Evolution of phase boundaries by configurational forces

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Abstract

In this article an initial-boundary value problem modeling the evolution of a surface of strain discontinuity driven by configurational forces is studied. Starting from a sharp interface model the problem is transformed into a problem with an evolution equation for the order parameter, which has similarities with a hyperbolic balance law. It is proved that in one space dimension global solutions exist. The method of transformation suggests that solutions of this evolution equation are approximated by solutions of a viscous Hamilton-Jacobi equation. If the approximation is valid then the initial-boundary value problem to this Hamilton-Jacobi eqution is a phase field model regularizing the sharp interface model.

1 Introduction and statement of main results

Changes of the morphology of material structure are often caused by configurational forces. In crystalline materials for example, discontinuous changes of the crystal structure generate configurational forces, which can move the discontinuity surface. This is observed in superalloys, which may exist in two different phases. In the two phases the lattice constants of the crystal lattices differ slightly, resulting in a strain discontinuity at the phase interface. The configurational forces generated by this discontinuity together with diffusion lead to the evolution of the microstructure generated by phase changes, cf. [14, 22, 27, 28, 19]. Another example for a configurational force moving discontinuities of the material structure is the Peach-Köhler force cf. [18, 24], which drives the glide of dislocations and leads to plastic deformation.

In this article we study an initial-boundary value problem which models the evolution of a surface of strain discontinuity driven by configurational forces. This problem has been thoroughly formulated in [1]; other discussions of this problem and applications in mechanics and material sciences can be found in [14, 23, 25, 26, 27], for example. The theory of configurational forces and, more generally, configurational mechanics is an intensively studied field with a large number of publications; we only mention here [4, 5, 12, 16, 20, 21].

The goal of our investigations is twofold. In the introduction we set the initial-boundary value problem into the general context of phase transformation models and compare it to other such models. In the main part of our investigations we study the mathematical structure and show that in one space dimension the problem is well posed and has solutions. We explain this more precisely:

The model formulated in [1] is of sharp interface type. In an attempt to avoid the difficulties connected with sharp interface models it has been discovered in [2] that this model can be transformed rigorously into an initial-boundary value problem with a partial differential equation governing the evolution of an order parameter characterising the different phases. In the following we call this partial differential equation the evolution equation for the order parameter. This is an unusual equation, which has similarities with a scalar hyperbolic balance law. In fact, in [1] the surfaces of strain discontinuity are called material shocks. Solutions of the sharp interface model are also solutions of the new initial-boundary value problem, but the new problem allows for more general solutions with the order parameter belonging to the space BV of functions with bounded variation.

The numerical solution of initial-boundary value problems, which can have such general discontinuous solutions, presents difficulties. Because of this one often prefers phase field models with smooth solutions. The results in [2] suggest that the initial-boundary value problem can be approximated by a problem with the evolution equation replaced by a well known Hamilton-Jacobi transport equation, which has smooth solutions. If this approximation is valid then the initial-boundary value problem with the Hamilton-Jacobi equation is a phase field model regularising the sharp interface model.

In this article we study the initial-boundary value problem with the evolution equation for the order parameter and show that solutions exist to several classes of initial data, however only in one space dimension. Some of our methods used in the proof are similar to methods used in the theory of hyperbolic conservation laws, but in the final steps, where we use weak convergence of measures to construct solutions, completely new ideas are needed.

We next state the initial-boundary value problem with the evolution equation for the order parameter and discuss the connection to the original sharp interface model. This motivates the form of the evolution equation and shows how the Hamilton-Jacobi equation arises. We also relate our model to the well known models of Allen-Cahn and Cahn-Hilliard. Finally, our main mathematical existence results proved in Sections 2 – 5 are stated at the end of the introduction.

Let Ω be an open bounded set in $\mathbb{R}^3$. It represents the material points of a solid body. The material of this body can exist in two different phases. We denote by $\gamma(t)$ the subset of Ω, which consists of all points, at which at time t the material is in the matrix phase. $\gamma'(t)$ denotes the subset of all points, at which at time t the material is in the second phase. For $\gamma = \bigcup_{t\geq 0}(\gamma(t)\times\{t\})$ and $\gamma' = \bigcup_{t\geq 0}(\gamma'(t)\times\{t\})$

we thus have

$$\gamma \cup \gamma' = \Omega \times [0,\infty), \qquad \gamma \cap \gamma' = \emptyset.$$

We set

$$\tilde{\gamma} = \Omega \cap \overline{\gamma} \cap \overline{\gamma'}, \qquad \tilde{\gamma}(t) = \{x \in \Omega \mid (x,t) \in \tilde{\gamma}\}.$$

$\tilde{\gamma}$ is the interface between the γ and γ' phases. Let $S : \Omega \times [0,\infty) \to \mathbb{R}$ denote the characteristic function of the set γ', hence

$$S(x,t) = \begin{cases} 0, & (x,t) \in \gamma \\ 1, & (x,t) \in \gamma'. \end{cases}$$

S is the order parameter, which characterizes the γ'-phase.

We assume that the values which the linear strain tensor would have if the material would be unstressed differ between the two phases. The difference is the misfit or transformation strain. $\tilde{\gamma}(t)$ is thus a surface of strain discontinuity. It is assumed that the configurational force generated by the misfit strain transforms by some process the material along $\tilde{\gamma}(t)$ from one phase to the other. This leads to an evolution of the phase interface. The goal is to compute this interface evolution.

The initial-boundary value problem which we use to model this material behavior is based on the assumption that only small strains occur. The unknowns are the order parameter $S(x,t) \in \mathbb{R}$, the displacement $u(x,t) \in \mathbb{R}^3$ of the material point $x \in \Omega$ at time t, and the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$. Here $\mathcal{S}^3$ denotes the set of symmetric 3×3–matrices. These unknowns must satisfy the quasi-static equations

$$-\mathrm{div}_x T(x,t) = b(x,t), \tag{1.1}$$

$$T(x,t) = D\left(\varepsilon(\nabla_x u(x,t)) - \bar{\varepsilon} S(x,t)\right), \tag{1.2}$$

$$|S_t(x,t)| = c\left|\mathrm{div}_x C(\nabla_x u(x,t), S(x,t)) - (\nabla_x u(x,t))^T b(x,t)\right| \tag{1.3}$$

in $\Omega \times (0,\infty)$ and the boundary and initial conditions

$$u(x,t) = f(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.4}$$

$$S(x,0) = S^{(0)}(x), \quad x \in \Omega. \tag{1.5}$$

Moreover, the Clausius-Duhem inequality

$$\frac{\partial}{\partial t}\psi(\varepsilon(\nabla_x u), S) - \mathrm{div}_x(T u_t) - b \cdot u_t \le 0 \tag{1.6}$$

must hold in $\Omega \times (0,\infty)$. Here $\nabla_x u(x,t)$ denotes the 3×3–matrix of first order derivatives of u, the deformation gradient, $(\nabla_x u(x,t))^T$ denotes the transposed matrix and

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\left(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\right) \in \mathcal{S}^3$$

is the strain tensor. $\bar{\varepsilon} \in \mathcal{S}^3$ is a given matrix, the misfit strain, and $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite matrix, the elasticity tensor. Given are the volume force $b : \Omega \times [0,\infty) \to \mathbb{R}^3$, the boundary displacement $f : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$ and the initial data $S^{(0)} : \Omega \to \mathbb{R}$.

(1.3) is the evolution equation for the order parameter S. In this equation c is a positive constant and $C = C(\nabla_x u(x,t), S(x,t))$ denotes the Eshelby tensor defined by

$$C(\nabla_x u(x,t), S(x,t)) = \psi\big(\varepsilon(\nabla_x u(x,t)), S(x,t)\big)I - \big(\nabla_x u(x,t)\big)^T T(x,t). \quad (1.7)$$

Here $(\nabla_x u)^T T$ denotes the matrix product, I is the unit matrix in $\mathcal{S}^3$ and

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big) \cdot (\varepsilon - \bar{\varepsilon}S) + \psi_1(S) \quad (1.8)$$

is the free energy. For the function S defined above only the values of ψ_1 at $S = 0$ and $S = 1$ matter. However, as explained next, we also consider order parameters which vary smoothly between 0 and 1. For $\psi_1 \in C^1(\mathbb{R}, [0,\infty))$ we therefore choose a double well potential with minima at 0 and 1.

The evolution equation (1.3) must allow for solutions (u, T, S) with S being the characteristic function of the set γ'. For such S the derivatives S_t and S_{x_i} are measures. Therefore (1.3) is understood in the sense of measures: We seek a solution (u, T, S) such that $S \in BV^{\text{loc}}(\Omega \times (0,\infty), \mathbb{R})$ and such that to the distributional derivative $\operatorname{div}_x C(\nabla_x u, S)$ there is a Radon measure μ and a μ–measureable function $\sigma : \Omega \times (0,\infty) \to R^3$ with $|\sigma(x,t)| = 1$, μ almost everywhere, satisfying

$$\sigma\mu = \operatorname{div}_x C(\nabla_x u, S) - (\nabla_x u(x,t))^T b(x,t).$$

The measure μ is denoted by $|\operatorname{div}_x C - (\nabla_x u)^T b|$, and $|S_t|$ denotes the variation measure of the measure S_t. Of course, this definition allows for solutions with S differing from a characteristic function. Piecewise smooth S are allowed, for example. This completes the formulation of the initial-boundary value problem.

The sharp interface model. Next we introduce the sharp interface model and explain how the initial-boundary value problem (1.1) – (1.6) is derived from it.

In the sharp interface model the unknowns u, T, S satisfy the equation (1.1) expressing conservation of momentum, the equation (1.2) stating the linear stress-strain relation, and the boundary and initial conditions (1.4), (1.5). Equation (1.3) is replaced by an equation for the normal speed of the phase interface. To formulate such an equation we first study the restrictions imposed by the second law of thermodynamics, i.e. by the Clausius-Duhem inequality (1.6).

It is shown in [1, 2] that (1.6) holds if and only if at every point $x \in \tilde{\gamma}(t)$ the dissipation inequality

$$s(x,t)\Big(n(x,t) \cdot \big[C\big(\nabla_x u(x,t), S(x,t)\big)\big] n(x,t)\Big) \geq 0 \quad (1.9)$$

is satisfied, where $n(x,t) \in \mathbb{R}^3$ denotes a unit normal vector to the interface $\tilde{\gamma}(t)$ oriented such that the jump of S at $\tilde{\gamma}(t)$ in the direction of n is positive, $s(x,t)$ is the normal speed of the interface measured positive in the direction of n, and

$[C(\nabla_x u, S)]$ denotes the jump of the Eshelby tensor accross $\tilde{\gamma}(t)$ in the direction of n. This inequality implies that s and $n \cdot [C]n$ must have the same sign. This suggests to consider $n \cdot [C]n$ as driving force for the interface and to require s to be a function of this configurational force such that (1.9) is satisfied. The simplest equation guaranteeing this is

$$s(x,t)\,[S(x,t)] = c\,n(x,t) \cdot \big[C(\nabla_x u(x,t), S(x,t))\big] n(x,t), \tag{1.10}$$

with a positive constant c. By the above definitions the jump of S satisfies $[S] = 1$; this term could thus be dropped. We included it since later we allow for jumps smaller than one, in which case we need the term.

(1.10) is a constitutive equation for the normal speed of the phase interface. It has been suggested in [1] that (1.1), (1.2), (1.10) and the boundary and initial conditions (1.4), (1.5) form a closed system of equations, which allows to compute the movement of this interface. This is the sharp interface model.

The derivation of the evolution equation (1.3) from the equation (1.10) for the normal speed is based on a result proved in [2]: Assume that $(u, T, S) : \Omega \times (0, \infty) \to \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}$ is a piecewise smooth solution of (1.1) and (1.2) with a jump along a piecewise smooth manifold $\tilde{\gamma}$. The function S can vary smoothly away from $\tilde{\gamma}$ and needs not to be piecewise constant. Then if (1.3), (1.6) hold it follows that along $\tilde{\gamma}$ the jump condition (1.10) must be satisfied, whereas in regions where (u, T, S) is smooth (1.3), (1.6) reduce to the Hamilton-Jacobi transport equation

$$S_t(x,t) = -c\,\psi_S(\varepsilon(\nabla_x u(x,t)), S(x,t))|\nabla_x S(x,t)|. \tag{1.11}$$

The necessity to combine (1.3) with the Clausius-Duhem inequality (1.6) is seen here, since from (1.3) alone we can only deduce that the absolute values of both sides of the equations (1.10) and (1.11) are equal. (1.6) is thus needed to fix the signs.

In fact, if S is piecewise smooth and (u, T, S) solves (1.1) and (1.2), then (1.10) and (1.11) hold if and only if (1.3) and (1.6) are satisfied. This can be shown by a slight extension of the investigations in [2]; for one space dimension it is proved in Corollary 2.3 in the next section.

It is clear that a piecewise constant function S satisfies the transport equation (1.11) away from the jumps. Consequently, a piecewise smooth function (u, T, S) with piecewise constant S is a solution of the sharp interface model (1.1), (1.2), (1.10), (1.4), (1.5) if and only if it satisfies the relations (1.1) – (1.6). Therefore, since (1.3) is well defined even if the order parameter S is not piecewise constant, the initial-boundary value problem (1.1) – (1.6) generalizes the sharp interface model. Moreover, if the order parameter S in the solution is smooth then the evolution equation (1.3) reduces to the simpler and well known Hamilton-Jacobi equation (1.11). The idea suggests itself to force the solution to stay smooth by replacing (1.11) with the equation (1.13) derived below, which is obtained from the Hamilton-Jacobi equation by adding a small viscosity term. The hope is that when the viscosity term tends to zero, the order parameter converges to a solution of the initial-boundary value problem (1.1) – (1.6). In this case the Hamilton-Jacobi equation with the small viscosity term can be used as a phase field model regularizing the sharp interface model.

Our results on existence of solutions of the initial-boundary value problem (1.1) – (1.6) in one space dimension contribute to the problem of convergence, since one expects of course that this convergence takes place only when the limit problem (1.1) – (1.6) has solutions. Still, the problem of convergence of solutions of the model with the Hamilton-Jacobi equation when the viscosity tends to zero remains open.

Comparison to other phase field models and properties of the evolution equation for the order parameter. To compare the model discussed in this article to other models for phase transformation problems we sketch the usual derivation of these phase field models, cf. [8, 14, 10, 3]: For $\nu \geq 0$ consider the modified free energy

$$\hat{\psi}(\varepsilon(\nabla_x u), S, \nabla_x S) = \psi(\varepsilon(x,t), S(x,t)) + \nu \frac{1}{2} |\nabla_x S(x,t)|,$$

with ψ defined in (1.8). We assume that (u, T, S) is a smooth solution of the equations (1.1), (1.2). The second law of thermodynamics requires that (1.6) is satisfied with ψ replaced by $\hat{\psi}$. We integrate (1.6) over Ω and employ the Divergence Theorem to obtain

$$\frac{d}{dt} \int_\Omega \hat{\psi}(\varepsilon, S, \nabla_x S)\, dx - \int_{\partial\Omega} (Tn) \cdot u_t \, d\sigma_x - \int_\Omega b \cdot u_t \, dx \leq 0. \tag{1.12}$$

(1.2) yields $\psi_\varepsilon = T$. From the symmetry of T we thus obtain

$$\hat{\psi}_t = \psi_\varepsilon \cdot \varepsilon_t + \psi_S S_t + \nu \nabla_x S \cdot \nabla_x S_t = T \cdot \nabla_x u_t + \psi_S S_t + \nu \nabla_x S \cdot \nabla_x S_t \,.$$

We insert this equation into (1.12), use the Divergence Theorem, assume a suitable boundary condition for S and note (1.1) to deduce

$$\int_\Omega \psi_S S_t + \nu \nabla_x S \cdot \nabla_x S_t \, dx = \int_\Omega \big(\psi_S - \nu \Delta_x S\big) S_t \, dx \leq 0.$$

The standard method to ensure that this inequality holds is to postulate

$$S_t = -c(\psi_S - \nu \Delta_x S),$$

which is an evolution equation for S, the Allen-Cahn equation with terms coupling to the equations (1.1) and (1.2). However, this inequality is as well satisfied if we instead postulate

$$S_t = -c(\psi_S - \nu \Delta_x S) |\nabla_x S|, \tag{1.13}$$

which for $\nu = 0$ is the Hamilton-Jacobi equation (1.11). The Allen-Cahn equation is used when diffusion playes an important role, whereas the Hamilton-Jacobi equation is the right equation when the interfaces are driven by configurational forces. This is seen from the above investigations. Thus, the indeterminateness in the standard method allows to formulate phase field models for both situations.

The Cahn-Hilliard equation is used when diffusion is the dominating process and the order parameter is conserved. It is derived in a similar, but slightly more complicated way than the Allen-Cahn equation. Just as above we can modify this derivation and introduce the term $|\nabla_x S|$ in the evolution equation. This suggests that the resulting equation is a model valid when the interfaces are driven by configurational forces and the order parameter is conserved. We do not dwell on this question here, but only mention for comparison that the model consisting of the Cahn-Hilliard equation coupled to the equations (1.1), (1.2) and related models are formulated or investigated mathematically in [19, 9, 14, 15, 7, 11], for example.

We surmise that solutions of the equations (1.1), (1.2), (1.13) with $\nu > 0$ are smooth and approximate solutions of the system (1.1) – (1.3), (1.6) for $\nu \to 0$. These three equations would thus form a phase field model regularizing the sharp interface model (1.1), (1.2), (1.10).

Of course, it is not immediately obvious whether it is really necessary to add the term $c\nu\, \Delta_x S\, |\nabla_x S|$ to the Hamilton-Jacobi equation for getting smooth solutions. Namely, it is tempting to prove existence of smooth solutions for the initial-boundary value problem (1.1), (1.2), (1.11), (1.4), (1.5) by using the method of viscosity solutions to solve (1.11), combined with methods for elliptic systems to solve the other equations. Yet, since ψ_1 in (1.8) is a double well potential, the function $S \mapsto \psi_S(\varepsilon, S)$ in (1.11) is not monotone; therefore the assumptions needed to apply comparison arguments and to prove existence of continuous viscosity solutions of (1.11) are not satisfied. Instead, simple examples show that S develops discontinuities even if the initial data are smooth. Consequently, the theory of discontinuous viscosity solutions has to be used. It turns out, however, that the standard definition of discontinuous viscosity solutions (cf. [17, 6]) allows too much freedom for the propagation speed of the phase interfaces modelled by jump discontinuities of S.

This can be seen best if we study jump discontinuities for a problem in one space dimension. Note first that for a piecewise smooth solution (u, T, S) of (1.1) – (1.6) we have

$$[\nabla_x u(x,t)]\, \tau(x,t) = [T(x,t)]\, n(x,t) = 0, \quad (x,t) \in \tilde{\gamma},$$

for all tangential vectors $\tau(x,t)$ to $\tilde{\gamma}$. With these equations the right hand side of (1.10) can be simplified by a short computation to obtain

$$s[S] = c\, n \cdot [C]\, n = \frac{c}{2}\Big([T] \cdot \langle \varepsilon \rangle - \langle T \rangle \cdot [\varepsilon] - [T \cdot \bar{\varepsilon} S] \Big) + c\, [\psi_1(S)], \tag{1.14}$$

with $\langle \varepsilon \rangle = \frac{1}{2}(\varepsilon(\nabla_x u+) + \varepsilon(\nabla_x u-))$. Here $w(x,t)+ = \lim_{y \to x,\ y \in \gamma'(t)} w(y,t)$ and $w(x,t)- = \lim_{y \to x,\ y \in \gamma(t)} w(y,t)$ are the limit values on both sides of $\tilde{\gamma}(t)$. It can be seen from Lemma 2.1 in Section 2 that if we reduce (1.1) – (1.6) to a problem in one space dimension with a scalar function T then T is continuous across the phase interface. This implies $[\varepsilon] = [u_x] = \bar{\varepsilon}\, [S]$. If we denote by $\bar{s}$ the speed of propagation of discontinuities measured in the positive x–direction, we thus obtain from (1.14)

$$\overline{s} = c\left(\frac{[\psi_1]}{[S]} - T\cdot\bar{\varepsilon}\right)\frac{S^+ - S^-}{|S^+ - S^-|}. \tag{1.15}$$

Here S^+ and S^- are the values of S to the right and to the left of the jump discontinuity. On the other hand, noting that in (1.11)

$$c\,\psi_S(\varepsilon, S) = c\left(\psi_1'(S) - T\cdot\bar{\varepsilon}\right), \tag{1.16}$$

the definition of discontinuous viscosity solutions implies that any jump discontinuity is allowed whose normal speed $\overline{s}$ satisfies the two inequalities

$$c\Big(\psi_1'(S^-) - T\cdot\bar{\varepsilon}\Big)\frac{S^+ - S^-}{|S^+ - S^-|} \geq \overline{s} \geq c\Big(\psi_1'(S^+) - T\cdot\bar{\varepsilon}\Big)\frac{S^+ - S^-}{|S^+ - S^-|}\,.$$

Since $c\left(\psi_1'(S) - T\cdot\bar{\varepsilon}\right)$ is the speed of characteristics of (1.11), these two inequalities require that the characteristic curves must end in the jump discontinuity on both sides, and thus allow for any normal speed of the discontinuity between the two charateristic speeds to the left and to the right of the discontinuity. Therefore discontinuities in viscosity solutions do not need to have the velocity given by (1.15). Yet, if (1.15) is not satisfied then phase interfaces are not modeled correctly. This implies that to construct discontinuous viscosity solutions we must use a construction procedure which automatically selects the right speed of propagation. We surmise that the usual construction procedure based on Perron‘s method does not satisfy this requirement.

