A geometrically exact derived Cosserat-plate including size effects, avoiding degeneracy in the thin plate limit. Modelling and mathematical analysis.

Patrizio Neff
Department of Mathematics
University of Technology
Darmstadt

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Abstract

This contribution is concerned with the consistent dimensional reduction of a previously introduced finite three-dimensional Cosserat micropolar elasticity model to the two-dimensional situation of thin plates and shells. The resulting membrane energy turns out to be a quadratic, elliptic, first order, non degenerate energy in contrast to classical approaches, the standard bending contribution is augmented with a term representing an additional stiffness of the Cosserat model and the corresponding system of balance equations remains of second order. The model includes size effects, transverse shear resistance, thickness stretch and drilling degrees of freedom. The thin shell limit is non-degenerate due to the additional Cosserat bending stiffness.

It is shown that the dimensionally reduced formulation is well-posed along the same line of argument which showed the well posedness of the three-dimensional model [Nef03a]. Decisive use is made of a dimensionally reduced version of an extended Korn's first inequality recently proved by the author [Nef02].

Key words: shells, plates, membranes, thin films, polar materials, non-simple materials, solid mechanics, elliptic systems, variational methods.
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Contents

1 Introduction

1.1 Generalities on shells

The dimensional reduction of a given model is already an old and mature subject and it has seen many 'solutions'. The different approaches toward elastic shell theory proposed in the literature and relevant references thereof are, therefore, too numerous to list here. In any case our own proposal falls within the so called **derived approach**, i.e., reducing a given three-dimensional model via (physically) reasonable constitutive assumptions on the kinematics to a two-dimensional model[1] as opposed to either the **intrinsic approach** which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **asymptotic methods** which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small parameter. The intrinsic approach is closely related to the **direct approach**[2] which takes the shell to be a two-dimensional directed medium in the sense of a **restricted Cosserat-surface** [CC09].[3] Two-dimensional equilibrium in appropriate new stress and strain variables is postulated ab-initio independent of three-dimensional considerations, cf. [Ant95, GNW65, ET58].

A detailed presentation of the classical shell theories can be found in [Nag72]. A thorough mathematical analysis of linear, infinitesimal shell theory, based on asymptotic methods is to be found in [Cia98a] and the extensive references therein, see also [Cia97, Cia99, Ant95, DS96, Dik82]. Excellent reviews and insightful discussions of the modelling and finite element implementation may be found in [SB92, San95, SB98, GSW89, GT92, BGS96, BR92] and in the series of papers [SF89, SFR89, SFR90, SRF90, SK92, SF92]. Recently, new C^1-conforming implementations for thin Kirchhoff-Love shells have been proposed in [COS00, CO01]. Properly invariant elastic plate theories are derived by formal asymptotic methods in [FRS93].

Let us sketch first the apparent areas of agreement in the development of the elastic case. The various shell models based on linearized three-dimensional elasticity proposed in the literature have been rigorously justified in those cases, where some normality assumption is introduced, either a priori or as a result of an asymptotic analysis, see notably the extensive work of Ciarlet and his co-workers [Cia97, Cia99]. Membrane and bending equations are identified as leading order terms of asymptotic expansions of the three-dimensional solution. Convergence of the computed solution (and error estimates) to the 'exact' solution of linearized three-dimensional elasticity is established in all relevant cases if various scaling assumptions on the data are made.

The situation is slightly less clear as far as infinitesimal restricted Cosserat models (Reissner-Mindlin plate, Timoshenko beam etc.) are considered. Here the convergence as the thickness tends to zero of some director to the (linearized) normal of the surface poses additional difficulties, but can be overcome, see e.g. [Ebe99] for the plate bending problem. It is known, that the solution of the infinitesimal Reissner-Mindlin model for various values of the shear correction factor κ converges to the solution of the infinitesimal Kirchhoff-Love model for vanishing thickness.

Already in the infinitesimal case it becomes apparent that **a model, involving membrane and bending simultaneously cannot be obtained by formal asymptotic methods** but is a result of careful modelling. One such successful model, the Koiter model [Koi70] is simply the sum of the correctly identified membrane and bending contribution, properly scaled with the thickness. The mathematical analysis establishing the well-posedness of all these infinitesimal models is fairly well established and will not be our principal concern. Though analytically understood, the numerical implementation of these infinitesimal, linear shell models is still an area of very active research, mostly because of intricacies related to the singular character of the considered systems as the thickness tends to zero. In the engineering community, the infinitesimal Reissner-Mindlin model is usually preferred numerically to which witnesses the uncountable proposals of new implementation variants,[4] since one only needs to solve a second order problem with standard C^0-finite elements for an augmented field instead of a fourth order problem with difficult to handle C^1-finite elements in the Kirchhoff-Love model. Moreover, the Reissner-Mindlin model allows for **transverse shear**, which may occur at free or loaded edges of the three-dimensional plate. However, the infinitesimal Reissner-Mindlin FEM-implementation is notoriously ill-conditioned without further provision while the underlying mathematical problem is well-posed. Membrane

[1] This line of thought is expressed by W.T. Koiter [Koi69, p.93]:"Any two-dimensional theory of thin shells is necessarily of an approximate character. An exact two-dimensional theory of shells cannot exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional. ... Since the theory we have to deal with is approximate in character, we feel that extreme rigour in its development is hardly desirable. ... Flexible bodies like thin shells require a flexible approach."

[2] The philosophy behind the direct approach is best framed by P.M. Naghdi[GN69, p.58]: "The theory of Cosserat is exact, but shell theory derived from the three-dimensional equations is approximate. It may be a matter of taste, but we prefer to regard an exact theory as more fundamental. The Cosserat theory of shells (Cosserat surface) is on a comparable footing with any exact three-dimensional continuum theory." This remark remains partly true today: while properly invariant derived shell models are now available, they do not necessarily guarantee invertibility.

[3] Restricted, since no material length scale usually enters the direct approach, only the relative thickness h appears in the model.

[4] There is a certain discrepancy between the effort put into the investigations of the **infinitesimal Reissner-Mindlin** model and its physical significance, given that the model is **not frame-indifferent** and for that matter, strictly speaking, irrelevant.

and shear **locking**, roughly meaning that the calculated solution on coarse meshes only poorly approximates the exact solution has motivated the search for locking free implementations and has stipulated to some extent the development of nonconforming elements and of discontinuous Galerkin methods (cf. references in [LNSO02]) which in principal should not suffer from locking. In this respect we mention also the **hierarchical plate models** [Sch96] which are a direct outcome of the finite element methods applied to thin structures. The idea there is to discretely minimize the three-dimensional energy functional over some thickness-restricted ansatz-space, preferably a polynomial approximation in thickness direction.

In the finite, elastic case, mostly based on the Saint Venant-Kirchhoff (SVK) free energy, the formal asymptotic methods are still successful in that they identify again leading membrane and bending terms. As far as the occurring membrane contribution is concerned, it is W_{mp} in (7.83) which is given in [GKM96, FRS93]. However, methods based on variational Γ-convergence [DR95a] suggest a fundamentally different membrane term which leads to a nonresistance of the membrane shell in compression. It should be noted, that the widely accepted membrane term of (7.83) shows the characteristic apparent change of the Lamé-moduli for the two-dimensional structure. As far as the bending term is concerned, some agreement has been obtained that the term consistent with the 3D-SVK energy is a quadratic expression in the second fundamental form of the surface. Nevertheless, the coefficients of this quadratic form give still room for some discussion: the Hamiltonian based derivation in [GKM96] differs from the results obtained by formal asymptotic analysis in [FRS93, Cia97] precisely in whether there is the same apparent change of the elastic moduli as occurs in the membrane case. This difference is immaterial as regards the mathematical analysis and can be explained by the use of a linear kinematical ansatz in thickness direction in [GKM96] whereas a quadratic ansatz in a Hamiltonian framework would yield the same result as in [FRS93, Cia97] and W_{bend} in (7.83).

It must be noted, that proceeding by asymptotic analysis is based itself on certain a priori assumptions, namely that all appearing quantities indeed admit an expansion in terms of a small parameter and satisfy certain scaling assumptions. No rigorous justification of the formal asymptotic approach has been given so far for finite elasticity, precisely because of the lack of some encompassing theory which guarantees the well-posedness of the three-dimensional problem. The application of formal asymptotic methods has never led to basically new plate or shell models, it seems to be restricted to an a posteriori justification of existing models. By contrast, the equations obtained by a variational approach i.e. energy projection and those for a Cosserat surface are independent of scaling assumptions.

We wish to remark that in the finite regime, no 'unique' elastic three-dimensional model exists: we have always to make constitutive choices for the bulk behaviour which has consequences for the reduced theory. In this case, making additional, physically sound, constitutive assumptions on the two-dimensional response itself, seems to be just another viable step in the modelling procedure. However, for infinitesimal strains we know the isotropic elastic bulk behaviour exactly[5] and subsequently it is reasonable to establish the convergence for vanishing relative thickness h to precisely one model without additional constitutive two-dimensional assumptions. This remark constitutes a strong justification for the asymptotic method in the infinitesimal case.

It has already been observed that the leading order term without additional provisions on the data is either a membrane or a bending term. But in applications, there are usually regions of a shell where membrane effects dominate while in others, bending is dominant. A fully three-dimensional resolution of a thin shell problem remains elusive notwithstanding the increased computer power. Hence, there is still a need to come up with a sound finite model, combining both effects in one system of equations, as does the Koiter model in the infinitesimal case.

Since we have in mind the future extension of the herein presented plate theories to multiplicative plasticity let us add that the picture is all the more complicated as far as elasto-plastic extensions are concerned, in part because of the (limited) state of the art of finite elasto-plasticity itself and in part because it is not straight forward to transform an existing 3D-model to its 2D-counterpart, see [BS99, BW91, SRF90] for representative examples. It is technically difficult to carry through the program of the formal asymptotic methods and in fact such a development seems not to have been undertaken in the finite case.

In order to get two-dimensional limit equations for plasticity despite these difficulties of some sort anyhow, additional mechanical assumptions on the stress distribution in the shell are usually introduced (e.g. **plane stress**, zero normal stress $S_1.e_3 = 0$ or less demanding: **zero normal tractions on the midsurface**, $S_{2,33} = 0$), moreover, the implementation of generally smooth, higher order shell elements is at variance with the lack of regularity either in (finite) plasticity or for very thin rigid shells. More problematic from a mathematical point of view, in many cases not an underlying self-consistent two-dimensional mathematical shell model is discretized

[5] If we assume that $\mu_c = 0$ in the finite three-dimensional Cosserat model, then the linearization coincides in fact with the classical infinitesimal model and the three-dimensional bulk has a unique infinitesimal response! The very possibility of $\mu_c = 0$ for a fully invariant three-dimensional finite Cosserat model has been considered impossible in the Cosserat community, since in effect, no infinitesimal, linear Cosserat bulk model would exist. While we keep completely track of $\mu_c > 0$ and $\mu_c = 0$ simultanuously, it is our belief that $\mu_c = 0$ is physically the correct choice.

in actual computations, but the shell like behaviour is enforced on the implementational element level only (this is the so called **degenerated solid approach**). There, evolution laws for plasticity are fully three-dimensionally integrated and elastic equilibrium is computed through numerical integration over the thickness. Only the two-dimensional kinematical constitutive ansatz for the total shell deformation reduces the problem. One has termed this method 2.5−dimensional. This applies to both elasticity and elasto-plasticity but, as already mentioned, the resulting problem is not consistent with any really two-dimensional definite model and a mathematical analysis for such a numerically motivated approach seems to be out of reach at present.

The Hamiltonian based, variational approach, which we will follow in disguise, has the distinctive advantage of being flexible enough to treat simultaneously finite elasticity, finite Cosserat models as well as finite elasto-plasticity in the framework of the multiplicative decomposition. This is to be contrasted with classical approaches for shells in curvilinear coordinates and indicial notation which must remain a mystery for all those not initiated.

The classical models proposed in the literature lead to effective numerical schemes only if the relative thickness h of the structure is still appreciable, i.e. classical bending terms are present and regularize the computation. However, there is an abundance of new applications where very thin structures are used, e.g. very thin metal layers on a substrate (in computer hardware, for the characteristic relative thickness $h \leq 5 \cdot 10^{-4}$). In these cases, classical bending energy, which comes with a factor of h^2 compared with the membrane energy contribution, cannot play a preponderant role for non-vanishing membrane energy. See also [BJ99] for an application to thin films. But the membrane terms e.g. in a finite, invariant Kirchhoff-Love plate or finite Reissner-Mindlin model are non-elliptic and the remaining (minimization) problem is not well-posed even if bending is included.

1.2 Outline and scope of this contribution

We therefore face several problems: first, there is no as yet generally accepted finite, properly invariant, elastic plate and shell model (and perhaps there cannot be); second, classical finite shell models are in general insufficient to account for very thin structures, the thin plate limit is degenerated; third, non-classical size effects, which cannot be neglected for very thin structures [CCC+03] are usually not accounted for; fourth, classical infinitesimal or finite shell models predict unrealistically high levels of smoothness, typically at least $C^{0,\alpha}(\omega)$ for the midsurface.

We propose therefore a new shell model for very thin almost rigid materials in addition to those already established which should remedy some of the aforementioned shortcomings with a view towards a subsequent stringent mathematical analysis and possible numerical implementation. We want to provide a model which is both theoretically and physically sound, such that the numerical implementation can concentrate on real convergence issues.

We view the obtained two-dimensional models as models in its own right: rather than trying to establish convergence results of the underlying three-dimensional model to its two-dimensional counterpart for vanishing thickness (which seems to be elusive given the appearing nonlinearities) we focus in a first attempt on the intrinsic mathematical problems inherent in the reduced models.

After introducing the underlying parent three-dimensional finite Cosserat model with **size effects** and **independent microrotations** and recalling the obtained existence results for this model, we proceed by considering a quadratic kinematical ansatz over the thickness where the director is automatically related to the rotations. Using generalized zero normal tractions on the transverse boundary, the two unknown leading coefficients in the quadratic ansatz can be determined in analytical form. The three-dimensional energy is then evaluated for the assumed form of plate deformation and analytically integrated over the thickness, this constitutes the energy projection. Boundary conditions are consistently reduced. The full minimization problem for the plate is gathered in section 4. The new model has six degrees of freedom (6 dof), including naturally one-drilling degree and allows for transverse shear. It is shown that the membrane part is uniformly Legendre-Hadamard elliptic at given rotations. This finishes the Cosserat modelling part.

Following, we derive a new Korn's first inequality for plates and elasto-plastic shells which is decisive for the mathematical treatment of the new models in a variational context. Depending on material constants and boundary conditions, different mathematical existence theorems are proposed. Generically, we obtain for the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$, i.e. the midsurface must not necessarily be continuous. It is shown, that the limit of vanishing relative thickness $h \to 0$ in the new model is non-degenerate. The limits $\mu_c \to \infty$ and the zero internal length limit $L_c \to 0$, as well as the pure membrane limit $h \to 0$, $L_c \to 0$ and the pure bending for vanishing internal length, are also described. We propose as well a modification of the new plate model which ensures local invertibility of the reconstructed deformation gradient and allows for large stretch. This modification takes place on the two-dimensional level only which implies that there need not exist any underlying three-dimensional model. Nevertheless, the modified two-dimensional model is shown to be physically more plausible than the preceeding model.

For comparison, we next present a derivation of a rather classical finite, invariant Reissner-Mindlin model with one independent director and of the finite, invariant Kirchhoff-Love plate model. It is shown that both finite models exhibit a certain unphysical response. A modification of the Kirchhoff-Love model in view of expected small strain behaviour allows to establish the existence of minimizers. However, the obtained regularity is unrealistically high and the implementational cost is known to be very large thus limiting in effect the usefulness of the Kirchhoff-Love model. The pure bending problem based on either Reissner-Mindlin or Kirchhoff-Love is shown to admit minimizers and to coincide with the pure bending problem obtained from the new Cosserat model.

In the appendix we introduce the relevant notation, detail the treatment of external loads and present the observed scaling relations. Generalized convexity conditions are recalled and macroscopic shear failure for plates is defined, including a Baker-Ericksen inequality for plates.

In order to relate the new finite Cosserat plate model to more traditional approaches, we show, that a linearization of the new model basically results in the classical infinitesimal Reissner-Mindlin model (without extra size effects) and **shear correction factor** $\kappa = 1$.

2 The underlying finite three-dimensional Cosserat model in variational form

In [Nef03a] a finite, fully frame-invariant Cosserat model is introduced. The problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) \in \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R}$ satisfying

$$
\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi\rangle - \langle M, \overline{R}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}\rangle \,\mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}),\\
&\qquad \overline{U} = \overline{R}^T F, \quad F = \nabla\varphi, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}\\
&\qquad \overline{R}_{|_\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}\,, & \text{rigid prescription}\\ \mathrm{polar}(\nabla\varphi)\,, & \text{consistent coupling} \Rightarrow S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U}) \in \mathrm{Sym} \text{ on } \Gamma \end{cases}\\
&W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12}\,(1 + \alpha_4\, L_c^q\, \|\mathfrak{K}\|^q)\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,,\\
&\qquad \mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} = \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right), \quad \text{third order } \textbf{curvature tensor}\,.
\end{aligned}
\tag{2.1}
$$

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ depends on the deformation gradient $F = \nabla\varphi$ and microrotations $\overline{R}$ together with their space derivatives. In general, the **micropolar stretch tensor** $\overline{U}$ is not symmetric. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. The external volume force is f and M takes on the role of external volume couples. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the **Cosserat couple modulus** and $L_c > 0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. **Consistent coupling** ensures that no non-classical effects are artificially introduced at the Dirichlet boundary.[6]

2.1 The different three-dimensional cases

We distinguish five completely different situations:

I: $\mu_c > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$, unconditional elastic macro-stability, local first order Cosserat micropolar, unqualified existence, microscopic specimens, non-zero Cosserat couple modulus. Fracture excluded.

II: $\mu_c = 0$, $\alpha_4 > 0$, $p \geq 1$, $q > 1$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded.

III: $\mu_c = \infty$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$, unconditional elastic macro-stability, the constrained gradient Cosserat micropolar problem (indeterminate couple stress model). Compatible Dirichlet boundary conditions: $\varphi_{|\Gamma} = g_{\mathrm{d}}$, $\operatorname{polar}(\nabla\phi)_{|\Gamma} = \operatorname{polar}(\nabla g_{\mathrm{d}})_{|\Gamma}$.

IV: $\mu_c = 0$, $\alpha_4 = 0$, $0 < p \leq 1$, $q = 0$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $\varphi \notin W^{1,1}(\Omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including **fracture** in multiaxial situations.

V: $\mu_c = 0, L_c = 0$, elastic pre-stability, finite elasticity with free rotations and microstructure. Weak solutions of a corresponding finite elasticity model are stationary points of this minimization problem. Allowing for **sharp interfaces**.

We refer to $0 < p < 1$, $q \geq 0$ as the **sub-critical case**, $p = 1$, $q \geq 0$ as the **critical case** and $p \geq 1$, $q > 1$ as the **super-critical case**. In [Nef03a] the first three cases are mathematically treated and case V is indeed shown to allow for sharp interfaces.

2.2 The coercive inequality in three-dimensions

The decisive analytical tool for the treatment of case II (super-critical) is the following non-trivial novel coercive inequality:

Theorem 2.1 (Extended 3D-Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$. Then

$$\exists\, c^+ > 0\ \forall\, \phi \in H_\circ^{1,2}(\Omega,\Gamma): \quad \|\nabla\phi F_p^{-1}(x) + F_p^{-T}(x)\nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Proof. The proof has been presented in [Nef02]. Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. However, in general, F_p is **incompatible** giving rise to a **non-riemannian manifold** structure. Compare to [CG01] for an interpretation and the physical relevance of the quantity $\operatorname{Curl} F_p$. ■

Motivated by the investigations in [Nef02], it has been shown recently by my colleague W. Pompe [Pom03] that the extended Korn's inequality can be viewed as a special case of a general class of coercive inequalities for quadratic forms. He was able to show that indeed $F_p \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for (2.1) to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ and $\operatorname{Curl} F_p \in L^{3+\delta}(\Omega)$ should suffice, whereas $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [Pom03].

In view of the important role of the extended Korn's first inequality let us agree in saying that a bulk-material is **elastically pre-stable**, whenever

$$\begin{aligned} &\exists\, H \in \mathbb{M}^{3\times3},\ H \neq 0: \quad D_F^2 W(x,F).(H,H) = 0 \\ &\exists\, c^+ > 0\ \exists\, G \in \mathrm{GL}^+(3,\mathbb{R})\ \forall\, H \in \mathbb{M}^{3\times3}: \ D_F^2 W(x,F).(H,H) \geq c^+\|G(x)^T H + H^T G(x)\|^2\,. \end{aligned} \tag{2.3}$$

[6] If, instead, we assume for the stretch energy

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U})\|^2 + \lambda\left(\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right) + \beta^+\|\operatorname{Cof}\overline{U} - 1\!\!1\|^2, \tag{2.2}$$

then $W_{\mathrm{mp}}(\overline{U})$ is polyconvex w.r.t. F and local invertibility of the deformation φ can be guaranteed. However, basing the dimensional reduction on this modification, would lead to excessive formulas.

In this terminology, infinitesimal classical elasticity is pre-stable with $G = 1\!\!1$ due to the classical Korn's first inequality and the extended Korn's first inequality links the smoothness of G to the positive definiteness of the elastic tangent stiffness tensor.

2.3 Mathematical results for the three-dimensional problem

Using the extended Korn's inequality, in [Nef03a] the following has been shown:

Theorem 2.2 (Existence for 3D-finite elastic Cosserat model: case I.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\rm d} \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times 3})$. Then (2.1) with material constants conforming to case I admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.

Theorem 2.3 (Existence for 3D-finite elastic Cosserat model: case II.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\rm d} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times 3})$. Then (2.1) with material constants conforming to case II admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.

Theorem 2.4 (Existence for 3D-finite elastic Cosserat model with consistent boundary coupling)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\mathrm{polar}(\nabla g_{\rm d}) \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times 3})$. Then (2.1) with material constants conforming to case I/II and the **consistent coupling condition**

$$\overline{R}_{|_\Gamma} = \mathrm{polar}(\nabla\varphi)_\Gamma \,, \tag{2.4}$$

admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.

3 Dimensional reduction of the Cosserat model

3.1 The three-dimensional problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably 'exact' 3D-theory to 2D. The problem (2.1) will now be adapted to a shell like theory. Let us assume that we are given a three-dimensional **absolutely thin domain**

$$\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2 \,, \tag{3.5}$$

with **transverse boundary** $\partial\Omega_h^{\rm trans} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **lateral boundary** $\partial\Omega_h^{\rm lat} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the thickness, and a deformation φ and **microrotation** $\overline{R}^{3d}$

$$\varphi : \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \,, \qquad \overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3,\mathbb{R}) \,, \tag{3.6}$$

solving the following minimization problem on Ω_h:

$$\begin{aligned}
&\int_{\Omega_h} W_{\rm mp}(\overline{U}) + W_{\rm curv}(\mathfrak{K}) - \langle f, \varphi\rangle \, \mathrm{dV} - \int_{\Omega_h^{\rm trans} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} \langle N, \varphi\rangle \, \mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}), \\
&\qquad \overline{U} = \overline{R}^T F, \quad \varphi_{|_{\Gamma_0^h}} = g_{\rm d}, \quad \Gamma_0^h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}], \quad \gamma_0 \subset \partial\omega, \ \gamma_s \cap \gamma_0 = \emptyset \\
&\quad \overline{R}_{|_{\Gamma_0^h}} = \mathrm{polar}(\nabla\varphi) \,, \quad \text{consistent coupling} \qquad\qquad (3.7)\\
&W_{\rm mp}(\overline{U}) = \mu \, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2} \, \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 \\
&W_{\rm curv}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12} \, (1 + \alpha_4 \, L_c^q \, \|\mathfrak{K}\|^q) \left(\alpha_5 \, \|\mathrm{sym}\, \mathfrak{K}\|^2 + \alpha_6 \, \|\mathrm{skew}\, \mathfrak{K}\|^2 + \alpha_7 \, \mathrm{tr}\, [\mathfrak{K}]^2\right)^{\frac{1+p}{2}} \,, \\
&\qquad \mathfrak{K} = \overline{R}^T \mathrm{D_x} \overline{R} = \left(\overline{R}^T \nabla(\overline{R}.e_1), \overline{R}^T \nabla(\overline{R}.e_2), \overline{R}^T \nabla(\overline{R}.e_3)\right), \quad \text{third order } \textbf{curvature tensor} \,.
\end{aligned}$$

We want to find a reasonable approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi, \overline{R}^{3d})$ involving only two-dimensional quantities. The reduction is based on assumed kinematics and energy projection.

3.2 Enriched quadratic Cosserat kinematics

In the engineering shell community it is well known [Che80, Sch85, Pie85] that the ansatz over the thickness should at least be quadratic[7] in order to avoid the so called **Poisson thickness locking**[8] and to fully capture the three-dimensional kinematics without artificial modification of the material laws[9], see the detailed discussion of this point in [BR00] and compare with [BR92, BBR94, RR96, BR97, SB98].

For a Cosserat theory for small elastic strains[10] we assume therefore the **quadratic ansatz** in the thickness direction for the (reconstructed) finite deformation $\varphi_s : \mathbb{R}^3 \mapsto \mathbb{R}^3$ of the shell like structure

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\varrho_b(x,y) \right) \cdot \overline{R}_{s,3}(x,y,0)\,, \tag{3.8}$$

where $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ takes on the role of the deformation of the midsurfaces of the shell viewed as a parametrized surface, the (reconstructed) rotation $\overline{R}_s : \Omega \mapsto \mathrm{SO}(3,\mathbb{R})$ and with yet indeterminate functions $\varrho_m,\, \varrho_b : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$ allowing for **thickness stretch** ($\varrho_m \neq 1$) and **transverse shear** ($\overline{R}_{s,3} \neq \vec{n}$).[11] The (reconstructed) rotations in the thin shell are assumed to be constant over the thickness

$$\overline{R}_s(x,y,z) = \overline{R}(x,y)\,. \tag{3.9}$$

This is then a kind of plate formulation since for the moment the unstressed reference configuration ω was assumed to lie in the plane. This implies for the (reconstructed) deformation gradient of the shell (plate)

$$F_s = \nabla\varphi_s(x,y,z) = (\nabla m|\,\varrho_m\,\overline{R}_3) + z\cdot(\nabla(\varrho_m\,\overline{R}_3)|\varrho_b\,\overline{R}_3) + \frac{z^2}{2}(\nabla(\varrho_b\,\overline{R}_3)|0) = \tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r\,. \tag{3.10}$$

It should be noted that the augmented ansatz changes already the term which is linear in the transverse direction.[12] Invertibility of the shell deformation (as a physical requirement) entails

$$\forall\, z \in [-h/2, h/2]:\ \det[\nabla\varphi_s(x,y,z)] > 0 \Rightarrow \varrho_m(x,y) > 0\,, \tag{3.11}$$

and we must guarantee that $\varrho_m : \omega \mapsto \mathbb{R}^+$. The three-dimensional local part of the elastic Cosserat energy in (2.1) has the form

$$W(F,\overline{R}) = \frac{\mu}{4}\,\|\overline{R}^T F + F^T\overline{R} - 2\mathbb{1}\|^2 + \frac{\mu_c}{4}\,\|\overline{R}^T F - F^T\overline{R}\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[\overline{R}^T F + F^T\overline{R} - 2\mathbb{1}\right]^2. \tag{3.12}$$

The equilibrium equations of the three-dimensional Cosserat problem given in [Nef03a] show that on the transverse boundary (exact)

$$\begin{aligned} S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= N^{\mathrm{trans}}(x,y,+h/2) \\ S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= N^{\mathrm{trans}}(x,y,-h/2)\,, \end{aligned} \tag{3.13}$$

where N^{trans} are the prescribed tractions N on the transverse boundary given globally in the basis (e_1,e_2,e_3). This implies (exact)

$$\begin{aligned} \overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= \overline{R}^{3d}(x,y,+h/2))^T N^{\mathrm{trans}}(x,y,+h/2) \quad (3.14)\\ \overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= \overline{R}^{3d}(x,y,-h/2))^T N^{\mathrm{trans}}(x,y,-h/2)\,. \end{aligned}$$

Abbreviate

$$N^+ := N^{\mathrm{trans}}(x,y,+h/2), \quad N^- := N^{\mathrm{trans}}(x,y,-h/2)\,, \tag{3.15}$$

[7] This mimics the (1|1|2)-hierarchic plate models: linear in-plane displacement and quadratic transverse displacement, instead of (1|1|0)-plate models with constant transverse displacement. The dimensional reduction is achieved by energy projection on the enriched ansatz space. In this sense, we propose a $(\infty|\infty|2)$-model.

[8] Meaning that the bending stiffness of the reduced theory would tend to ∞ as the Poisson-number $\nu \to \frac{1}{2}$.

[9] Let us quote from [Sch85]: "Due to bending this change of length is generally **asymmetric** about (the midsurface) and leads to a shift of the original midsurfaces.... This asymmetry requires at least a **quadratic** representation of the (deformation in thickness direction)."

[10] Which captures already shells with large in plane rigidity and high transverse flexibility.

[11] This leads at first glance to a 8 'dof' theory: 3 components of the membrane deformation, 3 degrees of freedom for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$, including naturally one **drilling degree** of freedom for in-plane rotations, 2 degrees of freedom over the thickness. However, the two thickness coefficients ϱ_m, ϱ_b will be eliminated, leaving us finally with a 6 'dof' model. Already in the classical elasticity context the beneficial influence of drill rotations for the numerical implementation has been investigated in the linear case in [HB89] and in the finite case in [SFH92].

