On computability on $\mathcal{I}$–minimal models.

V. PUZARENKO*

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Abstract

A description of the computable principles on $\mathcal{I}$–minimal admissible sets is given. It is shown that the reduction and total extension properties do not hold and the properties of separation and existence of a universal function are preserved from ideals.

1 Introduction

1.1 On computability and e–reducibility

The main results in the computability theory can be found e.g. in [1]. Here we give the notions which are applied in this paper.

An equality by definition is denoted by $\rightleftharpoons$. We write the set containing all natural numbers by ω.

As usual, *the join* $A \oplus B$ of subsets $A, B \subseteq \omega$ is defined by

$$\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}.$$

Let A_0, A_1, …, A_k, $k \geqslant 0$, be subsets of naturals. Then $\bigoplus_{i\leqslant k} A_i \rightleftharpoons A_0$ if $k=0$; and $\bigoplus_{i\leqslant k} A_i \rightleftharpoons (\bigoplus_{i\leqslant k-1} A_i) \oplus A_k$, otherwise.

Given a set X, we denote the power set of X by $\mathcal{P}(X)$.

For any n–ary predicate R, $\mathrm{Pr}_k(R)$, $k \leqslant n$, are the projections on the corresponding coordinates.

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Functions are often identified with their graphs. If φ is a partial function then we write its domain and range as $\delta\varphi$ and $\rho\varphi$ respectively. We will denote the graph of φ by Γ_φ.

By W_n we denote the n–th computably enumerable (c.e.) set in the Post numbering. Recall that this numbering is principal, that is, for any computable sequence $\{A_n\}_{n\in\omega}$ of c.e. sets there is a computable function such that $A_n = W_{f(n)}$ for every $n \in \omega$. Given $A \subseteq \omega$, by W_n^A we denote the n–th set which is computably enumerable with the oracle set A. The numbering $n \mapsto W_n^A$ is principal for the class of all the A–computable numberings. By D_n we denote the n–th finite set defined as follows: $D_n \risingdotseq \{a_1 < \ldots < a_k\}$ for $n = \sum_{i=1}^k 2^{a_i} > 0$; and $D_0 \risingdotseq \varnothing$. Notice that the relation $x \in D_m$ and the function $m \mapsto |D_m|$ are computable. A sequence $\{A_n\}_{n\in\omega}$ of finite sets is called *strongly computable* if the relation $x \in A_m$ and the function $m \mapsto |A_m|$ are computable.

Let A, B be sets of naturals. We say that A is *enumerably reducible* (in symbols, $A \leqslant_e B$) if

$$\exists n \forall t \, (t \in A \Leftrightarrow \exists m \, (\langle t, m\rangle \in W_n \ \& \ D_m \subseteq B)).$$

Define *enumeration operators* Φ_n, $n \in \omega$, as

$$\Phi_n(S) \quad = \quad \{x \mid \exists m \, (\langle x, m\rangle \in W_n \ \& \ D_m \subseteq S)\}.$$

Then we can give another definition of e–reducibility:

$$A \leqslant_e B \Leftrightarrow \exists n(\Phi_n(B) = A).$$

In this case, we say Φ_n is *given by* W_n. The following properties of enumeration operators play an important role:

monotonicity: $A \subseteq B \Rightarrow \Phi_n(A) \subseteq \Phi_n(B)$;

continuity: $x \in \Phi_n(A) \Rightarrow \exists X \subseteq A \, (\mathtt{card}\,(X) < \omega \ \& \ x \in \Phi_n(X))$.

We say that a collection Θ_n of enumeration operators *satisfies a property* P, if a collection A_n of c.e. sets giving Θ_n does. E.g., Θ_n is a computable sequence if A_n is.

One can also consider enumeration operators having several arguments. Likewise we define enumeration operators Φ_n, $n \in \omega$, with l set variables:

$$\Phi_n(S_0, S_1, \ldots, S_{l-1}) =$$
$$\{x \mid \exists m_0 \exists m_1 \ldots \exists m_{l-1} (\langle x, m_0, m_1, \ldots, m_{l-1}\rangle \in W_n \& \bigwedge_{i=0}^{l-1} D_{m_i} \subseteq S_i)\}.$$

This operator also satisfies the continuity and monotonicity properties for all the arguments. Furthermore, given a number n of such an operator it can be effectively found n' such that $\Phi_n(S_0, S_1, \dots, S_{l-1}) = \Phi_{n'}(S_0 \oplus S_1 \oplus \dots \oplus S_{l-1})$.

$\leqslant_e$ is a preorder on $\mathcal{P}(\omega)$ which induces an order on the set of e–degrees ${}^{\mathcal{P}(\omega)}/_{\equiv_e}$ where $A \equiv_e B \Leftrightarrow A \leqslant_e B \& B \leqslant_e A$. An associated order is denoted so as e–reducibility. Given $A \subseteq \omega$, we denote the e–degree containing A by $d_e(A)$. Notice that the set of all e–degrees with the order $\leqslant_e$ is an upper semilattice with a least element (we write it as L_e). Moreover, $d_e(A) \sqcup d_e(B) = d_e(A \oplus B)$ where $\mathbf{a} \sqcup \mathbf{b}$ is the sup of $\mathbf{a}$ and $\mathbf{b}$. $\mathbf{0}$ is the e–degree consisting of all c.e. sets.

A non–empty collection $\mathcal{I}$ of e–degrees is called an *e–ideal* (or, simply, an *ideal*) if the following conditions hold:

1. $\mathbf{a} \leqslant \mathbf{b} \,\&\, \mathbf{b} \in \mathcal{I} \Rightarrow \mathbf{a} \in \mathcal{I}$;

2. $\mathbf{a}, \mathbf{b} \in \mathcal{I} \Rightarrow \mathbf{a} \sqcup \mathbf{b} \in \mathcal{I}$.

The collection of all the ideals of L_e is denoted by $\mathcal{J}(L_e)$. Given an ideal $\mathcal{I}$, we let $\mathcal{I}^+ = \{S \subseteq \omega \mid S \neq \varnothing,\, d_e(S) \in \mathcal{I}\}$, $\mathcal{I}^* = \mathcal{I}^+ \cup \{\varnothing\}$.

1.2 On admissible sets theory

We use the theory developed in in [2]. Here we will give only definitions and propositions from [3].

A KPU–model $\mathbb{A}$ in a finite signature $\sigma \supseteq \{\mathrm{U}^1, \in^2, \varnothing\}$ is called an *admissible set* if it is well-ordered by $\in$. The relations U, $\in$ are interpreted as collection of all urelements, membership–relation respectively; and $\varnothing$ as the empty set. Admissible sets are denoted by $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$. If $\mathfrak{M}$ is an arbitrary model then its domain is denoted by $\mathrm{dom}(\mathfrak{M})$. We define computably enumerable (computable) sets on admissible sets as subsets being definable by formulas of a special kind, — Σ formulas (Σ and Π formulas simultaneously). Computably enumerable and computable subsets are called Σ and Δ subsets respectively. Collections of all Σ and Δ subsets on an admissible structure $\mathbb{A}$ are denoted by $\Sigma(\mathbb{A})$ and $\Delta(\mathbb{A})$ respectively.

