text{if } \left|s - \frac{1}{q}\right| < \tau_1,$$
$$g(x, D_x)c(x) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \quad \text{if } |s| < \tau_1.$$

Proof: First of all, let $\theta' \in (\theta, \min(1, \tau_2))$ such that $s + m - \theta' > \max(-\frac{1}{q'}, -\tau_2)$. Because of (3.1), $s + m - \theta' + \varepsilon < \tau_2 - \theta'$ for $\varepsilon > 0$ sufficiently small, and $\|cf\|_{L^q(\mathbb{R}_+, x_n^{\theta' q})} \leq \|c\|_{C^{\theta'}(\mathbb{R})}\|f\|_q$ if $c|_{x_n=0} = 0$, we have

$$c(x) \colon B^{s+m-\theta'+\varepsilon}_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \to B^{s+m-\theta'+\varepsilon}_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{\theta' q})).$$

Moreover, using Lemma 4.6, Corollary 4.10, and Theorem 3.4,

$$t(x', D_x) \colon B^{s+m-\theta'}_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{\theta' q})) \to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \quad \text{if } \left|s - \frac{1}{q}\right| < \tau_1,$$
$$g(x, D_x) \colon B^{s+m-\theta'}_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{\theta' q})) \to B^s_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \quad \text{if } |s| < \tau_1,$$
$$g(x, D_x) \colon B^{s+m-\theta'+\varepsilon}_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{\theta' q})) \to B^\varepsilon_q(\mathbb{R}^{n-1}; B^s_q(\mathbb{R}_+)) \quad \text{if } 0 < s < \tau_1$$

for $\varepsilon > 0$ sufficiently small. Hence choosing $\varepsilon > 0$ sufficiently small the case $s + m - \theta \geq 0$ is proved because of (2.13)-(2.14) and (2.2).

Now let $-\frac{1}{q'} < s + m - \theta < 0$ and $\theta' \in (\theta, \min(1, \tau_2))$ such that $-\frac{1}{q'} < s + m - \theta' < 0$. Then we use that

$$|(cf, g)_{L^2(\mathbb{R}_+)}| \leq C\|f\|_{B^{-s'}_q(\mathbb{R}_+)}\|c\|_{C^{\tau_2}(\mathbb{R})} \left(\|g\|_{L^{q'}(\mathbb{R}_+, x_n^{[\theta' - s']_+ q'})} + \|g\|_{B^{s'}_{q'}(\mathbb{R}_+, x_n^{\theta' q'})}\right)$$

for $s' \in (0, \frac{1}{q'})$ with $s' < \tau_2$ and $f, g \in C_0^\infty(\mathbb{R}_+)$. Therefore

$$c(x)\colon L^q(\mathbb{R}^{n-1}; B_q^{-s'}(\mathbb{R}_+)) \to L^q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{-[\theta'-s']+q}) + B_q^{-s'}(\mathbb{R}_+, x_n^{-\theta' q})).$$

Because of Remark 4.7 and Theorem 3.4, we conclude for $s' = -s - m - \theta' + \varepsilon$, $\varepsilon > 0$ sufficiently small,

$$t(x', D_x)\colon B_q^{-\varepsilon}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{-[\theta'-s']+q}) + B_q^{-s'}(\mathbb{R}_+, x_n^{-\theta' q})) \to B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1})$$

if $\left| s - \frac{1}{q} \right| < \tau_1$ and

$$g(x, D_x)\colon B_q^{-\varepsilon}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{-[\theta'-s']+q}) + B_q^{-s'}(\mathbb{R}_+, x_n^{-\theta' q})) \to B_q^{s}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)),$$
$$g(x, D_x)\colon B_q^{-\varepsilon/2}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+, x_n^{-[\theta'-s']+q}) + B_q^{-s'}(\mathbb{R}_+, x_n^{-\theta' q})) \to B_q^{\varepsilon/2}(\mathbb{R}^{n-1}; B_q^{s}(\mathbb{R}_+))$$

if $|s| < \tau_1$. Therefore by (2.2)

$$t(x', D_x)c(x)\colon L^q(\mathbb{R}^{n-1}; B_q^{s+m-\theta}(\mathbb{R}_+)) \to B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1}) \qquad \text{if } \left| s - \frac{1}{q} \right| < \tau_1,$$
$$g(x, D_x)c(x)\colon L^q(\mathbb{R}^{n-1}; B_q^{s+m-\theta}(\mathbb{R}_+)) \to B_q^{s}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \quad \text{if } |s| < \tau_1,$$
$$g(x, D_x)c(x)\colon L^q(\mathbb{R}^{n-1}; B_q^{s+m-\theta}(\mathbb{R}_+)) \to L^q(\mathbb{R}^{n-1}; B_q^{s}(\mathbb{R}_+)) \quad \text{if } 0 < s < \tau_1,$$

which finishes the proof. ■

5 Truncated Pseudodifferential Operators

5.1 Definition and Consequences

Recall that $\mathcal{H}_d$, $d \in \mathbb{Z}$, denotes the space of all smooth functions $f\colon \mathbb{R} \to \mathbb{C}$ which admit an asymptotic expansion $f(t) \sim s_d t^d + s_{d-1} t^{d-1} + \dots$ in the sense that for all k, l, and $N \in \mathbb{N}_0$

$$\left| \partial_t^l \left[t^k f(t) - \sum_{j=d-N}^{d} s_j t^{j+k} \right] \right| \le C_{k,l,N} (1 + |t|)^{d-N-1+k-l} \quad \text{as } |t| \to \infty.$$

It is important that $\mathcal{H}_{-1} = \mathcal{H}^+ \oplus \mathcal{H}_{-1}^-$, where $\mathcal{H}^+$ and $\mathcal{H}_{-1}^-$ are the subspaces of all $f \in \mathcal{H}_{-1}$ which can be extended holomorphically to the lower, upper complex plane, resp., and

$$\mathcal{H}^+ = \mathcal{F}[e^+ \mathcal{S}(\overline{\mathbb{R}}_+)], \qquad \mathcal{H}_{-1}^- = \mathcal{F}[e^- \mathcal{S}(\overline{\mathbb{R}}_-)],$$

where $e^\pm f$ denotes the extension by zero of a function f defined on $\overline{\mathbb{R}}_\pm$, see [12, Chapter II, Section 2.2] for details. Moreover, $h^+ = \mathcal{F} e^+ r^+ \mathcal{F}^{-1}$ and $h_{-1}^- = \mathcal{F} e^- r^- \mathcal{F}^{-1}$ are continuous projections on $\mathcal{H}^+$ and $\mathcal{H}_{-1}^-$, resp. Here $r^\pm$ denotes the restriction to $\mathbb{R}_\pm$ and $e^\pm$ the extension by zero from $\mathbb{R}_\pm$ to $\mathbb{R}$. We use the convention $\mathcal{H}_r^- = \mathcal{H}_{-1}^- \oplus \mathbb{C}_r[t]$, $r \in \mathbb{N}_0$, where $\mathbb{C}_r[t]$ denotes the set of all complex polynomials of degree r. Moreover, $h_{-1}\colon \mathcal{H}_d \to \mathcal{H}_{-1}$ is the projection with range $\mathcal{H}_{-1}$ and kernel $\mathbb{C}_d[t]$.

Remark 5.1 As in the standard calculus the Poisson, trace, and singular Green operator defined in the last section can be described with the aid of their *symbols*:

$$
\begin{aligned}
k(x,D_x)a &= \mathcal{F}^{-1}_{\xi\mapsto x}\left[k(x,\xi)\acute{a}(\xi')\right], \quad t(x',D_x)f = \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\int^+ t(x',\xi)\hat{f}(\xi)\,d\!\!{}^-\xi_n\right],\\
g(x,D_x)f &= \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\int^+ g(x,\xi,\eta_n)\hat{f}(\xi',\eta_n)\,d\!\!{}^-\eta_n\right],
\end{aligned}
$$

where $k(x,\xi) = \mathcal{F}_{y_n\mapsto\xi_n}[e^+_{y_n}\tilde{k}(.,y_n)]$, $t_0(x',\xi) = \bar{\mathcal{F}}_{y_n\mapsto\xi_n}[e^+_{y_n}\tilde{t}_0(.,y_n)]$, $g_0(x,\xi,\eta_n) = \mathcal{F}_{y_n\mapsto\xi_n}\bar{\mathcal{F}}_{z_n\mapsto\eta_n}[e^+_{y_n}e^+_{z_n}\tilde{g}_0(.,y_n,z_n)]$,

$$
t(x',\xi) = \sum_{j=0}^{r-1} s_j(x',\xi')\xi_n^j + t_0(x',\xi), \quad g(x,\xi) = \sum_{j=0}^{r-1} k_j(x,\xi')\xi_n^j + g_0(x',\xi)
$$

cf. [12, Section 2.3], and where $\bar{\mathcal{F}}[f](x) := \mathcal{F}[f](-x)$ denotes the conjugate Fourier transformation. Here $k_j(x,\xi)$ is the symbol of the Poisson operator $k_j(x,D_x)$ and $\int^+$ is the "plus-integral", cf. [12, Section 2.2].

Finally, $t(x,D_x)$ and $g(x,D_x)$ are said to be of class $-m$, $m\in\mathbb{N}$, if $t(x,\xi)\in\mathcal{H}_{-m-1}$ w.r.t. ξ_n, $g(x,\xi,\eta_n)\in\mathcal{H}_{-m-1}$ w.r.t. η_n, respectively.

The following transmission condition assures that $p(x,D_x)_+ = r^+p(x,D_x)e^+$ is continuous between Bessel potential spaces and Besov spaces on the half-space $\mathbb{R}^n_+$.

Definition 5.2 *Let $p\in C^\tau S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, $d\in\mathbb{Z}$. Then p satisfies the* global transmission condition *– simply called transmission condition in the following – if there are functions $s_{k,\alpha}(x,\xi')$ smooth in ξ' and in C^τ w.r.t. x such that for any $\alpha\in\mathbb{N}_0^n$ and $l\in\mathbb{N}_0$*

$$
\left\|\xi_n^l D_\xi^\alpha p(.,\xi) - \sum_{k=-l}^{d-|\alpha|} s_{k,\alpha}(.,\xi')\xi_n^{k+l}\right\|_{C^\tau(\mathbb{R}^n)} \le C_{k,l,\alpha}\langle\xi'\rangle^{d+1+l-|\alpha|}|\xi_n|^{-1} \tag{5.1}
$$

when $|\xi_n|\ge\langle\xi'\rangle$.

