Large Eddy Simulation. Existence of Stationary Solutions to a Dynamical Model

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Abstract

We consider the existence of stationary solutions to the Germano Model - equations describing turbulent flow of fluids. The model comes from Large Eddy Simulation techniques yielding modified Navier-Stokes Equations with an additional nonlocal term. On one hand this nonlocalness disturbs monotonicity, but on the other hand it is helpful for compactness arguments. Thus we combine the methods of monotone operators and smoothing properties of convolutions in passing to the limit in the approximate problem.

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1 Introduction

We are interested in existence of stationary weak solutions to the system describing turbulent flow in the three-dimensional torus $\mathbb{T}^3$

$$\begin{aligned} &v \cdot \nabla v - \operatorname{div}\, (\tilde{c}(v)|\nabla^s v|\nabla^s v) - \nu \Delta v + \nabla q = f, \\ &\operatorname{div} v = 0, \end{aligned} \tag{1}$$

where $v : \mathbb{T}^3 \longrightarrow \mathbb{R}^3$ is the velocity, $\nabla^s u = \frac{1}{2}(\nabla u + \nabla u^T)$ denotes the symmetric part of the gradient and $q : \mathbb{T}^3 \longrightarrow \mathbb{R}$ is the pressure. The operator $\tilde{c}$ is a nonlocal operator described in Section 2.3.

We briefly introduce the physical motivation for the above equations. The idea of Large Eddy Simulation (LES) has its origin in numerics. Typical for turbulent flows are very different scales, which lead to an increase of the number of numerical operations needed to compute the solution. The LES technique bases on choosing the scales for which the exact solution is computed directly (large scales, resolved) and the scales for which the solution is modelled (small scales, subgrid). Therefore the velocity u is decomposed into the mean part $\bar{u}$ and turbulent fluctuations u', i.e., $u = \bar{u} + u'$. The fluctuations are first smoothed out and then modelled. Selection of the scales is done by filtering, i.e. mollifying the velocity with some function (filter).

Filters

Different filters based on convolutions can be used. In general the convolution is done with respect to the space variable

$$\bar{u}(t,x) = u * \varphi_\delta(t,x) = \int\limits_{\mathbb{R}^3} u(t,y)\varphi_\delta(x-y)dy,$$

where the index δ denotes the filter width (so-called cut-off length). In general a filter is assumed to be a function of total mass one. In case of a bounded domain $\Omega \subset \mathbb{R}^3$ the filtered value $\bar{u}$ is defined by

$$\bar{u}(t,x) = \int\limits_{\Omega} u(t,y)\varphi_\delta(x-y)dy.$$

Then the problem of filtering near the boundary and of boundary values of $\bar{u}$ occurs. In this paper we concentrate on the problem with periodic boundary conditions. These nonphysical boundary conditions allow to ignore the problem of filtering near the boundary. Nevertheless, they are interesting from the mathematical point of view, because all other analytical difficulties remain unchanged.

Modelling

By convoluting the Navier Stokes equations with a filter one obtains

$$\begin{aligned}
&\bar{u}_t + \operatorname{div}(\overline{u \otimes u}) - \nu\Delta\bar{u} + \nabla\bar{p} = \bar{f},\\
&\operatorname{div}\bar{u} = 0.
\end{aligned}$$

Because of the nonlinearity in the equations the scales cannot be considered separately. Furthermore, looking for solutions representing the resolved scales, the interactions with the subgrid scales have to be taken into consideration. Therefore we express the convoluted convective term as a difference of the convective term in terms of $\bar{u}$ and of a so-called subgrid stress tensor $\tau = \bar{u} \otimes \bar{u} - \overline{u \otimes u}$ representing the contribution of small scales into the system. There has to be added some constitutive relation closing the system. In LES we find a wide range of closure models for the tensor τ. The most classical and still often used one is the Smagorinsky model where

$$\tau = (c\delta)^2|\nabla^s\bar{u}|\nabla^s\bar{u},$$

and $c > 0$ is constant. This leads to the following initial boundary value problem

$$\begin{aligned}
&\bar{u}_t + \operatorname{div}(\bar{u}\otimes\bar{u}) - \operatorname{div}(c\delta^2|\nabla^s\bar{u}|\nabla^s\bar{u}) - \nu\Delta\bar{u} + \nabla\bar{p} = \bar{f},\\
&\operatorname{div}\bar{u} = 0,\\
&\bar{u}(0,x) = \bar{u}_0(x),\\
&+ \text{ some boundary conditions.}
\end{aligned} \tag{2}$$

Existence and uniqueness to (2) have been shown with use of Galerkin approximation and monotone operator methods. For classical results in this field we refer to [Lio69, Lad70].

