Simple connectedness of the geometry of nondegenerate subspaces of a symplectic space over arbitrary fields

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November 18, 2003

Abstract

K.M. Das [4], [5] has shown that the geometry of nondegenerate subspaces of a symplectic space over a finite field is simply connected. The purpose of the present article is to provide a short, direct and general proof of that result for arbitrary fields. Some standard amalgam-theoretic consequences for the groups $Sp_{2n}(\mathbb{F})$ are given as well.

1 Introduction

Let $\mathbb{F}$ be a field and let V be a $2n$-dimensional vector space over $\mathbb{F}$ endowed with a nondegenerate alternating bilinear form. Let $\Delta(n, \mathbb{F})$ be the geometry of rank $n-1$ whose elements are the nondegenerate proper subspaces of V with natural incidence. Then the following holds.

Theorem 1. *Let $n \geq 4$ and let $\mathbb{F}$ be a field. Then the geometry $\Delta(n, \mathbb{F})$ is 2-simply connected.*

The rank of $\Delta(3, \mathbb{F})$ is two, whence it cannot be simply connected (the incidence graph of $\Delta(3, \mathbb{F})$ is not a tree), so the above result is sharp.

The group $Sp_{2n}(\mathbb{F})$ acts flag-transitively on $\Delta(n, \mathbb{F})$. Choose a canonical basis $e_1, f_1, \ldots, e_n, f_n$ of V. Then $\langle e_1, f_1\rangle, \langle e_1, f_1, e_2, f_2\rangle, \ldots, \langle e_1, f_1, \ldots, e_{n-1}, f_{n-1}\rangle$ is a chamber (i.e., a maximal flag) of $\Delta(n, \mathbb{F})$. The maximal parabolics P_i, $1 \leq i \leq n-1$ with respect to this chamber are of the form $Sp_{2i}(\mathbb{F}) \times Sp_{2n-2i}(\mathbb{F})$. A consequence of Theorem 1 is the following.

Theorem 2. *Let $n \geq 4$ and let $\mathbb{F}$ be a field. Then $Sp_{2n}(\mathbb{F})$ is the universal completion of the amalgam of at least three maximal parabolics P_i.*

*The author would like to express his gratitude to the organizers for the funding of his trip to the USA and his stay at the conference.

Denote by $\mathcal{A}_2(n,\mathbb{F})$ the amalgam of 1- and 2-parabolics of $Sp_{2n}(\mathbb{F})$ with respect to the chamber $\langle e_1, f_1\rangle$, $\langle e_1, f_1, e_2, f_2\rangle$, $\ldots$, $\langle e_1, f_1, \ldots, e_{n-1}, f_{n-1}\rangle$. That is, $\mathcal{A}_2(n,\mathbb{F})$ is the union of all intersections $\cap_{i\in J} P_i$ where J is a subset of $\{1,\ldots,n-1\}$ of co-cardinality at most two. Then we have the following.

Theorem 3. *Let $n \geq 4$ and let $\mathbb{F}$ be a field. Then $Sp_{2n}(\mathbb{F})$ is the universal completion of the amalgam $\mathcal{A}_2(n,\mathbb{F})$.*

For finite fields Theorem 1 has been proved independently first by K.M. Das ([4] and [5], for all finite fields) and later by C. Hoffman, S. Shpectorov and the author ([7], for all finite fields of size at least four). While Das achieved this result as a consequence of his study of Quillen complexes of finite symplectic groups based on work of Aschbacher [1] and Quillen [14], the proof by Hoffman, Shpectorov and the author makes use of the simple connectedness of another finite geometry that is closely related to $\Delta(n,\mathbb{F})$. The purpose of this paper is to present a short, direct and general proof of the simple connectedness of $\Delta(n,\mathbb{F})$ for an arbitrary field $\mathbb{F}$. The investigation of geometries like $\Delta(n,\mathbb{F})$ is motivated by the desire for a systematic approach to Phan-type theorems similar to the results of [12] and [13]. For an outline of the idea, using chamber systems and buildings, as well as precise definitions of *flips* and *flipflop geometries* refer to [3]. (Note that those concepts are not relevant for the understanding of the present article.) Example 1b and Example 2b of [3] give a description of similar geometries. However one big difference remains: while the examples of [3] can be described using *flips* and *opposite chamber systems*, one cannot obtain the geometry $\Delta(n,\mathbb{F})$ in this way; for, a *flip* induced by a symplectic form does not admit a chamber that is mapped onto its opposite, so the resulting *flipflop geometry* is empty. The most remarkable property of $\Delta(n,\mathbb{F})$ is that the simple connectedness holds and can be shown completely independent of the field. Neither the geometry from Example 1b nor the geometry from Example 2b of [3] have this property; they are not simply connected for low rank over the field of two elements.