It is an important fact that the symbols $s_{k,\alpha}(x,\xi')$ have to fit together under termwise differentiation as it is in the smooth coefficient case. In particular, they have to be zero after a finite number of differentiations in ξ'. Hence $s_{k,\alpha}(x,\xi')$ has to be a polynomial in ξ' with coefficients in $C^\tau(\mathbb{R}^n)$.

Remark 5.3 In contrast to the transmission condition for a smooth symbol $p\in S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, cf. e.g. [12, Definition 2.2.7], in the latter non-smooth version a condition not only at $x_n=0$ is posed. – Therefore it is called global transmission condition. – It is motivated by applications, where $p(x,\xi)=q(A(x)\xi)$, $A\in C^\tau(\mathbb{R}^n)^{n\times n}$, and $q\in S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ satisfies the transmission condition for symbols in $S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$. Of course it can be relaxed since only the behavior of $p(x,\xi)$ near $x_n=0$ plays a role in order to prove the continuity of the truncated pseudodifferential operator. However the latter condition is simple and sufficient for our purposes.

Finally, a Green operator of order $m \in \mathbb{R}$, class $r \in \mathbb{N}_0$, and with coefficients in C^τ, $\tau > 0$, is an operator of the form (1.1), where $p(x, D_x)$, $g(x, D_x)$, $k(x, D_x)$, $t(x', D_x)$, and $s(x', D_{x'})$ are pseudodifferential operators, singular Green, Poisson, and trace operators, resp., of order m such that $m \in Z$ if $p \neq 0$, $p(x, D_x)$ satisfies the transmission condition, and $g(x, D_x)$ and $t(x', D_x)$ are of class r. The boundary symbol operator $a(x, \xi', D_n)$ is the Green operator, which is obtained from the corresponding symbols and symbol-kernels by fixing (x', ξ') and considering all operators as one-dimensional operators acting only in x_n. Moreover, $p(x, \xi)$ is called the interior symbol of $a(x, D_x)$.

Lemma 5.4 *Let $p \in C^\tau S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $d \in \mathbb{Z}$, that satisfies the transmission condition. Then $r^+ p(x, D_x)\delta_0 \otimes a = k(x, D_x)a$ for all $a \in \mathcal{S}(\mathbb{R}^{n-1})$, where $\tilde{k} \in C^\tau S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ is a Poisson symbol-kernel of order $d+1$ and δ_0 denotes the delta distribution w.r.t. x_n.*

Proof: First of all by (5.1) $h_{-1}[\xi_n^l D_\xi^\alpha p(., \xi)] = \xi_n^l D_\xi^\alpha p(., \xi) - \sum_{k=-l}^{d-|\alpha|-j} s_{k,\alpha,j}(., \xi')\xi_n^{k+l}$. Therefore

$$\|h_{-1,\xi_n}[\xi_n^l D_\xi^\alpha p(., \xi)]\|_{C^\tau(\mathbb{R}^n)} \leq \begin{cases} C_{l,\alpha}\langle\xi'\rangle^{d+l+1-|\alpha|}|\xi_n|^{-1} & \text{when } |\xi_n| \geq \langle\xi'\rangle, \\ C_{l,\alpha}\langle\xi'\rangle^{d+l-|\alpha|} & \text{when } |\xi_n| < \langle\xi'\rangle, \end{cases} \tag{5.2}$$

where we have used (5.1) for $|\xi_n| \geq \langle\xi'\rangle$ and the symbol estimates for $|\xi_n| < \langle\xi'\rangle$. This implies

$$\|h_{-1,\xi_n}[\xi_n^l D_\xi^\alpha p(., \xi', .)]\|_{C^\tau(\mathbb{R}^n; L^2_{\xi_n}(\mathbb{R}))} \leq C_{k,\alpha}\langle\xi'\rangle^{d+\frac{1}{2}+l-|\alpha|} \tag{5.3}$$

by an elementary calculation, cf. the proof of [12, Theorem 2.2.10].

Since $r^+ p(x, \xi', D_{x_n})\delta_0 \otimes a = r^+ \mathcal{F}^{-1}_{\xi_n \mapsto x_n}[h^+ p(x, \xi)a]$ for all $a \in \mathbb{C}$, we have $\tilde{k}(x, \xi', y_n) = r^+ \mathcal{F}^{-1}_{\xi_n \mapsto y_n}[h^+ p(x, \xi)] \in \mathcal{S}(\overline{\mathbb{R}}_+)$ w.r.t. y_n. Hence the previous estimate implies

$$\|D_{\xi'}^{\alpha'} y_n^l D_{y_n}^{l'} \tilde{k}(., \xi', y_n)\|_{C^\tau(\mathbb{R}^n; L^2_{y_n}(\mathbb{R}_+))} \leq C_{l,l',\alpha'}\langle\xi'\rangle^{d+\frac{1}{2}-l+l'-|\alpha'|}, \tag{5.4}$$

which proves the lemma. ■

In connection with Lemma 5.4 the identity

$$[D^k_{x_n}, e^+] = -i \sum_{j=0}^{k-1} D^{k-1-j}_{x_n} \delta_0 \otimes \gamma_j, \tag{5.5}$$

cf. [12, (2.2.39)], where $[A, B] := AB - BA$, will often be used.

Let $p \in C^\tau S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then we denote

$$G^+(p(x, D_x)) := r^+ p(x, D_x) e^- J, \qquad G^-(p(x, D_x)) := J r^- p(x, D_x) e^+,$$

where $(Jf)(x) := f(x', -x_n))$, $x \in \mathbb{R}^n$.

Lemma 5.5 *Let* $p \in C^\tau S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $d \in \mathbb{Z}$, *satisfy the transmission condition. Then* $G^+(p(x, D_x)) = g^+(p)(x, D_x)$ *and* $G^-(p(x, D_x)) = g^-(p)(x, D_x)$, *where* $\tilde{g}^\pm(p) \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$ *are singular Green symbol-kernels of order* d. *Moreover,* $\tilde{g}^\pm(p)(x, \xi', y_n, z_n) = r^\pm \mathcal{F}^{-1}_{\xi_n \mapsto t}[p(x,\xi)]\big|_{t=\pm y_n \pm z_n}$.

Proof: For every fixed $x \in \mathbb{R}^n$ the symbol $p_x(\xi) := p(x,\xi)$ is a smooth symbol of order d satisfying the transmission condition. Hence the stated identities are direct consequences of the corresponding statements in the smooth case, cf. e.g. [12, Theorem 2.6.10]. Moreover, the estimates to show $\tilde{g}^\pm(p) \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$ are proved in the same way as in the proof of [12, Theorem 2.6.10], where the regularity in x does not play any role. ■

In order to consider $p(x, D_x)_+$ as operator-valued pseudodifferential operator on $\mathbb{R}^{n-1}$, we will use:

Lemma 5.6 *Let* $1 < q < \infty$ *and let* $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{Z}$, *satisfy the transmission condition. Then*

$$p(x, D_x)_+ = \sum_{j=0}^m s_j(x, D_{x'}) D^j_{x_n} + p'(x, D_x)_+,$$

where $p'(x, D_x) = \mathrm{OP}'(p'(x, \xi', D_n))$ *with*

$$p'(x, \xi', D_n)_+ \in C^{\tau-\tau'} S^{m+\theta}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H^{s-\theta}_q(\mathbb{R}_+), H^s_q(\mathbb{R}_+)))$$

for all $0 < \tau' < \tau$, $|s| < \tau'$, *and* $\theta \in [0,1]$ *with* $s - \theta > -\frac{1}{q'}$ *and where* $s_j(x, D_{x'})$ *are differential operators of order* $m - j$ *with coefficients in* $C^\tau(\mathbb{R}^n)$. *Moreover,* $\gamma_0 p(x, D_x)_+ = \sum_{j=0}^m s_j(x', 0, D_{x'})\gamma_j + t_0(x', D_{x'})$, *where* $\tilde{t}_0(x', \xi', y_n) \in C^\tau S^m_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$.

Proof: Let $p'(x,\xi) := h_{-1,\xi_n}[p(x,\xi)]$. As seen in the proof of Lemma 5.4, $p'(x,\xi) = p(x,\xi) - \sum_{j=0}^m s_j(x,\xi')\xi_n^j$, where $s_j(x,\xi') \in C^\tau S^{m-j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ are the symbols due to (5.1) for $\alpha = l = 0$. Because of (5.1) and the symbol estimates

$$\|\partial^\alpha_\xi p'(.,\xi)\|_{C^\tau(\mathbb{R}^n)} \le C_\alpha \langle \xi' \rangle^{m+\theta-|\alpha'|} \langle \xi_n \rangle^{-\theta-\alpha_n}, \qquad \xi \in \mathbb{R}^n, \tag{5.6}$$

for all $\alpha \in \mathbb{N}^n_0$ and $\theta \in [0,1]$. Because of the latter estimate, (3.1), and Theorem 3.3, we conclude that

$$\|\partial^{\alpha'}_{\xi'} p'(., x_n, \xi', D_n)\|_{C^{\tau-\tau'}(\mathbb{R}^{n-1}; \mathcal{L}(H^{s-\theta}_q(\mathbb{R}), H^s_q(\mathbb{R})))} \le C_{s,\alpha} \langle \xi' \rangle^{m+\theta-|\alpha'|}$$

for all $|s| < \tau'$, $\alpha' \in \mathbb{N}^{n-1}_0$.

Now, if $s - \theta \in (-\frac{1}{q'}, \frac{1}{q})$, then $e^+\colon H^s_q(\mathbb{R}_+) \to H^s_q(\mathbb{R})$ is a continuous mapping and therefore the latter estimate implies the statement in this case.

