# Finite multiplicative elastic-plastic Cosserat micropolar theory for polycrystals with grain rotations including fracture. Modelling and mathematical analysis.

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**Abstract**

We investigate geometrically exact generalized continua of Cosserat micropolar type. The variational form of these models is introduced and consistently extended to cover finite elasto-plasticity based on the multiplicative decomposition of the deformation gradient only. The decisive stress is the Eshelby energy momentum tensor. It is motivated that the traditional Cosserat couple modulus $\mu_c$ can and should be set to zero for macroscopic specimens liable to fracture in shear, still leading to a complete consistent Cosserat theory with independent rotations in the geometrically exact finite case in contrast to the infinitesimal, linearized model.

Depending on material constants different mathematical existence theorems in Sobolev-spaces are given for the resulting nonlinear boundary value problems in the elastic case. These are the first such results known to the author. Various assumptions on the magnitude of deformations and microrotations lead to simplified models which are all analysed mathematically.

Partial focus is set to the possible regularization properties of micropolar models compared to classical continuum models in the macroscopic case of materials failing in shear. The mathematical analysis heavily uses an extended Korn's first inequality (Neff, Proc.Roy.Soc.Edinb.A, 2002) discovered by the author recently. The methods of choice are the direct methods of the calculus of variations.

**Key words:** plasticity, visco-plasticity, polar-materials, non-simple materials, microstructure, structured continua, solid mechanics, elliptic systems, variational methods, fracture, shear failure.

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# Contents

# 1 Introduction

## 1.1 The development of Cosserat models, motivation and applications

This article addresses the modelling and mathematical analysis of **geometrically exact**[1] generalized continua of **Cosserat micropolar** type in the elastic as well as elasto-plastic case. General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [CC09]. In fact, their original motivation came from the theory of surfaces, where the moving three-frame (Gauss frame) had been used successfully.

Their development has been largely forgotten for decades only to be rediscovered in the beginning of the sixties [Osh55, Gün58, AK61, ES64, Eri68, Tou62, Tou64, GR64, MT62, Sch67, TN65]. At that time theoretical investigations on non-classical continuum theories were the main motivation [Krö68]. The Cosserat concept has been generalized in various directions, for an overview of these so called **microcontinuum** theories we refer to [EK76, Eri99, CG77, Cap89].

Among the first contributions extending the Cosserat framework to (infinitesimal, geometrically linear) elasto-plasticity we have to mention [Saw67, Lip69, Bes74]. More recent (infinitesimal) elastic-plastic formulations have been investigated in [dB92, DSW93, IW98, RV96]. These models directly comprise joint elastic and plastic Cosserat effects. Lately, the models have been extended to a finite elastic-plastic setting, see e.g. [GT01, San98a, San98b, San99, Ste94a, Gra03, FCS97] and references therein. Most of these finite extensions directly comprise of joint elastic and plastic Cosserat effects as well together with an additional split of the curvature into elastic and plastic parts but their physical and mathematical significance is at present much more difficult to asses than models in which Cosserat effects are restricted to the elastic response of the material [FCS97] and references therein. Our own contribution will be of the second type.

It is generally accepted that couple stresses, understood here as the presence of non-symmetric parts of the Cauchy stress, exist in real elastic material [TN65]. Discrepancies between classical linearized elasticity theory and experiments are observed predominantly for high gradients which occur by stress concentrations in the neighbourhood of holes, notches and cracks as predicted by classical elasticity. Indeed, the measured stresses around cracks are smaller than the predicted ones.

However, the source, magnitude and significance of couple stresses is still being discussed. A group of researchers [Voi87, HK65, Koi64], supported by experimental evidence [GJ75, Gau82, Sch66, ES67] admits **elastic** couple stresses in elasto statics on a **macroscale** only due to (very small) nonlocal effects such that an infinitesimal elastic Cosserat micropolar theory is meaningless: the infinitesimal continuum rotations must coincide with the infinitesimal microrotations and moreover, couple stresses are altogether neglected since they are supposedly small.[2] Despite this situation, infinitesimal elasto-plastic extensions of the indeterminate couple-stress theory have been investigated in [FMAH94, RV96].

Another group of researchers uses the infinitesimal Cosserat micropolar model, admitting non symmetric infinitesimal constitutive Cauchy stresses as a first order effect due to independent infinitesimal microrotations [Ste94a, IW98, Müh89, dB91, GT01]. Apparently, both views exclude each other. [3]

Notwithstanding, we present a model reconciling both views: the difference of opinion is due to the uncritical use of the infinitesimal, linear Cosserat model but disappears for a geometrically exact description of the Cosserat theory. The **Cosserat couple modulus** $\mu_c$ (modulus of local rotational stiffness, Cosserat shear modulus, torsional rigidity cf. (2.6)) appearing in both the infinitesimal and geometrically exact description can be **set** to **zero**, still there is a **nonlocal coupling** together with independent finite rotations, while the linearization of this theory has lost all elastic Cosserat effects.

Another eminent source for couple stresses are granular material [Osh55, MV87, Müh89, MH96, BP91, Bar94, Bar98] where individual grains are supposed to be in contact and to transmit forces by contact couples. Here, effects of couple stresses cannot usually be neglected, however, numerical simulations including a description of the contact mechanics still suggest that they are of second order [Bar98].

Using scaling arguments it is clear, that material length scale effects become the more accentuated the smaller the geometrical dimensions of the specimen are. This suggests the future application of Cosserat models for microscopic specimens or in such fields as thin films and micro actuators.

We remark that it has never really be admitted that Cosserat effects played a role as long as traditional engineering materials in their elastic range on a macroscale were considered. Since a micropolar model is

[1] Fully frame-indifferent

[2] In [HK65, p.339] we read: "Momentenspannungen sind merklich erst in Bereichen vorhanden, in denen normalerweise nicht nur die Anwendung der linearen, sondern auch der nichtlinearen Elastizitätstheorie nicht mehr sinnvoll ist."

[3] The experimental results of [FMAH94] on the torsion of thin copper wires revealed a strong **geometrical length scale effect** of the **plastic** behaviour: the thinnest wires displayed comparatively the strongest response up into the plastic range. Whether this is due to a genuine Cosserat effect cannot be ascertained. It must be noted that in their experiments, also **grain size effects** interfered, which have nothing to do with geometrical size effects.

considerably more difficult analytically, the infinitesimal[4] linear elastic Cosserat model was partially abandoned in the early seventies.

Renewed interest in Cosserat models arose with the advent of the computer allowing to circumvent analytical details. Today, apart from the theoretical development, the Cosserat type models are increasingly advocated as a means to **regularize** the pathological **mesh size dependence** of localization computations where shear failure[5] are mechanisms [CH85, MV87, Müh89, BP91, Bar94] play a dominant role, for applications in plasticity see the non-exhaustive list [IW98, DSW93, RV96, dB91, dBS91, dB92]. The occurring mathematical difficulties reflect the physical fact that upon localization of the deformation within narrow bands the validity limit [6] of the classical models is reached. In models without any internal length the deformation should be homogeneous on the scale of a representative volume element of the material [MA91].

Of course, there are many other possibilities available to overcome this deficiency, we mention only higher gradient theories [Aif98] and references therein, nonlocal models [BC84, Eri83] using integral kernels in the constitutive law or incremental variational formulations [Lam02, LMD03, ML03a, ML03b] and viscoplastic regularizations [Nef03a]. While all these models successfully regularize the mesh-dependency, nonlocal and viscoplastic approaches are localization limiters but do not necessarily introduce a specific geometric size effect. The stress-strain diagram in an incremental variational approach is mesh-independent, while the thickness of the localization zone is given by the size of the smallest finite element. Apart from incremental methods all regularizing approaches need additional material parameters. This is a distinctive advantage of incremental methods.

The incorporation of a length scale, which is natural in a Cosserat theory, in principal has the power to remove the mesh sensitivity. The presence of the internal length scale causes the localization zones to have finite width. In [DSW93, IW98] it is explicitly shown in the infinitesimal elasto-plastic context, that mode II failure (shear failure) is ruled out while the formation of holes (mode I failure) is still possible. However, the actual characteristic length scale of a material is difficult to establish experimentally and theoretically [Lak95], and remains basically an open question as is the determination of other additionally appearing material constants in the Cosserat framework: the employed non-zero value of the Cosserat couple modulus $\mu_c$ remains unmotivated in most of these regularization procedures.

Let us summarize at this stage the gathered circumferential evidence of Cosserat effects for metallic and granular materials:

[4] By infinitesimal we mean arbitrary small displacements and not just small displacements, certainly far below 1 percent elongation.

[5] In short: shear failure means for us that Legendre-Hadamard ellipticity is violated while the Baker-Ericksen inequalities are satisfied, cf. section (9.3).