Therefore we use another method to prove existence of solutions of the initial-boundary value problem in one space dimension, which is based on the similarity of equation (1.3) to a hyperbolic balance law. For the problem in one space dimension the similarity becomes even greater, cf. (1.19) below. The main difference to a hyperbolic balance law lies in the absolute value signs on both sides of (1.19). The mapping which assigns to the measures S_t and $C_1(u_x, S)_x - u_x\cdot b$ the variation measures is nonlinear, and thus is discontinuous with respect to weak convergence, in general. Thus, while in the investigation of conservation laws the main difficulties are connected with the function $S \mapsto C_1(u_x, S)$, which is nonlinear, whence discontinuous with respect to weak convergence, new difficulties arise in the investigation of (1.19) due to the variation measures. In our existence proof we use ideas from the shock tracking method in hyperbolic conservation laws to construct a sequence of approximate solutions, but because of this difficulty completely new ideas are needed when going to the limit.

Statement of the main results. In the remainder of this article we assume that all functions in the initial-boundary value problem (1.1) – (1.6) only depend on the x_1 and t variables, but are independent of the x_2 and x_3 variables. To simplify the notation we therefore write x instead of x_1, and assume that $\Omega = (a, b) \subset \mathbb{R}$ is a bounded open interval. By T_e we denote a positive number (time of existence), and we set

$$Z_{T_e} = (a, b)\times(0, T_e), \quad Z = (a, b)\times(0, \infty).$$

We still allow that the material points can be displaced in three space directions, hence $u(x,t) \in \mathbb{R}^3$, $T(x,t) \in \mathcal{S}^3$, $S \in \mathbb{R}$. If we denote the first column of the matrix $T(x,t)$ by $T_1(x,t)$ and set

$$\begin{aligned} \varepsilon(u_x) &= \frac{1}{2}\left((u_x,0,0)+(u_x,0,0)^T\right) \in \mathcal{S}^3, \\ C_1(u_x,S) &= \psi(\varepsilon(u_x),S) - u_x \cdot T_1\,, \end{aligned}$$

then (1.1) – (1.6) can be written in the slightly simplified form:

$$-T_{1x} = b, \tag{1.17}$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon}S), \tag{1.18}$$

$$|S_t| = c\,|C_1(u_x,S)_x - u_x \cdot b|\,, \tag{1.19}$$

$$u(a,t) = f(a,t), \quad u(b,t) = f(b,t), \quad t \geq 0, \tag{1.20}$$

$$S(x,0) = S^{(0)}(x), \quad x \in [a,b], \tag{1.21}$$

$$\frac{\partial}{\partial t}\psi(\varepsilon(u_x),S) - (T_1 \cdot u_t)_x - b \cdot u_t \leq 0. \tag{1.22}$$

For this initial-boundary value problem we prove existence of solutions to three different classes of initial data. To formulate these existence results in the next two lemmas and in Theorem 1.3 we need solutions of the boundary value problem of linear elasticity theory in one space dimension. This problem is

$$-\sigma_{1x}(x) = \hat{b}(x), \quad a < x < b, \tag{1.23}$$

$$\sigma(x) = D\varepsilon(w_x(x)), \quad a < x < b, \tag{1.24}$$

$$w(a) = \hat{f}(a), \quad w(b) = \hat{f}(b). \tag{1.25}$$

Let $H_i(W)$ be the usual Sobolev spaces of functions with quadratically integrable weak derivatives up to order i, where $W \subset \mathbb{R}^n$ is a Lebesgue measurable set. The norms of these spaces are denoted by $\|v\|_{i,W}$. The L^2–norm is $\|v\|_{0,W} = \|v\|_W$.

Lemma 1.1 (Piecewise constant initial data) *Let $b, f : Z \to \mathbb{R}^3$ satisfy $b \in H_2(Z_{T_e}, \mathbb{R}^3)$ and $f \in H_2(\{a,b\} \times [0,T_e], \mathbb{R}^3)$ for all $T_e > 0$. Assume that $S^{(0)} : [a,b] \to [0,1]$ is piecewise constant with finitely many jumps, which all lie in the interior of $[a,b]$.*

Then there is a weak solution $(u,T,S) : Z \to \mathbb{R}^3 \times \mathcal{S}^3 \times [0,1]$ of (1.17) – (1.22). The function S in this solution is piecewise constant and belongs to the space $BV(Z_{T_e})$. Moreover, (u,T) satisfies

$$\begin{aligned} u(x,t) &= u^*\left(\int_a^x S(y,t)dy - \frac{x-a}{b-a}\int_a^b S(y,t)dy\right) + w(x,t), \\ T(x,t) &= D(\varepsilon^* - \bar{\varepsilon})S(x,t) - D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy + \sigma(x,t), \end{aligned}$$

where $u^* \in \mathbb{R}^3$, $\varepsilon^* \in \mathcal{S}^3$ *only depend on the misfit strain* $\overline{\varepsilon}$, *and where* $(w(t), \sigma(t))$ *is the unique solution of the boundary value problem* (1.23) – (1.25) *to the data* $\hat{b} = b(t)$, $\hat{f} = f(t)$ *for every* $t > 0$. *This solution satisfies*

$$(w, \sigma) \in \bigcap_{i=0}^{2} H_{2-i}\big((0, T_e), H_{2+i}((a,b), \mathbb{R}^3) \times H_{1+i}((a,b), \mathcal{S}^3)\big).$$

for all $T_e > 0$.

Lemma 1.2 (Monotonic initial data) *Let* b *and* f *satisfy the assumptions of the preceding lemma. Assume that* $S^{(0)} : [a,b] \to [0,1]$ *is a continuous monotonic function. Then there is a weak solution* $(u, T, S) : Z \to \mathbb{R}^3 \times \mathcal{S}^3 \times [0,1]$ *of* (1.17) – (1.22), *where* (u, T) *is of the same form as in the preceding lemma, and where* $S \in BV(Z_{T_e})$. *Moreover, the function* $S(t) : [a,b] \to [0,1]$ *defined by* $x \mapsto S(x,t)$ *is monotonic and satisfies*

$$\operatorname{var}(S(t)) \le \operatorname{var}(S^{(0)})$$

for almost all $t \in [0, T_e]$.

Our main result is

Theorem 1.3 (Nonmonotone initial data) *Assume that* $b \in H_2(Z_{T_e}, \mathbb{R}^3)$ *and* $f \in H_2(\{a,b\} \times [0,T_e], \mathbb{R}^3)$ *are given functions. Let* $a < c < b$ *and assume that* $S^{(0)} \in C([a,b],[0,1])$ *is increasing in* $[a,c]$ *and decreasing in* $[c,b]$. *For every* $t > 0$ *let* $(w(t), \sigma(t))$ *be the solution of the boundary value problem* (1.23) – (1.25) *to the data* $\hat{b} = b(t)$, $\hat{f} = f(t)$. *If there is a constant* $M_1 > 0$ *such that this solution satisfies*

$$-\overline{\varepsilon} \cdot \sigma(x,t) \ge M_1 + \max_{0 \le s \le 1} |\psi_1'(s)| \tag{1.26}$$

for almost all (x,t), *then there is a weak solution* (u, T, S) *of* (1.17) – (1.22), *for which* $S \in BV(Z_{T_e})$ *and for which* (u, T) *is of the same form as in Lemma 1.1.*

Remark. We surmise that the result of Theorem 1.3 holds without condition (1.26), and that we need this condition only for technical reasons. This condition guarantees that the characteristic speeds of (1.3) and the speeds of jump discontinuities in solutions of (1.3) are bounded away from 0. Moreover, it guarantees that these discontinuities are directed such that the embedded phase asymptotically vanishes for $t \to \infty$, i.e. that S tends asymptotically to the value $\min_{a \le x \le b} S^{(0)}(x)$. If the first row of the matrix $D\overline{\varepsilon}$ does not vanish then the condition can always be guaranteed to hold by choosing suitable data b and f.

If the scalar product $\overline{\varepsilon} \cdot \sigma(x,t)$ is zero for all (x,t) and all boundary data $\hat{f}$ and right hand sides $\hat{b}$ in (1.23) – (1.25), then the order parameter S in the solution is independend of the boundary tractions and the volume force, hence the phase evolution is independent of the exterior forces. This is in accordance with the experimental observation that the phase evolution depends on the misfit strain $\overline{\varepsilon}$ in relation to the direction of the exterior stress field. Note that by considering a one dimensional problem we have intrinsically fixed a direction for the exterior forces.

2 Piecewise constant initial data

Here we prove Lemma 1.1. The proof is based on the observation that the jump condition (1.10), which must hold along any jump curve of S, yields a differential equation in time for this jump curve. To determine the finitely many jump curves of S we must therefore solve a coupled system of differential equations. This system contains the unknown function T and is therefore not closed. To close it we observe that if the function $S(t)$ is known for a fixed time t, then the equations (1.17), (1.18) and (1.20) form a boundary value problem for the functions $u(t)$ and $T(t)$, a slight extension of the boundary value problem of linear elasticity theory, which in one space dimension can be solved explicitly. Insertion of the explicit solution formulas into the system of ordinary differential equations closes the system.

In the first step of the proof we thus derive the explicit solution formulas for (1.17), (1.18), (1.20). Subsequently we derive the system of ordinary differential equations and discuss the construction of the jump curves. In the last step we verify that the function (u, T, S) constructed in this way satisfies all the equations (1.17) – (1.22) and thus is a solution of this initial-boundary value problem.

We begin with some notations: Let $\hat{\mathcal{S}}^3$ denote the subspace of all matrices $A \in \mathcal{S}^3$ with $A_{ij} = 0$ for $i, j = 2, 3$. The orthogonal space to $\hat{\mathcal{S}}^3$ is denoted by $\tilde{\mathcal{S}}^3$. It consists of all $A \in \mathcal{S}^3$ satisfying $A_{i1} = A_{1i} = 0$ for $i = 1, 2, 3$. Note that $\varepsilon(u_x(x,t)) \in \hat{\mathcal{S}}^3$. For the canonical projection of $\mathcal{S}^3$ onto $\hat{\mathcal{S}}^3$ we write $\hat{P}$. Since $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a positive definite linear mapping, $\langle \sigma, \tau \rangle = \sigma \cdot D\tau$ defines a scalar product on $\mathcal{S}^3$. The projection of $\mathcal{S}^3$ onto $\hat{\mathcal{S}}^3$, which is orthogonal with respect to this scalar product is denoted by $\hat{Q}$. These definitions imply

$$\ker \hat{Q} = D^{-1}\tilde{\mathcal{S}}^3 = D^{-1} \ker \hat{P}. \tag{2.1}$$

Lemma 2.1 *Let $\overline{\varepsilon} \in \mathcal{S}^3$, $b \in H_2(Z_{T_e}, \mathbb{R}^3)$, $f \in H_2(\{a, b\} \times [0, T_e], \mathbb{R}^3)$ and the measurable function $S : Z_{T_e} \to [0, 1]$ be given. Define the matrix $\varepsilon^* \in \hat{\mathcal{S}}^3$ and the vector $u^* \in \mathbb{R}^3$ by*

$$\varepsilon^* = \hat{Q}\overline{\varepsilon}, \qquad u^* = (\varepsilon^*_{11}, 2\varepsilon^*_{21}, 2\varepsilon^*_{31}). \tag{2.2}$$

Then the boundary value problem

$$\begin{aligned} -T_1(x,t)_x &= b(x,t), & (2.3)\\ T(x,t) &= D(\varepsilon(u_x(x,t)) - \bar{\varepsilon}S(x,t)), & (2.4)\\ u(a,t) &= f(a,t), \quad u(b,t) = f(b,t), & (2.5)\end{aligned}$$

has a unique solution (u,T) *in* Z_{T_e} *given by*

$$\begin{aligned}
u(x,t) &= u^*\left(\int_a^x S(y,t)dy - \frac{x-a}{b-a}\int_a^b S(y,t)dy\right) + w(x,t), && (2.6)\\
\varepsilon(u_x(x,t)) &= \varepsilon^*\left(S(x,t) - \frac{1}{b-a}\int_a^b S(y,t)dy\right) + \varepsilon(w_x(x,t)), && (2.7)\\
T(x,t) &= D(\varepsilon^* - \bar\varepsilon)S(x,t) - D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy + \sigma(x,t), && (2.8)\\
\hat P T(x,t) &= \hat P\left(\sigma(x,t) - D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy\right). && (2.9)
\end{aligned}$$

Here $(w(t),\sigma(t))$ *is the unique solution of the boundary value problem* (1.23) – (1.25) *to the data* $\hat b = b(t)$, $\hat f = f(t)$ *for every* $t > 0$. *We have*

$$(w,\sigma) \in \bigcap_{i=0}^{2} H_{2-i}\big((0,T_e), H_{2+i}((a,b),\mathbb{R}^3) \times H_{1+i}((a,b),\mathcal{S}^3)\big).$$

Proof. We define v and τ by

$$\begin{aligned}
v(x,t) &= u^*\left(\int_a^x S(y,t)dy - \frac{x-a}{b-a}\int_a^b S(y,t)dy\right), && (2.10)\\
\tau(x,t) &= D(\varepsilon(v_x(x,t)) - \overline{\varepsilon}S(x,t)). && (2.11)
\end{aligned}$$

The definition of u^* in (2.2) implies

$$\varepsilon(v_x(x,t)) = \varepsilon^*\left(S(x,t) - \frac{1}{b-a}\int_a^b S(y,t)dy\right), \tag{2.12}$$

whence, from (2.11)

$$\tau(x,t) = D(\varepsilon^* - \bar\varepsilon)S(x,t) - D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy. \tag{2.13}$$

(2.1) implies $\hat P D(\hat Q - I)\mathcal{S}^3 = \hat P D \ker \hat Q = \{0\}$, hence the definition of ε^* in (2.2) yields $\hat P D(\varepsilon^* - \overline{\varepsilon})S = \hat P D(\hat Q - I)\overline{\varepsilon}S = 0$. Application of $\hat P$ to (2.13) thus results in

$$\hat P\tau(x,t) = -\hat P D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy\,, \tag{2.14}$$

which is constant with respect to x. Thence $\tau_1(x,t)$ is constant with respect to x. Consequently, the function $(v(t),\tau(t))$ solves the system

$$\begin{aligned}
-\tau_{1x} &= 0,\\
\tau &= D(\varepsilon(v_x) - \bar\varepsilon S),\\
v(a,t) &= v(b,t) = 0.
\end{aligned}$$

It is immediately seen and well known that for $\hat{b} = b(t) \in H_2([a,b],\mathbb{R}^3)$ and for $\hat{f} = f(t)$ the system (1.23) – (1.25) has a unique solution $(w(t),\sigma(t)) \in H_4([a,b],\mathbb{R}^3) \times H_3([a,b],\mathcal{S}^3)$. Clearly, $(u,T) = (v+w,\tau+\sigma)$ is the unique solution of (2.3) – (2.5). Equations (2.6) – (2.8) follow from (2.10), (2.12) and (2.13), equation (2.9) is implied by (2.14), noting that $T = \tau + \sigma$. This proves the lemma.

The jump condition in one space dimension. In the jump condition (1.10) the speed of propagation s of the discontinuity is measured positive in the direction of increasing S. Thus, if for a function w with a jump we write $[w] = w^+ - w^-$, where w^+, w^- are the values to the right and to the left of the jump, and if we denote by $\overline{s}$ the velocity measured positive in positive x–direction, we obtain $[S]s = |[S]|\overline{s}$. Using this equation and the definition of C_1 we see that in one space dimension (1.10) becomes

$$|[S]|\overline{s} = c\,[C_1(u_x,S)]. \tag{2.15}$$

The function (u,T) in this formula is the solution of the boundary value problem (2.3) – (2.5) to the function $S(t)$. Since $u_x \cdot T_1 = \varepsilon(u_x) \cdot T$, the computation which leads to (1.14) remains valid in one space dimension and can be used to evaluate the jump $[C_1]$. To determine an explicit formula for this jump we use (2.6) – (2.9): Since $\sigma \in H_2(Z_{T_e},\mathcal{S}^3)$, it follows from (2.9) and from the Sobolev embedding theorem that $\hat{P}T(t)$ is a continuous function of x, whence $[\hat{P}T] = 0$. From this relation and from $\varepsilon(u_x(x,t)) \in \hat{\mathcal{S}}^3$, which implies $\langle\varepsilon\rangle = \frac{1}{2}(\varepsilon(u_x^+) + \varepsilon(u_x^-)) \in \hat{\mathcal{S}}^3$, we obtain

$$[T]\cdot\langle\varepsilon\rangle = [T]\cdot\langle\hat{P}\varepsilon\rangle = [\hat{P}T]\cdot\langle\varepsilon\rangle = 0, \quad \langle T\rangle\cdot[\varepsilon] = \langle T\rangle\cdot[\hat{P}\varepsilon] = \hat{P}T\cdot[\varepsilon].$$

Using these equations and $[\varepsilon] = \varepsilon^*[S]$, which is implied by (2.7), we obtain from (1.14) by insertion of (2.8) and (2.9) that

$$\begin{aligned}
[C_1] &= \frac{1}{2}\big([T]\cdot\langle\varepsilon\rangle - \langle T\rangle\cdot[\varepsilon] - [T\cdot\bar{\varepsilon}S]\big) + [\psi_1(S)] \\
&= -\frac{1}{2}\big([T\cdot\bar{\varepsilon}S] + \hat{P}T\cdot\varepsilon^*[S]\big) + [\psi_1(S)] \\
&= -\frac{1}{2}\bar{\varepsilon}\cdot D(\varepsilon^* - \bar{\varepsilon})[S^2] \qquad\qquad (2.16)\\
&\quad + \frac{1}{2}(\varepsilon^* + \bar{\varepsilon})\cdot\left(D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy - \sigma\right)[S] + [\psi_1(S)] = [\Psi]\,.
\end{aligned}$$

Here Ψ is a new potential, which we can write in the form

$$\begin{aligned}
\Psi\big(x,t,S,\textstyle\int_a^b S(y,t)dy\big) &= \frac{1}{2}(\bar{\varepsilon} - \varepsilon^*)\cdot D(\bar{\varepsilon} - \varepsilon^*)\,S^2 + \psi_1(S) \qquad\qquad (2.17)\\
&\quad + \left(\varepsilon^*\cdot D\varepsilon^*\frac{1}{b-a}\int_a^b S(y,t)dy - \bar{\varepsilon}\cdot\sigma(x,t)\right)S\,.
\end{aligned}$$

To see this note that $\varepsilon^* = \hat{Q}\overline{\varepsilon}$, that the projection $\hat{Q}$ is orthogonal with respect to the scalar product $\langle \cdot, \cdot\cdot \rangle$ and that $\sigma = D\varepsilon(w_x)$, which relations together imply

$$\begin{aligned}
&\overline{\varepsilon} \cdot D(\overline{\varepsilon} - \varepsilon^*) = \langle \overline{\varepsilon}, (I - \hat{Q})\overline{\varepsilon} \rangle = \langle (I - \hat{Q})\overline{\varepsilon}, (I - \hat{Q})\overline{\varepsilon} \rangle = (\overline{\varepsilon} - \varepsilon^*) \cdot D(\overline{\varepsilon} - \varepsilon^*) \,, \\
&(\overline{\varepsilon} + \varepsilon^*) \cdot D\varepsilon^* = \langle \overline{\varepsilon} + \varepsilon^*, \hat{Q}\varepsilon^* \rangle = \langle \hat{Q}(\overline{\varepsilon} + \varepsilon^*), \varepsilon^* \rangle = 2\varepsilon^* \cdot D\varepsilon^* \,, \\
&\varepsilon^* \cdot \sigma = \langle \hat{Q}\overline{\varepsilon}, \varepsilon(w_x) \rangle = \langle \overline{\varepsilon}, \hat{Q}\varepsilon(w_x) \rangle = \langle \overline{\varepsilon}, \varepsilon(w_x) \rangle = \overline{\varepsilon} \cdot \sigma \,.
\end{aligned}$$

From (2.15) and (2.16) we finally obtain for the speed of propagation

$$\overline{s}(x,t) = c\, \frac{\Big[\Psi(x,t,S,\int_a^b S(y,t)dy)\Big]}{|[S]|}. \tag{2.18}$$

