A geometrically exact micromorphic elastic solid. Modelling and existence of minimizers.

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Abstract

We investigate geometrically exact generalized continua of micromorphic type in the sense of Eringen. The two-field problem for the macrodeformation φ and the microdeformation $\overline{P} \in \mathrm{SL}(3,\mathbb{R})$ in the quasistatic, conservative case is investigated in a variational form.

Depending on material constants different mathematical existence theorems in Sobolev-spaces are given for the resulting nonlinear boundary value problems including as a special case an existence theorem for a geometrically exact Cosserat micropolar model. These are the first results known to the author for geometrically exact microcontinuum formulations.

In order to treat external loads a new condition, called bounded external work, has to be included, overcoming the conditional coercivity of the formulation. The mathematical analysis heavily uses an extended Korn's first inequality (Neff, Proc.Roy.Soc.Edinb.A, 2002) discovered by the author recently. The methods of choice are the direct methods of the calculus of variations.

Key words: polar-materials, microstructure, micromorphic,
structured continua, solid mechanics, variational methods.

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Contents

1 Introduction

This article addresses the modelling and mathematical analysis of **geometrically exact**[1] generalized continua of **micromorphic** type in the sense of Eringen in the elastic case. General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [12] at the beginning of the last century.

Their development has been largely forgotten for decades only to be rediscovered in the early sixties [44, 26, 1, 20, 18, 51, 52, 28, 38, 47, 53]. At that time theoretical investigations on non-classical continuum theories were the main motivation [36]. Since then, the Cosserat concept has been generalized in various directions, for an overview of these so called **microcontinuum** theories we refer to [19, 17, 6, 5, 7, 29]. Recently, in [9, 10], the **micromorphic balance equations** derived by Eringen have been **formally justified** as a more realistic continuum model based on molecular dynamics and ensemble averaging. The micromorphic model includes in a natural way **size effects**, i.e. small samples behave comparatively stiffer than large samples. These effects have recently received much attention in conjunction with nano-devices.

The mathematical analysis of general micromorphic solids is at present restricted to the infinitesimal, linear elastic models, see e.g. [31, 15, 30, 24, 25] for linear micropolar models and [34, 32, 33] for linear microstretch models. The major difficulty of the mathematical treatment in the finite strain case is related to the geometrically exact formulation of the theory and the appearance of **nonlinear manifolds** necessary for the description of the microstructure. No general existence theorems for finite micromorphic models are known to the author. The simpler, geometrically exact nonlinear micropolar case has been dealt with in [42].

This contribution is organized as follows: first, we shortly review the basic concepts of the geometrically exact elastic micromorphic theories in a variational context, i.e. we formulate the quasistatic conservative case as a minimization problem. We then provide the balance equations and investigate the influence of material constants on the ellipticity of the force balance equation.

More mathematically inclined readers may start directly in the analytical section 3. There, the complete problem statement of the geometrically exact elastic micromorphic case in a variational context is repeated. Since the **two-field** variational problem is only **conditionally coercive** we need to introduce a modification for the applied loads in order to ensure first that the functional to be minimized is bounded below and second that the curvature contribution can be controlled. This modification of the loads, herein called principle of **"bounded external work"** expresses nothing than the physical fact that by arbitrarily moving the solid in a force field only a finite amount of work can be gained. Such a condition is, however, unnecessary in classical elasticity.

With this preparation existence of minimizers in Sobolev-spaces is then established using the direct methods of variations and a novel extended Korn's first inequality. In the appendix we introduce the relevant notation and detail the formal derivation of the nonlinear balance equation which themselves, however, do not play a prominent role in our development. In addition we reconsider the linearization of the introduced finite micromorphic models.

2 A finite elastic micromorphic model

Let us now motivate the finite micromorphic approach.[2] For our development we choose a strictly Lagrangean description. We first introduce an independent kinematical field of **microdeformations** $\overline{P} \in \mathrm{SL}(3,\mathbb{R})$ together with its polar decomposition

$$\overline{P} = \overline{R}_p \cdot \overline{U}_p = \mathrm{polar}(\overline{P}) \cdot \overline{U_p}\,, \tag{2.1}$$

with $\overline{R}_p \in \mathrm{SO}(3,\mathbb{R})$ and $\overline{U_p} \in \mathrm{PSym}(3,\mathbb{R})$. The microdeformations $\overline{P}$ are meant to represent a substructure of the material which can **rotate**, **stretch** and **shear**. We refer to $\overline{R}_p$ as **microrotations**.

The micromorphic theory can formally be obtained by introducing the multiplicative decomposition of the macroscopic deformation gradient F into **independent microdeformation** $\overline{P}$ and the **micromorphic, nonsymmetric stretch tensor** $\overline{U}$ with

$$F = \overline{P} \cdot \overline{U}\,, \quad \overline{U} \in \mathrm{GL}^+(3,\mathbb{R}), \tag{2.2}$$

[1] Fully frame-indifferent.

[2] Following Eringen [17, p.13] we distinguish the general **micromorphic case**: $\overline{P} \in \mathrm{GL}^+(3,\mathbb{R}) = \mathbb{R}^+ \cdot \mathrm{SL}(3,\mathbb{R})$ with 9 additional **degrees of freedom** (dof), the **micro-incompressible micromorphic case**: $\overline{P} \in \mathrm{SL}(3,\mathbb{R})$ with 8 dof, the **microstretch case**: $\overline{P} \in \mathbb{R}^+ \cdot \mathrm{SO}(3,\mathbb{R})$ with 4 dof and the **micropolar case**: $\overline{P} \in \mathrm{SO}(3,\mathbb{R})$ with only 3 additional dof.

leading altogether to a **micro-incompressible, micromorphic formulation**.[3]

The notion **micro**morphic is prone to misunderstandings: the microdeformation $\overline{P}$ must be considered as a macroscopic (average) quantity as the deformation gradient and the resulting model is still phenomenological. However, geometrical features of the real substructure to be modelled, determine the choice of geometric manifolds for $\overline{P}$.

In the **quasistatic** case, the micromorphic theory is now derived from a **two-field** variational principle by postulating the following **'action euclidienne'** [12, p.156] for the finite macroscopic deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the independent microdeformation $\overline{P} : [0,T] \times \overline{\Omega} \mapsto \mathrm{SL}(3,\mathbb{R})$:

$$I(\varphi,\overline{P}) = \int_\Omega W(F,\overline{P},\mathrm{D_x}\overline{P}) - \Pi_f(\varphi) - \Pi_M(\overline{P})\,\mathrm{dV} - \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} - \int_{\Gamma_C} \Pi_{M_c}(\overline{P})\,\mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi,\overline{P}),$$
$$\overline{P}_{|_\Gamma} = \overline{P}_\mathrm{d}\,, \quad \varphi_{|_\Gamma} = g_\mathrm{d}(t)\,. \tag{2.3}$$

The elastically stored energy density W depends on the macroscopic deformation gradient F as usual but in addition on the microdeformation $\overline{P}$ together with their first order space derivatives, represented through the third order tensor $\mathrm{D_x}\overline{P}$. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g, \overline{P}_\mathrm{d}$ for displacements and microdeformations, respectively, can be prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. The potential of external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples.[4] In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of applied surface couples Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$, corresponding natural boundary conditions for φ and $\overline{P}$ apply, obtained automatically in the variational process.

Variation of the action I with respect to φ yields the traditional equation for balance of linear momentum and variation of I with respect to $\overline{P}$ yields balance of moment of momentum.

The standard conclusion from **frame-indifference** (here: invariance of the free energy under **s**uperposed **r**igid **b**ody **m**otions (**SRBM**) not merely **observer-invariance** of the model [50, 4, 39]: $\forall Q \in \mathrm{SO}(3,\mathbb{R})$: $W(F,\overline{P},\mathrm{D_x}\overline{P}) = W(QF,Q\overline{P},\mathrm{D_x}[Q\overline{P}])$ leads to the reduced representation of the energy (specify $Q = \overline{R}_p$):

$$W(F,\overline{P},\mathrm{D_x}\overline{P}) = W(\overline{R}_p^T F,\overline{R}_p^T\overline{P},\overline{R}_p^T\mathrm{D_x}\overline{P}) = W(\overline{U}_p\overline{U},\overline{U}_p,\overline{R}_p^T\mathrm{D_x}\overline{P}) = W(\overline{U},\overline{U}_p,\mathfrak{K})\,, \tag{2.4}$$

where

$$\mathfrak{K} := \overline{R}_p^T\mathrm{D_x}\overline{P} = \left(\overline{R}_p^T\nabla(\overline{P}.e_1),\overline{R}_p^T\nabla(\overline{P}.e_2),\overline{R}_p^T\nabla(\overline{P}.e_3)\right) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3}\,, \tag{2.5}$$

coincides with one specific representation[5] of the third order right **micropolar curvature tensor** (or torsion-curvature tensor, wryness tensor, second Cosserat deformation tensor, bending-twist tensor etc.) if $\overline{P} \in \mathrm{SO}(3,\mathbb{R})$. For a geometrically exact isotropic theory we assume in the following an additive split of the total free energy density into micromorphic local stretch and micromorphic curvature part according to

$$W = W_\mathrm{mp}(\overline{U},\overline{U}_p) + W_\mathrm{curv}(\mathfrak{K})\,, \tag{2.6}$$

since a possible coupling between $\overline{U}$ and $\mathfrak{K}$ for centrosymmetric bodies can be ruled out [43, p.14].

2.1 The elastic micromorphic strain energy density

For a **small elastic strain** theory, which should already cover most cases of physical interest, we require that $W_\mathrm{mp}(\overline{U},\overline{U}_p)$ is a non-negative isotropic quadratic form. We assume moreover the stretch energy density normalized to

$$W_\mathrm{mp}(1\!\!1,1\!\!1) = 0, \quad D_{\overline{U}}W_\mathrm{mp}(\overline{U},\overline{U}_p)_{|_{\overline{U}=1\!\!1,\overline{U}_p=1\!\!1}} = 0\,, \quad D_{\overline{U}_p}W_\mathrm{mp}(\overline{U},\overline{U}_p)_{|_{\overline{U}=1\!\!1,\overline{U}_p=1\!\!1}} = 0\,. \tag{2.7}$$

[3]The strain measure $\overline{U}$ which is induced by this definition corresponds to $\mathfrak{C}^T_{\mathrm{KL}}$ presented in $(1.5.11)_1$ of [17, p.15].

[4]appearing in a non-mechanical context e.g. as influence of a magnetic field on the polarization of a substructure of the bulk.