The Smagorinsky model has a lot of disadvantages. In order to adapt it better to local flow structures a dynamical procedure is applied - the Germano model. It bases on applying a second filter (test filter) to the Navier-Stokes equations. Denoting the width of the first filter (grid filter) as δ_1, the test filter φ_{δ_2} must have a different width δ_2, with $\delta_2 > \delta_1$ usually chosen $\delta_2 = 2\delta_1$. Applying this second filter extracts a test field from the resolved scales. The idea is the following: The smallest resolved scales are sampled to give information for modelling the subgrid scales (notation: $\tilde{u} = u * \varphi_{\delta_2}$). The next step is to use the so-called *Germano identity*, (which in fact is quite obvious), i.e.

$$L = T - \tilde{\tau}, \tag{3}$$

where τ and T are the subgrid tensors

$$\begin{aligned} \tau &= \bar{u} \otimes \bar{u} - \overline{u \otimes u}, \\ T &= \tilde{\bar{u}} \otimes \tilde{\bar{u}} - \widetilde{\overline{u \otimes u}} \end{aligned}$$

and

$$L = \tilde{\bar{u}} \otimes \tilde{\bar{u}} - \widetilde{\bar{u} \otimes \bar{u}}$$

is a *Leonard tensor*. The L tensor can be computed from the resolved field since it is associated with scales of motion between the grid and test scales. In the next step both subgrid tensors are modelled in a similar way as in Smagorinsky's model (the coefficient c is a square of the original quantity). The crucial simplification is that they can be modelled with the same $c = c(t, x)$, i.e.,

$$\begin{aligned} \tau &= 2c\delta_1^2 |\nabla^s \bar{u}| \nabla^s \bar{u}, \\ T &= 2c\delta_2^2 |\nabla^s \tilde{\bar{u}}| \nabla^s \tilde{\bar{u}}. \end{aligned}$$

Substituting it into (3)

$$L = 2c\delta_2^2 |\nabla^s \tilde{\bar{u}}| \nabla^s \tilde{\bar{u}} - \left(2c\delta_1^2 \widetilde{|\nabla^s \bar{u}| \nabla^s \bar{u}} \right)$$

(the tilde sign applies to the whole term in brackets) and assuming the additional simplification

$$(c\delta_1^2 \widetilde{|\nabla^s \bar{u}| \nabla^s \bar{u}}) = c \left(\delta_1^2 \widetilde{|\nabla^s \bar{u}| \nabla^s \bar{u}} \right)$$

(note: $c = c(t, x)$!) the following equation is obtained

$$L = 2cM \quad \text{with} \quad M = \delta_2^2 |\nabla^s \tilde{\bar{u}}| \nabla^s \tilde{\bar{u}} - \delta_1^2 \widetilde{|\nabla^s \bar{u}| \nabla^s \bar{u}}.$$

The above equation is in fact an overdetermined system of six equations for the coefficient c. Therefore the error $Q = (L - 2cM)^2$ is minimized by the least squares method, i.e., $\frac{\partial Q}{\partial c} = 0$, yielding

$$c = \frac{1}{2} \frac{L : M}{M : M}. \tag{4}$$

This c is substituted into the Smagorinsky system (2). Then $v = u$ and $q = p$ define a solution to the model equations

$$v_t + \operatorname{div}(v \otimes v) - \operatorname{div}(c|\nabla^s v|\nabla^s v) - \nu\Delta v + \nabla q = \bar{f},$$

$$\operatorname{div} v = 0.$$

For more details on modelling we refer to [GPMC91, Lil92, Jim95, Sag01].

Modifications

There can easily be found examples of initial data such that the matrix L cannot be estimated with help of the matrix M (see formula (4) for function c). Thus, if the denominator equals zero, there is no possibility to extend the operator c to a function defined for these values. This motivates some necessary modifications of this coefficient. We will not propose any new formula for c, only denote in general the mathematical assumptions we put. They are clearly assembled in Section 2.3.

2 Notation, Function Spaces

2.1 Basic Notation

In the following the subset of symmetric matrices in $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{S}^n$. Let $u, v \in \mathbb{R}^n$. We will use the following notation for scalar product of vectors, scalar product of matrices and tensor product, respectively

$$u \cdot v = \sum_{i=1}^{n} u_i v_i \qquad A \cdot B = \sum_{i,j=1}^{n} a_{ij} b_{ij} \qquad u \otimes u = (u_i u_j)_{i,j=1}^{n}$$

where $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$. For the simplicity of notation the product sign will often be omitted.
The set of smooth functions on the torus $\mathbb{T}^3$ can be identified with the set of periodic smooth functions with some period $L \in (0, \infty)$. Therefore in the whole paper $\Omega = (0, L)^3$ is a cube of period L in $\mathbb{R}^3$.
Before we give the definition of the weak solution let us introduce the spaces of divergence free periodic functions.
By $C^\infty_{\text{per}}(\mathbb{R}^3)$ we denote the set of functions from $C^\infty(\mathbb{R}^3)$, which are periodic in each ith direction with a period $L > 0$, i.e., $u(x + Le_i) = u(x)$, $i = 1, 2, 3$, for $u \in C^\infty_{\text{per}}(\mathbb{R}^3)$. Then let

$$\mathcal{V} \equiv \{u \,:\, u \in C^\infty_{\text{per}}(\mathbb{R}^3),\ \operatorname{div} u = 0, \int_\Omega u\, dx = 0\},$$

and let V be the closure of $\mathcal{V}$ with respect to norm $\|u\|_V = \left(\int_\Omega |\nabla u|^3 dx\right)^{\frac{1}{3}}$. Its dual space will be denoted by V'. We will use the notation $(\cdot,\cdot)$ for the scalar product in L^2 and $\langle\cdot,\cdot\rangle_{X,X'}$ for the dual pairing between the space X and its dual. In particular for the dual pairing between V and V' the notation $\langle\cdot,\cdot\rangle$ will be used. All $L^p, W^{1,p}-$ functions are meant to be periodic in each ith direction with period L and with vanishing mean on Ω. Note additionally that for divergence-free vector field $u : \Omega \longrightarrow \mathbb{R}^3$ it holds

$$\operatorname{div}(u \otimes u) = u \cdot \nabla u.$$

In the whole paper (except for Lemma 3.3) we will denote a subsequence (v^{n_k}) of the further subsequence of a sequence (v^n) also by (v^n).