[6] The occurring high deformation gradients in a shear band suggest that infinitesimal elasticity should not be used for a physically consistent description of shear bands. The same remark applies to calculations of stress concentrations in the vicinity of cracks: certainly, deformations are not any longer infinitesimal small.

| | **infinitesimal elastic** | **infinitesimal elastic-plastic** | **finite elastic** | **finite elasto-plastic** |
|---|---|---|---|---|
| **macroscopic** specimens, **heterogeneous** on a small scale, rather **homogeneous** on a large scale | **no** significant **Cosserat effects** reported: Ellis, Gauthier, Koiter, Schijve etc., $\Rightarrow \mu_c = 0$ | Cosserat models due to: Besdo, de Borst, Fleck, Forest, Iordache, Lippmann, Mühlhaus, Ristinmaa, Steinmann etc., value $\mu_c > 0$ mostly not motivated, inconsistent with infinitesimal elasticity | **significant Cosserat effects** reported for strains $> 4$ percent in localizations within shear bands: Bardet, Cambou, Mühlhaus etc. | **geometrical size effect** of plastic hardening, grain size effects, Hall-Petch relation, pronounced in **torsion**, small for stretch: Arzt, Fleck, Forest, Stölken, Tsakmakis etc. |
| **microscopic** specimen, more or less **homogeneous** | relevant size: atomic bonds, **no** appreciable **size effect** expected: Arzt, Koiter, Kröner, Voigt etc., $\Rightarrow \mu_c = 0$ | applied to simulate lattice rotations, dislocations in real single crystals, Forest, $\mu_c \gg 1$, Fleck, $\mu_c = \infty$, inconsistent with infinitesimal elasticity | relevant size: atomic bonds, **no** appreciable **size effect** expected? | microbend tests, plasticity of thin films, single crystals: Forest, Grammenoudis, Tsakmakis etc. |
| **remarks** | resulted in **premature** partial **abandoning** of linear infinitesimal indeterminate couple stress theory and linear Cosserat theory altogether | **regularize** pathological **mesh-dependence** in large scale numerical calculations, **no** intrinsic experimental evidence | Cosserat models are used to **regularize shear failure** of granular materials, based on experiments indicating a definite width of the bands | alternative: higher gradient plasticity theory, parabolic flow rule, nonlocal models, incremental variational formulations, |

If regularization of widely accepted classical continuum models on the macroscopic scale is our aim, the regularizing effects should be incorporated such that essential salient features of the classical model are still present. Otherwise we face the danger of over-regularization.[7] In the following we will see that such unwanted behaviour is in part linked to the value of certain material constants, notably the Cosserat couple modulus $\mu_c$.

The mathematical analysis of Cosserat micropolar models is at present restricted to the infinitesimal, linear elastic models, see e.g. [Ies71, Duv70, HH69, Ghe74a, Ghe74b]. The major difficulty of the mathematical treatment in the finite strain case is related to the geometrically exact formulation of the theory and the appearance of finite rotations. No general existence theorems are known to the author.

## 1.2 Outline and scope of this contribution

This contribution is organized as follows: first, we review the basic concepts of the geometrically exact elastic Cosserat micropolar theories in a variational context. In contrast to other contributions we keep the third-order tensor character of the curvature tensor $\mathfrak{K}$ and do not use a reduced second order format $\hat{\mathfrak{K}}$ based on the axial representation.

An investigation into the constitutive relations for a geometrically exact Cosserat theory apparently has never been done. We highlight therefore the striking constitutive consequences of the choice for the Cosserat couple modulus, $\mu_c > 0$, $\mu_c = 0$, respectively, in the force balance equation and it is easily seen that $\mu_c > 0$ is **not** a necessary constitutive assumption for the geometrically exact theory. Moreover, it is shown that, contrary to the infinitesimal case, the exact theory does not necessarily reduce to the classical elasticity theory in the limit of a vanishing internal length scale without further provision. To conclude this part, we provide a classification scheme of finite micropolar elasticity and motivate a new boundary condition for microrotations, which we call **consistent coupling** condition.

The elastic formulation is then consistently extended to finite multiplicative plasticity with **non-dissipative** micropolar effects. The decisive stress tensor is nothing else than the elastic **Eshelby energy momentum tensor**. Due to the third order curvature representation $\mathfrak{K}$ we retain also the Eshelby format for the curvature

[7]If the classical model fails in shear by fracture along a slip line, we impose at least that the onset of fracture is correctly reported.

part of the stresses. The obtained general plasticity model is then specialized to a macroscopic case of a polycrystal with grain rotations. It is motivated that for such a model the most natural choice is to set the Cosserat couple modulus $\mu_c = 0$. Various reductions of the geometrically exact model are possible, for conciseness we restrict attention to the infinitesimal micropolar elasto-plastic model, operative however, only for $\mu_c > 0$.

More mathematically inclined readers can safely skip the modelling part and start directly in the analytical section 4. There, the complete problem statement of the geometrically exact elastic Cosserat case in a variational context is repeated.

Existence of minimizers in Sobolev-spaces is established using the direct methods of variations and a novel extended Korn's first inequality. Similar methods allow to treat the various reduced situations as well with stronger results depending on the reductions made. However, only in the completely reduced well known infinitesimal case existence, uniqueness and continuous dependence on the data can be established.

Finally, various alternative forms of the curvature part are investigated and it is argued that Cosserat models can be superior in regularizing shear failure mechanisms than simply taking the quasiconvex hull or numerical approximations of it as a mere computational localization limiter.

In the appendix we provide missing arguments for the Boltzmann axiom and define what we mean by shear failure. In a detailed analysis of simple shear (simple glide) we derive analytical solutions which allow to contrast the different models and underline the merrits of the new approach allowing for sharp interfaces in the limit of vanishing internal length.

## 2 The finite elastic Cosserat micropolar model

Let us now motivate the finite Cosserat approach. The relevant notation will be found in the appendix. For our development we choose a strictly Lagrangean description. First, in the purely elastic case, a Cosserat theory can be obtained by introducing the multiplicative decomposition of the macroscopic deformation gradient $F$ into **independent microrotation** $\overline{R}$ (Cosserat rotation tensor) and **micropolar stretch tensor** $\overline{U}$ (or first Cosserat deformation tensor) with

$$F = \overline{R} \cdot \overline{U}\,, \tag{2.1}$$

where $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and $\overline{U} \in \mathrm{GL}^+(3,\mathbb{R})$ but $\overline{U} \notin \mathrm{PSym}(3)$ such that (2.1) is not necessarily the polar decomposition of $F$. The notion **micro**polar is prone to misunderstandings: $\overline{R}$ must be considered as a macroscopic (average) quantity as the deformation gradient and the resulting model is still phenomenological.

In the quasistatic case, the Cosserat theory is now derived from a variational principle by postulating the following **'action euclidienne'** [CC09, p.156] for the finite macroscopic deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the independent microrotation $\overline{R} : \overline{\Omega} \mapsto \mathrm{SO}(3)$:

$$\begin{aligned} I(\varphi, \overline{R}) &= \int_\Omega W(F, \overline{R}, \mathrm{D_x}\overline{R}) - \langle f, \varphi\rangle - \langle M, \overline{R}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}\rangle \,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (\varphi, \overline{R}), \\ \overline{R}_{|_\Gamma} &= \overline{R}_{\mathrm{d}}\,, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}(t)\,. \end{aligned} \tag{2.2}$$

The elastically stored energy $W$ depends on the deformation gradient as usual but in addition on the microrotations together with their space derivatives. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g, \overline{R}_{\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions $N$ are applied with $\Gamma \cap \Gamma_S = \emptyset$. The external volume force is $f$ and $M$ takes on the role of external volume couples.[8] In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where surface couples $M_c$ are applied with $\Gamma \cap \Gamma_C = \emptyset$. Variation of the action $I$ with respect to $\varphi$ yields the equation for balance of linear momentum and variation of $I$ with respect to $\overline{R}$ yields balance of angular momentum.

