ON THE GEOMETRY ON THE NONDEGENERATE SUBSPACES OF ORTHOGONAL SPACE

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ABSTRACT. The present article is part of the program described in [2]. Here we study the Phan-theoretic flipflop geometries related to the flip induced by a nondegenerate orthogonal form on a vector space over an arbitrary field of characteristic distinct from two. We obtain amalgam results in the spirit of Phan's theorems [6], [7] for fields that do not admit a quadratic extension and for real closed fields.

1. INTRODUCTION

Let $n \geq 1$ and let V be an $(n+1)$-dimensional vector space over some field $\mathbb{F}$ of characteristic distinct from two endowed with some nondegenerate orthogonal form $f = (\cdot,\cdot)$. By $\Gamma = \Gamma_n(\mathbb{F}, f)$ we denote the pregeometry on the proper subspaces of V that are nondegenerate with respect to $(\cdot,\cdot)$ with symmetrized containment as incidence and the vector space dimension as the type. It is easily seen that $\Gamma_n(\mathbb{F}, f)$ is a geometry, cf. Proposition 2.1. Our first main result is the simple connectedness of that geometry:

Theorem 1. *Let $n \geq 3$ and let $\mathbb{F}$ be an arbitrary field of characteristic not two distinct from $\mathbb{F}_3$ and $\mathbb{F}_5$. Then the geometry $\Gamma_n(\mathbb{F}, f)$ is simply connected.*

For sufficiently large n, say $n \geq 7$, the geometry $\Gamma_n(\mathbb{F}, f)$ is also simply connected over the fields $\mathbb{F}_3$ and $\mathbb{F}_5$. We do not know whether the geometries in smaller dimension actually are not simply connected or just are not covered by our particular proof. We did not invest too much energy into that problem as the geometries fail to be flag-transitive and hence Tits' Lemma 4.2 does not apply anyway. The flag-transitive geometries are given in the following theorem.

Theorem 2. *Let V be an $(n+1)$-dimensional vector space over some field $\mathbb{F}$ of characteristic distinct from two. The group $SO_{n+1}(\mathbb{F}, f)$ acts flag-transitively on the geometry $\Gamma_n(\mathbb{F}, f)$ if and only if $\mathbb{F}$ does not admit a quadratic extension.*

As usual we want to combine Theorems 1 and 2 by Tits' Lemma 4.2. As mentioned before this lemma does not apply in case of intransitive geometries. Fortunately, there is a method to construct a flag-transitive subgeometry of $\Gamma_n(\mathbb{F}, f)$. Let as before $\Gamma_n(\mathbb{F}, f) = (X, \mathrm{typ}, *)$ be the geometry on the nondegenerate proper subspaces of V and let $F = (x_i)_{1 \leq i \leq n}$ be some flag of Γ (not necessarily maximal). Define the geometry $\Delta_n^F(\mathbb{F}, f) = (Y, \mathrm{typ}_{|Y}, *_{|Y\times Y})$ over $\mathrm{typ}_{|Y}(Y)$ with $Y = \{x \in X \mid x \in F^g \text{ for some } g \in SO_{n+1}(\mathbb{F}, f)\}$.

Theorem 3. *Let V be an $(n+1)$-dimensional vector space over some field of characteristic distinct from two and let F be a flag of $\Gamma_n(\mathbb{F}, f)$. Then the group $SO_{n+1}(\mathbb{F}, f)$ acts flag-transitively on the geometry $\Delta_n^F(\mathbb{F}, f)$.*

The proof of Theorem 3 relies on Witt's theorem. In general the above construction does not lead to a flag-transitive geometry.

Of course, by passing to a flag-transitive subgeometry $\Delta_n^F(\mathbb{F}, f)$ from an intransitive geometry $\Gamma_n(\mathbb{F}, f)$ we have lost elements of our geometry, so in the worst case we may end up with a geometry that is not simply connected any more. However, in some cases one can prove that the smaller geometry still is simply connected as in the following setting.

Theorem 4. *Let $m, n \geq 0$ such that one of m and n is greater than or equal to three and the sum of m and n is greater than or equal to four. Let R be a real closed field and let $V \cong R^{m+n}$ be endowed with a nondegenerate symmetric bilinear form with isometry group $SO_R(m,n)$. If F is a flag of $\Gamma_{m+n-1}(R, f)$ containing anisotropic one-, two-, and three-dimensional subspaces of V, then $\Delta_{m+n-1}^F(R, f)$ is simply connected.*

Combining Theorem 1 and Theorem 2 we get the following.