Next we prove the statement for $s-\theta \in (k-\frac{1}{q'}, k+\frac{1}{q})$, $k \in \mathbb{N}_0$, with $|s| < \tau'$. Then the general case is obtain by interpolation. Using $1 = \frac{1}{q(\xi)} + \sum_{j=1}^n \frac{\xi_j^k}{q(\xi)} \xi_j^k$ with $q(\xi) = 1 + \sum_{j=1}^k \xi_j^{2k}$,

$$p'(x,\xi',D_n)_+ = \sum_{j=1}^{n-1} p_j(x,\xi',D_n)_+ \xi_j^k + p_n(x,\xi',D_n)_+ D_{x_n}^k + r^+ p_n(x,\xi',D_n)[D_{x_n}^k, e^+],$$

where $p_j(x,\xi) \in C^\tau S_{1,0}^{m-k}(\mathbb{R}^n \times \mathbb{R}^n)$, $j = 1,\ldots,n$, satisfy the transmission condition. Since $s-\theta-k \in (-\frac{1}{q'}, \frac{1}{q})$, $e^+ D_{x_n}^k \colon H_q^{s-\theta}(\mathbb{R}_+) \to H_q^{s-\theta-k}(\mathbb{R})$ and we can apply the first part on $p_j(x,\xi',D_n)$. Finally,

$$r^+ p_n(x,\xi',D_n) D_{x_n}^{k-1-j} \delta_0 \otimes \gamma_j \in C^{\tau-\theta} S_{1,0}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H_q^{s-\theta}(\mathbb{R}_+), H_q^s(\mathbb{R}_+)))$$

by Lemma 5.4, Corollary 4.10, and $\gamma_j \colon H_q^s(\mathbb{R}_+) \to \mathbb{C}$ if $j \le k-1$. Hence using (5.5) we obtain the same statement for $r^+ p_n(x,\xi',D_n)[D_{x_n}^k, e^+]$.

The identity for $\gamma_0 p(x,D_x)_+$ is obvious and $\tilde{t}_0(x',\xi',y_n) = \bar{\mathcal{F}}^{-1}_{\xi_n \mapsto y_n}[p'(x',0,\xi)] \in C^\tau S_{1,0}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ is proved by the same estimates as in the proof of Lemma 5.4. ■

Remark 5.7 If $p \in C^\tau S_{1,0}^{-m}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \ge 0$ and $\tau > 0$, then

$$p(x,\xi',D_n) \in C^{\tau-\tau'} S_{1,0}^{-m_1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H_q^{s-m_2}(\mathbb{R}), H_q^s(\mathbb{R})))$$

for all $|s| < \tau' < \tau$ and $m_1, m_2 \ge 0$ with $m_1 + m_2 = m$. Moreover, if $m \in \mathbb{N}_0$ and p satisfies the transmission condition, it can be proved as above that

$$p(x,\xi',D_n)_+ \in C^{\tau-\tau'} S_{1,0}^{-m_1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H_q^{s-m_2}(\mathbb{R}_+), H_q^s(\mathbb{R}_+)))$$

for all $|s| < \tau' < \tau$ with $s - m_2 > -\frac{1}{q'}$.

Let $p \in C^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $d \in \mathbb{R}$, $\tau > 0$. When discussing the compositions of $p(x,D_x)_+$ with Poisson, trace, and singular Green operators, the following Taylor expansion will be useful:

$$p(x,\xi) = \sum_{j=0}^k \frac{x_n^j}{j!} \partial_{x_n}^j p(x',0,\xi) + x_n^k q_k(x,\xi), \tag{5.7}$$

where $q_k \in C^{\tau-k} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $k = 0,\ldots,[\tau]$, and $q_k(x',0,\xi) = 0$.

5.2 Composition of Truncated Pseudodifferential Operators with Poisson, Trace, and Singular Green Operators

In the following we study the compositions of truncated pseudodifferential operators with Poisson, trace, and singular Green operators satisfying the following assumption:

Assumption 5.8 *Let* $\tilde{k}_2 \in C^{\tau_2} S^{m_2-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{t}_1 \in C^{\tau_1} S^{m_1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{g}_1 \in C^{\tau_1} S^{m_1-1}_{1,0}(\mathbb{R}^{n} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $\tilde{g}_2 \in C^{\tau_2} S^{m_2-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$ *for* $\tau_j > 0$ *and* $m_j \in \mathbb{R}$, $j = 1, 2$. *Moreover, let* $p_j \in C^{\tau_j} S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *satisfy the transmission condition, where we assume in the following that* $m_j \in \mathbb{Z}$ *if a composition with* p_j *is considered. We will denote by*

$$p_j(x, D_x) = \sum_{k=0}^{m_j} s_{j,k}(x, D_{x'}) D^k_{x_n} + p'_j(x, D_x) \tag{5.8}$$

the decomposition due to Lemma 5.6. Finally, let $\theta \in (0, \tau_2)$, $\theta \notin \mathbb{N}_0$, *and set* $\tau := \min(\tau_1, \tau_2 - [\theta])$, $m := m_1 + m_2$.

We study the following compositions:

$$p_1(x, D_x)_+ a_2(x', D_x) = (p_1 \#'_{[\theta]} a_2)(x, D_x) + R'_\theta(p_1, a_2)$$

$$(p_1 \#'_{[\theta]} a_2)(x, \xi', D_n) = \sum_{|\alpha'| \le [\theta]} \frac{1}{\alpha'!} \partial^{\alpha'}_{\xi'} p_1(x, \xi', D_n)_+ D^{\alpha'}_{x'} a_2(x', \xi', D_n) \tag{5.9}$$

$$a_1(x, D_x) p_2(x, D_x)_+ = (a_1 \#_{[\theta]} p_2)(x, D_x) + R_\theta(a_1, p_2) \quad \text{where}$$

$$(a_1 \#_{[\theta]} p_2)(x, \xi', D_n) = \sum_{|\alpha| \le [\theta]} \frac{1}{\alpha!} D^{\alpha'}_{\xi'} a_1(x, \xi', D_n) x_n^{\alpha_n} \partial^\alpha_x p_2(x', 0, \xi', D_n)_+ \tag{5.10}$$

for $a_1 = g_1, t_1$, $a_2 = k_2, g_2$.

Because of the composition rules of boundary symbol operators (in the smooth case) $(p_1 \#'_{[\theta]} k_2)(x, D_x)$, $(p_1 \#'_{[\theta]} g_2)(x, D_x)$, $(g_1 \#_{[\theta]} p_2)(x, D_x)$, and $(t_1 \#_{[\theta]} p_2)(x', D_x)$ are Poisson, singular Green, and trace operators, resp., of order m with coefficients in C^τ, cf. Remark 4.3. Here $(p_1 \#'_{[\theta]} g_2)(x, D_x)$ is of class 0 and $(g_1 \#_{[\theta]} p_2)(x, D_x)$, $(t_1 \#_{[\theta]} p_2)(x', D_x)$ are of class $\max(0, m_2)$.

THEOREM 5.9 *Let* $1 < q < \infty$, $s \in \mathbb{R}$, *and let* $\tilde{k}_2, \tilde{t}_1, \tilde{g}_j$, *and* p_j *be as in Assumption 5.8.*

1. *If* $|s| < \tau$, $s - \theta > -\tau_2$, $-\tau_2 + \theta < s + m_1 < \tau_2$, *and* $s + m_1 - \theta > -\frac{1}{q'}$, *then*

$$R'_\theta(p_1, k_2) \colon B^{s+m-\frac{1}{q}-\theta}_q(\mathbb{R}^{n-1}) \to B^s_q(\mathbb{R}^n_+),$$

$$R'_\theta(p_1, g_2) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \qquad \text{if } s + m - \theta > -\frac{1}{q'},$$

$$R_\theta(g_1, p_2) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \qquad \text{if } s + m - \theta, s + m_1 - \theta > -\frac{1}{q'}.$$

2. *If* $\left| s - \frac{1}{q} \right| < \tau$, $s - \frac{1}{q} - \theta > -\tau_2$, $-\tau_2 + \theta < s + m_1 < \tau_2$, $s + m_1 - \theta > -\frac{1}{q'}$, *and* $s + m - \theta > -\frac{1}{q'}$, *then* $R_\theta(t_1, p_2) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1})$.

The theorem will be proved at the end of this section.

Remark 5.10 Using the Taylor expansion (5.7) for p_1 with $k = [\theta']$ and $\theta' \in (0, \tau_1)$, $\theta' \notin \mathbb{N}_0$, the formulae can be reduced to Poisson, trace, and singular Green operators with x_n-independent coefficients as in the smooth case, cf. Remark 4.3.1. The new remainder terms are easily estimated using Theorem 4.11. But the remainder term will be of order $m_1 + m_2 - \theta'$ with θ' arbitrarily close to τ_1. Hence in that case there is loss of accuracy of the formulae if $\tau_1 < \tau_2$.

Lemma 5.11 *Let $\tilde{k}_2, \tilde{t}_1, \tilde{g}_j$, p'_j be as in Assumption 5.8. Moreover, let $R'_\theta(a_1, p'_2) := a_1(x, D_x)p'_2(x, D_x)_+ - \mathrm{OP}'(a_1(., D_n)\#'_{[\theta]}p_2(., D_n))$ for $a_1 = g_1, t_1$.*

1. *If $|s| < \tau$, $s - \theta > -\tau_2$, and $-\tau_2 + \theta < s + m_1 < \tau_2$, then*

$$R'_\theta(p'_1, k_2)\colon B_q^{s+m-\frac{1}{q}-\theta}(\mathbb{R}^{n-1}) \to B_q^s(\mathbb{R}^n_+),$$
$$R'_\theta(p'_1, g_2), R'_\theta(g_1, p'_2)\colon B_q^{s+m-\theta}(\mathbb{R}^n_+) \to B_q^s(\mathbb{R}^n_+) \qquad \text{if } s + m - \theta > -\frac{1}{q'}.$$

2. *If $|s - \frac{1}{q}| < \tau$, $s - \frac{1}{q} - \theta > -\tau_2$, $-\tau_2 + \theta < s + m_1 < \tau_2$, and $s + m - \theta > -\frac{1}{q'}$, then $R'_\theta(t'_1, p'_2)\colon B_q^{s+m-\theta}(\mathbb{R}^n_+) \to B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1})$.*

Proof: First of all, because of Remark 4.12 and $s + m - \theta > -\frac{1}{q'}$, it is sufficient to prove the mapping properties with $B_q^{s+m-\theta}(\mathbb{R}^n_+)$ replaced by $B_q^{s+m-\theta+s''}(\mathbb{R}^{n-1}; H_q^{-s'}(\mathbb{R}_+))$ for $s' \in [0, \frac{1}{q'})$.