The standard conclusion from frame-indifference (here: invariance of the free energy under **s**uperposed **r**igid **b**ody **m**otions (**SRBM**) not merely observer invariance of the model [SB97, BS01, Mur03]: $\forall\, Q \in \mathrm{SO}(3,\mathbb{R})$ : $W(F, \overline{R}, \mathrm{D_x}\overline{R}) = W(QF, Q\overline{R}, \mathrm{D_x}[Q\overline{R}])$ leads to the reduced representation of the energy

$$W(F, \overline{R}, \mathrm{D_x}\overline{R}) = W(\overline{R}^T F, 1\!\!1, \overline{R}^T \mathrm{D_x}\overline{R}) = W(\overline{U}, \overline{R}^T \mathrm{D_x}\overline{R}) = W(\overline{U}, \mathfrak{K})\,, \tag{2.3}$$

[8] appearing in a non-mechanical context e.g. as influence of a magnetic field on the polarization of a substructure of the bulk.

where $\mathfrak{K} := \overline{R}^T \mathrm{D}_x \overline{R} = \left(\overline{R}^T \nabla(\overline{R}.e_1), \overline{R}^T \nabla(\overline{R}.e_2), \overline{R}^T \nabla(\overline{R}.e_3)\right)$ is one specific representation[9] of the third order right **micropolar curvature tensor** (or torsion-curvature tensor, wryness tensor, second Cosserat deformation tensor, bending-twist tensor etc.). For a geometrically exact isotropic[10] theory we assume in the following an additive split of the total free energy into local stretch and curvature part according to

$$W = W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K})\,. \tag{2.4}$$

## 2.1 The elastic stretch energy

For a **small elastic strain** theory, which should already cover most cases of physical interest, we require that $W_{\mathrm{mp}}(\overline{U})$ is a non negative isotropic quadratic form [11] normalized to

$$W_{\mathrm{mp}}(1\!\!1) = 0, \quad D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})_{|_{\overline{U}=1\!\!1}} = 0\,. \tag{2.5}$$

The most general form of $W_{\mathrm{mp}}$ consistent with (2.5) is cf.(10.151)

$$W_{\mathrm{mp}}(\overline{U}) = \alpha_1 \,\|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\|\operatorname{skew}(\overline{U} - 1\!\!1)\|^2 + \alpha_3 \operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2, \tag{2.6}$$

with material constants $\alpha_1, \mu_c, \alpha_3$ such that $\alpha_1,\, 3\alpha_3 + \alpha_1, \mu_c \geq 0$ from non negativity [Eri99]. By consistency with the classical continuum model without microrotations we can take $\alpha_1 = \mu$, $\alpha_3 = \frac{\lambda}{2}$ with $\mu, \lambda > 0$ the classical Lamé constants and the **Cosserat couple modulus** $\mu_c$ remains for the moment unspecified but $\mu_c = 0$ is physically possible since the **micropolar reaction stress** $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T$ is not symmetric in general, i.e. the problem does not decouple, cf. (2.12). For comparison, in [Eri99, p.111] for the infinitesimal case, the elastic moduli are taken to be $\alpha_1 = \mu + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$, $\alpha_3 = \frac{\lambda}{2}$ but in this formula, $\mu$ cannot be regarded as one of the Lamé constants. [12] In [DSW93, Ste94a, Ste97, FCS97, DFC98, EDV98a] the abbreviation $\mu_c$ is used while in [Gra03] it is $\mu_c = \alpha$ and $\mu_c = G_c$ in [IW98]. [13]

## 2.2 The elastic curvature energy

For the curvature term, to be specific, we assume the general form

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12}\, (1 + \alpha_4\, L_c^q\, \|\mathfrak{K}\|^q) \left(\alpha_5 \,\|\operatorname{sym} \mathfrak{K}\|^2 + \alpha_6 \,\|\operatorname{skew} \mathfrak{K}\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}, \tag{2.7}$$

where $L_c$ is setting an internal length scale with units of length, $\alpha_4 \geq 0, p > 0, q \geq 0$ are additional material constants, the factor $\frac{1}{12}$ only for convenience and $\alpha_5 > 0, \alpha_6, \alpha_7 \geq 0$ as a minimal requirement. We mean $\operatorname{tr}[\mathfrak{K}]^2 = \|\operatorname{tr}[\mathfrak{K}]\|^2$ by abuse of notation. This choice for $W_{\mathrm{curv}}$ does not presuppose any knowledge of the magnitude of curvature [14] in the material and is non-degenerate in the origin $\|\mathfrak{K}\| = 0$, which is not essential however.

In [FCS97, DFC98] the following set of parameters $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, p) = (0, \beta, \gamma, \alpha, 1)$ is used, while in [BGdB98, EDV98a] the reduced set $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, p) = (0, \mu, \mu, \alpha, 1)$ is taken and $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, p) = (0, \gamma, \delta, \beta, 1)$ is used in [Gra03]. [15]

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[9] Note that $\mathfrak{K}^i = \overline{R}^T \nabla(\overline{R}.e_i) \notin \mathfrak{so}(3,\mathbb{R})$. Another representation of $\mathfrak{K}$ is given by $\overline{\mathfrak{K}} := \left(\overline{R}^T \partial_x \overline{R}, \overline{R}^T \partial_y \overline{R}, \overline{R}^T \partial_z \overline{R}\right)$. Since $\partial_x(\overline{R}^T \overline{R}) = 0$ for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ it holds that $\overline{\mathfrak{K}} \in \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})$. It is therefore possible to base all considerations of curvature on a more compact expression like $\hat{\mathfrak{K}} := \left(\operatorname{axl}(\overline{R}^T \partial_x \overline{R}) | \operatorname{axl}(\overline{R}^T \partial_y \overline{R}) | \operatorname{axl}(\overline{R}^T \partial_z \overline{R})\right) \in \mathbb{M}^{3\times 3}$. This is the traditional approach, see e.g. [San99, FBC00, Gra03]. We do not use $\hat{\mathfrak{K}}$ since it looses its advantages over $\mathfrak{K}$ if we want to consider **micromorphic** extensions of the theory, e.g. if we would allow for $\overline{R} \in \mathrm{SL}(3,\mathbb{R})$. By extending the theory to multiplicative plasticity it becomes apparent that $\mathfrak{K}$ is a natural representation.

[10] The complete structure for anisotropic infinitesimal formulations has been given in [Kes64].

[11] Hencky-type energies defined on $\ln \overline{U}$ are useless, since $\overline{U} \neq \mathrm{PSym}$ in general. The stretch part could depend in principle on $C = \overline{U}^T \overline{U}$, but would then fail to be altogether quadratic in $\overline{U}$. The same argument excludes a dependence on $U$. In addition, a possible coupling between $\overline{U}$ and $\mathfrak{K}$ for centrosymmetric bodies can be ruled out [Now86, p.14].

[12] A simple definition of the Lamè constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where $E$ and $\nu$ are uniquely determined from uniform traction. Eringens nomenclature unfortunately led to some confusion.

[13] In the remainder, from a modelling point of view, we need to carefully distinguish between micropolar moduli for microscopic specimens and effective moduli for macroscopic specimens, depending on the application aimed at.

[14] The following question merits attention: Is it experimentally possible to determine not only the value of the internal length scale $L_c$ but also to determine the relevant exponents $p, q$ independent of mathematical convenience. Dispersion experiments are the obvious candidate.

[15] Note that this identification is based on the representation of $\mathfrak{K}$ in terms of the axial representation $\hat{\mathfrak{K}}$. All results in the mathematical section hold without modification for $\hat{\mathfrak{K}}$ as well.

The form (2.7) is motivated by consistency with an expansion for a corresponding shell theory. But care has to be exerted in the finite regime: $W_{\text{curv}}$ should preferably be **coercive** in the sense that we impose pointwise

$$\exists\, c^+ > 0\, \exists\, r > 1 : \ \forall\, \mathfrak{K} \in \mathfrak{T}(3) : \quad W_{\text{curv}}(\mathfrak{K}) \geq c^+ \, \|\mathfrak{K}\|^r , \tag{2.8}$$

or less demanding

$$\exists\, r > 1 : \quad \frac{W_{\text{curv}}(\mathfrak{K})}{\|\mathfrak{K}\|^r} \to \infty \quad \text{as } \|\mathfrak{K}\| \to \infty , \tag{2.9}$$

which implies necessarily $\alpha_6 > 0$ in (2.8). This is at variance with the infinitesimal case (4.60), where $\alpha_6 = 0$ is still acceptable. A major step forward in the subsequent mathematical treatment would be to show that giving up (2.9), i.e. $\alpha_6 = 0$, still yields a well posed geometrically exact finite micropolar theory.

If it is known in advance that the curvature remains small, i.e. $L_c \cdot \|\mathfrak{K}\| \ll 1$, then $\alpha_4 = 0$, $p = 1$ may be a reasonable choice [GT01, BGdB98, EDV98a, FCS97, Ste94a] but we will see that this leads to a loss of control (in the critical case $\mu_c = 0$) in the corresponding finite boundary value problem that can be overcome by taking $\mu_c > 0$, which will be seen below. Consistent with this observation (but not based on), in [Sch62, Bes74, San99] the parameter $\mu_c$ has been set to $\mu_c = \mu$ such that

$$W_{\text{mp}}(\overline{U}) = \mu \, \|\overline{U} - 1\!\!1\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\overline{U} - 1\!\!1\right]^2 , \tag{2.10}$$

superficially coinciding with the functional form of the classical free energy defined on the positive definite right stretch tensor $U$. In [Ste94a, Ste94b] the strain energy

$$W_{\text{mp}}(\overline{U}) = \mu \, \langle \frac{\overline{U}}{\det[\overline{U}]^{1/3}} - 1\!\!1, 1\!\!1 \rangle + h(\det[\overline{U}]) \tag{2.11}$$

is proposed. Upon linearization, this corresponds as well to the choice $\mu = \mu_c$.