Now we give two important reducibilities on admissible sets.

(Yu.L. Ershov) A model $\mathfrak{M}$ in some finite relation signature $\{P_1^{n_1}, \dots, P_k^{n_k}\}$ is said Σ*–definable* in an admissible set $\mathbb{A}$ (in symbols, $\mathfrak{M} \leqslant_\Sigma \mathbb{A}$) if there exists a map $\nu : \mathrm{dom}(\mathbb{A}) \overset{onto}{\to} \mathrm{dom}(\mathfrak{M})$ such that $\nu^{-1}(=)$, $\nu^{-1}(P_1^{\mathfrak{M}})$, …, $\nu^{-1}(P_k^{\mathfrak{M}})$ are Δ predicates on $\mathbb{A}$.

(A.S. Morozov) We say that an admissible set $\mathbb{B}$ is Σ*–reducible* in an admissible set $\mathbb{A}$ (in symbols, $\mathbb{B} \sqsubseteq_\Sigma \mathbb{A}$) if there exists a map $\nu : \mathrm{dom}(\mathbb{A}) \overset{onto}{\to} \mathrm{dom}(\mathfrak{M})$ such that $\nu^{-1}(\Sigma(\mathbb{B})) \subseteq \Sigma(\mathbb{A})$.

It follows immediately from definitions that $\mathbb{B} \sqsubseteq_\Sigma \mathbb{A}$ implies $\mathbb{B} \leqslant_\Sigma \mathbb{A}$. However, the converse proposition doesn't hold.

An important subclass of admissible sets is ones of hereditarily finite sets. A hereditarily finite set over M can be defined as follows: $HF_0(M) = M \cup \{\varnothing\}$; $HF_{n+1}(M) = HF_n(M) \cup \mathcal{P}_\omega(HF_n(M))$; $HF(M) = \bigcup\limits_{n<\omega} HF_n(M)$; where $\mathcal{P}_\omega(X)$ is collection of all the finite subsets of X. If $\mathfrak{M}$ is a model in some finite relation signature σ and $\sigma \cap \{\varnothing, \in^2, U^1\} = \varnothing$ then it can be defined a model $\mathbb{HF}(\mathfrak{M})$ in the signature $\sigma^* = \sigma \cup \{\varnothing, \in^2, U^1\}$ with the domain $HF(M)$ and $U^{\mathbb{HF}(\mathfrak{M})} = M$. The model is called the *hereditarily finite set over* $\mathfrak{M}$.

Notice that $\omega \subseteq \mathrm{Ord}\,\mathbb{A}$ and ω is a Δ subset of $\mathbb{A}$, for any admissible set $\mathbb{A}$. All the collections having form $\{B \subseteq \omega \mid B \in \Sigma(\mathbb{A})\}$ for some admissible set $\mathbb{A}$ were described in [3].

Theorem 1.1 *1. Given an arbitrary admissible set $\mathbb{A}$, collection of all Σ subsets of $\omega \subseteq A$ is represented as $\mathcal{I}^*$ for some e–ideal $\mathcal{I}$.*

2. For every e–ideal $\mathcal{I}$ there exists a model $\mathfrak{M}$ in some finite signature such that $\mathcal{I}^*$ coincides with collection of all Σ subsets of ω on $\mathbb{HF}(\mathfrak{M})$. Moreover, this model can be chosen so that $\mathtt{card}\,(\mathfrak{M}) = \mathtt{card}\,(\mathcal{I}^*)$.

Let $\mathbb{A}$ be an admissible set. By $\mathcal{I}_e(\mathbb{A})$ we denote $\{d_e(B) \mid B \subseteq \omega,\ B \in \Sigma(\mathbb{A})\}$.

A collection $S \subseteq \mathcal{P}(\mathrm{dom}(\mathbb{A}))$ is called *computable on* $\mathbb{A}$ if $S \cup \{\varnothing\} = \{\Phi^{\mathbb{A}}[a, x] \mid a \in A\}$ for some Σ formula $\Phi(x_0, x_1)$, possibly with parameters. We will consider computable families of subsets of naturals. By $S_\omega(\mathbb{A})$ we denote class of all the computable on $\mathbb{A}$ collections of subsets of ω.

These classes preserve under the reducibilities on admissible sets mentioned above.

Proposition 1.1 *1.* (A.S. Morozov) *If $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$ then $S_\omega(\mathbb{A}) \subseteq S_\omega(\mathbb{B})$. In particular, $\mathcal{I}_e(\mathbb{A}) \subseteq \mathcal{I}_e(\mathbb{B})$.*

2. (Yu.L. Ershov; [2]) $\mathfrak{M} \leqslant_\Sigma \mathbb{A} \Leftrightarrow \mathbb{HF}(\mathfrak{M}) \sqsubseteq_\Sigma \mathbb{A}$.

To prove the proposition 2 of theorem 1.1 several classes of models were constructed. We give only ones constructed by author.

Let $\langle U, \Lambda\rangle$ be a pair consisting of some non–empty collection U of non–empty subsets of ω and some sequence of non–zero cardinals $\Lambda = \langle \alpha_S \mid S \in U\rangle$. We define a model $\mathfrak{M}_{\langle U,\Lambda\rangle}$ in a signature $\{Q^3, s^2, 0\}$ as follows:

$\mathrm{dom}(\mathfrak{M}_{\langle U,\Lambda\rangle}) \leftrightharpoons \omega \cup \{\langle S,\gamma\rangle \mid \gamma < \alpha_S,\ S \in U\} \cup \{\langle S,\gamma,n\rangle \mid n \in S,\ \gamma < \alpha_S,\ S \in U\}$;

$0^{\mathfrak{M}_{\langle U,\Lambda\rangle}} \leftrightharpoons \varnothing \in \omega$; $s^{\mathfrak{M}_{\langle U,\Lambda\rangle}} \leftrightharpoons \{\langle n, n+1\rangle \mid n \in \omega\}$;

$Q^{\mathfrak{M}_{\langle U,\Lambda\rangle}} \leftrightharpoons \{\langle\langle S,\gamma,n\rangle, \langle S,\gamma\rangle, n\rangle \mid n \in S,\ \gamma < \alpha_S,\ S \in U\}$.

Notice that the natural correspondence δ between ω and $\mathrm{Ord}\,\mathbb{HF}(\mathfrak{M}_{\langle U,\Lambda\rangle})$ is Σ function. It allows to identify the corresponding elements of these sets.