By Lemma 5.6

$$p'_j(x, \xi', D_n)_+ \in C^{\tau_j - \tau'} S^{m_j}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(B_q^s(\mathbb{R}_+)) \cap \mathcal{L}(H_q^s(\mathbb{R}_+))) \tag{5.11}$$
$$\cap\, C^{\tau_j} S^{m_j+s'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H_q^{-s'}(\mathbb{R}_+), L^q(\mathbb{R}_+))) \tag{5.12}$$

for all $0 < \tau' < \tau_j$, $|s| < \tau'$ with $s > -\frac{1}{q'}$, and $s' \in [0, \frac{1}{q'})$. Moreover, if $m_j < 0$, $p_j(x, D_x) = p'_j(x, D_x)$ and by Remark 5.7

$$p'_j(x, \xi', D_n)_+ \in C^{\tau_j - \tau'} S^0_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(B_q^{s+m_j}(\mathbb{R}_+), B_q^s(\mathbb{R}_+))), \tag{5.13}$$

for all $0 < \tau' < \tau_j$, $|s| < \tau'$, $s + m_j > -\frac{1}{q'}$.

Using (5.11), (5.12), Remark 4.7, and Corollary 4.10 we can apply Theorem 3.6 to obtain

$$R'_\theta(p'_1, k_2)\colon B_q^{s+m-\theta-\frac{1}{q}}(\mathbb{R}^{n-1}) \to B_q^s(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))$$
$$R'_\theta(p'_1, g_2), R'_\theta(g_1, p'_2)\colon B_q^{s+m-\theta+s'}(\mathbb{R}^{n-1}; H_q^{-s''}(\mathbb{R}_+)) \to B_q^s(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))$$
$$R'_\theta(t_1, p'_2)\colon B_q^{s+m-\theta+s'}(\mathbb{R}^{n-1}; H_q^{-s''}(\mathbb{R}_+)) \to B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1})$$

for $s' \in [0, \frac{1}{q'})$ and under the same restrictions on s as in the theorem. Hence, if $s \le 0$, the lemma is proved because of (2.14).

Moreover, if $m_1 \ge 0$ and $s > 0$, we use that by Remark 4.7

$$k_2(x', \xi', D_n) \in C^{\tau_2} S^{m_2+s-\frac{1}{q}}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, B^s_q(\mathbb{R}_+)))$$
$$g_2(x', \xi', D_n) \in C^{\tau_2} S^{m_2+s+s'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H^{-s'}_q(\mathbb{R}_+), B^s_q(\mathbb{R}_+)))$$

for $s' \in [0, \frac{1}{q'})$. Hence, if $m_1 \ge 0$, $s > 0$, $s' \in [0, \frac{1}{q'})$ and $\varepsilon > 0$ is sufficiently small,

$$R'_\theta(p'_1, k_2) \colon B^{s+m-\frac{1}{q}-\theta+\varepsilon}_q(\mathbb{R}^{n-1}) \to B^\varepsilon_q(\mathbb{R}^{n-1}; B^s_q(\mathbb{R}_+)) \tag{5.14}$$
$$R'_\theta(p'_1, g_2) \colon B^{s+m-\theta+s'+\varepsilon}_q(\mathbb{R}^{n-1}; H^{-s'}_q(\mathbb{R}_+)) \to B^\varepsilon_q(\mathbb{R}^{n-1}; B^s_q(\mathbb{R}_+)) \tag{5.15}$$

by Theorem 3.6 and (5.11). Here the assumptions of Theorem 3.6 are satisfied for sufficiently small $\varepsilon > 0$ since $m_1 < s + m_1 < \tau_2$ and $m_1 \ge 0$. Because of (2.2) and (2.13), this implies the statements for $p'_1(x, D_x)_+ k_2(x', D_x)$ and $p'_1(x, D_x)_+ g_2(x', D_x)$ in this case.

If $m_1 < 0$ and $s > 0$, we use (5.13) and

$$k_2(x', \xi', D_n) \in C^{\tau_2} S^{s+m-\frac{1}{q}}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, B^{s+m_1}_q(\mathbb{R}_+)))$$
$$g_2(x', \xi', D_n) \in C^{\tau_2} S^{s+m+s'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(H^{-s'}_q(\mathbb{R}_+), B^{s+m_1}_q(\mathbb{R}_+)))$$

for $s' \in [0, \frac{1}{q'})$. Then Theorem 3.6 yields (5.14)-(5.15) again.

For $R'_\theta(g_1, p'_2)$ we simply use (5.12) and $g_1(., D_n) \in C^{\tau-\tau'} S^{s+m_1}_{1,0}(\mathcal{L}(L^q(\mathbb{R}_+), B^s_q(\mathbb{R}_+)))$ for $0 < s < \tau' < \tau$ to conclude that

$$R'_\theta(g_1, p_2) \colon B^{s+m-\theta+s'+\varepsilon}_q(\mathbb{R}^{n-1}; H^{-s'}_q(\mathbb{R}_+)) \to B^\varepsilon_q(\mathbb{R}^{n-1}; B^s_q(\mathbb{R}_+))$$

for $s' \in [0, \frac{1}{q'})$ and $\varepsilon > 0$ sufficiently small, which finishes the proof. ■

Lemma 5.12 *Let $p \in C^{\tau_2} S^{m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m_2 \in \mathbb{Z}$, $\tau_2 > 0$, with $p(x', 0, \xi) = 0$ satisfy the transmission condition and let $p'(x, \xi)$ be as in Lemma 5.6. Moreover, let $\tilde{t} \in C^{\tau_1} S^{m_1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ and $\tilde{g} \in C^{\tau_1} S^{m_1-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$, $m_1 \in \mathbb{R}$ and set $\tau := \min(\tau_1, \tau_2)$, $m := m_1 + m_2$. Then for every $\theta \in (0,1)$ with $\theta < \tau_2$ and $s \in \mathbb{R}$ with $s + m - \theta > -\frac{1}{q'}$*

$$\mathrm{OP}'(t(x', \xi', D_n) p'(x, \xi', D_n)_+) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \quad \text{if } -\tau + \theta < s - \frac{1}{q} < \tau,$$
$$\mathrm{OP}'(g(x, \xi', D_n) p'(x, \xi', D_n)_+) \colon B^{s+m-\theta}_q(\mathbb{R}^n_+) \to B^s_q(\mathbb{R}^n_+) \quad \text{if } -\tau + \theta < s < \tau$$

are bounded operators.

Proof: By Lemma 4.6, Corollary 4.10, and interpolation

$$t(x',\xi',D_n) \in C^{\tau_1} S^{m_1+\frac{1}{q}-\theta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(L^q(\mathbb{R}_+,x_n^{-\theta q}),\mathbb{C})), \tag{5.16}$$

$$g(x,\xi',D_n) \in C^{\tau_1-\tau'} S^{m_1+\frac{1}{q}+s'-\theta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(L^q(\mathbb{R}_+,x_n^{-\theta q}),B^{s'}_q(\mathbb{R}_+))) \tag{5.17}$$

for $0<s'<\tau'<\tau_1$, where it is clear from the proof of Lemma 4.6 that in the case of $L^q(\mathbb{R}_+,x_n^{-\theta q})$ the restriction $\theta<\frac{1}{q'}$ is not necessary. Together with (5.12) for $p'_j=p'$ this yields

$$\begin{aligned} t(x',\xi',D_n)p'(x,\xi',D_n)_+ &\in C^{\tau'} S^{m+\frac{1}{q}+s''}_{1,0}(\mathcal{L}(H^{-s''}_q(\mathbb{R}_+),\mathbb{C}))\\ g(x,\xi',D_n)p'(x,\xi',D_n)_+ &\in C^{\tau'-s'} S^{m+s'+s''}_{1,0}(\mathcal{L}(H^{-s''}_q(\mathbb{R}_+),B^{s'}_q(\mathbb{R}_+)))\end{aligned}$$

for all $s''\in[0,\frac{1}{q'})$, $s'>0$ and $\tau'\in(0,\min(\tau_1-s,\tau_2))$.

Moreover, since $p'(x',0,\xi)=0$, $x_n^{-\theta}p'(x,\xi)\in C^{\tau_2-\tau'}(\mathbb{R}^{n-1};C^{\tau'-\theta}(\mathbb{R}))$ w.r.t. x for $0<\theta<\tau'<\min(1,\tau)$, (5.6) yields

$$\|\partial^\alpha_\xi x_n^{-\theta}p'(.,\xi)\|_{C^{\tau_2-\tau'}(\mathbb{R}^{n-1};C^{\tau'-\theta}(\mathbb{R}))} \le C\|\partial^\alpha_\xi p'(.,\xi)\|_{C^{\tau_2}} \le C_\alpha\langle\xi'\rangle^{m_2+s-|\alpha'|}\langle\xi_n\rangle^{-s-\alpha_n}$$

for all $\alpha\in\mathbb{N}_0^n$, $s\in[0,1]$, and $\theta\in(0,1)$ with $\theta<\tau_2$. Then the latter estimate implies

$$\begin{aligned} &x_n^{-\theta}p'(x,\xi',D_n)_+ \in C^{\tau_2-\tau'}S^{m_2+s''}_{1,0}(\mathcal{L}(H^{-s''}_q(\mathbb{R}_+),L^q(\mathbb{R}_+)))\\ \Leftrightarrow\ &p'(x,\xi',D_n)_+ \in C^{\tau_2-\tau'}S^{m_2+s''}_{1,0}(\mathcal{L}(H^{-s''}_q(\mathbb{R}_+),L^q(\mathbb{R}_+,x_n^{-\theta q})))\end{aligned}$$

for all $s''\in[0,\frac{1}{q'})$ and $\tau'\in(\theta,\tau_2)$. Hence (5.16)-(5.17) yields

$$\begin{aligned} t(x',\xi',D_n)p'(x,\xi',D_n)_+ &\in C^{\tau_\theta} S^{m+\frac{1}{q}+s''-\theta}_{1,0}(\mathcal{L}(H^{-s''}_q(\mathbb{R}_+),\mathbb{C}))\\ g(x,\xi',D_n)p'(x,\xi',D_n)_+ &\in C^{\tau_\theta} S^{m+s'+s''-\theta}_{1,0}(\mathcal{L}(H^{-s''}_q(\mathbb{R}_+),B^{s'}_q(\mathbb{R}_+)))\end{aligned}$$

for all $0\le s''<\frac{1}{q'}$ and $0<s'<\tau$, where $0<\tau_\theta<\min(\tau_1-s',\tau_2-\theta)$.