2.2 Preliminary facts

It will be useful to introduce the trilinear form b and to establish its properties

$$b(u,v,w) := \int_\Omega (u \cdot \nabla v) \cdot w \, dx.$$

Lemma 2.1 *Let b be defined as above. Then*

(a) b is a well-defined continuous trilinear form on $V \times V \times V$ (in particular).

(b) $b(u,v,v) = 0$ for all $u,v,w \in V$.

(c) b is antisymmetric, i.e. $b(u,v,w) = -b(u,w,v)$ for all $u,v,w \in V$.

Showing these properties is analogous as for the nonlinearity in the Navier-Stokes equations (cf. [Tem77, MNRR96]). Therefore the lemma is left without proof.

As a filter we choose a $C^\infty_{\text{per}}(\mathbb{R}^3)$- function φ such that $\int_\Omega \varphi dx = 1$. Then for $v \in L^p$ the following properties of the filtering of v, i.e. of $\tilde{v}(x) = \int_\Omega v(y)\varphi(x-y)\, dy$, are meaningful:

(i) $\|\tilde{v}\|_{L^p} \le \|v\|_{L^p}\|\varphi\|_{L^1}$ and $\|\tilde{v}\|_{L^p} \le \|v\|_{L^1}\|\varphi\|_{L^p}$,

(ii) $D^\alpha \tilde{v}(x) = \int_\Omega D^\alpha \varphi(x-y) v(y)\, dy$, where $D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}\, \partial x_3^{\alpha_3}}$ with multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$,

(iii) $\tilde{v} \in C^\infty_{\text{per}}(\mathbb{R}^3)$.

The proof of the above properties for filtered values is analogous to the case of the classical convolution on the whole $\mathbb{R}^3$ (cf. [Bre99]).
In order to work on the symmetric parts of a gradients we will recall a necessary tool - the Korn's inequality (cf. [Fu94]).

Lemma 2.2 (Korn's inequality) *Let $1 < p < \infty$ and $v \in W^{1,p}(\Omega)$, where $\Omega = (0,L)^3$, $L > 0$. Then there exists a constant $k = k(p,\Omega)$ such that*

$$\|v\|_{W^{1,p}} \le k \|\nabla^s v\|_{L^p}.$$

2.3 Formulation of the Problem

We formulate exactly the problem and define the solutions. Let $\Omega \subset \mathbb{R}^3$ be denoted as above. We consider the initial-boundary value problem derived in the previous section with periodic boundary conditions. It will allow to omit the problem of filtering near the boundary, when the domain of the filter does not overlap with the set Ω. For a given external force f and initial value v_0 we are looking for a velocity $v : (0,T) \times \Omega \longrightarrow \mathbb{R}$ and a pressure $q : (0,T) \times \Omega \longrightarrow \mathbb{R}$ solving the system

$$
\begin{aligned}
& v_t + v \cdot \nabla v - \operatorname{div}\,(\tilde{c}(v)|\nabla^s v|\nabla^s v) - \nu \Delta v + \nabla q = f, \\
& \operatorname{div} v = 0 \qquad\qquad (5) \\
& v(0,x) = v_0(x)
\end{aligned}
$$

with periodic boundary conditions ($i = 1,2,3$)

$$
\begin{aligned}
v(t, x + Le_i) &= v(t,x), \\
q(t, x + Le_i) &= q(t,x).
\end{aligned} \qquad (6)
$$

where $\{e_i\}_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$, L is the period in all directions and ν is the constant positive viscosity. We denote for brevity

$$\tilde{c}(v) := c(\tilde{v}, \widetilde{vv}, \nabla^s \tilde{v}, \widetilde{|\nabla^s v|\nabla^s v}).$$

The properties of the operator c are the following

$(C1)$ c is a function of $\tilde{v}, \widetilde{vv}, \nabla^s \tilde{v}, \widetilde{|\nabla^s v|\nabla^s v}$, continuous with respect to all four variables.

$(C2)$ c satisfies the condition

$$0 < \alpha \leq \tilde{c}(v) \leq \beta < \infty. \qquad (7)$$

We will look for stationary solutions to this problem. Then the velocity v is independent of time, i.e.

$$v(t,x) = v(x) \quad \text{for all} \quad t \geq 0.$$

Also the external forces f must be independent of time. Then the system (5) reduces to

$$
\begin{aligned}
& v \cdot \nabla v - \operatorname{div}\,(\tilde{c}(v)|\nabla^s v|\nabla^s v) - \nu \Delta v + \nabla q = f, \\
& \operatorname{div} v = 0
\end{aligned} \qquad (8)
$$

with the boundary conditions ($i = 1,2,3$)

$$
\begin{aligned}
v(x + Le_i) &= v(x), \\
q(x + Le_i) &= q(x).
\end{aligned} \qquad (9)
$$

We define weak solutions and formulate the result on the existence of these solutions.