Construction of the solution. Let $S^{(0)}$ be the piecewise constant initial data given in Lemma 1.1 and assume that $S^{(0)}$ has jumps at the points $x_1, \cdots, x_n$ with $a < x_1 < \cdots < x_n < b$. We set $x_0 = a, x_{n+1} = b$. Let S_i^+ and S_i^- be the constant values of $S^{(0)}$ to the right and to the left of x_i for $i = 1, 2, \cdots, n$. Obviously we have $S_i^+ = S_{i+1}^-$ for $i = 1, 2, \cdots, n-1$. In the domain $[a,b] \times [0, t_1]$ with a suitable time $t_1 > 0$ to be determined below the component S of the solution (u, T, S) will be piecewise constant with jumps along curves given by the graphs of continuously differentiable functions $\alpha_i : [0, t_1] \to [a, b]$, $i = 1, 2, \cdots, n$. The curve α_i starts at the discontinuity $(x_i, 0)$ of $S^{(0)}$ and we have

$$a < \alpha_i(t) < \alpha_{i+1}(t) < b, \quad 0 \le t < t_1, \quad i = 1, 2, \cdots, n-1.$$

The values of S are defined by

$$S(x,t) = \begin{cases} S_1^-, & a \le x < \alpha_1(t), \\ S_i^+, & \alpha_i(t) \le x < \alpha_{i+1}(t), \quad i = 1, 2, \cdots, n-1, \\ S_n^+, & \alpha_n(t) \le x \le b. \end{cases} \tag{2.19}$$

The discontinuities α_i have the speed of propagation given by (2.18). Thus,

$$\frac{d}{dt}\alpha_i(t) = c\, \frac{\Psi\Big(\alpha_i(t), t, S_i^+, \int_a^b S(y,t)dy\Big) - \Psi\Big(\alpha_i(t), t, S_i^-, \int_a^b S(y,t)dy\Big)}{|S_i^+ - S_i^-|}, \tag{2.20}$$

for $i = 1, \ldots, n$. If we note that by (2.19)

$$\int_a^b S(y,t)dy = \sum_{i=0}^{n} \big(\alpha_{i+1}(t) - \alpha_i(t)\big) S_i^+,$$

where we use the notations $\alpha_0(t) = a,\ \alpha_{n+1}(t) = b,\ S_0^+ = S_1^-$, we see that (2.20) is a system of ordinary differential equations for the functions $\alpha_1, \ldots, \alpha_n$. Lemma 2.1 implies that the function σ in (2.17) satisfies $\sigma_x \in H_2(Z_{T_e}, \mathcal{S}^3)$. The Sobolev imbedding theorem thus yields that $\sigma \in C(Z_{T_e})$. From this fact and from the definition of Ψ in (2.17) it follows that the right hand side of this system is defined for $a \leq \alpha_1, \ldots, \alpha_n \leq b$, that it is continuous with respect to $(t, \alpha_1, \ldots, \alpha_n) \in [0,\infty)\times[a,b]^n$ and that it satisfies a Lipschitz condition with respect to $(\alpha_1, \ldots, \alpha_n)$. By the Theorem of Picard-Lindelöf it thus follows that there exists a unique solution $(\alpha_1(t), \ldots, \alpha_n(t))$. The solution is continuously differentiable with respect to t. Let

$$t_1 = \sup\big\{t > 0 \mid a < \alpha_1(t) < \alpha_2(t) < \ldots < \alpha_n(t) < b\big\}$$

and define S in $[a,b] \times [0, t_1)$ by (2.19) and (2.20). If $t_1 < \infty$ let

$$S^{(1)}(x) = \begin{cases} \lim_{y\downarrow x} S(y,t_1), & a \leq x < b, \\ \lim_{y\uparrow b} S(y,t_1), & x = b. \end{cases}$$

We define S in the region $[a,b] \times [t_1, t_2)$ with a suitable time $t_2 > t_1$ by repeating the above construction, using $S^{(1)}$ as initial data.

The number of curves in the domain $[a,b] \times [t_1, t_2)$, along which S jumps, is smaller than in the domain $[a,b] \times [0,t_1)$. Therefore, after further repetition of this process we find that there is a largest time t_m such that the next step yields the solution in all of the domain $[a,b]\times[t_m,\infty)$, or that $S^{(m)} : [a,b] \to [0,1]$ is constant. In this case we set S equal to this constant value in all of $[t_m,\infty)$. This completes the construction of the function $S : [a,b] \times [0,\infty) \to [0,1]$.

Let (u,T) be the unique solution of the problem (2.3) – (2.5) to the function S. In the remainder of this section we show that the function $(u,T,S) : [a,b]\times[0,\infty) \to \mathbb{R}^3 \times \mathcal{S}^3 \times [0,1]$ thus defined is a solution of the initial-boundary value problem (1.17) – (1.22).

The measure valued derivatives. Since S has jumps, the first distributional derivatives of S and of other functions depending on S are measures. To study these measures we introduce some notations: Let α be one of the continuously differentiable curves along which S jumps. We identify this curve with the function $\alpha : [t_i, t_{i+1}] \to (a,b)$ which parametrizes α, and with the graph of α, a subset of Z. Any such curve is called a jump curve. By $\mathcal{J}$ we denote the finite set of all jump curves, and we define

$$J = \bigcup_{\alpha\in\mathcal{J}} \alpha \subseteq Z.$$

The one-dimensional Hausdorff measure $\mathcal{H}^1$ restricted to J is denoted by $\mathcal{H}_J$. Hence,

$$\mathcal{H}_J(V) = \mathcal{H}^1(J \cap V)$$

for every measurable subset $V \subseteq Z$. If $g : J \to \mathbb{R}$ is locally $\mathcal{H}_J$-summable and if K is compact we write

$$(g\,\mathcal{H}_J)(K) = \int_K g\,d\,\mathcal{H}_J.$$

For a function $v : Z \to \mathbb{R}$, which has jumps along the curves $\alpha \in \mathcal{J}$ and has weak L^2-derivatives in $Z\backslash J$, we denote by v_x, v_t the distributional derivatives and by v'_x, v'_t the L^2-derivatives away from the jump curves.

Finally, if (x,t) is a point of a jump curve different from the starting point and the endpoint we denote by $n(x,t) = (n'(x,t),\, n''(x,t)) \in \mathbb{R}^2$ the unit normal vector with $n'(x,t) > 0$.

Lemma 2.2 *Let $\tilde{S}$ be a piecewise continuously differentiable function with jumps along the curves in $\mathcal{J}$, and let $(\tilde{u}, \tilde{T})$ be the solution of the problem* (2.3) – (2.5) *to the function $\tilde{S}$ and to $b \in H_2(Z_{T_e}, \mathbb{R}^3)$, $f \in H_2(\{a,b\} \times [0,T_e], \mathbb{R}^3)$, for all $T_e > 0$. Then the distributional derivatives $\tilde{S}_t$, $C_1(\varepsilon(\tilde{u}_x), \tilde{S})_x - \tilde{u}_x \cdot b$ and $\psi(\varepsilon(\tilde{u}_x), \tilde{S})_t - (\tilde{T}_1 \cdot \tilde{u}_t)_x - b \cdot \tilde{u}_t$ are measures on Z and satisfy*

$$
\begin{aligned}
\tilde{S}_t &= n''[\tilde{S}]\, \mathcal{H}_J + \tilde{S}'_t \lambda, & (2.21)\\
C_{1x} - \tilde{u}_x \cdot b &= n'[C_1]\, \mathcal{H}_J + \psi_S \tilde{S}'_x\, \lambda, & (2.22)\\
\psi_t - (\tilde{T}_1 \cdot \tilde{u}_t)_x - b \cdot \tilde{u}_t &= -\frac{d\alpha}{dt}\, n'[C_1]\, \mathcal{H}_J + \psi_S\, \tilde{S}'_t\, \lambda, & (2.23)
\end{aligned}
$$

where λ is the Lebesgue measure.

Proof. (2.21) is immediately obtained by partial integration. To prove (2.22) observe first that away from the jump curves of $\tilde{S}$ the function $x \mapsto \tilde{u}(x,t)$ has two weak L^2-derivatives and $x \mapsto \tilde{T}(x,t)$ has one weak L^2-derivative, by Lemma 2.1. Thus, if $\tilde{S}$ is continuously differentiable in a neighborhood of (x,t), then

$$
\begin{aligned}
C_1(\tilde{u}_x, \tilde{S})'_x &= \big(\psi(\varepsilon(\tilde{u}_x), \tilde{S}) - \tilde{u}_x \cdot T_1\big)'_x = \psi_\varepsilon \cdot \varepsilon(\tilde{u}_{xx}) + \psi_S \tilde{S}'_x - \tilde{u}_{xx} \cdot \tilde{T}_1 - \tilde{u}_x \cdot \tilde{T}_{1x}\\
&= \tilde{T} \cdot \varepsilon(\tilde{u}_{xx}) - \tilde{u}_{xx} \cdot \tilde{T}_1 + \tilde{u}_x \cdot b + \psi_S \tilde{S}'_x = \tilde{u}_x \cdot b + \psi_S \tilde{S}'_x\,, \qquad (2.24)
\end{aligned}
$$

where we used that $\psi_\varepsilon = \tilde{T}$ and that $\varepsilon(\tilde{u}_{xx}) \cdot \tilde{T} = \tilde{u}_{xx} \cdot \tilde{T}_1$. We also applied (2.3).

Now let $\varphi \in \mathring{C}_1(Z, \mathbb{R})$. Partial integration and application of (2.24) yields

$$
\begin{aligned}
&\int_Z \big(-C_1(\tilde{u}_x, \tilde{S})\, \varphi_x - \tilde{u}_x \cdot b\, \varphi\big) d\lambda\\
&\quad = \int_Z \big(C_1(\tilde{u}_x, \tilde{S})'_x - \tilde{u}_x \cdot b\big)\varphi\, d\lambda + \int_Z (C_1^+ - C_1^-) n' \varphi\, d\mathcal{H}_J\\
&\quad = \int_Z \psi_S \tilde{S}'_x \varphi\, d\lambda + \int_Z n'[C_1] \varphi\, d\mathcal{H}_J\,.
\end{aligned}
$$

This implies that $C_{1x} - \tilde{u}_x \cdot b$ is a measure given by the right hand side of (2.22).

To prove (2.23) note that Lemma 2.1 implies $\tilde{u}_{tx} = \tilde{u}_{xt}$ away from the jumps of $\tilde{S}$. We thus obtain

$$
\begin{aligned}
&\psi\big(\varepsilon(\tilde{u}_x), \tilde{S}\big)'_t - (\tilde{T}_1 \cdot \tilde{u}_t)'_x - b \cdot \tilde{u}_t\\
&\quad = \psi_\varepsilon \cdot \varepsilon(\tilde{u}_{xt}) + \psi_S \tilde{S}'_t - \tilde{T}_{1x} \cdot \tilde{u}_t - \tilde{T}_1 \cdot \tilde{u}_{tx} - b \cdot \tilde{u}_t \qquad (2.25)\\
&\quad = \tilde{T} \cdot \varepsilon(\tilde{u}_{tx}) - \tilde{T}_1 \cdot \tilde{u}_{tx} + \psi_S \tilde{S}'_t = \psi_S \tilde{S}'_t\,,
\end{aligned}
$$

where we again used that $\psi_\varepsilon = T$ and that $\tilde{T} \cdot \varepsilon(\tilde{u}_{tx}) = \tilde{T}_1 \cdot \tilde{u}_{tx}$. We also used (2.3). Since $n = (n', n'')$ denotes the normal vector to a jump curve $\alpha \in \mathcal{J}$ with $n' > 0$, we conclude from

$$\frac{d\alpha}{dt} = -\frac{n''}{n'} \tag{2.26}$$

that $\operatorname{sign} n'' = -\operatorname{sign} \frac{d\alpha}{dt}$. This implies for $(x,t) \in \alpha$ that

$$\lim_{r \nearrow t} \psi(x,r) = \begin{cases} \psi^+, & n'' < 0\,, \\ \psi^-, & n'' > 0\,. \end{cases} \tag{2.27}$$

Using this equation and (2.25) we obtain for $\varphi \in \overset{\circ}{C}_1(Z,\mathbb{R})$ with $\varphi \geq 0$

$$\begin{aligned} &\int_Z \Big(-\psi\big(\varepsilon(\tilde{u}_x), \tilde{S}\big)\varphi_t + (\tilde{T}_1 \cdot \tilde{u}_t)\varphi_x - b \cdot \tilde{u}_t\, \varphi \Big)\, d\lambda \\ &\quad = \int_Z \big(\psi_t' - (\tilde{T}_1 \cdot \tilde{u}_t)_x' - b \cdot \tilde{u}_t\big)\varphi\, d\lambda + \int_Z \big([\psi] n'' - [\tilde{T}_1 \cdot \tilde{u}_t] n'\big)\varphi\, d\mathcal{H}_J \\ &\quad = \int_Z \psi_S \tilde{S}_t' \varphi\, d\lambda + \int_Z \big([\psi] n'' - \tilde{T}_1 \cdot [\tilde{u}_t] n'\big) d\mathcal{H}_J\,. \end{aligned} \tag{2.28}$$

We used that as a consequence of (2.9) we have $[\hat{P}\tilde{T}] = 0$, whence $[\tilde{T}_1] = 0$. To determine $[\tilde{u}_t]$ in this equation we employ (2.6), which shows that if $\tilde{S}$ is continuously differentiable in a neighborhood of the point $(x,t) \in (a,b) \times (0,\infty)$, then the time derivative $\tilde{u}_t$ exists and is given by

$$\begin{aligned} \tilde{u}_t(x,t) \;=\; & u^*\Big(\int_a^x \tilde{S}_t'(y,t) dy - \sum_{i=1}^{\ell} \frac{d}{dt}\alpha_i(t)\, [\tilde{S}]\big(\alpha_i(t), t\big) \Big) \\ & - u^* \frac{x-a}{b-a} \Big(\int_a^b \tilde{S}_t'(y,t) dt - \sum_{i=1}^{k} \frac{d}{dt}\alpha_i(t)\, [\tilde{S}]\, \big(\alpha_i(t), t\big) \Big) + w_t(x,t)\,. \end{aligned} \tag{2.29}$$

Here $\{\alpha_i\}_{i=1}^k$ is the set of jump curves intersecting the line segment $(a,b) \times \{t\}$, and ℓ is chosen such that

$$\alpha_1(t) < \ldots < \alpha_\ell(t) < x < \alpha_{\ell+1}(t) < \ldots < \alpha_k(t)\,.$$

Thus, if α is a jump curve of $\tilde{S}$ we obtain from (2.29) by considering the limit $\lim_{x \searrow \alpha(t)} \tilde{u}_t(x,t) - \lim_{x \nearrow \alpha(t)} \tilde{u}_t(x,t)$ that

$$[\tilde{u}_t]\big(\alpha(t), t\big) = -\frac{d}{dt}\alpha(t)\, u^*[\tilde{S}]\big(\alpha(t), t\big) = -\frac{d}{dt}\alpha(t)\, [\tilde{u}_x]\big(\alpha(t), t\big)\,,$$

hence, together with (2.26),

$$\begin{aligned} [\psi] n'' - \tilde{T}_1 \cdot [\tilde{u}_t] n' \;&=\; -[\psi]\, \frac{d\alpha}{dt}\, n' + \tilde{T}_1 \cdot [\tilde{u}_x]\, \frac{d\alpha}{dt}\, n' \\ &=\; -\big([\psi] - [\tilde{T}_1 \cdot \tilde{u}_x]\big) \frac{d\alpha}{dt}\, n' = -[C_1]\, \frac{d\alpha}{dt}\, n'\,. \end{aligned}$$

(2.23) follows by insertion of this relation into (2.28). The proof of Lemma 2.2 is complete.

Corollary 2.3 *Let $(\tilde{u}, \tilde{T}, \tilde{S})$ satisfy the assumptions of the preceding lemma. Then the evolution equation* (1.19) *and the Clausius-Duhem inequality* (1.22) *hold if and only if*

$$\frac{d\alpha}{dt}\,|[\tilde{S}]|\,\mathcal{H}_J = c\,[C_1]\,\mathcal{H}_J\,, \tag{2.30}$$

$$\tilde{S}'_t = -c\,\psi_S\,|\tilde{S}'_x|\,. \tag{2.31}$$

Proof: From (2.21), (2.22) and (2.26) we obtain for the variation measures

$$\begin{aligned} |\tilde{S}_t| = |n''[\tilde{S}]|\,\mathcal{H}_J + |\tilde{S}'_t|\,\lambda &= n'|\frac{d\alpha}{dt}|\,|[\tilde{S}]|\,\mathcal{H}_J + |\tilde{S}'_t|\,\lambda\,, \\ |C_{1x} - u_x \cdot b| &= n'|[C_1]|\,\mathcal{H}_J + |\psi_S \tilde{S}'_x|\,\lambda\,. \end{aligned}$$

Consequently, (1.19) holds if and only if the equations

$$|\frac{d\alpha}{dt}|\,|[\tilde{S}]|\,\mathcal{H}_J = c\,|[C_1]|\,\mathcal{H}_J \qquad \text{and} \qquad |\tilde{S}'_t| = |c\psi_S \tilde{S}'_x| \tag{2.32}$$

are satisfied. Moreover, since by our convention $n' > 0$, we see from (2.23) that (1.22) holds if and only if the inequalities

$$-\frac{d\alpha}{dt}\,[C_1]\,\mathcal{H}_J \le 0 \qquad \text{and} \qquad \psi_S \tilde{S}'_t \le 0 \tag{2.33}$$

are fulfilled. It is immediately seen that (2.32), (2.33) are equivalent to the pair of equations (2.30), (2.31).

End of the proof of Lemma 1.1: We defined the function (u, T, S) such that (2.3) – (2.5) are satisfied, hence this function satisfies (1.17), (1.18) and (1.20). Moreover, (1.21) is satisfied by construction. Consequently it remains to show that also (1.19) and (1.22) are fulfilled. By Corollary 2.3 these equations hold if (u, T, S) satisfies (2.30) and (2.31). The second equation is obviously satisfied since S is piecewise constant, hence $S'_t = S'_x = 0$. The first equation holds because (2.20) and (2.16) yield for the speed of any jump discontinuity

$$\frac{d\alpha}{dt} = c\,\frac{[\Psi]}{|[S]|} = c\,\frac{[C_1]}{|[S]|}\,. \tag{2.34}$$

3 Monotonically increasing initial data

In the proof of Lemma 1.2 given in this section we use Lemma 1.1 to construct a sequence (u_n, T_n, S_n) of solutions to (1.17) – (1.22) to piecewise constant monotonic initial data $S_n^{(0)}$ such that $S_n^{(0)} \to S^{(0)}$. The function S_n is piecewise constant and $x \mapsto S_n(x, t)$ is monotonic. As will be shown, this implies that (u_n, T_n, S_n) satisfies the evolution equation (1.19) without the absolute value signs. Thus, if we select a converging subsequence for which S_{n_m} and $C_1(u_{n_m,x}, S_{n_m})_x - u_{n_m,x} \cdot b$ converge weakly, the limit function satisfies (1.19) without the absolute value signs. Consequently, the limit function satisfies (1.19).

To select a converging subsequence we need bounds for the BV–norms of the approximating sequences. We begin by deriving such bounds.

The geometry of the discontinuities and the BV–norms. For definiteness assume that the function $S^{(0)}$ is monotonically increasing. We choose a sequence $\{S_n^{(0)}\}_{n=1}^{\infty}$ of monotonically increasing, piecewise constant functions $S_n^{(0)} : [a, b] \to [0, 1]$ with finitely many jumps in (a, b), such that $S_n^{(0)}(a) = S^{(0)}(a)$, $S_n^{(0)}(b) = S^{(0)}(b)$, and

$$\lim_{n\to\infty} \sup_{a\le x\le b} |S^{(0)}(x) - S_n^{(0)}(x)| = 0. \tag{3.1}$$

Define $(u_n, T_n, S_n) : Z \to \mathbb{R}^3 \times \mathcal{S}^3 \times [0, 1]$ to be the solution of the initial-boundary value problem (1.17) - (1.22) to the initial data $S_n^{(0)}$ constructed as in the proof of Lemma 1.1. We denote by $\mathcal{J}_n$ the set of all jump curves of S_n. For $\alpha \in J_n$ we denote the constant values of S_n to the left and to the right of α by $S_n(\alpha-)$ and $S_n(\alpha+)$, respectively. We also write $[S_n](\alpha) = S_n(\alpha+) - S_n(\alpha-)$.