Therefore Lemma 3.5 implies that

$$\mathrm{OP}'(t(.,D_n)p'(.,D_n)_+)\colon B^{s+m+s''-\theta}_q(\mathbb{R}^{n-1};H^{-s''}_q(\mathbb{R}_+))\to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1})$$

if $-\tau+\theta<s-\frac{1}{q}<\tau$, $0\le s''<\frac{1}{q'}$, and

$$\begin{aligned} \mathrm{OP}'(g(.,D_n)p'(.,D_n)_+)\colon B^{s+m+s''-\theta+\varepsilon}_q(\mathbb{R}^{n-1};H^{-s''}_q(\mathbb{R}_+)) &\to B^s_q(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))\\ \mathrm{OP}'(g(.,D_n)p'(.,D_n)_+)\colon B^{s+m+s''-\theta+\varepsilon}_q(\mathbb{R}^{n-1};H^{-s''}_q(\mathbb{R}_+)) &\to L^q(\mathbb{R}^{n-1};B^s_q(\mathbb{R}_+))\end{aligned}$$

if $-\tau+\theta<s<\tau$ are bounded operators where $\varepsilon>0$. This implies the statement of the lemma since $\theta\in(0,\min(1,\tau_2))$ can be chosen arbitrarily. ∎

Proof of Theorem 5.9: First of all, let $p''_j(x,D_x):=p_j(x,D_x)-p'_j(x,D_x)\equiv\sum_{|\alpha|\le m_j}c_{j,\alpha}(x)D^\alpha_x$ denote the differential operator part of the decomposition (5.8).

Then the compositions of $p_1''(x,D_x)$ with $k_2(x',D_x)$ and $g_2(x',D_x)$ reduce to the composition with D_x^α. But the composition with $D_{x_n}^{\alpha_n}$ is trivial and the composition with $D_{x'}^{\alpha'}$ can be treated with Theorem 4.13. Moreover, the compositions of $p_1'(x,D_x)$ and $k_2(x',D_x)$, $g_2(x',D_x)$, resp., were dealt with in Lemma 5.11. Hence the statements on $R_\theta(p_1,k_2)$ and $R_\theta(p_1,g_2)$ are proved.

Similarly, the composition of $g_1(x,D_x)$ and $t_1(x',D_x)$ with $p_2''(x,D_x)$ reduces to the composition with $c_{2,\alpha}(x)$, where $c_{2,\alpha}(x)=\sum_{j=0}^{[\theta]}\frac{x_n^j}{j!}\partial_{x_n}^j c_{2,\alpha}(x',0)+x_n^{[\theta]}c_{2,\alpha}'(x)$ with $c_{2,\alpha}'(x',0)=0$ is used. Here $g_1(x,D_x)x_n^j$ and $t_1(x',D_x)x_n^j$ are singular Green, trace operators, resp., of order m_1-j and the composition with $\partial_{x_n}^j c_{2,\alpha}(x',0)$ can be treated with Theorem 4.13. Moreover,

$$\begin{aligned} g_1(x,D_x)x_n^{[\theta]}c_{2,\alpha}'(x)\colon B_q^{s+m_1-\theta}(\mathbb{R}^n_+)&\to B_q^s(\mathbb{R}^n_+) && \text{if } |s|<\tau_1\\ t_1(x',D_x)x_n^{[\theta]}c_{2,\alpha}'(x)\colon B_q^{s+m_1-\theta}(\mathbb{R}^n_+)&\to B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1}) && \text{if } \left|s-\frac{1}{q}\right|<\tau_1, \end{aligned}$$

by Lemma 4.15 where $s\in\mathbb{R}$ such that $-\tau_2+\theta<s+m_1<\tau_2$ and $s+m_1-\theta>-\frac{1}{q'}$.

Finally, because of Lemma 5.11, it remains to consider

$$\sum_{|\alpha'|\le[\theta]}\frac{1}{\alpha'!}\operatorname{OP}'(\partial_{\xi'}^{\alpha'}a_1(x,\xi',D_n)D_{x'}^{\alpha'}p_2'(x,\xi',D_n)_+)$$

with $a_1=g_1,t_1$. By (5.7),

$$D_{x'}^{\alpha'}p_2'(x,\xi',D_n)=\sum_{j=0}^{[\theta]-|\alpha'|}\frac{x_n^j}{j!}D_{x'}^{\alpha'}\partial_{x_n}^j p_2'(x',0,\xi',D_n)+x_n^{[\theta]-|\alpha'|}q_{2,\alpha}(x,\xi',D_n)$$

where $q_{2,\alpha}\in C^{\tau_2-[\theta]}S_{1,0}^{m_2}(\mathbb{R}^n\times\mathbb{R}^n)$ with $q_{2,\alpha}(x',0,\xi')=0$ satisfies the transmission condition. Hence we can apply Lemma 5.12 to $\partial_{\xi'}^{\alpha'}a_1(x,\xi',D_n)x_n^{[\theta]-|\alpha'|}q_{2,\alpha}(x,\xi',D_n)_+$, $a_1=g_1,t_1$, to finish the proof. ■

5.3 The "Left-Over" Operator $L(p_1(x,D_x),p_2(x,D_x))$

In the following let $L(P_1,P_2):=(P_1P_2)_+-P_{1,+}P_{2,+}$.

THEOREM 5.13 *Let $p_j\in C^{\tau_j}S_{1,0}^{m_j}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$, $m_j\in\mathbb{Z}$, satisfy the transmission condition. Then for every $\theta\in(0,\tau_1)$, $\theta\notin\mathbb{N}$,*

$$p_1(x,D_x)_+p_2(x,D_x)_+=(p_1\#_{[\theta]}p_2)(x,D_x)_+-l_\theta(p_1,p_2)(x,D_x)+R_\theta$$

where $l_\theta(p_1,p_2)(x,D_x)$ is a singular Green operator of order m_1+m_2 and class $\max(m_2,0)$ *with*

$$l_\theta(p_1,p_2)(x,\xi',D_n)=\sum_{|\alpha|\le[\theta]}\frac{1}{\alpha!}L(\partial_{\xi'}^{\alpha'}p_1(x,\xi',D_n),x_n^{\alpha_n}D_{x'}^{\alpha'}\partial_{x_n}^{\alpha_n}p_2(x',0,\xi',D_n))\quad(5.18)$$

and $R_\theta\colon B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+)\to B_q^s(\mathbb{R}^n_+)$ *if* $|s|<\tau, s-\theta>-\tau_2,\ -\tau_2+\theta<s+m_1<\tau_2$, *and* $s+m_1+m_2-\theta>-\frac{1}{q'}$.

Proof: First we consider the case $s+m_1+m_2-\theta\in(-\frac{1}{q'},\frac{1}{q})$. Then by Theorem 3.6 and the continuity of $e^+\colon B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+)\to B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n)$

$$p_1(x,D_x)_+p_2(x,D_x)_+=(p_1\#_{[\theta]}p_2)(x,D_x)_++L(p_1(x,D_x),p_2(x,D_x))+R_{\theta,+},$$

where $R_{\theta,+}\colon B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+)\to B_q^s(\mathbb{R}^n_+)$. Using (5.8),

$$L(p_1(x,D_x),p_2(x,D_x))=\sum_{k=0}^{m_2}L(p_1'(x,D_x),s_{2,k}(x,D_{x'})D_{x_n}^k)+L(p_1'(x,D_x),p_2'(x,D_x)),$$

since $L(s_{1,k}(x,D_{x'})D_{x_n}^k,p_2(x,D_x))=0$. Moreover,

$$s_{2,k}(x,\xi')=\sum_{\alpha_n=0}^{[\theta]}\frac{x_n^{\alpha_n}}{\alpha_n!}\partial_{x_n}^{\alpha_n}s_{2,k}(x',0,\xi')+x_n^{[\theta]}r_{2,k}(x,\xi'),$$

where $r_{2,k}(x,\xi')$ is again the symbol of a differential operator and $r_{2,k}(x',0,\xi')=0$. Since $\gamma_j x_n^{[\theta]}r_{2,k}(x,\xi')=0$ for $j=0,\dots,[\theta]$, and because of (5.5), where $k-1\le m_2-1\le[\theta]$, we conclude that $L(p_1'(x,D_x),x_n^{[\theta]}r_{2,k}(x,D_{x'})D_{x_n}^k)=0$. Hence

$$L(p_1'(x,D_x),s_{2,k}(x,D_{x'})D_{x_n}^k)\;=\;\sum_{\alpha_n=0}^{[\theta]}\frac{1}{\alpha_n!}L(p_1'(x,D_x),x_n^{\alpha_n}D_{x_n}^k)\partial_{x_n}^{\alpha_n}s_{2,k}(x',0,D_{x'}),$$

where $L(p_1'(x,D_x),x_n^{\alpha_n}D_{x_n}^k)$ is a singular Green operator of order $m_1-\alpha_n+k$ and class $k-\alpha_n$. Therefore, by Theorem 4.13 and Remark 4.14,

$$L(p_1'(x,D_x),s_{2,k}(x,D_{x'})D_{x_n}^k)=\sum_{|\alpha|\le[\theta]}\frac{1}{\alpha!}\operatorname{OP}'(L(D_{\xi'}^{\alpha'}p_1'(.,D_n),x_n^{\alpha_n}\partial_x^\alpha s_{2,k}(x',0,\xi')D_{x_n}^k)).$$

On the other hand, by Lemma 5.5, Theorem 4.11, Theorem 4.11 and Theorem 4.13,

$$\begin{aligned}
L(p_1'(x,D_x),p_2'(x,D_x))&=g^+(p_1)(x,D_x)g^-(p_2)(x,D_x)\\
&=\sum_{\alpha_n=0}^{[\theta]}\frac{1}{\alpha_n!}g^+(p_1)(x,D_x)x_n^{\alpha_n}\partial_{x_n}^{\alpha_n}g^-(p_2)(x',0,D_x)+R_{1,\theta}\\
&=\sum_{|\alpha|\le[\theta]}\frac{1}{\alpha!}\operatorname{OP}'(D_{\xi'}^{\alpha'}g^+(p_1)(x,\xi',D_n)x_n^{\alpha_n}\partial_x^\alpha g^-(p_2)(x',0,\xi',D_n))+R_{2,\theta},
\end{aligned}$$

where $R_{j,\theta}\colon B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+)\to B_q^s(\mathbb{R}^n_+)$, $j=1,2$. Hence we proved the theorem for the case $s+m_1+m_2-\theta\in(-\frac{1}{q'},\frac{1}{q})$.