## 2.3 The balance equations

For the choices we have made we note the resulting material form of the field equations on the reference configuration (with $\alpha_4 = 0$, $p = 1$) which can be obtained after some algebraic manipulations.

$$\begin{aligned} 0 &= \operatorname{Div}\left( S_1(F, \overline{R}) + 2\,\mu_c\, \overline{R}\, \operatorname{skew}(\overline{R}^T F) \right) + f_{\mathbb{R}^3}, \qquad \textbf{force balance} \\ 0 &= \operatorname{skew}(D_{\overline{U}} W_{\text{mp}}(\overline{U})\, \overline{U}^T) + \operatorname{skew}\left( \overline{R}^T \operatorname{Div}\left[\overline{R}\, D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})\right]\right) + \operatorname{skew}\left(D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K}) \mathfrak{K}^T\right) \\ &\quad + \operatorname{skew}(\overline{R}^T M)_{\mathbb{M}^{3\times 3}}, \qquad \textbf{angular momentum balance}, \end{aligned} \tag{2.12}$$

where $S_1$ is the first Piola-Kirchhoff stress (for $\mu_c = 0$) with the functional form

$$S_1(F, \overline{R}) = \overline{R}\left[ \mu(F^T \overline{R} + \overline{R}^T F - 2 1\!\!1) + \lambda \operatorname{tr}\left[F^T \overline{R} - 1\!\!1\right] 1\!\!1 \right] , \tag{2.13}$$

as in [Nef03a, (P3)] and $D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})$ is the material **micropolar moment tensor** (or **couple-stress tensor**). A similar form of the unconventional[16] balance of angular momentum equation has been given in [Cap89, p.63]. In our variationally based development, the balance equations will not play a prominent role.

## 2.4 Constitutive consequences of the value for the Cosserat couple modulus

Looking at (2.6) with $\mu_c > 0$ we see that the implication of this choice at a first glance is an inoccuous rise in the macroscopic elastic strain energy $W_{\text{mp}}(\overline{U})$ if $\overline{R} \neq \operatorname{polar}(F)$, but $\overline{R}$ is generically assumed to be independent of $\operatorname{polar}(F)$. The choice $\mu_c > 0$ acts like a local 'elastic spring' between both continuum rotations and microrotations.

[16]because we have not transformed the tensor equation into a related vector format, which is usually preferred. Following [Cap89] we can identify an external volume couple $b_c$ in the equilibrium vector-format with $\operatorname{axl}(\operatorname{skew}(\overline{R}^T M))$. Then $b_c$ is a volume couple which is not a dead load. We note that a term $\operatorname{skew}\left(D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})\, \mathfrak{K}^T\right)$ does not directly appear in derivations based on $\hat{\mathfrak{K}}$ since e.g. $\hat{\mathfrak{K}}^1 = \operatorname{axl}(\overline{R}^T \partial_x \overline{R})$ and variation along the one-parameter group introduced in (10.154) yields

$$\delta \hat{\mathfrak{K}}^1 = \operatorname{axl}((A\overline{R})^T \partial_x \overline{R} + \overline{R}^T \partial_x [A\overline{R}]) = \operatorname{axl}(-\overline{R}^T A \partial_x \overline{R} + \overline{R}^T (\partial_x A) \overline{R} + \overline{R}^T A \partial_x \overline{R}) = \operatorname{axl}(\overline{R}^T (\partial_x A) \overline{R}) .$$

This is not at variance with $(2.12)_2$ since differentiation is carried out differently. Observe that $\operatorname{skew}\left(D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})\, \mathfrak{K}^T\right) = 0$ if $\alpha_5 = \alpha_6$, $\alpha_7 = 0$, i.e. if couple stresses are proportional to the curvature tensor.

Let us consider the mathematical implications of $\mu_c = 0$ and $0 < \mu_c \le \mu$, respectively, in more detail. It is readily verified that for the elasticity tensors (differentiating the stretch energy $W_{\rm mp}(\overline{R}^T F)$ at fixed $\overline{R}$ w.r.t. $F$)

$$\begin{aligned} \mu_{\mathbf{c}} > 0 \Rightarrow \quad & \forall H \in \mathbb{M}^{3\times 3}: \quad D_F^2 W_{\rm mp}(\overline{R}^T F).(H,H) \ge 2\,\mu_c\,\|H\|^2 \\ \mu_{\mathbf{c}} = 0 \Rightarrow \quad & \forall H \in \mathbb{M}^{3\times 3}: \quad D_F^2 W_{\rm mp}(\overline{R}^T F).(H,H) \ge 2\,\mu \|\frac{1}{2}(\overline{R}^T H + H^T \overline{R})\|^2 . \end{aligned} \tag{2.14}$$

Hence the choice $\mu_c > 0$ leads to **uniform convexity** of $W_{\rm mp}(\overline{R}^T F)$ w.r.t. $F$ and **unconditional elastic stability** on the macroscopic level: regardless of what distribution of microrotations $\overline{R}(x)$ is given, the macroscopic equation of balance of linear momentum is uniquely solvable and this equation is insensible to any deterioration of the spatial features of the microstructure. Uniform convexity is difficult to accept from a constitutive point of view since it is impossible for a geometrically exact description in the framework of a classical macroscopic continuum but clear from the above discussion: the additional elastic spring between micro- and continuum rotation extremely rigidifies the material and completely changes the type of the mathematical boundary value problem compared with the classical finite theory.[17]

Fortunately, such a far reaching unsatisfactory conclusion does not hold for $\mu_c = 0$, in which case we have

$$D_F^2 W_{\rm mp}(\overline{R}^T F).(\xi\otimes\eta,\xi\otimes\eta) = \mu\Big(\|\overline{R}^T\xi\otimes\eta\|^2 + \langle \overline{R}^T\xi\otimes\eta, \eta\otimes\overline{R}^T\xi\rangle\Big) = \mu\Big(\|\overline{R}^T\xi\otimes\eta\|^2 + \langle \overline{R}^T\xi,\eta\rangle^2\Big),$$

which shows the physically much more appealing inequality

$$D_F^2 W_{\rm mp}(\overline{R}^T F).(\xi\otimes\eta,\xi\otimes\eta) \ge \mu\,\|\xi\|^2\cdot\|\eta\|^2 , \tag{2.15}$$

expressing nothing but uniform **Legendre-Hadamard ellipticity** of the acoustic-tensor with ellipticity constant $\mu$ **independent** of $\overline{R}$. The Legendre-Hadamard condition has the most convincing physical basis [Ant95, p.461] in that it implies the reality of wave speeds and the Baker-Ericksen inequalities (stress increases with strain, [MH83, p.19]). The choice $\mu_c = 0$ leads to the strain energy density proposed in [Nef03a, (P3)] and [NW03, M1] if the appearing independent viscoelastic rotations there are identified with the independent elastic Cosserat microrotations here. [18]

## 2.5 The Boltzmann axiom

In the absence of volume couples and curvature terms, i.e. without internal length scale, $L_c = 0$, the second equation in (2.12) reduces to the classical symmetry condition [SFH92, (6)], the so called **Boltzmann axiom**,

$$D_{\overline{U}} W_{\rm mp}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \Leftrightarrow \mathrm{skew}\left(D_F W(\overline{R}^T F)\,F^T\right) = 0 \Leftrightarrow S_2 := F^{-1} S_1(F,\overline{R}) \in \mathrm{Sym}, \tag{2.16}$$

**postulating the symmetry of the second Piola-Kirchhoff stress** $S_2$ and we note that trivially $\left[\overline{U} \in \mathrm{Sym} \Leftrightarrow \overline{R} = \mathrm{polar}(F)\right] \Rightarrow D_{\overline{U}} W_{\rm mp}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$. However, for the converse we state a first result:

**Lemma 2.1 (Limit rotations with zero internal length scale)**
*Let $W_{\rm mp}$ be defined as in (2.6). If $\alpha_1 = \mu_c$ and $\mathrm{tr}\left[\overline{U}\right] < 3 + \frac{2\mu_c}{\alpha_3}$ then*

$$D_{\overline{U}} W_{\rm mp}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \Rightarrow \left[\overline{U} \in \mathrm{Sym} \Leftrightarrow \overline{R} = \mathrm{polar}(F)\right] . \tag{2.17}$$

*Otherwise, $D_{\overline{U}} W_{\rm mp}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ alone does not imply $\overline{U} \in \mathrm{Sym}$. In other words: symmetry of the Cauchy stresses $T = \frac{1}{\det[F]} F\,S_2\,F^T$ does not imply that microrotations coincide with continuum rotations.*

**Proof.** The proof is given in (9.2). This discrepancy between the fulfilment of the Boltzmann axiom and the symmetry of the microstretch $\overline{U}$ does not appear in the infinitesimal linear case, see (4.62). ■

We mention that an argument relating to the general case of $W_{\rm mp}$ taken as an isotropic scalar valued function of $\overline{U}$ has been given e.g. in [San99, p.29] and [SB95]. No conditions on the coefficients or the magnitude of

[17] In the analytical section we will see that $\mu_c > 0$ implies that $\varphi \in W^{1,1}(\Omega,\mathbb{R}^3)$ irrespective of $\overline{R}$, thus **excluding fracture**.