Definition 2.1 *The function $v \in V$ is a weak solution to problem (8), (9) if the equation*

$$\int_\Omega (v \nabla v \phi + \tilde{c}(v)|\nabla^s v|\nabla^s v \nabla^s \phi + \nabla v \nabla \phi - f\phi)\, dx = 0 \qquad (10)$$

is satisfied for all $\phi \in V$.

Theorem 2.3 (Existence) *Let $f \in V'$ and let c satisfy conditions (C1)-(C2). Then there exists a weak solution in the sense of Definition 2.1 to the stationary problem (8),(9).*

3 Proof of Theorem 2.3

Let $\{\omega_r\}$ be the set of eigenvectors of the Stokes operator in Ω, for the definition and properties of the Stokes operator in Ω see [MNRR96]. We define for each fixed $n \in \mathbb{N}$ an approximate solution to (10) by

$$v^n(x) = \sum_{r=1}^{n} \lambda_r^n \omega_r, \quad \lambda_r^n \in \mathbb{R}, \tag{11}$$

where $v^n \in V^n = \text{lin}\{\omega_1, \omega_2, ..., \omega_n\}$ solves the system of equations

$$b(v^n, v^n, \omega_r) + \int_\Omega \tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n \cdot \nabla^s \omega_r dx + \nu(\nabla v^n, \nabla \omega_r) = \langle f, \omega_r \rangle \tag{12}$$

for $r = 1, ..., n$. The existence of solutions to this approximate problem is a consequence of the Brouwer fixed point theorem, and in particular of the following lemma (cf. [Tem77, Eva98]), which follows from that theorem.

Lemma 3.1 *Let X be a finite dimensional space with scalar product $[\cdot,\cdot]$ and norm $|\cdot|$ and let $P : X \longrightarrow X$ be a continuous mapping such that*

$$[P(x), x] > 0 \quad for \quad |x| = K, \; K > 0.$$

Then there exists $x \in X$ with $|x| \le K$ such that $P(x) = 0$.

Now we will use this tool to prove the following theorem.

Theorem 3.2 *For given $f \in V'$ there exists a solution $\lambda_1^n, ..., \lambda_n^n$ (and therefore $v^n \in V^n$) to the approximate problem (11), (12).*

Proof
Note that V^n is a finite dimensional Hilbert space. Let P be defined as

$$[P(v), \phi] = \int_\Omega \tilde{c}(v)|\nabla^s v|\nabla^s v \nabla^s \phi dx + \nu(\nabla v, \nabla \phi) + b(v, v, \phi) - \langle f, \phi \rangle \quad \forall\, v, \phi \in V^n.$$

The above mapping is continuous w.r.t $v \in V^n$. Let us check the last assumption of Lemma 3.1. By Korn's inequality and assumption $(C2)$

$$\begin{aligned} [P(v), v] &= \nu\|\nabla v\|_{L^2}^2 + \int_\Omega c(\tilde{v})|\nabla^s v|^3\, dx - \langle f, v \rangle \overset{(C2)}{\ge} \nu\|\nabla v\|_{L^2}^2 \\ &+ \alpha \int_\Omega |\nabla^s v|^3 dx - \|f\|_{V'}\|v\|_V \overset{\text{Korn}}{\ge} \nu\|\nabla v\|_{L^2}^2 + k\|v\|_{W^{1,3}}^3 - \|f\|_{V'}\|v\|_V \\ &\ge \nu\|\nabla v\|_{L^2}^2 + \|v\|_V(k\|v\|_V^2 - \|f\|_{V'}). \end{aligned}$$

Thus for K big enough, i.e., $K > \left(\frac{\|f\|_{V'}}{k}\right)^{\frac{1}{2}}$, we get $[P(v), v] > 0$ for all $v \in V^n$, $\|v\| = K$. Therefore there exists a v^n such that the equation $P(v^n) = 0$ is satisfied.

∎

Energy estimate

We will show that the approximate solutions converge to a solution of the original problem. Multiplying the equations (12) by λ_r^n and summing over r one obtains by Lemma 2.1 (b)

$$\int_\Omega \tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n \nabla^s v^n \, dx + \nu \|\nabla v^n\|_{L^2}^2 = \langle f, v^n \rangle. \tag{13}$$

Estimates done for the turbulent term in a proof of Theorem 3.2 yield that

$$\int_\Omega \tilde{c}(v^n)|\nabla^s v^n|^3 \, dx \geq k\|v^n\|_V^3$$

Estimating the RHS with Young's inequality

$$\langle f, v^n \rangle \leq \|f\|_{V'}\|v^n\|_V \leq K\|f\|_{V'}^{\frac{3}{2}} + \frac{k}{2}\|v^n\|_V^3$$

with $K = K(k)$ we obtain

$$\frac{k}{2}\|v^n\|_V^3 + \nu\|\nabla v^n\|_{L^2}^2 \leq K\|f\|_{V'}^{\frac{3}{2}}. \tag{14}$$