Now let $s+m_1+m_2-\theta \in (k-\frac{1}{q'}, k+\frac{1}{q})$ for $k \in \mathbb{N}$. – Then the case $s+m_1+m_2-\theta = k-\frac{1}{q'}$, $k \in \mathbb{N}$, is obtained by interpolation. – Since $(p_1(x,D_x)p_2(x,D_x))_+$ may not be well-defined, we use an order reducing operator. – Note that, if e.g. $f \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$ with $\gamma_0 f \not\equiv 0$, then $e^+ f \in B_2^{s+m_1+m_2}(\mathbb{R}^n_+)$ implies $s+m_1+m_2 \leq \frac{1}{2}$. Hence, if $\tau_2 < \frac{1}{2}$ and $m_2 \geq 1$, there is no $s \in \mathbb{R}$ with $s+m_1 > -\tau_2$ and $s+m_1+m_2 \leq \frac{1}{2}$. Therefore $p_2(x,D_x)e_+ f$ is not well-defined in general.

Let $p_\beta(\xi) \in S^{-k}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the transmission condition such that

$$I = \sum_{|\beta|\leq k} p_\beta(D_x) D_x^\beta. \tag{5.19}$$

Then $p_2(x,D_x) = \sum_{|\beta|\leq k} p_{2,\beta}(x,D_x)D_x^\beta$, where $p_{2,\beta}(x,\xi) := p_2(x,\xi)p_\beta(\xi) \in C^{\tau_2}S^{m_2-k}_{1,0}$ satisfies the transmission condition. Therefore

$$p_2(x,D_x)_+ = \sum_{|\beta|\leq k} p_{2,\beta}(x,D_x)_+ D_x^\beta + \sum_{|\beta|\leq k} L(p_{2,\beta}(x,D_x), D_x^\beta),$$

where, because of (5.5) and Lemma 5.4, $L(p_{2,\beta}(x,D_x),D_x^\beta) = r^+ p_{2,\beta}(x,D_x)[D_x^\beta, e^+]$ is a singular Green operator of order m_2, class k, and with coefficients in C^{τ_2}. Since $s+m_1+m_2-k \in (-\frac{1}{q'}, \frac{1}{q})$, we can apply the theorem for this case proved above to conclude

$$\begin{aligned}
p_1(x,D_x)_+ p_{2,\beta}(x,D_x)_+ D_x^\beta &= (p_1 \#_{[\theta]} p_{2,\beta})(x,D_x)_+ D_x^\beta - l_\theta(p_1,p_{2,\beta})(x,D_x)D_x^\beta + R_\theta \\
&= ((p_1 \#_{[\theta]} p_{2,\beta})(x,D_x)D_x^\beta)_+ - L((p_1 \#_{[\theta]} p_{2,\beta})(x,D_x), D_x^\beta) \\
&\quad - l_\theta(p_1,p_{2,\beta})(x,D_x)D_x^\beta + R_\theta
\end{aligned}$$

where $R_\theta \colon B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+) \to B_q^s(\mathbb{R}^n_+)$. Using (5.7) for p_2 we conclude

$$\begin{aligned}
L((\partial_\xi^\alpha p_1 D_x^\alpha p_{2,\beta})(x,D_x), D_x^\beta) &= \sum_{j=0}^{[\theta]-|\alpha|} \frac{1}{j!} r^+ \operatorname{OP}(\partial_\xi^\alpha p_1 x_n^j D_x^\alpha \partial_{x_n}^j p_{2,\beta}(x',0,\xi))[D_x^\beta, e^+] \\
&\quad + r^+ \operatorname{OP}(\partial_\xi^\alpha p_1 x_n^{[\theta]-|\alpha|} D_x^\alpha q_{2,\beta})[D_x^\beta, e^+],
\end{aligned}$$

where $q_{2,\beta} \in C^{\tau_2-[\theta]}S^{m_2-k}_{1,0}$ with $q_{2,\beta}|_{x_n=0} = 0$. If $\tau_2 - [\theta] \leq \tau_1$, Theorem 4.11 yields

$$r^+ \operatorname{OP}(\partial_\xi^\alpha p_1(x,\xi) x_n^{[\theta]-|\alpha|} D_x^\alpha q_{2,\beta}(x,\xi))[D_x^\beta, e^+] \colon B_q^{s+m_1+m_2-\theta}(\mathbb{R}^n_+) \to B_q^s(\mathbb{R}^n_+)$$

if s satisfies the assumptions of the theorem. If $\tau_2 - [\theta] > \tau_1$, Lemma 5.14 below implies the same statement.

Therefore

$$\begin{aligned}
&L((p_1 \#_{[\theta]} p_{2,\beta})(x,D_x), D_x^\beta) \\
&= \sum_{|\alpha|\leq[\theta]} \sum_{j=0}^{[\theta]-|\alpha|} \frac{1}{\alpha! j!} L(\operatorname{OP}'(x_n^j \partial_\xi^\alpha p_1(x,\xi',D_n) D_x^\alpha \partial_{x_n}^j p_{2,\beta}(x',0,\xi',D_n)), D_x^\beta) + R_\theta \\
&= \sum_{|\gamma|\leq[\theta]} \frac{1}{\gamma!} L(\operatorname{OP}'(\partial_{\xi'}^{\gamma'} p_1(x,\xi',D_n) x_n^{\gamma_n} D_x^{\gamma'} \partial_{x_n}^{\gamma_n} p_{2,\beta}(x',0,\xi',D_n)), D_x^\beta) + R_\theta
\end{aligned}$$

by an elementary calculation. Moreover, by Theorem 5.9 and Theorem 4.11

$$\begin{aligned} &p_1(x,D_x)_+ L(p_{2,\beta}(x,D_x),D_x^\beta) \\ &= \sum_{|\alpha|\le[\theta]} \frac{1}{\alpha!} \mathrm{OP}'(\partial_{\xi'}^{\alpha'} p_1(x,\xi',D_n)_+ L(x_n^{\alpha_n} D_{x'}^{\alpha'} \partial_{x_n}^{\alpha_n} p_{2,\beta}(x',0,\xi',D_n),D_x^\beta) + R_\theta. \end{aligned}$$

Using the identity $L(P_1,P_2Q) = L(P_1P_2,Q) + L(P_1,P_2)Q_+ - P_{1,+}L(P_2,Q)$ for the boundary symbol operator and the calculations above, it is elementary to check that

$$\begin{aligned} l_\theta(p_1,p_2)(x,D_x) &= \sum_{|\beta|\le k} \big[L((p_1\#_{[\theta]}p_{2,\beta})(x,D_x),D_x^\beta) + l_\theta(p_1,p_{2,\beta})(x,D_x)D_x^\beta \\ &\quad -p_1(x,D_x)_+L(p_{2,\beta}(x,D_x),D_x^\beta)\big] + R_\theta, \end{aligned}$$

which finishes the proof. ∎

Lemma 5.14 *Let $p_j \in C^{\tau_j} S^{m_j}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, $m_j\in\mathbb{Z}$, $j=1,2$, $0<\tau_1<\tau_2\le 1$, satisfy the transmission condition with $p_2(x',0,\xi)=0$. Moreover, let $k(x,\xi',D_n)a := r^+\mathrm{OP}_n(p_1(x,\xi)p_2(x,\xi))\delta_0\otimes a$ for $a\in\mathbb{C}$. Then for every $\theta<\tau_2$ and $0<s<\tau_1$*

$$k(x,\xi',D_n)\in C^\tau S_{1,0}^{m_1+m_2+s+1-\frac{1}{q}-\theta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},B^s_q(\mathbb{R}_+))),$$

where $0<\tau<\min(\tau_1-s,\tau_2-\theta)$ and therefore $k(x,D_x)\colon B_q^{s+m_1+m_2+1-\frac{1}{q}-\theta}(\mathbb{R}^{n-1})\to B^s_q(\mathbb{R}^n_+)$ is a bounded linear operator if $-\min(\tau_1,\tau_2-\theta)<s<\tau_1$.

Proof: We can assume w.l.o.g. that $m_2=-1$ and $\theta\ge s$. Moreover, as in the proof of Lemma 4.9 it is sufficient to prove the statement for $B^s_{q,\infty}(\mathbb{R}_+)$ instead of $B^s_q(\mathbb{R}_+)$. Then

$$h_{-1,\xi_n}\left[p_1(x,\xi)p_2(x,\xi)\right] = h_{-1,\xi_n}[p_1(x,\xi)]p_2(x,\xi) + \sum_{j=0}^{m_1} s_{j,1}(x,\xi')h_{-1,\xi_n}\left[\xi_n^j p_2(x,\xi)\right],$$

where $s_{j,1}(x,\xi')$ are the terms in the expansion due to Definition 5.2 for p_1 with $\alpha=l=0$. The terms $r^+\sum_{j=0}^{m_1}\mathrm{OP}_n\left(s_{j,1}(x,\xi')h_{-1,\xi_n}\left[\xi_n^j p_2(x,\xi)\right]\right)\delta_0\otimes a$ are easily estimated with the aid of Lemma 4.9. and $\|fg\|_{B^s_{q,\infty}}\le C\left(\|f\|_{C^s}\|g\|_{L^q}+\|f\|_{L^\infty}\|g\|_{B^s_{q,\infty}}\right)$. Therefore we can assume for the rest of the proof that $p_1(x,\xi)\in\mathcal{H}_{-1}$ w.r.t. ξ_n.