[5]Note that $\mathfrak{K}^i = \overline{R_p}^T\nabla(\overline{P}.e_i) \notin \mathfrak{so}(3,\mathbb{R})$. Another representation of $\mathfrak{K}$ is given by $\overline{\mathfrak{K}} := \left(\overline{R}_p^T\partial_x\overline{P},\overline{R}_p^T\partial_y\overline{P},\overline{R}_p^T\partial_z\overline{P}\right) \in \mathfrak{T}(3)$. Since $\partial_x(\overline{R}_p^T\overline{P}) = 0$ for $\overline{P} = \overline{R}_p \in \mathrm{SO}(3,\mathbb{R})$, it holds that $\overline{\mathfrak{K}} \in \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})$ in this case. It is therefore possible to base all considerations of curvature in the **micropolar** case on a more compact expression like $\hat{\mathfrak{K}} := \left(\mathrm{axl}(\overline{R}_p^T\partial_x\overline{R}_p)|\,\mathrm{axl}(\overline{R}_p^T\partial_y\overline{R}_p)|\,\mathrm{axl}(\overline{R}_p^T\partial_z\overline{R}_p)\right) \in \mathbb{M}^{3\times3}$. This is the traditional micropolar approach, see e.g. [46, 21, 27]. For us it is, however, not possible to use $\hat{\mathfrak{K}}$ since we allow for $\overline{P} \in \mathrm{SL}(3,\mathbb{R})$.

The most general form of W_{mp} consistent[6] with (2.7) is

$$W_{\text{mp}}(\overline{U}) = \alpha_1 \|\operatorname{sym}(\overline{U} - \mathbb{1})\|^2 + \mu_c \|\operatorname{skew}(\overline{U} - \mathbb{1})\|^2 + \alpha_3 \operatorname{tr}\left[\operatorname{sym}(\overline{U} - \mathbb{1})\right]^2 + \beta_1 \|\overline{U}_p - \mathbb{1}\|^2 + \beta_2 \operatorname{tr}\left[\overline{U}_p - \mathbb{1}\right]^2 , \tag{2.8}$$

with material constants $\alpha_1, \mu_c, \alpha_3, \beta_1, \beta_2$ such that $\alpha_1,\ 3\alpha_3 + \alpha_1, \mu_c \geq 0,\ \ \beta_1, 3\beta_2 + \beta_1 \geq 0$ from non negativity [17]. By consistency with the classical continuum model without micromorphic behaviour we can take $\alpha_1 = \mu$, $\alpha_3 = \frac{\lambda}{2}$ with $\mu, \lambda > 0$ the classical Lamé constants and the so called **Cosserat couple modulus** μ_c remains for the moment unspecified but $\mu_c = 0$ is physically possible since the **micromorphic reaction stress** $D_{\overline{U}} W_{\text{mp}}(\overline{U}, \overline{U}_p) \cdot \overline{U}^T$ is not symmetric in general, i.e. the problem does not decouple, cf. (2.12). For comparison, in [17, p.111] for the infinitesimal micropolar case, the elastic moduli are taken to be $\alpha_1 = \mu + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$, $\alpha_3 = \frac{\lambda}{2}$, but in this formula, μ cannot be regarded as one of the Lamé constants.[7] [8] In [14, 48, 49, 22, 13, 16] the abbreviation μ_c is used while in [27] it is $\mu_c = \alpha$ and $\mu_c = G_c$ in [35] for the micropolar theory.

By formal similarity with the classical formulation we may as well introduce $\beta_1 = \mu^{\text{m}}$ and $\beta_2 = \frac{\lambda^{\text{m}}}{2}$, calling $\mu^{\text{m}}, \lambda^{\text{m}}$ the **microscopic Lamé moduli**.

2.2 The elastic micromorphic curvature energy density

For the curvature term, to be specific, we assume the general form

$$W_{\text{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 L_c^q \|\mathfrak{K}\|^q\right) \left(\alpha_5 \|\operatorname{sym}\mathfrak{K}\|^2 + \alpha_6 \|\operatorname{skew}\mathfrak{K}\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{K}]^2\right)^{\frac{1+p}{2}} , \tag{2.9}$$

where $L_c > 0$ is setting an internal length scale with units of length, $\alpha_4 \geq 0, p > 0, q \geq 0$ are additional material constants, the factor $\frac{1}{12}$ only for convenience and $\alpha_5 > 0, \alpha_6, \alpha_7 \geq 0$ as a minimal requirement. We mean $\operatorname{tr}[\mathfrak{K}]^2 = \|\operatorname{tr}[\mathfrak{K}]\|^2$ by abuse of notation. This choice for W_{curv} does not presuppose any knowledge of the magnitude of the micromorphic curvature in the material and is non-degenerate in the origin $\|\mathfrak{K}\| = 0$, which is, however, not essential to our subsequent mathematical analysis.

But care has to be exerted in the finite regime: W_{curv} should preferably be **coercive** in the sense that we impose pointwise

$$\exists\, c^+ > 0\ \exists\, r > 1 :\ \forall\, \mathfrak{K} \in \mathfrak{T}(3) : \quad W_{\text{curv}}(\mathfrak{K}) \geq c^+ \|\mathfrak{K}\|^r, \tag{2.10}$$

or less demanding

$$\exists\, r > 1 : \quad \frac{W_{\text{curv}}(\mathfrak{K})}{\|\mathfrak{K}\|^r} \to \infty \quad \text{as } \|\mathfrak{K}\| \to \infty\, , \tag{2.11}$$

which implies necessarily $\alpha_6 > 0$ in (2.10). Observe that our formulation of the micromorphic curvature tensor is convenient in the sense that $\|\mathfrak{K}\| = \|\overline{R}_p^T \mathrm{D_x} \overline{P}\| = \|\mathrm{D_x}\overline{P}\|$ provides pointwise control of all first derivatives of $\overline{P}$ independent of the values of $\overline{P}$ itself.[9]

Note that the presented formulation includes a finite Cosserat micropolar model as a special case if we set $\overline{P} = \overline{R} \in \mathrm{SO}(3, \mathbb{R})$.

[6] Mixed products like $\langle \overline{U} - \mathbb{1}, \overline{U}_p - \mathbb{1} \rangle$ and $\operatorname{tr}\left[\overline{U} - \mathbb{1}\right] \cdot \operatorname{tr}\left[\overline{U}_p - \mathbb{1}\right]$ are excluded by non-negativity.

[7] A simple definition of the Lamè constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where E and ν are uniquely determined from uniform traction. Eringens nomenclature unfortunately led to some confusion.

[8] Uniform traction and uniform compression do not activate rotations, hence the classical identification of the Lamé constants is achieved **independent** of μ_c. Uniform traction alone allows already to determine the Young modulus E and the Poisson ratio ν [11, p.126]. Contrary to [23, p.411] we do not see the possibility to define a specific "micropolar Young modulus" or "micropolar Poisson ratio".

[9] Thus is not true for other possible basic invariant curvature expressions like $\overline{P}^{-1} \mathrm{D_x} \overline{P}$ or $\overline{P}^T \mathrm{D_x} \overline{P}$ or $F^T \mathrm{D_x} \overline{P}$, see [17, 1.5.4,1.5.11].

2.3 The micromorphic balance equations

For the choices we have made we note the resulting material form of the nonlinear field equations on the reference configuration (with $\alpha_4 = 0$, $p = 1$) which can be obtained after some algebraic manipulations (we have gathered the influence of the external potentials in $\Pi(x, \varphi, \overline{P})$):

$$
\begin{aligned}
0 &= \operatorname{Div}\left(S_1(F, \overline{P}) + 2\,\mu_c\,\overline{P}^{-T}\operatorname{skew}(\overline{P}^{-1}F)\right) + D_\varphi \Pi(x, \varphi(x), \overline{P})_{\mathbb{R}^3}, \qquad \textbf{force balance} \qquad (2.12)\\
0 &= \operatorname{skew}(\overline{U}_p^{-1} D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}, \overline{U}_p)\,\overline{U}^T \overline{U}_p^T) + \operatorname{skew}\left(\overline{R}_p^T \operatorname{Div}\left[\overline{R}_p\, D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{U}_p\right) + \operatorname{skew}\left(D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\mathfrak{K}^T\right)\\
&\quad + \operatorname{skew}\left(\overline{R}_p^T D_{\overline{P}} \Pi(x, \varphi(x), \overline{P})\overline{U}_p\right)_{\mathbb{M}^{3\times 3}}, \qquad \textbf{balance of angular momentum},\\
0 &= \operatorname{dev}\operatorname{sym}\left(\overline{U}_p^{-1} D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}, \overline{U}_p)\,\overline{U}^T \overline{U}_p^T\right) - \operatorname{dev}\operatorname{sym}\left(D_{\overline{U}_p} W_{\mathrm{mp}}(\overline{U}, \overline{U}_p)\,\overline{U}_p^T\right)\\
&\quad + \operatorname{dev}\operatorname{sym}\left(\overline{R}_p^T \operatorname{Div}\left[\overline{R}_p\, D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{U}_p\right) + \operatorname{dev}\operatorname{sym}\left(\overline{R}_p^T D_{\overline{P}} \Pi(x, \varphi(x), \overline{P})\overline{U}_p\right),\\
&\qquad \textbf{remaining balance of moment of momentum},
\end{aligned}
$$

where S_1 is the first Piola-Kirchhoff stress (for $\mu_c = 0$) with the functional form

$$
S_1(F, \overline{P}) = \overline{P}^{-T}\left[\mu(F^T \overline{P}^{-T} + \overline{P}^{-1}F - 2\mathbb{1}) + \lambda \operatorname{tr}\left[F^T \overline{P}^{-T} - \mathbb{1}\right]\mathbb{1}\right], \qquad (2.13)
$$

similar to [41, (P3)] and $D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})$ is the material **micromorphic moment tensor** (or **couple-stress tensor**). A similar form of the unconventional balance of angular momentum equation has been given in [5, p.63] for the micropolar case. In our subsequent variationally based development, the balance equations will not play a prominent role.

2.4 Constitutive consequences of the value for the Cosserat couple modulus

Looking at (2.8) with $\mu_c > 0$ we see that the implication of this choice at a first glance is an inoccuous rise in the macroscopic elastic strain energy $W_{\mathrm{mp}}(\overline{U}, \overline{U}_p)$ if $\overline{R}_p \neq \operatorname{polar}(F)$ and $\overline{U}_p = \mathbb{1}$, but $\overline{R}_p$ is generically assumed to be independent of $\operatorname{polar}(F)$. The choice $\mu_c > 0$ acts like a local 'elastic spring' between both continuum rotations and microrotations.