Lemma 3.1 (i) *To $\alpha \in \mathcal{J}_n$ there exist jump points x_α and y_α of $S_n^{(0)}$ with $a < x_\alpha \le y_\alpha < b$ such that*

$$S_n(\alpha-) = S_n^{(0)}(x_\alpha-), \qquad S_n(\alpha+) = S_n^{(0)}(y_\alpha+). \tag{3.2}$$

(ii) *If α intersects the line segment $(a, b)\times\{t\}$ and if $\beta \in \mathcal{J}_n$ is the next discontinuity to the left of α which intersects this line segment, then*

$$x_\beta \le y_\beta < x_\alpha \le y_\alpha. \tag{3.3}$$

(iii) *There is no jump discontinuity of $S_n^{(0)}$ between y_β and x_α.*

Proof: (i) The discontinuity α starts either at the initial line segment $(a, b) \times \{0\}$, or at the point of intersection of several discontinuities $\beta_1, \dots, \beta_m \in \mathcal{J}_n$. Let β_1 be the leftmost of these dicontinuities and β_m be the rightmost discontinuity. By construction S_n satisfies

$$S_n(\alpha-) = S_n(\beta_1-), \quad S_n(\alpha+) = S_n(\beta_m+).$$

If β_1 does not start on the initial line segment, it starts at the point of intersection of several discontinuities. The value of S_n to the left of the leftmost of these discontinuities is $S_n(\alpha-)$. We follow the leftmost discontinuity backwards in time and repeat the process until we reach a point $(x_\alpha, 0) \in (a, b) \times \{0\}$. The function $S_n^{(0)}$ has a jump at x_α, and since the value of S_n to the left of the last discontinuity is $S_n(\alpha-)$, we deduce

$$S_n(\alpha-) = S_n^{(0)}(x_\alpha-).$$

Similarly, we start at α and follow backwards in time at every point of intersection the rightmost discontinuity until we reach a point $(y_\alpha, 0) \in (a, b) \times \{0\}$ with $y_\alpha \ge$

x_α. The function $S_n^{(0)}$ jumps at y_α, and since the value of S_n to the right of the discontinuity is $S_n(\alpha+)$, we obtain

$$S_n(\alpha+) = S_n^{(0)}(y_\alpha+).$$

(ii) We start at β and follow backwards in time at every intersection point the rightmost discontinuity until we reach the point $(y_\beta, 0)$ on the initial line segment. As above we have

$$S_n(\beta+) = S_n^{(0)}(y_\beta+).$$

The path from β to the point $(y_\beta, 0)$ does never intersect the path from α to $(x_\alpha, 0)$, since from every point of intersection of discountinuities at most one discontinuity emerges forward in time. Therefore we have $y_\beta < x_\alpha$.

(iii) There is no jump of $S_n^{(0)}$ between y_β and x_α. For, since $S_n^{(0)}$ is increasing, we would otherwise have $S_n^{(0)}(y_\beta+) < S_n^{(0)}(x_\alpha-)$, whence

$$S_n(\beta+) < S_n(\alpha-)\,. \tag{3.4}$$

Yet, by assumption no jump curve of S_n intersects the line segment $(a, b) \times \{t\}$ between α and β, hence $S_n(\beta+) = S_n(\alpha-)$. This contradicts (3.4), whence statement (iii) must be true.

Corollary 3.2 *For every n and for every jump curve $\alpha \in \mathcal{J}_n$ we have*

$$[S_n](\alpha) = S_n^{(0)}(y_\alpha+) - S_n^{(0)}(x_\alpha-) > 0\,,$$

whence $x \mapsto S_n(x, t)$ is increasing. Moreover, S_n satisfies for every $t \geq 0$

$$S^{(0)}(a) \leq S_n(a, t), \quad S_n(b, t) \leq S^{(0)}(b)\,, \tag{3.5}$$

$$\operatorname{var} S_n(\cdot, t) \leq \operatorname{var} S_n^{(0)} = \operatorname{var} S^{(0)} \leq 1\,. \tag{3.6}$$

Proof: Since $S_n^{(0)}$ is increasing, we obtain from (3.2) for every jump curve $\alpha \in \mathcal{J}_n$

$$[S_n](\alpha) = S_n(\alpha+) - S_n(\alpha-) = S_n^{(0)}(y_\alpha+) - S_n^{(0)}(x_\alpha-) > 0\,.$$

(3.5) results from (3.2) and from

$$S_n^{(0)}(x_\alpha-) \geq S_n^{(0)}(a) = S^{(0)}(a), \quad S_n^{(0)}(y_\alpha+) \leq S_n^{(0)}(b) = S^{(0)}(b)\,.$$

To verify (3.6) we use that $x \mapsto S_n(x, t)$ is increasing and apply (3.5) to conclude

$$\operatorname{var} S_n(\cdot, t) = S_n(b, t) - S_n(a, t) \leq S^{(0)}(b) - S^{(0)}(a) \leq 1\,.$$

The corollary is proven.

Now we can show that the variation measures of S_n , of the Eshelby tensor and of the free energy are uniformly bounded over Z_{T_e} for every $T_e > 0$ with respect to n. We use that for $f \in L^1(Z_{T_e}, \mathbb{R})$ the variation measures $|f_x|$ and $|f_t|$ satisfy

$$\begin{aligned} |f_x|(Z_{T_e}) &= \sup\Big\{ \int_{Z_{T_e}} f\,\varphi_x\,dx \;\Big|\; \varphi \in \overset{\circ}{C}_1(Z_{T_e}, \mathbb{R}), \quad |\varphi| \le 1 \Big\}, \\ |f_t|(Z_{T_e}) &= \sup\Big\{ \int_{Z_{T_e}} f\,\varphi_t\,dx \;\Big|\; \varphi \in \overset{\circ}{C}_1(Z_{T_e}, \mathbb{R}), \quad |\varphi| \le 1 \Big\}, \end{aligned}$$

cf. [13, p.170]. By definition, the function $f \in L^1(Z_{T_e}, \mathbb{R})$ belongs to the space $BV(Z_{T_e})$ if $|f_x|(Z_{T_e}) + |f_t|(Z_{T_e}) < \infty$. We also define

$$\begin{aligned} &|C_1(u_{n,x}, S_n)_x - u_{n,x}\cdot b|\,(Z_{T_e}) \\ &= \sup\Big\{ \int_{Z_{T_e}} \big(C_1(u_{n,x}, S_n)\,\varphi_x + u_{n,x}\cdot b\,\varphi\big) d(x,t) \;\Big|\; \varphi \in \overset{\circ}{C}_1(Z_{T_e}, \mathbb{R}),\ |\varphi| \le 1 \Big\}, \\ &|\psi\big(\varepsilon(u_{n,x}), S_n\big)_t - (T_{1,n}\cdot u_{n,t})_x - b\cdot u_{n,t}|\,(Z_{T_e}) \\ &= \sup\Big\{ \int_{Z_{T_e}} \Big(-\psi\big(\varepsilon(u_{n,x}), S_n\big)\varphi_t + T_{1,n}\cdot u_{n,t}\,\varphi_x - b\cdot u_{n,t}\,\varphi \Big)\, d(x,t) \\ &\qquad\qquad \Big|\; \varphi \in \overset{\circ}{C}_1(Z_{T_e}, \mathbb{R}),\ |\varphi| \le 1 \Big\}. \end{aligned}$$

Lemma 3.3 *For all n and for all $T_e > 0$ we have $S_n \in BV(Z_{T_e})$. There is a constant $A > 0$, which only depends on T_e and is an increasing function of this parameter, such that for the constant c from* (1.19)

$$|S_{n,x}|(Z_{T_e}) \le T_e \operatorname{var} S_n^{(0)}, \qquad |S_{n,t}|(Z_{T_e}) \le cA\,T_e \operatorname{var} S_n^{(0)}, \tag{3.7}$$

$$|C_1(u_{n,x}, S_n)_x - u_{n,x}\cdot b|\,(Z_{T_e}) \le AT_e \operatorname{var} S_n^{(0)}, \tag{3.8}$$

$$|\psi\big(\varepsilon(u_{n,x}), S_n\big)_t - (T_{1,n}\cdot u_{n,t})_x - b\cdot u_{n,t}|\,(Z_{T_e}) \le cA^2 T_e \operatorname{var} S_n^{(0)}. \tag{3.9}$$

Proof: For $\varphi \in \overset{\circ}{C}_1(Z_{T_e}, \mathbb{R})$ with $|\varphi| \le 1$ we obtain as in the proof of (2.22) for the piecewise constant function S_n that

$$-\int_{Z_{T_e}} S_n \varphi_x\,dx = \int_{Z_{T_e}} \varphi\, n'\,[S_n]\,d\mathcal{H}_{J_n} = \sum_{\alpha\in\mathcal{J}_n} \int_\alpha \varphi\,[S_n](\alpha)\,n' ds \tag{3.10}$$

$$= \sum_{\alpha\in\mathcal{J}_n} \int_0^{T_e} \chi_\alpha(t)\,\varphi\big(\alpha(t), t\big)\,[S_n](\alpha)\,dt = \int_0^{T_e} \sum_{\alpha\in\mathcal{J}_n} \chi_\alpha(t)\,[S_n](\alpha)\,\varphi\big(\alpha(t), t\big)\,dt,$$

where $\chi_\alpha : \mathbb{R} \to [0,1]$ denotes the characteristic function of the domain $D_\alpha \subseteq [0,\infty)$ of the parametrization $\alpha : D_\alpha \to [a,b]$ of the jump curve α. We also used that $n'ds = dt$. Noting

$$\sum_{\alpha\in\mathcal{J}_n} \chi_\alpha(t)\,[S_n](\alpha) = \operatorname{var} S_n(\cdot, t), \tag{3.11}$$

we conclude from (3.10) and Corollary 3.2 that

$$|S_{n,x}|(Z_{T_e}) \leq \int_0^{T_e} \operatorname{var} S_n(\cdot,t)dt \leq T_e \operatorname{var} S_n^{(0)}\,. \tag{3.12}$$

Similarly, since $S'_{nt} = 0$ we deduce from (2.21)

$$-\int_{Z_{T_e}} S_n\,\varphi_t\,dt = \int_{Z_{T_e}} \varphi\, n''[S_n]\,d\mathcal{H}_{J_n} = \sum_{\alpha\in\mathcal{J}_n} \int_\alpha [S_n](\alpha)\,\varphi\, n'' ds$$

$$= \int_0^{T_e} \sum_{\alpha\in\mathcal{J}_n} \chi_\alpha(t)[S_n](\alpha)\frac{n''}{n'}\varphi(\alpha(t),t)\,dt\,. \tag{3.13}$$

To estimate the right hand side of this equation we infer from $\frac{n''}{n'} = -\frac{d\alpha}{dt}$ and from (2.20) that

$$\left|\frac{n''}{n'}\right| = \left|\frac{d\alpha}{dt}\right| \leq cA, \tag{3.14}$$

where $A = \max\left\{\left|\frac{d}{ds}\Psi(y,t,s,r)\right| \,\middle|\, (y,t,s,r)\in[a,b]\times[0,T_e]\times[0,1]\times[0,b-a]\right\}$. Since $\sigma\in H_2(Z_{T_e})$ is continuous, it follows from the definition of Ψ in (2.17) that the maximum A exists. Of course, A depends on T_e. We use (3.14) in (3.13) and obtain together with (3.11) that

$$|S_{n,t}|(Z_{T_e}) \leq \int_0^{T_e} \operatorname{var} S_n(\cdot,t)cA\,dt \leq cAT_e \operatorname{var} S_n^{(0)}.$$

This estimate and (3.12) together yield (3.7). To verify (3.8) we note that (2.22) implies for $\varphi\in \overset{\circ}{C}_1(Z_{T_e},\mathbb{R})$ with $|\varphi|\leq 1$ that

$$-\int_{Z_{T_e}} C_1(u_{n,x},S_n)\,\varphi_x + u_{n,x}\cdot b\,\varphi\,d(x,t) = \int_{Z_{T_e}} \varphi\, n'[C_1]\,d\mathcal{H}_{J_n} \tag{3.15}$$

$$= \int_0^{T_e} \sum_{\alpha\in\mathcal{J}_n} \chi_\alpha(t)[C_1](\alpha)\varphi\big(\alpha(t),t\big)\,dt\,.$$

(2.16) yields $|\,[C_1]\,| = |\,[\Psi]\,| \leq \max\left|\frac{d}{ds}\Psi(y,t,s,r)\right|\,|[S_n]| = A[S_n]$, whence

$$\Big|\sum_{\alpha\in\mathcal{J}_n}\chi_\alpha(t)[C_1](\alpha)\Big| \leq A\sum_{\alpha\in\mathcal{J}_n}\chi_\alpha(t)[S_n](\alpha) = A\operatorname{var} S_n(\cdot,t).$$

Insertion of this inequality into (3.15) results in

$$|C_{1,x} - u_{n,x}\cdot b|\,(Z_{T_e}) \leq \int_0^{T_e} A\operatorname{var} S_n(\cdot,t)\,dt \leq AT_e\operatorname{var} S_n^{(0)}\,.$$

This is (3.8). Finally (2.23) yields

$$\int_{Z_{T_e}} \Big(-\psi\big(\varepsilon(u_{n,x}),S_n\big)\,\varphi_t + T_{1,n}\cdot u_{n,t}\,\varphi_x - b\cdot u_{n,t}\,\varphi\Big)\,d(x,t)$$

$$= -\int_{Z_{T_e}} \frac{d\alpha}{dt}\,n'\,[C_1]\,\varphi\,d\mathcal{H}_{J_n} \leq cA^2T_e\operatorname{var} S_n^{(0)}\,,$$

where we used (3.14) again and proceeded as in (3.15). This proves (3.9).

Lemma 3.4 *The function* (u_n, T_n, S_n) *satisfies*

$$S_{n,t} = -c\big(C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b\big) \tag{3.16}$$

on Z *in the sense of measures.*

Proof: Corollary 3.2 implies that $[S_n](\alpha) > 0$ for every jump curve $\alpha \in \mathcal{J}_n$. Thus, if we apply Lemma 2.2 to the function (u_n, T_n, S_n), use that $S'_{n,t} = S'_{n,x} = 0$ and employ (2.26) and (2.34), we obtain

$$\begin{aligned} S_{n,t} = n''\,[S_n]\,\mathcal{H}_{J_n} = -\frac{d\alpha}{dt}\,n'\,[S_n]\,\mathcal{H}_{J_n} &= -c\,\frac{[C_1]}{|[S_n]|}\,n'\,|[S_n]|\,\mathcal{H}_{J_n} \\ = -cn'\,[C_1]\,\mathcal{H}_{J_n} &= -c(C_{1,x} - u_{n,x} \cdot b)\,. \end{aligned}$$

End of the proof of Lemma 1.2: The proof is in three steps:

Claim 1: The sequence $\{(u_n, T_n, S_n)\}_n$ has a subsequence, again denoted by $\{(u_n, T_n, S_n)\}_n$, which converges in $L^2(Z_{T_e}, \mathbb{R}^3 \times \mathcal{S}^3) \times L^p(Z_{T_e}, \mathbb{R})$ to a function (u, T, S), which satisfies (2.3) – (2.5), for every $1 \le p < \infty$ and all $T_e > 0$. The function S belongs to $BV(Z_{T_e})$, the function u belongs to $H_1(Z_{T_e}, \mathbb{R}^3)$ and $u_{n,x} \to u_x$ strongly in $L^2(Z_{T_e}, \mathbb{R}^3)$, $u_{n,t} \rightharpoonup u_t$ weakly in $L^2(Z_{T_e}, \mathbb{R}^3)$ for all $T_e > 0$.

Proof: To see this, note that the inequality $0 \le S_n \le 1$ and Lemma 3.3 together with $\operatorname{var} S_n^{(0)} = \operatorname{var} S^{(0)}$ imply

$$\|S_n\|_{Z_{T_e}} + |S_{n,x}|(Z_{T_e}) + |S_{n,t}|(Z_{T_e}) \le T_e\big((b-a) + (1 + cA)\operatorname{var} S^{(0)}\big)\,,$$

where $\|S_n\|_{Z_{T_e}}$ denotes the L^1-norm. Therefore, if we set $T_e = m \in \mathbb{N}$, for every m we can select a subsequence of $\{S_n\}_{n=1}^{\infty}$, which converges in $L^1(Z_m, \mathbb{R})$ to a limit function $S \in BV(Z_m)$, cf. [13, p. 176]. By the usual argument the diagonal sequence, again denoted by $\{S_n\}_n$, converges to S in $L^1(Z_{T_e}, \mathbb{R})$ for every $T_e > 0$. Noting that $0 \le S_n \le 1$ we infer that this sequence converges to S even in $L^p(Z_{T_e}, \mathbb{R})$ for all $1 \le p < \infty$.

Let (u, t) be the solution of the problem (2.3) – (2.5) to the function S, for every $T_e > 0$. The difference $(u_n - u, T_n - T)$ is a solution of the boundary value problem (2.3) – (2.5) to the data $b = 0$, $f = 0$ and to the function $S_n - S$. From (2.6) and (2.8) we thus obtain

$$\begin{aligned} (u_n - u)(x,t) &= u^*\Big(\int_a^x (S_n - S)(y,t)dy - \frac{x-a}{b-a}\int_a^b (S_n - S)(y,t)dy\Big) \\ (T_n - T)(x,t) &= D(\varepsilon^* - \overline{\varepsilon})(S_n - S)(x,t) - D\varepsilon^*\,\frac{1}{b-a}\int_a^b (S_n - S)(y,t)dy\,. \end{aligned}$$

it follows immediately from these formulas and from $S_n \to S$ in $L^2(Z_{T_e}, \mathbb{R})$ that $(u_n, u_{n,x}, T_n) \to (u, u_x, T)$ in $L^2(Z_{T_e}, \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S}^3)$. To verify that $u_{n,t} \rightharpoonup u_t$ we use (2.29), applied to $\tilde{u} = u_n$ and to the piecewise constant function $\tilde{S} = S_n$, and obtain for $(x,t) \in Z_{T_e}$ that

$$
\begin{aligned}
|u_{n,t}(x,t)| \;&=\; \Big| u^* \Big(\frac{x-a}{b-a} \sum_{i=1}^{k} \frac{d\alpha_i}{dt}(t)\, [S_n]\big(\alpha_i(t),t\big) - \sum_{i=1}^{\ell} \frac{d\alpha_i}{dt}(t)\, [S_n]\big(\alpha_i(t),t\big) \Big) \\
&\qquad + w_t(x,t) \Big| \le 2|u^*|\, cA \operatorname{var} S_n(\cdot,t) + |w_t(x,t)|\,.
\end{aligned}
$$

Here cA is the bound for $|\frac{d\alpha}{dt}|$ in Z_{T_e} from (3.14). Since $\operatorname{var} S_n(\cdot,t) \le 1$, by Corollary 3.2, and $w_t \in L^2(Z_{T_e}, \mathbb{R}^3)$, by Lemma 2.1, we conclude that $\{u_{n,t}\}_n$ is bounded in $L^2(Z_{T_e}, \mathbb{R}^3)$. Hence it has a weakly converging subsequence. By the usual arguments we infer that the weak derivative u_t exists in $L^2(Z_{T_e}, \mathbb{R}^3)$ and that $u_{n,t} \rightharpoonup u_t$ for all $T_e > 0$. Since $u_x \in L^2(Z_{T_e})$ we obtain $u \in H_1(Z_{T_e}, \mathbb{R}^3)$. This finishes the proof of the claim.

Claim 2: The limit function (u,T,S) satisfies the equation

$$
S_t = -c\big(C_1(u_x,S)_x - u_x \cdot b\big) \tag{3.17}
$$

on Z in the sense of measures.

Proof: The claim follows from Lemma 3.4 if we show that the measures $S_{n,t}$ and $C_1(u_{n,x},S_n)_x - u_{n,x} \cdot b$ on both sides of (3.16) weak–$*$ converge to S_t and to $C_1(u_x,S)_x - u_x \cdot b$, respectively.

By definition, $S_{n,t} \overset{*}{\rightharpoonup} S_t$ if $\int_Z \varphi\, dS_{n,t} \to \int_Z \varphi\, dS_t$ for all $\varphi \in \mathring{C}(Z)$. Since $\mathring{C}_1(Z)$ is dense in $\mathring{C}(Z)$, it follows that $S_{n,t} \overset{*}{\rightharpoonup} S_t$ if

$$
\sup_{n\in\mathbb{N}} |S_{n,t}|(Z_{T_e}) < \infty \tag{3.18}
$$

for all $T_e > 0$ and if

$$
\int_Z \varphi\, dS_{n,t} = -\int_Z S_n \varphi_t\, d(x,t) \to -\int_Z S\varphi_t\, d(x,t) = \int_Z \varphi\, dS_t \tag{3.19}
$$

for all $\varphi \in \mathring{C}_1(Z)$. Equation (3.18) is a consequence of (3.7) since $\operatorname{var} S_n^{(0)} \le 1$, and (3.19) immediately follows from the fact that $S_n \to S$ in $L^2(Z_{T_e})$ for every $T_e > 0$.