[12] The corresponding stress field through the thickness $\overline{R}_s^T S_1(\nabla\varphi_s(x,y,z),\overline{R}_s).e_3$ is at least linear in the transverse variable z and not constant, as would be the case in a first order (linear) ansatz for the deformation.

and define

$$N_{\rm res} := N^{\rm trans}(x,y,+h/2) + N^{\rm trans}(x,y,-h/2)\,,\; N_{\rm diff} := \frac{1}{2}\left[N^{\rm trans}(x,y,+h/2) - N^{\rm trans}(x,y,-h/2)\right]\,. \quad (3.16)$$

Then also (exact)

$$\begin{aligned}
&\langle \overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3,e_3\rangle = \langle N^+,\overline{R}^{3d}(x,y,+h/2)).e_3\rangle \quad (3.17)\\
&\langle \overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).e_3,e_3\rangle = -\langle N^-,\overline{R}^{3d}(x,y,-h/2)).e_3\rangle\,.
\end{aligned}$$

We determine ϱ_m, ϱ_b from the corresponding requirement in terms of the assumed kinematics $(\varphi_s,\overline{R}_s)$, yielding

$$\begin{aligned}
&\langle \overline{R}_s^T(x,y,\pm h/2) S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}_s).e_3,e_3\rangle = \pm\langle N^{\rm trans}(x,y,\pm h/2),\overline{R}_s(x,y,\pm h/2).e_3\rangle \quad \Rightarrow\\
&\qquad \langle \overline{R}^T S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}).e_3,e_3\rangle = \pm\langle N^{\rm trans}(x,y,\pm h/2),\overline{R}.e_3\rangle\,, \quad (3.18)
\end{aligned}$$

which condition reduces to **zero normal tractions on the transverse free boundary** (in the absence of tractions $N^{\rm trans}$) in the classical continuum limit of $\overline{R} = \mathrm{polar}(\nabla\varphi)$. Since

$$S_1(F,\overline{R}) = \overline{R}\left[\mu\left(F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right) + 2\mu_c\,\mathrm{skew}(\overline{R}^T F) + \frac{\lambda}{2}\,\mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right]\mathbb{1}\right]\,, \quad (3.19)$$

the requirement $\langle \overline{R}^T S_1(\nabla\varphi_s(x,y,z),\overline{R}).e_3,e_3\rangle = \pm\langle N^{\rm trans}(x,y,\pm h/2),\overline{R}.e_3\rangle$ turns into

$$\begin{aligned}
&\pm\langle N^{\rm trans}(x,y,\pm h/2),\overline{R}.e_3\rangle = \mu\,(2(\varrho_m-1) + 2z\,\varrho_b)\\
&\quad + \lambda\left(\langle \overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m + z\,\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle + z\,\varrho_b - 3 + \frac{z^2}{2}\,\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\right)\,, \quad (3.20)
\end{aligned}$$

independent of the Cosserat couple modulus μ_c. Let us evaluate the last equation for $z = \pm h/2$. This yields two **linear** equations in ϱ_m,ϱ_b

$$\begin{aligned}
\langle N^+,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m-1) + h\,\varrho_b) \quad (3.21)\\
&\quad + \lambda\left(\langle \overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m + h/2\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle + h/2\,\varrho_b - 3 + \frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\right)\\
-\langle N^-,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m-1) - h\,\varrho_b)\\
&\quad + \lambda\left(\langle \overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m - h/2\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle - h/2\,\varrho_b - 3 + \frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\right)\,.
\end{aligned}$$

The exact solution is given by

$$\begin{aligned}
\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix} &= \frac{1}{(2\mu+\lambda)^2\,h - \frac{\lambda^2 h^3}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle^2}\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda h^2}{8}\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle & (2\mu+\lambda)\end{pmatrix}\\
&\qquad \begin{pmatrix}\langle N_{\rm diff},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\rm res},\overline{R}_3\rangle\end{pmatrix}\,, \quad (3.22)
\end{aligned}$$

which will be approximated through

$$\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix} \approx \frac{1}{(2\mu+\lambda)^2\,h}\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda h^2}{8}\langle\nabla\overline{R}_3|0),\overline{R}\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0),\overline{R}\rangle & (2\mu+\lambda)\end{pmatrix}\begin{pmatrix}\langle N_{\rm diff},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\rm res},\overline{R}_3\rangle\end{pmatrix}\,. \quad (3.23)$$

Hence the leading terms[13] are:

$$\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\rm diff},\overline{R}_3\rangle}{(2\mu+\lambda)} - \frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\,\langle N_{\rm res},\overline{R}_3\rangle\\
\varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle + \frac{\langle N_{\rm res},\overline{R}_3\rangle}{(2\mu+\lambda)\,h} - \frac{\lambda}{2(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\langle N_{\rm diff},\overline{R}_3\rangle \quad (3.24)\\
&\quad \frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\,.
\end{aligned}$$

[13]Note that ϱ_m,ϱ_b have different units. ϱ_m is dimensionless, whereas $[\varrho_b] = \mathrm{m}^{-1}$.

The term $\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]$ represents a **nonlinear coupling** between midsurface in-plane strain and normal curvature, an artefact of the derivation not present in the underlying three-dimensional theory where only products of deformationgradient and rotations occur, we therefore neglect this term.[14] Moreover, for a **almost rigid** material with $\lambda \gg 1$ we have $\frac{\lambda}{(2\mu+\lambda)^2} \ll 1$, leading finally to the reduced expressions:

$$\begin{aligned}\varrho_m &= 1-\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]+\frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}\,,\\ \varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle+\frac{\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\,.\end{aligned}\tag{3.25}$$

The formula (3.25) shows the physically reasonable behaviour that to first order, **fibers will be elongated by opposite transverse tractions** and **in-plane stretch leads to thickness reduction**.

Having obtained the general form of the relevant coefficients ϱ_m, ϱ_b, it is expedient to base the expansion of the three-dimensional elastic Cosserat energy on a further simplified expression, namely

$$F_s=\nabla\varphi_s(x,y,z)\approx(\nabla m|\,\varrho_m\,\overline{R}_3)+z\cdot(\nabla\overline{R}_3|\varrho_b\,\overline{R}_3)=A_m+z\,A_r=\overline{F}_s\,,\quad A_m=\tilde{A}_m\,.\tag{3.26}$$

This modification has only consequences as far as the resulting bending contribution is concerned and is motivated by our

Remark 3.1 (Guiding principle of reduction)

(G1.) The reduced model should at no place contain mixed products of normal curvature $\overline{R}^T(\nabla\overline{R}_3|0)$ *and midsurface in-plane strain* $\langle\overline{R}^T(\nabla m|0),-\rangle 2$*, since such a coupling is not present in the underlying three-dimensional model.*

(G2.) The reduced model should at no place contain space derivatives of the thickness stretch ϱ_m*, since in the underlying three-dimensional Cosserat model curvature is only present through the third order curvature tensor* $\mathfrak{K}$ *related only to rotations* $\overline{R}$*.*

The use of (3.26) excludes (up to order h^3) exactly those terms which would violate our principle had we used (3.10) instead. A simple but tedious calculation reveals that

$$\begin{aligned}&\frac{\mu}{4}\|\overline{R}^TA_r+A_r^T\overline{R}\|^2+\frac{\lambda}{8}\text{tr}\left[\overline{R}^TA_r+A_r^T\overline{R}\right]^2\\ &\quad=\mu\|\,\text{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2+\frac{\mu\lambda}{2\mu+\lambda}\text{tr}\left[\text{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2+\frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\,.\end{aligned}\tag{3.27}$$

Exactly the same computations as for the bending term allows us to conclude that

$$\begin{aligned}&\frac{\mu}{4}\|\overline{R}^TA_r+A_r^T\overline{R}-2\mathbb{1}\|^2+\frac{\lambda}{8}\text{tr}\left[\overline{R}^TA_r+A_r^T\overline{R}\right]^2\\ &\quad=\mu\|\,\text{sym}(\overline{R}^T(\nabla m|\overline{R}_3))-\mathbb{1}\|^2+\frac{\mu\lambda}{2\mu+\lambda}\text{tr}\left[\text{sym}(\overline{R}^T(\nabla m|\overline{R}_3))-\mathbb{1}\right]^2+\frac{\langle N_{\text{diff}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)}\,.\end{aligned}\tag{3.28}$$

3.3 Dimensionally reduced energy: energy projection

Now we perform the analytical integration over the thickness in terms of the reduced kinematics. We insert the result $\overline{F}_s$ (3.26) and $\overline{R}_s$ instead of F and $\overline{R}^{3d}$ into (3.7). Since

$$\begin{aligned}\|\,\text{sym}(\overline{R}_s^T\overline{F}_s)-\mathbb{1}\|^2&=\frac{1}{4}\|A_m^T\overline{R}+\overline{R}^TA_m+z\,A_r^T\overline{R}+z\,\overline{R}^TA_r-2\mathbb{1}\|^2\\ &=\frac{1}{4}\|A_m^T\overline{R}+\overline{R}^TA_m-2\mathbb{1}\|^2+z\,\langle A_m^T\overline{R}+\overline{R}^TA_m-2\mathbb{1},A_r^T\overline{R}\rangle+\frac{z^2}{4}\|A_r^T\overline{R}+\overline{R}^TA_r\|^2\,.\end{aligned}\tag{3.29}$$

[14]It would be possible to base all further considerations indeed on the exact solution of ϱ_m,ϱ_b and it seems that the resulting two-dimensional model would allow an existence proof. However, the much more involved expressions are not easily interpreted and do not reduce to the classical expressions upon linearization.

and a similar expression for the trace and skew part, we obtain by explicitly integrating over the (absolutely thin plate like referential) domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$

$$\begin{aligned}
\int_\omega \int_{-\frac{h}{2}}^{\frac{h}{2}} W_{\mathrm{mp}}(\overline{F}_s, \overline{R}_s)\,\mathrm{dV} &= \int_\omega h\left(\frac{\mu}{4}\|A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1}\|^2 + \frac{\mu_c}{4}\|A_m^T\overline{R} - \overline{R}^T A_m\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1}\right]^2\right) \mathrm{d}\omega \\
&\quad + 0 + \int_\omega \frac{h^3}{12}\left(\frac{\mu}{4}\|A_r^T\overline{R} + \overline{R}^T A_r\|^2 + \frac{\mu_c}{4}\|A_r^T\overline{R} - \overline{R}^T A_r\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[A_r^T\overline{R} + \overline{R}^T A_r\right]^2\right) \mathrm{d}\omega \\
&= \int_\omega h\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}_3))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\right]^2\right. \\
&\quad \left. + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)}\right) \mathrm{d}\omega \, + \qquad (3.30) \\
&\quad \int_\omega \frac{h^3}{12}\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\right) \mathrm{d}\omega \\
&= \int_\omega h\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}_3))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\right]^2\right. \\
&\quad \left. + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)} + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{24\,(2\mu+\lambda)}\right) \mathrm{d}\omega \\
&\quad + \int_\omega \frac{h^3}{12}\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\right) \mathrm{d}\omega\,,
\end{aligned}$$

and we may call $(A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1})$ the **membrane part** and $(A_r^T\overline{R} + \overline{R}^T A_r)$ the **bending part**. The influence of

$$h\left(\frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)} + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{24\,(2\mu+\lambda)}\right)\,, \tag{3.31}$$

in the reduced energy is of higher order than the comparable influence of the assumed resultant loading, cf. (10.102). Moreover, for large tractions, the influence of this term in the energy would coerce the component $\overline{R}_3$ to adjust orthogonal to tractions N instead of presumably parallel. Since $2\mu + \lambda \gg 1$ for a rigid material it is therefore suggested to neglect this contribution as well. This is all the more necessary, since (3.31) would be a **non-frame-indifferent** contribution to the plate elastic energy.

3.4 Reduction of the curvature

Similarly the Cosserat curvature term is integrated over the thickness. Consider

$$\mathfrak{K}_s = \overline{R}_s^T \mathrm{D_x}\overline{R}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \tag{3.32}$$

the **reduced** third order **curvature tensor**. Integration over the domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ yields

$$\int_\omega \int_{-\frac{h}{2}}^{\frac{h}{2}} W_{\mathrm{curv}}(\mathfrak{K}_s)\,\mathrm{dV} = \int_\omega \mu \frac{h\,L_c^{1+p}}{12}\,\left(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}_s\|^q\right)\,\left(\alpha_5\,\|\,\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}} \mathrm{d}\omega\,. \tag{3.33}$$

3.5 Reduction/deduction of the boundary conditions

Taking the Dirichlet boundary conditions for φ into account and the kinematical ansatz, we have

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\varrho_b(x,y)\right)\cdot\overline{R}_{s,3}(x,y,0)\,, \qquad \varphi_s(x,y,z)_{|\Gamma_0} = g_{\mathrm{d}}(x,y,z)\,. \tag{3.34}$$

Evaluating for $\pm h/2$ yields two vector equations:

$$g_{\mathrm{d}}(x,y,+h/2) = m(x,y) + \left(h/2\,\varrho_m(x,y) + \frac{h^2}{8}\varrho_b(x,y)\right)\cdot \overline{R}_{s,3}(x,y,0) \tag{3.35}$$
$$g_{\mathrm{d}}(x,y,-h/2) = m(x,y) + \left(-h/2\,\varrho_m(x,y) + \frac{h^2}{8}\varrho_b(x,y)\right)\cdot \overline{R}_{s,3}(x,y,0)\,.$$

Adding and subtracting shows

$$g_{\mathrm{d}}(x,y,+h/2) + g_{\mathrm{d}}(x,y,-h/2) = 2m(x,y) + \frac{h^2}{4}\varrho_b(x,y)\cdot \overline{R}_{s,3}(x,y,0) \tag{3.36}$$
$$g_{\mathrm{d}}(x,y,+h/2) - g_{\mathrm{d}}(x,y,-h/2) = h\,\varrho_m(x,y)\,\overline{R}_{s,3}(x,y,0) \Rightarrow\; \nabla g_{\mathrm{d}}(x,y,0).e_3 = \varrho_m(x,y)\,\overline{R}_{s,3}(x,y,0) + o(h)\,.$$

This implies

$$m(x,y) = \frac{1}{2}\left(g_{\mathrm{d}}(x,y,+h/2) + g_{\mathrm{d}}(x,y,-h/2)\right) \approx g_{\mathrm{d}}(x,y,0)\,. \tag{3.37}$$

In order get a boundary condition for the rotation we use the best available information of the three-dimensional theory: consider the three-dimensional **consistent coupling** boundary condition $\Gamma_0^h \subset \partial\Omega$:

$$\overline{R}^{3d}(x,y,z) = \mathrm{polar}(\nabla\varphi(x,y,z)) = \mathrm{polar}((\partial_x\varphi(x,y,z)|\partial_y\varphi(x,y,z)|\partial_z\varphi(x,y,z)))\,. \tag{3.38}$$

Since g_{d} is given on Γ_0^h, it holds that

$$g_{\mathrm{d}}(x,y,+h/2) = \varphi(x,y,+h/2)$$
$$g_{\mathrm{d}}(x,y,-h/2) = \varphi(x,y,-h/2) \Rightarrow \partial_z\varphi(x,y,0) = \nabla g_{\mathrm{d}}(x,y,0).e_3 + o(h)\,. \tag{3.39}$$

Hence

$$\begin{aligned}\overline{R}^{3d}(x,y,0) &= \mathrm{polar}(\nabla\varphi(x,y,0)) = \mathrm{polar}((\partial_x\varphi(x,y,0)|\partial_y\varphi(x,y,0)|\partial_z\varphi(x,y,0)))\\ &= \mathrm{polar}((\partial_x\varphi(x,y,0)|\partial_y\varphi(x,y,0)|\nabla g_{\mathrm{d}}(x,y,0).e_3))\,,\end{aligned} \tag{3.40}$$

which, in view of the assumed kinematics necessitates the **consistent coupling for plates**

$$\overline{R}_{|_{\gamma_0}}(x,y) = \mathrm{polar}((\nabla m(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3))\,. \tag{3.41}$$

This condition disposes us from the need to motivate rather artificially any boundary conditions for the rotations. Observe that this last boundary condition **does not imply** that the **rigid plate prescription**

$$\overline{R}_{3|_{\gamma_0}} = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \tag{3.42}$$

holds, which would correspond to a form of clamping[15] and which can be seen as a consequence of (3.36). Note, however, that (3.41) implies (3.42) in the limit of small-strain: i.e. if $(\nabla m|\nabla g_{\mathrm{d}}.e_3)_{|_{\gamma_0}} \in \mathrm{SO}(3,\mathbb{R})$. In this sense, (3.42) is a small strain approximation of (3.41).

[15] We reserve the notion **clamped**, meaning that $\vec{n}_m = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\nabla g_{\mathrm{d}}(x,y,0).e_3}$ on γ_0 to traditional fourth order Kirchhoff-Love models (7.83).

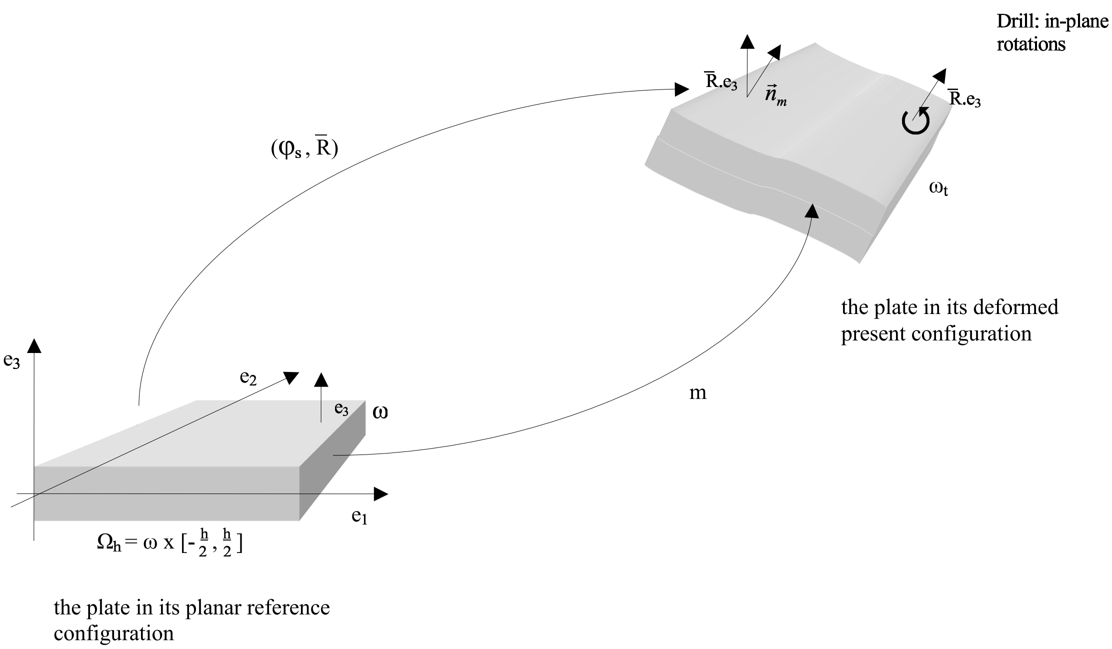


Figure 1: The assumed Cosserat plate kinematics incorporating transverse shear ($\overline{R}_3 \neq \vec{n}$), thickness stretch ($\varrho_m \neq 1$) and drill-rotations. Reconstructed three-dimensional deformation φ_s, midsurface deformation m, microrotation $\overline{R}$.

4 The new finite Cosserat thin plate model with size effects

Gathering our results we have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$
\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min .\ \text{w.r.t. } (m, \overline{R}),\\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m | \overline{R}_3), \quad F_s = (\nabla m | \varrho_m \overline{R}_3) \quad \text{reconstructed deformation gradient}\\
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \overbrace{\frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}}^{\text{non-invariant}} = \underbrace{1 - \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{U} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}}_{\text{first order thickness stretch}}\\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported (fixed)} \qquad\qquad (4.43)\\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m | \nabla g_{\mathrm{d}}(x,y,0).e_3))_{|\gamma_0}\,, \quad \text{reduced consistent coupling}\\
\overline{R}_{3_{|\gamma_0}} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \qquad \text{alternatively: rigid prescription}\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2\\
&= \mu\, \|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2)\|^2 + \mu_c\, \|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2\\
&\quad + \underbrace{\frac{\kappa(\mu+\mu_c)}{2}\left(\langle \overline{R}_3, m_x\rangle^2 + \langle \overline{R}_3, m_y\rangle^2\right)}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2)\right]^2\\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu \frac{L_c^{1+p}}{12}\left(1 + \alpha_4\, L_c^q\, \|\mathfrak{K}_s\|^q\right)\left(\alpha_5\, \|\mathrm{sym}\, \mathfrak{K}_s\|^2 + \alpha_6\, \|\mathrm{skew}\, \mathfrak{K}_s\|^2 + \alpha_7\, \mathrm{tr}\left[\mathfrak{K}_s\right]^2\right)^{\frac{1+p}{2}}\,,\\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \quad \text{reduced third order } \textbf{curvature tensor}\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\, \|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2\\
\mathfrak{K}_b &= \overline{R}^T(\nabla \overline{R}_3 | 0) = \mathfrak{K}_s^3\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned}
$$

The (relative) thickness of the plate (shell) is $h > 0$. The total elastically stored energy due to **membrane**, **curvature** and **bending**

$$
W = h\, W_{\mathrm{mp}} + h\, W_{\mathrm{curv}} + \frac{h^3}{12} W_{\mathrm{bend}}\,, \tag{4.44}
$$

depends on the midsurface deformation gradient ∇m and microrotations $\overline{R}$ together with their space derivatives only through $\overline{U}$ and $\mathfrak{K}_s$. The **micropolar stretch tensor** $\overline{U}$ of the plate is in general non-symmetric. Here $\omega \subset \mathbb{R}^2$ is a domain with boundary $\partial\omega$ and $\gamma_0 \subset \partial\omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{3,\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed. The reduced external loading functional $\Pi(m, \overline{R}_3)$ is a linear form in $(m, \overline{R}_3)$ defined in (10.102) in terms of the underlying three-dimensional loads. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the **Cosserat couple modulus** and $L_c > 0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal and which is responsible for the size effects. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. We have included the so called **shear correction factor** κ $(0 < \kappa \leq 1)$ to keep in line with infinitesimal models, in our derivation however, we obtain $\kappa = 1$. The model is fully **frame-indifferent**, meaning that

$$
\forall\, Q \in \mathrm{SO}(3,\mathbb{R}): \quad W(Q\hat{F}, Q\overline{R}) = W(\hat{F}, \overline{R})\,. \tag{4.45}
$$

The non-invariant term ϱ_m is only needed to reconstruct the 3D-deformation, which of course depends on the non-invariant loading.[16] Strain and curvature parts are additively decoupled, as in the underlying parent model (3.7).

[16]Of course, if the external tractions are rotated as well, we obtain invariance: $\langle Q.N_{\mathrm{diff}}, Q.\overline{R}_3\rangle = \langle N_{\mathrm{diff}}, \overline{R}_3\rangle$.

4.1 The different cases for the Cosserat plate

As in the three-dimensional case, we may distinguish five different situations: (different values of p, q compared with the three-dimensional case)

I: $\mu_{\mathbf{c}} > \mathbf{0}$, $\alpha_{\mathbf{4}} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{1}$, $\mathbf{q} \geq \mathbf{0}$, unconditional elastic macro-stability, local first order Cosserat micropolar, unqualified existence, microscopic specimens, non-zero Cosserat couple modulus. Fracture excluded.

II: $\mu_{\mathbf{c}} = \mathbf{0}$, $\alpha_{\mathbf{4}} = \mathbf{0}$, $\mathbf{p} > \mathbf{1}$, $\mathbf{q} = \mathbf{0}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded.

III: $\mu_{\mathbf{c}} = \infty$, $\alpha_{\mathbf{4}} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{1}$, $\mathbf{q} \geq \mathbf{0}$, unconditional elastic macro-stability, the constrained gradient Cosserat micropolar plate problem (indeterminate couple stress plate model (4.61)). Compatible Dirichlet boundary conditions: $m_{|\gamma_0} = g_{\rm d}$, $\operatorname{polar}((\nabla m|\varrho_m \vec{n}_m))_{|\gamma_0} = \operatorname{polar}(\nabla g_{\rm d})_{|\gamma_0}$.

IV: $\mu_{\mathbf{c}} = \mathbf{0}$, $\alpha_{\mathbf{4}} = \mathbf{0}$, $\mathbf{0} < \mathbf{p} \leq \mathbf{1}$, $\mathbf{q} = \mathbf{0}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $m \notin W^{1,1}(\omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including **fracture** in multiaxial situations.

V: $\mu_{\mathbf{c}} = \mathbf{0}, \mathbf{L}_{\mathbf{c}} = \mathbf{0}$, elastic pre-stability, finite elasticity with free rotations and microstructure. Weak solutions of corresponding finite elasticity are stationary points of this minimization problem. Allowing for **sharp interfaces**.

We refer to $0 < p < 1$, $q \geq 0$ as the **sub-critical case**, $p = 1$, $q \geq 0$ as the **critical case** and $p \geq 1$, $q > 1$ as the **super-critical case**. We will mathematically treat the first three cases.

4.2 Constitutive consequences of the value for the Cosserat couple modulus

Looking at the membrane energy $W_{\rm mp}$ with $\mu_c > 0$ we see that the implication of this choice at a first glance is an inoccuous rise in the macroscopic elastic membrane strain energy $W_{\rm mp}(\overline{U})$ of the plate if $\overline{R} \neq \operatorname{polar}(\nabla m|\overline{R}_3))$. The choice $\mu_c > 0$ acts like a local 'elastic spring' between both continuum rotations and microrotations.

Let us consider the mathematical implications of $\mu_c = 0$ and $0 < \mu_c \leq \mu$, respectively, for the membrane, in more detail. We compute the second derivative of the membrane strain energy $W_{\rm mp}(\overline{R}^T \hat{F})$ at fixed $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$ w.r.t. $\nabla m \in \mathbb{M}^{2\times 3}$. For $H \in \mathbb{M}^{2\times 3}$ we have

$$D^2_{\nabla m} W_{\rm mp}(\overline{R}^T \hat{F}).(H,H) \geq D^2_{\nabla m}\left(\mu \|\operatorname{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\|^2 + \mu_c \|\operatorname{skew}(\overline{R}^T(\nabla m|\overline{R}))\|^2\right).(H,H) \tag{4.46}$$

$$= 2\mu \|\operatorname{sym}(\overline{R}^T(H|0))\|^2 + 2\mu_c \|\operatorname{skew}(\overline{R}^T(H|0))\|^2 = \begin{cases} \geq 2\mu_c\, \|\overline{R}^T(H|0))\|^2 = 2\mu_c\, \|(H|0))\|^2 & \text{if } \mu_c > 0 \\ = 2\mu\, \|\operatorname{sym}(\overline{R}^T(H|0))\|^2 & \text{if } \mu_c = 0 \end{cases}.$$

Hence the choice $\mu_c > 0$ leads to **uniform convexity** of $W_{\rm mp}(\overline{R}^T \hat{F})$ w.r.t. ∇m and **unconditional elastic stability** on the macroscopic level: regardless of what distribution of microrotations $\overline{R}(x)$ is given, the macroscopic equation of balance of linear momentum is uniquely solvable and this equation is insensible to any deterioration of the spatial features of the microstructure. Uniform convexity is difficult to accept from a constitutive point of view since it is impossible for a geometrically exact description in the framework of a classical macroscopic continuum but clear from the above discussion: the additional elastic spring between micro- and continuum rotation extremely rigidifies the material and completely changes the type of the mathematical boundary value problem compared with the classical finite theory.[17]

Fortunately, such a far reaching unsatisfactory conclusion does not hold for $\mu_c = 0$. Choose $\xi \in \mathbb{R}^3$ and $\eta = (\eta_1, \eta_2, 0)^T$. Then consider $(H|0) = \xi \otimes \eta \in \mathbb{M}^{3\times 3}$ and

$$D^2_{\nabla m} W_{\rm mp}(\overline{R}^T \hat{F}).(\xi\otimes\eta, \xi\otimes\eta) = \mu\left(\|\overline{R}^T \xi\otimes\eta\|^2 + \langle \overline{R}^T \xi\otimes\eta, \eta\otimes\overline{R}^T\xi\rangle\right) = \mu\left(\|\overline{R}^T\xi\otimes\eta\|^2 + \langle \overline{R}^T\xi, \eta\rangle^2\right),$$

which shows the physically much more appealing inequality

$$D^2_{\nabla m} W_{\rm mp}(\overline{R}^T \hat{F}).(\xi\otimes\eta, \xi\otimes\eta) \geq \mu\, \|\xi\|^2_{\mathbb{R}^3} \cdot \|\eta\|^2_{\mathbb{R}^2}, \tag{4.47}$$

expressing nothing but uniform **Legendre-Hadamard ellipticity** of the membrane acoustic-tensor with ellipticity constant μ **independent** of $\overline{R}$. The Legendre-Hadamard condition has the most convincing physical basis [Ant95, p.461] in that it implies the reality of wave speeds and the Baker-Ericksen inequalities (stress increases with strain, [MH83, p.19]). The choice $\mu_c = 0$ is consistent with the three-dimensional strain energy density proposed in [Nef03b, (P3)] and [NW03, M1] if the appearing independent viscoelastic rotations there are identified with the independent elastic Cosserat microrotations here.

[17] In the analytical section we will see that $\mu_c > 0$ implies that $m \in W^{1,1}(\omega, \mathbb{R}^3)$ irrespective of $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$, thus **excluding fracture**.