Let $\mathrm{Code}(\mathfrak{M}_{\langle U,\Lambda\rangle}) \rightleftharpoons \{\langle S,\gamma\rangle \mid S \in U,\ \gamma < \alpha_S\}$ and $\lambda_{\mathfrak{M}_{\langle U,\Lambda\rangle}}$ a map from $\mathrm{Code}(\mathfrak{M}_{\langle U,\Lambda\rangle})$ to the power of ω defined by $\langle S,\gamma\rangle \mapsto S$. Then Σ function

$$\gamma_0(x) = \begin{cases} x, & \text{if } x \in \mathrm{Code}(\mathfrak{M}_{\langle U,\Lambda\rangle}); \\ \langle \delta(n), z\rangle, & \text{if } \langle x,z,n\rangle \in Q^{\mathfrak{M}_{\langle U,\Lambda\rangle}}; \\ \delta(x), & \text{if } x \in \omega; \\ \langle \varnothing, \{\gamma_0(z) \mid z \in x\}\rangle, & \text{otherwise}; \end{cases}$$

is embedding of $HF(\mathrm{dom}(\mathfrak{M}_{\langle U,\Lambda\rangle}))$ into $HF(\mathrm{Code}(\mathfrak{M}_{\langle U,\Lambda\rangle}))$. By γ_0^* we denote the inverse of γ_0.

A sequence $\Lambda = \langle \alpha_S^\Lambda \mid S \in U\rangle$ will be called *apposite* if $\alpha_S^\Lambda \geqslant \omega$ for any $S \in U$. It is showed in [?] that $\mathcal{I}_e(\mathbb{HF}(\mathfrak{M}_{\langle \mathcal{I}^+,\Lambda\rangle})) = \mathcal{I}$ for any ideal $\mathcal{I}$ and every apposite sequence Λ. Given any ideal $\mathcal{I}$, class of models $\{\mathfrak{M}_{\langle \mathcal{I}^+,\Lambda\rangle} \mid \Lambda \text{ is apposite}\}$ corresponding to the e–ideal $\mathcal{I}$ will be denoted by $\mathcal{K}_{\mathcal{I}}$.

Theorem 1.2 [**3**] *Given any admissible set* $\mathbb{A}$*, there exists a model* $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}_e(\mathbb{A})}$ *such that* $\mathfrak{M}_0 \leqslant_\Sigma \mathbb{A}$*.*

Theorem 1.3 [**3**] *Let* $\mathcal{I}$ *be an e–ideal and* $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$*. Then* $S \subseteq \mathcal{P}(\omega)$ *is computable on* $\mathbb{HF}(\mathfrak{M}_0)$ *iff* $S \cup \{\varnothing\} = \{\Theta_n(R, A) \mid n \in \omega,\ R \in \mathcal{I}^*\}$ *for some* $A \in \mathcal{I}^*$ *and a computable sequence* $\{\Theta_n\}_{n\in\omega}$ *of enumeration operators.*

A class $\mathcal{R}$ of admissible sets will be called $\mathcal{I}$*–minimal* if it satisfies the following conditions:

- for any admissible set $\mathbb{A}$ with $\mathcal{I}_e(\mathbb{A}) = \mathcal{I}$ there exists an admissible set $\mathbb{B} \in \mathcal{R}$ such that $\mathbb{B} \sqsubseteq_\Sigma \mathbb{A}$;
- $\mathbb{B}_0 \equiv \mathbb{A}_0$ for any admissible sets $\mathbb{B}_0, \mathbb{B}_1 \in \mathcal{R}$.

A sequence $\{\mathcal{R}_\mathcal{I}\}_{\mathcal{I}\in\mathcal{J}(L_e)}$ of $\mathcal{I}$–minimal classes will be called *uniform* if it satisfies the following conditions:

- all the classes of the sequence contain models in the same signature;
- if $\mathcal{I}_0,\, \mathcal{I}_1 \in \mathcal{J}(L_e)$ satisfy $\mathcal{I}_0 \leqslant \mathcal{I}_1$ then for every model $\mathbb{A}_0 \in \mathcal{R}_{\mathcal{I}_0}$ there exists $\mathbb{A}_1 \in \mathcal{R}_{\mathcal{I}_1}$ such that $\mathbb{A}_0 \leqslant \mathbb{A}_1$;
- if $\mathcal{I}_0,\, \mathcal{I}_1 \in \mathcal{J}(L_e)$ satisfy $\mathcal{I}_0 \leqslant \mathcal{I}_1$ then for every model $\mathbb{A}_1 \in \mathcal{R}_{\mathcal{I}_1}$ there exists $\mathbb{A}_0 \in \mathcal{R}_{\mathcal{I}_0}$ such that $\mathbb{A}_0 \leqslant \mathbb{A}_1$.

Notice that the sequence $\{\{\mathbb{HF}(\mathfrak{M}) \mid \mathfrak{M} \in \mathcal{K}_\mathcal{I}\}\}_{\mathcal{I}\in\mathcal{J}(L_e)}$ is uniform.

2 A description of Σ–subsets

First we give some description of Σ–subsets on hereditarily finite sets over models from $\mathcal{K}_\mathcal{I}$. Notice that it coincides with the Rice–Shapiro description of index sets.

Let X, Y be sets. We say that X *is approximately equal* to Y (in symbols, $X \approx Y$) if $(X \setminus Y) \cup (Y \setminus X)$ is finite. Recall that $\lambda_{\mathfrak{M}_0}$ is the natural map from $\mathrm{Code}(\mathfrak{M}_0)$ to the power of ω.

Proposition 2.1 *Let $\mathcal{I}$ be an e–ideal, $\mathfrak{M}_0 \in \mathcal{K}_\mathcal{I}$ and $k \geqslant 0$. Then the following conditions are equivalent:*

1. *$X \subseteq \mathrm{Code}(\mathfrak{M}_0)$ is definable in $\mathbb{HF}(\mathfrak{M}_0)$ by some Σ–formula $\Phi(x_0)$ with parameters $s_0, \ldots, s_{k-1} \in \mathrm{Code}(\mathfrak{M}_0)$;*
2. *$X \cup \{s_0, \ldots, s_{k-1}\} = \{x \in \mathrm{Code}(\mathfrak{M}_0) \mid \exists u \in \Theta(\lambda_{\mathfrak{M}_0}(s_0) \oplus \ldots \oplus \lambda_{\mathfrak{M}_0}(s_{k-1}))(D_u \subseteq \lambda_{\mathfrak{M}_0}(x))\} \cup \{s_0, \ldots, s_{k-1}\}$ for some enumeration operator Θ.*

Moreover, a number of some formula Φ (some enumeration operator Θ) can be effectively found by the number of enumeration operator Θ (formula Φ).

Proof. Let $\mathcal{I}$ and $\mathfrak{M}_0$ be objects from the proposition. For any $A \in \mathcal{I}^*$, we let $Y_A = \{x \in \mathrm{Code}(\mathfrak{M}_0) \mid \exists u \in A(D_u \subseteq \lambda_{\mathfrak{M}_0}(x))\}$.