Theorem 5. *Let V be an $(n+1)$-dimensional vector space over some field $\mathbb{F}$ of characteristic distinct from two that does not admit any quadratic extension. Let F be a maximal flag of $\Gamma_n(\mathbb{F}, f)$ and let $\mathcal{A}_{(2)}$ be the amalgam of all rank two parabolics, i.e., stabilizers in $SO_{n+1}(\mathbb{F}, f)$ of subflags of F of corank two. Then $SO_{n+1}(\mathbb{F}, f)$ is the universal completion of $\mathcal{A}_{(2)}$.*

Finally, Theorem 3 and Theorem 4 imply an analogous result.

Theorem 6. *Let $m, n \geq 0$ such that one of m and n is greater than or equal to three and the sum of m and n is greater than or equal to four. Let R be a real closed field and let $V \cong R^{m+n}$ be endowed with a nondegenerate symmetric bilinear form with isometry group $SO_R(m,n)$ and let F be a flag of $\Gamma_{m+n-1}(R, f)$ of rank at least three consisting of all positive definite (negative definite) subspaces of V. Let $\mathcal{A}_{(2)}$ be the amalgam of all rank two parabolics in $SO_R(m,n)$ with respect to the maximal flag F of $\Delta_{m+n-1}^F(R, f)$. Then $SO_R(m,n)$ is the universal completion of $\mathcal{A}_{(2)}$.*

This paper is organized as follows. In Section 2 we study the connectedness and residual connectedness of $\Gamma_n(\mathbb{F}, f)$. In Section 3 we turn our attention to the simple connectedness of $\Gamma_n(\mathbb{F}, f)$ and provide a proof of Theorem 1. Section 4 deals with transitivity properties of $\Gamma_n(\mathbb{F}, f)$ and proofs of Theorem 2 and Theorem 5. Finally, Section 5 focuses on flag-transitive subgeometries of $\Gamma_n(\mathbb{F}, f)$ and provides proofs of Theorems 3, 4, and 6.

2. Nondegenerate subspaces of orthogonal space

Our geometric notions are standard. As a reference see [3] or [4]. We will remind the reader of relevant notions as they occur. Let $n \geq 1$ and let V be an $(n+1)$-dimensional vector space over some field $\mathbb{F}$ of characteristic distinct from two endowed with some nondegenerate orthogonal form $f = (\cdot,\cdot)$. By $\Gamma = \Gamma_n(\mathbb{F}, f)$ we denote the pregeometry on the proper subspaces of V that are nondegenerate with respect to $(\cdot,\cdot)$ with symmetrized containment as incidence and the vector space dimension as the type. Recall that the difference between a geometry and a pregeometry over the type set $\{1,\ldots,n\}$ is that in the former each flag is contained in a chamber, i.e., a flag of type $\{1,\ldots,n\}$, while in the latter this need not necessarily be the case.

Proposition 2.1. *The pregeometry $\Gamma_n(\mathbb{F}, f)$ is a geometry.*

Proof: We have to prove that each flag can be embedded in a flag of cardinality n. To this end let $F = \{x_1,\ldots,x_t\}$ be a flag of Γ. We can assume that the nondegenerate subspace x_1 of V has dimension one. Indeed, if it has not, then we can find a nondegenerate one-dimensional subspace x_0 of x_1 and study the flag $F' = F \cup \{x_0\}$ instead. Now observe that the residue of the nondegenerate one-dimensional subspace x_1 is isomorphic to $\Gamma_{n-1}(\mathbb{F}, f')$ for some induced form f' via the map that sends an element U of the residue of x_1 to $U \cap x_1^{\perp}$. Hence induction applies. ■

Lemma 2.2. *If l is a line and p is a point not on l, then there are at most two points of Γ on l which are not collinear to p.*

Proof: This follows immediately from the fact that at most two two-dimensional subspaces of $\langle p, l\rangle$ containing p are degenerate with respect to $(\cdot,\cdot)$. ■

The collinearity graph of a pregeometry Γ is the graph on the points of Γ in which two vertices are adjacent if and only if the corresponding points of Γ are collinear.

Proposition 2.3. *Let $n \geq 2$. The collinearity graph of $\Gamma_n(\mathbb{F}, f)$ has diameter two.*

Proof: Suppose $n \geq 3$, then the dimension of the vector space V is at least 4. Now fix two points $\langle a\rangle$ and $\langle b\rangle$, which are not collinear. Two points $\langle a\rangle$ and $\langle b\rangle$ are not collinear if and only if the space $\langle a, b\rangle$ is singular with respect to $(\cdot,\cdot)$. However $\langle a, b\rangle$ is a two-dimensional subspace of V which is not totally singular, because (a, a) and (b, b) are distinct from zero. Therefore the radical of $\langle a, b\rangle$ is a one-dimensional space. The dimension of $\langle a, b\rangle^{\perp}$ is greater or equal to two, as $n \geq 3$. Consequently, the orthogonal complement of $\langle a, b\rangle$ contains a point, say $\langle c\rangle$. Now consider the two two-dimensional subspaces $\langle a, c\rangle$ and $\langle b, c\rangle$. Since $\langle a\rangle$ and $\langle b\rangle$ are perpendicular to $\langle c\rangle$ both $\langle a, c\rangle$ and $\langle b, c\rangle$ are lines. The distance between $\langle a\rangle$ and $\langle c\rangle$ is one and so is the distance between $\langle c\rangle$ and $\langle b\rangle$. Thus the distance between

$\langle a \rangle$ and $\langle b \rangle$ is two. Certainly Γ contains a pair of noncollinear points, so we are done.