Passing to the limit

A direct consequence of the boundedness of the sequence (v^n) in V is the existence of a subsequence, such that for $n \to \infty$

$$v^n \rightharpoonup v \text{ in } V. \tag{15}$$

Passing to the limit in the trilinear form b does not produce any difficulties. Notice that in the three-dimensional case $W^{1,3}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $1 \leq q < \infty$. Therefore

$$v^n \longrightarrow v \quad \text{in } L^q(\Omega) \quad \text{for } 1 \leq q < \infty. \tag{16}$$

Then the convergence

$$b(v^n, v^n, \phi) \longrightarrow b(v, v, \phi) \quad \text{for all } \phi \in V \tag{17}$$

follows from

$$\begin{aligned} &\int_\Omega (v^n \nabla v^n - v\nabla v)\, \phi dx \\ &= \int_\Omega (v^n - v)\nabla v^n \phi dx + \int_\Omega v(\nabla v^n - \nabla v)\phi dx, \end{aligned}$$

where both integrals in the sum converge to zero due to weak convergence (15) and strong convergence (16). The biggest problems appear in passing to the limit in the nonlinear turbulent term. From the equation we conclude that $\operatorname{div}\{\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n\}$ is bounded in V' and therefore there exists a (further) subsequence and a $\xi \in V'$ such that for $n \to \infty$

$$\operatorname{div}\{\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n\} \rightharpoonup \xi \quad \text{in } V'. \tag{18}$$

If instead of the turbulent term in this form we had the term div $(|\nabla^s v|\nabla^s v)$, which is monotone, then passing to the limit could easily be done. But in this case there appears a product of a monotone operator and of c, which is a nonlocal operator dependent on filtered values $\tilde{v}, \widetilde{vv}, \nabla^s\tilde{v}, \widetilde{|\nabla^s v|\nabla^s v}$. Therefore, there is no chance for the whole term to be monotone. However the properties of convolutions improve the convergence of c, which will allow - with help of the Minty-Browder trick (cf. [Eva90]) and strong convergence of gradients - to pass to the limit in the whole turbulent term. First we formulate a lemma concerning the convergence of c.

Lemma 3.3 *Let $c = c(\cdot,\cdot,\cdot,\cdot)$ be the function satisfying conditions (C1)-(C2). Then for each sequence $(v^n)_{n\in\mathbb{N}}$ such that $v^n \rightharpoonup v$ in V there exists $\chi \in L^{\frac{3}{2}}$ and it holds:*

(i) For $n \longrightarrow \infty$ the following sequences converge strongly in $L^\infty(\Omega)$

$$\begin{aligned} \widetilde{v^n} &\longrightarrow \tilde{v}, \\ \widetilde{v^n v^n} &\longrightarrow \widetilde{vv}, \\ \nabla^s \widetilde{v^n} &\longrightarrow \nabla^s \tilde{v}. \end{aligned}$$

We can extract a further subsequence (v^{n_k}) of (v^n) such that

$$\widetilde{|\nabla^s v^{n_k}|\nabla^s v^{n_k}} \longrightarrow \tilde{\chi}.$$

(ii) Moreover

$$c(\widetilde{v^{n_k}}, \widetilde{v^{n_k}v^{n_k}}, \nabla\widetilde{v^{n_k}}, \widetilde{|\nabla^s v^{n_k}|\nabla^s v^{n_k}}) \longrightarrow c(\tilde{v}, \widetilde{vv}, \nabla\tilde{v}, \tilde{\chi}) \quad \text{in } L^\infty(\Omega). \tag{19}$$

Proof
Let us start with showing the convergences from assertion (i). The filter φ is a smooth function. From linearity of the convolution one can conclude

$$\|\widetilde{v^n} - \tilde{v}\|_{W^{1,\infty}} \leq (\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty})\, \|v^n - v\|_{L^1}$$

and also

$$\|\widetilde{v^n v^n} - \widetilde{vv}\|_{L^\infty} \leq \|\varphi\|_{L^\infty} \|v^n v^n - vv\|_{L^1}.$$

Hence the strong convergence (16) implies the first three convergences in assertion (i). The last step deals with the convergence of $\widetilde{|\nabla^s v^n|\nabla^s v^n}$. Due to the a priori estimate the term $|\nabla v^n|\nabla v^n$ and also $|\nabla^s v^n|\nabla^s v^n$ are bounded in $L^{\frac{3}{2}}$. Therefore we can extract a subsequence such that

$$|\nabla^s v^{n_k}|\nabla^s v^{n_k} \rightharpoonup \chi \quad \text{in } L^{\frac{3}{2}}(\Omega). \tag{20}$$

However now it cannot be claimed yet that $\chi = |\nabla^s v|\nabla^s v$. This result will be obtained after finishing all the steps of the proof. Note that in particular $L^{\frac{3}{2}} \subset\subset W^{-1,\frac{3}{2}}$ so

$$|\nabla^s v^{n_k}|\nabla^s v^{n_k} \longrightarrow \chi \quad \text{in } W^{-1,\frac{3}{2}}(\Omega). \tag{21}$$