[18] The preferred value $\mu_c = 0$ for the macroscopic case can as well be motivated by the following consideration: Consider the Green strains $F^T F - 1\!\!1 = (\overline{U} - 1\!\!1)^T(\overline{U} - 1\!\!1) + 2\,\mathrm{sym}(\overline{U} - 1\!\!1)$. Therefore $\frac{\mu}{4}\|F^T F - 1\!\!1\|^2 = \mu\|\,\mathrm{sym}\,\overline{U} - 1\!\!1\|^2 + O(\|\overline{U} - 1\!\!1\|^3)$. Hence $\mu_c = 0$ provides the correct first order approximation to a classical St. Venant-Kirchhoff material. With $\mu_c = 0$ we exclusively recover the fact of the classical continuum theory that $W$ isotropic implies symmetry of the Biot stress tensor: $D_U W(U) \in \mathrm{Sym}$. If we expand $\overline{R} = 1\!\!1 + \overline{A} + \ldots$ with $\overline{A} \in \mathfrak{so}(3)$ and write $F = 1\!\!1 + \nabla u$, then the Cosserat effects dissappear to first order for $\mu_c = 0$. In this sense, $\mu_c = 0$ is close to classical elasticity.

tr $[\overline{U}]$ are involved, which raises some questions. However, the conclusion in [San99, p.29] is true in some cases [19], which may be seen for

$$W_{\rm mp}(\overline{U}) = \mu\|\overline{U} - 1\!\!1\|^2 + \mu\left(\det[\overline{U}] + \frac{1}{\det[\overline{U}]} - 2\right)^2 = \mu\,\|\overline{U} - 1\!\!1\|^2 + \mu\left(\det[F] + \frac{1}{\det[F]} - 2\right)^2$$
$$D_{\overline{U}} W_{\rm mp}^{\rm shear}(\overline{U}) \cdot \overline{U}^T = 2\mu\,\left(\overline{U} - 2 1\!\!1\right)\overline{U}^T\,. \tag{2.18}$$

Since $\det[\overline{U}] = \det[\overline{R}^T F] = \det[F]$ is independent of $\overline{R}$, balance of angular momentum is only affected through the shear contribution and independent of the volumetric response. Therefore the symmetry condition in the second equation in (2.12) reduces to $2\mu\,\overline{U}\,\overline{U}^T - 2\mu\,\overline{U}^T \in \mathrm{Sym}$, which implies already $\overline{U} \in \mathrm{Sym}$. The same reasoning applies to (2.11). Incidentally, this could be a reasonable assumption for a generalized finite Cosserat micropolar theory as well: why should the microrotations $\overline{R}$ affect the volumetric response of the material on the macroscopic level at all? We would be led to assume that the stretch energy has the form

$$W_{\rm mp}(\overline{U}) = W_{\rm mp}^{\rm shear}(\overline{U}) + h(\det[\overline{U}])\,, \tag{2.19}$$

decoupling shear behaviour from volumetric response.

The result of Lemma 2.1 is noteworthy: It shows that **symmetry** of $\overline{U}$ is an **independent** additional **assumption** generally not implied by balance of angular momentum (2.5) in the absence of an internal length scale (arbitrary large samples) in the finite regime.

## 2.6 Classification of elastic Cosserat micropolar models

Let us summarize at this stage the proposed geometrically exact finite elastic Cosserat micropolar model: the task is to find $(\varphi, \overline{R}) \in \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ such that

$$\int_\Omega W_{\rm mp}(\overline{U}) + W_{\rm curv}(\mathfrak{K}) - \langle f, \varphi\rangle - \langle M, \overline{R}\rangle\,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle\,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}\rangle\,\mathrm{dS} \mapsto \min\,.\ \text{w.r.t.}\ (\varphi, \overline{R}),$$
$$\overline{U} = \overline{R}^T F, \quad F = \nabla\varphi, \quad \mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}, \quad \overline{R}_{|_\Gamma} = \overline{R}_{\rm d}, \quad \varphi_{|_\Gamma} = g_{\rm d} \tag{2.20}$$
$$W_{\rm mp}(\overline{U}) = \mu\,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2$$
$$W_{\rm curv}(\mathfrak{K}) = \mu\frac{L_c^{1+p}}{12}\,\left(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q\right)\,\left(\alpha_5\,\|\,\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,.$$

In [Ste97] a classification of isotropic micropolar theories is given. We refer to the geometrically exact case therein and put our models in this framework at the same time extending it.

0. **rigidity in torsion**: $\mu_c = \infty$, $L_c = \infty$. Classical infinitesimal elasticity in tension. $L_c = \infty$ keeps the microrotations constant and $\mu_c = \infty$ implies that continuum- and microrotations coincide.

1. **gradient type, constrained Cosserat micropolar theory** (or **indeterminate couple-stress theory**, a special case of an **elastic material of grade two**,[Tou62, Tou62, Min64, Gri60, Koi64, Eri68, Cap85]): $\mu_c = \infty$, $\mathfrak{K} = \mathrm{polar}(\nabla\varphi)^T \mathrm{D_x}\,\mathrm{polar}(\nabla\varphi)$, $\overline{U} = U = \mathrm{polar}(\nabla\varphi)^T\nabla\varphi$, variation of the action functional $I$ only with respect to $\varphi$: the field equations are of fourth order, microrotations coincide with macrorotations, local contribution to the infinitesimal Cauchy stress tensor $\sigma^{\rm loc}$ and local contribution[20] to the finite second Piola-Kirchhoff stress tensor $S_2^{\rm loc} := F^{-1}\,D_F W_{\rm mp}(\overline{U})$ are **symmetric**, "trièdre caché" in the original Cosserat terminology [CC09, p.30], only the non-local part (called the hyperstress) is responsible for the overall antisymmetric stresses. The antisymmetric part of the total stress is therefore not determined by the local value of the deformation field alone.

2. **regularized gradient theory**: $\mu_c > 0, \mu_c \to \infty$, $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}$, independent variation with respect to $(\varphi, \overline{R})$, field equations of second order, independent microrotations, a subclass of case (3), $\mu_c \to \infty$ as a penalty parameter, yields in the limit constrained gradient theory (indeterminate couple stress theory), provides a 'cheap' numerical approximation to (1).

[19] It would be useful to obtain general necessary and sufficient conditions on the free energy function $W$ such that $[\overline{U} \in \mathrm{Sym} \Leftrightarrow \overline{R} = \mathrm{polar}(F)] \;\Leftrightarrow\; D_{\overline{U}} W_{\rm mp}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$. A further problem is then to characterize the classical continuum as a certain limit of the finite Cosserat model for vanishing internal length. We might want to conjecture that the classical Boltzmann continuum is the $\Gamma$-limit [Mas92] of (2.2) for $L_c \to 0$ and $\mu_c = 0$, with appropriate boundary conditions $\overline{R}_{|_\Gamma} = \mathrm{polar}(\nabla\varphi)$ preventing the situation in (9.85). A proof of this conjecture is beyond the scope of this investigation.

[20] For us, stress denotes the sum of local and nonlocal stresses in the force balance equations. Usually, what we call local stress is denoted simply with (constitutive) stress whereas our nonlocal stress is said to be the hyperstress in the case of the constrained gradient model.

3. **first order Cosserat micropolar**: $\mu_c > 0$, $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}$, independent variation with respect to $(\varphi, \overline{R})$, field equations of second order, independent microrotations, strong local coupling of first order between continuum rotations and microrotations, infinitesimal (constitutive) Cauchy stress tensor $\sigma$ and finite second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U})$ are **non-symmetric**, "trièdre mobiles" in the original Cosserat terminology. Appropriate for rather rigid microscopic specimens, fracture excluded since $\mu_c > 0$.

3.1 **traditional infinitesimal Cosserat micropolar**: $\mu_c > 0$, $\kappa = \mathrm{D_x}\overline{A}$, $\overline{R} = \exp(\overline{A})$, $\overline{A} \in \mathfrak{so}(3)$, independent variation w.r.t. displacement and infinitesimal microrotations $(u, \overline{A})$, linearization of (3). The next three cases are our own contribution:

4. **second order Cosserat micropolar** (or **relaxed micropolar theory**): $\mu_c = 0$, $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}$, independent variation with respect to $(\varphi, \overline{R})$, field equations of second order, independent microrotations, weak non-local coupling of second order, infinitesimal (linearized) Cauchy stress tensor $\sigma$ is still **symmetric**, second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U})$ is **non-symmetric**. The antisymmetric part of the stresses is determined. Appropriate for macroscopic specimens, in principle allowing for **fracture**.