Now assume $n = 2$ and let $\langle a \rangle$ and $\langle b \rangle$ be two arbitrary points in V. If the space $l = \langle a, b \rangle$ is a line then the distance between $\langle a \rangle$ and $\langle b \rangle$ is one. Otherwise pick a point $\langle \tilde{a} \rangle$ in $\langle a \rangle^{\perp}$. The space $\langle a, \tilde{a} \rangle$ is a line and the point $\langle b \rangle$ is not on $\langle a, \tilde{a} \rangle$. The point $\langle b \rangle$ is collinear with at least two points on $\langle a, \tilde{a} \rangle$ by Lemma 2.2. Pick one of these points, say the point $\langle c \rangle$. The distance between $\langle a \rangle$ and $\langle c \rangle$ is one, because the space $\langle a, c \rangle$ is the line $\langle a, \tilde{a} \rangle$. The distance between $\langle b \rangle$ and $\langle c \rangle$ is one as well, because $\langle c \rangle$ and $\langle b \rangle$ are collinear. This implies that the distance between point $\langle a \rangle$ and point $\langle b \rangle$ is two. ■

Recall that a pregeometry is called residually connected if each residue of a flag of corank at least two is connected and each residue of a flag of corank one is non-empty.

Corollary 2.4. *Let $n \geq 2$. Then $\Gamma_n(\mathbb{F}, f)$ is residually connected.*

Proof: Each residue of $\Gamma_n(\mathbb{F}, f)$ with respect to some flag of corank at least two is of the form $\oplus\Gamma_m(\mathbb{F}, f')$, i.e., the direct sum of geometries $\Gamma_m(\mathbb{F}, f')$ for suitable m and suitable nondegenerate orthogonal forms f'. If $\oplus\Gamma_m(\mathbb{F}, f')$ consists of a unique direct summand, this summand is connected by Proposition 2.3. If $\oplus\Gamma_m(\mathbb{F}, f')$ has more than one direct summand then it is connected anyway. ■

3. Simple connectedness

Recall the definition of the fundamental group of a connected geometry Δ. A **path** of length k in the geometry is a sequence of elements $x_0, \ldots, x_k$ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. A **cycle** based at an element x is a path in which $x_0 = x_k = x$. Two paths are **homotopically equivalent** if one can be obtained from the other via the following operations called **elementary homotopies**: inserting or deleting a repetition (i.e., a cycle of length 1), a return (i.e., a cycle of length 2), or a triangle (i.e., a cycle of length 3). The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the **fundamental group** of Δ and denoted by $\pi_1(\Delta, x)$. A geometry is called **simply connected** if its fundamental group is trivial. Notice that in order to prove that Δ is simply connected it is enough to prove that any cycle based at x is homotopically equivalent to the cycle of length 0. A cycle with this property is called **null-homotopic**, or **homotopically trivial**. We refer the reader to [8] or [9] for more detailled information.

Recall that the incidence graph of some geometry is the graph on the elements of that geometry in which two distinct elements are adjacent if and only if they are incident. This means the fundamental group of a rank n geometry is nothing else than the fundamental group of its incidence graph considered as a $(n-1)$-dimensional simplicial complex.

Lemma 3.1. *Let $n \geq 1$. Every cycle $\gamma = x_0 x_1 \ldots x_{k-1} x_0$ in the incidence graph of $\Gamma_n(\mathbb{F}, f)$ is homotopically equivalent to a cycle γ' touching only points and lines.*

Proof: This follows by a standard argument using the residual connectedness of Γ, see Lemma 5.1 of [5]. ■

If $n = 2$, then the vector space V has dimension three. Thus, the geometry $\Gamma_2(\mathbb{F}, f)$ contains only elements of type one or two. In the incidence graph of $\Gamma_2(\mathbb{F}, f)$, only points and lines are adjacent but never two different points or two different lines. Therefore, the incidence graph of $\Gamma_2(\mathbb{F}, f)$ cannot be decomposed into triangles. We have proved the following.