Let us consider the mathematical implications of $\mu_c = 0$ and $0 < \mu_c \leq \mu$, respectively, in more detail. It is readily verified that for the elasticity tensors (differentiating the stretch energy density $W_{\mathrm{mp}}(\overline{P}^{-1}F, \overline{U}_p)$ at fixed $\overline{P}$ w.r.t. F)

$$
\begin{aligned}
\mu_{\mathbf{c}} > 0 \Rightarrow \quad \forall H \in \mathbb{M}^{3\times 3}: \quad & D_F^2 W_{\mathrm{mp}}(\overline{P}^{-1}F, \overline{U}_p).(H, H) \geq 2\,\mu_c\,\|\overline{P}^{-1}H\|^2 \geq 2\mu_c\,\lambda_{\min}(\overline{P}^{-T}\overline{P}^{-1})\,\|H\|^2\\
\mu_{\mathbf{c}} = 0 \Rightarrow \quad \forall H \in \mathbb{M}^{3\times 3}: \quad & D_F^2 W_{\mathrm{mp}}(\overline{P}^{-1}F, \overline{U}_p).(H, H) \geq 2\,\mu\,\|\frac{1}{2}(\overline{P}^{-1}H + H^T \overline{P}^{-T})\|^2. \qquad (2.14)
\end{aligned}
$$

Hence the choice $\mu_c > 0$ leads to **uniform convexity** of $W_{\mathrm{mp}}(\overline{P}^{-1}F, \overline{U}_p)$ w.r.t. F if $\overline{P} \in L^\infty(\Omega, \mathrm{SL}(3, \mathbb{R}))$ and **unconditional elastic stability** on the macroscopic level: regardless of what distribution of microdeformations $\overline{P}(x)$ is given, the macroscopic equation of balance of linear momentum would then bei uniquely solvable and this equation is insensible to any deterioration of the spatial features of the microstructure as long as $\overline{P}$ is merely essentially bounded. Uniform convexity is difficult to accept from a constitutive point of view since it is impossible for a geometrically exact description in the framework of a classical macroscopic continuum but clear from the above discussion: the additional elastic spring between micro- and continuum rotation extremely rigidifies the material and completely changes the type of the mathematical boundary value problem compared with the classical finite theory.

Fortunately, such a far reaching unsatisfactory conclusion does not hold for $\mu_c = 0$, in which case we have for $\xi, \eta \in \mathbb{R}^3$:

$$
\begin{aligned}
D_F^2 W_{\mathrm{mp}}(\overline{P}^{-1}F, \overline{U}_p).(\xi \otimes \eta, \xi \otimes \eta) &= \mu\left(\|\overline{P}^{-1}\xi \otimes \eta\|^2 + \langle \overline{P}^{-}\xi \otimes \eta, \eta \otimes \overline{P}^{-1}\xi\rangle\right)\\
&= \mu\left(\|\overline{P}^{-1}\xi \otimes \eta\|^2 + \langle \overline{P}^{-1}\xi, \eta\rangle^2\right), \qquad (2.15)
\end{aligned}
$$

which shows the physically much more appealing inequality

$$
D_F^2 W_{\mathrm{mp}}(\overline{P}^{-1}F, \overline{U}_p).(\xi \otimes \eta, \xi \otimes \eta) \geq \mu\,\lambda_{\min}(\overline{P}^{-T}\overline{P}^{-1})\,\|\xi\|^2 \cdot \|\eta\|^2, \qquad (2.16)
$$

expressing nothing but **uniform Legendre-Hadamard ellipticity** of the acoustic-tensor with ellipticity constant $\mu\,\lambda_{\min}(\overline{P}^{-T}\overline{P}^{-1})$ **independent** of $\overline{P}$ for bounded $\overline{P}\in L^\infty(\Omega,\mathrm{SL}(3,\mathbb{R}))$. The Legendre-Hadamard condition has the most convincing physical basis [2, p.461] in that it implies the **reality of wave speeds** and the **Baker-Ericksen inequalities** (stress increases with strain, [37, p.19]).[10]

3 Mathematical analysis

3.1 Statement of the finite elastic micromorphic problem in variational form

Let us gather the obtained two-field problem posed in a variational form. The task is to find a pair $(\varphi,\overline{P}):\Omega\subset\mathbb{R}^3\mapsto\mathbb{R}^3\times\mathrm{SL}(3,\mathbb{R})$ of **macroscopic** deformation φ and **independent microdeformation** $\overline{P}$ minimizing the functional I with

$$I(\varphi,\overline{P})=\int_\Omega W_{\mathrm{mp}}(\overline{P}^{-1}\nabla\varphi,\overline{U}_p)+W_{\mathrm{curv}}(\overline{R}_p^T\mathrm{D_x}\overline{P})-\Pi_f(\varphi)-\Pi_M(\overline{P})\,\mathrm{dV}$$
$$-\int_{\Gamma_S}\Pi_N(\varphi)\,\mathrm{dS}-\int_{\Gamma_C}\Pi_{M_c}(\overline{P})\,\mathrm{dS}\mapsto\min.\ \text{w.r.t.}\ (\varphi,\overline{P})\,, \tag{3.1}$$

under the constraints

$$\overline{U}_p=\overline{R}_p^T\overline{P},\quad \overline{R}_p=\mathrm{polar}(\overline{P})\,, \tag{3.2}$$

and the boundary conditions

$$\overline{P}_{|_\Gamma}=\overline{P}_{\mathrm{d}}\Rightarrow\overline{R}_{p_{|_\Gamma}}=\overline{R}_{p_{\mathrm{d}}},\quad\overline{U}_{p_{|_\Gamma}}=\overline{U}_{p_{\mathrm{d}}},\quad\varphi_{|_\Gamma}=g_{\mathrm{d}}\,. \tag{3.3}$$

Here, the constitutive assumptions on the densities are

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{U},\overline{U}_p)&=\mu\,\|\mathrm{sym}(\overline{U}-1\!\!1)\|^2+\mu_c\,\|\mathrm{skew}(\overline{U})\|^2+\frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}-1\!\!1)\right]^2\\
&\quad+\mu^{\mathrm{m}}\,\|\overline{U}_p-1\!\!1\|^2+\frac{\lambda^{\mathrm{m}}}{2}\,\mathrm{tr}\left[\overline{U}_p-1\!\!1\right]^2,\\
W_{\mathrm{curv}}(\mathfrak{K})&=\mu\frac{L_c^{1+p}}{12}\,(1+\alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}\|^2+\alpha_6\,\|\mathrm{skew}\,\mathfrak{K}\|^2+\alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,,\\
\mathfrak{K}&=\overline{R}_p^T\mathrm{D_x}\overline{P}=\left(\overline{R}_p^T\nabla(\overline{P}.e_1),\overline{R}_p^T\nabla(\overline{P}.e_2),\overline{R}_p^T\nabla(\overline{P}.e_3)\right),\quad\text{third order \textbf{curvature tensor}}\,.
\end{aligned} \tag{3.4}$$

The total elastically stored energy $W=W_{\mathrm{mp}}+W_{\mathrm{curv}}$ depends on the deformation gradient $F=\nabla\varphi$ and microdeformations $\overline{P}$ together with their space derivatives. Here $\Omega\subset\mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma\subset\partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}},\overline{P}_{\mathrm{d}}$ for displacements and microdeformations, respectively, are prescribed while $\Gamma_S\subset\partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma\cap\Gamma_S=\emptyset$. In addition, $\Gamma_C\subset\partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} are applied with $\Gamma\cap\Gamma_C=\emptyset$. On the free boundary $\partial\Omega\setminus\{\Gamma\cup\Gamma_S\cup\Gamma_C\}$ corresponding natural boundary conditions for $(\varphi,\overline{P})$ apply. The potential of external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples.

The parameters μ, $\lambda>0$ are the Lamé constants of classical elasticity, $\mu_c\geq 0$ is called the **Cosserat couple modulus**, μ^{m}, $\lambda^{\mathrm{m}}>0$ are the Lamé constants of the substructure and $L_c>0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. If not stated otherwise, we assume that $\alpha_5>0,\alpha_6>0,\alpha_7\geq 0$. A finite Cosserat micropolar theory is included in the formulation (3.1), (3.2),(3.4) by restricting $\overline{P}\in\mathrm{SO}(3,\mathbb{R})$ or setting $\mu^{\mathrm{m}}=\infty$, formally.

[10] The preferred value $\mu_c=0$ for the macroscopic case can as well be motivated by the following consideration: Consider the Green strains $F^TF-1\!\!1=(\overline{U}-1\!\!1)^T(\overline{U}-1\!\!1)+2\,\mathrm{sym}(\overline{U}-1\!\!1)$. Therefore $\frac{\mu}{4}\|F^TF-1\!\!1\|^2=\mu\|\mathrm{sym}\,\overline{U}-1\!\!1\|^2+O(\|\overline{U}-1\!\!1\|^3)$. Hence $\mu_c=0$ provides the correct first order approximation to a classical St. Venant-Kirchhoff material. With $\mu_c=0$ we exclusively recover the fact of the classical continuum theory that W isotropic implies symmetry of the Biot stress tensor: $D_UW(U)\in\mathrm{Sym}$. If we expand $\overline{R}=1\!\!1+\overline{A}+\ldots$ with $\overline{A}\in\mathfrak{so}(3)$ and write $F=1\!\!1+\nabla u$, then the micropolar effects dissappear to first order for $\mu_c=0$. In this sense, $\mu_c=0$ is close to classical elasticity.

3.2 The external potentials

In the conservative, dead load case one would have traditionally for the potentials of applied loads

$$\Pi_f(\varphi) = \langle f, \varphi \rangle\,, \quad \Pi_M(\overline{P}) = \langle M, \overline{P} \rangle\,, \quad \Pi_N(\varphi) = \langle N, \varphi \rangle\,, \quad \Pi_{M_c}(\overline{P}) = \langle M_c, \overline{P} \rangle\,, \tag{3.5}$$

with functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times 3})$.

For our treatment, we need to assume, however, that the external potentials, describing the configuration dependent applied loads are continuous with respect to the topology of $L^1(\Omega)$, $L^1(\Gamma_S)$, $L^1(\Gamma_C)$, respectively and satisfy in addition the condition

$$\begin{aligned} &\exists\, C^+ > 0 \quad \forall\, \varphi \in L^1(\Omega, \mathbb{R}^3),\, \overline{P} \in L^1(\Omega, \mathrm{SL}(3,\mathbb{R}))\ : \\ &\int_\Omega \Pi_f(\varphi) - \Pi_M(\overline{P})\,\mathrm{dV}, \quad \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS}, \quad \int_{\Gamma_C} \Pi_{M_c}(\overline{P})\,\mathrm{dS} \le C^+\,. \end{aligned} \tag{3.6}$$

While continuity is satisfied e.g. for dead loads $\Pi_f(\varphi) = \langle f, \varphi \rangle$ and $f \in L^\infty(\Omega)$, the second condition (3.6) restricts attention to **"bounded external work"**. If we want to describe a situation corresponding to the classical dead load case we could take

$$\Pi_f(\varphi) = \frac{1}{1 + [\|\varphi(x)\| - K^+]_+}\,\langle f(x), \varphi(x) \rangle\,, \tag{3.7}$$

for some large positive constant K^+ and $[\cdot]_+$ the positive part of a scalar argument. It suffices now that $f \in L^1(\Omega)$, then $\int_\Omega \Pi_f(\varphi)\,\mathrm{dV} \le C^+$, independent of $\varphi \in L^1(\Omega)$.