Also, we have $C_1(u_{n,x},S_n)_x - u_{n,x} \cdot b \overset{*}{\rightharpoonup} C_1(u_x,S)_x - u_x \cdot b$ if

$$
\sup_{n\in\mathbb{N}} |C_1(u_{n,x},S_n)_x - u_{n,x} \cdot b|(Z_{T_e}) < \infty \tag{3.20}
$$

for all $T_e > 0$ and

$$
\int_Z \big(C_1(u_{n,x},S_n)\varphi_x + u_{n,x} \cdot b\varphi\big) d(x,t) \to \int_Z \big(C_1(u_x,S)\varphi_x + u_x \cdot b\varphi\big) d(x,t) \tag{3.21}
$$

for all $\varphi \in \overset{\circ}{C}_1(Z)$. Equation (3.20) is a consequence of (3.8). To prove (3.21) observe that

$$C_1(u_{n,x}, S_n) = \frac{1}{2} T_n \cdot \big(\varepsilon(u_{n,x}) - \overline{\varepsilon} S_n\big) + \psi_1(S_n) - u_{n,x} \cdot T_{1,n} \,. \tag{3.22}$$

Since $T_n \to T$ in $L^2(Z_{T_e})$, $u_{n,x} \to u_x$ in $L^2(Z_{T_e})$ and $S_n \to S$ in $L^2(Z_{T_e})$ we conclude that

$$\begin{aligned} &\frac{1}{2} T_n \cdot \varepsilon(u_{n,x})\, \varphi_x \to \frac{1}{2} T \cdot \varepsilon(u_x)\, \varphi_x \,, && \frac{1}{2} T_n S_n\, \varphi_x \to \frac{1}{2} T S\, \varphi_x \,, \\ &\psi_1(S_n)\, \varphi_x \to \psi_1(S)\, \varphi_x \,, && u_{n,x} \cdot T_{1,n}\, \varphi_x \to u_x \cdot T_1\, \varphi_x \,, \end{aligned}$$

where the convergence is in $L^1(Z, \mathbb{R})$, since φ has compact support. From (3.22) we thus obtain

$$C_1(u_{n,x}, S_n)\varphi_x \to C_1(u_x, S)\varphi_x$$

in $L^1(Z, \mathbb{R})$. Relation (3.21) is implied by this relation together with $u_{n,x} \cdot b\varphi \to u_x \cdot b\varphi$ in $L^1(Z, \mathbb{R})$, which again follows from the convergence of $u_{n,x}$ to u_x.

Claim 3: (u, T, S) satisfies the equations (1.17) – (1.22).

Proof: By Claim 1 the function (u, T, S) satisfies the equations (2.3) – (2.5), which coincide with (1.17), (1.18), (1.20). Equation (1.19) follows from (3.17) by taking the variation measures on both sides. To show that the Clausius-Duhem inequality (1.22) holds it suffices to prove that in the sense of measures

$$\psi\big(\varepsilon(u_{n,x}), S_n\big)_t - (T_{1,n} \cdot u_{n,t})_x - b \cdot u_{n,t} \overset{*}{\rightharpoonup} \psi\big(\varepsilon(u_x), S\big)_t - (T_1 \cdot u_t)_x - b \cdot u_t \,, \tag{3.23}$$

since (u_n, T_n, S_n) satisfies (1.22). Because the right hand side of (3.9) is uniformly bounded by the constant $cA^2\, T_e \mathrm{var}\, S^{(0)}$, we infer just as in the proof of (3.17) that (3.23) holds if

$$\int_Z \psi\big(\varepsilon(u_{n,x}), S_n\big)\varphi_t \, d(x,t) \;\to\; \int_Z \psi\big(\varepsilon(u_x), S\big)\varphi_t \, d(x,t) \tag{3.24}$$

$$\int_Z T_{1,n} \cdot u_{n,t}\, \varphi_x \, d(x,t) \;\to\; \int_Z T_1 \cdot u_t\, \varphi_x \, d(x,t) \tag{3.25}$$

$$\int_Z b \cdot u_{n,t}\, \varphi \, d(x,t) \;\to\; \int_Z b \cdot u_t\, \varphi \, d(x,t) \tag{3.26}$$

for all $\varphi \in \overset{\circ}{C}_1(Z, \mathbb{R})$. Yet, the convergence (3.24) follows exactly as in the proof of (3.17) and (3.25), (3.26) are implied by the convergence relations $T_{1,n} \to T_1$ in $L^2(Z_{T_e}, \mathbb{R}^3)$, $u_{n,t} \rightharpoonup u_t$ weakly in $L^2(Z_{T_e}, \mathbb{R}^3)$, which hold by Claim 1.

To verify the initial condition (1.21) we first extend the signed measures $S_{n,t}$ and S_t on Z_{T_e} to measures on the set $Z_{-\infty,T_e} = (a, b) \times (-\infty, T_e)$ by defining for $B \subseteq Z_{-\infty,T_e}$

$$S_{n,t}(B) = S_{n,t}(B \cap Z_{T_e}), \qquad S_t(B) = S_t(B \cap Z_{T_e}),$$

provided $B \cap Z_{T_e}$ is $S_{n,t}$–measurable or S_t–measurable. In the proof of Claim 2 we showed that $\{S_{n,t}\}_n$ converges weak–$*$ to S_t on Z_{T_e}. Here we show that the sequence of extended measures $\{S_{n,t}\}_n$ converges weak–$*$ to S_t on $Z_{-\infty,T_e}$.

To this end note that if δ is a constant satisfying $0 < \delta \le T_e$ and if we apply (3.7) with Z_{T_e} replaced by Z_δ, then we obtain for the extended measure

$$|S_{n,t}|(Z_{-\infty,\delta}) = |S_{n,t}|(Z_\delta) \le cA\delta \operatorname{var} S_n^{(0)} \le cA\delta\,. \tag{3.27}$$

A can be chosen independent of $\delta \le T_e$, since it is an increasing function of this parameter. From (3.27) we obtain in particular that $|S_{n,t}|(Z_{-\infty,T_e}) \le cAT_e$. Consequently, there is a subsequence $\{S_{n_j,t}\}_j$, which converges weak–$*$ to a measure μ on $Z_{-\infty,T_e}$. From the properties of weak–$*$ convergence we know that $\mu(B) = S_t(B)$ for $B \subseteq Z_{T_e}$. Thus, if we show that $\mu(B) = 0$ for all sets $B \subseteq \big((a,b)\times(-\infty,0]\big)$, it follows that μ is equal to the extended measure S_t, and this implies for the extended measures that $S_{n,t} \overset{*}{\rightharpoonup} S_t$.

Thus, let $B \subseteq \big((a,b)\times(-\infty,0]\big)$. Then B is a subset of the open set $Z_{-\infty,\delta}$ for any $0 < \delta \le T_e$, hence (3.27) implies

$$|\mu|(B) \le |\mu|(Z_{-\infty,\delta}) \le \liminf_{j\to\infty} |S_{n_j,t}|(Z_{-\infty,\delta}) \le cA\delta\,, \tag{3.28}$$

cf. [13, p. 54]. This yields the desired result $|\mu|(B) = 0$.

In the second step of the proof we use that $S_{n,t} \overset{*}{\rightharpoonup} S_t$ on $Z_{-\infty,T_e}$ implies for $\varphi \in \mathring{C}(Z_{-\infty,T_e},\mathbb{R})$

$$\lim_{n\to\infty}\int_Z \varphi\, dS_{n,t} = \lim_{n\to\infty}\int_{Z_{-\infty,T_e}} \varphi\, dS_{n,t} = \int_{T_{-\infty,T_e}} \varphi\, dS_t = \int_Z \varphi\, dS_t\,. \tag{3.29}$$

Because S_n is piecewise constant with finitely many jumps we deduce by partial integration for $\varphi \in \mathring{C}_1(Z_{-\infty,T_e},\mathbb{R})$ that

$$\int_a^b S_n^{(0)}\varphi(0)\,dx = -\int_Z S_n\varphi_t\, d(x,t) - \int_Z \varphi\, dS_{n,t}\,. \tag{3.30}$$

Since $S_n \to S$ in $L^2(Z_{T_e},\mathbb{R})$ and since $S_n^{(0)}$ satisfies (3.1), we obtain by taking the limits on both sides of (3.30), observing (3.29), that

$$\int_a^b S^{(0)}\varphi(0)dx = -\int_Z S\,\varphi_t\, d(x,t) - \int_Z \varphi\, dS_t\,.$$

By the trace theorem for BV-functions (cf. [13, p.177]), this equation implies that $S^{(0)}$ coincides with the uniquely defined trace of $S \in BV(Z_{T_e})$ on $(a,b)\times\{0\}$. Therefore the initial condition (1.21) is satisfied.

This completes the proof of Lemma 1.2 for increasing initial data $S^{(0)}$. For decreasing $S^{(0)}$ the proof is almost the same. The only essential difference is that in this case $[S_n](\alpha) < 0$ for all jumps of the approximate solutions, which implies that instead of (3.17) the function S satisfies

$$S_t = c\big(C_1(u_x,S)_x - u_x\cdot b\big)\,.$$

4 Nonmonotone initial data

This section is devoted to the proof of Theorem 1.3. For initial data increasing in the interval $[a, c]$ and decreasing in $[c, b]$ it is not possible to construct a solution, which satisfies the equation (1.19) without the absolute value signs, as we could do this for monotone initial data. Instead, we have to deal with all the difficulties arising from the variation measures in (1.19).

The existence proof uses a convergent sequence $\{(u_n, T_n, S_n)\}_n$ of solutions to piecewise constant initial data constructed as in the proof of Lemma 1.2. The arguments of the preceding section can be repeated to show that the limit function satisfies (1.17), (1.18), (1.20) – (1.22). The main difficulty in the proof that the evolution equation (1.19) is satisfied lies in the verification of $|C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b| \overset{*}{\rightharpoonup} |C_1(u_x, S)_x - u_x \cdot b|$. To prove this we decompose $\nu_n = C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b$ into the positive and negative part $\nu_n^{\pm}$. For the weak–$*$ limits we have $\lim_{n\to\infty} |\nu_n| = \lim_{n\to\infty}(\nu_n^+ + \nu_n^-) = \nu^+ + \nu^-$. In general, $\nu^+ + \nu^-$ is different from the variation measure of $C_1(u_x, S)_x - u_x \cdot b$. However, we can show that in our situation equality holds, which proves the desired result. The central idea used to show this is contained in the proof of Proposition 4.9, which is given in Sect. 5.

Construction of the solution. We choose a sequence $\{S_n^{(0)}\}_n$ of piecewise constant functions $S_n^{(0)} : [a, b] \to [0, 1]$ with finitely many jumps in $(a, c) \cup (c, b)$, such that $S_n^{(0)}$ is increasing on (a, c), decreasing on (c, b), and such that

$$\lim_{n\to\infty} \sup_{a \le x \le b} |S^{(0)}(x) - S_n^{(0)}(x)| = 0\,. \tag{4.1}$$

Define $(u_n, T_n, S_n) : Z \to \mathbb{R}^3 \times \mathcal{S}^3 \times [0, 1]$ to be the solution of the initial-boundary value problem (1.17) – (1.22) to the initial data $S_n^{(0)}$ constructed as in the proof of Lemma 1.1. For the functions S_n the statements (i) and (ii) of Lemma 3.1 hold, with the same proof. This allows to deduce the following uniform estimate:

Lemma 4.1 *The function S_n satisfies*

$$\operatorname{var} S_n(\cdot, t) \le \operatorname{var} S_n^{(0)} \le 2,$$

for every n and all $t > 0$.

Proof: From Lemma 3.1 (i) we obtain for every n and for every jump curve $\alpha \in \mathcal{J}_n$ that

$$|[S_n](\alpha)| \le |S_n^{(0)}(y_\alpha+) - S_n^{(0)}(x_\alpha-)| \le \operatorname{var}\left(S_n^{(0)} \;\middle|\; [x_\alpha-, y_\alpha+]\right), \tag{4.2}$$

where x_α and y_α are defined as in that lemma and where

$$\operatorname{var}\left(S_n^{(0)} \;\middle|\; [x_\alpha-, y_\alpha+]\right) = \lim_{\eta \searrow 0} \operatorname{var}\left(S_n^{(0)} \;\middle|\; [x_\alpha - \eta, y_\alpha + \eta]\right).$$

Moreover, if $\alpha_1, \ldots, \alpha_k \in \mathcal{J}_n$ are the jump curves intersecting the line segment $(a, b) \times \{t\}$, ordered such that $\alpha_1(t) < \alpha_2(t) < \ldots < \alpha_k(t)$, then Lemma 3.1 (ii) implies

$$x_{\alpha_1} \le y_{\alpha_1} < x_{\alpha_2} \le y_{\alpha_2} < \ldots < x_{\alpha_k} \le y_{\alpha_k}\,.$$

Noting these inequalities we infer from (4.2)

$$\operatorname{var} S_n(\cdot,t) = \sum_{i=1}^{k} |[S_n](\alpha_i)| = \sum_{i=1}^{k} |S_n^{(0)}(y_{\alpha_i}+) - S_n^{(0)}(x_{\alpha_i}-)| \le \operatorname{var} S_n^{(0)} \le 2\,.$$

The proof is complete.

Based on the estimate in this lemma we can repeat the proof of Lemma 3.3 for the functions (u_n, T_n, S_n), with minor changes. Consequently, the inequalities (3.7) – (3.9) hold for (u_n, T_n, S_n). Exactly as in the proofs of Claim 1 and Claim 3 in the preceding section we thus obtain

Lemma 4.2 *The sequence $\{(u_n, T_n, S_n)\}_n$ has a subsequence, again denoted by $\{(u_n, T_n, S_n)\}_n$, which converges in the norm of the space $L^2(Z_{T_e}, \mathbb{R}^3 \times \mathcal{S}^3) \times L^p(Z_{T_e}, \mathbb{R})$ to a function*

$$(u, T, S) \in H_1(Z_{T_e}, \mathbb{R}^3) \times L^2(Z_{T_e}, \mathcal{S}^3) \times BV(Z_{T_e}, \mathbb{R})\,,$$

for every $T_e > 0$ and all $1 \le p < \infty$. Moreover, $u_{n,x} \to u_x$ in $L^2(Z_{T_e}, \mathbb{R}^3)$ and $u_{n,t} \rightharpoonup u_t$, weakly in $L^2(Z_{T_e}, \mathbb{R}^3)$, for all $T_e > 0$. The function (u, T, S) satisfies the equations (1.17), (1.18), (1.20), *the initial condition* (1.21) *and the Clausius-Duhem inequality* (1.22).

Convergence of the variation measures $|S_{n,t}|$. The remainder of this article is devoted to the proof that (u, T, S) satisfies the evolution equation (1.19). Since (u_n, T_n, S_n) satisfies (1.19), it suffices for the proof to show that $|S_{n,t}| \overset{*}{\rightharpoonup} |S_t|$ and $|C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b| \overset{*}{\rightharpoonup} |C_1(u_x, S)_x - u_x \cdot b|$. To prove the first of these relations we first study the jump curves of S_n and state some estimates used in later parts of our investigation.

Lemma 4.3 *Assume that condition* (1.26) *holds.*
(i) *The jump of C_1 along any jump curve $\alpha \in \mathcal{J}_n$ satisfies*

$$[C_1](\alpha) = f(\alpha, S_n, \sigma)\,[S_n](\alpha), \tag{4.3}$$

where the function

$$\begin{aligned} f(\alpha, S_n, \sigma) \;&=\; (\overline{\varepsilon} - \varepsilon^*) \cdot D(\overline{\varepsilon} - \varepsilon^*)\,\langle S_n \rangle(\alpha) \\ &\quad + \varepsilon^* \cdot D\,\varepsilon^* \,\frac{1}{b-a} \int_a^b S_n(y,t)dy - \overline{\varepsilon} \cdot \sigma(x,t) + \frac{[\psi_1](\alpha)}{[S_n](\alpha)} \end{aligned}$$

can be estimated by

$$M_1 \le f(\alpha, S_n, \sigma) \le M_2\,. \tag{4.4}$$

Here $M_1 > 0$ is the constant in (1.26) *and*

$$M_2 = (\overline{\varepsilon} - \varepsilon^*) \cdot D(\overline{\varepsilon} - \varepsilon^*) + \varepsilon^* \cdot D\varepsilon^* + \|\overline{\varepsilon} \cdot \sigma\|_{L^\infty(Z,\mathbb{R})} + \sup_{0 \le s \le 1} |\psi_1'(s)|\,.$$

(ii) *With the constant $c > 0$ from (1.19) let $\mathcal{V}_1 = cM_1$ and $\mathcal{V}_2 = cM_2$. For every jump curve $\alpha \in \mathcal{J}_n$ the speed of propagation satisfies*

$$0 < \mathcal{V}_1 \leq \pm \frac{d\alpha}{dt}(t) \leq \mathcal{V}_2 \,, \tag{4.5}$$

where the plus sign holds if $[S_n](\alpha) > 0$ and the minus sign is valid if $[S_n](\alpha) < 0$.

Proof: (4.3) follows from (2.16), (2.17) noting that $\frac{1}{2}\,[S^2] = \frac{1}{2}\,(S^+ - S^-)(S^+ + S^-) = [S]\langle S\rangle$ and that $0 \leq S_n \leq 1$. The inequality (4.4) follows from (1.26) by a direct computation, and (4.5) is a consequence of (4.4) and of (2.20), (2.16), which yield

$$\frac{d\alpha}{dt} = c\,\frac{[C_1](\alpha)}{|[S_n](\alpha)|} = c\, f(\alpha, S_n, \sigma)\,\mathrm{sign}[S_n](\alpha).$$

Corollary 4.4 *If* (1.26) *holds then $S_{n,t} \leq 0$ for every n. Therefore the variation measure satisfies $|S_{n,t}| = -S_{n,t}$. For the limit function S the distributional derivative S_t is a measure and, in the sense of measures, $S_{n,t} \stackrel{*}{\rightharpoonup} S_t$. Thus,*

$$S_t \leq 0, \quad |S_t| = -S_t, \quad |S_{n,t}| = -S_{n,t} \stackrel{*}{\rightharpoonup} -S_t = |S_t|.$$

Proof: Equation (2.21), applied to the piecewise constant function S_n, and (2.26) together yield

$$S_{n,t} = n''[S_n]\,\mathcal{H}_J = -n'\,\frac{d\alpha}{dt}\,[S_n]\,\mathcal{H}_J. \tag{4.6}$$

Here $n' > 0$, by our choice of the normal vector (n', n''). From (4.5) we thus infer that $-n'\frac{d\alpha}{dt}[S_n](\alpha) < 0$ for all jump curves $\alpha \in \mathcal{J}_n$, whence $S_{n,t} \leq 0$, by (4.6). The definition of the variation measure now immediately yields $|S_{n,t}| = -S_{n,t}$.

$S_{n,t} \stackrel{*}{\rightharpoonup} S_t$ follows as in the proof of Claim 2 in Section 3. This convergence implies for $\varphi \in \overset{\circ}{C}_1(Z, \mathbb{R})$ with $\varphi \geq 0$ that

$$-\int_Z S\,\varphi_t\,d(x,t) = -\lim_{n\to\infty}\int_Z S_n\,\varphi_t\,d(x,t) = \lim_{n\to\infty}\int_Z \varphi\,dS_{n,t} \leq 0\,.$$

Since $\overset{\circ}{C}_1(Z, \mathbb{R})$ is dense in $\overset{\circ}{C}(Z, \mathbb{R})$, this equation yields $S_t \leq 0$. The remaining statements in the corollary are now obvious.

The positive and negative parts of the measure $C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b$. The proof that $|C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b| \stackrel{*}{\rightharpoonup} |C_1(u_x, S)_x - u_x \cdot b|$ cannot be based on the simple idea used to verify $|S_{n,t}| \stackrel{*}{\rightharpoonup} |S_t|$. For, since the initial data $S_n^{(0)}$ are increasing on $[a, c]$ and decreasing on $[c, b]$ it follows that $[S_n](\alpha)$ has negative and positive values, depending on the jump curve α. Because (4.3) and (4.4) together imply $\mathrm{sign}[C_1](\alpha) = \mathrm{sign}[S_n](\alpha)$, also $[C_1](\alpha)$ has negative and positive values, hence the measure

$$C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b = n'[C_1]\,\mathcal{H}_J$$

does not have a sign. The last equation is obtained from (2.22), applied to the piecewise constant function S_n. To prove convergence also in this situation we introduce the positive and negative parts of this measure:

Definition 4.5 *Let $\alpha \in \mathcal{J}_n$ be a jump curve. For $C_1 = C_1(u_{n,x}, S_n)$ we set*

$$[C_1]_+(\alpha) = \frac{1}{2}\big(|\,[C_1](\alpha)\,| + [C_1](\alpha)\big)\,, \qquad [C_1]_-(\alpha) = \frac{1}{2}\big(|\,[C_1](\alpha)\,| - [C_1](\alpha)\big)\,,$$

$$\nu_n = n'[C_1]\,\mathcal{H}_J = C_{1x} - u_{n,x}\cdot b\,, \qquad \nu_n^{\pm} = n'[C_1]_{\pm}\,\mathcal{H}_J\,.$$

The measures $\nu_n^{\pm}$ are the positive and negative parts of the measure ν, and we have

$$\nu_n^{\pm} \geq 0\,, \qquad \nu_n = \nu_n^{+} - \nu_n^{-}\,, \qquad |\nu_n| = \nu_n^{+} + \nu_n^{-}\,. \tag{4.7}$$

Lemma 4.6 (i) *For the limit function (u, T, S) the distributional derivative $C_1(u_x, S)_x - u_x \cdot b$ is a measure, which we denote by ν. We have $\nu_n \overset{*}{\rightharpoonup} \nu$.*
(ii) *There is a subsequence of $\{(u_n, T_n, S_n)\}_n$, again denoted by $\{(u_n, T_n, S_n)\}_n$, such that the corresponding subsequences $\{\nu_n^{+}\}_n$, $\{\nu_n^{-}\}_n$ converge weak–$*$ to measures ν^{+} and ν^{-}, respectively. These measures satisfy $\nu^{+}, \nu^{-} \geq 0$.*

Proof: Above we remarked that Lemma 3.3 holds for (u_n, T_n, S_n). We therefore obtain from (3.8) and from (4.7)

$$\nu_n^{\pm}(Z_{T_e}) \leq |\nu_n|(Z_{T_e}) \leq AT_e \operatorname{var} S_n^{(0)} \leq 2AT_e\,, \tag{4.8}$$

for every $T_e > 0$. The last inequality sign in this estimate follows from Lemma 4.1. Using this estimate for ν_n we can show exactly as in the proof of Claim 2 in Section 3 that $\nu = C_1(u_x, S)_x - u_x \cdot b$ is a measure and that $\nu_n \overset{*}{\rightharpoonup} \nu$. Also, since by (4.8) $\sup_n \nu_n^{\pm}(Z_{T_e}) < \infty$, the sequences of Radon measures $\{\nu_n^{\pm}\}_n$ have subsequences, which converge weak–$*$ to Radon measures $\nu^{\pm}$, cf. [13, p.55]. This proves the lemma.