4.3 The coercive inequality in two-dimensions

In this section we show how to use the three-dimensional extended Korn's first inequality Theorem 2.1 in our reduced two-dimensional context of plates and shells in order to improve Legendre-Hadamard ellipticity to uniform positivity. In order to show that the elastic membrane energy is uniformly convex we look at the second differential of $W_{\mathrm{mp}}(\overline{R}^T\hat{F})$ with respect to m

$$D^2_{\nabla m}W_{\mathrm{mp}}(\overline{R}^T\hat{F}).(\nabla\phi,\nabla\phi) \geq \frac{\mu}{2}\|(\nabla\phi|0)^T\overline{R}+\overline{R}^T(\nabla\phi|0)\|^2\,. \tag{4.48}$$

Set for simplicity $\mu = 2$ and consider the slightly more general quadratic form (appropriate for elastic shells and elasto-plastic shells)

$$\begin{aligned}\|F_p^{-T}(\nabla\phi|0)^T\overline{R}_e+\overline{R}_e^T(\nabla m|0)F_p^{-1}\|^2 &= \|\overline{R}_e\left(F_p^{-T}(\nabla\phi|0)^T\overline{R}_e+\overline{R}_e^T(\nabla\phi|0)F_p^{-1}\right)\overline{R}_e^T\|^2\\ &= \|(\overline{R}_eF_p)^{-T}(\nabla\phi|0)^T+(\nabla\phi|0)(\overline{R}_eF_p)^{-1}\|^2\,,\end{aligned} \tag{4.49}$$

where $\phi:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and $\phi_{|\gamma_0}=0$ for $\gamma_0\subset\partial\omega$. Extend now ϕ by $\bar{\phi}:\mathbb{R}^3\mapsto\mathbb{R}^3$ through

$$\bar{\phi}(x,y,z) := \phi(x,y) \Rightarrow \bar{\phi}(x,y,z)_{\gamma_0\times[-\frac{h}{2},\frac{h}{2}]} = 0. \tag{4.50}$$

This extension implies

$$\nabla_{(x,y,z)}\bar{\phi}(x,y,z) = (\nabla_{(x,y)}\phi|0). \tag{4.51}$$

For $\bar{\phi}$ it is possible to use the 3D-extended Korn's first inequality Theorem 2.1. To this end consider $\Omega_h = \omega\times[-\frac{h}{2},\frac{h}{2}]$ and the lateral Dirichlet boundary $\Gamma_0^h = \gamma_0\times[-\frac{h}{2},\frac{h}{2}]\subset\partial\Omega_h$. Then Γ_0^h has non-vanishing 2-dimensional Lebesgue measure. Set by abuse of notation $F_p = (\overline{R}_eF_p)$ for the moment. With smooth enough, invertible F_p it holds on applying Theorem 2.1 that

$$\begin{aligned}&\int_{\omega\times[-\frac{h}{2},\frac{h}{2}]}\|\nabla\bar{\phi}^TF_p^{-1}+F_p^{-T}\nabla\bar{\phi}\|^2\,\mathrm{dV} \geq c_{3D}^+\cdot\int_{\omega\times[-\frac{h}{2},\frac{h}{2}]}\|\bar{\phi}\|^2+\|\nabla\bar{\phi}\|^2\,\mathrm{dV} \Rightarrow\\ &\int_\omega\int_{-\frac{h}{2}}^{\frac{h}{2}}\|\nabla\bar{\phi}^TF_p^{-1}+F_p^{-T}\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\,\mathrm{dz} \geq c_{3D}^+\cdot\int_\omega\int_{-\frac{h}{2}}^{\frac{h}{2}}\|\bar{\phi}\|^2+\|\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\,\mathrm{dz}\,.\end{aligned} \tag{4.52}$$

Since $\bar{\phi}$ is independent of z we get, however,

$$\int_\omega\|\nabla\bar{\phi}^TF_p^{-1}+F_p^{-T}\nabla\bar{\phi}\|^2\,\mathrm{d}\omega \geq c_{3D}^+\cdot\int_\omega\|\bar{\phi}\|^2+\|\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\,, \tag{4.53}$$

or back in terms of ϕ

$$\int_\omega\|(\nabla\phi|0)^TF_p^{-1}+F_p^{-T}(\nabla\phi|0)\|^2\,\mathrm{d}\omega \geq c_{3D}^+\cdot\int_\omega\|\phi\|^2+\|(\nabla\phi|0)\|^2\;\mathrm{d}\omega. \tag{4.54}$$

Observe that the constant c_{3D}^+ is **independent of the thickness** h which might be surprising at first glance. This observation allows one to bound $m\in H_\circ^{1,2}(\omega,\mathbb{R}^3;\gamma_0)$ independent of the relative thickness h only in terms of the membrane energy $\int_\omega W(\nabla m,\overline{R})\,\mathrm{d}\omega$ if $\overline{R}\in\mathrm{SO}(3,\mathbb{R})$ is smooth enough. Thus we have finally proved

Theorem 4.1 (Extended Korn´s first inequality for rigid shells)
Let $\omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and let $\gamma_0\subset\partial\omega$ be a smooth part of the boundary with non vanishing 1-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\omega,\mathbb{R}^3;\gamma_0) := \{\phi\in H^{1,2}(\omega),\ \phi:\omega\mapsto\mathbb{R}^3\mid\phi_{|\gamma_0}=0\}$ and let $F_p,F_p^{-1}\in C^1(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\mathrm{Curl}\,F_p\in C^1(\overline{\omega},\mathbb{M}^{3\times3})$. *Then*

$$\exists\,c^+>0\quad\forall\,\phi\in H_\circ^{1,2}(\omega,\mathbb{R}^3;\gamma_0):\ \|(\nabla\phi|0)F_p^{-1}(x)+F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)} \geq c^+\,\|\phi\|^2_{H^{1,2}(\omega)}\,. \qquad\blacksquare$$

Based on the strengthening proposed in [Pom03] we get immediately the following

Corollary 4.2 (Improved Korn's inequality for rigid shells)
Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\gamma_0 \subset \partial\omega$ be a part of the boundary with non vanishing 1-dimensional Lebesgue measure. Define $H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0) := \{\phi \in H^{1,2}(\omega),\ \phi : \omega \mapsto \mathbb{R}^3 \mid \phi_{|\gamma_0} = 0\}$ and let $F_p \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3, \mathbb{R}))$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0) : \ \|(\nabla\phi|0)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)} \geq c^+ \ \|\phi\|^2_{H^{1,2}(\omega)}\,, \tag{4.55}$$

and the constant is bounded away from zero for F_p bounded in $W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3, \mathbb{R}))$.

Proof. The Sobolev embedding shows that $F_p \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3, \mathbb{R}))$ may be identified with a continuous function. A contradiction argument as in [Nef03c] shows that the constant is bounded away from zero since $W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3, \mathbb{R}))$ is compactly embedded in $C(\overline{\omega}, \mathrm{GL}(3, \mathbb{R}))$. ■

However, taking the special structure of the extended Korn's inequality into account, work in progress suggests that even continuity is not really necessary: instead $F_p \in L^\infty(\omega, \mathrm{GL}(3, \mathbb{R}))$ and $\mathrm{Curl}\, F_p \in L^{N+\delta}(\omega)$ with $N = \dim(\omega)$ should suffice, whereas $F_p \in L^\infty(\omega, \mathrm{GL}(3, \mathbb{R}))$ alone is not sufficient, see the counterexample presented in [Pom03].

4.4 Mathematical analysis of the two-dimensional problem

The following results are the first existence theorems for geometrically exact[18] derived elastic Cosserat plate models known to the author:[19]

Theorem 4.3 (Existence for 2D-finite elastic Cosserat model: case I.)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\omega, \mathrm{SO}(3, \mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (10.102). Then (4.43) with material constants conforming to case I admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p}(\omega, \mathrm{SO}(3, \mathbb{R}))$.

Proof. We apply the direct methods of variations. First, the requirement on the data shows that

$$\Pi(m, \overline{R}_3) \leq C \cdot \left(\|m\|_{L^2(\omega)} + \|\overline{R}_3\|_\infty\right) = C\ \left(\|m\|_{L^2(\omega)} + 1\right)\,. \tag{4.56}$$

With the prescription of $(g_{\mathrm{d}}, \overline{R}_{\mathrm{d}})$ it is clear that $I < \infty$ for some pair $(m, \overline{R})$. Observe first that the micropolar curvature term $\mathfrak{K}_s$ controls $\overline{R} \in W^{1,1+p}(\omega, \mathrm{SO}(3, \mathbb{R}))$, since $\|\mathfrak{K}_s\| = \|\overline{R}^T \mathrm{D_x}\overline{R}\| = \|\mathrm{D_x}\overline{R}\|$, pointwise and $\alpha_5, \alpha_6 > 0$. Moreover, $\mathrm{SO}(3, \mathbb{R})$ is weakly closed in the topology of $W^{1,1+p}(\omega)$. We omit to show that I is bounded below: this will turn out not to be necessary. We may choose decreasing (infimizing) sequences of pairs $(m^k, \overline{R}^k)$. The curvature contribution together with the appropriate boundary conditions and Poincarés inequality yields boundedness of $\overline{R}^k \subset W^{1,1+p}(\omega, \mathrm{SO}(3, \mathbb{R}))$. We may extract a subsequence again denoted by $\overline{R}^k$ converging strongly in $L^{1+p}(\omega)$ to an element $\tilde{\overline{R}} \in W^{1,1+p}(\omega, \mathrm{SO}(3, \mathbb{R}))$ since $p > 0$ by assumption. Because $\mu_c > 0$, it is immediate that $(\nabla m_k|\overline{R}^k) = \hat{F}^k$ is **bounded** in $L^2(\omega, \mathbb{M}^{3\times 3})$, **independent** of $\overline{R}^k$ on account of

$$W_{\mathrm{mp}}(\overline{R}^{k,T}\hat{F}^k) \geq \mu_c\, \|\overline{R}^{k,T}\hat{F}^k - 1\!\!1\|^2 = \mu_c\ \left(\|\hat{F}^k\|^2 - 2\langle \hat{F}^k, \overline{R}^k\rangle + 3\right) \geq \mu_c\ \left(\|\hat{F}^k\|^2 - 2\sqrt{3}\|\hat{F}^k\| + 3\right)\,, \tag{4.57}$$

and

$$\begin{aligned}
\infty > \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) + h\, W_{\mathrm{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m_k, \overline{R}^k_3) &\geq \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) - \Pi(m_k, \overline{R}^k_3)\, \mathrm{d}\omega \\
\geq \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k)\, \mathrm{d}\omega - C\, \left(\|m_k\|_{L^2(\omega)} + 1\right) & \qquad (4.58)\\
\geq \mu_c\, h\, \|\hat{F}^k\|^2_{L^2(\omega)} - 2\sqrt{3}\, \mu_c\, h\, \|\hat{F}^k\|_{L^2(\omega)} - C\|m_k\|_{H^{1,2}(\omega)} + 3\mu_c\, h - C & \\
\geq \mu_c\, h\, \|\nabla m_k\|^2_{L^2(\omega)} - 2\sqrt{3}\, \mu_c\, h\, \|\nabla m_k\|_{L^2(\omega)} - C\, \|m_k\|_{H^{1,2}(\omega)} + 3\mu_c\, h - C & \\
\geq \mu_c\, h\, \|\nabla m_k\|^2_{L^2(\omega)} - C\, \|m_k\|_{H^{1,2}(\omega)} - C \geq \mu_c\, c_{\mathrm{P}}^+\, h\, \|v_k\|^2_{H^{1,2}(\omega)} - C_1\, \|v_k\|_{H^{1,2}(\omega)} + C_2\,, &
\end{aligned}$$

[18] same as frame-indifferent

[19] The proposed finite results determine the macroscopic midsurface plate deformation $m \in H^1(\omega, \mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure). If $\mu_c > 0$ fracture is effectively ruled out, which is unrealistic. All results remain true for arbitrary shear correction factor $\kappa > 0$.

where we made use of the appropriate boundary conditions for $m_k = x + v_k(x)$, and applied Poincarés inequality to u_k since it has zero boundary values on γ_0. This yields the boundedness of v_k, thus m_k is bounded in $H^1(\omega, \mathbb{R}^3)$. Hence we may extract a subsequence, not relabelled, such that $m_k \rightharpoonup \tilde{m} \in H^1(\omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of $(m_k, \overline{R}^k)$ such that $\overline{U}_k = \overline{R}^{k,T} \hat{F}^k$ converges weakly in $L^2(\omega)$ to an element $\tilde{\overline{U}}$ on account of the boundedness of the stretch energy and $\mu_c > 0$.

For $p \geq 1$ we have as well that $\overline{R}^k$ converges indeed strongly in $L^2(\omega)$ to an element $\tilde{\overline{R}} \in H^{1,2}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Thus $\overline{R}^{k,T} \hat{F}^k$ converges weakly to $\tilde{\overline{R}}^T \hat{F}$ in $L^1(\omega)$. The weak limit in $L^1(\omega)$ must coincide with the weak limit of $\overline{U}_k$ in $L^2(\omega)$. Hence, $\tilde{\overline{U}} = \tilde{\overline{R}}^T (\nabla \tilde{m} | \tilde{\overline{R}}_3)$.

Since the total energy is convex in $(\overline{U}, \mathfrak{K}_s, \mathfrak{K}_b)$ and $(\hat{F}, D\overline{R})$, we get

$$\begin{aligned} I(\tilde{m}, \tilde{\overline{R}}) &= \int_\omega h\, W_{\mathrm{mp}}(\tilde{\overline{U}}) + h\, W_{\mathrm{curv}}(\tilde{\mathfrak{K}}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\tilde{\mathfrak{K}}_b)\, \mathrm{d}\omega - \Pi(\tilde{m}, \tilde{\overline{R}}_3) \\ &\leq \liminf_{k\to\infty} \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) + h\, W_{\mathrm{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_{b,k})\, \mathrm{d}\omega - \Pi(m_k, \overline{R}_3^k) = \lim_{k\to\infty} I(m_k, \overline{R}^k)\,, \end{aligned} \tag{4.59}$$

which implies that the limit pair is a minimizer. Note that the limit microrotations $\tilde{\overline{R}}$ may fail to be continuous if $p \leq 2$ (non-existence or limit case of Sobolev embedding). Moreover, uniqueness cannot be ascertained, since $\mathrm{SO}(3,\mathbb{R})$ is a nonlinear manifold (and the considered problem is indeed nonlinear), such that convex combinations of rotations are not rotations in general. Since the functional I is differentiable the minimizing pair is a stationary point and therefore a solution of the corresponding field equations. Note again that the limit microrotations are trivial in $L^\infty(\omega)$ but may fail to be continuously distributed in space. That under these unfavourable circumstances a minimizing solution may nevertheless be found is entirely due to $\mu_c > 0$ and $p \geq 1$. ■

We continue with the (more realistic) super-critical case.

Theorem 4.4 (Existence for 2D-finite elastic Cosserat model: case II.)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (10.102). Then (4.43) with material constants conforming to case II admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$.

Proof. We repeat the argument of case I. However, the boundedness of infimizing sequences is not immediately clear. Boundedness of the rotations $\overline{R}^k$ holds true in the space $W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$ with $1+p+q > N = 3$, hence we may extract a subsequence, not relabelled, such that $\overline{R}^k$ converges strongly to $\hat{\overline{R}} \in C^0(\overline{\omega}, \mathrm{SO}(3,\mathbb{R}))$ in the topology of $C^0(\overline{\omega}, \mathrm{SO}(3,\mathbb{R}))$ on account of the Sobolev-embedding theorem. Along such strongly convergent sequence of rotations, the corresponding sequence of midsurface-deformations m^k is also bounded in $H^1(\omega, \mathbb{R}^3)$. However, this is not due to a basically simple pointwise estimate as in case I, but only true after integration over the domain: at face value we only control certain mixed symmetric expressions in the reconstructed deformation gradient. More precisely, we have

$$\begin{aligned} \infty &> \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) + h\, W_{\mathrm{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m_k, \overline{R}_3^k) \geq \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) - \Pi(m_k, \overline{R}_3^k)\, \mathrm{d}\omega \\ &\geq \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k)\, \mathrm{d}\omega - C\left(\|m_k\|_{L^2(\omega)} + 1\right) \qquad (4.60) \\ &\geq \int_\omega h\, \frac{\mu}{4} \|\overline{R}_k^T (\nabla m_k | \overline{R}_3) + (\nabla m_k | \overline{R}_3)^T \overline{R}_k - 2 1\!\!1\|^2\, \mathrm{d}\omega - Ch\, \left(\|m_k\|_{H^{1,2}(\omega)} + 1\right) \\ &\geq \int_\omega h\, \frac{\mu}{4}\, \|\overline{R}_k^T (\nabla m_k | 0) + (\nabla m_k | 0)^T \overline{R}_k\|^2\, \mathrm{d}\omega - C_1\, \|m_k\|_{H^{1,2}(\omega)} + C_2 \\ &= \int_\omega h\, \frac{\mu}{4}\, \|(\overline{R}_k - \hat{R} + \hat{R})^T (\nabla v_k | 0) + (\nabla v_k | 0)^T (\overline{R}_k - \hat{R} + \hat{R})\|^2\, \mathrm{d}\omega - C_1\, \|v_k\|_{H^{1,2}(\omega)} + C_2 \\ &\geq \oint h\, \frac{\mu}{4} \underbrace{\|\hat{R}^T (\nabla v_k | 0) + (\nabla v_k | 0)^T \hat{R}\|^2}_{\text{combinations of derivatives}}\, \mathrm{d}\omega - C_3\, \|\hat{R} - \overline{R}_k\|_\infty\, \|v_k\|^2_{H^{1,2}(\omega)} \end{aligned}$$

$$
\begin{aligned}
&- (C_1 + 2\,\|\hat{R} - \overline{R}_k\|_\infty)\,\|v_k\|_{H^{1,2}(\omega)} + C_2 \\
&\geq (\frac{\mu}{4}\, c_K^+ - C_3\,\|\hat{R} - \overline{R}_k\|_\infty)\,\|v_k\|^2_{H^{1,2}(\omega)} - (C_1 + 2\,\|\hat{R} - \overline{R}_k\|_\infty)\,\|v_k\|_{H^{1,2}(\omega)} + C_2\,,
\end{aligned}
$$

where we made use of the appropriate boundary conditions for $m^k = x + u_k$ and applied the extended Korn's inequality (2.1) in the improved version of [Pom03] yielding the positive constant c_K^+ for the continuous microrotation $\hat{R}$. Since $\|\hat{R} - \overline{R}_k\|_\infty \to 0$ we conclude the boundedness of v_k in $H^1(\omega)$. Hence, m_k is bounded as well in $H^1(\omega)$. Now we obtain that $\overline{U_k} \rightharpoonup \tilde{\overline{U}} = \hat{\overline{R}}^T\, \nabla\tilde{m}$ by construction with the notations as in case I.

The remainder proceeds as in case I. This finishes the argument. The limit microrotations $\hat{\overline{R}}$ are indeed found to be continuous. However, for mixed boundary conditions, the midsurface deformation m cannot be shown to be smooth for lack of elliptic regularity. ■

4.5 The limit problem for infinite Cosserat couple modulus $\mu_c \to \infty$: the Biot-plate

As in the three-dimensional case, a constrained plate model is obtained by setting formally $\mu_c = \infty$ in (4.43). This implies that $\overline{U} = \overline{R}^T(\nabla m|\overline{R}_3) \in \mathrm{Sym}$, which entails $\overline{R}_3 = \vec{n}_m$ and the constraint rotation $\overline{R} = \mathrm{polar}(\nabla m|\vec{n}_m)$. Moreover, $\mathfrak{K}_b \in \mathrm{Sym}$ is enforced. Independent variation is only possible w.r.t. m and (4.43) turns into the **constrained** minimization problem on ω:

$$
\begin{aligned}
I &= \int_\omega h\,W_{\mathrm{mp}}(\overline{U}) + h\,W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\,W_{\mathrm{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m,\vec{n}_m) \mapsto \min .\ \text{w.r.t. } m\,, \\
\overline{U} &= \overline{R}^T\hat{F} = \sqrt{(\nabla m|\vec{n})^T(\nabla m|\vec{n})} \in \mathrm{Sym}, \quad \hat{F} = (\nabla m|\vec{n}_m), \quad F_s = (\nabla m|\varrho_m\vec{n}_m), \quad \overline{R} = \mathrm{polar}(\nabla m|\vec{n}) \\
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{U} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \vec{n}\rangle}{(2\mu+\lambda)}\,, \quad \text{first order thickness stretch} \\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (4.61) \\
\mathrm{polar}(\nabla m|\vec{n}) &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|\gamma_0}\,, \quad \text{reduced consistent coupling} \\
\vec{n}_{|\gamma_0} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \quad \text{alternatively: classical rigid condition} \\
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\overline{U} - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]^2 \\
&= \mu\,\|\sqrt{I_m} - 1\!\!1_2\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\sqrt{I_m} - 1\!\!1_2\right]^2, I_m\text{: } \textbf{first fundamental form} \text{ of the surface} \\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}_s\|^q)\,\left(\alpha_5\,\|\,\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}}\,, \\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \quad \text{reduced third order } \textbf{curvature tensor} \\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\,\|\mathfrak{K}_b\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[\mathfrak{K}_b]^2 = \mu\,\|\nabla\vec{n}\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\overline{R}^T(\nabla\vec{n}|0)\right]^2 \\
\mathfrak{K}_b &= \mathrm{polar}((\nabla m|\vec{n}))^T(\nabla\vec{n}|0)\,, \quad \text{second order, } \textbf{weighted, bending tensor} \\
\mathfrak{K}_b &\in \mathrm{Sym} \Leftrightarrow \overline{U}^{-1}\widehat{II}_m \in \mathrm{Sym}\,, \quad \text{symmetry constraint} \Leftrightarrow \langle\overline{R}_{1,y}, \vec{n}\rangle = \langle\overline{R}_{2,x}, \vec{n}\rangle \quad \text{for smooth fields} \\
\widehat{II}_m &: \quad \text{extended } \textbf{second fundamental form} \text{ of the surface}\,.
\end{aligned}
$$

Let us therefore define the set of admissible deformations $\mathcal{A} := \{m \in H^{1,2}(\omega,\mathbb{R}^3)\,|\,\langle\mathrm{polar}(\nabla m|\vec{n})_{1,y}, \vec{n}\rangle = \langle\mathrm{polar}(\nabla m|\vec{n})_{2,x}, \vec{n}\rangle\,\}$. This set is not empty: pure bending situations $(\nabla m|\vec{n}) \in \mathrm{SO}(3,\mathbb{R})$ and deformations, where $U \in \mathrm{diag}$ and $II_m \in \mathrm{diag}$ are contained in $\mathcal{A}$.

Theorem 4.5 (Existence for 2D-constrained Cosserat plate model: case III)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega,\mathbb{R}^3)$ and $\mathrm{polar}(\nabla g_{\mathrm{d}}) \in W^{1,1+p}(\omega,\mathrm{SO}(3,\mathbb{R}))$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\omega,\mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega,\mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s,\mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega,\mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s,\mathbb{R}^3)$, see (10.102). If $I < \infty$ over $\mathcal{A}$ then problem (4.61) with $p \geq 1$ admits at least one minimizing solution $m \in H^{1,2}(\omega,\mathbb{R}^3)$.

Proof. The proof mimics case I since the sequence of infimizing rotations $\overline{R}_k$ is constrained to the orthogonal part $\mathrm{polar}(\hat{F}_k)$ of the corresponding sequence of deformation gradients F_k. Due to the extra Cosserat curvature control, the rotations $\overline{R}_k = \mathrm{polar}(\nabla m_k|\vec{n}_k)$ can be chosen such that they converge weakly in $H^1(\omega, \mathrm{SO}(3,\mathbb{R}))$ and such weak limit lies in $\mathcal{A}$. ■

Remark 4.6
Complete higher regularity of m in the constrained Cosserat model, i.e. $m \in H^{2,2}(\omega,\mathbb{R}^3)$ cannot be ascertained in general since we only control certain second derivatives of m in the curvature term. One might wonder therefore, whether the additional C^1-continuity in treating the fourth order indeterminate couple stress problem numerically is worth the effort.

4.6 The limit problem for vanishing relative thickness $h \to 0$

While it does not make much sense to let $h \to 0$ at fixed in-plane elongation $L > 0$, since there is an absolute lower bound on the thickness in terms of the internal length L_c, we may consider a sequence of plates, whose absolute thickness is fixed, but whose in-plane elongation L is increased. This implies that the relative thickness h tends to zero. In a formal sense then, the thin plate limit problem is obtained by neglecting the h^3- bending tensor contribution and giving up the possibility/necessity to prescribe microrotations $\overline{R}_3$ at the Dirichlet boundary γ_0. In view of the expected limit behaviour of $\mathrm{skew}(\overline{U}) = 0 \Rightarrow \overline{R}_3 = \vec{n}_m$ we consider $\mu_c = 0$ only. The two-dimensional limit problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the thin plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solves formally the following minimization problem on ω:

$$
\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s)\,\mathrm{d}\omega - \Pi(m,\overline{R}_3) \mapsto \min.\ \text{w.r.t. } (m,\overline{R}),\\
&m_{|\gamma_0} = g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad (4.62)\\
&\overline{R}_{|\gamma_0} = \mathrm{polar}((\nabla m|\varrho_m\,\overline{R}_3))_{|\gamma_0}\,, \quad \text{reduced consistent coupling} \Rightarrow \overline{R}_{3_{|\gamma_0}} = \vec{n}_m, \quad \text{free}\\
&W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\mathrm{sym}(\overline{U}-1\!\!1)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}-1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}_s) = \mu\frac{L_c^{1+p}}{12}\,(1+\alpha_4\,L_c^q\,\|\mathfrak{K}_s\|^q)\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}}\,,\\
&\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \quad \text{reduced third order } \textbf{curvature tensor}\,.
\end{aligned}
$$

Theorem 4.7 (Existence for 2D-finite Cosserat limit model for vanishing relative thickness)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega,\mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega,\mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s,\mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega,\mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s,\mathbb{R}^3)$, see (10.102). *Then* (4.62) *admits at least one minimizing solution pair $(m,\overline{R}) \in H^1(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$.*

Proof. Exactly the same proof as for case II applies since the decisive control is afforded by W_{curv} and not W_{bend}. ■

Conjecture 4.8 (Γ-limit)
The Γ-limit for $h \to 0$ of suitably rescaled energies in (3.7) *and $\mu_c \geq 0$ is given by the variational problem* (4.62) *with $\mu_c = 0$.* ■

4.7 The limit problem for vanishing internal length $L_c \to 0$

This limit is practically encountered if very large, relatively thin plates are considered. The difference to the case $h \to 0$ from above is clear: we consider a sequence of ever larger plates with the **same relative thickness**. A scaling argument (10.3.3) shows easily that the respectively transformed L_c on a unit domain ω will tend to zero. We obtain formally the following two-dimensional minimization problem for the deformation of the midsurface $m:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R}:\omega\subset\mathbb{R}^2\mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$
\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m,\overline{R}_3) \mapsto \min . \text{ w.r.t. } (m,\overline{R}),\\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3), \quad F_s = (\nabla m|\varrho_m \overline{R}_3)\\
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{U}-1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}, \quad \text{first order thickness stretch}\\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (4.63)\\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3)_{|\gamma_0}\,, \quad \text{reduced consistent coupling}\\
\overline{R}_{3_{|\gamma_0}} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \quad \text{rigid prescription}\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U}-1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}-1\!\!1)\right]^2\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\,\|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\,\|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2\\
\mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0)\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned}
$$

For $\mu_c = 0$ this is case V of our classification. In this form, the problem is not completely determined since the remaining bending term only controls the 'director' $\overline{R}_3$ but leaves **in plane rotations free**. However, anticipating that $\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1$ is small (appropriate for almost rigid materials), a modification of the bending term is suggested: we modify

$$
\mathfrak{K}_b = \overline{R}^T(\nabla\overline{R}_3|0) \notin \mathrm{Sym} \Rightarrow \begin{pmatrix} -\|R_{1,x}\| & -\|R_{1,y}\| & 0\\ -\|R_{2,x}\| & -\|R_{2,y}\| & 0\\ 0 & 0 & 0\end{pmatrix} \notin \mathrm{Sym}\,. \qquad (4.64)
$$

Remark 4.9 (Motivation)
The motivation of this modification for relatively thin Cosserat shells is as follows: either the **membrane energy is non-zero***, in which case it dominates and the* **bending contribution can be neglected** *or the* **membrane energy is zero** *($\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1 = 0$) in which case the* **non-symmetric bending tensor** *of* (4.63) **coincides** *with the symmetric expression of* (4.64)*, see Lemma 11.8.*

A formulation based on this modification supports an existence theorem if $\mu_c > 0$, notwithstanding the inherent nonlinearity along the same lines as in Theorem 4.3. The more interesting case of $\mu_c = 0$ must remain open at present, since the limit rotations in $H^{1,2}(\omega, \mathrm{SO}(3,\mathbb{R})))$ must not necessarily be smooth.

4.8 The limit problem for vanishing L_c: the pure bending case.

Assume that the boundary conditions for the Cosserat plate support multiple finite bending modes, i.e. $\nabla m^T\nabla m = 1\!\!1_2$ and the membrane energy $W_{\mathrm{mp}}(\overline{U})$ is zero. What can we say about the corresponding degenerated minimization problem? The variational problem for the Cosserat bending plate is then to find a deformation of the midsurface $m:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R}:\omega\subset\mathbb{R}^2\mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$
\begin{aligned}
I &= \int_\omega \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega \mapsto \min . \text{ w.r.t. } (m,\overline{R}) \text{ such that } \nabla m^T\nabla m = 1\!\!1_2 \text{ and } W_{\mathrm{mp}}(\overline{U}) = 0,\\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3) \quad m_{|\gamma_0} = g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (4.65)\\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3)_{|\gamma_0}\,, \quad \text{reduced consistent coupling}\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U}-1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}-1\!\!1)\right]^2\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\,\|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\,\|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2\\
\mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0)\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned}
$$

It is easily seen, that $\nabla m^T \nabla m = 1\!\!1_2$ and $W_{\mathrm{mp}}(\overline{U}) = 0$ already constrains the microrotations to $\overline{R} = (\nabla m|\vec{n}_m) \in \mathrm{SO}(3,\mathbb{R})$ for $\mu_c > 0$[20] and $\mu_c = 0$.[21] This implies $\mathfrak{K}_b = \overline{R}^T(\nabla \overline{R}_3|0) = (\nabla m|\vec{n}_m)^T(\nabla \vec{n}|0)$ and the in general non-symmetric bending tensor $\mathfrak{K}_b$ coincides with the symmetric second fundamental form of the midsurface m. The resulting minimization **coincides** with the bending problem based on the Kirchhoff-Love theory (7.88) and admits therefore a solution $m \in H^2(\omega,\mathbb{R}^3)$, from which we recover $\overline{R} = (\nabla m|\vec{n}) \in H^{1,2}(\omega, \mathrm{SO}(3,\mathbb{R}))$.

4.9 The limit problem for vanishing L_c and vanishing h: the pure membrane.

The problem for vanishing relative thickness h and without consideration of the internal length L_c leads to the **pure membrane dominated** limit problem[22] for the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the thin plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ on ω:

$$\int_\omega h\, W_{\mathrm{mp}}(\overline{U})\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min.\ \text{w.r.t. } (m, \overline{R}), \quad m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0), \tag{4.66}$$

which is equivalent to

$$\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{U})\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min.\ \text{w.r.t. } m \text{ at given } \overline{R} \in \mathrm{SO}(3,\mathbb{R}).\,, \quad m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0), \qquad (4.67)\\
&\overline{U} = \overline{R}^T(\nabla m|\overline{R}_3)\\
&0 = \underbrace{\mathrm{skew}\left(\overline{R}^T(\overline{M}|0|0)\right) + h\ \mathrm{skew}\left(\overline{R}^T D_{\overline{R}} W_{\mathrm{mp}}(\nabla m, \overline{R})\right)}_{\text{2D-balance of angular momentum}}, \quad \text{local, algebraic condition } \Leftrightarrow\\
&0 = \underbrace{\mathrm{skew}\left(\overline{R}^T(\overline{M}|0|0)\right) + h\ \mathrm{skew}\left(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T\right)}_{\text{thickness integrated 3D-balance of angular momentum}} - h\ \mathrm{skew}\left((0|0|D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3)\right),
\end{aligned}$$

see (10.102) for the definition of $\overline{M}$. The local condition comes from locally minimizing w.r.t. $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$, it is balance of angular momentum for the plate in disguise.[23] Note that at given $\overline{R}$, the membrane minimization problem w.r.t. m is still uniformly Legendre-Hadamard elliptic. However, coercivity w.r.t. m depends crucially on the smoothness of $\overline{R}$ if $\mu_c = 0$. There is no reason to expect $\overline{R}$ to be smoothly distributed. Existence to this problem is open: we expect therefore sharp interfaces.