The new condition (3.6) can be rephrased as saying that only a finite amount of work can be performed against the external loads, regardless of the magnitude of translation and microdeformation. This is certainly true for any real field of applied loads.[11]

3.3 The different cases

We distinguish three different situations:

I: $\mu_c > 0$, $\alpha_4 \ge 0$, $p \ge 1$, $q \ge 0$, elastic macro-stability, local first order micromorphic. **Fracture excluded**.

II: $\mu_c = 0$, $\alpha_4 > 0$, $p \ge 1$, $q > 1$, elastic pre-stability, nonlocal second order micromorphic, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. **Fracture excluded**.

III: $\mu_c = 0$, $\alpha_4 = 0$, $0 < p \le 1$, $q = 0$, elastic pre-stability, nonlocal second order micromorphic theory, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $\varphi \notin W^{1,1}(\Omega, \mathbb{R}^3)$, due to lack of elastic coercivity, **including fracture** in multiaxial situations.

We refer to $0 < p < 1$, $q \ge 0$ as the **sub-critical case**, $p = 1$, $q \ge 0$ as the **critical case** and $p \ge 1$, $q > 1$ as the **super-critical case**. We will mathematically treat the first two cases.

3.4 The coercivity inequality

The decisive analytical tool for the treatment of case II (super-critical) is the following new non-trivial inequality establishing coercivity:

Theorem 3.1 (Extended 3D-Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Hausdorff measure. Define $H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0\ \forall\, \phi \in H^{1,2}_\circ(\Omega, \Gamma): \quad \|\nabla\phi F_p^{-1}(x) + F_p^{-T}(x)\nabla\phi^T\|^2_{L^2(\Omega)} \ge c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

[11] In classical finite elasticity, such a condition is not necessary since the elastic energy density is assumed a priori to verify a **coercivity condition** of the type $W(F) \ge c^+\|F\|^q - C$, $q > 1$, which, together with Dirichlet conditions and Poincarés inequality controls the $L^q(\Omega)$ part of the deformation. Fields satisfying (3.6) are e.g. the gravity field of a finite mass, the electric field of a finite charge etc. Remark as well that (3.6) does not exclude local, integrable singularities. The dead load case in (3.5) must rather be interpreted as linearization of the finite external potential: $\Pi(x, \varphi(x)) = \Pi(x, x + u(x)) = \Pi(x,x) + \langle D_\varphi \Pi(x,x), u \rangle + \ldots = \text{const.} + \langle f, u \rangle + \ldots$ with $f(x) = D_\varphi \Pi(x,x)$. The author is not aware of a previous introduction of a condition similar to (3.6).

Proof. The proof has been presented in [40]. Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. However, in general, F_p is **incompatible** giving rise to a **non-riemannian manifold** structure. Compare to [8] for an interpretation and the physical relevance of the volume dislocation density tensor $\operatorname{Curl} F_p$.

Motivated by the investigations in [40], it has been shown recently by Pompe [45] that the extended Korn's inequality can be viewed as a special case of a general class of coercivity inequalities for quadratic forms. He was able to show that indeed $F_p, F_p^{-1} \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for (3.1) to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p, F_p^{-1} \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ and $\operatorname{Curl} F_p \in L^{3+\delta}(\Omega)$ should suffice, whereas $F_p, F_p^{-1} \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [45]. ■

In view of the important role of the extended Korn's first inequality let us agree in saying that an inhomogeneous material characterized by a free energy density $W : \mathbb{R}^3 \times \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ is **elastically pre-stable**, whenever

$$\begin{aligned} &\exists\, H \in \mathbb{M}^{3\times3},\ H \neq 0: \quad D_F^2 W(x,F).(H,H) = 0 \\ &\exists\, c^+ > 0\ \exists\, G \in \mathrm{GL}^+(3,\mathbb{R})\ \forall H \in \mathbb{M}^{3\times3} :\ D_F^2 W(x,F).(H,H) \geq c^+ \|G(x)^T H + H^T G(x)\|^2 . \end{aligned} \tag{3.8}$$

In this terminology, infinitesimal classical elasticity is pre-stable with $G = 1\!\!1$ and the extended Korn's first inequality links the smoothness of G to the positive definiteness of the elastic tangent stiffness tensor.

3.5 The geometrically exact elastic micromorphic model

The following results are the first existence theorems for geometrically exact elastic micromorphic models known to the author:[12]

Theorem 3.2 (Existence for elastic micromorphic model: case I.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_\mathrm{d} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{P}_\mathrm{d} \in W^{1,1+p}(\Omega, \mathrm{SL}(3,\mathbb{R}))$. Moreover, let the applied external potentials satisfy (3.6). Then (3.1) with material constants conforming to case I admits at least one minimizing solution pair $(\varphi, \overline{P}) \in H^1(\Omega,\mathbb{R}^3)\times W^{1,1+p}(\Omega, \mathrm{SL}(3,\mathbb{R}))$.

Proof. We apply the direct methods of variations. The influence of the external potentials is gathered in writing $\Pi(\varphi,\overline{P})$. With the prescription of $(g_\mathrm{d},\overline{P}_\mathrm{d})$ it is clear that $I < \infty$ for exactly this pair of functions. Since (3.6) is assumed, it is also clear that I is bounded below for all $\varphi \in L^1(\Omega,\mathbb{R}^3)$ and $\overline{P} \in L^1(\Omega, \mathrm{SL}(3,\mathbb{R}))$.

We may choose decreasing (infimizing) sequences of pairs $(\varphi^k, \overline{P}^k) \in H^1(\Omega,\mathbb{R}^3)\times W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))$. The curvature contribution is bounded independent of the number k again on account of (3.6).[13]

Observe now that the micromorphic curvature term $\mathfrak{K}$ controls $\overline{P} \in W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))$, since $\|\mathfrak{K}\| = \|\overline{R}_p^T \mathrm{D_x}\overline{P}\| = \|\mathrm{D_x}\overline{P}\|$, pointwise and $\alpha_5,\alpha_6 > 0$. This fact, together with the appropriate boundary conditions and Poincarés inequality yields boundedness of the sequence $\{\overline{P}^k\}_{k=1}^\infty \subset W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))$.

We may extract a subsequence again denoted by $\overline{P}^k$ converging strongly in $L^{1+p}(\Omega)$ to an element $\widehat{P} \in W^{1,1+p}(\Omega,\mathbb{M}^{3\times3})$ since $p > 0$ by assumption. Moreover, a further subsequence can be found, such that $\mathfrak{K}_k$ converges weakly to some $\widehat{\mathfrak{K}}$ in $L^{1+p}(\Omega)$. The embedding $W^{1,2}(\Omega) \subset L^{6-\delta}(\Omega)$ for three space dimensions is compact and shows that the subsequence $\overline{P}^k$ can be chosen such that it converges indeed strongly in the topology of $L^{6-\delta}(\Omega)$ since $p \geq 1$ which implies immediately that $\widehat{P} \in W^{1,1+p}(\Omega,\mathrm{SL}(3,\mathbb{R}))$. Because $\mu_c > 0$, we have the simple algebraic estimate

$$\begin{aligned} W_{\mathrm{mp}}(\overline{P}^{-1,k}F^k, \overline{U}_p^k) &\geq \mu_c\,\|\overline{P}^{-1,k}F^k - 1\!\!1\|^2 = \mu_c\left(\|\overline{P}^{-1,k}F^k\|^2 - 2\langle \overline{P}^{-1,k}F^k, 1\!\!1\rangle + 3\right) \\ &\geq \mu_c\left(\|\overline{U}_k\|^2 - 2\sqrt{3}\|\overline{U}_k\| + 3\right), \end{aligned} \tag{3.9}$$

[12]The proposed finite results determine the macroscopic deformation $\varphi \in H^1(\Omega,\mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure). If $\mu_c > 0$ fracture is effectively ruled out, which is unrealistic.

[13]If (3.6) does not hold, one might have infimizing sequences with unbounded curvature. The geometrically exact micromorphic formulation is only **conditionally coercive**.

implying the boundedness of $\overline{U}_k = \overline{P}^{-1,k} F^k$ in $L^2(\Omega)$. Moreover, by Hölder's inequality, we obtain

$$\|F^k\|_{s,\Omega} = \|\overline{P}^k \overline{P}^{-1,k} F^k\|_{s,\Omega} \le \|\overline{P}^k\|_{r_1,\Omega} \|\overline{P}^{-1,k} F^k\|_{r_2,\Omega}\,, \quad \frac{1}{s} = \frac{1}{r_1} + \frac{1}{r_2}\,. \tag{3.10}$$

Since $\overline{P}^k$ is bounded in $L^6(\Omega)$ and $\overline{P}^{-1,k} F^k$ is bounded in $L^2(\Omega)$ we may choose $r_1 = 6$, $r_2 = 2$ to obtain boundedness of $F^k = \nabla\varphi_k$ in $L^s(\Omega)$, $s = \frac{3}{2}$. Using the boundary conditions for φ_k and the generalized Poincaré inequality we get

$$\|\varphi_k\|_{W^{1,s}(\Omega,\mathbb{R}^3)} \le Const.\,. \tag{3.11}$$

By the boundedness of φ^k in $W^{1,s}(\Omega,\mathbb{R}^3)$, we may extract a subsequence, not relabelled, such that $\varphi^k \rightharpoonup \hat{\varphi} \in W^{1,s}(\Omega,\mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of $(\varphi^k, \overline{P}^k)$ such that $\overline{U}_k = \overline{P}^{k,T} F^k$ converges weakly in $L^2(\Omega)$ to an element $\widehat{\overline{U}}$ on account of the boundedness of the stretch energy and $\mu_c > 0$.

We have already shown that for $p \ge 1$ the sequence $\overline{P}^k$ converges indeed strongly in $L^{1+p}(\Omega)$ to an element $\widehat{P} \in W^{1,1+p}(\Omega, \mathrm{SL}(3,\mathbb{R}))$. Thus $\overline{P}^{k,T} F^k$ converges certainly weakly to $\widehat{P}^T F$ in $L^1(\Omega)$ on account of Hölder's inequality (not sharp). The weak limit in $L^1(\Omega)$ must coincide with the weak limit of $\overline{U}_k$ in $L^2(\Omega)$. Hence, $\widehat{\overline{U}} = \widehat{P}^T \nabla\hat{\varphi}$.