First let $0<h\le x_n$. Then

$$\begin{aligned} &h^{-s}|\tilde{k}(x',x_n+h,\xi',y_n)-\tilde{k}(x',x_n,\xi',y_n)| \\ &\le h^{-s}\|p_1(x',x_n+h,\xi',.)-p_1(x,\xi',.)\|_{L^2(\mathbb{R})}\|p_2(x',x_n+h,\xi',.)\|_{L^2(\mathbb{R})} \\ &\quad +h^{-s}\|p_1(x,\xi',.)\|_{L^2(\mathbb{R})}\|p_2(x',x_n+h,\xi',.)-p_2(x,\xi',.)\|_{L^2(\mathbb{R})}, \end{aligned}$$

where

$$h^{-s}\|p_1(x',x_n+h,\xi',.)-p_1(x,\xi',.)\|_{L^2(\mathbb{R})}\le C\|p_1(.,\xi',.)\|_{C^{\tau_1}(\mathbb{R}^n;L^2(\mathbb{R}))}\le C\langle\xi'\rangle^{m_1+\frac12}$$
$$h^{-s}\|p_2(x',x_n+h,\xi',.)-p_2(x,\xi',.)\|_{L^2(\mathbb{R})}\le Cx_n^{\theta-s}\langle\xi'\rangle^{m_2+\frac12}$$
$$\|p_1(x,\xi',.)\|_{L^2(\mathbb{R})}\le C\langle\xi'\rangle^{m_1+\frac12}\quad \|p_2(x',x_n+h,\xi',.)\|_{L^2(\mathbb{R})}\le Cx_n^{\theta}\langle\xi'\rangle^{m_2+\frac12}$$

by (5.3) and since $p_2(x',0,\xi)=0$. Hence

$$h^{-s}x_n^{s-\theta}|\tilde{k}(x',x_n+h,\xi',y_n)-\tilde{k}(x,\xi',y_n)|\le C\langle\xi'\rangle^{m_1+m_2+1}.$$

By the same calculations as above, it can be shown that

$$h^{-s}x_n^{s-\theta}\left|y_n^l\partial_{y_n}^{l'}\partial_{\xi'}^{\alpha'}\left(\tilde{k}(x',x_n+h,\xi',y_n)-\tilde{k}(x,\xi',y_n)\right)\right|\le C_{l,l',\alpha'}\langle\xi'\rangle^{m_1+m_2+1-l+l'-|\alpha'|}$$

for $l,l'\in\mathbb{N}_0$, $\alpha'\in\mathbb{N}_0^{n-1}$. This implies

$$h^{-s}\left|x_n^{s'}\partial_{\xi'}^{\alpha'}\left(\tilde{k}(x',x_n+h,\xi',x_n+h)-\tilde{k}(x,\xi',x_n)\right)\right|\le C_{s',\alpha'}\langle\xi'\rangle^{m_1+m_2+1-\theta+s-s'-|\alpha'|}$$

for $s'\ge 0$, $\alpha'\in\mathbb{N}_0^{n-1}$.

In the case $h>x_n$, one can use $\left|x_n^{s'-\theta}\partial_{\xi'}^{\alpha'}\tilde{k}(x,\xi',x_n)\right|\le C\langle\xi'\rangle^{m_1+m_2+1-s'-|\alpha'|}$ for $\alpha'\in\mathbb{N}_0^{n-1}$, $s'\ge 0$, to prove the latter estimate. Hence

$$\sup_{h>0}h^{-s}\left\|\partial_{\xi'}^{\alpha'}\left(\tilde{k}(x',.+h,\xi',.+h)-\tilde{k}(x',.,\xi',.)\right)\right\|_q\le C_{\alpha'}\langle\xi'\rangle^{m_1+m_2+1-\frac1q-\theta+s-|\alpha'|}$$

by (4.4). In a similar way one estimates $\|\partial_{\xi'}^{\alpha'}\tilde{k}(x',.,\xi',.)\|_q$. Thus

$$\|\partial_{\xi'}^{\alpha'}k(x,\xi',D_n)\|_{\mathcal{L}(\mathbb{C},B^s_{q,\infty}(\mathbb{R}_+))}\le C_{\alpha'}\langle\xi'\rangle^{m_1+m_2+1-\frac1q-\theta-|\alpha'|}.$$

Finally, replacing $\tilde{k}(x,\xi',y_n)$ by $|h'|^{-\tau}(\Delta_{h'}\tilde{k})(x,\xi',y_n):=\tilde{k}(x'+h',x_n,\xi',y_n)-\tilde{k}(x,\xi',y_n)$, $h'\in\mathbb{R}^{n-1}$, it can be proved as above that

$$\|\partial_{\xi'}^{\alpha'}k(.,\xi',D_n)\|_{C^\tau(\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},B^s_{q,\infty}(\mathbb{R}_+)))}\le C_{\alpha'}\langle\xi'\rangle^{m_1+m_2+1-\frac1q-\theta+s-|\alpha'|}.$$

Then the continuity of $k(x,D_x)$ is proved as in the proof of Theorem 4.11. ■

As a consequence of the composition rules we obtain:

THEOREM 5.15 *Let $p\in C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, $m\in\mathbb{Z}$, satisfy the transmission condition. Then*

$$p(x,D_x)_+\colon H_q^{s+m}(\mathbb{R}^n_+)\to H_q^s(\mathbb{R}^n_+)$$

is a continuous operator for all $|s|<\tau$ with $s+m>-\frac{1}{q'}$.

Proof: The proof is done by the same scheme as in Lemma 5.5 and Theorem 5.13. The case $s+m \in (-\frac{1}{q'}, \frac{1}{q})$ is trivial since $e^+\colon H^{s+m_q(\mathbb{R}^n_+)} \to H^s_q(\mathbb{R}^n_+)$. Then the case $s+m \in (k-\frac{1}{q'}, k+\frac{1}{q})$, $k \in \mathbb{N}$, is reduced to the first case using

$$p(x,D_x)_+ = \sum_{|\alpha|\le k} p_\alpha(x,D_x)_+ D_x^\alpha + \sum_{|\alpha|\le k} L(p_\alpha(x,D_x), D_x^\alpha),$$

where $p_\alpha \in C^\tau S^{m-k}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, cf. (5.19). ■

5.4 Negative Classes and Proofs of the Main Theorems

The concept of negative classes easily carries over to the non-smooth situation since it is only a matter of the behavior of the symbols w.r.t. ξ_n, η_n, resp., cf. Remark 5.1. As in the smooth coefficient case it holds that

$$t(x',D_x) \text{ is of class } -m \quad\Leftrightarrow\quad t(x',D_x)D^m_{x_n} \text{ is of class } 0, \tag{5.20}$$
$$g(x,D_x) \text{ is of class } -m \quad\Leftrightarrow\quad g(x,D_x)D^m_{x_n} \text{ is of class } 0, \tag{5.21}$$

cf. [12, (2.8.2)].

Moreover, as in [12, Definition 2.8.2] we say that $p(x,D_x)_+ + g(x,D_x)$ is of class $r \in \mathbb{N}_0$ if $g(x,D_x)$ is of class r and that $p(x,D_x)_+ + g(x,D_x)$ is of class $r=-m$, $m \in \mathbb{N}$ if

$$(p(x,D_x)_+ + g(x,D_x))D^m_{x_n} = p'(x,D_x)_+ + g'(x,D_x) \text{ with } g'(x,D_x) \text{ of class } 0.$$

Then $a(x,D_x)$ is said to be of class $r \in \mathbb{Z}$ if $p(x,D_x)_+ + g(x,D_x)$ and $t(x',D_x)$ are of class r.

Finally, it remains to prove our main theorems:

Proof of Theorem 1.1: Because of Theorem 4.8, Theorem 4.11, and Theorem 5.15, the case $r \in \mathbb{N}_0$ is proved. By the same arguments as in in [12, Theorem 2.8.3] it is easy to prove

$$p(x,D_x)_+ + g(x,D_x)\colon H^{s+m}_q(\mathbb{R}^n_+) \to H^s_q(\mathbb{R}^n_+) \qquad \text{if } |s| < \tau,$$
$$t(x',D_x)\colon H^{s+m}_q(\mathbb{R}^n_+) \to B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \quad \text{if } \left|s-\frac{1}{q}\right| < \tau$$

for the general class $r \in \mathbb{Z}$ by using the statement if the class is 0. Hence the theorem is proved. ■

Proof of Theorem 1.2: First of all, since $\theta \in (0,\tau_2)$, $\theta \notin \mathbb{N}$, is arbitrary, the Bessel potential spaces can be replaced by Besov spaces using (2.2). Hence it only remains to extend the statements of Theorem 4.13, Theorem 5.9, and Theorem 5.13 to arbitrary classes $r_j \in \mathbb{Z}$. As mentioned in Remark 4.14, the compositions with

$g_j(x, D_x)$ and $t_j(x', D_x)$ of class $r_j \in \mathbb{N}_0$ reduce to compositions with γ_j and operators of class 0. Using $\gamma_j = \gamma_0 D^j_{x_n}$, the compositions with γ_j can be reduced to Theorem 5.9, Theorem 5.13, and Lemma 5.6. Finally, if $r_j \in \mathbb{Z}$, it only remains to check that $t(x', D_x)$ and $p(x, D_x)_+ + g(x, D_x)$ are of class $\max(r_1 + m_2, r_2)$, which can be done by using the definitions and (5.20)-(5.21) directly or by the same argument as in [12, Remark 2.8.4]. ■

6 Parametrix Construction

In this last section we apply Theorem 1.2 to construct a parametrix to elliptic Green operators. In the following we will assume that the symbols of the operators are *polyhomogeneous*, i.e., there is an asymptotic expansion in homogeneous terms of decreasing order. The precise definition is completely analogous to the definition in the smooth case, cf. e.g. [9], where we assume that the coefficients of $g(x, D_x)$ and $k(x, D_x)$ are independent of x_n in order to have a uniquely defined principle part. The principle part of $a(x, D_x)$ will be denoted by $a_0(x, D_x)$.