4.1 **second order consistent Cosserat micropolar**: as case (4), but independent Dirichlet boundary condition for the microrotations $\overline{R}_{|\Gamma} = \overline{R}_{\mathrm{d}}$ replaced by **consistent coupling** requirement $\overline{R}_{|\Gamma} = \mathrm{polar}(\nabla\varphi)_{|\Gamma}$.

4.2 **finite elasticity with free rotations and microstructure**: $\mu_c = 0$, $L_c = 0$, independent variation w.r.t. $(\varphi, \overline{R})$, no internal length scale. **Symmetry** of the second Piola-Kirchhoff tensor $S_2$ is a **local side condition** coming from balance of angular momentum which does not imply that $\overline{R} = \mathrm{polar}(\nabla\varphi)$. Local minimization of rotations. Weak solutions of classical finite elasticity are automatically stationary solutions of this minimization problem. In this sense encompassing classical finite elasticity.

5. **classical finite elasticity**: $\mu_c = 0$, $L_c = 0$, independent variation only w.r.t. $\varphi$, no internal length scale. The second Piola-Kirchhoff tensor $S_2$ is automatically **symmetric**.

6. **classical infinitesimal elasticity**: $\mu_c = 0$, $L_c = 0$, variation w.r.t. $\varphi$, no internal length scale. The infinitesimal Cauchy stress $\sigma$ is **symmetric**.

One may be inclined to think that case (1) is closest to classical elasticity. This is not true. To the contrary, the influence of the curvature part on the deformation is much more pronounced since the spatial variation of the continuum rotations is directly penalized. Such a model tends to systematically maximize the material length scale effects, see e.g. [Eri99, p.222] where stress concentration factors are computed for the different cases based on the infinitesimal theory. Use of (1) as a model in its own right has been put into question on theoretical grounds [Eri68, p.698] and rejected by Koiter [Koi64] as well who, however, prematurely concluded that couple stresses altogether played no prominent role. If such a model is intended to approximate classical linear elasticity, then the appearing length scale $L_c$ must be chosen significantly smaller than the length scale $L_c$, which appears in the Cosserat micropolar models with independent rotations.

If we assume that classical infinitesimal elasticity is a correct approximation to material behaviour under very small loads but that couple stresses may nevertheless occur in a material [TN65, p.398], we prefer for applications within the elastic range defined on a macroscopic level the case (4) of weak nonlocal coupling $(\mu_c = 0)$ without excluding the other (more microscopically) cases from our mathematical analysis.

It is clear that if couple stresses are assumed to be second order effects[21], then they should not appear in the infinitesimal treatment in the first place which is exactly what we obtain for $\mu_c = 0$ subsequently. Experimental evidence [Eri99, p.165 ] suggests already that $\mu_c \approx 0.0039\,\mu$ for the infinitesimal theory, orders of magnitude smaller than the classical shear modulus. This value is consistent with results [DFC98] obtained from calculations on discrete networks with rigid substructure. In the same paper, it is shown that if the representative volume element is increased (macroscopic case) then $\mu_c \to 0$ while the geometrical size effect is still present.[22]

The case of $\mu_c > 0$ might be however, suitable for computations on a microscale where internal material length scales are of the order of the geometrical dimensions of the specimen. This is e.g. the case in models for single crystals where the microrotations conceptually should closely follow the lattice vectors for a proper physical definition of them. Then $\mu_c \gg 1$ may be advisable since we do not expect fracture.

[21] "In classical elasticity couple stresses are to be interpreted as a non-local effect intimately connected with the range of the atomic forces. The couple stresses are of a higher order in this range than force-stresses and can therefore usually be neglected." [HK65]

[22] In the same volume [DDC98] a numerical investigation of contact couples for granular media arrives at the conclusion: "The effect of contact couples appears only by a second order term which is not considered by the (infinitesimal) Cosserat approach." (infinitesimal) my addition. In [Bar94, Fig.2] it is shown that for strains up to 4 percent the effects of particle rotations in idealized granular materials are practically absent, whereas for higher strains particle rotation significantly decreases the failure stress. The role of particle rotations is especially important in shear bands.(ibidem)

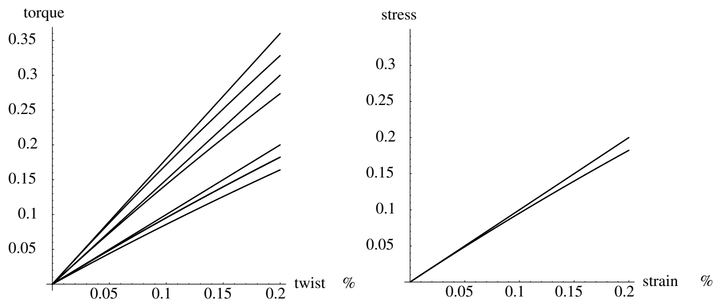


Figure 1: Left: Qualitative influence of $\mu_c$ and $L_c$ on the **elastic** behaviour in **torsion**. From bottom to top: finite elasticity (5), second order Cosserat (4), infinitesimal elasticity (6), first order Cosserat (3), linearized first order Cosserat (3´), indeterminate couple stress theory (1), linearized indeterminate couple stress theory, rigid behaviour (vertical axes) (0). Right: finite elasticity and infinitesimal linear elasticity coincide in tension for ideally homogeneous bulk material. No Cosserat effects. If inhomogeneity is present, a small Cosserat effect will appear, lower curve. If $\mu_c > 0$, the elastic response in **torsion** would be **stiffer** than expected from calculations with linear elasticity based on material parameters obtained from measurements in tension, uniform traction and uniform compression already for arbitrary small twist. The smaller $\mu_c$, the larger one may choose $L_c$ in order to still approximate classical infinitesimal elasticity.

We can gain some feeling as regards the influence of the Cosserat couple modulus $\mu_c$ and the characteristic length $L_c$ on the finite Cosserat model by looking at extremal values: $L_c = 0$ corresponds to the physically possible limit case of arbitrary large samples, $L_c = \infty$ corresponds to the limit case of arbitrary small samples.

| | $L_c = 0$ | $L_c = 0.1$ | $L_c = \infty$ |
|---|---|---|---|
| $\mu_c = 0$ | **finite elasticity with free rotations and microstructure**, case (4.2), encompassing classical **finite elasticity** model, case (5) | new **second order Cosserat** model, case (4), linear: case (6) | classical **infinitesimal elasticity** model, case (6) |
| $\mu_c = \frac{\mu}{2}$ | in certain! cases: classical **finite elasticity** model, case (5) | finite **first order Cosserat** model, case (3), linear: case (3.1) | **linear elasticity** in **torsion** (7), **inconsistent** torsional tangent **modulus**, classical **infinitesimal elasticity** in **tension** |
| $\mu_c = \infty$ | classical **finite elasticity** model, case (5) | traditional **indeterminate couple stress** theory, gradient constrained model, case (1) | totally **rigid** behaviour in **torsion**, case (0), classical **infinitesimal elasticity** in tension |

The only row where each entry does not conflict with experiments is the first one. It is to be observed that $L_c = \infty$ linearizes the behaviour with respect to a given rigid configuration. [23]

For completeness we state the **finite gradient constrained Cosserat micropolar (indeterminate couple stress theory)** problem as well, formally corresponding to $\mu_c = \infty$. Given the boundary value

[23] Uniform traction and uniform compression do not activate rotations, hence the classical identification of the Lamé constants is achieved independent of $\mu_c$. Uniform traction alone allows already to determine the Young modulus $E$ and the Poisson ratio $\nu$ [Cia88, p.126]. Contrary to [Gau82, p.411] we do not see the possibility to define a specific "micropolar Young modulus" or "micropolar Poisson ratio".