Since the mapping polar : $\mathrm{GL}^+(3,\mathbb{R}) \mapsto \mathrm{SO}(3,\mathbb{R})$ is a bounded continuous function on invertible matrices with positive determinant, it generates a nonlinear superposition operator

$$\mathrm{polar}(\cdot) : L^{1+p}(\Omega, \mathrm{SL}(3,\mathbb{R})) \mapsto L^{1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))\,, \tag{3.12}$$

which, moreover, is continuous [3, p.101,Th.3.7]. Thus $\overline{R}_k = \mathrm{polar}(\overline{P}_k) \to \widehat{R} = \mathrm{polar}(\widehat{P})$ strongly in $L^{1+p}(\Omega)$ and a similar argument as for $\overline{U}_k$ shows that $\mathfrak{K}_k \rightharpoonup \widehat{\mathfrak{K}} = \mathrm{polar}(\widehat{P})^T \mathrm{D_x}\widehat{P}$ in $L^{1+p}(\Omega)$, weakly.

Since the total energy is convex in $(\overline{U}, \overline{U}_p, \mathfrak{K})$, and the external potential Π is continuous w.r.t. strong convergence in $L^1(\Omega)$ on account of (3.6), we get

$$\begin{aligned} I(\hat{\varphi}, \widehat{P}) &= \int_\Omega W_{\mathrm{mp}}(\widehat{\overline{U}}, \widehat{\overline{U}}_p) + W_{\mathrm{curv}}(\widehat{\mathfrak{K}})\,\mathrm{dV} - \Pi(\hat{\varphi}, \widehat{P}) \\ &\le \liminf_{k\to\infty} \int_\Omega W_{\mathrm{mp}}(\overline{U}_k, \overline{U}_p^k) + W_{\mathrm{curv}}(\mathfrak{K}_k)\,\mathrm{dV} - \Pi(\varphi_k, \overline{P}_k) \\ &= \lim_{k\to\infty} I(\varphi^k, \overline{P}^k) = \inf_{\substack{\varphi \in L^1(\Omega,\mathbb{R}^3) \\ \overline{P} \in L^1(\Omega, \mathrm{SL}(3,\mathbb{R}))}} I(\varphi, \overline{P})\,, \end{aligned} \tag{3.13}$$

which implies that the limit pair $(\hat{\varphi}, \widehat{P})$ is a minimizer. Note that the limit microdeformations $\widehat{P}$ may fail to be continuous if $p \le 2$ (non-existence or limit case of Sobolev embedding). Moreover, uniqueness cannot be ascertained, since $\mathrm{SL}(3,\mathbb{R})$ is a nonlinear manifold (and the considered problem is indeed nonlinear), such that convex combinations in $\mathrm{SL}(3,\mathbb{R})$ may leave $\mathrm{SL}(3,\mathbb{R})$. Since the functional I is differentiable the minimizing pair is a stationary point and therefore a solution of the field equations (2.12). Note again that the limit microdeformations may fail to be continuously distributed in space. That under these unfavourable circumstances a minimizing solution may nevertheless be found is entirely due to $\mu_c > 0$ and $p \ge 1$. The proof simplifies considerably in the geometrically exact Cosserat micropolar case $\overline{P} \in \mathrm{SO}(3,\mathbb{R})$. ■

We continue with the super-critical case appropriate for macroscopic situations and closer to classical elasticity.

Theorem 3.3 (Existence for elastic micromorphic model: case II.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{P}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SL}(3,\mathbb{R}))$. Moreover, let the applied external potentials satisfy (3.6). Then (3.1) with material constants conforming to case II admits at least one minimizing solution pair $(\varphi, \overline{P}) \in H^1(\Omega,\mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SL}(3,\mathbb{R}))$.

Proof. We repeat the argument of case I. However, the boundedness of infimizing sequences is not immediately clear. Boundedness of the microdeformations $\overline{P}^k$ holds true in the space $W^{1,1+p+q}(\Omega, \mathrm{SL}(3,\mathbb{R}))$

with $1+p+q>N=3$, hence we may extract a subsequence, not relabelled, such that $\overline{P}^k$ converges strongly to $\widehat{P}\in C^0(\overline{\Omega},\mathrm{SL}(3,\mathbb{R}))$ in the topology of $C^0(\overline{\Omega},\mathrm{SL}(3,\mathbb{R}))$ on account of the Sobolev-embedding theorem. Since $\widehat{P}^{-1}=\frac{1}{\det[\widehat{P}]}\operatorname{Adj}\widehat{P}$ we obtain as well $\widehat{P}^{-1}\in C^0(\overline{\Omega},\mathrm{SL}(3,\mathbb{R}))$.

Along such strongly convergent sequence of microdeformations, the sequence of deformations φ^k is also bounded in $H^1(\Omega,\mathbb{R}^3)$. However, this is not due to a basically simple estimate as in case I, but only true after integration over the domain: at face value we only control certain mixed symmetric expressions in the deformation gradient. More precisely, we have

$$
\begin{aligned}
\infty &> \int_\Omega W_{\mathrm{mp}}(\overline{U}_k,\overline{U}_p^k)+W_{\mathrm{curv}}(\mathfrak{K}_k)\,\mathrm{dV}-\Pi(\varphi_k,\overline{P}^k)\geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k)\,\mathrm{dV}-\Pi(\varphi_k,\overline{P}^k)\\
&\geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k,\overline{U}_p^k)\,\mathrm{dV}-C\\
&\geq \int_\Omega \frac{\mu}{4}\|\overline{P}^{-1,k}\nabla\varphi_k+\nabla\varphi_k^T\overline{P}^{-T,k}-2\mathbb{1}\|^2\,\mathrm{dV}-C \qquad (3.14)\\
&\geq \int_\Omega \frac{\mu}{4}\|\overline{P}^{-1,k}\nabla u_k+\nabla u_k^T\overline{P}^{-T,k}\|^2\,\mathrm{dV}-C_1(\|\overline{P}\|_\infty^k)\,\|u_k\|_{H^{1,2}(\Omega)}+C_2\\
&= \int_\Omega \frac{\mu}{4}\|(\overline{P}^{-1,k}-\widehat{P}^{-1}+\widehat{P}^{-1})\nabla u_k+\nabla u_k^T(\overline{P}^{-1,k}-\widehat{P}^{-1}+\widehat{P}^{-1})^T\|^2\,\mathrm{dV}-C_1\,\|u_k\|_{H^{1,2}(\Omega)}+C_2\\
&\geq \int_\Omega \frac{\mu}{4}\underbrace{\|\widehat{P}^{-1}\nabla u_k+\nabla u_k^T\widehat{P}^{-T}\|^2}_{\text{combinations of derivatives}}\,\mathrm{dV}-C_3\,\|\widehat{P}^{-1}-\overline{P}^{-1,k}\|_\infty\,\|u_k\|^2_{H^{1,2}(\Omega)}\\
&\qquad -(C_1+2\,\|\widehat{P}^{-1}-\overline{P}^{-1,k}\|_\infty)\,\|u_k\|_{H^{1,2}(\Omega)}+C_2\\
&\geq (\frac{\mu}{4}c_K-C_3\,\|\widehat{P}^{-1}-\overline{P}^{-1,k}\|_\infty)\,\|u_k\|^2_{H^{1,2}(\Omega)}-(C_1+2\,\|\widehat{P}^{-1}-\overline{P}^{-1,k}\|_\infty)\,\|u_k\|_{H^{1,2}(\Omega)}+C_2\,,
\end{aligned}
$$

where we made use of the appropriate boundary conditions for $\varphi^k=x+u_k$ and applied the extended Korn's inequality (3.1) in the improved version of [45] yielding the positive constant c_K for the **continuous** microdeformation $\widehat{P}^{-1}$. Since $\|\widehat{P}^{-1}-\overline{P}^{-1,k}\|_\infty=\|\frac{1}{\det[\widehat{P}]}\operatorname{Adj}\widehat{P}-\frac{1}{\det[\overline{P}^k]}\operatorname{Adj}\overline{P}^k\|_\infty=\|\operatorname{Adj}\widehat{P}-\operatorname{Adj}\overline{P}^k\|_\infty\to 0$ by strong convergence of $\overline{P}^k$, we conclude the boundedness of u_k in $H^1(\Omega)$. Hence, φ_k is bounded in $H^1(\Omega)$. Now we obtain that $\overline{U_k}\rightharpoonup\widehat{\overline{U}}=\widehat{P}^{-1}\nabla\hat{\varphi}$ by construction with the notations as in case I. The remainder proceeds as in case I. This finishes the argument. The limit microdeformations $\widehat{P}$ are indeed found to be continuous. ■

4 Final remarks

The presented variational micromorphic problem fits neatly in the framework of the direct methods of variations. The coercivity part for the deformation is, however, highly nontrivial and for the value of the Cosserat couple modulus $\mu_c=0$, additional difficulties arise which can only be circumvented by the use of the generalized Korn's first inequality. In both cases I/II, more realistic assumptions on the applied external loads Π are necessary to establish a lower bound for the energy I and an independent control of the curvature.

Altogether, the quasistatic finite micromorphic theory is established on firm mathematical grounds. With the same methods, the geometrically exact microstretch case can also be treated.

An extension of the method to other choices of strain and curvature measures needs to be done, however, this might be a non-trivial task.

The open case III allows in principle for discontinuous macroscopic deformations and might therefore be a model problem allowing to describe fracture. The presented variational framework is ideally suited for subsequent numerical treatment by the finite element method.

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References

[1] E.L. Aero and E.V. Kuvshinskii. Fundamental equations of the theory of elastic media with rotationally interacting particles. *Soviet Physics-Solid State*, 2:1272–1281, 1961.

[2] S. Antman. *Nonlinear Problems of Elasticity*, volume 107 of *Applied Mathematical Sciences*. Springer, Berlin, 1995.

[3] J. Appell and P. Zabrejko. *Nonlinear Superposition Operators.*, volume 95 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.

[4] A. Bertram and B. Svendsen. On material objectivity and reduced constitutive equations. *Arch. Mech.*, 53:653–675, 2001.

[5] G. Capriz. *Continua with Microstructure*. Springer, Heidelberg, 1989.

[6] G. Capriz and P. Podio Guidugli. Formal structure and classification of theories of oriented media. *Ann. Mat. Pura Appl., Ser. IV*, 115:17–39, 1977.

[7] G. Capriz and P. Podio-Guidugli. Structured continua from a lagrangian point of view. *Ann. Mat. Pura Appl. Ser IV*, 135:1–25, 1983.