Proposition 3.2. *Let $n = 2$. The geometry $\Gamma_2(\mathbb{F}, f)$ is not simply connected.* ■

In the remainder of this section we will prove the simple connectedness of $\Gamma_n(\mathbb{F}, f)$ for $n \geq 3$. Since every closed path based on an arbitrary element in the incidence graph of Γ is homotopically equivalent to a cycles based on a point and passing only points and lines, there is, for every cycle in the incidence graph, a homotopically equivalent closed path in the point-line-incidence graph which implies that it suffices to study the point-line-incidence graph. Moreover, since Γ is a partial linear space, each line is uniquely determined by two of its points, so it is enough to study the collinearity graph of Γ.

In the nondegenerate vector space V, let $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$ be different points and the three two-dimensional spaces $\langle a, b \rangle$, $\langle a, c \rangle$, and $\langle b, c \rangle$ be lines. We call the 3-cycle $\langle a \rangle \langle b \rangle \langle c \rangle \langle a \rangle$ a **nondegenerate triangle** or **good triangle** if $\langle a, b, c \rangle$ is a nondegenerate vector subspace of V. Otherwise $\langle a \rangle \langle b \rangle \langle c \rangle \langle a \rangle$ is a **degenerate triangle** or **bad triangle**.

Since the diameter of the collinearity graph of Γ is two, in order to prove simple connectedness it suffices to prove that we can decompose triangles, quadrangles and pentagons in the collinearity graph into products of good triangles. Let's start with pentagons:

Proposition 3.3. *Let $n \geq 3$ and let $|\mathbb{F}| \geq 5$. Every pentagon in the collinearity graph of Γ can be decomposed into a product of triangles and quadrangles.*

Proof: Let $\gamma = \langle a \rangle \langle b \rangle \langle c \rangle \langle d \rangle \langle e \rangle \langle a \rangle$ be an arbitrary 5-cycle in the collinearity graph of Γ. Since $|\mathbb{F}| \geq 5$, the line $\langle c, d \rangle$ contains at least four points of Γ, so by Lemma 2.2 it contains a point of Γ collinear to $\langle a \rangle$, say $\langle y \rangle$. Since $\langle a \rangle$ is collinear to $\langle y \rangle$ the space $\langle a, y \rangle$ is a line. We have decomposed the 5-cycle γ into a product of 4-cycles and 3-cycles. ■

Now we deal with 4-cycles.

Proposition 3.4. *Let $n \geq 3$ and let $|\mathbb{F}| \geq 7$. Every quadrangle in the collinearity graph of Γ can be decomposed into a product of triangles.*

Proof: Let $\gamma = \langle a\rangle\langle b\rangle\langle c\rangle\langle d\rangle\langle a\rangle$ be an arbitrary 4-cycle in the collinearity graph of Γ. Since $|\mathbb{F}| \geq 7$, the line $\langle a, b\rangle$ contains at least six points of Γ. By Lemma 2.2 of those six points at least four are collinear to $\langle c\rangle$, and, by Lemma 2.2 again, of those four points at least two are collinear to $\langle d\rangle$ decomposing the 4-cycle γ into 3-cycles. ■

We have decomposed pentagons and quadrangles into products of triangles. However, those triangles may be bad. For that reason we finish the proof of the simple connectivity of the geometry Γ by showing that a bad triangle in the collinearity graph of Γ can be decomposed in a product of good triangles. In the sequel we will distinguish between $n = 3$ and $n \geq 4$.

Proposition 3.5. *For $n \geq 4$ every degenerate triangle can be decomposed into a product of nondegenerate triangles.*

Proof: Let $\gamma = \langle a\rangle\langle b\rangle\langle c\rangle\langle a\rangle$ be an arbitrary 3-cycle in the collinearity graph of Γ such that $U = \langle a, b, c\rangle$ is singular. Since U contains the lines $\langle a, b\rangle$, $\langle a, c\rangle$ and $\langle b, c\rangle$, the radical of U has dimension one. But $\dim(U^\perp) \geq 2$ and therefore the space $U^\perp$ contains a one-dimensional subspace $\langle d\rangle$ of V with $(d, d) \neq 0$. The two-dimensional spaces $\langle a, d\rangle$, $\langle b, d\rangle$ and $\langle c, d\rangle$ are lines as $(a, d) = (b, d) = (c, d) = 0$. Now we have to prove that $\langle a, b, d\rangle$, $\langle a, c, d\rangle$ and $\langle b, c, d\rangle$ are nondegenerate vector subspaces of V. The Gram matrix $G_{\langle a,b,d\rangle}$ is $\begin{pmatrix} (a,a) & (a,b) & (a,d) \\ (b,a) & (b,b) & (b,d) \\ (a,d) & (b,d) & (d,d) \end{pmatrix} = \begin{pmatrix} (a,a) & (a,b) & 0 \\ (b,a) & (b,b) & 0 \\ 0 & 0 & (d,d) \end{pmatrix}$. The determinant of $G_{\langle a,b,d\rangle}$ is $\det(G_{\langle a,b\rangle}) \cdot (d, d) \neq 0$ because $\det(G_{\langle a,b\rangle}) \neq 0$ and $(d, d) \neq 0$, which shows that $\langle a, b, d\rangle$ is a nondegenerate vector subspace. The same argument holds for the spaces $\langle a, c, d\rangle$ and $\langle b, c, d\rangle$. ■