$(2 \Rightarrow 1)$ Notice that if Y is Σ–subset on $\mathbb{HF}(\mathfrak{M}_0)$ and $X \approx Y$ then X is too. Let $A \in \mathcal{I}^*$ and $X \approx Y_A$. Then it follows from $\mathcal{I}_e(\mathbb{HF}(\mathfrak{M}_0)) = \mathcal{I}$ and

$$x \in Y_A \Leftrightarrow \exists u[u \in A \wedge \forall t \in D_u \exists z\, Q(z, x, t)]$$

that Y_A is Σ subset on $\mathbb{HF}(\mathfrak{M}_0)$ and hence X is Σ too.

($1 \Rightarrow 2$) Let X be definable by some Σ formula $\Phi(x_0, s_0, \ldots, s_{k-1})$ with parameters $s_0, \ldots, s_{k-1}$, $k \geqslant 0$, from $\mathrm{dom}(\mathfrak{M}_0)$ (we assume this list is empty for $k = 0$). We can suppose that all the parameters are from $\mathrm{Code}(\mathfrak{M}_0)$. First we prove two lemmas.

Lemma 2.1 *Let $x, y \in \mathrm{Code}(\mathfrak{M}_0)$ be such that $x \in X$, $\lambda_{\mathfrak{M}_0}(x) \subseteq \lambda_{\mathfrak{M}_0}(y)$ and $\{x, y\} \cap \{s_0, \ldots, s_{k-1}\} = \varnothing$. Then $y \in X$.*

To prove the lemma we construct an auxiliary model $\mathfrak{M}_1$ in the signature $\{0, s^2, Q^3\}$ as follows: given $n \in \lambda_{\mathfrak{M}_0}(y) \setminus \lambda_{\mathfrak{M}_0}(x)$ we take $z_n \notin \mathrm{dom}(\mathfrak{M}_0)$ so that $z_{n_1} \neq z_{n_2}$ for $n_1 \neq n_2$. Now we let $\mathrm{dom}(\mathfrak{M}_1) = \mathrm{dom}(\mathfrak{M}_0) \cup \{z_n \mid n \in \lambda_{\mathfrak{M}_0}(y) \setminus \lambda_{\mathfrak{M}_0}(x)\}$, $s^{\mathfrak{M}_1} = s^{\mathfrak{M}_0}$, $0^{\mathfrak{M}_1} = 0^{\mathfrak{M}_0}$, $Q^{\mathfrak{M}_1} = Q^{\mathfrak{M}_0} \cup \{\langle z_n, x, n\rangle \mid n \in \lambda_{\mathfrak{M}_0}(y) \setminus \lambda_{\mathfrak{M}_0}(x)\}$. It can be easily verified that $\mathbb{HF}(\mathfrak{M}_0) \leqslant_{\mathrm{end}} \mathbb{HF}(\mathfrak{M}_1)$ and hence $\mathbb{HF}(\mathfrak{M}_1) \models \Phi(x, s_0, \ldots, s_{k-1})$. Furthermore, there is an isomorphism $f : \mathbb{HF}(\mathfrak{M}_1) \to \mathbb{HF}(\mathfrak{M}_0)$ satisfying $f(x) = y$ and $f(s_i) = s_i$ for any $i < k$. Hence, $\mathbb{HF}(\mathfrak{M}_0) \models \Phi(y, s_0, \ldots, s_{k-1})$. Thus $y \in X$. $\Box$

Lemma 2.2 *Let $x \in X$ be such that $x \notin \{s_0, \ldots, s_{k-1}\}$. Then there is $y \in X \setminus \{s_0, \ldots, s_{k-1}\}$ such that $\lambda_{\mathfrak{M}_0}(y)$ is finite and $\lambda_{\mathfrak{M}_0}(y) \subseteq \lambda_{\mathfrak{M}_0}(x)$.*

Since Φ is Σ formula and $\mathbb{HF}(\mathfrak{M}_0) \models \Phi(x, s_0, \ldots, s_{k-1})$ there exists a finitely generated (in this case, a finite) model $\mathfrak{M}_0' \leqslant \mathfrak{M}_0$ such that $\mathbb{HF}(\mathfrak{M}_0') \models \Phi(x, s_0, \ldots, s_{k-1})$. We can suppose that $\omega \cap |\mathfrak{M}_0'|$ is an initial segment of ω. Let $y \in \mathrm{Code}(\mathfrak{M}_0) \setminus \mathrm{Code}(\mathfrak{M}_0')$ be such that $\lambda_{\mathfrak{M}_0}(y) = \lambda_{\mathfrak{M}_0'}(x)$. Then there exists an embedding $f' : \mathbb{HF}(\mathfrak{M}_0') \to \mathbb{HF}(\mathfrak{M}_0)$ satisfying $f'(x) = y$ and $f' \upharpoonright (|\mathfrak{M}_0'| \setminus \{x\}) = id_{(|\mathfrak{M}_0'| \setminus \{x\})}$. It is obvious that $f'(\mathbb{HF}(\mathfrak{M}_0')) \leqslant_{\mathrm{end}} \mathbb{HF}(\mathfrak{M}_0)$ and hence $\mathbb{HF}(\mathfrak{M}_0) \models \Phi(y, s_0, \ldots, s_{k-1})$. Thus $y \in X$. $\Box$

Return to proof of the proposition 2.1. Consider collection of all the finite models $\mathfrak{M}^0$ in the signature $\{Q^3, s^2, 0\}$ satisfying the following conditions:

- $\omega \cap |\mathfrak{M}^0|$ is a proper initial segment of ω;
- the symbols 0 and s are interpreted as above;
- $\forall x(x \in \omega \leftrightarrow \exists z \exists n Q(z, n, x))$;
- $\forall x(x \notin \omega \leftrightarrow \exists z \exists n \in \omega(Q(z, x, n) \vee Q(x, z, n)))$;
- $\forall x(\exists z \exists n Q(x, z, n) \to (\exists! z \exists! n Q(x, z, n) \wedge \forall z \forall n \neg Q(z, x, n)))$.

It is evident that the collection is strongly computable, that is, there is a strongly computable sequence of domains of models with uniformly computable signature relations.