[8] P. Cermelli and M.E. Gurtin. On the characterization of geometrically necessary dislocations in finite plasticity. *J. Mech. Phys. Solids*, 49:1539–1568, 2001.

[9] Y. Chen and J.D. Lee. Connecting molecular dynamics to micromorphic theory. I. Instantaneous and averaged mechanical variables. *Physica A*, 322:359–376, 2003.

[10] Y. Chen and J.D. Lee. Connecting molecular dynamics to micromorphic theory. II. Balance laws. *Physica A*, 322:376–392, 2003.

[11] Philippe G. Ciarlet. *Three-Dimensional Elasticity*, volume 1 of *Studies in Mathematics and its Applications*. Elsevier, Amsterdam, first edition, 1988.

[12] E. Cosserat and F. Cosserat. *Théorie des corps déformables*. Librairie Scientifique A. Hermann et Fils, Paris, 1909.

[13] R. Dendievel, S. Forest, and G. Canova. An estimation of overall properties of heterogeneous Cosserat materials. In A. Bertram and F. Sidoroff, editors, *Mechanics of Materials with Intrinsic Length Scale: Physics, Experiments, Modelling and Applications.*, Journal Physique IV France 8, pages 111–118. EDP Sciences, France, 1998.

[14] A. Dietsche, P. Steinmann, and K. William. Micropolar elastoplasticity and its role in localization. *Int. J. Plasticity*, 9:813–831, 1993.

[15] G. Duvaut. Elasticité linéaire avec couples de contraintes. Théorèmes d'existence. *J. Mec. Paris*, 9:325–333, 1970.

[16] W. Ehlers, S. Diebels, and W. Volk. Deformation and compatibilty for elasto-plastic micropolar materials with applications to geomechanical problems. In A. Bertram and F. Sidoroff, editors, *Mechanics of Materials with Intrinsic Length Scale: Physics, Experiments, Modelling and Applications.*, Journal Physique IV France 8, pages 127–134. EDP Sciences, France, 1998.

[17] A. C. Eringen. *Microcontinuum Field Theories*. Springer, Heidelberg, 1999.

[18] A.C. Eringen. Theory of Micropolar Elasticity. In H. Liebowitz, editor, *Fracture. An advanced treatise.*, volume II, pages 621–729. Academic Press, New York, 1968.

[19] A.C. Eringen and C.B. Kafadar. Polar Field Theories. In A.C. Eringen, editor, *Continuum Physics*, volume IV: Polar and Nonlocal Field Theories, pages 1–73. Academic Press, New York, 1976.

[20] A.C. Eringen and E.S. Suhubi. Nonlinear theory of simple micro-elastic solids. *Int. J. Engrg. Sci.*, 2:189–203, 1964.

[21] S. Forest, F. Barbe, and G. Cailletaud. Cosserat modelling of size effects in the mechanical behaviour of polycrystals and multi-phase materials. *Int. J. Solids Struct.*, 37:7105–7126, 2000.

[22] S. Forest, G. Cailletaud, and R. Sievert. A Cosserat theory for elastoviscoplastic single crystals at finite deformation. *Arch. Mech.*, 49(4):705–736, 1997.

[23] R.D. Gauthier. Experimental investigations on micropolar media. In O. Brulin and R.K.T. Hsieh, editors, *Mechanics of Micropolar Media*, pages 395–463. CISM Lectures, World Scientific, Singapore, 1982.

[24] V. Gheorghita. On the existence and uniqueness of solutions in linear theory of Cosserat elasticity. I. *Arch. Mech.*, 26:933–938, 1974.

[25] V. Gheorghita. On the existence and uniqueness of solutions in linear theory of Cosserat elasticity. II. *Arch. Mech.*, 29:355–358, 1974.

[26] W. Günther. Zur Statik und Kinematik des Cosseratschen Kontinuums. *Abh. Braunschweigische Wiss. Gesell.*, 10:195–213, 1958.

[27] P. Grammenoudis. *Mikropolare Plastizität*. Ph.D-Thesis Mechanics. TU Darmstadt, http://elib.tu-darmstadt.de/diss/000312, 2003.

[28] A.E. Green and R.S. Rivlin. Multipolar continuum mechanics. *Arch. Rat. Mech. Anal.*, 17:113–147, 1964.

[29] M.E. Gurtin and P. Podio-Guidugli. On the formulation of mechanical balance laws for structured continua. *Z. Angew. Math. Phys.*, 43:181–190, 1992.

[30] I. Hlavacek and M. Hlavacek. On the existence and uniqueness of solutions and some variational principles in linear theories of elasticity with couple-stresses. I: Cosserat continuum. II: Mindlin's elasticity with micro-structure and the first strain gradient. *J. Apl. Mat.*, 14:387–426, 1969.

[31] D. Iesan. Existence theorems in micropolar elastostatics. *Int. J. Eng. Sci.*, 9:59, 1971.

[32] D. Iesan and A. Pompei. On the equilibrium theory of microstretch elastic solids. *Int. J. Eng. Sci.*, 33:399–410, 1995.

[33] D. Iesan and R. Quintanilla. Existence and continuous dependence results in the theory of microstretch elastic bodies. *Int. J. Eng. Sci.*, 32:991–1001, 1994.

[34] D. Iesan and A. Scalia. On Saint-Venants principle for microstretch elastic bodies. *Int. J. Eng. Sci.*, 35:1277–1290, 1997.

[35] M.M. Iordache and K. William. Localized failure analysis in elastoplastic Cosserat continua. *Comp. Meth. Appl. Mech. Engrg.*, 151:559–586, 1998.

[36] E. Kröner. *Mechanics of Generalized Continua.* Proceedings of the IUTAM-Symposium on the generalized Cosserat continuum and the continuum theory of dislocations with applications in Freudenstadt, 1967. Springer, Heidelberg, 1968.

[37] J.E. Marsden and J.R. Hughes. *Mathematical Foundations of Elasticity.* Prentice-Hall, Englewood Cliffs, New Jersey, 1983.

[38] R.D. Mindlin and H.F. Tiersten. Effects of couple stresses in linear elasticity. *Arch. Rat. Mech. Anal.*, 11:415–447, 1962.

[39] A.I. Murdoch. Objectivity in classical continuum physics: a rational for discarding the 'principle of invariance under superposed rigid body motions' in favour of purely objective considerations. *Cont. Mech. Thermo.*, 15:309–320, 2003.

[40] P. Neff. On Korn's first inequality with nonconstant coefficients. *Proc. Roy. Soc. Edinb.*, 132A:221–243, 2002.

[41] P. Neff. Finite multiplicative plasticity for small elastic strains with linear balance equations and grain boundary relaxation. *Cont. Mech. Thermodynamics*, 15(2)(DOI 10.1007/s00161-002-0190-x):161–195, 2003.

[42] P. Neff. Finite multiplicative elastic-viscoplastic Cosserat micropolar theory for polycrystals with grain rotations. Modelling and mathematical analysis. *Preprint 2297, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/pp03.html, submitted to Arch. Rat. Mech. Anal.*, 9/2003.

[43] W. Nowacki. *Theory of Asymmetric Elasticity.* Pergamon Press, Oxford, 1986.

[44] N. Oshima. Dynamics of granular media. In K. Kondo, editor, *Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry*, volume 1, Division D-VI, pages 111–120 (563–572). Gakujutsu Bunken Fukyo-Kai, 1955.

[45] W. Pompe. Korn's first inequality with variable coefficients and its generalizations. *Comment. Math. Univ. Carolinae*, 44,1:57–70, 2003.

[46] C. Sansour. *Ein einheitliches Konzept verallgemeinerter Kontinua mit Mikrostruktur unter besonderer Berücksichtigung der finiten Viskoplastizität.* Habilitation-Thesis, Shaker-Verlag, Aachen, 1999.

[47] H. Schaefer. Das Cosserat-Kontinuum. *ZAMM*, 47:485–498, 1967.

[48] P. Steinmann. A micropolar theory of finite deformation and finite rotation multiplicative elastoplasticity. *Int. J. Solids Structures*, 31(8):1063–1084, 1994.

[49] P. Steinmann. A unifying treatise of variational principles for two types of micropolar continua. *Acta Mech.*, 121:215–232, 1997.

[50] B. Svendsen and A. Bertram. On frame-indifference and form-invariance in constitutive theory. *Acta Mechanica*, 132:195–207, 1997.

[51] R.A. Toupin. Elastic materials with couple stresses. *Arch. Rat. Mech. Anal.*, 11:385–413, 1962.

[52] R.A. Toupin. Theory of elasticity with couple stresses. *Arch. Rat. Mech. Anal.*, 17:85–112, 1964.

[53] C. Truesdell and W. Noll. The non-linear field theories of mechanics. In S. Fluegge, editor, *Handbuch der Physik*, volume III/3. Springer, Heidelberg, 1965.

6 Appendix

6.1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and by $\mathfrak{T}(3)$ the set of all third order tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \;|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|X^TX = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \;|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \;|\operatorname{tr}[X] = 0\}$ of traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$. The operator $\operatorname{axl} : \mathfrak{so}(3,\mathbb{R}) \mapsto \mathbb{R}^3$ is the canonical identification. We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)\,U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. For a second order tensor X we define the third order tensor $\mathfrak{h} = \mathrm{D}_x X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3}$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}\left[\mathfrak{h}^1\right], \operatorname{tr}\left[\mathfrak{h}^2\right], \operatorname{tr}\left[\mathfrak{h}^3\right]) \in \mathbb{R}^3$. Moreover, for any second order tensor X we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$ correspondingly. Quantities with a bar, e.g. the micropolar rotation $\overline{R}_p$, represent the micropolar replacement of the corresponding classical continuum rotation R. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$ and we use ∇ in general only for column-vectors in $\mathbb{R}^3$. Furthermore, $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$, respectively. Sometimes we use also $\partial_X W(X)$ to denote the first derivative of W with respect to X. We employ the standard notation of Sobolev spaces, i.e.

$L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. For $X \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ we define $\operatorname{Curl} X(x)$ and $\operatorname{Div} X(x)$ as the operation curl and Div applied row wise, respectively. For $\mathfrak{h} \in \mathfrak{T}(3)$ we define $\operatorname{Div}\mathfrak{h} = \left(\operatorname{Div}\mathfrak{h}^1 | \operatorname{Div}\mathfrak{h}^2 | \operatorname{Div}\mathfrak{h}^3\right)^T \in \mathbb{M}^{3\times3}$. We define $H_\circ^{1,2}(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$. Finally, w.r.t. abbreviates with respect to.