$g_{\rm d} \in H^1(\Omega,\mathbb{R}^3)$, $\nabla g_{\rm d} \in \mathrm{GL}^+(3,\mathbb{R})$ *a.e.* we look for the deformation $\varphi : [0,T]\times\overline{\Omega} \mapsto \mathbb{R}^3$ satisfying

$$
\begin{aligned}
I(\varphi) =& \int_\Omega W_{\rm mp}(\mathrm{polar}(\nabla\varphi)^T\nabla\varphi) + W_{\rm curv}(\mathrm{polar}(\nabla\varphi)^T \mathrm{D_x}\,\mathrm{polar}(\nabla\varphi)) - \langle f,\varphi\rangle - \langle M, \mathrm{polar}(\nabla\varphi)\rangle \,\mathrm{dV} \\
& - \int_{\Gamma_S} \langle N,\varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \mathrm{polar}(\nabla\varphi)\rangle \,\mathrm{dS} \mapsto \min.\ \text{w.r.t. } \varphi, \\
W_{\rm mp}(U) =& \mu\,\|U-1\!\!1\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[U-1\!\!1\right]^2 \qquad (2.21)\\
S_2^{\rm loc} =& F^{-1}D_F W_{\rm mp}(U) \in \mathrm{Sym}\,,\ \text{constitutive stress} \\
W_{\rm curv}(\mathfrak{K}) =& \mu\frac{L_c^{1+p}}{12}\,\left(1+\alpha_4\,L_c^q\,\|\mathfrak{K}\|^q\right)\left(\alpha_5\,\|\,\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\left[\mathfrak{K}\right]^2\right)^{\frac{1+p}{2}}\,, \\
\varphi_{|_\Gamma} =& g_{\rm d}(t)\,, \quad \mathrm{polar}(\nabla\varphi)_{|_\Gamma} = \mathrm{polar}(\nabla g_{\rm d})_{|_\Gamma}\,.
\end{aligned}
$$

Balance of angular momentum is, as the consequence of invariance of the action under rigid rotations, automatically satisfied and is but a defining equation for the moment stress tensor and not recorded here. Observe that the first order, local contribution $W_{\rm mp}(U)$ is **uniformly convex** in $U$, the continuum right stretch tensor, but that $F \mapsto W_{\rm mp}(U)$ fails to be Legendre-Hadamard elliptic with respect to $F$ and is as such **not quasiconvex** but satisfies the **Baker-Ericksen inequalities**, cf. section 9.3. Whether or not the external couples $M, M_c$ should be non-zero is a modelling choice. Observe as well that this special elastic material of grade two **does not** completely control the second derivatives of the deformation $\varphi$ which shows, that a simple compactness argument does not suffice to overcome the nonlinearity and non-quasiconvexity in the first order stretch term. At first sight it is therefore not obvious why such a model can have a regularizing effect in the finite case.

**Remark 2.2 (Consistent Dirichlet boundary coupling conditions for the rotations)**
*For all presented models with internal length scale the microrotations on the part of the boundary $\Gamma$ can in principle be specified arbitrarily. This implies five degrees of freedom: 3 components of the deformation $\varphi$ and two orthogonal vectors, the third vector of the rotation is then defined, we call this the* **rigid Dirchlet case**. *However, if we want to describe a basically classical situation, where only $\varphi_{|_\Gamma} = g_{{\rm d}|_\Gamma}$ is specified, we may as well impose a* **consistent coupling condition**

$$\overline{R}_{|_\Gamma} = \mathrm{polar}(\nabla\varphi)_{|_\Gamma}\,, \qquad (2.22)$$

*which prevents non-compatibility between micro- and macrorotations on the Dirichlet boundary $\Gamma$ and allows for otherwise impossible homogeneous solutions. It leads to three essential degrees of freedom at the Dirichlet boundary and disposes us from the need to motivate any independent boundary condition for $\overline{R}$. In addition, the Boltzmann axiom is not violated on $\Gamma$. It is mathematically possible to leave the microrotations entirely free on $\Gamma$, however, this does not seem to correspond to any physical situation in a real body.*

# 3 Non-dissipative Cosserat micropolar elasto-plasticity

## 3.1 Extension to finite micropolar elasto-plasticity

Now we extend the formulation of micropolar elasticity to cover finite plasticity. It should be clear that there exists various ways of obtaining such an extension, for an overview of the competing models we refer to the instructive survey article [FS03]. Incidentally, the Cosserats themselves [CC09, p.5] already envisaged the application of their general theory to plasticity and fracture. For conciseness we take $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$ for the curvature term in this part.

In a first preliminary step we consider the elastic problem defined over a transformed domain $\Omega_\xi = \Theta(\Omega)$ where $\Theta$ is a diffeomorphism. With respect to $\Omega_\xi$ we assume the micropolar decomposition

$$\nabla_\xi\varphi(\xi) = F_\xi = \overline{R}_\xi\cdot\overline{U}_\xi\,, \qquad (3.23)$$

such that the Cosserat problem on $\Omega_\xi$ reads

$$
\begin{aligned}
I_\xi(\varphi_\xi,\overline{R}_\xi) =& \int_{\Omega_\xi} W_{\rm mp}(\overline{R}_\xi^T\nabla_\xi\varphi_\xi) + W_{\rm curv}(\overline{R}_\xi^T\mathrm{D}_\xi\overline{R}_\xi) - \langle f_\xi,\varphi_\xi\rangle - \langle M_\xi,\overline{R}_\xi\rangle \,\mathrm{d}\xi \\
& - \int_{\Gamma_{S,\xi}} \langle N_\xi,\varphi_\xi\rangle\,\mathrm{dS}_\xi - \int_{\Gamma_{C,\xi}} \langle M_{\xi,c},\overline{R}_\xi\rangle\,\mathrm{dS}_\xi \mapsto \min.\ \text{w.r.t. } (\varphi_\xi,\overline{R}_\xi), \qquad (3.24)\\
\overline{R}_{\xi|_{\Gamma_\xi}} =& \mathrm{polar}(\nabla g_\xi)_{|_{\Gamma_\xi}}\,, \quad \varphi_{\xi|_{\Gamma_\xi}} = g_\xi(\xi)\,.
\end{aligned}
$$

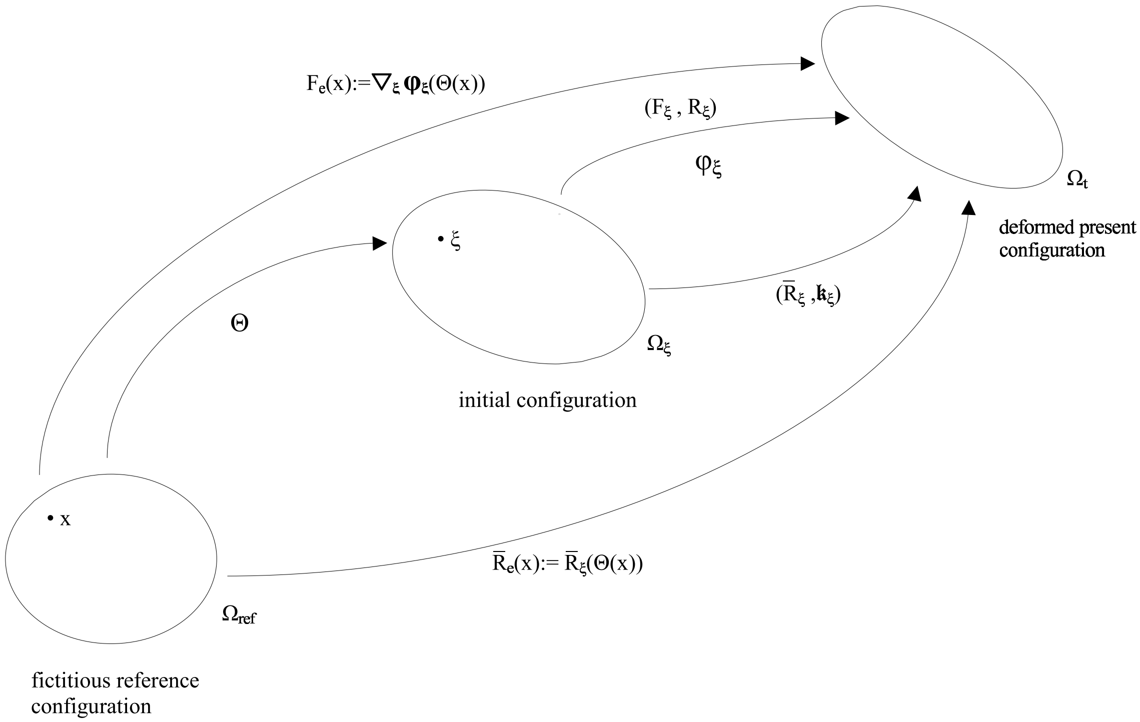


Figure 2: Schematic picture of the transformed elastic Cosserat kinematics.