This yields

$$\| \, \widetilde{|\nabla^s v^{n_k}|\nabla^s v^{n_k}} - \tilde{\chi}\|_{L^\infty} \le \|\varphi\|_{W^{1,3}} \| \, |\nabla^s v^{n_k}|\nabla^s v^{n_k} - \chi\|_{W^{-1,\frac{3}{2}}},$$

and therefore

$$\widetilde{|\nabla^s v^{n_k}|\nabla^s v^{n_k}} \longrightarrow \widetilde{\chi} \quad \text{in } L^\infty(\Omega).$$

Because c is a continuous function of all its arguments, the assertion (ii) of the lemma holds.

■

Coming back to the convergence (18) one notices that Lemma 3.3 brings some more information on the limit ξ. Then for a (further) subsequence (19) and (20) imply that

$$\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n \rightharpoonup c(\tilde{v}, \widetilde{vv}, \nabla^s\tilde{v}, \tilde{\chi})\, \chi \quad \text{in } L^{\frac{3}{2}}. \tag{22}$$

For brevity we denote

$$\tilde{c}(v,\chi) := c(\tilde{v}, \widetilde{vv}, \nabla^s\tilde{v}, \tilde{\chi}).$$

Passing to the limit in (12) we obtain the limit identity

$$b(v,v,\phi) + \int_\Omega \tilde{c}(v,\chi)\, \chi \nabla^s\phi \, dx + \nu(\nabla v, \nabla\phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in V. \tag{23}$$

Again for the simplicity of notation we introduce $a(\xi) := |\xi|\,\xi$. Recall the following useful properties of a:

- a is strongly monotone, i.e. there exists a positive constant k such that

$$\langle a(\xi) - a(\eta), \xi - \eta\rangle_{L^{\frac{3}{2}},L^3} \ge k\|\xi - \eta\|^3_{L^3} \quad \text{for all} \quad \xi,\eta \in L^3.$$

- a is hemicontinuous, i.e., $\forall\, u,v,w \in V$ the function $t \mapsto (a(u+tv), w)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

Minty-Browder trick

Let us analyze for $z \in V$ the integral

$$\begin{aligned} I^n &= \langle \tilde{c}(v^n)\, a(\nabla^s v^n) - \tilde{c}(v,\chi)\, a(\nabla^s z), \nabla^s v^n - \nabla^s z\rangle_{L^{\frac{3}{2}},L^3} \\ &= \langle [\tilde{c}(v^n) - \tilde{c}(v,\chi)] a(\nabla^s v^n), \nabla^s v^n - \nabla^s z\rangle_{L^{\frac{3}{2}},L^3} \\ &+ \langle \tilde{c}(v,\chi)[a(\nabla^s v^n) - a(\nabla^s z)], \nabla^s v^n - \nabla^s z\rangle_{L^{\frac{3}{2}},L^3} = I_1^n + I_2^n. \end{aligned}$$

The first integral can be estimated by

$$|I_1^n| \le \|\tilde{c}(v^n) - \tilde{c}(v,\chi)\|_{L^\infty} \|\nabla v^n\|^2_{L^3} \|\nabla^s v^n - \nabla^s z\|_{L^3}.$$

Because of condition $(C2)$ and the strong monotonicity of a the second integral can be estimated by

$$I_2^n \ge \alpha\langle a(\nabla^s v^n) - a(\nabla^s z), \nabla^s v^n - \nabla^s z\rangle_{L^{\frac{3}{2}},L^3} \ge \alpha\|\nabla^s v^n - \nabla^s z\|^3_{L^3}.$$

This allows to conclude that

$$I^n \geq \gamma^n,$$

where

$$\gamma^n = \alpha \|\nabla^s v^n - \nabla^s z\|_{L^3}^3 - \|\tilde{c}(v^n) - \tilde{c}(v,\chi)\|_{L^\infty} \|\nabla v^n\|_{L^3}^2 \|\nabla^s v^n - \nabla^s z\|_{L^3}.$$

The function γ^n does not have to be positive for all n. But the property

$$\liminf_{n\to\infty} \gamma^n \geq 0 \quad \text{for all } z \in V,$$

which is a consequence of Lemma 3.3, will be useful in the following.
From equation (13) we know that

$$\langle \tilde{c}(v^n) a(\nabla^s v^n), \nabla^s v^n \rangle_{L^{\frac{3}{2}},L^3} = \langle f, v^n \rangle - \nu \|\nabla v^n\|_{L^2}^2.$$

Substituting it into I^n one obtains

$$\begin{aligned} I^n &= \langle f, v^n \rangle - \nu \|\nabla v^n\|_{L^2}^2 \\ &- \langle \tilde{c}(v^n) a(\nabla^s v^n), \nabla^s z \rangle_{L^{\frac{3}{2}},L^3} - \langle \tilde{c}(v,\chi) a(\nabla^s z), \nabla^s v^n - \nabla^s z \rangle_{L^{\frac{3}{2}},L^3}. \end{aligned}$$