By $\mathcal{S}$ we denote the collection of models with the effective structure given on them. Then a relation $\mathbb{HF}(\mathfrak{M}^0) \models \Phi(x, a_0, \ldots, a_{k-1})$ will be computably enumerable in respect to $\mathfrak{M}^0 \in \mathcal{S}$, Σ formula Φ and elements x, a_0, …, a_{k-1} of $\mathrm{dom}(\mathfrak{M}^0)$. Now we introduce one more auxiliary notion. Let $\mathfrak{M}$, $\mathfrak{M}'$ be models in a signature σ and $\sigma' \subseteq \sigma$. A homomorphism $\phi : \mathfrak{M} \to \mathfrak{M}'$ will be called σ'*–embedding* if $\phi : \mathfrak{M} \restriction \sigma' \to \mathfrak{M}' \restriction \sigma'$ is embedding. We say that $\mathfrak{M}$ is σ'*–embeddable* into $\mathfrak{M}'$ (in symbols, $\mathfrak{M} \hookrightarrow_{\sigma'} \mathfrak{M}'$) if there exists σ'–embedding $\phi : \mathfrak{M} \to \mathfrak{M}'$. Let A be

$$\begin{aligned}&\{u : \exists \mathfrak{M}^0 \in \mathcal{S} \exists x \in |\mathfrak{M}^0|[\mathbb{HF}(\mathfrak{M}^0) \models \Phi(x, u_0, \ldots, u_{k-1}) \wedge \\ \wedge &\bigwedge_{i=0}^{k-1} \neg(x = u_i) \wedge ((\mathfrak{M}^0, u_0, \ldots, u_{k-1}) \hookrightarrow_{\{s\}} (\mathfrak{M}_0, s_0, \ldots, s_{k-1})) \wedge \\ &\wedge(\forall t \in |\mathfrak{M}^0|((\exists z \in |\mathfrak{M}^0|\langle z, x, t\rangle \in Q^{\mathfrak{M}^0}) \leftrightarrow t \in D_u))]\}.\end{aligned} \tag{1}$$

It can be easily established that $(\mathfrak{M}^0, u_0, \ldots, u_{k-1}) \hookrightarrow_{\{s\}} (\mathfrak{M}_0, s_0, \ldots, s_{k-1})$ iff $\forall i < k[D_{u_i} \subseteq \lambda_{\mathfrak{M}_0}(s_i)]$, so, by (1), we obtain $A \leqslant_e \lambda_{\mathfrak{M}_0}(s_0) \oplus \ldots \oplus \lambda_{\mathfrak{M}_0}(s_{k-1})$. To prove $X \approx Y_A$ it suffices for every $\exists$–formula φ in the signature $\{Q, s, 0\}$ to find some $\exists$–formula ψ with positive occurence of Q which is equivalent to φ in respect of $\mathfrak{M}_0$ and models from $\mathcal{S}$. But it can be made, by induction on complexity of formulas, from the following relation:

$$\begin{aligned}\neg Q(x_0, x_1, x_2) \equiv \exists x_3 \exists x_4 (Q(x_3, x_0, x_4) \vee (Q(x_3, x_4, x_0) \vee (Q(x_1, x_3, x_4) \vee \\ (Q(x_3, x_4, x_1) \vee (Q(x_2, x_3, x_4) \vee (Q(x_3, x_2, x_4) \vee \\ (Q(x_0, x_3, x_4) \wedge (\neg(x_1 = x_3) \vee \neg(x_2 = x_4)))))))).\end{aligned}$$

□

Corollary 2.1 *Let X be definable in $\mathbb{HF}(\mathfrak{M}_0)$ by some Σ formula with parameters $s_0, \ldots, s_{k-1}$ from $\mathrm{Code}(\mathfrak{M}_0)$ $\langle a_0, a_1, \ldots, a_{l-1}\rangle \in X$, $l \geqslant 1$ where $a_i \in \mathrm{Code}(\mathfrak{M}_0)$, $0 \leqslant i < l$; $a_i \neq a_j$ for $0 \leqslant i < j < l$; $s_q \neq a_r$, $0 \leqslant q < k$, $0 \leqslant r < l$. Then there is a finite $F \subseteq \omega$ such that $\langle b_0, b_1, \ldots, b_{l-1}\rangle \in X$ for every $b_i \in \mathrm{Code}(\mathfrak{M}_0)$, $0 \leqslant i < l$, with the following properties: $F \subseteq \lambda_{\mathfrak{M}_0}(b_i)$, $0 \leqslant i < l$; $b_i \neq b_j$ for $0 \leqslant i < j < l$; $s_q \neq b_r$, $0 \leqslant q < k$, $0 \leqslant r < l$.*

3 On principles on $\mathcal{I}$–minimal admissible sets

We say that an admissible set $\mathbb{A}$ satisfies the *reduction property* if for any $B, C \in \Sigma(\mathbb{A})$ there are distinct $B_0, C_0 \in \Sigma(\mathbb{A})$ with $B_0 \subseteq B$, $C_0 \subseteq C$, $B_0 \cup C_0 = B \cup C$. The results concerning this property can be seen in [2, 4].

Proposition 3.1 *Let $\mathcal{I}$ be an e–ideal and $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$. Then the reduction principle doesn't hold on $\mathbb{HF}(\mathfrak{M}_0)$.*

Proof. Let $X_0 = \{x \in \text{Code}(\mathfrak{M}_0) \mid 0 \in \lambda_{\mathfrak{M}_0}(x)\}$ and $X_1 = \{x \in \text{Code}(\mathfrak{M}_0) \mid 1 \in \lambda_{\mathfrak{M}_0}(x)\}$. These sets are Σ definable. Let $Y_i \subseteq X_i$, $i = 0, 1$, be Σ subsets and satisfy $X_0 \cup X_1 = Y_0 \cup Y_1$. Then, by proposition 2.1, there is $a \in Y_0 \cap Y_1$ such that $\lambda_{\mathfrak{M}_0}(a) = \{0, 1\}$. $\square$

We say that an admissible set $\mathbb{A}$ satisfies the *total extension property* if for every partial Σ function φ there is a total Σ function f extending φ, that is, $\Gamma_\varphi \subseteq \Gamma_f$.

Proposition 3.2 *Let $\mathcal{I}$ be an e–ideal and $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$. Then $\mathbb{HF}(\mathfrak{M}_0)$ doesn't satisfy the total extension property.*

Proof. Suppose that this proposition doesn't hold for some e–ideal $\mathcal{I}$. Take $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$ for such an ideal $\mathcal{I}$; consider a partial Σ–function $\varphi(x, y)$ such that $\varphi(x, y) = z \Leftrightarrow Q(z, y, x)$. Let $f(x, y)$ be a total Σ–function extending φ. Notice that if $\lambda_{\mathfrak{M}_0}(a) = A$ then $n \notin A \Leftrightarrow \exists z[\neg Q(z, a, n) \wedge (f(n, a) = z)]$ so it is necessary that $\mathcal{I}$ would closed under the jump operation. Now let $s_0, s_1, \ldots, s_{k-1}$ be parameters from $\text{Code}(\mathfrak{M}_0)$ being used in a definition Γ_f. Then we take any $b \in \text{Code}(\mathfrak{M}_0)$ such that $\lambda_{\mathfrak{M}_0}(b) \neq \omega$ and b is not a parameter. By corollary 2.1, we have $n \notin \omega$ for $n \notin \lambda_{\mathfrak{M}_0}(b)$, contradictory. $\square$