In the absence of external loads, the remaining symmetry condition

$$\mathrm{skew}\left(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T\right) = \mathrm{skew}\left((0|0|D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3)\right) \tag{4.69}$$

is satisfied, if $\overline{U} \in \mathrm{Sym}$, which itself implies $\overline{R} = \mathrm{polar}(\nabla m|\vec{n})$. Nevertheless, considered as a local condition, the remaining symmetry condition does not automatically imply the symmetry of $\overline{U}$, see the discussion of a similar problem in the three-dimensional case in [Nef03a]. Such a discrepancy does not occur in the infinitesimal Reissner-Mindlin model (10.127).

[20] In this case, we could dispose of the requirement $\nabla m^T \nabla m = 1\!\!1_2$.

[21] $\mathrm{sym}(\overline{U} - 1\!\!1) = 0$ implies immediately $\overline{R}_3 = \vec{n}$. Write $\overline{R}_i = a_i^1 m_x + a_i^2 m_y$, $i = 1,2$. Using $\langle m_x, m_y\rangle = 0$ the result follows. Whether one can do without $\nabla m^T \nabla m = 1\!\!1_2$ in case $\mu_c = 0$ is open, since $\mathrm{sym}(\overline{U} - 1\!\!1) = 0$ for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and $(\nabla m|\vec{n}) \in \mathrm{GL}^+(3,\mathbb{R})$ considered **without gradient constraint** on m has nontrivial solutions.

[22] Observe that the Cosserat model does not automatically endow a thin plate limit with additional stiffness, since it is physically not possible to let $h \to 0$ and keep the in-plane elongations L constant.

[23] To see the equivalence of the two local statements in (4.67), consider variation of $\overline{R}$ along a one-parameter group of rotations $\frac{\mathrm{d}}{\mathrm{dt}}\overline{R} = A(t)\cdot\overline{R}$, $A \in \mathfrak{so}(3,\mathbb{R})$ and evaluate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} W_{\mathrm{mp}}(\overline{R}(\nabla m|\overline{R}_3)) &= \langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}), (\delta\overline{R})^T(\nabla m|\overline{R}_3) + \overline{R}^T(0|0|(\delta\overline{R})_3)\rangle = \langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}), (A\overline{R})^T(\nabla m|\overline{R}_3) + \overline{R}^T(0|0|(A\overline{R})_3)\rangle\\
&= -\langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T, \overline{R}^T A\overline{R}\rangle + \langle \overline{R} D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}), (0|0|A\overline{R}.e_3)\rangle = -\langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T, \overline{R}^T A\overline{R}\rangle + \langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3, \overline{R}^T A\overline{R}.e_3\rangle\\
&= -\langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T, \overline{R}^T A\overline{R}\rangle + \langle (0|0|D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3), \overline{R}^T A\overline{R}\rangle\,. \qquad (4.68)
\end{aligned}$$

5 A modified finite Cosserat thin plate for large stretch and local invertibility

While the preceeding models have been derived from a three-dimensional model which itself is appropriate only for small strain and large rotations, let us present a modified model,[24] which in principle allows for arbitrary large stretch and which automatically preserves local invertibility if the reconstructed deformation is smooth. It is clear that such an extension is by no means unique. The model reads

$$
\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min.\ \text{w.r.t. } (m, \overline{R}), \\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m | \overline{R}_3), \quad F_s = (\nabla m | \varrho_m \overline{R}_3) \\
\varrho_m &= \frac{1}{1 + \frac{\lambda}{2\mu+\lambda}(\det[\overline{U}] - 1)} + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3 \rangle}{(2\mu + \lambda)} \\
m_{|\gamma_0} &= g_{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported} \qquad (5.70) \\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m | \nabla g_{\mathrm{d}}(x, y, 0).e_3))_{|\gamma_0}\,, \quad \text{reduced consistent coupling} \\
\overline{R}_{3_{|\gamma_0}} &= \frac{\nabla g_{\mathrm{d}}(x, y, 0).e_3}{\|\nabla g_{\mathrm{d}}(x, y, 0).e_3\|}\,, \quad \text{alternatively: rigid prescription} \\
W_{\mathrm{mp}}(\overline{U}) &= \mu \, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \frac{1}{2} \left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2 \right) \\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 L_c^q \|\mathfrak{K}_s\|^q\right) \left(\alpha_5 \, \|\mathrm{sym}\, \mathfrak{K}_s\|^2 + \alpha_6 \, \|\mathrm{skew}\, \mathfrak{K}_s\|^2 + \alpha_7 \, \mathrm{tr}\, [\mathfrak{K}_s]^2 \right)^{\frac{1+p}{2}}\,, \\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0) \right)\,, \quad \text{reduced third order } \textbf{curvature tensor} \\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu \, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \mathrm{tr}\, [\mathrm{sym}(\mathfrak{K}_b)]^2 \\
\mathfrak{K}_b &= \overline{R}^T (\nabla \overline{R}_3 | 0) = \mathfrak{K}_s^3\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned}
$$

Let us summarize the salient features of this model. First, $W_{\mathrm{mp}}(\overline{U}) \to \infty$ if $\det[\overline{U}] \to 0$. Thus, if minimizers exist, then $\det[\overline{U}] > 0$ a.e. and the minimizing surface is locally regular. The modified energy contribution is **polyconvex** w.r.t ∇m and thus Legendre-Hadamard elliptic. If $\overline{R}_3 = \vec{n}$, then

$$
\det[\overline{U}] = \|\,\mathrm{Cof}\,(\nabla m|0)\|, \quad \|\,\mathrm{Cof}\,(\nabla m|0)\|^2 = \|m_x \times m_y\|^2 = \|m_x\|^2 \|m_y\|^2 - \langle m_x, m_y \rangle^2 = \det[I_m]\,, \qquad (5.71)
$$

a pure, intrinsic measure of the surface stretch. If $W_{\mathrm{mp}}(\overline{U}) = 0$ then $\overline{U} = 1\!\!1$ although $\mu_c = 0$. The thickness stretch ϱ_m has such a form, that at finite energy one has $0 < \varrho_m < \infty$ without restriction on the kinematics and transverse fibers will be elongated upon action of opposite tractions. Moreover, $\varrho_m \equiv 1$ for $\lambda = 0$ (extreme compressibility, $\nu = 0$) and $\varrho_m = \frac{1}{\det[\overline{U}]}$ for $\lambda = \infty$ (incompressibility, $\nu = \frac{1}{2}$) such that $\det[F_s] = \det[(\nabla m | \varrho_m \overline{R}_3)] \equiv 1$, i.e. exact incompressibility for the reconstructed deformation.

The formulation (5.70) has the same linearized behaviour as the initial model (4.43).[25] We can prove the following result

Theorem 5.1 (Existence for 2D-finite elastic Cosserat model with large stretch and invertibility) *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (10.102). Then (5.70) with material constants conforming to case II admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$ with $\det[(\nabla m|\overline{R}_3)] > 0$ a.e.*

Proof. The proof mimics the arguments of the preceeding existence results for case II. We only need to observe in addition, that the modified membrane energy is in fact polyconvex at given $\overline{R}$ w.r.t. ∇m. The modified term provides us with the information that $\det[(\nabla m_k|\overline{R}_3^k)]$ is uniformly bounded in $L^2(\omega)$ for minimizing sequences. Hence we may always choose a minimizing sequence, such that $\det[(\nabla m_k|\overline{R}_3^k)] \rightharpoonup \zeta \in L^2(\omega)$. We have as well $\overline{R}^k \to \overline{R} \in C^0(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, $\nabla m_k \rightharpoonup \nabla m \in L^2(\omega, \mathbb{M}^{2\times 3})$. Thus, $\det[(\nabla m_k|\overline{R}_3^k)] \to \det[(\nabla m|\overline{R}_3)]$

[24] It is clear that a modification to large stretch does not concern the bending term since bending only plays a role for small stretch.

[25] Because $\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right) = 2\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]^2 + O(\|\overline{U} - 1\!\!1\|^3)$.

strongly in the sense of distributions [Bal77, Th. 3.4]. This implies $\zeta = \det[(\nabla m|\overline{R}_3)]$. The remainder is standard. ∎

It is therefore believed that (5.70) represents an improvement over (4.43), although (5.70) itself is not strictly obtained from a parent model.[26]

6 The finite, invariant Reissner-Mindlin plate

To contrast the previous models, let us directly derive a new nonlinear, finite, properly invariant Reissner-Mindlin plate starting from the three-dimensional SVK elasticity model. Again, we assume a quadratic ansatz in the thickness direction for the (reconstructed) finite deformation $\varphi_s : \mathbb{R}^3 \mapsto \mathbb{R}^3$ of the shell like structure

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\varrho_b(x,y)\right)\cdot \vec{d}(x,y)\,, \tag{6.72}$$

where $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ takes on the role of the deformation of the midsurfaces of the shell viewed as a parametrized surface and $\vec{d} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$ is a **unit director** field; the functions $\varrho_m,\ \varrho_b : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$ allow for **thickness stretch** ($\varrho_m \neq 1$) and **transverse shear** ($\vec{d} \neq \vec{n}$). [27] This implies for the (reconstructed) deformation gradient of the shell (plate)

$$\begin{aligned} F_s = \nabla\varphi_s(x,y,z) &= (\nabla m|\,\varrho_m\,\vec{d}) + z\cdot(\nabla(\varrho_m\,\vec{d})|\varrho_b\,\vec{d}) + \frac{z^2}{2}(\nabla(\varrho_b\,\vec{d})|0) = \tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r \\ &\approx (\nabla m|\,\varrho_m\,\vec{d}) + z\cdot((\nabla\vec{d})|\varrho_b\,\vec{d}) = A_m + z\,A_r\,. \end{aligned} \tag{6.73}$$

The underlying three-dimensional Saint Venant Kirchhoff energy has the form

$$W_{\mathrm{SVK}}(F) = \frac{\mu}{4}\,\|F^TF - 1\!\!1\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F^TF - 1\!\!1\right]^2. \tag{6.74}$$

The equations of the three-dimensional finite elasticity problem show that on the transverse boundary (exact)

$$\begin{aligned} S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2)).e_3 &= N^{\mathrm{trans}}(x,y,+h/2) \\ S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2)).(-e_3) &= N^{\mathrm{trans}}(x,y,-h/2)\,, \end{aligned} \tag{6.75}$$

where N^{trans} are the prescribed tractions N on the transverse boundary.[28]

Following the steps which led to (3.18) we have (exact)

$$\langle F^{-1}(x,y,\pm h/2)S_1(\nabla\varphi(x,y,\pm h/2)).e_3,e_3\rangle = \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), F^{-T}(x,y,\pm h/2).e_3\rangle\,, \tag{6.76}$$

which condition reduces to **zero normal tractions on the transverse free boundary**:

$$S_{2,33}(\nabla\varphi(x,y,\pm h/2)) = 0\,, \tag{6.77}$$

in the absence of tractions N^{trans}. In view of the **assumed rigidity** ($\mu \gg 1$) we expect that $\nabla\varphi^T\nabla\varphi - 1\!\!1 \ll 1$ such that $\nabla\varphi^{-T} \approx \nabla\varphi$ and we determine $\varrho_m,\ \varrho_b$ from the corresponding modified requirement in terms of the assumed kinematics for φ_s, yielding

$$\begin{aligned} \langle F_s^{-1}(x,y,\pm h/2)S_1(\nabla\varphi_s(x,y,\pm h/2)).e_3,e_3\rangle &= \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), \overbrace{F_s(x,y,\pm h/2).e_3}^{\text{modified}}\rangle \\ &= \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), (\varrho_m + z\,\varrho_b)\,\vec{d}\rangle\,. \end{aligned} \tag{6.78}$$

Since $S_1 = F\left[\mu(F^TF - 1\!\!1) + \frac{\lambda}{2}\mathrm{tr}\left[F^TF - 1\!\!1\right]1\!\!1\right]$, we obtain the two **nonlinear** equations

$$\langle\left[\mu(F_s^TF_s - 1\!\!1) + \frac{\lambda}{2}\mathrm{tr}\left[F_s^TF_s - 1\!\!1\right]1\!\!1\right].e_3,e_3\rangle = \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), (\varrho_m + z\,\varrho_b)\,\vec{d}\rangle\,. \tag{6.79}$$

---

[26] There is a general danger of direct theories to postulate two-dimensional models from scratch without recours to any underlying parent model : while general two-dimensional balance principles are easily applied, it is not clear how to incorporate any three-dimensional information.

[27] This leads finally to a 5 'dof' theory: 3 components of the membrane deformation and 2 degrees of freedom for the unit director field, the coefficients ϱ_m, ϱ_b will again be eliminated.

[28] Using the approximated F_s in (6.73) leads to an affine linear reconstruction of the transverse shear stress $\langle S_2(F_s).e_3, e_i\rangle$, $i = 1, 2$.

There is no simple way to solve these equations exactly. To leading order in h we obtain for ϱ_m

$$\varrho_m = +\frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} \pm \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2 - 2] + \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\,, \tag{6.80}$$

and for ϱ_b

$$\varrho_b = -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle + \frac{1}{(2\mu+\lambda)\,h}\langle N_{\mathrm{res}}, \vec{d}\rangle + \frac{1}{\varrho_m\,(2\mu+\lambda)}\langle(\nabla m|\varrho_m\,\vec{d}), (\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle\,.$$

Since we do not want to consider space variations in the thickness-stretch ϱ_m we take finally

$$\begin{aligned}
\varrho_m &= \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}} \\
\varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle + \frac{\langle N_{\mathrm{res}}, \vec{d}\rangle}{(2\mu+\lambda)\,h}\,.
\end{aligned} \tag{6.81}$$

Note that if we identify $\vec{d} = \overline{R}_3$ then ϱ_b in the last formula coincides with the expression for ϱ_b found in (3.25) while ϱ_m is still different.

Following conceptually the same computation which starts after (3.26) we obtain after thickness integration the following minimization problem for the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the unit director field $\vec{d} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$ on ω:

$$\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{C}) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m, \vec{d}) \mapsto \min.\ \text{w.r.t. } (m, \vec{d}) \\
&\overline{C} = \hat{F}^T\hat{F}, \quad \hat{F} = (\nabla m|\vec{d}), \quad F_s = (\nabla m|\varrho_m\vec{d}) \\
&\varrho_m = \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}} \\
&\quad = \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\mathrm{tr}\left[\overline{C} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\,, \quad \text{first order thickness stretch} \\
&m_{|_{\gamma_0}} = g_{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported}, \quad \vec{d}_{|_{\gamma_0}} = \frac{\nabla g_{\mathrm{d}}(x, y, 0).e_3}{\|\nabla g_{\mathrm{d}}(x, y, 0).e_3\|}\,, \quad \text{rigid prescription} \\
&W_{\mathrm{mp}}(\overline{C}) = \frac{\mu}{4}\,\|\overline{C} - 1\!\!1\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\mathrm{tr}\left[\overline{C} - 1\!\!1\right]^2 \\
&\quad = \frac{\mu}{4}\,\|\underbrace{\nabla m^T\nabla m - 1\!\!1_2}_{\text{intrinsic energy}}\|^2 + \frac{\kappa\,\mu}{2}\underbrace{\left(\langle m_x, \vec{d}\rangle^2 + \langle m_y, \vec{d}\rangle^2\right)}_{\text{transverse shear energy}} + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\mathrm{tr}\left[\nabla m^T\nabla m - 1\!\!1_2\right]^2 \\
&W_{\mathrm{bend}}(\mathfrak{K}_b) = \mu\,\|\,\mathrm{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2, \quad \mathfrak{K}_b = (\nabla m|\vec{d})^T(\nabla\vec{d}|0)\,.
\end{aligned} \tag{6.82}$$

It is immediate, that the obtained model is **frame-indifferent** in the sense that $\forall\, Q \in \mathrm{SO}(3, \mathbb{R}) :\ W(Q(\nabla m|\vec{d}) = W((Q\nabla m|Q.\vec{d}) = W((\nabla Q.m|Q.\vec{d}) = W(\nabla m|\vec{d})$. The membrane part is **coercive** in $H^{1,4}(\omega, \mathbb{R}^3)$. However, the membrane part neither satisfies the Baker-Ericksen inequalities nor is it Legendre-Hadamard elliptic. It is not obvious, which type of control can be expected in the bending contribution. Drill rotations are absent, but the model allows for transverse shear (again, $\kappa = 1$ is the shear correction factor). Invertibility of the reconstructed deformation is not ensured. Nothing seems to be known on existence. No extra size effects enter the description. While ϱ_m shows the physically correct behaviour that small opposite transverse tractions will elongate fibers, for non-infinitesimal transverse tractions which 'presurrize' the plate, the fibers would as well be elongated instead of shrinked. Linearization of this model results in the classical infinitesimal Reissner-Mindlin Model (10.127) and restricting the director $\vec{d}$ to the unit normal of the surface simplifies the model into the following finite Kirchhoff-Love plate. In this sense, the model has some merits.

7 The finite, invariant Kirchhoff-Love plate

7.1 Variational formulation

Either by formal asymptotic analysis (and adding together the leading membrane and bending part) or a proper kinematical ansatz[29] or else by restricting the director $\vec{d}$ in (6.82) to the unit normal of the midsurface, a finite, properly invariant[30] Kirchhoff-Love plate problem in variational form can be written in the form of a minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ on ω:

$$\begin{aligned}
&\int_\omega h\, W_{\rm mp}(\overline{C}) + \frac{h^3}{12} W_{\rm bend}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min .\ \text{w.r.t. } m\\
&\overline{C} = \hat{F}^T \hat{F}, \quad \hat{F} = (\nabla m|\vec{n}_m), \quad F_s = (\nabla m|\varrho_m \vec{n}_m)\\
&\varrho_m = \frac{\langle N_{\rm diff}, \vec{n}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\mathrm{tr}\left[\overline{C} - 1\!\!1\right] + \frac{\langle N_{\rm diff}, \vec{n}\rangle^2}{(2\mu+\lambda)^2}}\,, \quad \text{first order thickness stretch}\\
&m_{|\gamma_0} = g_{\rm d}(x,y,0)\,, \qquad \text{simply supported}, \quad \vec{n}_{m_{|\gamma_0}} = \frac{\nabla g_{\rm d}(x,y,0).e_3}{\|\nabla g_{\rm d}(x,y,0).e_3\|}\,, \quad \text{clamped} \qquad (7.83)\\
&W_{\rm mp}(\overline{C}) = \frac{\mu}{4}\|\overline{C} - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{C} - 1\!\!1\right]^2 = \frac{\mu}{4}\|\nabla m^T\nabla m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\mathrm{tr}\left[\nabla m^T\nabla m - 1\!\!1_2\right]^2\\
&\qquad = \frac{\mu}{4}\|I_m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\mathrm{tr}\left[I_m - 1\!\!1_2\right]^2, \quad I_m\text{: } \textbf{first fundamental form} \text{ of the surface}\\
&W_{\rm bend}(\mathfrak{K}_b) = \mu\,\|\,\mathrm{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2 = \mu\,\|II_m\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[II_m\right]^2\\
&\mathfrak{K}_b = (\nabla m|\vec{n}_m)^T(\nabla\vec{n}_m|0) = II_m \in \mathrm{Sym}\,, \quad II_m\text{: } \textbf{second fundamental form} \text{ of the surface } m\,.
\end{aligned}$$

The reconstructed deformation $\varphi_s(x,y,z) = m(x,y) + \left(z\varrho_m + \frac{z^2}{2}\varrho_b\right)\vec{n}_m$ yields the plane stress condition $S_1(\nabla\varphi_s(x,y,0).e_3 = 0$, which is only consistent with three-dimensional equilibrium if there are no normal tractions at the transverse boundary. [31]

It is easily seen that the resultant membrane strain energy $W_{\rm mp}(\overline{C})$ is neither quasiconvex nor Legendre-Hadamard elliptic. Moreover, the resultant membrane strain energy does not satisfy the Baker-Ericksen inequalities in contrast to the Biot-plate model (4.61)! The significance of this statement can be seen as follows. Take $\omega = [-1,1]\times[-1,1]$ and consider zero external loads and boundary conditions for m on $\partial\omega$ which uniformly shrink the plate: $m_{|\partial\omega}(x,y) = \overline{B}.(x,y)^T$, $\overline{B} \in \mathrm{GL}^+(2,\mathbb{R})$. Now take a sequence of minimizing deformations m_k with $\vec{n}_m^k = e_3 = \text{const}$, i.e. $m_3^k(x,y) \equiv 0$. The sequence m_k is naturally bounded in $H^{1,4}(\omega)$. Hence a subsequence converges weakly: $m_k \rightharpoonup \tilde{m} \in H^{1,4}(\omega)$. The minimizing sequence can be chosen such that $\nabla\tilde{m} = \overline{B}$. However $I(m_k) \to 0$ but $I(\tilde{m}) > 0$. Thus the homogeneously shrinked plate is not energy-minimal, which it

[29] Or other constitutive requirements [LS98, p.476]. Indeed there is no general agreement as to what really constitutes an isotropic Kirchhoff-Love plate theory [LS98, p.xiii] and [Kil65]. One encompassing independent statement to obtain Kirchhoff-Love in an engineering context may read: **i. normals remain straight and normal to the midsurface (but may be extended), ii. plane stress, iii. the elastic plate energy is additively decoupled in membrane and curvature parts.** Formal energy projection would also yield indefinite mixed products like $\langle I_m - 1\!\!1, II_m\rangle$.

[30] not to be confused with the nonlinear, non frame-indifferent, Kirchhoff-Love plate model given in [Cia97, p. 318] and mathematically justified in [Mon03].

[31] In fact, the condition $\vec{d} = \vec{n}$ can also be motivated by eliminating locally the free, extensible director $\varrho_m \cdot \vec{d}$ from the finite Reissner-Mindlin model through taking

$$\begin{aligned}
&(\vec{d}, \varrho_m) := \mathrm{argmin}_{\varrho_m\in\mathbb{R}^+, \vec{d}\in\mathbb{S}^2, (\nabla m|\varrho_m\vec{d})\in\mathrm{GL}^+(3,\mathbb{R})} W_{\rm mp}(\overline{C})\,, \qquad \overline{C} = (\nabla m|\varrho_m\vec{d})^T(\nabla m|\varrho_m\vec{d}) \quad \Rightarrow\\
&\vec{d} = \vec{n}_m\,, \qquad \varrho_m^2 = 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right]\,. \qquad (7.84)
\end{aligned}$$

In doing so, no available three-dimensional information has been used. If instead, one defines a reduced membrane energy $W_0 : \mathbb{M}^{2\times 3} \to \mathbb{R}$ without recourse to a specific kinematical ansatz as in [DR95b, p.573] and **without invertibility constraint**

$$\begin{aligned}
&W_0(\nabla m) := \inf_{\eta\in\mathbb{R}^3} W_{\rm mp}((\nabla m|\eta)^T(\nabla m|\eta)) = \inf_{\widehat{\varrho}_m\in\mathbb{R}, \vec{d}\in\mathbb{S}^2} W_{\rm mp}((\nabla m|\widehat{\varrho}_m\vec{d})^T(\nabla m|\widehat{\varrho}_m\vec{d})) \Leftrightarrow\\
&\langle D_F W_{\rm mp}((\nabla m|\widehat{\varrho}_m\vec{d})^T(\nabla m|\widehat{\varrho}_m\vec{d})), (0|0|\delta)\rangle = 0 \quad \forall\,\delta\in\mathbb{R}^3 \Leftrightarrow \quad S_1((\nabla m|\widehat{\varrho}_m\vec{d})).e_3 = 0, \quad \textbf{plane stress} \Rightarrow \quad \vec{d} = \vec{n}\,, \qquad (7.85)\\
&\widehat{\varrho}_m = \begin{cases} \varrho_m & 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] \ge 0\,, \quad (\nabla m|\widehat{\varrho}_m\vec{n}) \in \mathrm{GL}^+(3,\mathbb{R}) \\ 0 & 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] < 0\,, \quad (\nabla m|\widehat{\varrho}_m\vec{n}) \notin \mathrm{GL}^+(3,\mathbb{R}) \end{cases}, \quad W_0(\nabla m) = W_{\rm mp}((\nabla m|\widehat{\varrho}_m\vec{n})^T(\nabla m|\widehat{\varrho}_m\vec{n}))\,,
\end{aligned}$$

then zero normal tractions $S_{2,33}(\nabla m|\widehat{\varrho}_m\vec{n}) = 0$ are not satisfied for $1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] < 0$, which shows the unphysical behaviour, cf. [DR96, DR95c, DR00].

clearly should be, given the stabilization inherent through $\vec{n}_m^k = e_3$. This deficiency must be seen as unphysical and will be called **in-plane failure**.

Thus it is motivated why it is not known whether minimization based on (7.83) does admit a solution for arbitrary data. Even the inclusion of the classical bending term might not be enough: the control of only certain second derivatives of m does not suffice to treat the highly nonlinear problem by a compactness argument and to pass to the limit by strong convergence in the non-elliptic membrane part. The above example suggests that the in-plane failure is somehow related to the absence of drill-rotations.

Moreover, the very feasibility of a Kirchhoff-Love ansatz with thickness stretch places a **restriction on the kinematics** in the sense that it must be guaranteed for the membrane deformation that

$$1 - \frac{\lambda}{(2\mu+\lambda)} \mathrm{tr}\left[\overline{C} - 1\!\!1\right] \geq 0 \Rightarrow \|\nabla m\|^2 \leq 3 + \frac{2\mu}{\lambda} \Leftrightarrow \mathrm{tr}\left[\overline{C} - 1\!\!1\right] < 1 + \frac{2\mu}{\lambda}, \tag{7.86}$$

in the absence of tractions. This condition figures in [Cia97, p.355] among others, under which the quasiconvex hull of the membrane energy $W_{\mathrm{mp}}(\overline{C})$ in (7.83) coincides with the energy itself. In our derivation, condition (7.86) is, as a mathematical consequence of a **physical requirement** from the three-dimensional problem (6.78), most natural. It has also appeared in [FRS93, p.180] where it is believed to be '...unduly restrictive' due to the shortcomings of the SVK energy. While the shortcomings of the SVK energy are well known, similar restrictions occur most natural also for our Cosserat model, there in the form $\varrho_m > 0$, dictated by invertibility of the reconstructed shell deformation. The physical significance of the computed solution is thus tied to $\varrho_m > 0$, which in turn expresses as well the physical (not mathematical) requirement $S_{2,33}(x, y, \pm h/2) = 0$. Looking for solutions with $\varrho_m = 0$ is, physically speaking, not realistic.[32]

Remark 7.1
The problem of the non-ellipticity in the case of the Kichhoff-Love theory has been dealt with in Le Dret and Raoult [DR95b, DR00]. They perform the thin shell limit analysis based on the St.Venant-Kirchhoff density. As a result, they get that the limit energy deformation (of the 3D-model) is independent of the transverse variable and minimizes a limit energy computed as the Γ-limit [Mas92] of the 3D-(St.Venant-Kirchhoff) energy. The limit stored energy is again that of a nonlinearly elastic 'membrane' shell, in the sense that it contains only first derivatives of the unknown deformation m of the midsurface. However, it turns out that the limit energy offers no resistance to compression, a feature that is appropriate only for 'soft' elastic materials like a deflated baloon or a sail but in our oppinion inacceptable for 'almost rigid' materials like metals or paper, the topic whe are interested in since the rigidity translates directly into the small strain assumption.[33] The non resistance to compression in the above analysis is related to the use of the quasiconvex hull [34] QW_0 of the reduced St.Venant Kirchhoff energy W_0 in (7.85), which, surprisingly enough, can be given in closed form [DR95c, HP96] and which shows to be in general positive but zero in the compression range [35] since St.Venant Kirchhoff typically looses ellipticity there. These remarks indicate that results based on Γ-convergence and global minimization are not in all cases the appropriate direction to take, certainly not for almost rigid materials.

However, given all these deficiencies of the finite Kirchhoff-Love model, anticipating that $\nabla m^T \nabla m - 1\!\!1_2$ is small (appropriate for almost rigid materials) as in (4.64), a modification of the bending term is suggested: we modify

$$\mathfrak{K}_b = (\nabla m|\vec{n}_m)^T (\nabla \vec{n}_m|0) = II_m \in \mathrm{Sym} \Rightarrow \begin{pmatrix} -\|m_{xx}\| & -\|m_{xy}\| & 0 \\ -\|m_{yx}\| & -\|m_{yy}\| & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}\,. \tag{7.87}$$

Remark 7.2 (Motivation)
The motivation of this modification for thin shells is as follows: either the **membrane energy is non-zero**, *in which case it dominates and the* **bending contribution can be neglected** *or the* **membrane energy is zero** *in which case the* **bending term** *of* (7.83) **coincides** *with that of* (7.87), *see Lemma 11.7 and compare to [GKM96].*

[32]One might be inclined to think that the apparent problem of non-ellipticity of the membrane expression is only related to the use of the non-elliptic parent SVK-energy. This is not the case. Proceeding by energy projection from a **polyconvex** Neo-Hooke energy, the resulting membrane energy is again non-elliptic. This is well known feature, [DR95b, p.560,iii].

[33]They remark [DR95b, p.550]: "...then the corresponding nonlinear membranes offer no resistance to crumpling. This is an empirical fact, witnessed by anyone who ever played with a deflated ballon."

[34]"... the fact that this function ($W_{\mathrm{mp}}(\overline{C})$) is not quasiconvex already implied that it had to be relaxed in order to give rise to a well posed problem."[DR95b, p.575].