With use of the lower semicontinuity of the norm we conclude from (15) and (22) that

$$\limsup I^n \leq \langle f, v \rangle - \nu \|\nabla v\|_{L^2}^2 - \langle \tilde{c}(v,\chi)\chi, \nabla^s z \rangle_{L^{\frac{3}{2}},L^3} - \langle \tilde{c}(v,\chi) a(\nabla^s v), \nabla^s v - \nabla^s z \rangle_{L^{\frac{3}{2}},L^3}.$$

Using the equation (23) and that $\limsup I^n \geq \liminf \gamma^n$ and $\liminf \gamma^n \geq 0$ it follows

$$\langle \tilde{c}(v,\chi)\chi - \tilde{c}(v,\chi) a(\nabla^s z), \nabla^s v - \nabla^s z \rangle_{L^{\frac{3}{2}},L^3} \geq 0 \quad \text{for all} \quad z \in V.$$

Taking $z = v - \lambda\phi$ with $\lambda > 0$ we obtain

$$\lambda \langle \tilde{c}(v,\chi)\chi - \tilde{c}(v,\chi) a(\nabla^s z), \nabla^s \phi \rangle_{L^{\frac{3}{2}},L^3} \geq 0.$$

Dividing by λ and then letting $\lambda \searrow 0$, the hemicontinuity of the operator a yields

$$\langle \tilde{c}(v,\chi)\chi - \tilde{c}(v,\chi) a(\nabla^s v), \nabla^s \phi \rangle_{L^{\frac{3}{2}},L^3} \geq 0$$

for all $\phi \in V$. Hence,

$$\langle \tilde{c}(v,\chi)\chi, \nabla^s \phi \rangle_{L^{\frac{3}{2}},L^3} = \langle \tilde{c}(v,\chi) a(\nabla^s v), \nabla^s \phi \rangle_{L^{\frac{3}{2}},L^3}$$

for all test functions $\phi \in V$. Since these test functions ϕ are divergence-free, the above result only yields

$$\operatorname{div} \tilde{c}(v,\chi)\chi = \operatorname{div} \tilde{c}(v,\chi) |\nabla^s v| \nabla^s v + \nabla q, \tag{24}$$

where the scalar function q will be interpreted as a pressure. This information already allows to improve the limit identity (23) to

$$b(v,v,\phi) + \int_\Omega \tilde{c}(v,\chi)\, |\nabla^s v| \nabla^s v \nabla^s \phi \, dx + \nu(\nabla v, \nabla \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in V. \tag{25}$$

Nevertheless, it still has to be proved that $\chi = |\nabla^s v|\nabla^s v$ in order to get (10).

Strong convergence of gradients

In the last step we will show that

$$\nabla^s v^n \longrightarrow \nabla^s v \quad \text{in } L^3(\Omega). \tag{26}$$

In (25) let us use as a test function $v^n \in V^n \subset V$ to obtain

$$\int_\Omega \tilde{c}(v,\chi)|\nabla^s v|\nabla^s v \nabla^s v^n \, dx + \nu(\nabla v, \nabla v^n) = \langle f, v^n\rangle - b(v,v,v^n). \tag{27}$$

Choosing the same test function in the approximate equation (12) we get

$$\int_\Omega \tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n \nabla^s v^n \, dx + \nu(\nabla v^n, \nabla v^n) = \langle f, v^n\rangle. \tag{28}$$

The above identities together with the fact that $\lim\limits_{n\to\infty} b(v,v,v^n) = 0$ allow to conclude that

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \left(\tilde{c}(v^n)|\nabla^s v^n|^3 + \nu|\nabla v^n|^2\right) dx &= \lim_{n\to\infty}\langle f, v^n\rangle \\ = \lim_{n\to\infty}\int_\Omega \left(\tilde{c}(v,\chi)|\nabla^s v|\nabla^s v \nabla^s v^n + \nu \nabla v \nabla v^n\right) dx &= \int_\Omega \left(\tilde{c}(v,\chi)|\nabla^s v|^3 + |\nabla v|^2\right) dx.\end{aligned}$$

Lemma 3.3 yields the estimate

$$\begin{aligned}\limsup_{n\to\infty} \left|\int_\Omega (\tilde{c}(v^n) - \tilde{c}(v,\chi))|\nabla^s v^n|^3 dx\right| &\le \limsup_{n\to\infty} \int_\Omega \left|(\tilde{c}(v^n) - \tilde{c}(v,\chi))|\nabla^s v^n|^3\right| dx \\ &\le \limsup_{n\to\infty} \|\tilde{c}(v^n) - \tilde{c}(v,\chi)\|_{L^\infty}\|\nabla^s v^n\|_{L^3} = 0.\end{aligned}$$

Therefore

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega \left(\tilde{c}(v^n)|\nabla^s v^n|^3 + |\nabla v^n|^2\right) dx \\ &= \lim_{n\to\infty}\left\{\int_\Omega (\tilde{c}(v^n) - \tilde{c}(v,\chi))|\nabla^s v^n|^3 dx + \int_\Omega \left(\tilde{c}(v,\chi)|\nabla^s v^n|^3 + |\nabla v^n|^2\right) dx\right\} \\ &= \lim_{n\to\infty}\int_\Omega \left(\tilde{c}(v,\chi)|\nabla^s v^n|^3 + |\nabla v|^2\right) dx.\end{aligned}$$