Now we turn to the case $n = 3$. The proof of Proposition 3.5 does not apply in case $n = 3$, because the orthogonal complement of a three-dimensional singular space in a four-dimensional space is equal to the radical of the three-dimensional space. Hence we have to construct the point $\langle d\rangle$ in another way.

Let $\langle a\rangle\langle b\rangle\langle c\rangle\langle a\rangle$ be a 3-cycle in the collinearity graph of Γ. We call $\langle a\rangle\langle b\rangle\langle c\rangle\langle a\rangle$ of **perpendicular type** if one of the equalities $(a, b) = 0$, $(a, c) = 0$, or $(b, c) = 0$ holds.

The idea is to show that every triangle can be decomposed into a product of triangles of perpendicular type and then that every triangle of perpendicular type can be decomposed again into a product of nondegenerate triangles.

For the first step assume $|\mathbb{F}| \geq 5$. Let $\gamma = \langle a\rangle\langle b\rangle\langle c\rangle\langle a\rangle$ be an arbitrary 3-cycle. If $\mathcal{C}$ is a cycle of perpendicular type then we have nothing to prove. Otherwise take the line $\langle a, c\rangle^\perp$ and pick a point $\langle d\rangle$ from that line, which is collinear with $\langle b\rangle$. Lemma 2.2 implies that such a point $\langle d\rangle$ exists. The resulting 3-cycles are of perpendicular type. We have proved the following.

Lemma 3.6. *Let $|\mathbb{F}| \geq 5$. Any 3-cycle can be decomposed into a product of 3-cycles of perpendicular type.* ■

Let $\langle a,b,c\rangle$ be a 3-space and take $\langle d\rangle$ to be a point in $\langle a,c\rangle^\perp$. We say $\langle d\rangle$ is **good** if the vector subspace $\langle c,b,d\rangle$ is nondegenerate; otherwise we call $\langle d\rangle$ **bad**.

Assume $|\mathbb{F}|\geq 7$ and let $\gamma=\langle a\rangle\langle b\rangle\langle c\rangle\langle a\rangle$ be a degenerate 3-cycle of perpendicular type, say a is perpendicular to b. The two-dimensional vector subspace $\langle a,c\rangle^\perp$ is a line and because $\langle a,b,c\rangle$ is singular, b is not an element of $\langle a,c\rangle^\perp$. Using Lemma 2.2, there exists a point $\langle d\rangle$ of $\langle a,c\rangle^\perp$ such that $\langle d\rangle$ and $\langle b\rangle$ are collinear. The point $\langle d\rangle$ can be good or bad with respect to the space $\langle b,c,d\rangle$. We claim that we can find a good point. Suppose $\langle d\rangle$ is a bad point. Then $U_d=\langle b,c,d\rangle$ is a singular space. Because the line $\langle b,c\rangle$ is properly contained in U_d, the radical of U_d has dimension one. Let $\langle s\rangle$ be the radical of U_d. Then $\langle s\rangle$ is contained in the space $\langle b,c\rangle^\perp$. It follows that $\langle b,c,s\rangle$ is a three-dimensional space contained in $\langle b,c,d\rangle$ which implies that $\langle b,c,s\rangle=\langle b,c,d\rangle$. We claim that there is an one-to-one correspondence between a bad point $\langle d\rangle$ and the radical of U_d. For, suppose for two different bad points $\langle d\rangle$ and $\langle \bar{d}\rangle$ we have $\mathrm{Rad}(U_d)=\mathrm{Rad}(U_{\bar{d}})=\langle s\rangle$, and hence $\langle b,c,d\rangle=\langle b,c,s\rangle=\langle b,c,\bar{d}\rangle$. Moreover, s, d and $\bar{d}$ are elements of $\langle c\rangle^\perp$, in fact $\langle s,d,\bar{d}\rangle\subseteq\langle c\rangle^\perp\cap\langle b,c,s\rangle$. The dimension of $\langle c\rangle^\perp\cap\langle b,c,s\rangle$ is two, which implies $\langle s,d,\bar{d}\rangle=\langle s,d\rangle=\langle s,\bar{d}\rangle$. Since $\langle s,d\rangle$ is singular, the space $\langle s,d\rangle$ is distinct from the space $\langle a,c\rangle^\perp$. Therefore the vector subspace $\langle s,d\rangle\cap\langle a,c\rangle^\perp=\langle s,\bar{d}\rangle\cap\langle a,c\rangle^\perp$ has dimension one and contains both point $\langle d\rangle$ and point $\langle\bar{d}\rangle$, which shows that the vector $\bar{d}$ is an element of $\langle d\rangle$, a contradiction to the hypothesis that $\langle d\rangle$ is distinct from $\langle\bar{d}\rangle$.