To prove further propositions we introduce effective encoding of elements of a hereditarily finite set $\mathbb{HF}(\mathfrak{M})$ by ordered pairs of form $\langle n, g\rangle$ where n range natural ordinals and g is a finite injective function with $\delta g \in \omega$ (this value will be effectively defined by n; see below) and $\rho g \subseteq \text{dom}(\mathfrak{M})$. A number n is a code in some bijective computation representation of constructions of elements of hereditarily finite sets. By *construction* we mean an arbitrary term in a signature $\{\varnothing, \cup^2, \{\ \}^1\}$ without fictitious variables; its variables range $\text{dom}(\mathfrak{M})$ only. Notice that the construction is independent of a choise of a hereditarily finite set. We say that constructions $t_1(u_0, u_1, \ldots, u_{k-1})$, $k \geqslant 0$, and $t_2(v_0, v_1, \ldots, v_{l-1})$, $l \geqslant 0$, are *equivalent* if $k = l$ and

$$t_1^{\langle HF(\mathrm{dom}(\mathfrak{M})), \varnothing, \cup, \{\,\}\rangle}(a_0, \ldots, a_{k-1}) = t_2^{\langle HF(\mathrm{dom}(\mathfrak{M})), \varnothing, \cup, \{\,\}\rangle}(a_{\pi(0)}, \ldots, a_{\pi(k-1)})$$

for some permutation π of $\{0, 1, \ldots, k-1\}$ and a tuple $\langle a_0, a_1, \ldots, a_{k-1}\rangle$ of pairwise distinct elements being from $\mathrm{dom}(\mathfrak{M})$ (if $k = 0$ then assume that the tuple of elements and the permutation are empty). Notice that the equivalence of constructions doesn't imply the equality of them as terms. We define δg as a number of variables of the construction (obviously, it is equal to a cardinal number of some term value support). Now we correspond a pair $\langle n, g\rangle$ to x if $x = t_n^{\langle HF(\mathrm{dom}(\mathfrak{M})), \varnothing, \cup, \{\,\}\rangle}(g(0), g(1), \ldots, g(\delta g - 1))$. The above reasoning allow to conclude that the function $\mathrm{Term}(n, g) \leftrightharpoons t_n^{\langle HF(\mathrm{dom}(\mathfrak{M})), \varnothing, \cup, \{\,\}\rangle}(g(0), g(1), \ldots, g(\delta g - 1))$ is Σ definable. Furthermore, we can provide that for the code $\langle n, g\rangle$ of x it can be effectively found collection of all the pair coding elements of a "set" x. To make this, it suffices to consider a hereditarily finite set $\mathbb{HF}(\mathfrak{N})$ over $\mathfrak{N} = \langle \omega, 0, s\rangle$ where s is the successor function on naturals. The main sense of construction is that given a number it can be effectively found the structure of the corresponding elements. It is important that this encoding is independent of a choise of hereditarily finite set, that is, it can be defined by the same formula in all the hereditarily finite set. The coding is one–to–one under the first coordinate and, in general, isn't one–to–one under the second ones. E.g., $\{a_0, a_1\} = \{a_1, a_0\}$ has two codes. However, there is a bijective correspondence between collection of all permutations π such that $t_n^{\langle HF(\mathrm{dom}(\mathfrak{M})), \varnothing, \cup, \{\,\}\rangle}(a_0, a_1, \ldots, a_{k-1}) = t_n^{\langle HF(\mathrm{dom}(\mathfrak{M})), \varnothing, \cup, \{\,\}\rangle}(a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(k-1)})$ and collection of all codes for every construction having a number n. Collection of all the permutations corresponding to n (we will denote it by S_n) is a group. Obviously, the sequence $\{S_n\}_{n\in\omega}$ is strongly computable.

Let $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$ for some e–ideal $\mathcal{I}$ and γ_0, γ_0^* functions defined in section 1.2. Now we introduce encoding of triples $\langle n, g_0, g_1\rangle$ satisfying the following conditions:

- $g_0 \cup g_1$ is a finite permutation with $\delta(g_0 \cup g_1) \in \omega$ and $\delta g_0 \in \omega$;
- $g_0 \cap g_1 = \varnothing$;
- $\delta g_0 < \mathrm{card}_n$ where card_n is a cardinal number of the support of a "set" having a number of construction n.

Here range of g_1 is a set of parameters; given a triple $\langle n, g_0, g_1\rangle$ we will construct an element $\mathrm{Term}(n, g)$ for some g satisfying $g^{-1}(\{s_0, \dots, s_{k-1}\}) \subseteq \rho g_1$ where s_i, $i < k$, are parameters; $g^{-1}(\mathrm{sp}(g) \setminus \{s_0, s_1, \dots, s_{k-1}\}) = \rho g_0$. A code of $\langle n, g_0, g_1\rangle$ is denoted by $\boldsymbol{\langle} n, g_0, g_1 \boldsymbol{\rangle}$.

Let $\mathcal{I}$ be an e–ideal. We say that $\mathcal{I}$ has an *universal function* (and we write $\mathbf{Uf}(\mathcal{I})$) if there is a computable sequence $\{\Theta_n\}_{n\in\omega}$ and a set A with $d_e(A) \in \mathcal{I}$ such that $\{\Theta_n(R \oplus A) \mid n \in \omega,\ d_e(R) \in \mathcal{I}\}$ contains exactly all the graphs of partial functions f with $d_e(\Gamma_f) \in \mathcal{I}$.

Let $\mathbb{A}$ be an admissible set. A partial Σ function $f(x, y)$ is called *universal* if $\{f(a, y) \mid a \in \mathrm{dom}(\mathbb{A})\}$ contains exactly all the partial Σ functions.

Examples of admissible sets with universal Σ functions and without ones were constructed in [2, 5].

Proposition 3.3 *Let an e–ideal $\mathcal{I}$ is that there is an universal in $\mathcal{I}$ function. Then $\mathbb{HF}(\mathfrak{M}_0)$ has an universal Σ function for any $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$.*

Proof. Let $f(x)$ be a partial Σ function on $\mathbb{HF}(\mathfrak{M}_0)$; then $\gamma_0 \circ f \circ \gamma_0^*$ is a partial Σ function on $\mathbb{HF}(\mathfrak{M}_0)$ mapping from $HF(\mathrm{Code}(\mathfrak{M}_0))$ to $HF(\mathrm{Code}(\mathfrak{M}_0))$. Furthermore, we can assume that the last function is definable by some Σ formula with parameters being from $\mathrm{Code}(\mathfrak{M}_0)$ (via γ_0^*). First we construct a Σ function, which is universal for class of all the partial Σ functions f with $\mathrm{Field}(f) \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$.

$\langle 1\rangle$ *Let f be a partial function with* $\mathrm{Field}(f) \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$*, which is definable by some Σ formula with parameters $s_0, \dots, s_{k-1}$ being from* $\mathrm{Code}(\mathfrak{M}_0)$*, and* $f(\mathrm{Term}(n, g)) = \mathrm{Term}(m, g')$*. Then the following hold:*

1. $\rho g' \subseteq \rho g \cup \{s_0, \dots, s_{k-1}\}$;
2. $\mathrm{Term}(m, \pi \circ g') = \mathrm{Term}(m, g')$ *for any permutation π of the set $\rho g \cup \{s_0, \dots, s_{k-1}\}$ satisfying $\pi \restriction (\rho g \setminus (\rho g' \cup \{s_0, \dots, s_{k-1}\})) = id$ and* $\mathrm{Term}(n, \pi \circ g) = \mathrm{Term}(n, g)$.