Finally

$$\lim_{n\to\infty}\int_\Omega \left(\tilde{c}(v,\chi)|\nabla^s v^n|^3 + \nu|\nabla v^n|^2\right) dx = \int_\Omega \left(\tilde{c}(v,\chi)|\nabla^s v|^3 + \nu|\nabla v|^2\right) dx.$$

Note that by (15)

$$\liminf_{n\to\infty} \int_\Omega |\nabla v^n|^2 dx \geq \int_\Omega |\nabla v|^2 dx$$

and

$$\nabla^s v^n \rightharpoonup \nabla^s v \quad \text{in } L^3_{\tilde{c}} \tag{29}$$

(because of equivalence of norms $\|\cdot\|_{L^3_{\tilde{c}}}$ and $\|\cdot\|_{L^3}$), which implies

$$\liminf_{n\to\infty} \int_\Omega \tilde{c}(v,\chi)|\nabla^s v^n|^3 dx \geq \int_\Omega \tilde{c}(v,\chi)|\nabla^s v|^3 dx.$$

then

$$\lim_{n\to\infty} \int_\Omega \tilde{c}(v,\chi)|\nabla^s v^n|^3 dx = \int_\Omega \tilde{c}(v,\chi)|\nabla^s v|^3 dx. \tag{30}$$

Let us now introduce a norm equivalent to the standard norm in L^3, namely

$$\|u\|_{L^3_{\tilde{c}}} = \left(\int_\Omega \tilde{c}(v,\chi)|u|^3 dx \right)^{\frac{1}{3}}.$$

The following fact will bring us to desired result.

Theorem 3.4 *Let a Banach space X with norm $\|\cdot\|$ be uniformly convex (i.e. for each ε with $0 < \varepsilon \leq 2$ there exists $\delta(\varepsilon)$ such that $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| \geq \varepsilon$ imply $\|u + v\| < 2(1 - \delta(\varepsilon))$. If $u^n \rightharpoonup u$ in X and $\|u^n\|_X \to \|u\|_X$, then $u^n \to u$ in X.*

For the proof see [Bar76]. The space L^3 with norm $\|\cdot\|_{L^3_{\tilde{c}}}$ is uniformly convex. We show this in a similar way as the uniform convexity of the spaces L^p for $1 < p < \infty$ with standard norm (cf. [Ada75, HS69]). We will consider the sequence of the symmetric parts of the gradients. From (29) and the convergence of norms $\|\nabla^s v^n\|_{L^3_{\tilde{c}}} \longrightarrow \|\nabla^s v\|_{L^3_{\tilde{c}}}$ shown in (30) due to Theorem 3.4 it holds

$$\|\nabla^s v^n - \nabla^s v\|_{L^3_{\tilde{c}}} \longrightarrow 0$$

Therefore also (for a further subsequence)

$$\nabla^s v^n \longrightarrow \nabla^s v \quad \text{strongly in } L^3 \text{ and a.e. in } \Omega.$$

Thus

$$|\nabla^s v^n|\nabla^s v^n \longrightarrow |\nabla^s v|\nabla^s v \quad \text{a.e in } \Omega,$$

and the information

$$\chi = |\nabla^s v|\nabla^s v \quad \text{a.e in } \Omega$$

allows to conclude that

$$\tilde{c}(v^n) \longrightarrow \tilde{c}(v) \quad \text{a.e in } \Omega.$$

Finally we are able to determine the limit of the turbulent term, namely

$$\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n \longrightarrow \tilde{c}(v)|\nabla^s v|\nabla^s v \quad \text{a.e in } \Omega.$$

This limit exists, it is finite a.e. in Ω and additionally the sequence $\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n$ is uniformly integrable, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_M \left|\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n\right| dx < \varepsilon \quad \forall M \subset \Omega,\ |M| < \delta,$$

because

$$\int_M \left|\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n\right| dx \le \beta \int_M |\nabla^s v^n|^2 dx \le \beta \left(\int_\Omega |\nabla^s v^n|^3 dx\right)^{\frac{2}{3}} \left(\int_M 1 dx\right)^{\frac{1}{3}}$$
$$\le \beta \|v^n\|_V^2 |M|^{\frac{1}{3}} \le k|M|^{\frac{1}{3}}.$$

Then Vitali's lemma (cf. [MNRR96]) yields

$$\int_\Omega \tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n \nabla^s \phi dx \longrightarrow \int_\Omega \tilde{c}(v)|\nabla^s v|\nabla^s v \nabla^s \phi dx \quad \forall \phi \in V.$$

This finishes the proof of the theorem. ■

Remark
Note that showing the strong convergence of gradients without the step using the Minty-Browder trick would not have been possible. The crucial fact to show convergence of norms (30) was the information that $\tilde{c}(v^n)|\nabla^s v^n|\nabla^s v^n$ converges weakly to $\tilde{c}(v,\chi)|\nabla^s v|\nabla^s v$, not only to $\tilde{c}(v,\chi)\chi$.

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