6.2 Derivation of the geometrically exact micromorphic balance equations

The derivation of the force balance equation is straight forward. Since we can write $\overline{P} = \overline{R}_p \cdot \overline{U}_p$ and $\overline{R}_p, \overline{U}_p$ can be prescribed arbitrarily, we may realize the variation of $\overline{P}$ through independent variation of the orthogonal and stretch part:

$$\overline{P} = \overline{R}_p \cdot \overline{U}_p \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}\overline{P} = \left[\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}_p\right]\overline{U}_p + \overline{R}_p\left[\frac{\mathrm{d}}{\mathrm{dt}}\overline{U}_p\right]. \tag{6.1}$$

Now take either $\frac{\mathrm{d}}{\mathrm{dt}}\overline{U}_p = 0$ or $\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}_p = 0$. In the first case, we have the variation

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{P} = \left[\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}_p\right]\overline{U}_p = A\,\overline{R}_p\,\overline{U}_p = A\,\overline{P}, \quad A \in \mathfrak{so}(3,\mathbb{R}), \tag{6.2}$$

and in the second case we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{P} = \overline{R}_p\left[\frac{\mathrm{d}}{\mathrm{dt}}\overline{U}_p\right] = \overline{R}_p\,T\,\overline{U}_p, \quad T \in \mathfrak{sl}(3,\mathbb{R}) \cap \operatorname{Sym}(3). \tag{6.3}$$

For the first case, we consider simultaneously in each space point a one parameter group of microdeformations $\frac{\mathrm{d}}{\mathrm{dt}}\hat{P}(x,t) = A(x,t)\cdot\hat{P}(x,t),\ \hat{P}(x,0) = \overline{P}(x),\ A \in C_0^\infty(\Omega, \mathfrak{sl}(3,\mathbb{R}))$. The corresponding stationarity condition is obtained from $\frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}} I(\varphi, \hat{P}(x,t)) = 0$. This yields three terms: the derivatives involving $W_{\mathrm{mp}}(F,\overline{P})$ and $\Pi(\overline{P})$ are straightforward, using the definition of the one parameter group, and yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}}\Pi(\hat{P}(x,t)) &= \langle D_{\overline{P}}\Pi(\hat{P}(x,t), \frac{\mathrm{d}}{\mathrm{dt}}\hat{P}(x,t)\rangle = \langle D_{\overline{P}}\Pi(\hat{P}(x,t), A(x,t)\cdot\hat{P}(x,t)\rangle = \langle D_{\overline{P}}\Pi(\overline{P})\overline{P}^T, A(x,t)\rangle \\
&= \langle D_{\overline{P}}\Pi(\overline{P})\overline{U}_p\overline{R}_p^T, A(x,t)\rangle = \langle \overline{R}_p\overline{R}_p^T\, D_{\overline{P}}\Pi(\overline{P})\overline{U}_p\overline{R}_p^T, A(x,t)\rangle = \langle \overline{R}_p \operatorname{skew}\left(\overline{R}_p^T\, D_{\overline{P}}\Pi(\overline{P})\overline{U}_p\right)\overline{R}_p^T, A(x,t)\rangle,
\end{aligned}\tag{6.4}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}} W_{\mathrm{mp}}(F,\hat{P}(x,t)) &= \langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p), \frac{\mathrm{d}}{\mathrm{dt}}\overline{U}\rangle = \langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p), \frac{\mathrm{d}}{\mathrm{dt}}[\hat{P}^{-1}F]\rangle = \langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p), [\frac{\mathrm{d}}{\mathrm{dt}}\hat{P}^{-1}]F\rangle \\
&= \langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p), -\hat{P}^{-1}[\frac{\mathrm{d}}{\mathrm{dt}}\hat{P}]\hat{P}^{-1}F\rangle = -\langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p), \hat{P}^{-1}[\frac{\mathrm{d}}{\mathrm{dt}}\hat{P}]\overline{U}\rangle \\
&= -\langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p), \hat{P}^{-1}A(x,t)\cdot\hat{P}(x,t)\overline{U}\rangle = -\langle D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p)\overline{U}^T, \hat{P}^{-1}A(x,t)\cdot\hat{P}(x,t)\rangle \\
&= -\langle \hat{P}^{-T} D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p)\overline{U}^T\hat{P}^T, A(x,t)\rangle = -\langle \overline{R}_p\overline{U}_p^{-1} D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p)\overline{U}^T\overline{U}_p\overline{R}_p^T, A(x,t)\rangle \qquad (6.5)\\
&= -\langle \overline{R}_p \operatorname{skew}\left(\overline{U}_p^{-1} D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p)\overline{U}^T\overline{U}_p\right)\overline{R}_p^T, A(x,t)\rangle = -\langle \overline{R}_p \operatorname{skew}\left(\overline{U}_p^{-1} D_{\overline{U}}W_{\mathrm{mp}}(\overline{U},\overline{U}_p)\overline{U}^T\overline{U}_p^T\right)\overline{R}_p^T, A(x,t)\rangle
\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ means additionally integration w.r.t. x. For the term containing the curvature part, we note

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}}\int_\Omega W_{\mathrm{curv}}(\mathfrak{K}(x,t))\,\mathrm{dV} &= \sum_{i=1}^3 \langle \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3), \overline{R}_p^T\nabla(A\overline{P}.e_i) + (A\overline{R}_p)^T\nabla(\overline{P}.e_i)\rangle_{\mathbb{M}^{3\times3}} \\
&= \sum_{i=1}^3 \langle \overline{R}_p\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3), \nabla(A\overline{P}.e_i)\rangle_{\mathbb{M}^{3\times3}} + \langle \overline{R}_p\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\overline{R}_p^T, A^T\rangle_{\mathbb{M}^{3\times3}} \qquad (6.6)\\
&= \sum_{i=1}^3 -\langle \operatorname{Div}\left[\overline{R}_p\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\right], A\overline{P}.e_i\rangle_{\mathbb{R}^3} + \langle \overline{R}_p\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\overline{R}_p^T, A^T\rangle \\
&= -\langle \operatorname{Div}\left[\overline{R}_p\,D_{\mathfrak{K}}W_{\mathrm{curv}}(\mathfrak{K})\right], A\overline{P}\rangle_{\mathbb{M}^{3\times3}} + \langle \overline{R}_p\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\,\overline{R}_p^T\right), A^T\rangle \\
&= -\langle \operatorname{Div}\left[\overline{R}_p\,D_{\mathfrak{K}}W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{P}^T, A\rangle + \langle \overline{R}_p\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\overline{R}_p^T, A^T\rangle \\
&= -\langle \overline{R}_p\,\overline{R}_p^T \operatorname{Div}\left[\overline{R}_p\,D_{\mathfrak{K}}W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{U}_p\overline{R}_p^T, A\rangle - \langle \overline{R}_p \operatorname{skew}\left(\sum_{i=1}^3\left(\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\right)\overline{R}_p^T, A\rangle \qquad (6.7)\\
&= -\langle \overline{R}_p \operatorname{skew}\left(\overline{R}_p^T \operatorname{Div}\left[\overline{R}_p\,D_{\mathfrak{K}}W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{U}_p\right)\overline{R}_p^T, A\rangle - \langle \overline{R}_p \operatorname{skew}\left(D_{\mathfrak{K}}W_{\mathrm{curv}}(\mathfrak{K})\mathfrak{K}^T\right)\overline{R}_p^T, A\rangle.
\end{aligned}$$

Since $A \in C_0^\infty(\Omega, \mathfrak{so}(3,\mathbb{R}))$ is arbitrary, equation $(2.12)_2$ follows. In order to obtain the remaining five equations for the five independent components of $\overline{U}_p \in \mathrm{SL}(3,\mathbb{R}) \cap \mathrm{PSym}(3)$ we consider the second possible independent variation of $\overline{P}$. With

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{P} = \overline{R}_p\,T\,\overline{U}_p, \quad T \in \mathfrak{sl}(3,\mathbb{R}) \cap \operatorname{Sym}(3), \tag{6.8}$$

we consider simultaneously in each space point a one parameter group of microdeformations $\frac{\mathrm{d}}{\mathrm{dt}}\hat{P}(x,t) = \overline{R}_p\,T\,\overline{U}_p,\ \hat{P}(x,0) = \overline{P}(x),\ T \in C_0^\infty(\Omega, \mathfrak{sl}(3,\mathbb{R}))$. The corresponding stationarity condition is obtained from $\frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}} I(\varphi, \hat{P}(x,t)) = 0$. This yields again three terms: the derivatives involving $W_{\rm mp}(F,\overline{P})$ and $\Pi(\overline{P})$ are straightforward, using the definition of the one parameter group, and yield

$$\begin{aligned}\frac{\rm d}{\rm dt}_{|t=0}\Pi(\hat{P}(x,t)) &= \langle D_{\overline{P}}\Pi(\hat{P}(x,t),\frac{\rm d}{\rm dt}\hat{P}(x,t)\rangle = \langle D_{\overline{P}}\Pi(\hat{P}(x,t),\overline{R}_pT(x,t)\cdot\overline{U}_p(x,t)\rangle = \langle\overline{R}_p^TD_{\overline{P}}\Pi(\overline{P})\overline{U}_p,T(x,t)\rangle \\ &= \langle\operatorname{dev}\operatorname{sym}\left(\overline{R}_p^TD_{\overline{P}}\Pi(\overline{P})\overline{U}_p\right),T(x,t)\rangle\,, \end{aligned} \tag{6.9}$$

and

$$\begin{aligned}\frac{\rm d}{\rm dt}_{|t=0}W_{\rm mp}(F,\hat{P}(x,t)) &= \langle D_{\overline{U}}W_{\rm mp}(\overline{U},\overline{U}_p),\frac{\rm d}{\rm dt}\overline{U}\rangle + \langle D_{\overline{U}_p}W_{\rm mp}(\overline{U},\overline{U}_p),\frac{\rm d}{\rm dt}\overline{U}_p\rangle \\ &= -\langle D_{\overline{U}}W_{\rm mp}(\overline{U},\overline{U}_p),\hat{P}^{-1}[\frac{\rm d}{\rm dt}\hat{P}]\overline{U}\rangle + \langle D_{\overline{U}_p}W_{\rm mp}(\overline{U},\overline{U}_p),T(x,t)\overline{U}_p\rangle \\ &= -\langle D_{\overline{U}}W_{\rm mp}(\overline{U},\overline{U}_p),\hat{P}^{-1}\overline{R}_pT(x,t)\cdot\overline{U}_p\overline{U}\rangle + \langle D_{\overline{U}_p}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}_p^T,T(x,t)\rangle \\ &= -\langle D_{\overline{U}}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}^T,\overline{U}_p^{-1}T(x,t)\overline{U}_p\rangle + \langle D_{\overline{U}_p}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}_p^T,T(x,t)\rangle \\ &= -\langle\overline{U}_p^{-1}D_{\overline{U}}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}^T\overline{U}_p,T(x,t)\rangle + \langle D_{\overline{U}_p}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}_p^T,T(x,t)\rangle \\ &= -\langle\operatorname{dev}\operatorname{sym}\left(\overline{U}_p^{-1}D_{\overline{U}}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}^T\overline{U}_p\right),T(x,t)\rangle + \langle\operatorname{dev}\operatorname{sym}\left(D_{\overline{U}_p}W_{\rm mp}(\overline{U},\overline{U}_p)\overline{U}_p^T\right),T(x,t)\rangle\,. \end{aligned} \tag{6.10}$$