Theorem 2 and Theorem 3 dealing with the presentation of $Sp_{2n}(\mathbb{F})$ by certain amalgams nowadays are standard corollaries of Theorem 1. A classification of those amalgams is not included in the present article. For a closely related characterization of $Sp_{2n}(\mathbb{F})$ refer to Theorem 4.4.25 of [6] or to the results of [8].

2 Connectedness properties of Δ

Let $\mathbb{F}$ be a field and let V be a $2n$-dimensional vector space over $\mathbb{F}$ endowed with a nondegenerate alternating bilinear form. Perpendicularity with respect to that form is denoted by $\perp$. Let $\Delta = \Delta(n,\mathbb{F})$ be the pregeometry of rank $n-1$ whose elements are the nondegenerate proper subspaces of V with symmetrized containment as incidence. The nondegenerate subspaces of V of dimension two are called *points*, those of dimension four *lines* and so on. Evidently, Δ is a geometry (i.e., maximal flags are chambers, which means they have size $n-1$).

Moreover, by induction on n using Witt's theorem, the geometry Δ is flag-transitive with the group $Sp_{2n}(\mathbb{F})$ as a flag-transitive group of automorphisms of Δ; see Lemma 7.3 of [7] for a proof.

Lemma 2.1. *Let $n \geq 2$ and let l, m be nondegenerate two-dimensional subspaces of V with $l \cap m = \{0\}$ and $\dim(l^\perp \cap m) \geq 1$. Then $\langle l, m \rangle$ is nondegenerate.*

Proof. The statement is obvious, if $m \subseteq l^\perp$. If $m \not\subseteq l^\perp$, then $m \cap l^\perp$ is a one-dimensional subspace of V, say $\langle p \rangle$. Choose a canonical basis a, b of l and choose $q \in m$, such that p, q form a canonical basis of m. Then the Gram matrix of the symplectic form restricted to $\langle l, m \rangle$ with respect to the basis a, b, p, q has the shape

$$\begin{pmatrix} 0 & 1 & 0 & * \\ -1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 \\ * & * & -1 & 0 \end{pmatrix},$$

which evidently has full rank, whence $\langle l, m \rangle$ is nondegenerate. □

Proposition 2.2. *Let $n \geq 3$. Then the collinearity graph of Δ has diameter two. In particular, Δ is connected.*

Proof. Let l, m be nondegenerate two-dimensional subspaces of V. Let p be an arbitrary one-dimensional subspace of $\langle l, m \rangle^\perp$ and let q be an arbitrary one-dimensional subspace of V that intersects $p^\perp$ trivially. Then $\langle p, q \rangle$ is a point of Δ (as $q \not\subseteq p^\perp$) which is collinear to l and m (by Lemma 2.1). □

Corollary 2.3. *Let $n \geq 3$. Then the geometry Δ is residually connected.* □

Recall the definition of the fundamental group of a connected geometry Γ.[1] A path of length k in the geometry is a sequence of elements $x_0, \ldots, x_k$ of Γ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. We do not allow repetitions; hence $x_i \neq x_{i+1}$. A *cycle* based at an element x is a path in which $x_0 = x_k = x$. Two paths are *homotopically equivalent* if one can be obtained from the other via the following operations (called *elementary homotopies*): inserting or deleting a return (i.e., a cycle of length 2) or a triangle (i.e., a cycle of length 3). The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the *fundamental group* of Γ and denoted by $\pi_1(\Gamma, x)$. A geometry is called *simply connected* if its fundamental group is trivial.[2] Notice that in order to prove

[1] Note that there are two concurrent notions of a fundamental group. One notion of a fundamental group considers the (incidence graph of the) geometry Γ as a one-dimensional simplicial complex (triangles are not null-homotopic), the other notion considers the (incidence graph of the) geometry as a two-dimensional simplicial complex (triangles are null-homotopic). In the present paper we use the latter notion.