It follows that the number of different bad points is equal to the number of different one-dimensional singular vector subspaces in $\langle b,c\rangle^\perp$, which is at most two as $\langle b,c\rangle$ is nondegenerate.

Since we assumed $\mathbb{F}$ to contain at least seven elements, we can find a good point $\langle d\rangle$. We know that $\langle a,c,d\rangle$ and $\langle b,c,d\rangle$ are nondegenerate vector subspaces. For the nondegeneracy of $\langle a,b,d\rangle$ we use the following argument. The Gram matrix $G_{\langle a,b,d\rangle}$ is $\begin{pmatrix}(a,a)&(a,b)&(a,d)\\(b,a)&(b,b)&(b,d)\\(a,d)&(b,d)&(d,d)\end{pmatrix}=\begin{pmatrix}(a,a)&0\\0&G_{\langle b,d\rangle}\end{pmatrix}$ and $\langle b,d\rangle$ is a line. The determinant of $G_{\langle a,b,d\rangle}$ is $(a,a)\cdot\det(G_{\langle b,d\rangle})\neq 0$. This shows that $\langle a,b,d\rangle$ is nondegenerate and proves the following proposition.

Proposition 3.7. *Let $|\mathbb{F}|\geq 7$. Each degenerate triangle of perpendicular type in the collinearity graph of $\Gamma_3(\mathbb{F},f)$ can be decomposed into nondegenerate triangles.* ∎

Altogether we have proved Theorem 1.

4. Flag transitivity

Let $\mathbb{F}$ be a field of characteristic distinct from two that does not admit any quadratic extension and let V be a nondegenerate orthogonal space over $\mathbb{F}$ of dimension $n+1$. The classification of nondegenerate orthogonal forms shows that each orthogonal form on V is isometric to the form whose Gram matrix is the identity matrix.

Proposition 4.1. *Let V be an $(n+1)$-dimensional vector space over some field $\mathbb{F}$ of characteristic distinct from two that does not admit any quadratic extension. The group $SO_{n+1}(\mathbb{F}, f)$ acts transitively on the points of Γ.*

Proof: The group $O_{n+1}(\mathbb{F}, f)$ acts transitively on the points of Γ by Witt's theorem, so for any pair p, q of points of Γ we can find an element of $O_{n+1}(\mathbb{F}, f)$ that maps p to q. On the other hand, the matrix $\begin{pmatrix} -1 & 0 \\ 0 & \mathrm{id}_{n\times n} \end{pmatrix}$ with respect to a basis whose first vector spans q has determinant -1 and stabilizes q. Therefore also $SO_{n+1}(\mathbb{F}, f)$ acts transitively on the points of Γ. ■

Proof of Theorem 2: One implication of the claim follows from Proposition 4.1 by induction on n using the isomorphism between the residue of a point in $\Gamma_n(\mathbb{F}, f)$ and $\Gamma_{n-1}(\mathbb{F}, f')$. The other implication is obvious. ■

In the present paper an **amalgam** $\mathcal{A}$ of groups is a set with a partial operation of multiplication and a collection of subsets $\{H_i\}_{i\in I}$, for some index set I, such that the following hold: (1) $\mathcal{A} = \cup_{i\in I} H_i$; (2) the product ab is defined if and only if $a, b \in H_i$ for some $i \in I$; (3) the restriction of the multiplication to each H_i turns H_i into a group; and (4) $H_i \cap H_j$ is a subgroup in both H_i and H_j for all $i, j \in I$. It follows that the groups H_i share the same identity element, which is then the only identity element in $\mathcal{A}$, and that $a^{-1} \in \mathcal{A}$ is well-defined for every $a \in \mathcal{A}$. We will call the groups H_i the **members** of the amalgam $\mathcal{A}$. A group H is called a **completion** of an amalgam $\mathcal{A}$ if there exists a map $\pi : \mathcal{A} \to H$ such that (1) for all $i \in I$ the restriction of π to H_i is a homomorphism of H_i to H; and (2) $\pi(\mathcal{A})$ generates H. Among all completions of $\mathcal{A}$ there is one "largest" which can be defined as the group having the presentation $U(\mathcal{A}) = \langle t_h \mid h \in \mathcal{A},\, t_x t_y = t_z, \text{ whenever } xy = z \text{ in } \mathcal{A}\rangle$. Obviously, $U(\mathcal{A})$ is a completion of $\mathcal{A}$ since one can take π to be the mapping $h \mapsto t_h$. Every completion of $\mathcal{A}$ is isomorphic to a quotient of $U(\mathcal{A})$, and because of that $U(\mathcal{A})$ is called the **universal completion**.