(4.7) implies for the weak–$*$ limits

$$\begin{aligned} \nu^{+} - \nu^{-} &= \lim_{n\to\infty} (\nu_n^{+} - \nu_n^{-}) = \lim_{n\to\infty} \nu_n = \nu, && (4.9)\\ \nu^{+} + \nu^{-} &= \lim_{n\to\infty} (\nu_n^{+} + \nu_n^{-}) = \lim_{n\to\infty} |\nu_n|, && (4.10) \end{aligned}$$

but in a general situation the measures ν^{+} and ν^{-} are not necessarily equal to the positive and negative part of ν; hence $\nu^{+} + \nu^{-}$ can be different from $|\nu|$. Therefore in the remainder our goal is to prove that in the present situation we indeed have $|\nu| = \nu^{+} + \nu^{-}$. From (4.10) we then obtain $|\nu_n| \overset{*}{\rightharpoonup} |\nu|$, which is our desired result.

The limit measures ν^{+} and ν^{-}. To simplify the notation we extend ν_n to a measure von $\mathbb{R}^2$ by defining $\nu_n(V) = \nu_n(V \cap Z)$ for $V \subseteq \mathbb{R}^2$. The same extension is used for the other measures. By $B(z)$ we denote an open ball in $\mathbb{R}^2$ with center $z = (x, t)$ and positive radius $r \leq 1$. To specify the radius we write $B_r(z)$. The numbers δ and η are assumed to belong to the countable set $\{\frac{1}{m} \mid m \in \mathbb{N}\}$.

Definition 4.7 *Let the sets $\tilde{E}, \tilde{F} \subseteq \mathbb{R}^2$ be defined by*

$$\begin{aligned}\tilde{E} &= \big\{z \in Z \mid \text{for all } \delta > 0 \text{ there is a ball } B(z) = \hat{B}(z,\delta) \text{ and} \\ &\qquad \text{a subsequence such that } \nu^-_{n_m}(B(z)) \le \delta \nu^+_{n_m}(B(z))\big\},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\tilde{F} &= \big\{z \in Z \mid \text{for all } \delta > 0 \text{ there is a ball } B(z) = \tilde{B}(z,\delta) \text{ and} \\ &\qquad \text{a subsequence such that } \nu^+_{n_m}(B(z)) \le \delta \nu^-_{n_m}(B(z))\big\}.\end{aligned} \tag{4.12}$$

Also, for $\delta > 0$ let the sets $\tilde{G}_\delta, \tilde{G} \subseteq \mathbb{R}^2$ be given by

$$\begin{aligned}\tilde{G}_\delta &= \big\{z \in Z \mid \text{there is } R = R(z) \text{ such that to all } 0 < r < R \text{ there is } n_0 \text{ with} \\ &\qquad \frac{1}{\delta}\nu^+_n(B_r(z)) > \nu^-_n(B_r(z)) > \delta \nu^+_n(B_r(z)), \quad n \ge n_0\big\}\end{aligned} \tag{4.13}$$

and by

$$\tilde{G} = \bigcup_{\delta > 0} \tilde{G}_\delta\,. \tag{4.14}$$

The sets $\tilde{E}$, $\tilde{F}$ and $\tilde{G}$ are not necessarily disjoint, but they satisfy

$$\tilde{E} \cup \tilde{F} \cup \tilde{G} = Z. \tag{4.15}$$

For, if $z \notin \tilde{E} \cup \tilde{F}$ then there is $\delta > 0$ such that for all balls $B(z)$ there is n_0 with

$$\nu^-_n(B(z)) > \delta \nu^+_n(B(z)), \quad \frac{1}{\delta}\nu^+_n(B(z)) > \nu^-_n(B(z)), \quad n \ge n_0\,.$$

This implies $z \in \tilde{G}_\delta \subseteq \tilde{G}$, hence (4.15) holds.

Let $\delta > 0$ and $\eta > 0$. By the Besicovitch Covering Theorem stated in the Appendix there are a number N and countable families $\mathcal{E}_\delta,\ \mathcal{F}_\delta,\ \mathcal{G}_{\delta_\eta}$,

$$\begin{aligned}\mathcal{E}_\delta &\subseteq \{\hat{B}(z,\delta) \mid z \in \tilde{E}\}, \\ \mathcal{F}_\delta &\subseteq \{\tilde{B}(z,\delta) \mid z \in \tilde{F}\}, \\ \mathcal{G}_{\delta\eta} &\subseteq \{B_r(z) \mid z \in \tilde{G}_\delta,\ r < \min(\eta, R(z))\},\end{aligned}$$

each one consisting of closure disjointed subfamilies $\mathcal{E}^{(i)}_\delta, \mathcal{F}^{(i)}_\delta, \mathcal{G}^{(i)}_{\delta\eta}, i = 1, \ldots, N$, such that

$$\tilde{E} \subseteq E_\delta := \bigcup_{B \in \mathcal{E}_\delta} B, \quad \tilde{F} \subseteq F_\delta := \bigcup_{B \in \mathcal{F}_\delta} B, \quad \tilde{G}_\delta \subseteq G_{\delta\eta} := \bigcup_{B \in \mathcal{G}_{\delta\eta}} B. \tag{4.16}$$

N depends only on the space dimension, in this case 2. We define

$$E = \bigcap_{\delta > 0} E_\delta, \quad F = \bigcap_{\delta > 0} F_\delta, \quad G_\delta = \bigcap_{\eta > 0} G_{\delta\eta}, \quad G = \bigcup_{\delta > 0} G_\delta. \tag{4.17}$$

The sets E_δ, F_δ and $G_{\delta\eta}$ are open, whence E, F and G_δ are Borel sets as countable intersections of open sets, and G is a Borel set as a countable union of Borel sets. (4.14), (4.16) and (4.17) imply

$$\tilde{E} \subseteq E, \quad \tilde{F} \subseteq F, \quad \tilde{G}_\delta \subseteq G_\delta, \quad \tilde{G} \subseteq G,$$

whence, by (4.15),

$$E \cup F \cup G = Z. \tag{4.18}$$

Lemma 4.8 *The limit measures ν^-, ν^+ satisfy*

$$\nu^-(E) = \nu^+(F) = 0 \quad \textit{and} \quad \nu^-(G) = \nu^+(G) = 0.$$

To prove this lemma we need the following result, whose proof is postponed to Sect. 5:

Proposition 4.9 *To every $\delta, \vartheta > 0$ there is $\eta_0 > 0$ such that for all $\eta \le \eta_0$ and for every finite collection $B_1, \ldots, B_l \in \mathcal{G}_{\delta\eta}$ with $B_i \subseteq Z_{T_e}$ there is k_0 such that for all $n \ge k_0$*

$$\nu_n^-\big(\bigcup_{i=1}^{l} B_i\big) \le \vartheta, \qquad \nu_n^+\big(\bigcup_{i=1}^{l} B_i\big) \le \vartheta.$$

Proof of Lemma 4.8: To prove that $\nu^-(E) = 0$ let $\delta > 0$ and let $B \in \mathcal{E}_\delta$. By definition of $\mathcal{E}_\delta$ the open ball $B = \hat{B}(z, \delta)$ satisfies the condition in (4.11), hence there is a subsequence such that $\nu_{n_m}^-(B) \le \delta\, \nu_{n_m}^+(B)$ holds for all m. Since $\nu_{n_m}^- \overset{*}{\rightharpoonup} \nu^-$ and $\nu_{n_m}^+ \overset{*}{\rightharpoonup} \nu^+$, it follows

$$\begin{aligned} \nu^-(B) &\le \liminf_{m\to\infty} \nu_{n_m}^-(B) \\ &\le \delta \liminf_{m\to\infty} \nu_{n_m}^+(B) \le \delta \limsup_{m\to\infty} \nu_{n_m}^+(B) \le \delta\, \nu^+(\overline{B}), \end{aligned} \tag{4.19}$$

cf. [13, p. 54].

For $r > 0$ we set $E(r) = \{z \in E \mid |z| < r\}$. Since $E(r) \subseteq E \subseteq E_\delta = \bigcup_{B\in\mathcal{E}_\delta} B$ and since we assumed that the radii of all balls in $\mathcal{E}_\delta$ are not greater than one we can select a subfamily $\mathcal{E}'_\delta$ of $\mathcal{E}_\delta$ such that

$$E(r) \subseteq \bigcup_{B\in\mathcal{E}'_\delta} B \subseteq \bigcup_{B\in\mathcal{E}'_\delta} \overline{B} \subseteq E(r+2).$$

Since $\mathcal{E}_\delta$ is composed of the subfamilies $\mathcal{E}_\delta^{(i)}, i = 1, \ldots, N$, we obtain from (4.19)

$$\begin{aligned} \nu^-(E(r)) &\le \nu^-\big(\bigcup_{B\in\mathcal{E}'_\delta} B\big) \le \sum_{B\in\mathcal{E}'_\delta} \nu^-(B) \le \delta \sum_{B\in\mathcal{E}'_\delta} \nu^+(\overline{B}) = \delta \sum_{i=1}^{N} \sum_{B\in\mathcal{E}'_\delta\cap\mathcal{E}_\delta^{(i)}} \nu^+(\overline{B}) \\ &= \delta \sum_{i=1}^{N} \nu^+\big(\bigcup_{B\in\mathcal{E}'_\delta\cap\mathcal{E}_\delta^{(i)}} \overline{B}\big) \le \delta \sum_{i=1}^{N} \nu^+(E(r+2)) = \delta\, N \nu^+(E(r+2)), \end{aligned}$$

where we used that the closed hulls of the balls in $\mathcal{E}_\delta^{(i)}$ are pairwise disjoint. This estimate holds for all $\delta > 0$, hence $\nu^-(E(r)) = 0$ for all $r > 0$, and so $\nu^-(E) = 0$.

The equation $\nu^+(F) = 0$ is verified in the same way, interchanging the roles of ν_n^+ and ν_n^-.

To prove that $\nu^-(G) = \nu^+(G) = 0$ let $\delta, \vartheta > 0$ and let $\eta = \eta_0(\delta, \vartheta) > 0$ be the number whose existence is assured in Proposition 4.9. Assume that $K \subseteq G_\delta$ is a compact subset. $G_{\delta\eta}$ is an open covering of K, since (4.17) implies $G_\delta \subseteq G_{\delta\eta} = \bigcup_{B \in \mathcal{G}_{\delta\eta}} B$. Therefore there exist finitely many $B_1, \ldots, B_l \in \mathcal{G}_{\delta\eta}$ such that $K \subseteq \bigcup_{i=1}^l B_i$. By Proposition 4.9 there is k_0 with

$$\nu_n^-\big(\bigcup_{i=1}^l B_i\big) \le \vartheta, \qquad \nu_n^+\big(\bigcup_{i=1}^l B_i\big) \le \vartheta$$

for all $n \ge k_0$. Since $\nu_n^- \stackrel{*}{\rightharpoonup} \nu^-$, $\nu_n^+ \stackrel{*}{\rightharpoonup} \nu^+$ and since $\bigcup_{i=1}^l B_i$ is open, we obtain

$$\nu^\pm(K) \le \nu^\pm\big(\bigcup_{i=1}^l B_i\big) \le \liminf_{n\to\infty} \nu_n^\pm\big(\bigcup_{i=1}^l B_i\big) \le \vartheta.$$

Since ϑ was chosen arbitrarily, it follows that $\nu^-(K) = \nu^+(K) = 0$. This holds for every compact subset K of G_δ. Since G_δ is a Borel set, we conclude that $\nu^-(G_\delta) = \nu^+(G_\delta) = 0$, cf. [Evans, p. 6]. Thus, $G = \bigcup_{m=1}^\infty G_{\frac{1}{m}}$ is a countable union of null sets, whence $\nu^-(G) = \nu^+(G) = 0$.

Corollary 4.10 *The measures ν^- and ν^+ satisfy $|\nu| = \nu^+ + \nu^-$. Moreover,*

$$|C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b| = |\nu_n| \stackrel{*}{\rightharpoonup} |\nu| = |C_1(u_x, S)_x - u_x \cdot b|.$$

Proof: From (4.18) we see that the complement $E' = Z \backslash E$ of E is a subset of $F \cup G$. Since ν^+ is a nonnegative measure we therefore obtain from Lemma 4.8 for every ν^+-measurable set R that

$$\nu^+(R) = \nu^+\left((R \cap E) \cup (R \cap E')\right) \le \nu^+(R \cap E) + \nu^+(F \cup G) = \nu^+(R \cap E),$$

hence $\nu^+(R) = \nu^+(R \cap E)$. Similarly, $\nu^-(R) = \nu^-(R \cap E')$. By definition of the variation measure $|\nu|$ we have

$$|\nu|(R) = \sup \sum_{i=1}^l |\nu(R_i)|,$$

where the supremum is taken over all finite collections $\{R_i\}$ of ν-measurable, pairwise disjoint sets with $R_i \subseteq R$. With $\{R_i\}$ also $\{R_i \cap E\} \cup \{R_i \cap E'\}$ is such a collection. Thus,

$$|\nu|(R) = \sup \sum_{i=1}^l |\nu(R_i)| \le \sup \sum_{i=1}^l \left(|\nu(R_i \cap E)| + |\nu(R_i \cap E')|\right) \le |\nu|(R).$$

Using (4.9) we thus conclude for any measurable subset R of Z that

$$\begin{aligned}
|\nu|(R) &= \sup \sum_{i=1}^{l} \big(|(\nu^+ - \nu^-)(R_i \cap E)| + |(\nu^+ - \nu^-)(R_i \cap E')| \big) \\
&= \sup \sum_{i=1}^{l} \big(|\nu^+(R_i)| + |\nu^-(R_i)| \big) = \sup \sum_{i=1}^{l} \big(\nu^+(R_i) + \nu^-(R_i) \big) \\
&= \sup \left(\nu^+ \big(\bigcup_{i=1}^{l} R_i \big) + \nu^- \big(\bigcup_{i=1}^{l} R_i \big) \right) = \nu^+(R) + \nu^-(R).
\end{aligned}$$

This proves that $|\nu| = \nu^+ + \nu^-$. The relation $|\nu_n| \overset{*}{\rightharpoonup} |\nu|$ follows from this equation and from (4.10). The proof is complete.

End of the proof of Theorem 1.3: By Lemma 4.2 the function (u, T, S) satisfies the equations and inequalities (1.17), (1.18), (1.20), (1.21) and (1.22). To see that also equation (1.19) is satisfied remember that by construction (u_n, T_n, S_n) fulfills this equation. From Corollary 4.4 and Corollary 4.10 we thus obtain for the weak–$*$ limits

$$|S_t| = \lim_{n\to\infty} |S_{n,t}| = \lim_{n\to\infty} c \left| C_1(u_{n,x}, S_n)_x - u_{n,x} \cdot b \right| = c \left| C_1(u_x, S)_x - u_x \cdot b \right| .$$

Consequently, (u, T, S) satisfies also the evolution equation (1.19).

5 Proof of Proposition 4.9

This section is devoted to the proof of Proposition 4.9. We start by stating and verifying several auxiliary lemmas. The idea of the proof of the proposition is explained at the beginning of that proof, and we advice the reader to study that part first.

Definition 5.1 *For a jump curve $\alpha \in \mathcal{J}_n$ let*

$$[S_n]_+(\alpha) = \frac{1}{2}\big(|\,[S_n](\alpha)\,| + [S_n](\alpha) \big), \quad [S_n]_-(\alpha) = \frac{1}{2}\big(|\,[S_n](\alpha)\,| - [S_n](\alpha) \big) .$$

With the Hausdorff measure $\mathcal{H}_J$ and with the first component $n' > 0$ of the unit normal vector (n', n'') to the jump curve α define

$$\mu_n = n'[S_n]\,\mathcal{H}_J\,, \qquad \mu_n^{\pm} = n'[S_n]_{\pm}\,\mathcal{H}_J\,.$$

μ_n is a signed measure and $\mu_n^{\pm}$ are Radon measures, the positive and negative part of μ_n. Lemma 4.3 yields

$$\nu_n = n'[C_1]\,\mathcal{H}_J = n' f [S_n]\,\mathcal{H}_J \quad \text{and} \quad \nu_n^{\pm} = n'[C_1]_{\pm}\,\mathcal{H}_J = n' f [S_n]_{\pm}\,\mathcal{H}_J, \, .$$

From (4.4) we therefore obtain for any measurable subset V

$$M_1\, \mu_n^{\pm}(V) \le \nu_n^{\pm}(V) \le M_2\, \mu_n^{\pm}(V). \tag{5.1}$$

This shows that the measures $\nu_n^{\pm}$ can be estimated above and below by the measures $\mu_n^{\pm}$. We use this to derive the inequalities for $\nu_n^{\pm}$ in Proposition 4.9 from analogous inequalities for $\mu_n^{\pm}$.

For a jump curve $\alpha \in \mathcal{J}_n$ satisfying $\alpha \cap Z_{T_e} \neq \emptyset$ we call the curve with graph $\alpha \cap Z_{T_e}$ a jump curve in Z_{T_e}. If $\alpha_1, \ldots, \alpha_l \in \mathcal{J}_k$ are jump curves in Z_{T_e} such that the endpoint of α_i coincides with the starting point of α_{i+1} for every $i = 1, \ldots, l-1$, we say that the curve with graph $\alpha_1 \cup \ldots \cup \alpha_l \subseteq Z_{T_e}$ is the composition of $\alpha_1, \ldots, \alpha_l$. The composed curve is said to pass over the jump curve α_j for all $j = 1, \ldots, l$. The composition is called of maximal length if there is no proper extension in Z_{T_e}.

Definition 5.2 *A composition of jump curves in Z_{T_e} of maximal length is called a chain. The set of chains is denoted by Δ_n. The subset of all chains with starting point on the line segment $(a, c) \times \{0\}$ is denoted by Δ_n^+, the subset of chains with starting point on $(c, b) \times \{0\}$ is Δ_n^-.*

Note that every chain α starts at the line segment $(a, b) \times \{0\}$ and ends at a point $(x_\alpha, t_\alpha) \in \partial Z_{T_e}$ with $t_\alpha > 0$, whence $\Delta_n = \Delta_n^+ \cup \Delta_n^-$. We always identify the chain α with its parametrization $\alpha : [0, t_\alpha] \to [a, b]$ and with the graph of this parametrization, a subset of $\overline{Z_{T_e}}$. Note that several different chains can pass over one and the same jump curve $\alpha \in \mathcal{J}_n$.

For chains α and β we write $\alpha \le \beta$ if the starting points $(x, 0)$ of α and $(y, 0)$ of β satisfy $x \le y$. If the graphs α and β are not disjoint, we call the point $(x_0, t_0) \in \alpha \cap \beta$ with

$$t_0 = \min\{t \mid (x, t) \in \alpha \cap \beta\}$$

the point of intersection of α and β. The construction of S_n in Sect. 2 implies that two chains coincide for $t \ge t_0$, hence they have at most one point of intersection.