[2] Standard topology shows that a geometry is simply connected if and only if it does not admit any proper simplicial cover. For combinatorial proofs see [9] or [10]; alternatively, see [15].

that Γ is simply connected it is enough to prove that any cycle based at x is homotopically equivalent to the cycle of length 0. A cycle with this property is called *null-homotopic*, or *homotopically trivial*.

Let us go back to the geometry $\Delta(n, \mathbb{F})$. We pick the base element x to be a point of Δ.

Lemma 2.4. *Let $n \geq 4$. Every cycle of Δ based at x is homotopically equivalent to a cycle passing only through points and lines.*

Proof. This follows immediately from the residual connectedness of Δ. See Lemma 5.1 of [7] for a proof; alternatively, see [2]. □

We can therefore restrict our attention to the point-line incidence graph of Δ. However, as Δ is not a partially linear geometry (there exist nondegenerate two-dimensional subspaces of V that are contained in more than one nondegenerate four-dimensional subspace of V, so a pair of collinear points does not necessarily admit a unique joining line), it is not immediately clear that we can restrict ourselves to the collinearity graph of Δ. The following lemma takes care of that problem.

Lemma 2.5. *Let $n \geq 4$. Any digon p, l, q, m of Δ consisting of points $p \neq q$ and lines $l \neq m$ is homotopically trivial.*

Proof. The dimension of $\langle l, m\rangle$ equals five, because l and m intersect in a three-dimensional space. The space $\langle x\rangle = l^\perp \cap \langle l, m\rangle$ is the radical of $\langle l, m\rangle$. Choosing any vector $y \in l^\perp \backslash x^\perp$, we obtain a nondegenerate six-dimensional space $\langle l, m, y\rangle$ that contains the digon p, l, q, m. Therefore the digon is homotopically trivial. □

The above lemma allows us to restrict our attention to the collinearity graph of Δ, since the homotopy type of a path γ passing through points and lines is independent of the particular choice of the joining lines of the pairs of collinear points occuring in γ. In order to determine the fundamental group of Γ we have to distinguish between *good* and *bad* triangles, i.e., triangles a, b, c whose points span a nondegenerate subspace $\langle a, b, c\rangle$ of V and triangles a, b, c whose points do not span a nondegenerate subspace $\langle a, b, c\rangle$ of V. Obviously, a good triangle is null-homotopic as it spans a line or a plane of the geometry Δ. For bad triangles we have the following lemma.

Lemma 2.6. *Let $n \geq 4$. Any triangle in the collinearity graph of Δ can be decomposed into good triangles.*

Proof. Let a, b, c be the nondegenerate two-dimensional subspaces of V that constitute the three points of some triangle. Choose some line l of Δ containing a and b. The space $l^\perp$ is nondegenerate and has dimension at least four. Therefore, to any one-dimensional subspace $p \subseteq \langle l, c\rangle^\perp$, we can find a one-dimensional subspace q of $l^\perp$ such that $\langle p, q\rangle =: d \perp l \supseteq \langle a, b\rangle$ is nondegenerate and intersects c trivially. By Lemma 2.1, the space $\langle c, d\rangle$ is nondegenerate, and we have decomposed the triangle a, b, c into triangles which admit two points that are perpendicular.