[35]Strictly speaking, the use of the quasiconvex hull leads to a so called **tension field theory** [Ste90]. Steigmann himself [Ste90, p.143] notes "A question then arises concerning the validity of tension filed theory as an approximation to a theory of shells with bending stiffness that is small in some sense. Evidently, the deformation is not well described, though the theory delivers solutions that approximate the average of the deformation observed in a real membrane containing many wrinkles. We conjecture that the stress is accurately predicted, however."

A formulation based on this modification supports an existence theorem.

Theorem 7.3 (Existence for finite almost rigid KL-plate)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^2(\omega, \mathbb{R}^3)$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (10.102). Then (7.83) with the modification (7.87) admits at least one minimizing solution $m \in H^2(\omega, \mathbb{R}^3)$.

Proof. We apply the direct methods of variations. The functional I is bounded above and below. We may choose an infimizing sequence m^k. Due to the boundary conditions and Poincaré's inequality the sequence is bounded in $H^2(\omega)$. The compact embedding $H^2(\omega) \subset W^{1,4}(\omega)$ shows that we may choose a weakly convergent subsequence, not relabelled, such that strongly $\nabla m^k \to \nabla m \in L^4(\omega)$. The weak limit is a minimizer since the bending term is convex in the second derivatives and the nonlinear, non-quasiconvex membrane term is handled by strong convergence. The modified bending term imparts as well additional control for in-plane deformations. ■

Such a theorem might not be of much practical value because it is precisely the level of smoothness we want to avoid and it must be noted that the proposed modification of the bending term is **not consistent** with the classical Kirchhoff-Love theory upon linearization!

7.2 The pure finite bending Kirchhoff-Love problem

Assume that the boundary conditions for the plate support multiple finite bending modes, i.e. the membrane energy is zero, hence $I_m = 1\!\!1$. What can we say about the corresponding degenerated minimization problem based on the remaining term involving only curvature? The variational problem for the clamped plate reads then

$$\inf\{ \int_\omega \frac{h^3}{12} \left[\mu \, \|II_m\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[II_m\right]^2 \right] \mathrm{d}\omega, \; m \in H^2(\omega, \mathbb{R}^3) : \nabla m^T \nabla m = 1\!\!1_2, \tag{7.88}$$
$$m_{|\gamma_0} = g_{\rm d}(x, y, 0), \, \partial_\nu m_{|\gamma_0} = \partial_\nu g_{{\rm d}|\gamma_0}, \, g_{\rm d} \in H^2(\omega, \mathbb{R}^3) : (\nabla g_{\rm d} | \vec{n}_{g_{\rm d}}) \in \mathrm{SO}(3)\} \,.$$

Here ∂_ν are normal derivatives at the boundary. The proposed system coincides with that previously derived by [GKM96, p.44] apart from a modified material parameter $\lambda \mapsto \frac{2\mu\lambda}{2\mu+\lambda}$. Note that under pure bending of a plate, we have for the Gauss curvature $K = 0$ and using (11.139) we get, by adding zero, equivalently

$$\frac{h^3}{12}\left(\mu \, \|II_m\|^2 + \frac{\lambda}{2} \frac{2\mu}{(2\mu+\lambda)} \operatorname{tr}\left[II_m\right]^2 + 2\mu \underbrace{\frac{\operatorname{tr}\left[II_m\right]^2 - \|II_m\|^2}{2 \det[I_m]}}_{K=0} \right) = \frac{h^3}{12}\left(\mu + \frac{\lambda}{2} \frac{2\mu}{(2\mu+\lambda)} \right) \operatorname{tr}\left[II_m\right]^2$$
$$= \frac{h^3}{12} 2\mu \frac{\mu+\lambda}{2\mu+\lambda} \operatorname{tr}\left[II_m\right]^2 = \frac{h^3}{12} \frac{1}{2} \frac{E}{1-\nu^2} \operatorname{tr}\left[II_m\right]^2 , \tag{7.89}$$

on using (11.187). Inserting the linearized quantity $\operatorname{tr}\left[II_m\right]^2 \approx \|\Delta v_3\|^2 + \ldots$ we obtain

$$\underbrace{\frac{h^3}{12} \frac{1}{2} \frac{E}{1-\nu^2}}_{\text{flexural rigidity}} \|\Delta v_3\|^2 , \tag{7.90}$$

the classical infinitesimal plate bending energy leading to the biharmonic equation. It is possible to show that the finite minimization problem admits at least one solution.

Theorem 7.4 (Existence for pure bending finite KL-plate)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^2(\omega, \mathbb{R}^3)$. Then (7.88) admits at least one minimizing solution $m \in H^2(\omega, \mathbb{R}^3)$.

Proof. The proof is based on the crucial observation that on the space of admissible functions, the energy coincides with the quadratic expression

$$\int_\omega \frac{h^3}{12} \left[\mu \, \|\nabla \vec{n}\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \left(\|m_{xx}\| + \|m_{yy}\| \right)^2 \right] \mathrm{d}\omega \,. \tag{7.91}$$

Standard arguments of the direct method of variations finish the proof. A detailed presentation was given in [Cia97, p.347]. ■

Again, the level of smoothness is discomforting.

Corollary 7.5 (Existence for pure bending finite RM-plate)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume that the boundary data $g_{\mathrm{d}} \in H^2(\omega, \mathbb{R}^3)$ supports bending modes. Then the pure finite Reissner-Mindlin bending problem based on (6.82) *admits at least one minimizing solution $m \in H^2(\omega, \mathbb{R}^3)$ and $\vec{d} = \vec{n}_m \in H^1(\omega, \mathbb{S}^2)$.*

Remark 7.6 (Pure bending problems)
Note that all presented pure finite bending problems **coincide** *for the new Cosserat model, the new finite Reissner-Mindlin model and the finite Kirchhoff-Love model. The last two results show that the classical finite bending terms provide enough control in pure bending for models, in which the membrane part would have been non-elliptic. However, the classical bending terms are insufficient to stabilize joint membrane and bending situations.*

8 Discussion and open problems

Starting from a fully invariant three-dimensional physically linear Cosserat theory with independent rotations and size effects, we have obtained a family of fully invariant, finite Cosserat plates by means of assumed kinematics and energy projection. The models include in a natural way drilling degrees of freedom and size effects (smaller samples with the same relative thickness are relatively stiffer than corresponding larger samples). Since the assumed kinematics is quadratic through the thickness, one avoids the so called Poisson thickness locking. In contrast to traditional reduced theories, the membrane part is uniformly elliptic and allows a non-degenerate passage to zero relative thickness. The balance equations for the midsurface are not only uniformly Legendre-Hadamard elliptic, but linear at given rotations.

For vanishing Cosserat couple modulus $\mu_c = 0$, the formulation is shown to be downwards compatible with traditional infinitesimal linear Reissner-Mindlin theories and shear-correction factor $\kappa = 1$.

A detailed mathematical analysis of the resulting two-dimensional models is proposed which closely follows the three-dimensional ideas. It is based on a correspondingly dimensionally reduced version of a new extended Korn's first inequality. We have achieved a surprising unification of two- and three-dimensional concepts.

From a mechanical point of view, compared to more traditional, non-elliptic finite Reissner-Mindlin and Kirchhoff-Love models, it seems to be the beneficial influence of the drill-rotations in conjunction with the internal length $L_c > 0$ which stabilizes the new Cosserat thin plate model.

Certain limit cases related to Sobolev-embedding theorems must remain open for the moment, notably the case IV including possible fracture of the plate. They leave a wide field of challenging new mathematical problems.

A modification of the new Cosserat plate model is also proposed, which ensures invertibility of the reconstructed deformation gradient and which allows as well for minimizers. This model shows the most reasonable physical behaviour, but is not easily seen to be obtained by direct descend from three-dimensions.

While we have large freedom of specifying boundary conditions for the rotations at the Dirichlet boundary, we prefer a generalization of the three-dimensional consistent coupling condition, which includes as a special case prescriptions corresponding to clamping.

A major conceptual advantage of the new proposed model is the appearance of rotations already in the three-dimensional **parent** model. There is no need to artificially introduce independent directors of the plate.

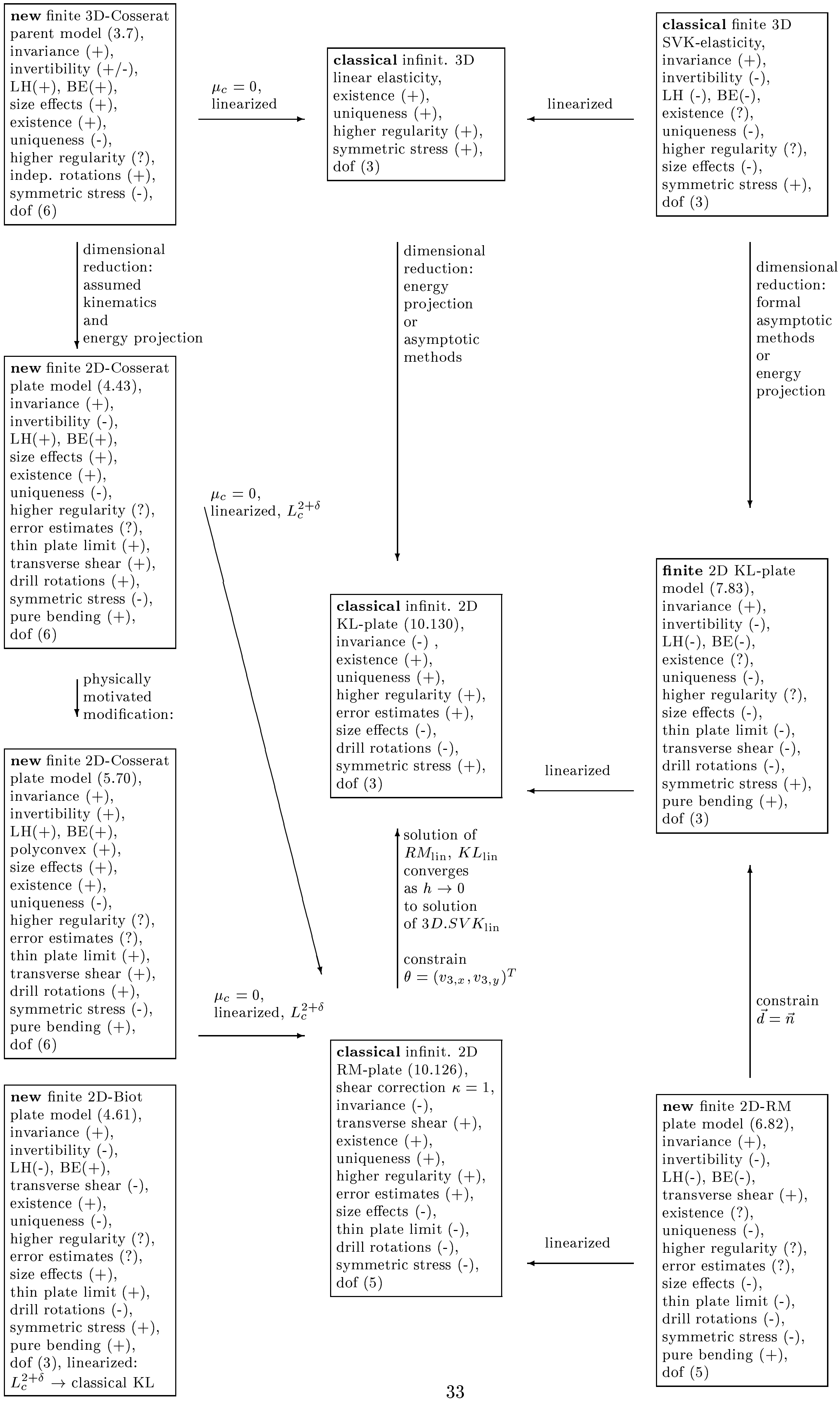
In a subsequent contribution, it will be shown that the proposed method can be easily extended to shells and multiplicative elasto-plasticity with the possibility of exactly the same mathematical analysis in the elastic case.

From a numerical point of view, the new Cosserat plate model offers the highly welcome perspective to use only C^0-conforming finite elements. When interpolating the midsurface deformation one order higher than the rotations, shear locking should be avoided.

It remains to completely justify the apparently sound, new finite Cosserat thin plate model by means of either a convergence proof for vanishing relative thickness to the underlying three-dimensional parent Cosserat model or by showing, that a suitably rescaled three-dimensional problem Γ-converges to one of the two-dimensional limit problems.

Let us summarize and relate some basic features of the obtained new plate models. We abbreviate LH: Legendre-Hadamard elliptic, BE: Baker-Ericksen inequalities, dof: degrees of freedom, invariance: fully frame-indifferent, v: midsurface displacement, $\vec{d}$: unit director, $\vec{n}$: unit normal of the midsurface, θ: infinitesimal director, invertibility: local invertibility of the reconstructed deformation in the sense of a strictly positive determinant of the deformation gradient almost everywhere, pure bending: the problem obtained by restricting considerations to locally length preserving deformations (inextensional).

It can be seen, that **linearization does not always commute with dimensional reduction**. From a modelling point of view it is clear, however, that linearization is the last step to be performed. The unifying role of setting $\mu_c = 0$ is also appreciated.


new finite 3D-Cosserat parent model (3.7), invariance (+), invertibility (+/-), LH(+), BE(+), size effects (+), existence (+), uniqueness (-), higher regularity (?), indep. rotations (+), symmetric stress (-), dof (6)
$\mu_c = 0$, linearized
classical infinit. 3D linear elasticity, existence (+), uniqueness (+), higher regularity (+), symmetric stress (+), dof (3)
linearized
classical finite 3D SVK-elasticity, invariance (+), invertibility (-), LH (-), BE(-), existence (?), uniqueness (-), higher regularity (?), size effects (-), symmetric stress (+), dof (3)
dimensional reduction: assumed kinematics and energy projection
dimensional reduction: energy projection or asymptotic methods
dimensional reduction: formal asymptotic methods or energy projection
new finite 2D-Cosserat plate model (4.43), invariance (+), invertibility (-), LH(+), BE(+), size effects (+), existence (+), uniqueness (-), higher regularity (?), error estimates (?), thin plate limit (+), transverse shear (+), drill rotations (+), symmetric stress (-), pure bending (+), dof (6)
$\mu_c = 0$, linearized, $L_c^{2+\delta}$
classical infinit. 2D KL-plate (10.130), invariance (-) , existence (+), uniqueness (+), higher regularity (+), error estimates (+), size effects (-), drill rotations (-), symmetric stress (+), dof (3)
finite 2D KL-plate model (7.83), invariance (+), invertibility (-), LH(-), BE(-), existence (?), uniqueness (-), higher regularity (?), size effects (-), thin plate limit (-), transverse shear (-), drill rotations (-), symmetric stress (+), pure bending (+), dof (3)
physically motivated modification:
linearized
new finite 2D-Cosserat plate model (5.70), invariance (+), invertibility (+), LH(+), BE(+), polyconvex (+), size effects (+), existence (+), uniqueness (-), higher regularity (?), error estimates (?), thin plate limit (+), transverse shear (+), drill rotations (+), symmetric stress (-), pure bending (+), dof (6)
solution of RM_{lin}, KL_{lin} converges as $h \to 0$ to solution of $3D.SVK_{\text{lin}}$
constrain $\theta = (v_{3,x}, v_{3,y})^T$
constrain $\vec{d} = \vec{n}$
$\mu_c = 0$, linearized, $L_c^{2+\delta}$
classical infinit. 2D RM-plate (10.126), shear correction $\kappa = 1$, invariance (-), transverse shear (+), existence (+), uniqueness (+), higher regularity (+), error estimates (+), size effects (-), thin plate limit (-), drill rotations (-), symmetric stress (-), dof (5)
new finite 2D-Biot plate model (4.61), invariance (+), invertibility (-), LH(-), BE(+), transverse shear (-), existence (+), uniqueness (-), higher regularity (?), error estimates (?), size effects (+), thin plate limit (+), drill rotations (-), symmetric stress (+), pure bending (+), dof (3), linearized: $L_c^{2+\delta} \to$ classical KL
linearized
new finite 2D-RM plate model (6.82), invariance (+), invertibility (-), LH(-), BE(-), transverse shear (+), existence (?), uniqueness (-), higher regularity (?), error estimates (?), size effects (-), thin plate limit (-), drill rotations (-), symmetric stress (-), pure bending (+), dof (5)

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10 Appendix A

10.1 Notation

10.1.1 Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \;|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|X^TX = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \;|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \;|\operatorname{tr}[X] = 0\}$ of traceless tensors. With $\operatorname{Adj} X$ we denote the tensor of transposed cofactors $\mathrm{Cof(X)}$ such that $\operatorname{Adj} X = \det[X]X^{-1} = \mathrm{Cof(X)}^{\mathrm{T}}$ if $X \in \mathrm{GL}(3,\mathbb{R})$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$.

We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)\,U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Furthermore, $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{dt}}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_FW(F).H$ and $D_F^2W(F).(H, H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we define $\operatorname{Curl} A(x)$ as the operation curl applied row wise. We define $H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$.

10.1.2 Notation for shells

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The thickness of the plate is taken to be $h > 0$ with dimension length (contrary to Ciarlet's definition of the thickness to be 2ε, which difference leads only to various different constants in the resulting formulas). We denote by $\mathbb{M}^{n\times m}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{2\times 3}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3\times 3}$ to denote the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. The identity tensor on $\mathbb{M}^{2\times 2}$ will be denoted by $1\!\!1_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$. Sometimes, this is also written as ∇m. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the displacement of the midsurface, such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is written $\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = \mathrm{dV} = \mathrm{d}\omega\,\mathrm{dz}$.

10.2 The treatment of external loads

10.2.1 Dead load body forces for the thin plate

In the three-dimensional theory the dead load body forces $f(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term

$$\int_{\Omega_h} f(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dV}\,. \tag{10.92}$$

Inserting the ansatz for the reconstructed deformation φ_s results in

$$\begin{aligned}\int_{\Omega_h} f(x,y,z) \cdot \varphi_s(x,y,z)\,\mathrm{dV} &\approx \int_{\Omega_h} f(x,y,z) \cdot \left[m(x,y) + z\,\varrho_m \overline{R}_3 + \frac{z^2}{2}\varrho_b\,\overline{R}_3\right]\mathrm{dV}\\ &= \int_\omega h\,\hat{f}(x,y) \cdot m(x,y)\,\mathrm{d}\omega + \int_\omega \left(\int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz}\right)\varrho_m \overline{R}_3\,\mathrm{d}\omega + \int_\omega \left(\int_{-h/2}^{h/2} \frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz}\right)\varrho_b \overline{R}_3\,\mathrm{d}\omega\end{aligned} \tag{10.93}$$

Let us define

$$\hat{f}_0(x,y) := \int_{-h/2}^{h/2} f(x,y,z)\,\mathrm{dz}\,, \quad \hat{f}_1(x,y) := \int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz}\,, \quad \hat{f}_2(x,y) := \int_{-h/2}^{h/2} \frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz}\,, \tag{10.94}$$

such that $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$ are the zero, first, second moment of f in thickness direction. Moreover

$$\begin{aligned}\int_{-h/2}^{h/2} \frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz} &= \int_{-h/2}^{h/2} \frac{z^2}{2}\,(f(x,y,0) + z\,\partial_z f(x,y,0) + \dots)\,\mathrm{dz} = \frac{h^3}{24} f(x,y,0) + O(h^5)\\ \int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz} &= \int_{-h/2}^{h/2} z\,(f(x,y,0) + z\,\partial_z f(x,y,0) + \dots)\,\mathrm{dz} = 0 + \frac{h^3}{12}\partial_z f(x,y,0) + O(h^5)\,.\end{aligned} \tag{10.95}$$

Therefore

$$\int_{\Omega_h} f(x,y,z) \cdot \varphi_s(x,y,z)\,\mathrm{dV} \approx \int_\omega \hat{f}_0(x,y) \cdot m(x,y)\,\mathrm{d}\omega + \int_\omega \hat{f}_1(x,y)\varrho_m \overline{R}_3\,\mathrm{d}\omega + \int_\omega \hat{f}_2(x,y)\varrho_b \overline{R}_3\,\mathrm{d}\omega \tag{10.96}$$

10.2.2 Traction boundary conditions for the thin plate

In the three-dimensional theory the traction boundary forces $N(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term

$$\int_{\partial\Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dS}\,. \tag{10.97}$$

Inserting our ansatz for the reconstructed deformation φ_s results in

$$\begin{aligned}\int_{\partial\Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z) \cdot \varphi_s(x,y,z)\;\mathrm{dS} \;\approx\; &\int_{\omega \times \{-\frac{h}{2},\frac{h}{2}\}} N(x,y,z) \cdot \left[m(x,y) + z\varrho_m \overline{R}_3 + \frac{z^2}{2}\varrho_b \overline{R}_3\right]\mathrm{dS}\\ &+ \int_{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]} N(x,y,z) \cdot \left[m(x,y) + z\varrho_m \overline{R}_3 + \frac{z^2}{2}\varrho_b \overline{R}_3\right]\mathrm{dS}.\end{aligned}$$

Let us define

$$\hat{N}_{\mathrm{lat},0}(x,y) := \int_{-h/2}^{h/2} N(x,y,z)\,\mathrm{dz}\,, \quad \hat{N}_{\mathrm{lat},1}(x,y) := \int_{-h/2}^{h/2} z\,N(x,y,z)\,\mathrm{dz}\,, \quad \hat{N}_{\mathrm{lat},2}(x,y) := \int_{-h/2}^{h/2} \frac{z^2}{2}\,N(x,y,z)\,\mathrm{dz}\,, \tag{10.98}$$

such that $\hat{N}_{\mathrm{lat},0}$, $\hat{N}_{\mathrm{lat},1}$, $\hat{N}_{\mathrm{lat},2}$ are the zero, first, second moment of the tractions N at the lateral boundary in thickness direction. Hence

$$\begin{aligned}\int_{\partial\Omega_h} N(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dS} &\approx \int_\omega [N(x,y,\frac{h}{2}) + N(x,y,-\frac{h}{2})] \cdot m(x,y)\,\mathrm{d}\omega + \int_\omega \frac{h}{2}[N(x,y,\frac{h}{2}) - N(x,y,-\frac{h}{2})]\varrho_m \overline{R}_3\,\mathrm{d}\omega\\ &+ \int_\omega [\frac{h^2}{8}N^+ + \frac{h^2}{8}N^-]\varrho_b \overline{R}_3\,\mathrm{d}\omega + \int_{\gamma_s} \hat{N}_{\mathrm{lat}}(x,y) \cdot m(x,y)\,\mathrm{ds} + \int_{\gamma_s} \hat{N}_{\mathrm{lat},1}(x,y)\,\varrho_m \overline{R}_3\,\mathrm{ds} + \int_{\gamma_s} \hat{N}_{\mathrm{lat},2}(x,y)\,\varrho_b \overline{R}_3\,\mathrm{ds}\\ &= \int_\omega N_{\mathrm{res}}(x,y) \cdot m(x,y)\;\mathrm{d}\omega + \int_\omega h\,N_{\mathrm{diff}}(x,y)\varrho_m \overline{R}_3\,\mathrm{d}\omega + \int_\omega \frac{h^2}{8} N_{\mathrm{res}}\varrho_b \overline{R}_3\,\mathrm{d}\omega\\ &+ \int_{\gamma_s} \hat{N}_{\mathrm{lat},0}(x,y) \cdot m(x,y)\,\mathrm{ds} + \int_{\gamma_s} \hat{N}_{\mathrm{lat},1}(x,y)\,\varrho_m \overline{R}_3\,\mathrm{ds} + \int_{\gamma_s} \hat{N}_{\mathrm{lat},2}(x,y)\,\varrho_b \overline{R}_3\,\mathrm{ds}\,,\end{aligned} \tag{10.99}$$

with

$$N_{\mathrm{res}} := [N(x,y,\frac{h}{2}) + N(x,y,-\frac{h}{2})]\,, \quad N_{\mathrm{diff}} := \frac{1}{2}[N(x,y,\frac{h}{2}) - N(x,y,-\frac{h}{2})]\,. \tag{10.100}$$

10.2.3 The external loading functional

Let us gather all influences of the external loading terms. It would be possible to account for all appearing influences, however, in view of a reasonable simplification we consider only those terms, which would have appeared, if we had made the restricted linear ansatz without thickness stretch $\varphi_s = m + z\,\overline{R}_3$. To leading order we have the

$$\begin{aligned}
\overline{f} &= \hat{f}_0 + N_{\mathrm{res}}\,, && \textbf{resultant body force}\\
\overline{M} &= \hat{f}_1 + h\,N_{\mathrm{diff}}\,, && \textbf{resultant body couple}\\
\overline{N} &= \hat{N}_{\mathrm{lat},0}\,, && \textbf{resultant surface traction}\\
\overline{M}_c &= \hat{N}_{\mathrm{lat},1}\,, && \textbf{resultant surface couple}\,.
\end{aligned} \tag{10.101}$$

The **resultant loading functional** Π is given by

$$\Pi(m,\overline{R}_3) = \int_\omega \langle \overline{f}, m\rangle + \langle \overline{M}, \overline{R}_3\rangle \,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, m\rangle + \langle \overline{M}_c, \overline{R}_3\rangle \,\mathrm{ds}\,. \tag{10.102}$$

If we denote the dependence of Π on the loads of the underlying three-dimensional problem as $\Pi(f,N;\,m,\overline{R}_3)$, then it is easily seen that frame-indifference of the external loading functional is satisfied in the sense that $\Pi(Q.f,Q.N;\,Q.m,Q.\overline{R}_3) = \Pi(f,N;\,m,\overline{R}_3)$ for all rigid rotations $Q \in \mathrm{SO}(3,\mathbb{R})$. It is possible to use the **same functional form** of the loading functional **for all finite and linearized models**. We only need to replace $(m,\overline{R}_3)$ by $(m,\vec{d})$, $(m,\vec{n})$, $(v,\overline{A}_3)$ for the different finite and linearized models, respectively.

10.3 Transformation of the domain and scaling

10.3.1 Classical finite elasticity

Set $\Omega_L^{\mathrm{rel.thin}} = [0,L[\mathrm{m}]]\times[0,L[\mathrm{m}]]\times[-\frac{h}{2}\cdot L, \frac{h}{2}\cdot L]$ with h a small parameter indicating the **relative thickness** of the domain, e.g. $h \in (0,\frac{1}{20}[\mathrm{m}]]$ with dimension length. The three-dimensional problem with respect to the relatively thin domain $\Omega_L^{\mathrm{rel.thin}}$ reads

$$\int_{\xi\in\Omega_L^{\mathrm{rel.thin}}} W_{3D}(\nabla_\xi \varphi_L(\xi)) - \langle f_L(\xi), \varphi_L(\xi)\rangle \,\mathrm{d}\xi - \int_{\partial\Omega_L^{\mathrm{rel.thin}}} \langle N_L, \varphi_L\rangle \,\mathrm{dS}_L \mapsto \min.\quad \text{w.r.t. } \varphi_L\,, \tag{10.103}$$

where we are looking for a function $\varphi_L : \Omega_L^{\mathrm{rel.thin}} \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$. Introducing the **scaling transformation**

$$\zeta : \Omega_h = [0,1[\mathrm{m}]]\times[0,1[\mathrm{m}]]\times[-\frac{h}{2},\frac{h}{2}] \subset \mathbb{R}^3 \mapsto \Omega_L^{\mathrm{rel.thin}} \subset \mathbb{R}^3\,, \qquad \zeta(x) = L\cdot x\,, \tag{10.104}$$

(note that L itself is non-dimensional here) this turns into

$$\int_{x\in\Omega_h} \left[W_{3D}(\nabla\zeta(x)\nabla\varphi(x)\nabla\zeta^{-1}(x)) - \langle f_L(\zeta(x)), L\cdot\varphi(x)\rangle\right] \det[\nabla\zeta(x)]\,\mathrm{dV} \tag{10.105}$$

$$- \int_{\partial\Omega_h} \langle N_L(\zeta(x), L\cdot\varphi(x)\rangle\, \|\,\mathrm{Cof}\,\nabla\zeta.e_3\|\, dS_h \mapsto \min.\quad \text{w.r.t. } \varphi.\,, \tag{10.106}$$

for a function $\varphi : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined implicitly through $\varphi_L(\xi) = \zeta(\varphi(\zeta^{-1}(\xi)))$. With $f(x) = L\cdot f_L(\zeta(x))$, $N(x) = N_L(\zeta(x))$ we have

$$\int_{x\in\Omega_h} \left[W_{3D}(\nabla\varphi) - \langle f,\varphi\rangle\right] L^3\,\mathrm{dV} - \int_{\partial\Omega_h} L\,\langle N,\varphi\rangle\, L^2\,\mathrm{dS} \mapsto \min.\quad \text{w.r.t. } \varphi.\,, \tag{10.107}$$

or equivalently

$$\int_{x\in\Omega_h} \left[W_{3D}(\nabla\varphi) - \langle f,\varphi\rangle\right] \mathrm{dV} - \int_{\partial\Omega_h} \langle N,\varphi\rangle\,\mathrm{dS} \mapsto \min.\quad \text{w.r.t. } \varphi.\,, \tag{10.108}$$

which shows how the scaling from a domain which is relatively thin to a domain which is absolutely thin is to be performed in order to apply the subsequent dimensional reduction procedure.

10.3.2 Scaling relations for finite Cosserat models with internal length

For completeness let us summarize the scaling relations appearing in a finite elastic Cosserat theory. Our goal is to relate the response of large and small samples of the same material and to asses the influence of the characteristic length L_c.

First, in our definition, the **characteristic length** L_c is a given material parameter, corresponding to the **smallest discernable distance** to be accounted for in the model. A simple consequence is that actual geometrical dimensions L of the bulk material must be larger than L_c, indeed for a continuum theory to apply L should be significantly larger than L_c.