Suppose a group $H \leq \mathrm{Aut}(\Gamma)$ acts flag-transitively on a geometry Γ. A **rank k parabolic** is the stabilizer in H of a flag of corank k from Γ. Parabolics of rank $n-1$ (where n is the rank of Γ) are called **maximal parabolics**. They are exactly the stabilizers in H of single elements of Γ.

Let F be a maximal flag in Γ, and let H_x denote the stabilizer in H of $x \in \Gamma$. The amalgam $\mathcal{A} = \mathcal{A}(F) = \cup_{x\in F} H_x$ is called the amalgam of maximal parabolics in H. Since the action of H is flag-transitive, this amalgam is defined uniquely up to conjugation in H. For a fixed flag F we can also use the notation M_i for the maximal parabolic H_x, where $x \in F$ is of type i. For a subset $J \subset I = \{1, 2 \ldots, n\}$, define M_J to be $\cap_{j\in J} M_j$, including $M_\emptyset = H$. Notice that M_J is a parabolic of rank $|I \setminus J|$; indeed, it is the stabilizer of the subflag of F of type J. Similarly to $\mathcal{A}$, we can define the amalgam $\mathcal{A}_{(s)}$ as the union of all rank s parabolics. With this notation we can write $\mathcal{A} = \mathcal{A}_{(n-1)}$. Moreover, according to our definition, $\mathcal{A}_{(n)} = H$.

Now we need to define coverings of geometries. Suppose Γ and $\hat{\Gamma}$ are two geometries over the same type set and suppose $\phi : \hat{\Gamma} \to \Gamma$ is a **morphism** of geometries, i.e., ϕ preserves the type and sends incident elements to incident elements. The morphism ϕ is called a **covering** if and only if for every non-empty flag $\hat{F}$ in $\hat{\Gamma}$ the mapping ϕ induces an isomorphism between the residue of $\hat{F}$ in $\hat{\Gamma}$ and the residue of $F = \phi(\hat{F})$ in Γ. Coverings of a geometry correspond to the usual topological coverings of its flag complex, see also [8] or [9]. In particular, by §55 of [8] or Theorem 1.1 of [9] a simply connected geometry (as defined in Section 3) admits no nontrivial covering.

The following lemma from [10] combines the topological structure of a geometry with amalgams obtained from flag-transitive groups of automorphisms.

Tits' Lemma 4.2. *Suppose a group H acts flag-transitively on a geometry Γ and let $\mathcal{A}$ be the amalgam of maximal parabolics associated with some maximal flag F. Then H is the universal completion of the amalgam $\mathcal{A}$ if and only if Γ is simply connected.*

Tits' Lemma together with Theorems 1 and 2 immediately implies that $SO_{n+1}(\mathbb{F}, f)$ is the universal completion of the amalgam of maximal parabolics in $SO_{n+1}(\mathbb{F}, f)$ with respect to some maximal flag of Γ. Theorem 5 follows from that observation by a standard induction argument using the residual connectedness of Γ and the simple connectedness of all residues of Γ as in the proof of Theorem 1 of [5].

5. Flag-transitive parts

What remains is a discrepancy between the fields that occur in Theorem 1 and the ones that occur in Theorem 2. The standard method to force flag-transitivity would be to study the orbit of one flag under the group $SO_{n+1}(\mathbb{F}, f)$ of isometries of the form $(\cdot, \cdot)$ on V. To be precise let as before $\Gamma_n(\mathbb{F}, f) = (X, \mathrm{typ}, *)$ be the geometry on the nondegenerate proper subspaces of V and let $F = (x_i)_{i \in J}$, $J \subseteq I = \{1, \ldots, n\}$ be a flag of Γ. Define the geometry $\Delta_n^F(\mathbb{F}, f) = (Y, \mathrm{typ}_{|Y}, *_{|Y \times Y})$ with $Y = \{x \in X \mid x \in F^g \text{ for some } g \in SO_{n+1}(\mathbb{F}, f)\}$.