Definition 5.3 *For a chain α we define the strength $|\alpha| : [0, t_\alpha] \to [0, \infty)$ as follows: Let $0 < t_1 < \ldots t_{m-1} < T_e$ with $m \ge 1$ be the times, where intersections of chains occur, and let $t_m = T_e$. For every chain $\alpha \in \Delta_n$ we set*

$$|\alpha|(t) = |[S_n^{(0)}](\alpha(0))|, \quad 0 \le t < t_1. \tag{5.2}$$

Let $1 \le i \le m-1$ and assume that $|\alpha|(t)$ is defined for every chain α and for every $0 \le t < \min(t_\alpha, t_i)$. Assume that the point (x, t_i) belongs to the graphs of the chains $\alpha_1, \ldots, \alpha_k \in \Delta_n^+$ and $\beta_1, \ldots, \beta_l \in \Delta_n^-$, and let

$$h = \sum_{j=1}^{k} |\alpha_j|(t_i-) - \sum_{j=1}^{l} |\beta_j|(t_i-). \tag{5.3}$$

If $h \ge 0$ choose h_j satisfying

$$0 \le h_j \le |\alpha_j|(t_i-), \quad \sum_{j=1}^{k} h_j = h \tag{5.4}$$

and define

$$|\alpha_j|(t) = h_j \quad \textit{for} \quad t_i \le t < \min(t_{i+1}, t_{\alpha_j}), \quad j = 1, \ldots, k,$$
$$|\beta_j|(t) = 0 \quad \textit{for} \quad t_i \le t < \min(t_{i+1}, t_{\beta_j}), \quad j = 1, \ldots, l.$$

If $h < 0$ choose h_j satisfying

$$0 \le h_j \le |\beta_j|(t_i-), \quad \sum_{j=1}^{k} h_j = |h| \tag{5.5}$$

and define

$$|\alpha_j|(t) = 0 \quad \textit{for} \quad t_i \le t < \min(t_{i+1}, t_{\alpha_j}), \quad j = 1, \ldots, k,$$
$$|\beta_j|(t) = h_j \quad \textit{for} \quad t_i \le t < \min(t_{i+1}, t_{\beta_j}), \quad j = 1, \ldots, l.$$

Lemma 5.4 (i) *The strength is a decreasing function satisfying*

$$0 \le |\alpha|(t) \le |\alpha|(0) = |[S_n^{(0)}](\alpha(0))| \,.$$

(ii) *Let (x,t) belong to the graphs of the chains $\alpha_1, \ldots, \alpha_k \in \Delta_n^+$ and $\beta_1, \ldots, \beta_l \in \Delta_n^-$. Then*

$$[S_n](x,t) = \sum_{j=1}^{k} |\alpha_j|(t) - \sum_{j=1}^{l} |\beta_j|(t). \tag{5.6}$$

(iii) *Moreover, either $|\alpha_j|(t) = 0$ for $j = 1, \ldots, k$ or $|\beta_j|(t) = 0$ for $j = 1, \ldots, l$.*

Proof: (i) follows immediately from Definition 5.3. To verify (ii) let $0 < t_1 < \ldots < t_{m-1}$ be the intersection times of chains. Let (x,t) with $t > 0$ belong to the graph of a jump curve $\chi \in \mathcal{J}_n$. If $t < t_1$ then χ starts at the line segment $(a,b) \times \{0\}$, and exactly one chain α passes over χ. The jump curve α belongs to Δ_n^+ or to Δ_n^-, respectively, if $a < \chi(0) = \alpha(0) < c$ or if $c < \chi(0) < b$, respectively. Since $S_n^{(0)}$ is increasing on (a,c) and decreasing on (c,b), we thus obtain from (5.2)

$$[S_n](x,t) = [S_n^{(0)}](\alpha(0)) = \begin{cases} \;\;\, |\alpha(t)|, & \text{if} \quad \alpha \in \Delta_n^+ \,, \\ -|\alpha(t)|, & \text{if} \quad \alpha \in \Delta_n^- \,. \end{cases}$$

This proves (5.6) for $t < t_1$. Assume next that $t_{i+1} > t > t_i$ and that (5.6) holds in Z_{t_i}. It follows that the point $(\chi(t_i), t_i)$ belongs to the graph of χ. If it is the starting point of χ then there are jump curves $\chi_1 \le \chi_2 \le \ldots \le \chi_m \in \mathcal{J}_n$ which all end at $(\chi(t_i), t_i)$. If $(\chi(t_i), t_i)$ is not the starting point of χ we can still consider it as the end point of the part of χ in the set Z_{t_i}. We denote this part by χ_1. In this case we have $m = 1$. The sets of chains $\alpha_1, \ldots, \alpha_k \in \Delta_n^+$ and $\beta_1, \ldots, \beta_l \in \Delta_n^-$ passing over (x,t) can be partitioned into subsets of chains passing over $\chi_1, \ldots, \chi_m$, and every chain passing over one of $\chi_1, \ldots, \chi_m$ also passes over (x,t). On the one hand, if χ_1 is the leftmost of the jump curves $\chi_1, \dots, \chi_m$ and χ_m the rightmost, then our construction of S_n in Sect. 2 implies

$$[S_n](x,t) = [S_n](\chi) = S_n(\chi_m+) - S_n(\chi_1-) = \sum_{j=1}^{m} [S_n](\chi_j). \tag{5.7}$$

On the other hand, by our assumption we have

$$\sum_{j=1}^{m} [S_n](\chi_j) = \sum_{p=1}^{k} |\alpha_p|(t_i-) - \sum_{q=1}^{l} |\beta_q|(t_i-) = \sum_{p=1}^{k} |\alpha_p|(t) - \sum_{q=1}^{l} |\beta_q|(t),$$

where we used (5.3) – (5.5) to get the last equality sign. This equation and (5.7) together imply (5.6).

(iii) is an immediate consequence of Definition 5.3. This completes the proof of the lemma.

It follows from statements (i) and (ii) of this lemma that

$$[S_n]_\pm(x,t) = \sum_{\substack{\alpha \in \Delta_n^\pm \\ \alpha(t)=x}} |\alpha|(t). \tag{5.8}$$

Thus, Definition 5.1 yields for every measurable set $V \subseteq Z_{T_e}$ with characteristic function χ_V that

$$\mu_n^\pm(V) = \sum_{\alpha \in \Delta_n^\pm} \int_\alpha \chi_V(\alpha(t),t)|\alpha|(t) n' ds = \sum_{\alpha \in \Delta_n^\pm} \int_0^{T_e} \chi_V(\alpha(t),t)|\alpha|(t) dt\,, \tag{5.9}$$

where we used that $n'ds = dt$, and extended the function $|\alpha|$ from the domain $[0, t_\alpha]$ to $[0, T_e]$ by zero. Consequently, the measure μ_n^+ vanishes on any set, which is not intersected by chains from Δ_n^+, and μ_n^- vanishes on any set not intersected by chains from Δ_n^-. (4.5) and (5.6) imply that the jump curve $\chi \in \mathcal{J}_n$ has positive slope if only chains from Δ_n^+ pass over χ, and negative slope if only chains from Δ_n^- pass over χ. In particular, chains from Δ_n^+ have positive slope until they intersect a chain from Δ_n^-, and vice versa.

There is at most one curve over which chains both from Δ_n^+ and Δ_n^- pass. Namely, let $\overline{\alpha}_n$ be the maximal chain from Δ_n^+, i.e. the chain $\overline{\alpha}_n \in \Delta_n^+$ satisfying $\alpha \le \overline{\alpha}_n$ for all $\alpha \in \Delta_n^+$, and let $\hat{\beta}_n$ be the minimal chain from Δ_n^-. For all $\alpha \in \Delta_n^+$, all $\beta \in \Delta_n^-$ and all t from the common domain of definition we then have

$$\alpha(t) \le \overline{\alpha}_n(t) \le \hat{\beta}_n(t) \le \beta(t),$$

hence to the left of $\overline{\alpha}_n$ there are only chains from Δ_n^+, and to the right of $\hat{\beta}_n$ there are only chains from Δ_n^-. Let (x_0, t_0) be the point of intersection of $\overline{\alpha}_n$ and $\hat{\beta}_n$. Both chains coincide from the point of intersection on. The jump curves, which compose the common part of $\overline{\alpha}_n$ and $\hat{\beta}_n$ are the only ones over which chains both from Δ_n^+ and Δ_n^- pass.

In the following the **separation curve** given by the graph of a Lipschitz continuous function

$$\omega : [0, t_\omega] \to [a, b].$$

plays an important role. To define this function note that (4.5) implies that the sequence $\{\overline{\alpha}_n : [0, t_{\overline{\alpha}_n}] \to [a, b]\}_n$ of parametrizations is uniformly Lipschitz continuous. Therefore we can select a subsequence of $\{\overline{\alpha}_n\}_n$, again denoted by $\{\overline{\alpha}_n\}_n$, which converges uniformly to a Lipschitz continuous function, which we take to be ω.

Henceforth we go over to this subsequence and, for example, instead of using the original sequences we always work with the corresponding subsequences of $\{\mu_n\}_n$, $\{\mu_n^\pm\}_n$, which we again denote by the same symbols.

Lemma 5.5 *The set $\tilde{G}$ satisfies $\tilde{G} \subseteq \omega$.*

Proof: Let Ω_1 be the set of all points of Z_{T_e} to the left of ω, let $\Omega_2 \subseteq Z_{T_e}$ be the set of all points to the right of ω and let $(x, t) \in \Omega_1$. Since Ω_1 is open, (x, t) has distance $R > 0$ to ω. Therefore, because $\overline{\alpha}_n$ converges uniformly to ω, there is n_0 such that (x, t) lies to the left of $\overline{\alpha}_n$ and has distance $\geq R/2$ to $\overline{\alpha}_n$ for all $n \geq n_0$. Consequently, for all $r < R/2$ and $n \geq n_0$ the ball $B_r(x, t)$ is not intersected by chains from Δ_n^-, hence by (5.1)

$$\nu_n^-(B_r(x, t)) \leq M_2\, \mu_n^-(B_r(x, t)) = 0.$$

This contradicts (4.13) for every $\delta > 0$, hence $(x, t) \notin \tilde{G}_\delta$, thence $\Omega_1 \cap \tilde{G}_\delta = \emptyset$ for all $\delta > 0$. This implies that $\Omega_1 \cap \tilde{G} = \Omega_1 \cap \bigcup_{\delta > 0} \tilde{G}_\delta = \emptyset$. In the same way it is shown that $\Omega_2 \cap \tilde{G} = \emptyset$. Since $Z_{T_e} = \Omega_1 \cup \omega \cup \Omega_2$ it follows $\tilde{G} \subseteq \omega$. The lemma is proven.

In the proof of Proposition 4.9 we need some auxiliary lemmas, which we state and prove now. To this end we need some more definitions and notations:

For a chain $\alpha \in \Delta_n$ the strength $|\alpha|$ is a decreasing function. The function α^κ is obtained from α by cutting the "tail" where $|\alpha|$ is small: If κ is a given number with $0 < \kappa \leq 1$ let α^κ be the restriction

$$\alpha^\kappa = \alpha|_{[0, t_{\alpha^\kappa})},$$

where $t_{\alpha^\kappa} = \sup\{t \mid |\alpha|(t) \geq \kappa|\alpha|(0)\}$. Clearly, this definition implies for all t from the domain of definition of α^κ that

$$|\alpha^\kappa|(t) \geq \kappa|\alpha^\kappa|(0) = \kappa|[S_n^{(0)}](\alpha(0))|. \tag{5.10}$$

The measure μ_n^+ is "generated" by the chains in Δ_n^+. We next define measures generated by subsets of Δ_n^+. To this end let

$$P : \mathbb{R}^2 \to \mathbb{R}, \quad P(x, t) = t$$

be the projection to the t–axis, let $\Gamma \subseteq \Delta_n^+$ and let $V \subseteq Z_{T_e}$ be a μ_n^+–measurable set. The Radon measure $\mu_{\Gamma,\kappa}^+$ is defined by

$$\mu_{\Gamma,\kappa}^+(V) = \sum_{\alpha\in\Gamma} \int_{P(\alpha^\kappa\cap V)} |\alpha^\kappa|(t)\, dt\,. \tag{5.11}$$

Of course, this measure satisfies $0 \le \mu_{\Gamma,\kappa}^+ \le \mu_n^+$.

Finally, we denote the Lebesgue measure of the one-dimensional set $P(\alpha^\kappa \cap V)$ by $\operatorname{meas} P(\alpha^\kappa \cap V)$.

The first auxiliary lemma is

Lemma 5.6 *Let $L > 0$, let $\Gamma \subseteq \Delta_n^+$ and let V be a finite union of balls such that* $\operatorname{meas} P(\alpha^\kappa \cap V) \ge L$ *for all $\alpha \in \Gamma$. Then*

$$\mu_{\Gamma,\kappa}^+(Z_{T_e}) \le \frac{T_e}{L\kappa}\, \mu_{\Gamma,\kappa}^+(V).$$

Proof:

$$\begin{aligned}
\mu_{\Gamma,\kappa}^+(Z_{T_e}) &= \sum_{\alpha\in\Gamma} \int_{P(\alpha^\kappa)} |\alpha^\kappa|(t)dt \le \sum_{\alpha\in\Gamma} T_e\, |\alpha^\kappa|(0) \le \sum_{\alpha\in\Gamma} \frac{T_e}{L} \int_{P(\alpha^\kappa\cap V)} |\alpha^\kappa|(0)dt \\
&\le \frac{T_e}{L\kappa} \sum_{\alpha\in\Gamma} \int_{P(\alpha^\kappa\cap V)} |\alpha^\kappa|(t)dt = \frac{T_e}{L\kappa}\, \mu_{\Gamma,\kappa}^+(V).
\end{aligned}$$

Lemma 5.7 *Let V be a finite union of balls and assume that*

$$\mu_n^+(V) \ge 3\,\vartheta$$

with $\vartheta > 0$. Let $L = \frac{\vartheta}{2}$, $\kappa = \frac{\vartheta}{2T_e}$, and let Γ be the set of all $\alpha \in \Delta_n^+$ such that $\operatorname{meas} P(\alpha^\kappa \cap V) \ge L$. *Then*

$$\mu_{\Gamma,\kappa}^+(V) \ge \vartheta.$$

Proof: Every $\alpha \in \Delta_n^+$ is a composition of α^κ and of a curve $\hat\alpha$, where $\hat\alpha$ satisfies $|\hat\alpha|(t) \le \kappa|\alpha|(0)$ for all t from the domain of $\hat\alpha$. Thus, with the definition of $\mu_{\Gamma,\kappa}^+(V)$ in (5.11),

$$\begin{aligned}
\mu_n^+(V) &= \sum_{\alpha\in\Delta_n^+} \int_{P(\alpha\cap V)} |\alpha|(t)dt \qquad (5.12)\\
&= \mu_{\Gamma,\kappa}^+(V) + \sum_{\alpha\in\Delta_n^+\setminus\Gamma} \int_{P(\alpha^\kappa\cap V)} |\alpha^\kappa|(t)dt + \sum_{\alpha\in\Delta_n^+} \int_{P(\hat\alpha\cap V)} |\hat\alpha|(t)dt
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{\alpha\in\Delta_n^+\setminus\Gamma} \int_{P(\alpha^\kappa\cap V)} |\alpha^\kappa|(t)dt &\le \sum_{\alpha\in\Delta_n^+\setminus\Gamma} \int_{P(\alpha^\kappa\cap V)} |\alpha|(0)dt \qquad (5.13)\\
&\le L \sum_{\alpha\in\Delta_n^+} |[S_n^{(0)}](\alpha(0))| \le L\operatorname{var} S_n^{(0)} \le 2L = \vartheta\,,
\end{aligned}$$

where we used that $P(\alpha^\kappa \cap V) < L$ for all $\alpha \in \Delta_n^+ \backslash \Gamma$. Also,

$$\sum_{\alpha\in\Delta_n^+} \int_{P(\hat\alpha\cap V)} |\hat\alpha|(t)dt \;\le\; \sum_{\alpha\in\Delta_n^+} \int_{P(\hat\alpha\cap V)} \kappa|\alpha|(0)dt \tag{5.14}$$

$$\le\; T_e\kappa \sum_{\alpha\in\Delta_n^+} |[S_n^{(0)}](\alpha(0))| \le T_e\kappa \,\mathrm{var}\, S_n^{(0)} \le 2T_e\kappa = \vartheta\,,$$

The statement of the lemma follows from (5.12) – (5.14).

Lemma 5.8 *For every ball $B(z)$ with center $z \in \omega$ there is a number n_0 such that for all $n \ge n_0$ the following holds: Let $\alpha, \beta \in \Delta_n^+$ with $\beta \le \alpha$, let $[0, t^{\alpha^\kappa}]$ be the domain of α^κ and assume that $B(z) \subseteq Z_{t^{\alpha^\kappa}}$. Then*

$$\mathrm{meas}\, P(\beta^\kappa \cap B(z)) \le \sqrt{1+4\mathcal{V}_2^2}\; \mathrm{meas}\, P(\alpha^\kappa \cap B(z)). \tag{5.15}$$

$\mathcal{V}_2$ is the constant from (4.5).

Proof: I.) First we show that for n_0 sufficiently large we can assume that the chain $\alpha \in \Delta_n^+$ and the center $z = (\tilde x, \tilde t) \in \omega$ of $B(z)$ satisfy

$$\alpha(\tilde t) < \tilde x\,. \tag{5.16}$$

For, ω is the uniform limit of a sequence of chains $\{\overline{\alpha}_n\}_n$ satisfying $\chi \le \overline{\alpha}_n$ for all $\chi \in \Delta_n^+$. It thus follows that to $\rho > 0$ there is n_0 such that for all $n \ge n_0$ and all $(x,t) \in \omega$

$$\alpha(t) \le \overline{\alpha}_n(t) < x + \rho.$$

If (5.16) does not hold we therefore have

$$\tilde x \le \alpha(\tilde t) < \tilde x + \rho.$$

Using that $|\frac{d\alpha}{dt}| \le \mathcal{V}_2$, by (4.5), we conclude from this inequality by a simple geometrical consideration that if ρ is less than the radius r of the ball $B(z)$ then $(\alpha(t), t) \in B(z)$ for all $t \in U = (\tilde t - h, \tilde t + h)$ with $h = \mathcal{V}_2(1+\mathcal{V}_2^2)^{-1}(r-\rho)$. Thus, $U \subseteq P(\alpha \cap B(z))$. Now we deform the curve α in this neighborhood U of $\tilde t$ such that (5.16) is satisfied by the deformed curve. From the value of h given above we immediately see that if we choose $\rho < \frac{\mathcal{V}_2^2}{1+2\mathcal{V}_2^2} r$ this can be done such that the deformed curve satisfies

$$\begin{aligned} &1.\quad \alpha(\tilde t) < \tilde x, \\ &2.\quad |\frac{d\alpha}{dt}| \le 2\mathcal{V}_2\,, \\ &3.\quad (\alpha(t), t) \in B(z) \text{ for all } t \in U. \end{aligned} \tag{5.17}$$

The third property implies that $P(\alpha \cap B(z))$ is not changed by the deformation. The same argument also shows that we can deform β, if necessary, such that the deformed curves α and β satisfy $\beta \leq \alpha$, with $P(\beta \cap B(z))$ unchanged. Therefore, since both sides of (5.15) are not changed by the deformation, it suffices to prove this inequality for the deformed curves satisfying (5.16) and (5.17). The number n_0 only depends on $\mathcal{V}_2^2(1+2\mathcal{V}_2^2)^{-1}r$, hence it only depends on the radius r of $B(z)$.

II.) We assume that (5.16) holds. Let $\mathcal{K}$ be a connected component of $\alpha^\kappa \cap B(z)$, and let $\beta_{\mathcal{K}}$ be the subset of all $z_\beta = (x_\beta, t_\beta) \in \beta^\kappa \cap B(z)$ with the property that the radius vector from the center z to z_β intersects $\mathcal{K}$. The requirement $B(z) \subseteq Z_{t^{\alpha^\kappa}}$ implies that every connected component $\mathcal{K}$ is an arc which starts and ends at the boundary $\partial B(z)$. This fact, (5.16) and $\beta \leq \alpha$ together imply that every radius vector ending at a point of $\beta^\kappa \cap B(z)$ intersects α^κ, hence

$$\bigcup_{\mathcal{K}} \beta_{\mathcal{K}} = \beta^\kappa \cap B(z)\,, \tag{5.18}$$

where the union is over all connected components of $\alpha^\kappa \cap B(z)$. For every $\mathcal{K}$ the set $P(\mathcal{K})$ is an open interval and we have

$$\sum_{\mathcal{K}} \operatorname{meas} P(\mathcal{K}) = \operatorname{meas} P(\alpha^\kappa \cap B(z)). \tag{5.19}$$

Claim: We have

$$\operatorname{meas} P(\beta_{\mathcal{K}}) \leq \sqrt{1+4\mathcal{V}_2^2}\, \operatorname{meas} P(\mathcal{K}). \tag{5.20}$$

To prove this claim fix $\mathcal{K}$ and assume that $P(\mathcal{K}) = (t_1, t_2)$. Since (5.20) is obvious if $\beta_{\mathcal{K}}$ is empty, we also assume that $\beta_{\mathcal{K}} \neq \emptyset$. Since the endpoints $z_1 = (\alpha(t_1), t_1)$ and $z_2 = (\alpha(t_2), t_2)$ of the arc $\mathcal{K}$ are boundary points of $B(z)$, it follows that the line segment d connecting z_1 to z_2 is a secant to the circle $\partial B(z)$. We denote by $B^l(z)$ that part of $B(z)$, which lies to the left of d. From (5.17) we infer that the arc $\mathcal{K}$ is contained in the parallelogram

$$Q = \{(x,t) \in Z_{t^{\alpha^\kappa}} \;\mid\; |x-\alpha(t_1)| \leq 2\mathcal{V}_2(t-t_1),\; |x-\alpha(t_2)| \leq 2\mathcal{V}_2(t_2-t)\},$$

whose diagonal is d. We denote by Q^r the triangular region of Q which lies to the right of d.