This follows immediately from corollary 2.1.

Now we correspond to every partial Σ function f considered in $\langle 1\rangle$ some Σ predicate R_f as follows: for every $x \in \delta f$ take the least number $\boldsymbol{\langle} n, g_0, g_1 \boldsymbol{\rangle}$ such that $x = \mathrm{Term}(n, (\theta_0 \cup \theta_1) \circ (g_0 \cup g_1)^{-1} \restriction |\mathrm{sp}(x)|)$ for $\theta_1 : i + |\mathrm{sp}(x) \setminus \{s_0, \dots, s_{k-1}\}| \mapsto s_i$, $i < k$, and some injective function

$$\theta_0 : |\mathrm{sp}(x) \setminus \{s_0, \dots, s_{k-1}\}| \mapsto \mathrm{sp}(x) \setminus \{s_0, \dots, s_{k-1}\};$$

further, if $f(x)=y$ and x is given by $\langle\!\langle n, g_0, g_1\rangle\!\rangle$ then for y take the least number $\langle\!\langle m, \varnothing, g_1'\rangle\!\rangle$ such that $y = \mathrm{Term}(m, (\theta_0\cup\theta_1)\circ {g_1'}^{-1} \restriction |\mathrm{sp}(y)|)$, $\delta(g_0\cup g_1) = \delta g_1'$ for θ_0, θ_1 from definition of $\langle\!\langle n, g_0, g_1\rangle\!\rangle$. In this case, $\langle\langle\!\langle n, g_0, g_1\rangle\!\rangle, \langle\!\langle m, \varnothing, g_1'\rangle\!\rangle\rangle \in R_f$. Notice that $x, x' \in \delta f$ have the same code iff their constructions over the parameters $s_0, \ldots, s_{k-1}$ coinside. Furthermore, $\langle x, y\rangle, \langle x', y'\rangle \in \Gamma_f$ have the same pair of codes iff their constructions over the parameters $s_0, \ldots, s_{k-1}$ coinside. It follows from properties of the constructions that R_f is a Σ–predicate.

$\langle 2\rangle$ *R_f is graph of some number Σ–function.* Assume that it doesn't hold. Then it follows from the description that there are found $\langle x, y\rangle, \langle x', y'\rangle \in \Gamma_f$ such that x, x' have the same construction over parameters $s_0, \ldots, s_{k-1}$ but the constructions $\langle x, y\rangle, \langle x', y'\rangle$ over parameters $s_0, \ldots, s_{k-1}$ are different. Let a be that the constructions of a, x over parameters $s_0, \ldots, s_{k-1}$ coinside and $\lambda_{\mathfrak{M}_0}(u) = \omega$ for every $u \in \mathrm{sp}(a) \setminus \{s_0, \ldots, s_{k-1}\}$. Then it follows from 2.1 that there exist b, b' such that $\langle a, b\rangle, \langle a, b'\rangle \in \Gamma_f$ and the constructions of $\langle a, b\rangle$, $\langle x, y\rangle$ and $\langle a, b'\rangle$, $\langle x, y'\rangle$ over parameters $s_0, \ldots, s_{k-1}$ coinside respectively, but this contradicts to f being a function.

Further, let $R \subseteq \mathrm{Ord}\,\mathbb{HF}(\mathfrak{M}_0)$ be a Σ–subset on $\mathbb{HF}(\mathfrak{M}_0)$. By $G_R[s_0, s_1, \ldots, s_{k-1}]$ where $s_0, s_1, \ldots, s_{k-1}$ are from $\mathrm{Code}(\mathfrak{M}_0)$ (they will play a role of parameters) we denote a Σ–predicate defined as follows:

let $\langle\langle\!\langle n, g_0, g_1\rangle\!\rangle, \langle\!\langle m, \varnothing, g_1'\rangle\!\rangle\rangle \in R$, satisfy the following conditions:

1. $|\delta g_1| = k$, $\delta(g_0 \cup g_1) = \delta g_1'$;

2. $\mathrm{Term}(m, (g_1')^{-1} \restriction \mathrm{card}_m) = \mathrm{Term}(m, ((g_1')^{-1} \circ \pi) \restriction \mathrm{card}_m)$ for every permutation $\pi : \delta g_1' \to \delta g_1'$ satisfying the conditions $\pi \restriction ((\delta g_0 \setminus \{m \mid g_1'(m) < \mathrm{card}_m\}) \cup \delta g_1) = id$ (here card_m is the cardinal number of support of element with construction number m) and $\mathrm{Term}(n, (g_0 \cup g_1)^{-1} \restriction \mathrm{card}_n) = \mathrm{Term}(n, ((g_0 \cup g_1)^{-1} \circ \pi) \restriction \mathrm{card}_n)$;

3. $\langle\!\langle n, g_0, g_1\rangle\!\rangle$, $\langle\!\langle m, \varnothing, g_1'\rangle\!\rangle$ are the least numbers among giving $\mathrm{Term}(n, (g_0 \cup g_1)^{-1} \restriction \mathrm{card}_n)$ and $\mathrm{Term}(n, (g_1')^{-1} \restriction \mathrm{card}_m)$ respectively;

then we put to $G_R[s_0, s_1, \ldots, s_{k-1}]$ all pairs $\langle x, y\rangle$ such that $x = \mathrm{Term}(n, (\theta_0 \cup \theta_1) \circ (g_0 \cup g_1)^{-1})$, $y = \mathrm{Term}(m, (\theta_0 \cup \theta_1) \circ (g_1')^{-1})$ where $\theta_1 : \delta g_0 + i \mapsto s_i$, $i < k$; $\delta(\theta_0 \cup \theta_1) = \delta g_1'$ and $\theta_0 \cup \theta_1$ is injective.

$\langle 3\rangle$ *If a partial function $f : HF(\mathrm{Code}(\mathfrak{M}_0)) \to HF(\mathrm{Code}(\mathfrak{M}_0))$ is definable by some Σ formula with parameters $s_0, s_1, \ldots, s_{k-1}$ being from $\mathrm{Code}(\mathfrak{M}_0)$, then $\Gamma_f \subseteq G_{R_f}[s_0, s_1, \ldots, s_{k-1}]$.* It follows immediately from the descriptions.

$\langle 4\rangle$ *If* $R \subseteq \operatorname{Ord}\mathbb{HF}(\mathfrak{M}_0)$ *is graph of some* Σ *function, then so* $G_R[s_0, \ldots, s_{k-1}]$ *is for any* $s_0, \ldots, s_{k-1}$ *being from* $\mathrm{Code}(\mathfrak{M}_0)$. First notice that if R is graph of some Σ function, then the set of pairs of R satisfying 1 – 3, is too, so assume that all elements of R satisfy 1 – 3. However, these conditions provide the desired property.