Proof of Theorem 3: Let x_1 and x_2 be elements of $\Delta_n^F(\mathbb{F}, f) \subseteq \Gamma_n(\mathbb{F}, f)$ with $x_1 * x_2$. This means there exist g_1, g_2 in $SO_{n+1}(\mathbb{F}, f)$ with $x_1 \in F^{g_1}$ and $x_2 \in F^{g_2}$ or, equivalently, $x_1^{g_1^{-1}} \in F$ and $x_2^{g_1^{-1}} \in F^{g_2 g_1^{-1}}$. Note that $x_1^{g_1^{-1}}$ is incident with both $x_2^{g_1^{-1}}$ and the element $y \in F$ of type $\mathrm{typ}(x_2^{g_1^{-1}})$. The subspaces y and $x_2^{g_1^{-1}}$ of V are isometric so by Witt's theorem applied to $x_1^{g_1^{-1}}$ if $\mathrm{typ}(x_1) > \mathrm{typ}(x_2)$, resp. $(x_1^{g_1^{-1}})^{\perp}$ if $\mathrm{typ}(x_1) < \mathrm{typ}(x_2)$ there exists an element of $SO_{n+1}(\mathbb{F}, f)$ stabilizing $x_1^{g_1^{-1}}$ that maps $x_2^{g_1^{-1}}$ onto y. Induction on $|J|$ shows that $SO_{n+1}(\mathbb{F}, f)$ acts flag-transitively on $\Delta_n^F(\mathbb{F}, f)$. ∎

Proof of Theorem 4: Let U be the three-dimensional space of the flag F. Notice that, as U is anisotropic, any cycle consisting of elements of U is null-homotopic. If p and q are points of $\Delta_{m+n-1}^F(R, f)$, then $p^\perp \cap q^\perp \cap U$ contains an anisotropic one-dimensional subspace r collinear to both p and q. Therefore the diameter of $\Delta_{m+n-1}^F(R, f)$ is two. The argument of Lemma 3.1 implies that it suffices to decompose triangles, quadrangles and pentagons in the collinearity graph of $\Delta_{m+n-1}^F(R, f)$. Pentagons decompose as for any point p and any line l there exists a point q in $p^\perp \cap l$ collinear to p. A quadrangle a, b, c, d decomposes by the following argument. Let p_{ab} be a point contained in $a^\perp \cap b^\perp \cap U$. Similarly, define p_{bc}, p_{cd}, p_{ad}. Certainly, the quadrangle p_{ab}, p_{bc}, p_{cd}, p_{ad} is null-homotopic. Therefore we have decomposed the original quadrangle into a null-homotopic quadrangle and a number of triangles. A triangle is decomposed in exactly the same way as a quadrangle. ∎

Tits' Lemma 4.2 together with Theorems 3 and 4 immediately implies that $SO_R(m, n)$ is the universal completion of the amalgam of maximal parabolics in $SO_R(m, n)$ with respect to some maximal flag of $\Delta_{m+n-1}^F(R, f)$. Theorem 6 follows from that observation by a standard induction argument using the residual connectedness of $\Delta_{m+n-1}^F(R, f)$ and the simple connectedness of all residues of $\Delta_{m+n-1}^F(R, f)$ as in the proof of Theorem 1 of [5].

References

[1] K. Altmann, *The geometry of nondegenerate subspaces of complex orthogonal space*, Diplomarbeit, TU Darmstadt, 2003.

[2] C.D. Bennett, R. Gramlich, C. Hoffman, S. Shpectorov, Curtis-Phan-Tits theory, In: *Groups, Combinatorics and Geometry; Proceedings of the Durham Conference on Groups and Geometry 2001* (edited by A.A. Ivanov, M.W. Liebeck, J. Saxl), World Scientific, New Jersey, 2003, 13 - 29.

[3] F. Buekenhout, *Handbook of incidence geometry*, North Holland, Amsterdam, 1995.

[4] F. Buekenhout, A.M. Cohen, *Diagram geometry*, in preparation, preprints of the Chapters 1 – 12, 14 can be found at `http://www.win.tue.nl/~amc/buek`

[5] R. Gramlich, C. Hoffman, S. Shpectorov, A Phan-type theorem for $Sp(2n, q)$, *J. Algebra* **264** (2003), 358–384.

[6] K. W. Phan, On groups generated by three-dimensional special unitary groups, I. *J. Austral. Math. Soc. Ser. A* **23** (1977), 67–77.

[7] K. W. Phan, On groups generated by three-dimensional special unitary groups. II, *J. Austral. Math. Soc. Ser. A* **23** (1977), 129–146.

[8] H. Seifert, W. Threlfall, *Lehrbuch der Topologie*, Chelsea Publishing Company, New York, 1934.

[9] D.B. Surowski, Covers of simplicial complexes and applications to geometry, *Geom. Dedicata* **16** (1984), 35–62.

[10] J. Tits, Ensembles Ordonnés, immeubles et sommes amalgamées, *Bull. Soc. Math. Belg. Sér. A* **38** (1986), 367–387.