Now let $\Omega_L = [0,L[\mathrm{m}]]\times[0,L[\mathrm{m}]\times[0,L[\mathrm{m}]]$ be the cube with (non-dimensional) edge length L, representing the bulk material. Consider a deformation $\varphi_L : \xi\in\Omega_L \mapsto \mathbb{R}^3$ and microrotation $\overline{R}_L(\xi) : \Omega_L \mapsto \mathrm{SO}(3,\mathbb{R})$ as solution of the simplified minimization problem

$$\int_{\xi\in\Omega_L} \mu\,\|\overline{R}_L^T(\xi)F(\xi) - 1\!\!1\|^2 + \mu\,L_c^q\,\|\mathrm{D}_\xi \overline{R}_L(\xi)\|^q \,\mathrm{d}\xi \mapsto \min.\,\text{w.r.t. } (\varphi_L,\overline{R}_L)\,. \tag{10.109}$$

The simple scaling transformation $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\zeta(x) = L\cdot x$ maps the unit cube $\Omega_1 = [0,1[\mathrm{m}]]\times[0,1[\mathrm{m}]]\times[0,1[\mathrm{m}]]$ into Ω_L. Defining the related deformation $\varphi : x\in\Omega_1 \mapsto \mathbb{R}^3$ and microrotation $\overline{R}(x) : \Omega_1 \mapsto \mathrm{SO}(3,\mathbb{R})$ as

$$\varphi(x) := \zeta^{-1}\left(\varphi_L(\zeta(x))\right), \quad \overline{R}(x) := \overline{R}_L(\zeta(x))\,, \tag{10.110}$$

shows

$$\nabla_x \varphi(x) = \frac{1}{L}\nabla_\xi \varphi_L(\zeta(x))\nabla_x\zeta(x) = \nabla_\xi\varphi_L(\xi)\,, \qquad \mathrm{D_x}\overline{R}(x) = \mathrm{D}_\xi \overline{R}_L(\zeta(x))\cdot\nabla_x\zeta(x) = \mathrm{D}_\xi\overline{R}_L(\xi)\cdot L\,. \tag{10.111}$$

Hence, the minimization problem can be transformed

$$\int_{\xi\in\Omega_L}\mu\,\|\overline{R}_L^T(\xi)\nabla_\xi\varphi_L(\xi) - 1\!\!1\|^2 + \mu\,L_c^q\,\|\mathrm{D}_\xi\overline{R}_L(\xi)\|^q\,\mathrm{d}\xi = \int_{x\in\Omega_1}\mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2\det[\nabla_x\zeta(x)] + \mu\,L_c^q\,\|\frac{1}{L}\mathrm{D_x}\overline{R}(x)\|^q\det[\nabla_x\zeta(x)]\,\mathrm{dx}$$

$$= \int_{x\in\Omega_1}\mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2\,L^3 + \mu\,L_c^q\,L^{3-q}\,\|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx}\,, \tag{10.112}$$

and we may consider at last the problem defined on the unit cube Ω_1:

$$\int_{x\in\Omega_1}\mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2 + \mu\,L_c^q\,L^{3-q-3}\,\|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx} \mapsto \ \min.\ \text{w.r.t.}\ (\varphi,\overline{R}). \tag{10.113}$$

Comparison of different sample sizes is afforded by transformation to the unit cube respectively, e.g. we compare two samples of the same material with bulk sizes $L_1 > L_2$. Transformation to the unit cube shows that the response of sample two is stiffer than the response of sample one.

It is plain to see that for L large compared to L_c, the influence of the rotations will be small and in the limit $\frac{L_c}{L}\to 0$, classical behaviour results. Otherwise, the larger $\frac{L_c}{L} < 1$, the more pronounced the Cosserat effects become and a small sample is relatively stiffer than a large one.

10.3.3 Scaling relations for finite Cosserat plates

As a consequence for relatively thin shells of the former development we consider the finite problem on the relative thin domain $\Omega_L^{\text{rel.thin}}$ in simplified form:

$$\int_{\xi\in\Omega_L^{\text{rel.thin}}}\mu\,\|\overline{R}_L^T(\xi)\nabla_\xi\varphi_L(\xi) - 1\!\!1\|^2 + \mu\,L_c^q\,\|D_\xi\overline{R}_L(\xi)\|^q\,\mathrm{d}\xi \mapsto \ \min.\ \text{w.r.t.}\ (\varphi_L,\overline{R}_L). \tag{10.114}$$

This implies on $\Omega_h = \omega\times[-\frac{h}{2},\frac{h}{2}]$ for the correspondingly transformed variables

$$\int_{x\in\Omega_h}\mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2 + \mu\,\frac{L_c^q}{L^q}\,\|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx} \mapsto \ \min.\ \text{w.r.t.}\ (\varphi,\overline{R}). \tag{10.115}$$

Inserting the reduced kinematics and integrating over the thickness we should consider on ω

$$\int_\omega \mu\,h\,\|\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1\|^2 + \frac{h^3}{12}\,\mu\|\overline{R}^T(\nabla\overline{R}_3|0)\|^2 + \mu\,h\,\frac{L_c^q}{L^q}\,\|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{d}\omega \mapsto \ \min.\ \text{w.r.t.}\ (m,\overline{R}). \tag{10.116}$$

Comparing domains with the **same relative thickness** $h>0$, but different in-plane elongation L, we see that the **smaller sample is relatively stiffer** for the same relative thickness.

For very large samples with the same relative thickness, the classical bending terms are retrieved.[36] In this sense, classical plate formulations represent the limit behaviour of ever larger, thin structures with the same relative thickness..

10.4 Generalized convexity conditions

For the convenience of the reader we collect some of the most useful convexity conditions. Let an elastic free energy density $W:\mathbb{M}^{n\times m}\mapsto\mathbb{R}$, $n\le m$ be given. We say that W considered in $\nabla\varphi = F\in\mathbb{M}^{n\times m}$ is

1. **uniformly stable**, if $D_F^2W(F).(H,H)\ge c^+\,\|H\|^2$, $H\in\mathbb{M}^{n\times m}$
2. **strictly Legendre elliptic**, if $D_F^2W(F).(H,H) > 0$, $\forall\,H\neq 0$
3. **pre-stable**, if $D_F^2W(x,F).(H,H)\ge c^+\,\|(H|0)^TG(x)+G(x)^T(H|0)\|^2$, $\forall\,(H|0)\in\mathbb{M}^{m\times m}$, $H\neq 0$ with $G\in\mathrm{GL}(m,\mathbb{R})$.
4. **polyconvex**, if there exists a convex function $P:\mathbb{M}^{n\times m}\times\mathbb{M}^{n\times m}\times\mathbb{R}\mapsto\mathbb{R}$ such that $W(F) = P(F,\mathrm{Minors}_{ij})$.
5. **quasiconvex**, if

$$\forall\,\hat{F}\in\mathbb{M}^{n\times m}\ :\ |D|\cdot W(\hat{F}) \le \int_D W(\hat{F}+\nabla\phi(x))\,\mathrm{dx}\quad\forall\,\phi\in C_0^\infty(D,\mathbb{R}^m)\,, \tag{10.117}$$

   which implies that the homogeneous deformation $\hat{F}$ is absolute minimizer to its own boundary conditions and excludes **internal failure**.
6. **uniformly Legendre-Hadamard elliptic**, if $D_F^2W(F).(\xi\otimes\eta,\xi\otimes\eta)\ge c^+\,\|\xi\|^2_{\mathbb{R}^m}\cdot\|\eta\|^2_{\mathbb{R}^n}$
7. **Legendre-Hadamard elliptic**, if $D_F^2W(F).(\xi\otimes\eta,\xi\otimes\eta)\ge 0$
8. **rank-one convex**, if $f(t):=W(F+t(\xi\otimes\eta))$ is convex in t for all $F\in\mathbb{M}^{n\times m}$.

It is known [Dac89] that

$$\textbf{convexity}\Rightarrow\textbf{polyconvexity}\Rightarrow\textbf{quasiconvexity}\Rightarrow\textbf{rank-one convexity}\Leftrightarrow\textbf{Legendre-Hadamard ellipticity}\,, \tag{10.118}$$

but the reverse implications are false in general. For the **scalar case** $\varphi(x_1,\ldots,x_n)\in\mathbb{R}$ and the **one dimensional case** $\varphi(x_1)\in\mathbb{R}^m$, all conditions coincide if correctly identified and simplify to the requirement of convexity of W.

[36] In plane rotations remain unspecified, they cannot be determined from $\overline{R}_3$ alone.

Definition 10.1 (Weak lower semicontinuity)
We say that a functional I defined on the Sobolev space $W^{1,p}(\Omega)$ is weakly lower semicontinuous, whenever $\varphi_k \rightharpoonup \varphi \in W^{1,p}(\Omega)$ implies

$$I(\varphi) \leq \liminf_k I(\varphi_k)\,. \tag{10.119}$$

If $I(\varphi) := \int_\Omega W(\nabla\varphi)\,\mathrm{dx}$, then weak lower semicontinuity is equivalent to quasiconvexity of W. This result is the cornerstone of the classical direct methods of variations.

10.5 Macroscopic elastic shear failure for plates

It is convenient to define what we mean by **shear failure** for plates in classical isotropic elasticity. Let $W((\nabla m|\vec{n})) = \hat{W}(\nabla m^T \nabla m)$ be the free elastic energy density of the membrane (intrinsic) part of the plate defined on the **first fundamental form** of the surface $\nabla m^T\nabla m = I_m \in \mathrm{Sym}$. If for some regular $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ with $(\nabla m|\vec{n}_m) \in \mathrm{GL}^+(3,\mathbb{R})$

$$\exists\, \xi,\eta \in \mathbb{R}^3: \quad D_F^2 W((\nabla m|\vec{n})).(\xi\otimes\eta, \xi\otimes\eta) < 0\,, \tag{10.120}$$

we say that the material fails or looses **Legendre-Hadamard ellipticity (LH)**, also called a **material instability**.[37] This failure can give rise to highly localized deformation patterns, subsumed under the notion of **microstructure**. Related is the possible emergence of discontinuous deformations since **Hadamard´s jump relations** are violated. However, loss of ellipticity may already occur for midsurface deformations which are not related to in-plane shear, e.g. uniaxial situations and pure in plane dilations. Thus we say that W suffers from **genuine elastic shear failure** whenever

$$\begin{aligned} &\exists\, F \in \mathrm{GL}^+(3,\mathbb{R}) \quad \exists\, \xi,\eta\in\mathbb{R}^3: \quad D^2W(F).(\xi\otimes\eta,\xi\otimes\eta) < 0\,, \quad \text{but} \\ &\forall\, F \in \mathrm{diag}(\lambda_1^+,\lambda_2^+,1) \quad \forall\, \xi,\eta\in\mathbb{R}^3: \quad D^2W(F).(\xi\otimes\eta,\xi\otimes\eta) \geq 0\,. \end{aligned} \tag{10.121}$$

It seems that failure of a material on a macroscale other than shear failure is unphysical and rather due to the idiosyncrasy of the constitutive equations, as long as the bulk is modelled as elastic. In fact, Legendre-Hadamard ellipticity for $F = \mathrm{diag}(\lambda_1^+,\lambda_2^+,1)$ of the membrane energy implies immediately the **Baker-Ericksen (BE)** inequalities [MH83, p.19] for the membrane and genuine elastic shear failure happens, if BE is satisfied but LH is violated.[38]

In this sense the following non exhaustive list of free energy terms should be avoided for the membrane since they are not only failing under shear (already BE is not satisfied): with $\overline{C} = (\nabla m|\vec{n})^T(\nabla m|\vec{n})$, $\overline{U} = \sqrt{\overline{C}}$, $F = (\nabla m|\vec{n})$ the list reads

$$\|\overline{C}-1\!\!1\|^2,\ \langle\overline{C}-1\!\!1,1\!\!1\rangle^2,\ \langle\ln\overline{C},1\!\!1\rangle^2,\ \langle\ln\overline{C},1\!\!1\rangle^2 + \|\mathrm{dev}\ln\overline{C}\|^2,\ \langle\ln\overline{U},1\!\!1\rangle^2,\ -\ln\det[F] + (\ln\det[F])^2,\ \|\frac{\overline{C}}{\det[\overline{C}]^{1/3}} - 1\!\!1\|^2\,, \tag{10.122}$$

and it is obvious that the membrane part of the finite Kirchhoff-Love plate model (7.83) and the finite Reissner-Mindlin model (6.82) is failing, not only in shear! Of course, combination with other terms could remove the problem. Terms which genuinely fail only in shear are e.g.

$$\|\overline{U}-1\!\!1\|^2,\ \langle\overline{U}-1\!\!1,1\!\!1\rangle^2,\ \|\frac{\overline{U}}{\det[\overline{U}]^{1/3}} - 1\!\!1\|^2,\ \mathrm{tr}\left[\frac{\overline{U}}{\det[\overline{U}]^{1/3}} - 1\!\!1_2\right]^2. \tag{10.123}$$

10.6 Linearized plate models

10.6.1 The classical infinitesimal Reissner-Mindlin model

Let us linearize a modification of case II ($\mu_c = 0$, $\alpha_4 = 0$, $q = 0$, $p > 1$) for situations of small midsurface deformations and small curvature. We write $m(x,y) = (x,y,0)^T + v(x,y)$, with the displacement of the midsurface of the plate $v : \omega \mapsto \mathbb{R}^3$ and $\overline{R} = 1\!\!1 + \overline{A} + \ldots$ with $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ the infinitesimal microrotation. For the boundary deformation we write $g_{\mathrm{d}}(x,y,z) = (x,y,z)^T + u^{\mathrm{d}}(x,y,z)$, with the consequence, that $\nabla g_{\mathrm{d}}.e_3 = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 1 + u^{\mathrm{d}}_{3,z})$. The curvature tensors are expanded as

$$\begin{aligned} \mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0) = (1\!\!1 + \overline{A} + \ldots)^T\,(\nabla[\overline{A}_3 + \overline{A}^2.e_3 + \ldots]|0) \approx (\nabla\overline{A}_3|0) + \ldots \\ \mathfrak{K}_s &\approx \big((\nabla(\overline{A}.e_1)|0), (\nabla(\overline{A}.e_2)|0), (\nabla(\overline{A}.e_3)|0)\big)\,, \end{aligned} \tag{10.124}$$

and the Cosserat micropolar plate stretch tensor expands like

$$\overline{U} = \overline{R}^T F_s = \overline{R}^T(\nabla m|\overline{R}_3) = (1\!\!1 + \overline{A} + \ldots)^T \left(\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} + \nabla v|(1\!\!1 + \overline{A} + \ldots).e_3)\right) \approx 1\!\!1 + (\nabla v|\overline{A}_3) + \,. \tag{10.125}$$

Since $p > 1$, the Cosserat curvature contribution has an exponent strictly bigger than two such that a linearization w.r.t zero curvature does not yield any contribution of this term. Moreover, for $\mu_c = 0$, in-plane rotations (drilling degrees of freedom) do

[37] Material instability should be carefully distinguished from **geometrical instabilities** occurring in buckling or necking and which are fully consistent with Legendre-Hadamard ellipticity. In this sense, **polyconvex** materials are **unconditionally** materially stable and certainly appropriate for rubber and soft-tissues [SN02, HN03].

[38] One version of the **BE**-inequalities for membranes can be stated as follows: for $\lambda_i^2 \geq 0$, $i = 1,2$, $\lambda_3^2 = 1$ the (generalized) principal stretches (here λ_i^2 are the eigenvalues of $(\nabla m|\vec{n})^T(\nabla m|\vec{n})$), the free energy $\Phi(\lambda_1,\lambda_2,1) := \hat{W}(\nabla m^T\nabla m)$ is separately convex in λ_i. No mathematical existence results based only on BE are known. Note also that BE is enough to effectively exclude phase-transformations, modelled with multi-well potentials.

not survive the linearization process! We are indeed left with the minimization problem for $v \in \mathbb{R}^3$ and $A_3 \in \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h\, \left(\mu\, \|\operatorname{sym}((\nabla v|\overline{A}_3))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}((\nabla v|\overline{A}_3))\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu\,\|\operatorname{sym}((\nabla \overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}((\nabla \overline{A}_3|0))\right]^2\right)\, d\omega - \Pi(v,\overline{A}_3) \mapsto \min.\ \text{w.r.t.}\ (v,\overline{A}_3), \\
v_{|\gamma_0} &= u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad (10.126)\\
\overline{A}_{|\gamma_0} &= \operatorname{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,, \quad \text{linearized consistent coupling} \Rightarrow \overline{A}_{3|\gamma_0} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T \\
\overline{A}_{3|\gamma_0} &= (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription}\,.
\end{aligned}$$

Abbreviating now $\theta = (\theta_1, \theta_2, 0)^T = -\overline{A}_3$, we are left with the following set of equations for the displacement of the midsurface of the plate $v : [0,T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the infinitesimal increment of the 'normal', $\theta : \omega \mapsto \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h \left(\mu\,\|\operatorname{sym}\nabla(v_1,v_2)\|^2 + \underbrace{\frac{\mu}{2}\,\|\nabla v_3 - \theta\|^2}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla(v_1,v_2)\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu\,\|\operatorname{sym}\nabla\theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla\theta\right]^2\right) d\omega - \Pi(v,-\theta) \mapsto \min.\ \text{w.r.t.}\ (v,\theta), \\
v_{|\gamma_0} &= u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad (10.127)\\
-\theta_{|\gamma_0} &= \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T, \quad \text{linearized consistent coupling} \\
-\theta_{|\gamma_0} &= (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription}\,.
\end{aligned}$$

A further reduction arises if we assume only normal displacements: $v_1 = v_2 = 0$. The resulting minimization problem is

$$\int_\omega h\,\frac{\mu}{2}\,\|\nabla v_3 - \theta\|^2 + \frac{h^3}{12}\left(\mu\,\|\operatorname{sym}\nabla\theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla\theta\right]^2\right) d\omega - \Pi(v_3 \cdot e_3, -\theta) \mapsto \min.\ \text{w.r.t.}\ (v_3,\theta), \qquad (10.128)$$

$$\begin{aligned}
v_{3|\gamma_0} &= u^{\mathrm{d}}_3\,, \qquad \text{simply supported} \\
-\theta_{|\gamma_0} &= \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T \qquad \text{linearized coupling}\,, \qquad -\theta_{|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \quad \text{rigid}\,.
\end{aligned}$$

The elastic free energy should be compared with

$$W_{\mathrm{RM,class}}(\nabla v_3, \theta) = h\,\frac{\kappa\,\mu}{2}\,\|\nabla v_3 - \theta\|^2 + \frac{h^3}{12}\left(\frac{\mu}{4}\,\|\nabla\theta^T + \nabla\theta\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\operatorname{tr}\left[\nabla\theta^T + \nabla\theta\right]^2\right), \qquad (10.129)$$

where $\kappa = \frac{5}{6}$ is the so called **shear correction factor**. In this last form, the Reissner-Mindlin problem can be found in many textbooks, e.g. [Bra92, p.281] or [Ste95]. It should be noted, however, that in our variationally based finite derivation with subsequent linearization there is no imminent reason to introduce $\kappa \neq 1$. In fact, the shear correction factor κ can be seen as a tuning parameter of the infinitesimal model which, for certain types of loading,[39] allows to **improve** the **order of convergence** of the infinitesimal Reissner-Mindlin solution to the three-dimensional linear elasticity solution [Rös99].[40]

10.6.2 The classical infinitesimal Kirchhoff-Love plate (Koiter model)

For the convenience of the reader we also supply the similar system of equations for the classical infinitesimal Kirchhoff-Love plate (also the Koiter model) which we derive as linearization of the finite Kirchhoff-Love plate. In terms of the midsurface displacement v we have to find a solution of the minimization problem for $v \in \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h\,\left(\mu\,\|\operatorname{sym}\nabla(v_1,v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla(v_1,v_2)\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu\,\|D^2 v_3\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[D^2 v_3\right]^2\right) d\omega - \Pi(v, -\nabla v_3) \mapsto \min.\ \text{w.r.t.}\ v, \\
v_{|\gamma_0} &= u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad (10.130)\\
-\nabla v_{3|\gamma_0} &= \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T, \quad \text{linearized consistent coupling} \Rightarrow \nabla v_3 = -1/2\,(u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \\
-\nabla v_{3|\gamma_0} &= (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription, linearized Kirchhoff}\,.
\end{aligned}$$

This energy can be obtained formally from (10.129) by setting $\theta = \nabla v_3$.

[39] Hence the shear correction factor κ shows some similarity to the Cosserat couple modulus μ_c, whose influence on the solution of the three-dimensional problem is also strongly dependent on boundary conditions. For rather thick plates, it is known that the shear energy in RM_{lin} is overestimated, therefore, one is led to reduce the shear energy contribution a posteriori by taking $\kappa < 1$.

[40] It would be interesting to know the optimal shear correction factor $0 < \kappa \le 1$ of the infinitesimal Reissner-Mindlin model with our reduced consistent coupling boundary condition. Such an optimized parameter should also be beneficial for the finite Cosserat plate!

Address:
Patrizio Neff
AG6, Fachbereich Mathematik
Darmstadt University of Technology
Schlossgartenstrasse 7
64289 Darmstadt, Germany
email: neff@mathematik.tu-darmstadt.de

11 Appendix B

11.1 Prerequisites from differential geometry

A given mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, describing a surface imbedded in the three-dimensional space is called **regular** whenever $\operatorname{rank}(\nabla m) = 2$. The vector

$$\vec{n} := \frac{m_x \times m_y}{\|m_x \times m_y\|}\,, \tag{11.131}$$

is the **Gauss unit normal field** on the surface. The map $n : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$ is called the **Gauss map** and the moving 3-frame $(m_x|m_y|n)$ is called the **Gauss frame** of the surface m which in general is not orthonormal. The matrix representation of the **first fundamental form (metric)** is given through

$$I_m := \nabla m^T\, \nabla m = \begin{pmatrix} \|m_x\|^2 & \langle m_x, m_y\rangle \\ \langle m_x, m_y\rangle & \|m_y\|^2 \end{pmatrix} \in \mathbb{M}^{2\times 2}, \quad I_m + e_3 \otimes e_3 = (\nabla m|n)^T(\nabla m|n) =: \widehat{I}_m \tag{11.132}$$
$$\det[I_m] = \det[\widehat{I}_m] = \det[(\nabla m|\vec{n})]^2\,.$$

The metric alone is not sufficient to describe the shape of a surface in the ambient three-dimensional Euclidean space, the curvature is also needed, although in the **rigid** case $(\nabla m|\vec{n}) \in \mathrm{SO}(3,\mathbb{R})$, the metric is indeed enough.

The matrix representation of the **second fundamental form** providing a measure for curvature of the surface is given by

$$II_m := -\nabla m^T\, Dn = -(m_x|m_y)^T \cdot (n_x|n_y) = -\begin{pmatrix} \langle m_x, D_x n\rangle & \langle m_x, D_y n\rangle \\ \langle m_y, D_x n\rangle & \langle m_y, D_y n\rangle \end{pmatrix} \in \mathbb{M}^{2\times 2} \tag{11.133}$$
$$(\nabla m|n)^T(D_x n|D_y n|0) = \begin{pmatrix} \langle m_x, D_x n\rangle & \langle m_x, D_y n\rangle & 0 \\ \langle m_y, D_x n\rangle & \langle m_y, D_y n\rangle & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \widehat{II}_m := -(\nabla m|\vec{n})^T(\nabla \vec{n}|\vec{n})\,, \quad \det[II_m] = \det[\widehat{II}_m]\,.$$

Since n is orthogonal to the tangent space $T_x m$, the relation $0 = \partial_x\langle m_y, \vec{n}\rangle = \partial_y\langle m_x, \vec{n}\rangle$ shows easily that II_m is **symmetric**. The **third fundamental form** of the surface in matrix representation is defined as

$$III_m := Dn^T\, Dn = \begin{pmatrix} \|D_x n\|^2 & \langle D_x n, D_y n\rangle \\ \langle D_y n, D_x n\rangle & \|D_y n\|^2 \end{pmatrix} \in \mathbb{M}^{2\times 2}\,, \quad \widehat{III}_m := (\nabla \vec{n}|\vec{n})^T(\nabla \vec{n}|\vec{n})\,. \tag{11.134}$$

The matrix representation of the **Weingarten map** (shape operator) L is given by

$$L(x,y) := -Dn(x,y)\, \nabla_\xi m^{-1}(m(x,y) \in \mathbb{M}^{3\times 3}, \quad L = -(\nabla \vec{n}|0)(\nabla m|n)^{-1}\,, \tag{11.135}$$

representing the variation of the normal in the metric of the surface. In order to see that $L = -(Dn|0)(\nabla m|n)^{-1}$ we extent m to $\mathbb{R}^3$ by setting $\Theta(x,y,z) = m(x,y) + z\, n(x,y)$. This yields $\Theta(x,y,0) = m(x,y)$ and $\nabla\Theta(x,y,0) = (\nabla m|n)$ while $\Theta^{-1}(\Theta(x,y,z)) = (x,y,z)^T$ and the chain rule shows $\nabla_\xi \Theta^{-1}(\Theta(x,y,z)]\nabla\Theta(x,y,z) = 1\!\!1$. Hence

$$\nabla_\xi \Theta^{-1}(\Theta(x,y,0))\nabla\Theta(x,y,0) = 1\!\!1$$

but $\nabla_\xi \Theta^{-1}(\Theta(x,y,0)) = \nabla_\xi m^{-1}(m(x,y))$ which finishes the argument. The **Gauss curvature** K of the surface is determined by

$$K(x,y) := \frac{\det[II_m]}{\det[I_m]} = \det[L] = \det[Dn\, \nabla_\xi m^{-1}(\xi)]\,, \tag{11.136}$$

and the **mean curvature** H through

$$2\, H(x,y) := \operatorname{tr}[L] = \operatorname{tr}\left[Dn\, \nabla_\xi m^{-1}(\xi)\right]\,. \tag{11.137}$$

The relation $III_m - 2H\, II_m + K\, I_m = 0$ ([Kli78, Prop. 3.5.6]) is a consequence of the Caley-Hamilton theorem and shows that III_m is not independent of I_m, II_m. The principal curvatures κ_1, κ_2 are the solutions of the characteristic equation of $-L$, i.e. $\kappa^2 - \operatorname{tr}[L]\kappa + \det[L] = \kappa^2 - 2H\kappa + K = 0$. The Caley-Hamilton theorem on $\mathbb{M}^{2\times 2}$ implies for the second fundamental form on account of its symmetry

$$II_m^2 - \operatorname{tr}[II_m]\, II_m + \det[II_m]\, 1\!\!1_2 = 0 \Rightarrow \|II_m\|^2 - \operatorname{tr}[II_m]^2 + 2\det[II_m] = 0 \quad \Rightarrow \operatorname{tr}[II_m]^2 - \|II_m\|^2 = 2\det[II_m]\,. \tag{11.138}$$

Thus the Gauss curvature K can be expressed equivalently as

$$K = \frac{\operatorname{tr}[II_m]^2 - \|II_m\|^2}{2\det[I_m]}\,. \tag{11.139}$$

Of major importance is the following classification

Definition 11.1 (Intrinsicality)
A property or a set of equations is **intrinsic** *whenever it can be reduced to the first fundamental form, i.e. depends only on local length and local angles on the surface. (Or the change of local length and local angles.) Intrinsic properties remain invariant under isometries.*

For example, the mean curvature H is not intrinsic, since bending a surface changes H but leaves length and angles invariant (bending belongs to the outer geometry of the surface); or take the normal of the surface n: this is not an intrinsic quantity, since bending changes the normals but leaves length and angles invariant.

Gauss' celebrated **Theorema Egregium** states that contrary to appearance (it involves the normals!), the Gauss curvature K is an intrinsic quantity: it can be computed through the first and second derivatives of the first fundamental form. The same is trivially true for $\|(\nabla m|\vec{n})^T(\nabla m|\vec{n}) - 1\!\!1\|^2$- it is a purely intrinsic strain measure (the dependence on $\vec{n}$ cancels out algebraically).

In the thin shell limit of $h \to 0$ it is expedient to get a model which is purely two-dimensional, i.e. intrinsic.

Theorem 11.2 (Fundamental theorem of surface theory)
Any two surfaces $m, \tilde{m} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, which have the same first and second fundamental form, differ only by an **isometry**, *i.e. $\tilde{m}(x,y) = Q.m(x,y),\ Q \in \mathrm{SO}(3)$.*

Proof. Well known in differential geometry, e.g.[Kli78, p.64]. ■

Lemma 11.3 (Developable surfaces)
A surface m with no planar points ($II_m \neq 0$) is developable (on the plane, without stretch) if and only if the Gauss curvature K vanishes.

Proof. Theorem 3.7.9 in [Kli78]. ■

Lemma 11.4 (Isometric surfaces)
Two surfaces with different Gauss curvature K cannot be mapped isometrically into each other.

Proof. Well known. ■

The following classification is standard. The surface m is locally

$$\begin{cases} \textbf{elliptic} \\ \textbf{parabolic} \\ \textbf{hyperbolic} \end{cases} \quad \text{at } (x,y) \in \omega \text{ if } \det[II_m(x,y)] \text{ is } \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} . \tag{11.140}$$

The surface m is locally **uniformly elliptic** if

$$\exists\, c^+ > 0 : \forall\, \xi \in \mathbb{R}^2 : \quad \langle II_m.\xi, \xi\rangle_{\mathbb{R}^2} = \langle \nabla m^T Dn.\xi, \xi\rangle_{\mathbb{R}^2} = \langle Dn.\xi, \nabla m.\xi\rangle_{\mathbb{R}^3} \geq c^+ \|\xi\|^2 . \tag{11.141}$$

Definition 11.5 (Christoffel symbols)
Let the regular surface m be given. The Christoffel symbols of the first kind of the surface are defined by

$$\Gamma^i_{jk} := \langle \partial_j a_k, a_i\rangle, \quad a_1 = m_x, a_2 = m_y, a_3 = n,\ j = 1,2,\ i,k = 1,2,3 . \tag{11.142}$$

They are **not independent of the choice of coordinates (not covariant), but intrinsic quantities**, *belonging to the inner geometry of the surface, see [Lau60, p.36].*

Let us look at $\|\nabla_x[\varrho^2]\|^2$. It is clear that this defines an intrinsic quantity, since it can be expressed as partial derivatives of the metric. We have

$$\nabla_x\|\nabla m\|^2 = \nabla_x\langle \nabla m^T\nabla m, 1\!\!1_2\rangle = \nabla_x(\|m_x\|^2 + \|m_y\|^2) = 2\begin{pmatrix} \langle m_x, m_{xx}\rangle + \langle m_y, m_{yx}\rangle \\ \langle m_x, m_{xy}\rangle + \langle m_y, m_{yy}\rangle \end{pmatrix} = 2\begin{pmatrix} \Gamma^1_{11} + \Gamma^2_{12} \\ \Gamma^1_{21} + \Gamma^2_{22} \end{pmatrix} . \tag{11.143}$$

Hence, $\|\nabla_x[\varrho^2]\|^2 = 4\,(\Gamma^1_{11} + \Gamma^2_{12})^2 + 4\,(\Gamma^1_{21} + \Gamma^2_{22})^2$.