Definition 6.1 *A polyhomogeneous Green operator $a(x, D_x)$ of order $m \in \mathbb{Z}$, class $r \in \mathbb{Z}$, and coefficients in C^τ, $\tau > 0$, is said to be* uniformly elliptic *if the principal interior symbol $p_0(x, \xi)\colon \mathbb{C}^N \to \mathbb{C}^N$ is invertible for every $x \in \mathbb{R}^n$, $|\xi| = 1$, and $p_0^{-1}(x, \xi)$ is uniformly bounded in $x \in \mathbb{R}^n$, $|\xi| = 1$, and principal boundary symbol operator*

$$a_0(x', 0, \xi', D_n)\colon H^r_2(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M \to H^{r-m}_2(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^{M'}$$

is invertible and $a_0(x', 0, \xi', D_n)^{-1}$ is uniformly bounded in $x', \xi' \in \mathbb{R}^{n-1}$ with $|\xi'| = 1$.

Since matrix inversion is smooth, $p_0^{-1}(x, \xi) \in C^\tau S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N)$ (suitably defined for $|\xi| \leq 1$). But it remains to prove that $a_0(x', 0, \xi', D_n)^{-1}$ is again a boundary symbol operator in the non-smooth symbol-kernel classes.

Since for every fixed $x'_0 \in \mathbb{R}^{n-1}$ the boundary symbol operator $a_{x'_0}(\xi', D_n) := a_0(x'_0, 0, \xi', D_n)$ belongs to the standard calculus, $a_{x'_0}(\xi', D_n)^{-1}$ is again a boundary symbol operator of order $-m$ and class $r - m$, cf. [5], [22], or [12]. Hence it remains to prove that $a_0(x', 0, \xi', D_n)^{-1}$ is in C^τ w.r.t. x' and satisfies the corresponding symbol-kernel estimates. As known from the proof in the smooth coefficient case, cf. [5], [22, Proposition 3.1.1.2.6], or [12, Theorem 3.1.7], the statement can be reduced to the inversion of $a(x', 0, \xi', D_n) = I + g(x', \xi', D_n)$, where $g(x', D_x)$ is a Green operator of order and class 0 with small operator norm in $\mathcal{L}(L^2(\mathbb{R}_+))$. This is done by composition with order reducing operators and other operators belonging to the calculus as well as inversion of matrix-valued pseudodifferential symbols. All these steps directly carry over to the non-smooth coefficient case. Finally, the next lemma treats the operator $I + g(x', \xi', D_n)$.

Lemma 6.2 *If* $\tilde{g} \in C^\tau S^{-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++})) \otimes \mathcal{L}(\mathbb{C}^N)$, $\tau > 0$, $N \in \mathbb{N}$, *with* $\|\tilde{g}(x', \xi', ., .)\|_{L^2(\mathbb{R}^2_{++})} \leq \frac{1}{2}$, *then* $I + g(x', \xi', D_n)$ *is invertible and there is a* $\tilde{g}' \in C^\tau S^{-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++})) \otimes \mathcal{L}(\mathbb{C}^N)$ *such that* $(I + g(x', \xi', D_n))^{-1} = I + g'(x', \xi', D_n)$.

Proof: The lemma can be proved by similar arguments as in the proof of [12, Proposition 3.2.1]. By the assumptions $\|g(x', \xi', D_n)\|_{\mathcal{L}(L^2(\mathbb{R}_+))} = \|\tilde{g}(x', \xi', ., .)\|_{L^2(\mathbb{R}^2_{++})} \leq \frac{1}{2}$. Hence $I + g(x', \xi', D_n)$ is invertible in $\mathcal{L}(L^2(\mathbb{R}_+))$ and $(I + g(x', \xi', D_n))^{-1} = \sum_{k=0}^\infty g(x', \xi', D_n)^k$, where $g(x', \xi', D_n)^k = g_k(x', \xi', D_n)$ with

$$\tilde{g}_k(x', \xi', x_n, y_n) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{g}(x', \xi', x_n, w_1) \tilde{g}(x', \xi', w_1, w_2) \cdots \tilde{g}(x', \xi', w_{k-1}, y_n) dw_1 dw_2 \cdots dw_{k-1}$$

for $k \geq 2$. Then it can be proved in a straight-forward manner that $\tilde{g}'(x', \xi', x_n, y_n) := \sum_{k=1}^\infty \tilde{g}_k(x', \xi', x_n, y_n) \in C^\tau S^{-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++})) \otimes \mathcal{L}(\mathbb{C}^N)$. ■

Corollary 6.3 *If* $a(x, D_x)$ *is a polyhomogeneous elliptic Green operator of order* $m \in \mathbb{Z}$, *class* $r \in \mathbb{Z}$, *and in* C^τ, $\tau > 0$, *w.r.t.* x, *then* $a_0(x', 0, \xi', D_n)^{-1}$ *is a boundary symbol operator of order* $-m$, *class* $r - m$, *and in* C^τ *w.r.t.* x'.

In order to construct a parametrix in the non-smooth coefficient case one has to take care of the restriction of the mapping properties due to the limited smoothness of the coefficients. If for instance $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, is elliptic and $q \in C^\tau S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $q(x, \xi) = p^{-1}(x, \xi)$ for $|\xi| \geq R > 0$, then by Theorem 3.3 $p(x, D_x)\colon H^{s+m}_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n)$ if $|s| < \tau$ but $q(x, D_x)\colon H^s_q(\mathbb{R}^n) \to H^{s+m}_q(\mathbb{R}^n)$ if $|s + m| < \tau$ for $1 < q < \infty$. Hence the restriction on s is too strong unless $m = 0$. In [1] the problem was solved by taking the parametrix in y-form instead of x-form. But, since we did not treat operators in y-form, we use order-reducing operators to the operator to order 0.

By [10, Proposition 4.2], there is a family of elliptic polyhomogeneous symbols $\lambda^m_-(\xi) \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the transmission condition such that $\lambda^m_-(\xi', D_n)_+$ is of class $-\infty$ and $\lambda^j_-(\xi', D_n)_+ \lambda^k_-(\xi', D_n)_+ = \lambda^{j+k}_-(\xi', D_n)_+$ for $j, k \in \mathbb{Z}$. Hence, if $a(x, D_x)$ is an elliptic Green operator of order m and class r, then

$$a'(x, D_x) := a(x, D_x) \begin{pmatrix} \lambda^{-m}_-(D_x)_+ & \\ & \langle D_{x'} \rangle^{-m} \end{pmatrix} \tag{6.1}$$

is an elliptic Green operator of order 0 and class $r - m$.

THEOREM 6.4 *Let* $a(x, D_x)$ *be an elliptic Green operator of order* $m \in \mathbb{Z}$, *class* $r \in \mathbb{Z}$, *and of regularity* C^τ, $\tau > 0$, *in* x. *Then for every* $\theta \in (0, \tau)$, $\theta \notin \mathbb{N}$, *there is a parametrix* $B = B_{[\theta]}$ *such that* $a(x, D_x)B = I + R_\theta$, *where*

$$R_\theta\colon H^{s-\theta}_q(\mathbb{R}^n_+)^N \times B^{s-\theta-\frac{1}{q}}_q(\mathbb{R}^{n-1})^{M'} \to H^s_q(\mathbb{R}^n_+)^N \times B^{s-\frac{1}{q}}_q(\mathbb{R}^{n-1})^M$$

if $-\tau+\theta < s < \tau - [\theta]$, $s-\theta > r-m-\frac{1}{q'}$, *and* $s-\frac{1}{q} > -\tau+\theta$ *when* $M \neq 0$ *or* $M' \neq 0$. *More precisely,* $B = \operatorname{diag}(\lambda_-(D_x)_+^{-m}, \langle D_{x'}\rangle^{-m}) b(x,D_x)$, *where* $b(x,D_x)$ *is a Green operator of order* 0 *and class* $r-m$.

Proof: In the following R_θ will denote an operator with mapping properties stated in the theorem. Because of (6.1), we can assume that $m=0$. Moreover, we consider for simplicity only the case that $a(x,D_x) = a_0(x,D_x)$.

In order to construct an inverse modulo terms of order $-\theta$, we make the Ansatz $b(x,D_x) = \sum_{j=0}^{[\theta]} b_j(x,D_x)$, where $b_j(x,D_x)$ are Green operators of order $-m-j$ with coefficients in $C^{\tau-j}$. Moreover, denote by $q_j(x,\xi)$ the interior symbol of $b_j(x,D_x)$. Then by Theorem 1.2

$$a(x,D_x)b_j(x,D_x) = (a\#_{[\theta]-j}b_j)(x,D_x) + R_\theta = \sum_{k=0}^{[\theta]-j} r_j^{(k)}(x,D_x) + R_\theta,$$

where $r_j^{(k)}(x,D_x)$ is a Green operator of order $-m-j-k$ with coefficients in $C^{\tau-j-k}$. Moreover, let $q_j^{(k)}(x,\xi)$ denote the interior symbol of $r_j^{(k)}(x,D_x)$. Then

$$r_j^{(0)}(x',0,\xi',D_n) = a_0(x',0,\xi',D_n)b_j(x',0,\xi',D_n), \quad q_j^{(0)}(x,\xi) = p_0(x,\xi)q_j(x,\xi).$$

Hence sorting the terms by their order $a(x,D_x)b(x,D_x) = \sum_{l=0}^{[\theta]} \sum_{k=0}^{l} r_{l-k}^{(k)}(x,D_x) + R_\theta$. In order to obtain $a(x,D_x)b(x,D_x) = I + R_\theta$, we determine $b_j(x,D_x)$, $j \geq 1$, successively by

$$\begin{aligned} b_0(x',0,\xi',D_n) &= a_0(x',0,\xi',D_n)^{-1}, \\ b_l(x',0,\xi',D_n) &= -a_0(x',0,\xi',D_n)^{-1} \sum_{k=1}^{l} r_{l-k}^{(k)}(x',0,\xi',D_n), \quad l = 1,\ldots,[\theta] \end{aligned}$$

for $|\xi'| \geq 1$ and $q_0(x,\xi) = p_0(x,\xi)$, $q_l(x,\xi) = -p(x,\xi)^{-1}\sum_{k=0}^{l} q_{l-k}^{(k)}(x,\xi)$, $l = 1,\ldots,[\theta]$, for $|\xi| \geq 1$. ■

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