So now assume we have a triangle a, b, c with $a \perp c$. If $a \cap b \neq \emptyset$ and $b \cap c \neq \emptyset$, then $b \subset \langle a, c \rangle$, and there is nothing to prove. Hence, by symmetry, we can assume that $b \cap c = \emptyset$. As above choose some line l of Δ containing a and b. The space $l^\perp$ is nondegenerate and has dimension at least four. To any one-dimensional subspace $p \subseteq \langle l, c \rangle^\perp$, we can find a one-dimensional subspace q of $l^\perp$ such that $\langle p, q \rangle =: d \perp l \supseteq \langle a, b \rangle$ is nondegenerate and intersects $\langle b, c \rangle$ trivially. This is possible as the dimension of $\langle b, c \rangle \cap l^\perp$ is at most two (since $\emptyset = b \cap b^\perp \supset b \cap l^\perp$). Notice that the triangles a, b, d and a, c, d are good (indeed, $d \perp l \supset a, b$ and $a \perp c, d$). Moreover, for hyperbolic bases x_b, y_b of b and x_c, y_c of c and p, y_d of d, we obtain the following Gram matrix for the alternating form on $\langle b, c, d \rangle$ with respect to x_b, y_b, x_c, y_c, p, y_d:

$$\begin{pmatrix} 0 & 1 & * & * & 0 & 0 \\ -1 & 0 & * & * & 0 & 0 \\ * & * & 0 & 1 & 0 & * \\ * & * & -1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & * & * & -1 & 0 \end{pmatrix}.$$

Evidently, this matrix has full rank if and only if its submatrix consisting of the first four rows and columns has full rank. But that submatrix is the Gram matrix of the form restricted to the nondegenerate space $\langle b, c \rangle$. Therefore we have decomposed the original triangle into good triangles. □

Lemma 2.7. *Let $n \geq 4$. Any quadrangle in the collinearity graph of Δ can be decomposed into triangles.*

Proof. Let a, b, c, d be the nondegenerate two-dimensional subspaces of V that constitute the four points of some quadrangle in the collinearity graph of Δ. Consider $W := \langle a, b \rangle^\perp$ and $U := \langle c, d \rangle^\perp$. If $W \perp U$, then $\langle c, d \rangle^\perp = U \subseteq W^\perp = \langle a, b \rangle$. Since the dimension of $\langle c, d \rangle^\perp$ is at least four and the dimension of $\langle a, b \rangle$ at most four, equality holds. But this implies that $a \perp c$ and $b \perp d$, yielding an immediate decomposition of the quadrangle into triangles. If $W \not\perp U$, we can find one-dimensional subspaces $p \subseteq W$, $q \subseteq U$ with $p \not\perp q$. The resulting nondegenerate two-dimensional space $e := \langle p, q \rangle$ is collinear to a, b, c, d by Lemma 2.1. (Notice that e cannot intersect a, b, c, or d nontrivially since $a, b \perp W \subseteq p$ and $c, d \perp U \subseteq q$, but $p \not\perp q$.) □

Lemma 2.8. *Let $n \geq 4$. Any pentagon in the collinearity graph of Δ can be decomposed into triangles and quadrangles.*

Proof. Let a, b, c, d, e be the nondegenerate two-dimensional subspaces of V that constitute the five points of some pentagon in the collinearity graph of Δ. Choose a line l of Δ passing through c and d. Then $l^\perp$ is nondegenerate and has dimension at least four. Let $p \subseteq \langle a, l \rangle^\perp$ be an arbitrary one-dimensional space. There exists a one-dimensional subspace of $l^\perp$ that together with p

spans a nondegenerate space f that intersects a trivially. We have $f \perp l \supseteq c, d$. Moreover, f also spans a nondegenerate space with a, by Lemma 2.1, decomposing the pentagon. □

Any cycle of greater length decomposes by the bound on the diameter of the collinearity graph of Δ from Lemma 2.2. Therefore Δ is simply connected. An induction argument as in [7] implies that Δ is 2-simply connected, and Theorem 1 follows. The proof of Theorem 2 is identical to the proof of Theorem 4 of [7]. Theorem 3 follows by Tits' lemma (Corollaire 1 of [16]) and a standard induction argument as in the proof of Theorem 1 of [7].

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