For the term containing the curvature part, we note

$$\begin{aligned}\frac{\rm d}{\rm dt}_{|t=0}\int_\Omega W_{\rm curv}(\mathfrak{K}(x,t))\,{\rm dV} &= \sum_{i=1}^3\langle\partial_{\mathfrak{K}^i}W_{\rm curv}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3),\overline{R}_p^T\nabla(\overline{R}_pT\overline{U}_p.e_i)+(\frac{\rm d}{\rm dt}\overline{R}_p)^T\nabla(\overline{P}.e_i)\rangle_{\mathbb{M}^{3\times3}} \\ &= \sum_{i=1}^3\langle\overline{R}_p\,\partial_{\mathfrak{K}^i}W_{\rm curv}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3),\nabla(T\overline{U}_p.e_i)\rangle_{\mathbb{M}^{3\times3}} = -\langle\operatorname{dev}\operatorname{sym}\left(\overline{R}_p^T\operatorname{Div}\left[\overline{R}_p\,D_{\mathfrak{K}}W_{\rm curv}(\mathfrak{K})\right]\overline{U}_p\right),T\rangle\,. \end{aligned} \tag{6.11}$$

Since $T\in C_0^\infty(\Omega,\mathfrak{sl}(3,\mathbb{R}))$ is arbitrary, equation $(2.12)_3$ follows. By splitting the possible variations of $\overline{P}\in{\rm SL}(3,\mathbb{R})$, we have implicitly used the Cartan Lie-algebra decomposition: $\mathfrak{sl}(3,\mathbb{R})=\mathfrak{so}(3,\mathbb{R})\oplus\mathfrak{p}$, $\mathfrak{p}=\{T\in{\rm sym}(3)\,|\,{\rm tr}\,[T]=0\}$.

6.3 The infinitesimal micromorphic elastic solid

Starting with the finite formulation (3.1) we may obtain a linear, infinitesimal microincompressible, micromorphic model by expanding all appearing variables to first order and keeping quadratic terms in the energy expression. Thus we write $F=1\!\!1+\nabla u$, $\overline{P}=1\!\!1+\overline{p}$, ${\rm tr}\,[\overline{p}]=0$ and the model turns into the problem of finding a pair $(u,\overline{p}):\Omega\subset\mathbb{R}^3\mapsto\mathbb{R}^3\times\mathfrak{sl}(3,\mathbb{R})$ of displacement u and **independent, infinitesimal microdeformation** $\overline{p}$ satisfying

$$\begin{aligned}&\int_\Omega W_{\rm mp}(\overline{\varepsilon},{\rm sym}\,\overline{p})+W_{\rm curv}(\mathfrak{k})-\langle f,u\rangle-\langle M,\overline{p}\rangle\,{\rm dV}-\int_{\Gamma_S}\langle N,u\rangle\,{\rm dS}-\int_{\Gamma_C}\langle M_c,\overline{p}\rangle\,{\rm dS}\mapsto\min.\ \text{w.r.t. } (u,\overline{p}),\\ &\overline{\varepsilon}=\nabla u-\overline{p},\quad \overline{p}_{|_\Gamma}=\overline{p}_{\rm d}\in\mathfrak{p},\quad \varphi_{|_\Gamma}=g_{\rm d}\\ &f(x)=D_\varphi\Pi_f(x,x),\quad M(x)=D_{\overline{P}}\Pi_M(x,1\!\!1),\quad N(x)=D_\varphi\Pi_N(x,x),\quad M_c(x)=D_{\overline{P}}\Pi_{M_c}(x,1\!\!1)\\ &W_{\rm mp}(\overline{\varepsilon},\overline{p})=\mu\,\|{\rm sym}\,\overline{\varepsilon}\|^2+\mu_c\,\|{\rm skew}\,\overline{\varepsilon}\|^2+\frac{\lambda}{2}\,{\rm tr}\,[{\rm sym}\,\overline{\varepsilon}]^2+\mu^{\rm m}\,\|{\rm sym}\,\overline{p}\|^2+\frac{\lambda^{\rm m}}{2}{\rm tr}\,[{\rm sym}\,\overline{p}]^2\\ &\qquad=\mu\,\|{\rm sym}\,\nabla u-{\rm sym}\,\overline{p}\|^2+\mu_c\,\|{\rm skew}(\nabla u-\overline{p})\|^2+\frac{\lambda}{2}\,{\rm tr}\,[{\rm sym}\,\nabla u]^2+\mu^{\rm m}\,\|{\rm sym}\,\overline{p}\|^2\\ &W_{\rm curv}(\mathfrak{k})=\mu\frac{L_c^2}{12}\,\left(\alpha_5\,\|{\rm sym}\,\mathfrak{k}\|^2+\alpha_6\,\|{\rm skew}\,\mathfrak{k}\|^2+\alpha_7\,{\rm tr}\,[\mathfrak{k}]^2\right)\,,\\ &\mathfrak{k}={\rm D_x}\overline{p}=(\nabla(\overline{p}.e_1),\nabla(\overline{p}.e_2),\nabla(\overline{p}.e_3))\,,\quad\text{third order, infinitesimal } \textbf{curvature tensor}\,. \end{aligned} \tag{6.12}$$

If $\mu,\mu^{\rm m}>0$ and $\mu_c,\lambda,\lambda^{\rm m}\geq 0$ it is an easy matter to show existence and uniqueness without any modification of the classical situation for the external loads. The energy functional can be easily shown to be bounded below as soon as we assume that $f,M\in L^2(\Omega)$, $N,M_c\in L^2(\partial\Omega)$.

6.4 The infinitesimal microstretch elastic solid

Such a model is obtained by assuming $\overline{P}=e^{\overline{\alpha}}\,\overline{R},\overline{\alpha}\in\mathbb{R},\overline{R}\in{\rm SO}(3,\mathbb{R})$ with independent variables $\overline{\alpha},\overline{R}$ and independent curvature parts $\mathfrak{K}=\overline{R}^T{\rm D_x}\overline{R}$ and $\nabla\overline{\alpha}(x,y,z)$. Inserting this assumption[14] into (3.1) and expanding $e^{\overline{\alpha}}=1+\overline{\alpha}+\dots$, $\overline{R}=$

[14]This is not the most general infinitesimal microstretch formulation.

$1\!\!1 + \overline{A} + \dots$ yields to first order the **three-field** problem

$$\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}, \overline{\alpha}) + W_{\mathrm{curv}}(\mathfrak{k}, \nabla\overline{\alpha}) - \langle f, u\rangle - \langle M, \overline{A} + \overline{\alpha}\cdot 1\!\!1\rangle \,\mathrm{dV}$$

$$- \int_{\Gamma_S} \langle N, u\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A} + \overline{\alpha}\cdot 1\!\!1\rangle \,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (u, \overline{A}, \overline{\alpha}),$$

$$\overline{\varepsilon} = \nabla u - \overline{A} - \overline{\alpha}\cdot 1\!\!1, \quad \overline{A}_{|_\Gamma} = \overline{A}_{\mathrm{d}} \in \mathfrak{so}(3,\mathbb{R}), \quad \overline{\alpha}_{|_\Gamma} = \overline{\alpha}_{\mathrm{d}\,|_\Gamma}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}} \tag{6.13}$$

$$\overline{U}_p = \sqrt{\overline{P}^T\overline{P}} = \sqrt{e^{2\overline{\alpha}}\overline{R}^T\overline{R}} = e^{\overline{\alpha}}\cdot 1\!\!1 = (1 + \overline{\alpha} + \dots)\cdot 1\!\!1 = 1\!\!1 + \overline{\alpha}\cdot 1\!\!1 + \dots$$

$$W_{\mathrm{mp}}(\overline{\varepsilon}, \overline{A}, \overline{\alpha}) = \mu\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\overline{\varepsilon}]^2 + \mu^{\mathrm{m}}\,\|\overline{\alpha}\cdot 1\!\!1\|^2 + \frac{\lambda^{\mathrm{m}}}{2}\mathrm{tr}\,[\overline{\alpha}\cdot 1\!\!1]^2$$

$$= \mu\,\|\mathrm{sym}\,\nabla u - \overline{\alpha}1\!\!1\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\nabla u - \overline{\alpha}\cdot 1\!\!1]^2 + \frac{3\overline{\alpha}^2}{2}\,(2\mu^{\mathrm{m}} + 3\lambda^{\mathrm{m}})$$

$$W_{\mathrm{curv}}(\mathfrak{k}, \nabla\overline{\alpha}) = \mu\frac{L_c^2}{12}\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{k}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{k}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{k}]^2 + \alpha_8\|\nabla\overline{\alpha}\|^2\right),$$

$$\mathfrak{k} = \mathrm{D_x}\overline{A} = \left(\nabla(\overline{A}.e_1), \nabla(\overline{A}.e_2), \nabla(\overline{A}.e_3)\right), \quad \text{third order, infinitesimal } \textbf{curvature tensor}\,.$$

6.5 The infinitesimal micropolar elastic solid

Such a model is obtained by setting $\overline{\alpha} \equiv 0$ in (6.13). We are left with the **two-field** problem

$$\int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\mathfrak{k}) - \langle f, u\rangle - \langle M, \overline{A}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N, u\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}\rangle \,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (u, \overline{A}),$$

$$\overline{\varepsilon} = \nabla u - \overline{A}, \quad \overline{A}_{|_\Gamma} = \overline{A}_{\mathrm{d}} \in \mathfrak{so}(3,\mathbb{R}), \quad \varphi_{|_\Gamma} = g_{\mathrm{d}} \tag{6.14}$$

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\overline{\varepsilon}]^2$$

$$= \mu\,\|\mathrm{sym}\,\nabla u\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\nabla u]^2$$

$$W_{\mathrm{curv}}(\mathfrak{k}) = \mu\frac{L_c^2}{12}\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{k}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{k}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{k}]^2\right),$$

$$\mathfrak{k} = \mathrm{D_x}\overline{A} = \left(\nabla(\overline{A}.e_1), \nabla(\overline{A}.e_2), \nabla(\overline{A}.e_3)\right), \quad \text{third order, infinitesimal } \textbf{curvature tensor}\,.$$