The nonempty set $\beta_{\mathcal{K}}$ consists of all points of $\beta^\kappa \cap B(z)$, whose straight connection to z intersects $\mathcal{K}$. Therefore $\beta_{\mathcal{K}}$ must be separated from z by the arc $\mathcal{K}$. Since z satisfies (5.16) and $\mathcal{K}$ is a subarc of α, this can only be if $\beta_{\mathcal{K}}$ is contained in the part of $B(z)$ to the left of $\mathcal{K}$ and z is contained in the part of $B(z)$ to the right. This implies

$$\beta_{\mathcal{K}} \subseteq B^l(z) \cup Q^r,$$

whence

$$P(\beta_{\mathcal{K}}) \subseteq P(B^l(z)) \cup P(Q^r) = P(B^l(z)). \tag{5.21}$$

To obtain the last equality we used that $P(Q^r) = (t_1, t_2) \subseteq P(B^l(z))$.

Thus, to estimate $\operatorname{meas} P(\beta_{\mathcal{K}})$ it suffices to estimate $\operatorname{meas} P(B^l(z))$. To this end we observe that the center z of $B(z)$ lies on the line g, which is orthogonal to the secant d of $\partial B(z)$ and intersects d in the middle. Since z is contained in the part of $B(z)$ to the right of $\mathcal{K}$, it follows that z belongs to that half line of g which is bounded at the left by the intersection of g with the arc $\mathcal{K}$. From $\mathcal{K} \subseteq Q$ we thus conclude that z also belongs to the half line g^r of g, whose left endpoint is the point of intersection z_g of g with the left boundary of the parallelogram Q. By $B(z_g)$ we denote the ball, whose boundary $\partial B(z_g)$ passes through the endpoints z_1 and z_2 of $\mathcal{K}$ and thus has secant d. It is immediately seen that $z \in g^r$ implies

$$B^l(z) \subseteq B^l(z_g).$$

From this relation and from (5.21) we conclude

$$\operatorname{meas} P(\beta_{\mathcal{K}}) \leq \operatorname{meas} P(B^l(z_g)) \leq \operatorname{meas} P(B(z_g)) = 2r_g, \tag{5.22}$$

where r_g is the radius of $B(z_g)$. To estimate this radius we use that $z_g = (x_g, t_g) \in \partial Q$. This implies that if t_0 denotes that one of the numbers t_1, t_2, which is closer to t_g, then

$$\begin{aligned} |t_g - t_0| &\leq \frac{1}{2}(t_2 - t_1), \\ |x_g - \alpha(t_0)| &\leq 2\mathcal{V}_2|t_g - t_0| \leq \mathcal{V}_2(t_2 - t_1). \end{aligned}$$

Thus,

$$r_g = |(x_g, t_g) - (\alpha(t_0), t_0)| \leq \frac{1}{2}\sqrt{1 + 4\mathcal{V}_2^2}\,(t_2 - t_1) = \frac{1}{2}\sqrt{1 + 4\mathcal{V}_2^2}\operatorname{meas} P(\mathcal{K}),$$

where we used that $P(\mathcal{K}) = (t_1, t_2)$. This estimate and (5.22) together yield (5.20) and prove the claim.

To finish the proof of the lemma we observe that (5.18) yields $P(\beta^\kappa \cap B(z)) = \bigcup_{\mathcal{K}} P(\beta_{\mathcal{K}})$. Together with (5.20) and (5.19) we thus infer

$$\begin{aligned} \operatorname{meas} P(\beta^\kappa \cap B(z)) &\leq \sum_{\mathcal{K}} \operatorname{meas} P(\beta_{\mathcal{K}}) \leq \sqrt{1 + 4\mathcal{V}_2^2} \sum_{\mathcal{K}} \operatorname{meas} P(\mathcal{K}) \\ &= \sqrt{1 + 4\mathcal{V}_2^2}\operatorname{meas} P(\alpha^\kappa \cap B(z)). \end{aligned}$$

The proof is complete.

Corollary 5.9 *For every finite union $V = \bigcup_{i=1}^k B(z_i)$ of balls $B(z_i) \in \mathcal{G}_{\delta\eta}$ there is a number n_0 such that for all $n \geq n_0$ the following holds: Let $\alpha, \beta \in \Delta_n^+$ with $\beta \leq \alpha$, let $[0, t^{\alpha^\kappa}]$ be the domain of α^κ and assume that $V \subseteq Z_{t^{\alpha^\kappa}}$. Then*

$$\operatorname{meas} P(\beta^\kappa \cap V) \leq N\sqrt{1 + 4\mathcal{V}_2^2}\operatorname{meas} P(\alpha^\kappa \cap V),$$

where N is the number of closure disjointed subfamilies $\mathcal{G}_{\delta\eta}^{(1)}, \ldots, \mathcal{G}_{\delta\eta}^{(N)}$ of $\mathcal{G}_{\delta\eta}$.

Proof: $\mathcal{G}_{\delta\eta}$ consists of balls with center in $\tilde{G}$. Since $\tilde{G} \subseteq \omega$, by Lemma 5.5, all balls in $\mathcal{G}_{\delta\eta}$ have center on ω. Therefore we can apply Lemma 5.8 and choose n_0 large enough such that the estimate (5.15) holds for all balls $B(z_i)$, $i = 1, \dots, k$. We group the balls $B(z_1), \dots, B(z_k)$ into N subfamilies $\{B_{ij}\}_i \subseteq \mathcal{G}_{\delta\eta}^{(j)}$ of disjoint balls and remark that if $B_{ij} \cap B_{lj} = \emptyset$ then also $P(\alpha^\kappa \cap B_{ij}) \cap P(\alpha^\kappa \cap B_{lj}) = \emptyset$, since P is a bijective mapping from the graph of the curve α^κ to $[0, \infty)$. From (5.15) we thus conclude

$$\begin{aligned}
&\operatorname{meas} P(\beta^\kappa \cap V) \leq \sum_{i=1}^{k} \operatorname{meas} P(\beta^\kappa \cap B(z_i)) \\
&\leq \sqrt{1 + 4\mathcal{V}_2^2} \sum_{i=1}^{k} \operatorname{meas} P(\alpha^\kappa \cap B(z_i)) = \sqrt{1 + 4\mathcal{V}_2^2} \sum_j \sum_i \operatorname{meas} P(\alpha^\kappa \cap B_{ij}) \\
&\leq \sqrt{1 + 4\mathcal{V}_2^2} \sum_j \operatorname{meas} P(\alpha^\kappa \cap V) = N \sqrt{1 + 4\mathcal{V}_2^2} \operatorname{meas} P(\alpha^\kappa \cap V).
\end{aligned}$$

Lemma 5.10 *Let $\alpha \in \Delta_n^+$ and let $\eta > 0$. Assume that $V = \bigcup_{i=1}^k B_i$ is a union of balls whose radii are all bounded by η and which satisfy $\alpha^\kappa \cap B_i \neq \emptyset$ for $i = 1, \dots, k$. Then*

$$\mu_n^-(V) \leq 2\sqrt{1 + \frac{1}{\mathcal{V}_1^2}}\,\eta, \tag{5.23}$$

where $\mathcal{V}_1$ is the constant from (4.5).

Proof: For $\beta \in \Delta_n^-$ let $t^* = \max\{0 \leq t \leq T_e \mid |\beta|(t) > 0\}$, and let $\beta^* = \beta|_{[0,t^*]}$. Then (5.9) implies

$$\begin{aligned}
\mu_n^-(V) &= \sum_{\beta \in \Delta_n^-} \int_{P(\beta^* \cap V)} |\beta^*|(t)dt \leq \sum_{\beta \in \Delta_n^-} \operatorname{meas} P(\beta^* \cap V) |\beta|(0) \qquad (5.24)\\
&\leq \max_{\beta \in \Delta_n^-} \operatorname{meas} P(\beta^* \cap V) \sum_{\beta \in \Delta_n^-} |[S_n^{(0)}](\beta(0))| \leq \max_{\beta \in \Delta_n^-} \operatorname{meas} P(\beta^* \cap V).
\end{aligned}$$

Here we used $\sum_{\beta \in \Delta_n^-} |[S_n^{(0)}](\beta(0))| = |S_n^{(0)}(b) - S_n^{(0)}(c)| \leq 1$.

It remains to estimate $\operatorname{meas} P(\beta^* \cap V)$. To this end note that relation (5.10) implies $|\alpha^\kappa|(t) \geq \kappa |\alpha|(0) > 0$ for all t from the domain of α^κ. By Lemma 5.4 (iii) and (5.8) we thus have $[S_n] = [S_n]_+ > 0$ along α^κ. Equation (4.5) therefore yields

$$\frac{d\alpha^\kappa}{dt}(t) \geq \mathcal{V}_1 > 0 \tag{5.25}$$

for all t from the domain of α^κ. By the same reasoning we obtain

$$\frac{d\beta^*}{dt}(t) \leq -\mathcal{V}_1 < 0 \tag{5.26}$$

for all t from the domain of β^*. Finally, observe that $\alpha^\kappa \cap B_i \neq \emptyset$, $i = 1, \dots, k$ implies

$$V \subseteq W_\alpha = \{z \in Z_{T_e} \mid \operatorname{dist}(z, \alpha^\kappa) < 2\eta\},$$

hence

$$\operatorname{meas} P(\beta^* \cap V) \leq \operatorname{meas} P(\beta^* \cap W_\alpha) \leq t_2 - t_1 \,, \tag{5.27}$$

with $t_2 = \sup P(\beta^* \cap W_\alpha)$, $t_1 = \inf P(\beta^* \cap W_\alpha)$. Using (5.25) and (5.26) we see by some geometrical considerations, which we leave to the reader, that

$$t_2 - t_1 \leq 2\sqrt{1 + \frac{1}{\mathcal{V}_1^2}}\,\eta.$$

Combining this inequality with (5.24) and (5.27) yields (5.23).

Proof of Proposition 4.9: We assume that the statement of this proposition for the measure ν_n^+ does not hold. Then there are numbers $\delta, \vartheta > 0$ such that for all $\eta_0 > 0$ there is $\eta \leq \eta_0$ and a finite collection $B_1, \dots, B_k \in \mathcal{G}_{\delta\eta}$ such that for every positive integer k_0 there is $n \geq k_0$ with

$$\nu_n^+\Big(\bigcup_{i=1}^k B_i\Big) > \vartheta. \tag{5.28}$$

In the following we write $V = \bigcup_{i=1}^k B_i$. We aim to prove an estimate of the form $\nu_n^-(V) < K\eta$, which together with the definition of $\mathcal{G}_{\delta\eta}$ would contradict (5.28) when η is small. However, because of the unknown location of the balls B_i the proof of such an estimate seems to be difficult. Our strategy for the proof therefore is to construct a chain $\hat{\alpha} \in \Delta_n^+$ and a subfamily of the balls from $B_1, \dots, B_k$ for which an estimate analogous to (5.28) holds and for which every ball intersects $\hat{\alpha}$. For such a subfamily of balls lined up along a curve $\hat{\alpha}$ with positive slope we have already proved an estimate for μ_n^- in Lemma 5.10. The desired estimate for ν_n^- is immediately obtained from that estimate via (5.1).

Therefore our first goal is to verify the following

Claim 1: If (5.28) holds there is a constant $\Theta > 0$ with the following property: For every η there is n_0 such that for all $n \geq \max(k_0, n_0)$ there is a subfamily $B_1', \dots, B_m'$ of $B_1, \dots, B_k$ and a chain $\hat{\alpha} \in \Delta_n^+$ with $B_j' \cap \hat{\alpha}^\kappa \neq \emptyset$, $j = 1, \dots, m$, and with

$$\nu_n^+\Big(\bigcup_{j=1}^m B_j'\Big) \geq \Theta > 0. \tag{5.29}$$

To prove this claim we first remember (5.1) and conclude from (5.28)

$$\mu_n^+(V) \geq \frac{\vartheta}{M_2}\,.$$

Let

$$L = \frac{\vartheta}{6M_2}, \qquad \kappa = \frac{\vartheta}{6M_2 T_e}\,, \tag{5.30}$$

and let Γ be the set of all $\alpha \in \Delta_n^+$ such that $\operatorname{meas} P(\alpha^\kappa \cap V) \geq L$. With Lemma 5.7 we obtain

$$\mu^+_{\Gamma,\kappa}(V) \geq \frac{\vartheta}{3M_2}\,. \tag{5.31}$$

Let p be the smallest integer with $\frac{L}{2}p \geq T_e$ and set $h = T_e/p$. Then

$$0 = t_0 < t_1 < \ldots < t_p = T_e$$

with $t_i = ih$ is a partition of the interval $[0, T_e]$ into p subintervals of length

$$h \leq \frac{L}{2}\,. \tag{5.32}$$

For $\alpha \in \Gamma$ let $[0, t^{\alpha^\kappa})$ be the domain of α^κ and for $q = 1, \ldots, p$ let

$$\Gamma_q = \{\alpha \in \Gamma \mid t_{q-1} < t^{\alpha^\kappa} \leq t_q\}.$$

Then $\Gamma = \bigcup_{q=1}^p \Gamma_q$, hence (5.31) yields

$$\mu^+_{\Gamma_1,\kappa}(V) + \ldots + \mu^+_{\Gamma_p,\kappa}(V) = \mu^+_{\Gamma,\kappa}(V) \geq \frac{\vartheta}{3M_2}\,.$$

Thus, there is at least one q with

$$\mu^+_{\Gamma_q,\kappa}(V) \geq \frac{\vartheta}{3pM_2}\,. \tag{5.33}$$

For every η and n we fix such a q. Let $\hat\alpha$ be the minimal element from Γ_q, i.e. $\hat\alpha \leq \alpha$ for all $\alpha \in \Gamma_q$. Also, let $B'_1, \ldots, B'_m$ be the subfamily of those balls from $B_1, \ldots, B_k$ which are contained in the set $Z_{t_{q-1}}$ and which are intersected by $\hat\alpha^\kappa$. We set $V' = \bigcup_{j=1}^m B'_j$. Finally, let V_q be the union of those balls from $B_1, \ldots, B_k$ which are not contained in $Z_{t_{q-1}}$. Since all balls belong to $\mathcal{G}_{\delta\eta}$ their radii are bounded by η. Therefore

$$V_q \subseteq \{(x,t) \in Z_{T_e} \mid t \geq t_{q-1} - 2\eta\}.$$

From $P(\hat\alpha^\kappa) = [0, t^{\hat\alpha^\kappa}] \subseteq [0, t_q]$ we thus obtain

$$P(\hat\alpha^\kappa \cap V_q) = P(\hat\alpha^\kappa) \cap P(V_q) \subseteq (t_{q-1} - 2\eta, t_q),$$

hence, together with (5.32)

$$\operatorname{meas} P(\hat\alpha^\kappa \cap V_q) \leq h + 2\eta \leq \frac{L}{2} + 2\eta. \tag{5.34}$$

We can assume that

$$\eta < \frac{L}{8}\,. \tag{5.35}$$

Since $\hat{\alpha} \in \Gamma$, we have $\operatorname{meas} P(\hat{\alpha}^\kappa \cap V) \geq L$. Moreover, $\hat{\alpha}^\kappa \cap V = \hat{\alpha}^\kappa \cap (V' \cup V_q)$, since $V' \cup V_q$ differs from V at most by a union of balls which are not intersected by $\hat{\alpha}^\kappa$. Together with (5.34) and (5.35) we thus obtain

$$\begin{aligned} L &\leq \operatorname{meas} P(\hat{\alpha}^\kappa \cap V) = \operatorname{meas} P(\hat{\alpha}^\kappa \cap (V' \cup V_q)) \\ &\leq \operatorname{meas} P(\hat{\alpha}^\kappa \cap V') + \operatorname{meas} P(\hat{\alpha}^\kappa \cap V_q) \leq \operatorname{meas} P(\hat{\alpha}^\kappa \cap V') + \frac{3}{4} L, \end{aligned}$$

hence

$$\operatorname{meas} P(\hat{\alpha}^\kappa \cap V') \geq \frac{1}{4} L.$$

We can now apply Corollary 5.9. Since V' is a union of balls in $\mathcal{G}_{\delta\eta}$ and satisfies $V' \subseteq Z_{t_{q-1}}$, since the interval $[0, t_{q-1}]$ belongs to the domains of all $\alpha \in \Gamma_q$ and since $\hat{\alpha}$ is the minimal element in Γ_q it follows from this corollary that there is $n_0 = n_0(\eta)$ such that for all $n \geq n_0$ and all $\alpha \in \Gamma_q$

$$\frac{L}{4} \leq \operatorname{meas} P(\hat{\alpha}^\kappa \cap V') \leq N\sqrt{1 + 4\mathcal{V}_2^2} \operatorname{meas} P(\alpha^\kappa \cap V').$$

This inequality shows that all $\alpha \in \Gamma_q$ satisfy the assumptions of Lemma 5.6. Together with (5.33) we thus conclude from this lemma that

$$\frac{\vartheta}{3pM_2} \leq \mu^+_{\Gamma_q,\kappa}(V) \leq \mu^+_{\Gamma_q,\kappa}(Z_{T_e}) \leq \frac{T_e}{L'\kappa} \mu^+_{\Gamma_q,\kappa}(V'), \tag{5.36}$$

where $L' = L(4N\sqrt{1 + 4\mathcal{V}_2^2})^{-1}$. Relation (5.1) implies

$$\mu^+_{\Gamma_q,\kappa}(V') \leq \mu^+_n(V') \leq \frac{1}{M_1} \nu^+_n(V').$$

This estimate and (5.36) together yield (5.29), where the constant Θ has the value

$$\Theta = \frac{1}{12} \frac{\kappa M_1 \vartheta L}{M_2 \sqrt{1 + 4\mathcal{V}_2^2}\, N T_e p} > 0.$$

κ and L are given in (5.30). This proves Claim 1.

Claim 2: For all $\eta > 0$ there is n_1 such that for all $n \geq n_1$

$$\nu^+_n\Big(\bigcup_{j=1}^m B'_j\Big) \leq 2M_2 N \frac{1}{\delta} \sqrt{1 + \frac{1}{\mathcal{V}_1^2}}\, \eta. \tag{5.37}$$

To verify this claim note that all balls B'_j belong to the family $\mathcal{G}_{\delta\eta}$ and thus satisfy (4.13), by definition of this family. Hence, there is n_1 with

$$\nu^+_n(B'_j) \leq \frac{1}{\delta} \nu^-_n(B'_j), \tag{5.38}$$

for all $n \geq n_1$ and all $j = 1, \ldots, m$. As in the proof of Corollary 5.9 we group the balls $B'_1, \ldots, B'_m$ into N subfamilies $\{B'_{ij}\}_i \subseteq \mathcal{G}^{(j)}_{\delta\eta}$ of disjoint balls, and obtain from (5.38) for $n \geq n_1$ that

$$\begin{aligned}
\nu_n^+\Big(\bigcup_{j=1}^m B'_j\Big) &\leq \sum_{j=1}^m \nu_n^+(B'_j) \leq \frac{1}{\delta}\sum_{j=1}^m \nu_n^-(B'_j) \\
&= \frac{1}{\delta}\sum_j \sum_i \nu_n^-(B'_{ij}) \leq \frac{1}{\delta}\sum_j \nu_n^-(V') = \frac{N}{\delta}\nu_n^-(V').
\end{aligned} \tag{5.39}$$

Since $V' = \bigcup_{j=1}^m B'_j$ is a union of balls whose radii are all bounded by η and which satisfy $\hat{\alpha}^\kappa \cap B'_j \neq \emptyset$ for $j = 1, \ldots, m$, we can apply Lemma 5.10. Together with (5.1) we discover that

$$\nu_n^-(V') \leq M_2 \mu_n^-(V') \leq 2M_2\sqrt{1 + \frac{1}{\mathcal{V}_1^2}}\,\eta.$$

Claim 2 follows by insertion of this estimate into (5.39).

End of the proof of Proposition 4.9: Choose η small enough such that the right hand side of (5.37) is less than the constant Θ in (5.29). Then for $n \geq \max(n_0, k_0, n_1)$ the inequalities (5.29) and (5.37) are in obvious contradiction. Consequently, our hypotheses must be false and the inequality stated in Proposition 4.9 for ν_n^+ must hold. The inequality for ν_n^- is proved in the same way by interchanging the roles of ν_n^+ and ν_n^- and by applying the second inequality in (4.13), which has not yet been used. The proof is complete.

A Appendix

Here we state the version of the Besicovitch Covering Theorem which we use in Sect. 4 to define the families $\mathcal{E}_\delta$, $\mathcal{F}_\delta$, $\mathcal{G}_{\delta\eta}$ and the sets E, F, G.

Definition A.1 *Let $\mathcal{B}$ be a family of open sets. $\mathcal{B}$ is called closure disjointed if every pair of sets $V_1, V_2 \in \mathcal{B}$ with $V_1 \neq V_2$ satisfies $\overline{V_1} \cap \overline{V_2} = \emptyset$.*

The following theorem is proved in exactly the same way as the version of the Besicovitch Covering Theorem for closed balls given in [13, pp. 30–35], [29, pp. 9–12].

Theorem A.2 (Besicovitch Covering Theorem) *Let $\mathcal{B}$ be a family of open balls in $\mathbb{R}^n$ with uniformly bounded radii. There are closure disjointed, countable subfamilies $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(N)}$ of $\mathcal{B}$, with $N > 1$ only depending on the dimension n, such that if A is the set of centers of balls in $\mathcal{B}$ then*

$$A \subseteq \bigcup_{i=1}^N \bigcup_{B \in \mathcal{B}^{(i)}} B.$$

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