Further, by $\mathbf{Uf}(\mathcal{I})$, there exists a Σ predicate Q, which is universal for class of graphs of all number Σ functions. Let Q_a be a set with a code a in Q. By S we denote a Σ predicate, which is universal for class of all Σ subsets on $\mathbb{HF}(\mathfrak{M}_0)$ and by S_b a set with a code b in S. Then a Σ predicate $T \rightleftharpoons \{\langle\langle a, b, s\rangle, c\rangle \mid s \in \mathrm{Code}(\mathfrak{M}_0)^{<\omega},\ a, b \in HF(M_0),\ c \in S_b \cap G_{Q_a}[s]\}$ is universal for class of graphs of all partial Σ functions f with $\mathrm{Field}(f) \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$.

Finally, if $g(x, y)$ is a Σ function, which is universal for class of all unary Σ functions f with $\mathrm{Field}(f) \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$, then $g^*(x, y) \rightleftharpoons \gamma_0^* g(x, \gamma_0(y))$ is an universal Σ function. $\square$

Let $\mathcal{I}$ be an e–ideal. We say that $\mathcal{I}$ satisfies the *separation* property (in symbols, $\mathbf{Separation}(\mathcal{I})$) if for all distinct sets B, C with $d_e(B) \in \mathcal{I}$, $d_e(C) \in \mathcal{I}$ there exists D with $d_e(D \oplus \overline{D}) \in \mathcal{I}$ such that $B \subseteq D \subseteq \omega \setminus C$.

Let $\mathbb{A}$ be an admissible set. We say that $\mathbb{A}$ satisfies the *separation* property if for all distinct sets B, $C \in \Sigma(\mathbb{A})$ there exists $D \in \Delta(\mathbb{A})$ such that $B \subseteq D \subseteq \mathrm{dom}(\mathbb{A}) \setminus C$.

First examples of admissible sets with the separation property were constructed by V.Rudnev [6].

Proposition 3.4 *Let* $\mathcal{I}$ *be an* e*–ideal satisfying the separation property. Then the separation principle holds in* $\mathbb{HF}(\mathfrak{M}_0)$ *for any* $\mathfrak{M}_0 \in \mathcal{K}_{\mathcal{I}}$.

Proof. Let $\mathcal{I}$ and $\mathbb{HF}(\mathfrak{M}_0)$ be from the proposition. Take distinct Σ subsets A and B and construct a Δ subset C such that $A \subseteq C \subseteq HF(M_0) \setminus B$.

We set $A_0 = \gamma_0(A)$, $B_0 = \gamma_0(B)$. Notice that $A_0 \cap B_0 = \varnothing$. First we construct a Δ subset $C_0 \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$ such that $A_0 \subseteq C_0 \subseteq HF(M_0) \setminus B_0$. To obtain it we introduce two set operators.

Given a subset $D \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$, which is definable by some Σ formula with parameters $s_0, s_1, \ldots, s_{k-1}$ being from $\mathrm{Code}(\mathfrak{M}_0)$, we find a Σ subset $S_D \subseteq \operatorname{Ord}\mathbb{HF}(\mathfrak{M}_0)$ by the following rules: for every $x \in D$ take the least number $\langle\!\langle n, g_0, g_1\rangle\!\rangle$ such that $x = \mathrm{Term}(n, (\theta_0 \cup \theta_1) \circ (g_0 \cup g_1)^{-1} \restriction |\mathrm{sp}(x)|)$ where $\theta_1 : i + |\mathrm{sp}(x) \setminus \{s_0, \ldots, s_{k-1}\}| \mapsto s_i$, $i < k$, and $\theta_0 : |\mathrm{sp}(x) \setminus \{s_0, \ldots, s_{k-1}\}| \mapsto \mathrm{sp}(x) \setminus \{s_0, \ldots, s_{k-1}\}$ is some injective function; then we put $\langle\!\langle n, g_0, g_1\rangle\!\rangle$ to S_D.

Further, given a Σ subset $H \subseteq \mathrm{Ord}\,\mathbb{HF}(\mathfrak{M}_0)$ we construct some Σ subset $V_H[s_0, \ldots, s_{k-1}] \subseteq HF(\mathrm{Code}(\mathfrak{M}_0))$ where $s_0, \ldots, s_{k-1}$ are from $\mathrm{Code}(\mathfrak{M}_0)$ and play a role of parameters: let $\langle n, g_0, g_1 \rangle \in H$ satisfy the following conditions:

1. $|\delta g_1| = k$;
2. $\langle n, g_0, g_1 \rangle$ is the least number giving $\mathrm{Term}(n, (g_0 \cup g_1)^{-1} \restriction \mathrm{card}_n)$;

then we put $\mathrm{Term}(n, (\theta_0 \cup \theta_1) \circ (g_0 \cup g_1)^{-1})$ to $V_H[s_0, \ldots, s_{k-1}]$ where $\theta_1 : \delta g_0 + i \mapsto s_i$, $i < k$, $\delta(\theta_0 \cup \theta_1) = \delta(g_0 \cup g_1)$ and $\theta_0 \cup \theta_1$ is injective.

Return to proof of the proposition. We can assume that A_0, B_0 are definable by some Σ formulas with the same parameters $s_0, s_1, \ldots, s_{k-1}$ being from $\mathrm{Code}(\mathfrak{M}_0)$. By corollary 2.1, $S_{A_0} \cap S_{B_0} = \varnothing$. Then there exists a Δ subset $R \subseteq \mathrm{Ord}\,\mathbb{HF}(\mathfrak{M}_0)$ such that $S_{A_0} \subseteq R \subseteq \omega \setminus S_{B_0}$, because of **Separation**$(\mathcal{I})$.

Therefore,

$$A_0 \subseteq V_{S_{A_0}}[s_0, \ldots, s_{k-1}] \subseteq V_R[s_0, \ldots, s_{k-1}]$$

and

$$B_0 \subseteq V_{S_{B_0}}[s_0, \ldots, s_{k-1}] \subseteq V_{\overline{R}}[s_0, \ldots, s_{k-1}].$$

Furthermore, it follows from the description that

$$V_R[s_0, \ldots, s_{k-1}] \cap V_{\overline{R}}[s_0, \ldots, s_{k-1}] = \varnothing,$$

$$V_R[s_0, \ldots, s_{k-1}] \cup V_{\overline{R}}[s_0, \ldots, s_{k-1}] = HF(\mathrm{Code}(\mathfrak{M}_0)) \supseteq \gamma_0(HF(M_0)).$$

To complete proof we set $C_0 = V_R[s_0, \ldots, s_{k-1}]$, $C = \gamma_0^*(C_0)$. $\square$

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