11.2 Additional material

Lemma 11.6 (Normality and polar decomposition)
Let $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ be regular and assume for some $R \in \mathrm{SO}(3)$ that $R = \mathrm{polar}(\nabla m|\,\varrho\, R.e_3)$, where $\varrho > 0$ is given. Then $R_3 = \vec{n}_m$ and $R = \mathrm{polar}(\nabla m|\varrho\,\vec{n})$.

Proof. Since $F = R\,U$ we must have $F^T R \in \mathrm{Sym}$. But

$$U = U^T = F^T R = (\nabla m|\,\varrho\, R_3)^T \cdot \mathrm{polar}(\nabla m|\,\varrho\, R_3) = (\nabla m|\,\varrho\, R_3)^T \cdot R = \begin{pmatrix} \langle m_x, R_1\rangle & \langle m_x, R_2\rangle & \langle m_x, \varrho\, R_3\rangle \\ \langle m_y, R_1\rangle & \langle m_y, R_2\rangle & \langle m_y, \varrho\, R_3\rangle \\ 0 & 0 & \varrho \end{pmatrix} , \tag{11.144}$$

which implies $\varrho\,\langle m_y, R_3\rangle = \varrho\,\langle m_x, R_3\rangle = 0$ by symmetry of U. Thus R_3 coincides with the unit normal $\vec{n}_m$ on m. ■

For $m = (x,y,0)^T + v(x,y)$ we have

$$(\nabla v|\overline{A}_3) = \begin{pmatrix} v_{1,x} & v_{1,y} & \beta \\ v_{2,x} & v_{2,y} & \gamma \\ v_{3,x} & v_{3,y} & 0 \end{pmatrix} , \quad \mathrm{sym}((\nabla v|\overline{A}_3)) = \begin{pmatrix} v_{1,x} & \frac{v_{1,y}+v_{2,x}}{2} & \frac{\beta+v_{3,x}}{2} \\ \frac{v_{1,y}+v_{2,x}}{2} & v_{2,y} & \frac{\gamma+v_{3,y}}{2} \\ \frac{\beta+v_{3,x}}{2} & \frac{\gamma+v_{3,y}}{2} & 0 \end{pmatrix} \tag{11.145}$$

$$\|\mathrm{sym}((\nabla v|\overline{A}_3))\|^2 = \|\mathrm{sym}_2(\nabla(v_1,v_2))\|^2 + \|\nabla v_3 - (-\overline{A}_3)\|^2 , \quad \mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3))\right]^2 = \mathrm{tr}\left[\mathrm{sym}(\nabla v|0)\right]^2 .$$

Lemma 11.7 (Rigidity coincidence II)
If $(\nabla m|n) \in \mathrm{SO}(3)$ then

$$\begin{aligned} \|II_m\|^2 &= \mathrm{tr}\,[III_m] = \|D_x n\|^2 + \|D_y n\|^2 \\ \mathrm{tr}\,[II_m]^2 &= (\|m_{xx}\| + \|m_{yy}\|)^2 \\ \|II_m\|^2 &= \|m_{xx}\|^2 + 2\|m_{xy}\|^2 + \|m_{yy}\|^2 . \end{aligned} \tag{11.146}$$

Proof.

$$(\nabla m|n)^T \cdot (D_x n|D_y n|0) = \begin{pmatrix} \langle m_x, D_x n\rangle & \langle m_x, D_y n\rangle & 0 \\ \langle m_y, D_x n\rangle & \langle m_y, D_y n\rangle & 0 \\ \underbrace{\langle n, D_x n\rangle}_{=0} & \underbrace{\langle n, D_y n\rangle}_{=0} & 0 \end{pmatrix}. \tag{11.147}$$

Therefore

$$\|II_m\|^2 = \|(\nabla m|n)^T (D_x n|D_y n|0)\|^2 = \|(D_x n|D_y n|0)\|^2 = \|D_x n\|^2 + \|D_y n\|^2 = \operatorname{tr}[III_m], \tag{11.148}$$

which finishes the first part. Now $\operatorname{tr}[II_m]^2 = (\langle m_x, n_x\rangle + \langle m_y, n_y\rangle)^2$. Using $0 = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_x, n\rangle = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_y, n\rangle$ we have $\operatorname{tr}[II_m]^2 = (\langle m_{xx}, n\rangle + \langle m_{yy}, n\rangle)^2$. But if $(\nabla m|n) \in \mathrm{SO}(3,\mathbb{R})$ we get in addition, on account of

$$0 = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_x, m_x\rangle, \quad 0 = \frac{\mathrm{d}}{\mathrm{dy}}\langle m_y, m_y\rangle, \quad 0 = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_x, m_y\rangle, \tag{11.149}$$

that $\langle m_{xx}, m_x\rangle = \langle m_{xx}, m_y\rangle = 0$ and $\langle m_{yy}, m_x\rangle = \langle m_{yy}, m_y\rangle = 0$ which implies that $m_{xx} = \|m_{xx}\|\, n$ and $m_{yy} = \|m_{yy}\|\, n$. The same reasoning implies altogether

$$(\nabla m|n)^T \cdot (D_x n|D_y n|0) = \begin{pmatrix} -\|m_{xx}\| & -\|m_{xy}\| & 0 \\ -\|m_{yx}\| & -\|m_{yy}\| & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.150}$$

■

Corollary 11.8 (Rigidity coincidence III)
If $\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1 = 0$ then $R = (\nabla m|\vec{n}_m) \in \mathrm{SO}(3)$ and $R_{1,y} = R_{2,x}$ together with

$$\overline{R}^T(\nabla \overline{R}_3|0) = II_m = \begin{pmatrix} -\|\overline{R}_{1,x}\| & -\|\overline{R}_{1,y}\| & 0 \\ -\|\overline{R}_{2,x}\| & -\|\overline{R}_{2,y}\| & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.151}$$

Moreover,

$$\|\operatorname{sym}\begin{pmatrix} -\|\overline{R}_{1,x}\| & -\|\overline{R}_{1,y}\| & 0 \\ -\|\overline{R}_{2,x}\| & -\|\overline{R}_{2,y}\| & 0 \\ 0 & 0 & 0 \end{pmatrix}\|^2 \geq \|\nabla \overline{R}_1\|^2 + \|\nabla \overline{R}_2\|^2. \tag{11.152}$$

Proof. The first part is a consequence of Lemma 11.7. The second part is an algebraic computation. ■

11.3 Linearized quantities

At various places we are interested in the linearization of the proposed systems with respect to the reference plane. Let therefore $m(x,y) = (x,y,0)^T + (v_1, v_2, v_3)^T$ and $g_{\mathrm{d}} = (x,y,z)^T + u^{\mathrm{d}}(x,y,z)^T$. Then upon expanding to first order

$$\begin{aligned}
\nabla m &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{1,x} & v_{1,y} \\ v_{2,x} & v_{2,y} \\ v_{3,x} & v_{3,y} \end{pmatrix} \\
m_x \times m_y &= e_3 + e_1 \times v_y + v_x \times e_2 + v_x \times v_y \\
n &\approx e_3 + (-v_{3,x}, -v_{3,y}, 0)^T + o(\|\nabla v\|) \\
\nabla g_{\mathrm{d}} &= 1\!\!1 + \nabla u^{\mathrm{d}}, \quad \nabla g_{\mathrm{d}}.e_3 = e_3 + \nabla u^{\mathrm{d}}.e_3 = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 1 + u^{\mathrm{d}}_{3,z})^T \\
\frac{\nabla g_{\mathrm{d}}.e_3}{\|\nabla g_{\mathrm{d}}.e_3\|} &= \nabla g_{\mathrm{d}}.e_3 \cdot \|\nabla g_{\mathrm{d}}.e_3\|^{-1} = (e_3 + \nabla u^{\mathrm{d}}.e_3) \cdot \|e_3 + \nabla u^{\mathrm{d}}.e_3\|^{-1} \\
&\approx (e_3 + \nabla u^{\mathrm{d}}.e_3)\left[\|e_3\|^{-1} + (-1)\|e_3\|^{-2}\langle \frac{e_3}{\|e_3\|}, \nabla u^{\mathrm{d}}.e_3\rangle + \dots\right] \\
&\approx (e_3 + \nabla u^{\mathrm{d}}.e_3)\left[1 - \langle e_3, \nabla u^{\mathrm{d}}.e_3\rangle\right] \approx e_3 + \nabla u^{\mathrm{d}}.e_3 - u^{\mathrm{d}}_{3,z} = e_3 + (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \\
(\nabla m|n) &\approx 1\!\!1 + \begin{pmatrix} v_{1,x} & v_{1,y} & -v_{3,x} \\ v_{2,x} & v_{2,y} & -v_{3,y} \\ v_{3,x} & v_{3,y} & 0 \end{pmatrix} + o(\|\nabla v\|) \\
(\nabla m|n)^T(\nabla m|n) - 1\!\!1 &\approx \begin{pmatrix} 2v_{1,x} & v_{1,y} + v_{2,x} & 0 \\ v_{2,x} + v_{1,y} & 2v_{2,y} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(\|\nabla v\|) \\
\|\nabla m\|^2 &\approx 2 + 2(v_{1,x} + v_{2,y}) + \|\nabla v\|^2 = 2 + 2\operatorname{Div} v + \|\nabla v\|^2 \\
\operatorname{polar}((\nabla m|n)) &\approx 1\!\!1 + \operatorname{skew}\begin{pmatrix} v_{1,x} & v_{1,y} & -v_{3,x} \\ v_{2,x} & v_{2,y} & -v_{3,y} \\ v_{3,x} & v_{3,y} & 0 \end{pmatrix} + o(\|\nabla v\|) \\
U((\nabla m|n)) &\approx 1\!\!1 + \frac{1}{2}\begin{pmatrix} 2v_{1,x} & v_{1,y} + v_{2,x} & 0 \\ v_{2,x} + v_{1,y} & 2v_{2,y} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(\|\nabla v\|) \\
(Dn|0) &\approx \begin{pmatrix} -v_{3,xx} & -v_{3,xy} & 0 \\ -v_{3,yx} & -v_{3,yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(\|\nabla v\|) = -D^2 v_3 + o(\|\nabla v\|)
\end{aligned} \tag{11.153}$$

$$(\nabla m|n)^T(Dn|0) \approx -D^2 v_3$$
$$\operatorname{tr}\left[(\nabla m|n)^T(Dn|0)\right]^2 \approx \operatorname{tr}\left[D^2 v_3\right]^2$$
$$\|(\nabla m|n)^T(Dn|0)\|^2 \approx \|D^2 v_3\|^2$$
$$\varrho_m^2(\nabla m) \approx 1 - \frac{2\lambda}{2\mu+\lambda}\operatorname{Div} v - \frac{\lambda}{2\mu+\lambda}\|\nabla v\|^2$$
$$\nabla_x[\varrho_m^2(\nabla m)] \approx -\frac{2\lambda}{2\mu+\lambda}\begin{pmatrix} v_{1,xx}+v_{2,yx} \\ v_{1,xy}+v_{2,yy}\end{pmatrix}$$
$$\|\nabla_x[\varrho_m^2(\nabla m)]\|^2 \approx \frac{4\lambda^2}{(2\mu+\lambda)^2}\left((v_{1,xx}+v_{2,yx})^2+(v_{1,xy}+v_{2,yy})^2\right)$$
$$\operatorname{Curl}(\nabla m|n) \approx \begin{pmatrix} -v_{3,xy} & v_{3,xx} & 0 \\ -v_{3,yy} & v_{3,yx} & 0 \\ 0 & 0 & 0\end{pmatrix}, \quad \|\operatorname{Curl}(\nabla m|n)\|^2 \approx v_{3,xx}^2 + 2v_{3,xy}^2 + v_{3,yy}^2$$
$$\overline{R} = 1\!\!1 + \overline{A} + \frac{1}{2}\overline{A}^2 + \ldots, \quad \overline{R}.e_3 \approx e_3 + \overline{A}.e_3$$

11.4 Detailed derivations

11.4.1 Detailed computations for the new Cosserat model

The equilibrium equations of the three-dimensional Cosserat problem given in [Nef03a] show that on the transverse boundary (exact)

$$\begin{aligned} S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= N^{\text{trans}}(x,y,+h/2) \\ S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= N^{\text{trans}}(x,y,-h/2)\,, \end{aligned} \tag{11.154}$$

where N^{trans} are the prescribed tractions N on the transverse boundary given globally in the basis (e_1,e_2,e_3). This implies (exact)

$$\begin{aligned} \overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= \overline{R}^{3d}(x,y,+h/2))^T N^{\text{trans}}(x,y,+h/2) \\ \overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= \overline{R}^{3d}(x,y,-h/2))^T N^{\text{trans}}(x,y,-h/2)\,. \end{aligned} \tag{11.155}$$

Abbreviate

$$N^+ := N^{\text{trans}}(x,y,+h/2), \quad N^- := N^{\text{trans}}(x,y,-h/2)\,, \tag{11.156}$$

and define

$$N_{\text{res}} := N^{\text{trans}}(x,y,+h/2) + N^{\text{trans}}(x,y,-h/2)\,,\ N_{\text{diff}} := \frac{1}{2}\left[N^{\text{trans}}(x,y,+h/2) - N^{\text{trans}}(x,y,-h/2)\right]\,. \tag{11.157}$$

Then also (exact)

$$\begin{aligned} \langle \overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3, e_3\rangle &= \langle N^+, \overline{R}^{3d}(x,y,+h/2)).e_3\rangle \\ \langle \overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).e_3, e_3\rangle &= -\langle N^-, \overline{R}^{3d}(x,y,-h/2)).e_3\rangle\,. \end{aligned} \tag{11.158}$$

We determine ϱ_m, ϱ_b from the corresponding requirement in terms of the assumed kinematics $(\varphi_s, \overline{R}_s)$, yielding

$$\begin{aligned} \langle \overline{R}_s^T(x,y,\pm h/2) S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}_s).e_3, e_3\rangle &= \pm\langle N^{\text{trans}}(x,y,\pm h/2), \overline{R}_s(x,y,\pm h/2).e_3\rangle \quad \Rightarrow \\ \langle \overline{R}^T S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}).e_3, e_3\rangle &= \pm\langle N^{\text{trans}}(x,y,\pm h/2), \overline{R}.e_3\rangle\,, \end{aligned} \tag{11.159}$$

which condition reduces to **zero normal tractions on the transverse free boundary** (in the absence of tractions N^{trans}) in the classical continuum limit of $\overline{R} = \operatorname{polar}(\nabla\varphi)$. We compute

$$\begin{aligned} \overline{R}^T F_s &= \overline{R}^T\left[(\nabla m|\varrho_m\overline{R}_3) + z(\nabla(\varrho_m\,\overline{R}_3)|\varrho_b\,\overline{R}_3) + \frac{z^2}{2}(\nabla(\varrho_b\,\overline{R}_3)|0)\right] \\ &= \overline{R}^T\left[(\nabla m|0) + (0|0|\varrho_m\overline{R}_3) + z(\nabla(\varrho_m\,\overline{R}_3)|0) + (0|0|\varrho_b\,\overline{R}_3) + \frac{z^2}{2}(\nabla(\varrho_b\,\overline{R}_3)|0)\right] \\ &= \overline{R}^T(\nabla m|0) + \varrho_m(0|0|e_3) + z\overline{R}^T(\nabla(\varrho_m\,\overline{R}_3)|0) + z\varrho_b(0|0|e_3) + \frac{z^2}{2}\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0) \\ F_s^T\overline{R} + \overline{R}^T F_s - 2 1\!\!1 &= \overline{R}^T(\nabla m|0) + (\nabla m|0)^T\overline{R} + 2\varrho_m(0|0|e_3) - 2 1\!\!1 \\ &\quad + z\left(\overline{R}^T(\nabla(\varrho_m\,\overline{R}_3)|0) + (\nabla(\varrho_m\,\overline{R}_3)|0)^T\overline{R} + 2\varrho_b(0|0|e_3)\right) \\ &\quad + \frac{z^2}{2}\left(\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0) + (\nabla(\varrho_b\,\overline{R}_3)|0)^T\overline{R}\right) \end{aligned} \tag{11.160}$$

$$\begin{aligned} \langle\left[F_s^T\overline{R} + \overline{R}^T F_s - 2 1\!\!1\right].e_3, e_3\rangle &= 2(\varrho_m - 1) + 2z\,\varrho_b \\ \operatorname{tr}\left[F_s^T\overline{R} + \overline{R}^T F_s - 2 1\!\!1\right] &= 2\Big(\langle\overline{R}^T(\nabla m|0), 1\!\!1\rangle + \varrho_m + z\langle(\nabla(\varrho_m\,\overline{R}_3)|0)^T\overline{R}, 1\!\!1\rangle + z\,\varrho_b - 3 \\ &\qquad + \frac{z^2}{2}\langle\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0), 1\!\!1\rangle\Big) \\ \langle(\nabla(\varrho_b\,\overline{R}_3)|0), \overline{R}\rangle &= \langle(\varrho_{b,x}\overline{R}_3|(\varrho_{b,y}\overline{R}_3|0), (\overline{R}_1|\overline{R}_2|\overline{R}_3)\rangle + \varrho_b\,\langle(\nabla\overline{R}_3)|0), \overline{R}\rangle = \varrho_b\,\langle(\nabla\overline{R}_3)|0), \overline{R}\rangle \\ \langle(\nabla(\varrho_m\,\overline{R}_3)|0), \overline{R}\rangle &= \varrho_m\,\langle(\nabla\overline{R}_3)|0), \overline{R}\rangle\,. \end{aligned}$$

Since

$$S_1(F,\overline{R}) = \overline{R}\left[\mu\left(F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right) + 2\mu_c\,\mathrm{skew}(\overline{R}^T F) + \frac{\lambda}{2}\,\mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right]\mathbb{1}\right], \tag{11.161}$$

the requirement $\langle \overline{R}^T S_1(\nabla\varphi_s(x,y,z),\overline{R}).e_3, e_3\rangle = \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), \overline{R}.e_3\rangle$ turns into

$$\begin{aligned}
&\pm\langle N^{\mathrm{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle = \mu\,(2(\varrho_m - 1) + 2z\,\varrho_b)\\
&\quad + \lambda\left(\langle \overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m + z\langle(\nabla(\varrho_m\,\overline{R}_3)|0)^T\overline{R},\mathbb{1}\rangle + z\,\varrho_b - 3 + \frac{z^2}{2}\langle\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0),\mathbb{1}\rangle\right) \Rightarrow\\
&\pm\langle N^{\mathrm{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle = \mu\,(2(\varrho_m - 1) + 2z\,\varrho_b)\\
&\quad + \lambda\left(\langle \overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m + z\,\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle + z\,\varrho_b - 3 + \frac{z^2}{2}\,\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\right),
\end{aligned} \tag{11.162}$$

independent of the Cosserat couple modulus μ_c. Let us evaluate the last equation for $z = \pm h/2$. This yields two **linear** equations in ϱ_m, ϱ_b

$$\begin{aligned}
\langle N^+,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m - 1) + h\,\varrho_b)\\
&\quad + \lambda\left(\langle\overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m + h/2\varrho_m\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle + h/2\,\varrho_b - 3 + \frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\right)\\
-\langle N^-,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m - 1) - h\,\varrho_b)\\
&\quad + \lambda\left(\langle\overline{R}^T(\nabla m|0),\mathbb{1}\rangle + \varrho_m - h/2\varrho_m\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle - h/2\,\varrho_b - 3 + \frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\right).
\end{aligned} \tag{11.163}$$

Adding and subtracting shows that

$$\begin{aligned}
&\varrho_m\,[2\mu+\lambda] + \varrho_b\left[\frac{\lambda\,h^2}{8}\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle\right] = \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{2} + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\
&\varrho_m\left[\lambda\,h\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle\right] + \varrho_b\,[(2\mu+\lambda)h] = \langle N_{\mathrm{res}},\overline{R}_3\rangle\\
&\begin{pmatrix} 2\mu+\lambda & \frac{\lambda\,h^2}{8}\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle\\ \lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle & (2\mu+\lambda)h\end{pmatrix}\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix} = \begin{pmatrix}\langle N_{\mathrm{diff}},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\mathrm{res}},\overline{R}_3\rangle\end{pmatrix}.
\end{aligned} \tag{11.164}$$

The exact solution is given by

$$\begin{aligned}
\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix} &= \frac{1}{(2\mu+\lambda)^2\,h - \frac{\lambda^2\,h^3}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle^2}\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda\,h^2}{8}\langle\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle & (2\mu+\lambda)\end{pmatrix}\\
&\qquad\begin{pmatrix}\langle N_{\mathrm{diff}},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\mathrm{res}},\overline{R}_3\rangle\end{pmatrix},
\end{aligned} \tag{11.165}$$

which will be approximated through

$$\begin{aligned}
\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix} &\approx \frac{1}{(2\mu+\lambda)^2\,h}\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda\,h^2}{8}\langle\nabla\overline{R}_3|0),\overline{R}\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0),\overline{R}\rangle & (2\mu+\lambda)\end{pmatrix}\begin{pmatrix}\langle N_{\mathrm{diff}},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\mathrm{res}},\overline{R}_3\rangle\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{(2\mu+\lambda)} & -\frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\\ -\frac{\lambda}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle & \frac{1}{(2\mu+\lambda)\,h}\end{pmatrix}\begin{pmatrix}\langle N_{\mathrm{diff}},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\mathrm{res}},\overline{R}_3\rangle\end{pmatrix}\\
&= \begin{pmatrix}1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} - \frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\,\langle N_{\mathrm{res}},\overline{R}_3\rangle\\ -\frac{\lambda}{2(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\langle N_{\mathrm{diff}},\overline{R}_3\rangle - \frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\\ +\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\end{pmatrix}.
\end{aligned} \tag{11.166}$$

Hence the leading terms[41] are:

$$\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} - \frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\,\langle N_{\mathrm{res}},\overline{R}_3\rangle\\
\varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h} - \frac{\lambda}{2(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\langle N_{\mathrm{diff}},\overline{R}_3\rangle\\
&\qquad \frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right].
\end{aligned} \tag{11.167}$$

The term $\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]$ represents a nonlinear coupling between midsurface deformationgradient and curvature, an artefact of the derivation not present in the underlying three-dimensional theory where only products of deformation-gradient and rotations occur, we therefore neglect this term.[42] Moreover, for a **rigid** material with $\lambda \gg 1$ we have $\frac{\lambda}{(2\mu+\lambda)^2} \ll 1$, leading finally to the reduced expressions:

$$\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)},\\
\varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}.
\end{aligned} \tag{11.168}$$

[41] Note that ϱ_m, ϱ_b have different units. ϱ_m is dimensionless, whereas $[\varrho_b] = \mathrm{m}^{-1}$.

[42] It would be possible to base all further considerations indeed on the exact solution of ϱ_m, ϱ_b and it seems that the resulting two-dimensional model would allow an existence proof. However, the much more involved expressions are not easily interpreted and do not reduce to the classical expressions upon linearization.

The formula (11.168) shows the physically reasonable behaviour that to first order, **fibers will be elongated by opposite transverse tractions.**

Having obtained the general form of the relevant coefficients ϱ_m, ϱ_b it is expedient to base the expansion of the three-dimensional elastic Cosserat energy, as far as its bending contribution is concerned, on a further simplified expression, namely

$$F_s = \nabla\varphi_s(x,y,z) \approx (\nabla m|\,\varrho_m\,\overline{R}_3) + z\cdot(\nabla\overline{R}_3|\varrho_b\,\overline{R}_3) = A_m + z\,A_r = \overline{F}_s\,. \tag{11.169}$$

This is motivated by our already anounced principle of reduction. The use of (3.26) excludes (up to order h^3) exactly those terms which would violate our principle had we used (3.10) instead. We compute further

$$\overline{R}^T A_r = \overline{R}^T(\nabla\overline{R}_3|\tilde{\varrho}_b\,\overline{R}_3) = \begin{pmatrix} \langle\overline{R}_1,\overline{R}_{3,x}\rangle & \langle\overline{R}_1,\overline{R}_{3,y}\rangle & 0\\ \langle\overline{R}_2,\overline{R}_{3,x}\rangle & \langle\overline{R}_2,\overline{R}_{3,y}\rangle & 0\\ \underbrace{\langle\overline{R}_3,\overline{R}_{3,x}\rangle}_{=0} & \underbrace{\langle\overline{R}_3,\overline{R}_{3,y}\rangle}_{=0} & \varrho_b \end{pmatrix},$$

$$\|\operatorname{skew}(\overline{R}^T A_r)\|^2 = \frac{1}{2}\left(\langle\overline{R}_1,\overline{R}_{3,y}\rangle - \langle\overline{R}_2,\overline{R}_{3,x}\rangle\right)^2 = \|\operatorname{skew}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2$$

$$\overline{R}^T A_r + A_r^T\overline{R} = \begin{pmatrix} 2\langle\overline{R}_1,\overline{R}_{3,x}\rangle & \langle\overline{R}_1,\overline{R}_{3,y}\rangle + \langle\overline{R}_2,\overline{R}_{3,x}\rangle & 0\\ \langle\overline{R}_1,\overline{R}_{3,y}\rangle + \langle\overline{R}_2,\overline{R}_{3,x}\rangle & 2\langle\overline{R}_2,\overline{R}_{3,y}\rangle & 0\\ 0 & 0 & 2\varrho_b \end{pmatrix}, \tag{11.170}$$

$$\varrho_b^2 = \frac{\lambda^2\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2}{(2\mu+\lambda)^2} - \frac{2\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}$$

$$\begin{aligned}
\|\overline{R}^T A_r + A_r^T\overline{R}\|^2 &= 4\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + 4\,\varrho_b^2\\
&= 4\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{4\lambda^2}{(2\mu+\lambda)^2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\\
&\quad - \frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{4\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 &= (2\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + 2\,\varrho_b)^2\\
&= 4\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + 8\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\varrho_b + 4\,\varrho_b^2 \qquad (11.171)\\
&= 4\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 - \frac{8\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{8\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\\
&\quad + \frac{4\lambda^2}{(2\mu+\lambda)^2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 - \frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{4\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}\\
&= \operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\cdot\left(4 - \frac{8\lambda}{2\mu+\lambda} + \frac{4\lambda^2}{(2\mu+\lambda)^2}\right) - \frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\\
&\quad + \frac{8\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{4\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}\,.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\mu}{4}\left(-\frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\right) + \frac{\lambda}{8}\left(-\frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\right.\\
&\quad\left.+\frac{8\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\right) = \left(\frac{-2\mu\lambda}{(2\mu+\lambda)^2\,h} - \frac{\lambda^2}{(2\mu+\lambda)^2\,h} + \frac{\lambda}{(2\mu+\lambda)\,h}\right)\langle N_{\text{res}},\overline{R}_3\rangle\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\\
&= \left(\frac{-(2\mu+\lambda)\lambda}{(2\mu+\lambda)^2\,h} + \frac{\lambda}{(2\mu+\lambda)\,h}\right)\langle N_{\text{res}},\overline{R}_3\rangle\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] = 0\,, \qquad (11.172)
\end{aligned}$$

therefore (the mixed term just cancels!)

$$\begin{aligned}
&\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R}\|^2 + \frac{\lambda}{8}\operatorname{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2\\
&= \mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda^2}{(2\mu+\lambda)^2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\\
&\quad + \frac{\lambda}{8}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\cdot\left(4 - \frac{8\lambda}{2\mu+\lambda} + \frac{4\lambda^2}{(2\mu+\lambda)^2}\right) + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2} \qquad (11.173)\\
&= \mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \left[\frac{\mu\lambda^2}{(2\mu+\lambda)^2} + \frac{\lambda}{8}\cdot\left(4 - \frac{8\lambda}{2\mu+\lambda} + \frac{4\lambda^2}{(2\mu+\lambda)^2}\right)\right]\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\\
&\quad + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\\
&= \mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\,.
\end{aligned}$$

A similar computation is now performed for the membrane contribution. Set

$$\varrho_m = 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle}{(2\mu+\lambda)} = 1 + \hat{\varrho}_m$$

$$\hat{\varrho}_m = -\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle}{(2\mu+\lambda)} = -\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|\overline{R}_3), \overline{R}\rangle - 3\right] + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle}{(2\mu+\lambda)}\,. \tag{11.174}$$

Then

$$\begin{aligned}
\hat{\varrho}_m^2 &= \frac{\lambda^2\,{\rm tr}\left[{\rm sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right]^2}{(2\mu+\lambda)^2} - \frac{2\lambda\,\langle N_{\rm diff}, \overline{R}_3\rangle}{(2\mu+\lambda)^2}{\rm tr}\left[{\rm sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right] + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle^2}{(2\mu+\lambda)^2} \\
\|\overline{R}^T A_m + A_m^T\overline{R} - 2 1\!\!1\|^2 &= 4\|\,{\rm sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\|^2 + 4\,\hat{\varrho}_m^2 \\
{\rm tr}\left[\overline{R}^T A_m + A_m^T\overline{R} - 2 1\!\!1\right]^2 &= (2{\rm tr}\left[{\rm sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right] + 2\,\hat{\varrho}_m)^2\,.
\end{aligned} \tag{11.175}$$

Exactly the same computations as for the bending term allows us to conclude that

$$\begin{aligned}
&\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R} - 2 1\!\!1\|^2 + \frac{\lambda}{8}{\rm tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 \\
&\quad = \mu\|\,{\rm sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}{\rm tr}\left[{\rm sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right]^2 + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)}\,.
\end{aligned} \tag{11.176}$$

11.4.2 Detailed computations for the new finite Reissner-Mindlin model

We compute

$$\begin{aligned}
F_s^T F_s &= \left(\tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r\right)^T\left(\tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r\right) \\
&= \tilde{A}_m^T\tilde{A}_m + z\,\tilde{A}_m^T\tilde{A}_r + \frac{z^2}{2}\tilde{A}_m^T\tilde{B}_r + z\,\tilde{A}_r^T\tilde{A}_m + z^2\,\tilde{A}_r^T\tilde{A}_r + \frac{z^3}{2}\tilde{A}_r^T\tilde{B}_r + \frac{z^2}{2}\tilde{B}_r^T\tilde{A}_m + \frac{z^3}{2}\tilde{B}_r^T\tilde{A}_r + \frac{z^4}{4}\tilde{B}_r^T\tilde{B}_r \\
&= \underbrace{\tilde{A}_m^T\tilde{A}_m}_{=:A} + z\underbrace{\left(\tilde{A}_m^T\tilde{A}_r + \tilde{A}_r^T\tilde{A}_m\right)}_{=:B} + \frac{z^2}{2}\underbrace{\left(\tilde{A}_m^T\tilde{B}_r + 2\,\tilde{A}_r^T\tilde{A}_r + \tilde{B}_r^T\tilde{A}_m\right)}_{=:C} + O(h^3)\,,
\end{aligned} \tag{11.177}$$

and further

$$\begin{aligned}
\tilde{A}_m^T\tilde{A}_m &= (\nabla m|\,\varrho_m\,\vec{d})^T(\nabla m|\,\varrho_m\,\vec{d}) = \begin{pmatrix} \|m_x\|^2 & \langle m_x, m_y\rangle & \varrho_m\langle m_x, \vec{d}\rangle \\ \langle m_x, m_y\rangle & \|m_y\|^2 & \varrho_m\langle m_y, \vec{d}\rangle \\ \varrho_m\langle m_x, \vec{d}\rangle & \varrho_m\langle m_y, \vec{d}\rangle & \varrho_m^2 \end{pmatrix} \\
{\rm tr}\,[B] &= {\rm tr}\left[\tilde{A}_m^T\tilde{A}_r + \tilde{A}_r^T\tilde{A}_m\right] = 2{\rm tr}\left[\tilde{A}_m^T\tilde{A}_r\right] = 2\langle\tilde{A}_m, \tilde{A}_r\rangle = 2\left(\langle(\nabla m|\varrho_m\,\vec{d}), (\nabla(\varrho_m\vec{d})|0)\rangle + \varrho_m\,\varrho_b\right) \\
\langle B.e_3, e_3\rangle &= 2\langle\tilde{A}_m^T\tilde{A}_r.e_3, e_3\rangle = 2\langle\tilde{A}_m.e_3, \tilde{A}_r.e_3\rangle = 2\langle\varrho_m\,\vec{d}, \varrho_b\,\vec{d}\rangle = 2\,\varrho_m\,\varrho_b\,.
\end{aligned} \tag{11.178}$$

We neglect the $O(h^3)$ contribution and insert $z = \pm h/2$. This yields two equations

$$\begin{aligned}
&\langle\left[\mu(A - 1\!\!1 + (h/2)B + h^2/4C) + \frac{\lambda}{2}{\rm tr}\left[A - 1\!\!1 + (h/2)B + h^2/4C\right]1\!\!1\right].e_3, e_3\rangle = \langle N^+, (\varrho_m + (h/2)\,\varrho_b)\,\vec{d}\rangle \\
&\langle\left[\mu(A - 1\!\!1 - (h/2)B + h^2/4C) + \frac{\lambda}{2}{\rm tr}\left[A - 1\!\!1 - (h/2)B + h^2/4C\right]1\!\!1\right].e_3, e_3\rangle = -\langle N^-, (\varrho_m - (h/2)\,\varrho_b)\,\vec{d}\rangle\,.
\end{aligned} \tag{11.179}$$

Adding and subtracting shows that

$$\begin{aligned}
2\langle\left[\mu(A - 1\!\!1 + h^2/4C) + \frac{\lambda}{2}{\rm tr}\left[A - 1\!\!1 + h^2/4C\right]1\!\!1\right].e_3, e_3\rangle &= \langle N^+ - N^-, \varrho_m\vec{d}\rangle + h/2\,\varrho_b\langle N^+ + N^-, \vec{d}\rangle \\
\langle\left[\mu(hB) + \frac{\lambda}{2}{\rm tr}\,[hB]1\!\!1\right].e_3, e_3\rangle &= \langle N^+ - (-N^-), \varrho_m\,\vec{d}\rangle + h/2\,\langle N^+ - N^-, \varrho_b\,\vec{d}\rangle\,,
\end{aligned} \tag{11.180}$$

and to leading order in both equations

$$\begin{aligned}
2\langle\left[\mu(A - 1\!\!1) + \frac{\lambda}{2}{\rm tr}\,[A - 1\!\!1]1\!\!1\right].e_3, e_3\rangle &= \langle N^+ - N^-, \varrho_m\,\vec{d}\rangle \\
\langle\left[\mu B + \frac{\lambda}{2}{\rm tr}\,[B]1\!\!1\right].e_3, e_3\rangle &= \frac{1}{h}\langle N^+ + N^-), \varrho_m\,\vec{d}\rangle
\end{aligned} \tag{11.181}$$

or

$$\langle\left[\mu(A - 1\!\!1) + \frac{\lambda}{2}{\rm tr}\,[A - 1\!\!1]1\!\!1\right].e_3, e_3\rangle = \langle N_{\rm diff}, \varrho_m\,\vec{d}\rangle, \quad \langle\left[\mu B + \frac{\lambda}{2}{\rm tr}\,[B]1\!\!1\right].e_3, e_3\rangle = \langle\frac{N_{\rm res}}{h}, \varrho_m\,\vec{d}\rangle\,. \tag{11.182}$$

This yields for ϱ_m

$$\varrho_m \langle N_{\text{diff}}, \vec{d}\rangle = \left[\mu(\varrho_m^2-1)+\frac{\lambda}{2}\left(\|\nabla m\|^2+\varrho_m^2-3\right)\right] \Rightarrow$$
$$\left[2\mu(\varrho_m^2-1)+\lambda\left(\|\nabla m\|^2+(\varrho_m^2-1)-2\right)\right] = 2\varrho_m \langle N_{\text{diff}}, \vec{d}\rangle$$
$$(\varrho_m^2-1)(2\mu+\lambda)+\lambda[\|\nabla m\|^2-2] = 2\varrho_m \langle N_{\text{diff}}, \vec{d}\rangle$$
$$(\varrho_m^2-1)-2\varrho_m\frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)}+\frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2-2]=0$$
$$\varrho_m^2-2\varrho_m\frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)}+\frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2-2]-1=0$$
$$\varrho_m = +\frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} \pm \sqrt{1-\frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2-2]+\frac{\langle N_{\text{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\,, \tag{11.183}$$

and for ϱ_b

$$\langle \left[\mu B+\frac{\lambda}{2}\operatorname{tr}[B]1\!\!1\right].e_3, e_3\rangle = \frac{1}{h}\langle N_{\text{res}}, \varrho_m\,\vec{d}\rangle$$
$$\frac{1}{h}\langle N_{\text{res}}, \varrho_m\,\vec{d}\rangle = 2\mu\varrho_m\,\varrho_b+\lambda\,\varrho_m\,\varrho_b+\lambda\,\langle(\nabla m|\varrho_m\,\vec{d}), (\nabla(\varrho_m\vec{d})|0)\rangle \Rightarrow \tag{11.184}$$
$$(2\mu+\lambda)\,\varrho_m\,\varrho_b+\lambda\,\langle(\nabla m|\varrho_m\,\vec{d}), (\varrho_m\nabla\vec{d}|0)+(\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle = \varrho_m\frac{1}{h}\langle N_{\text{res}}, \vec{d}\rangle \tag{11.185}$$
$$(2\mu+\lambda)\,\varrho_b+\lambda\,\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)+\frac{1}{\varrho_m}(\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle = \frac{1}{h}\langle N_{\text{res}}, \vec{d}\rangle$$
$$\varrho_b = -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle+\frac{1}{(2\mu+\lambda)\,h}\langle N_{\text{res}}, \vec{d}\rangle+\frac{1}{\varrho_m\,(2\mu+\lambda)}\langle(\nabla m|\varrho_m\vec{d}), (\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle\,.$$

Since we do not want to consider space variations in the thickness-stretch ϱ_m we take finally

$$\varrho_m = \frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)}+\sqrt{1-\frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2-2]+\frac{\langle N_{\text{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}$$
$$\varrho_b = -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle+\frac{\langle N_{\text{res}}, \vec{d}\rangle}{(2\mu+\lambda)\,h}\,. \tag{11.186}$$

11.5 Units and elastic constants

The body force f has units $[\mathrm{N/m^3}]$, the surface traction N has units $[\mathrm{N/m^2}]$, of course. Note that a typical value of the elastic moduli for steel is $\mu = 80.000[\mathrm{N/mm^2}] = 8\cdot 10^{10}[\mathrm{N/m^2}]$ and $\lambda = 100.000[\mathrm{N/mm^2}] = 10\cdot 10^{10}[\mathrm{N/m^2}] = 80.000MPa = 80GPa$.

The Youngs-modulus E and the Poisson number ν are defined in terms of the Lamé constants as follows:

$$E := \mu\,\frac{2\mu+3\lambda}{\mu+\lambda}, \quad \nu := \frac{\lambda}{2(\mu+\lambda)}, \quad \lambda\to\infty \Leftrightarrow \nu\to\frac{1}{2}$$
$$\lambda = \frac{E\,\nu}{(1+\nu)\,(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}\,. \tag{11.187}$$

This implies the well known relations

$$\frac{1}{2}\frac{E}{1-\nu^2} = 2\mu\,\frac{(2\mu+3\lambda)(\mu+\lambda)}{(2\mu+\lambda)(2\mu+3\lambda)} = 2\mu\,\frac{(\mu+\lambda)}{(2\mu+\lambda)}\,. \tag{11.188}$$

It is also useful to have the physical properties of various very thin samples at hand. For A4-paper ($80\mathrm{g/m^2}$), the thickness of a 20cm quadrangle is roughly 0.08mm which gives a characteristic value $h\approx\frac{1}{1000}$. Representative values for elastic moduli for isotropic standardized paper are $E = 5840[\mathrm{N/mm^2}]$, $\nu = 0.24$ or $\mu = 2.6GPa$, $\lambda = 2.34GPa$.

For kitchen plastic wrap one has the thickness 0.03mm which implies $h\approx\frac{3}{10000}$ and standard Aluminum foil has a thickness of 0.01mm implying $h\approx\frac{1}{10000}$. A typical thin film, for which we consider a 20mm rectangle with thickness as small as 5 micrometers ($5\cdot 10^{-6}$m) yields a characteristic thickness of $h\approx\frac{5}{10000}$. In the special case of e.g. a steel rod of length 1m and radius 2mm we obtain the characteristic variable $h\approx\frac{4}{1000}$. For such small values of h it seems to be clear that classical bending cannot play a prominent role.

11.6 The penalized finite Cosserat plate

While the treatment of the microrotations $\overline{R}$ is conceptually clear, any numerical implementation has the burden that the rotations live on a nonlinear manifold. In order to circumvent this difficulty, we propose a simplified variant of our new Cosserat plate model, where we relax the constraint of exact rotations and add a penalizing term. The new minimization problem reads: find the deformation of the midsurface $m:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and the relaxed 'microrotation' of the plate (shell) $\overline{R}:\omega\subset\mathbb{R}^2\mapsto\mathbb{M}^{3\times 3}$ solving on ω:

$$
\begin{aligned}
I = \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b) + \frac{\aleph}{4} \|\overline{R}^T\overline{R} - 1\!\!1\|^2 \,\mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min . \text{ w.r.t. } (m, \overline{R}),\\
\overline{U} = \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3), \quad \textbf{penalty:} \quad \aleph \to \infty\\
m_{|\gamma_0} = g_{\mathrm{d}}(x, y, 0)\,, \quad \overline{R}_{|\gamma_0} = \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x, y, 0).e_3))_{|\gamma_0}\,, \qquad (11.189)\\
W_{\mathrm{mp}}(\overline{U}) = \mu\, \|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\mu\lambda}{2\mu + \lambda}\, \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2, \quad W_{\mathrm{curv}}(\mathfrak{K}_s) = \mu \frac{L_c^2}{12}\, \|\mathfrak{K}_s\|^2\,,\\
\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\\
W_{\mathrm{bend}}(\mathfrak{K}_b) = \mu\, \|\,\mathrm{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu + \lambda}\, \mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2, \quad \mathfrak{K}_b = \overline{R}^T(\nabla \overline{R}_3|0) = \mathfrak{K}_s^3\,.
\end{aligned}
$$

It should be observed that the penalized model is still frame-indifferent, a welcome feature.

11.7 The partially linearized Cosserat plate

Another method of reducing the complexity of the ensuing model consists in **partially linearizing** the equations. Let us reduce a modification of case II ($\mu_c = 0$, $\alpha_4 = 0$, $q = 0$, $p = 1$, $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$) for situations in which we expect the curvature and microrotations to remain small but the midsurface deformations are unrestricted. We write $m(x, y) = (x, y, 0)^T + v(x, y)$, with the (finite) displacement of the midsurface of the plate $v : \omega \mapsto \mathbb{R}^3$ and $\overline{R} = 1\!\!1 + \overline{A} + \ldots$ with $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$ the infinitesimal microrotation. For the boundary deformation we write $g_{\mathrm{d}}(x, y, z) = (x, y, z)^T + u^{\mathrm{d}}(x, y, z)$, with the consequence, that $\nabla g_{\mathrm{d}}.e_3 = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 1 + u^{\mathrm{d}}_{3,z})$. The curvature tensors are expanded as

$$
\begin{aligned}
\mathfrak{K}_b &= \overline{R}^T(\nabla \overline{R}_3|0) = (1\!\!1 + \overline{A} + \ldots)^T\, (\nabla[\overline{A}_3 + \overline{A}^2.e_3 + \ldots]|0) \approx (\nabla \overline{A}_3|0) + \ldots\\
\mathfrak{K}_s &\approx \left((\nabla(\overline{A}.e_1)|0), (\nabla(\overline{A}.e_2)|0), (\nabla(\overline{A}.e_3)|0)\right)\,, \qquad (11.190)
\end{aligned}
$$

and the Cosserat micropolar plate stretch tensor expands like

$$
\begin{aligned}
\overline{U} &= \overline{R}^T F_s = \overline{R}^T(\nabla m|\overline{R}_3) = (1\!\!1 + \overline{A} + \ldots)^T \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \nabla v|(1\!\!1 + \overline{A} + \ldots).e_3) \right) \approx 1\!\!1 + (\nabla v|\overline{A}_3) + \overline{A}^T + \overline{A}^T(\nabla v|\overline{A}_3)\\
&\approx 1\!\!1 + (\nabla v|\overline{A}_3) + \overline{A}^T + \underbrace{\overline{A}^T(\nabla v|0)}_{\text{drill rotations}} + \overline{A}^T(0|0|\overline{A}_3)\,. \qquad (11.191)
\end{aligned}
$$

Neglecting the quadratic term $\overline{A}^T(0|0|\overline{A}_3)$ in view of the expected smallness of rotations, we are indeed left with the minimization problem for $v \in \mathbb{R}^3$ and $A \in \mathfrak{so}(3, \mathbb{R})$

$$
\begin{aligned}
&\int_\omega h \left(\mu\, \|\,\mathrm{sym}((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0))\|^2 + \frac{\mu\lambda}{2\mu + \lambda}\, \mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0))\right]^2 + \mu\, \frac{L_c^2}{12} \|\mathrm{D}_{\mathrm{x}}\overline{A}\|^2 \right)\\
&\quad + \frac{h^3}{12} \left(\mu\, \|\,\mathrm{sym}((\nabla \overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu + \lambda}\, \mathrm{tr}\left[\mathrm{sym}((\nabla \overline{A}_3|0))\right]^2 \right) \mathrm{d}\omega - \Pi(v, \overline{A}_3) \mapsto \min . \text{ w.r.t. } (v, \overline{A}),\\
&v_{|\gamma_0} = u^{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported} \qquad (11.192)\\
&\overline{A}_{|\gamma_0} = \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,, \quad \text{linearized consistent coupling} \Rightarrow \overline{A}_{3_{|\gamma_0}} = (\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0)^T\\
&\overline{A}_{3_{|\gamma_0}} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription}\,.
\end{aligned}
$$

The internal length $L_c > 0$ is seen to be necessary to control the **in-plane drill rotations** which appear only as a **second order effect** in the model. The membrane part can be shown to be coercive w.r.t. v in $H^1(\omega, \mathbb{R}^3)$ if $\overline{A} \in C^0(\overline{\omega}, \mathfrak{so}(3, \mathbb{R}))$, since the second derivative w.r.t. v can be estimated through $\|\,\mathrm{sym}((1\!\!1 + \overline{A})^T(\nabla\phi|0))\|^2$ and $1\!\!1 + \overline{A} \in \mathrm{GL}(3, \mathbb{R})$ for $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$. The corresponding field equations are **semilinear**, more precisely, balance of linear momentum is a uniformly Legendre-Hadamard elliptic linear system w.r.t. v at given $\overline{A}$ and balance of angular momentum is a uniformly Legendre-Hadamard elliptic linear system w.r.t. $\overline{A}$ at given v with constant coefficients. Nevertheless, the resulting model is nonlinear but not frame-indifferent. While it is not entirely clear how to show existence, the simplified model with drill rotations should prove to be easily implemented along the lines of traditional infinitesimal Reissner-Mindlin models taking into account all the available knowledge on non-locking approximations.

Finally, we now re-derive the classical Reissner-Mindlin model in the infinitesimal context, pointing out certain 'inconsistencies' usually encountered and give a short existence proof. Let us sketch briefly the 'direct' derivation of (10.129) in the infinitesimal context in order to understand some of the peculiarities of plate modelling.

11.8 Derivation of the classical infinitesimal Reissner-Mindlin bending plate

If ε is the symmetrized displacement gradient of the three-dimensional theory, the elastic free energy of an isotropic medium takes the form

$$
W_{\mathrm{infin}}(\varepsilon) = \mu\, \|\varepsilon\|^2 + \frac{\lambda}{2} \mathrm{tr}\,[\varepsilon]^2, \quad \sigma = 2\mu\, \varepsilon + \lambda\, \mathrm{tr}\,[\varepsilon]\, 1\!\!1\,, \qquad (11.193)
$$

where σ is the symmetric Cauchy stress tensor. If we assume zero normal traction **across** the thickness on $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, i.e. $\sigma_{33}(x, y, z) = 0$, $z \in [-\frac{h}{2}, \frac{h}{2}]$, then this implies immediately

$$
0 = \sigma_{33} = 2\mu\, \varepsilon_{33} + \lambda\, \mathrm{tr}\,[\varepsilon]\, 1 \Rightarrow \varepsilon_{33} = -\frac{\lambda}{2\mu + \lambda}\, (\varepsilon_{11} + \varepsilon_{22}) = -\frac{\lambda}{2\mu + \lambda}\, \langle \varepsilon, 1\!\!1_2 \rangle\,, \qquad (11.194)
$$

still on the three-dimensional level. Applying (11.194) and eliminating ε_{33} in the elastic energy (so called **condensation of the material law**) yields,

$$
\begin{aligned}
W^{\text{stress}}_{\text{infin}}(\varepsilon) &= \mu\,\|\varepsilon\|^2 + \frac{\lambda}{2}\text{tr}\,[\varepsilon]^2\\
&= \mu\,\|\varepsilon\,1\!\!1_2\|^2 + \mu\,(\varepsilon_{31}^2+\varepsilon_{32}^2) + \mu\,\frac{\lambda^2}{(2\mu+\lambda)^2}\langle\varepsilon,1\!\!1_2\rangle^2 + \frac{\lambda}{2}\left(\langle\varepsilon,1\!\!1_2\rangle^2 - 2\langle\varepsilon,1\!\!1_2\rangle\frac{\lambda}{2\mu+\lambda}\langle\varepsilon,1\!\!1_2\rangle + \frac{\lambda^2}{(2\mu+\lambda)^2}\langle\varepsilon,1\!\!1_2\rangle^2\right)\\
&= \mu\,\|\varepsilon\,1\!\!1_2\|^2 + \mu\,(\varepsilon_{31}^2+\varepsilon_{32}^2) + \frac{\lambda}{2}\frac{2\mu}{2\mu+\lambda}\langle\varepsilon,1\!\!1_2\rangle^2\,. \qquad (11.195)
\end{aligned}
$$

Now the **linear** kinematical ansatz $\varphi_s(x,y,z) = m + z\,\overline{R}.e_3$ together with $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, $\theta = -\overline{A}_3$ implies to leading order

$$
\varphi_s(x,y,z) = \begin{pmatrix} x+v_1(x,y)\\ y+v_2(x,y)\\ 0+v_3(x,y)\end{pmatrix} + z\begin{pmatrix}-\theta_1\\-\theta_2\\ z\end{pmatrix} + \ldots = \begin{pmatrix}x\\y\\z\end{pmatrix} + u(x,y,z) + \ldots\,,\quad u(x,y,z) = \begin{pmatrix} v_1(x,y) - z\theta_1(x,y)\\ v_2(x,y) - z\theta_2(x,y)\\ v_3(x,y)\end{pmatrix}. \qquad (11.196)
$$

Hence for $v_1, v_2 = 0$ we get

$$
\varepsilon = \frac{1}{2}(\nabla u^T + \nabla u) = \begin{pmatrix} -z\,\theta_{1,x} & -\frac{z}{2}\,(\theta_{1,y}+\theta_{2,x}) & \frac{v_{3,x}-\theta_1}{2}\\ -\frac{z}{2}\,(\theta_{1,y}+\theta_{2,x}) & -z\,\theta_{2,y} & \frac{v_{3,y}-\theta_2}{2}\\ \frac{v_{3,x}-\theta_1}{2} & \frac{v_{3,y}-\theta_2}{2} & 0\end{pmatrix}. \qquad (11.197)
$$

Explicitly integrating over the thickness with respect to z results in (10.129) with $\kappa = 1$. We note that this derivation seems to be not fully consistent: the linear kinematical ansatz yields $\varepsilon_{33} = 0$, while we use $\varepsilon_{33} = -\frac{\lambda}{2\mu+\lambda}\langle\varepsilon,1\!\!1_2\rangle \neq 0$ in evaluating the elastic free energy. The zero normal traction condition σ_{33} is true for the chosen kinematical ansatz only on the midsurface while in the derivation we have tacitly assumed it to hold uniformly over the thickness. However, the final result is correct.

11.9 The classical infinitesimal Kirchhoff bending plate

For the convenience of the reader we also supply the similar system of equations for the classical Kirchhoff bending plate. If only transverse deflections $v_3(x,y)$ are considered, the energy to be minimized is

$$
W^{\text{class}}_{\text{K}} = \frac{h^3}{12}\left(\frac{\mu}{4}\,\|\nabla(\nabla v_3)^T + \nabla(\nabla v_3)\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\text{tr}\left[\nabla(\nabla v_3)^T + \nabla(\nabla v_3)^T\right]^2\right) = \frac{h^3}{12}\left(\mu\,\|D^2 v_3\|^2 + \frac{\lambda}{2}\frac{2\mu}{2\mu+\lambda}\text{tr}\left[D^2 v_3\right]^2\right), \qquad (11.198)
$$

since $\nabla(\nabla v_3) = D^2 v_3$. This energy can be obtained formally from (10.129) by setting $\theta = \nabla v_3$, see [Bra92, p.266]. It should be clear, however, that these bending equations are only appropriate for deflections $v_3 \ll h$. For $v_3 \approx h$ combined membrane/bending needs to be used and for $v_3 \gg h$ the membrane effect dominates.

Let us turn quickly to the existence theory [Dav75] involved in the infinitesimal case:

Theorem 11.9 (Existence for infinitesimal Reissner-Mindlin)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\text{d}} \in H^2(\omega,\mathbb{R}^3)$. Moreover, let $\overline{f} \in L^2(\omega,\mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s,\mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega,\mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s,\mathbb{R}^3)$, see (10.102). Then problem (10.127) admits a ***unique*** *minimizing solution pair $(v,\theta) \in H^1(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^3)$.*

Proof. By the direct methods of variations it is a simple matter to establish the existence of a solution: Since the functional is bounded above, we may take infimizing sequences $(v_k,\theta_k) \in H^1(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^3)$ and establish weak convergence of $\theta_k \rightharpoonup \theta \in H^1(\omega,\mathbb{R}^3)$, strong in $L^2(\omega,\mathbb{R}^3)$. This implies the boundedness of $v_k \in H^1(\omega,\mathbb{R}^3)$ by Korn's first inequality and establishes as well that the functional is bounded below. We may extract a subsequence v_k not relabelled, converging weakly to $v \in H^1(\omega,\mathbb{R}^3)$. Overall convexity of the functional allows us to pass to the limit. The pair (v,θ) is a minimizer.

The general infinitesimal problem is easily seen to have a **unique** solution (v,θ) on account of the strict positivity of the second derivative of the energy $W_{\text{RM,infin}}$:

$$
D^2 W_{\text{RM,infin}}(\nabla v,\theta).((\nabla\phi,\delta\theta),(\nabla\phi,\delta\theta)) \geq h\,\mu\,\|\text{sym}((\nabla\phi|\delta\theta))\|^2 + \frac{h^3}{12}\mu\,\|\nabla\delta\theta\|^2\,, \qquad (11.199)
$$

on the linear space $H^1_0(\omega,\mathbb{R}^3)\times H^1_0(\omega,\mathbb{R}^3)$. Strict positivity is a consequence of the classical Korn's inequality for the membrane part and full control of the skew-symmetric increment in the bending part. In this case, the drill rotations, which are associated to α, remain unspecified. Since only two independent simple rotations are required to orient a unit director field, a distinctive feature of classical plate and shell theories is a rotation field defined in terms of only two independent degrees of freedom: rotations about the director itself-the so called drill rotations, are irrelevant and for that matter undefined in classical shell theory. ■

The analysis based on (10.128) is even simpler and can be done with Poincaré's inequality replacing Korn's inequality. Note, however, that a numerical implementation of the linearized models based on the presented setting (displacement approach) shows to perform badly on coarse meshes [Bra92] for small $h > 0$ due to **shear locking**.

11.10 Comparison of formulas for the thickness stretch

The different formulas for the thickness stretch ϱ_m in the plate models will be elucidated. We have

$$
\varrho_m := \begin{cases} 1 - \frac{\lambda}{2\mu+\lambda}\text{tr}\left[\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} & \text{new Cosserat plate (4.43)}\\ \frac{1}{1+\frac{\lambda}{2\mu+\lambda}(\det[(\nabla m|\overline{R}_3)]-1)} + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} & \text{modified new Cosserat plate (5.70)}\\ \frac{\langle N_{\text{diff}},\vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\text{diff}},\vec{d}\rangle^2}{(2\mu+\lambda)^2}} & \text{new finite Reissner-Mindlin plate (6.82)}\\ \frac{\langle N_{\text{diff}},\vec{n}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\text{diff}},\vec{n}\rangle^2}{(2\mu+\lambda)^2}} & \text{finite Kirchhoff-Love plate (7.83)}\,. \end{cases} \qquad (11.200)
$$

Now assume that $N_{\text{diff}} = 0$ and $\overline{R} = \text{polar}(\nabla m|\vec{n})$. Let $\lambda_1, \lambda_2 \geq 0$ be the eigenvalues of $\sqrt{\nabla m^T \nabla m}$. In terms of λ_1, λ_2 we distinguish the cases

$$\varrho_m^{\text{COSS}} = 1 - \frac{\lambda}{2\mu+\lambda}\left[\lambda_1 + \lambda_2 - 2\right], \quad \varrho_m^{\text{LARGE}} = \frac{1}{1 + \frac{\lambda}{2\mu+\lambda}\left[\lambda_1\,\lambda_2 - 1\right]}, \quad \varrho_m^{\text{KL}} = \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\left[\lambda_1^2 + \lambda_2^2 - 2\right]}\,. \tag{11.201}$$

To further simplify the exposition, take $\lambda = 2\mu$ and assume that $\lambda_1 = \lambda_2 = |\zeta|$. Then

$$\varrho_m^{\text{COSS}} = 2 - |\zeta|, \qquad \varrho_m^{\text{LARGE}} = \frac{2}{1+\zeta^2}, \qquad \varrho_m^{\text{KL}} = \sqrt{2-\zeta^2}\,. \tag{11.202}$$

All three formulas produce the same tangent at the identity $\zeta = 1$ (no in-plane stretch). In the Kirchhoff-Love model, evaluation of $\varrho_m^{\text{KL}} = \sqrt{2-\zeta^2}$ is only possible for $\zeta \leq \sqrt{2}$, a severe shortcoming of the model. In the new Cosserat plate model, evaluation of ϱ_m^{Coss} is possible for all $\zeta \in \mathbb{R}$ but does make sense only for $\zeta \leq 2$. Finally, the modified Cosserat model allows useful evaluation for all $\zeta \in \mathbb{R}$.

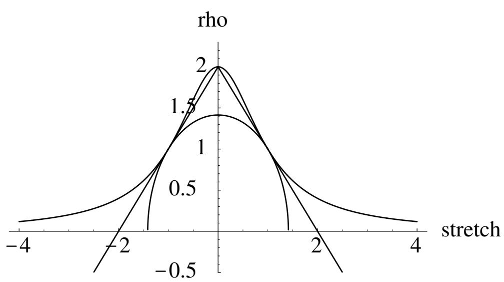

Figure 2: The different formulas for the thickness stretch ϱ_m: unphysical response of the Kirchhhoff-Love model and reasonable response $\varrho_m > 0$, $(\nabla m|\varrho_m \overline{R}_3) \in \text{GL}^+(3,\mathbb{R})$ of the modified Cosserat model.

11.11 Open questions

Show that

$$\int_\Omega \|\overline{R}^T \nabla\varphi + \nabla\varphi^T \overline{R} - 2 1\!\!1\|^2 \,\text{dV} \mapsto \min.\ \text{w.r.t.}\ (\varphi, \overline{R}).\,, \tag{11.203}$$

has the unique family of solutions $\overline{R} = Q = \text{const.}$, $\varphi(x) = Q.x + b$, $Q \in \text{SO}(3,\mathbb{R})$. Without gradient constraint on φ the solution is not unique. The same problem for plates and shells: show that

$$\int_\omega \|\overline{R}^T(\nabla m|\overline{R}_3) + (\nabla m|\overline{R}_3)^T \overline{R} - 2 1\!\!1\|^2 \,\text{d}\omega \mapsto \min.\ \text{w.r.t.}\ (m, \overline{R}).\,, \tag{11.204}$$

has the unique family of solutions $\overline{R} = Q = \text{const.}$, $m(x,y) = Q_1 \cdot x + Q_2 \cdot y + b$, $Q \in \text{SO}(3,\mathbb{R})$. Without gradient constraint on m the solution is not unique, but it can be seen that $\overline{R}_3 = \vec{n}$ must hold anyway. The same question turned around: assume that $\varphi \in H^1(\Omega,\mathbb{R}^3)$, $\overline{R} \in \text{SO}(3,\mathbb{R})$ and $\overline{R}^T \nabla\varphi + \nabla\varphi^T \overline{R} - 2 1\!\!1 = 0$. Show that this implies **rigidity**: $\overline{R} = Q = \text{const.}$, $\varphi(x) = Q.x + b$ and that we are dealing with a true strain measure.

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