Now we transform the problem to the fictitious reference configuration $\Omega$: the transformation of variables formula and Nansons relation yields

$$
\begin{aligned}
&\int_\Omega W_{\rm mp}(\overline{U}_\xi)\det[\nabla\Theta] + W_{\rm curv}(\mathfrak{K}_\xi)\det[\nabla\Theta] - \langle f,\varphi\rangle\det[\nabla\Theta] - \langle M,\overline{R}_\xi\rangle\det[\nabla\Theta]\,{\rm dV} \\
&\qquad - \int_{\Gamma_S}\langle N,\varphi\rangle\,\|\operatorname{Cof}\nabla\Theta.\vec{n}\|\,{\rm dS} - \int_{\Gamma_C}\langle M_c,\overline{R}_\xi\rangle\,\|\operatorname{Cof}\nabla\Theta.\vec{n}\|\,{\rm dS} \mapsto \min.\ \text{w.r.t. } (\varphi,\overline{R}_\xi), \\
\overline{R}_{\xi_{|\Gamma}} &= \operatorname{polar}(\nabla g_{\rm d}\cdot\nabla\Theta^{-1})_{|\Gamma}, \quad \varphi_{|\Gamma} = g_\xi(\Theta(x)) = g_{\rm d} \qquad (3.25)\\
W_{\rm mp}(\overline{U}_\xi) &= \mu\,\|\operatorname{sym}(\overline{U}_\xi - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U}_\xi)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{U}_\xi - 1\!\!1)\right]^2 \\
W_{\rm curv}(\mathfrak{K}_\xi) &= \mu\frac{L_c^{1+p}}{12}\,\left(1+\alpha_4\,L_c^q\,\|\mathfrak{K}_\xi\|^q\right)\,\|\mathfrak{K}_\xi\|^{1+p} \\
\nabla_\xi\varphi_\xi &= \nabla\varphi\cdot\nabla\Theta^{-1}, \quad \overline{U}_\xi = \overline{R}_\xi^T\nabla_\xi\varphi_\xi, \quad \nabla\Theta\in \mathrm{GL}^+(3,\mathbb{R}) \\
\mathfrak{K}_\xi &= \overline{R}_\xi^T{\rm D}_\xi\overline{R}_\xi = \overline{R}_\xi^T(\Theta(x))\left[{\rm D_x}\left[\overline{R}_\xi(\Theta(x))\right]\cdot\nabla\Theta^{-1}\right]\,.
\end{aligned}
$$

Conceptually, the plasticity model is obtained by **relaxing** the **compatibility** of $\nabla\Theta$: we replace $\nabla\Theta$ with $F_p$, identify $F_e(x)$ as $F_\xi = \nabla_\xi\varphi_\xi(\Theta(x))$, set $\overline{R}_e(x) := \overline{R}_\xi(\Theta(x))$ and need to supply a thermodynamical admissible flow rule for $F_p$. This is tantamount to consider directly the multiplicative decomposition of the elastic part of the deformation gradient

$$F_e = \overline{R}_e\cdot\overline{U}_e\,, \qquad (3.26)$$

which defines micropolar elastic rotations $\overline{R}_e$ and micropolar elastic stretch $\overline{U}_e$.

The thermodynamical admissible flow rule for $F_p$ can be obtained as follows. Consider the rate of change of the elastic energy density only due to the time variation of the incompatible configuration $F_p$ (variation of the "domain" of definition $F_p$): after some simple but tedious manipulations we obtain

$$
\begin{aligned}
&\frac{\rm d}{\rm dt}\left[W_{\rm mp}(\overline{U}_e)\det[F_p] + W_{\rm curv}(\mathfrak{K}_e)\det[F_p]\right] \\
&\quad = \langle\det[F_p]\left(\overline{U}_e^T D_{\overline{U}_e}W_{\rm mp}(\overline{U}_e) - W_{\rm mp}(\overline{U}_e)\,1\!\!1\right), F_p\frac{\rm d}{\rm dt}\left[F_p^{-1}\right]\rangle + \qquad (3.27)\\
&\quad + \langle\det[F_p]\left(\mathfrak{K}_e^T D_{\mathfrak{K}_e}W_{\rm curv}(\mathfrak{K}_e) - W_{\rm curv}(\mathfrak{K}_e)\,1\!\!1\right), F_p\frac{\rm d}{\rm dt}\left[F_p^{-1}\right]\rangle\,,
\end{aligned}
$$

where it is understood that $\mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\rm curv}(\mathfrak{K}_e) = \sum_{i=1}^3 \mathfrak{K}_e^{i,T} \cdot \partial_{\mathfrak{K}_e^i} W_{\rm curv}(\mathfrak{K}_e^1, \mathfrak{K}_e^2, \mathfrak{K}_e^3)$. A sufficient condition for the second law of thermodynamics to be fulfilled [Nef03a] is to guarantee the **reduced dissipation inequality**

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[W_{\rm mp}(\overline{U}_e)\det[F_p] + W_{\rm curv}(\mathfrak{K}_e)\det[F_p]\right] \le 0\,, \tag{3.28}$$

where $F, \overline{R}_e$ are held constant. This can be achieved by choosing the **left-invariant** flow rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] &\in -F_p^{-1}\cdot f(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\rm mp} + \Sigma_{E,\rm curv}\\
\Sigma_{E,\rm mp} &= \overline{U}_e^T D_{\overline{U}_e} W_{\rm mp}(\overline{U}_e)\det[F_p] - W_{\rm mp}(\overline{U}_e)\det[F_p]\,1\!\!1\\
\Sigma_{E,\rm curv} &= \mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\rm curv}(\mathfrak{K}_e)\det[F_p] - W_{\rm curv}(\mathfrak{K}_e)\det[F_p]\,1\!\!1\,,
\end{aligned} \tag{3.29}$$

where the **flow function** $f : \mathbb{M}^{3\times3} \mapsto \mathcal{P}(\mathbb{M}^{3\times3})$ governs the plastic evolution and must satisfy $\forall\,\Sigma:\ \langle f(\Sigma), \Sigma\rangle \ge 0$. Such a formulation can be called of **pre-monotone type** in the sense of the classification for infinitesimal elasto-plastic models in [Alb98].

Here $\Sigma_E$ denotes the **total elastic Eshelby stress tensor** (the driving force behind evolving inhomogeneities in the reference configuration [Mau99]) which may be reduced to $\Sigma_M = F_e^T D_{F_e} W(F_e, R_e)$, the **elastic Mandel stress tensor** in case of a deviatoric flow rule which preserves the incompressibility constraint $\det[F_p] = 1$. The Eshelby stress tensor has two different contributions: $\Sigma_{E,\rm mp}$ due to macro-stretch and $\Sigma_{E,\rm curv}$ due to micro torsion-curvature.

In the case of a simple material a similar derivation of the flow rules for multiplicative elasto-plasticity based on the Eshelby tensor has been given in [Nef03c, Nef03a]. Note that the multiplicative decomposition [Lee69, Man73] of the deformation gradient into elastic and plastic parts

$$\nabla\varphi = F = F_e \cdot F_p\,, \tag{3.30}$$

is a by-product of the derivation. [24]

In the quasi-static setting we are thus led to study the following system of coupled partial differential and evolution equations for the finite deformation $\varphi : [0,T]\times\overline{\Omega} \mapsto \mathbb{R}^3$, the plastic deformation $F_p : [0,T]\times\overline{\Omega} \mapsto \mathrm{GL}^+(3,\mathbb{R})$ and the independent Cosserat elastic microrotation $\overline{R}_e : [0,T]\times\overline{\Omega} \mapsto \mathrm{SO}(3)$

$$\begin{aligned}
&\int_\Omega W_{\rm mp}(\overline{U}_e)\det[F_p] + W_{\rm curv}(\mathfrak{K}_e)\det[F_p] - \langle f,\varphi\rangle\det[F_p] - \langle M,\overline{R}_e\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N,\varphi\rangle\,\|\,\mathrm{Cof}\,F_p.\vec{n}\|\,\mathrm{dS}\\
&\qquad - \int_{\Gamma_C}\langle M_c,\overline{R}_e\rangle\,\|\,\mathrm{Cof}\,F_p.\vec{n}\|\,\mathrm{dS} \mapsto \min\,.\,\text{w.r.t. } (\varphi,\overline{R}_e) \text{ at constant } F_p,\\
&\overline{R}_{e_{|\Gamma}} = \begin{cases}\mathrm{polar}(\nabla g_{\rm d}\cdot F_p^{-1})_{|\Gamma}, & \text{rigid}\\ \mathrm{polar}(\nabla\varphi\cdot F_p^{-1})_{|\Gamma}, & \text{consistent}\end{cases}, \quad \varphi_{|\Gamma} = g_{\rm d}(t)\\
&W_{\rm mp}(\overline{U}_e) = \mu\,\|\mathrm{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}_e - 1\!\!1)\right]^2\\
&W_{\rm curv}(\mathfrak{K}_e) = \mu\frac{L_c^{1+p}}{12}\,\left(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}_e\|^q\right)\,\|\mathfrak{K}_e\|^{1+p}\\
&F_e = \nabla\varphi\cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = \overline{R}_e^T[\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]\\
&\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] \in -F_p^{-1}\cdot f(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\rm mp} + \Sigma_{E,\rm curv}\\
&\Sigma_{E,\rm mp} = \overline{U}_e^T D_{\overline{U}_e} W_{\rm mp}(\overline{U}_e)\det[F_p] - W_{\rm mp}(\overline{U}_e)\det[F_p]\,1\!\!1\\
&\Sigma_{E,\rm curv} = \mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\rm curv}(\mathfrak{K}_e)\det[F_p] - W_{\rm curv}(\mathfrak{K}_e)\det[F_p]\,1\!\!1\\
&F_p^{-1}(0) = F_{p_0}^{-1}, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R})\,.
\end{aligned} \tag{3.31}$$

[24] While we continue to use the term **multiplicative decomposition** and **intermediate configuration** it is rather an **elastic isomorphism** in the sense of [Ber98]. Some authors use $P$ instead of $F_p^{-1}$, [CHM02]. Examples for classical finite plasticity formulations may be found in [Sim88, SO85, Mie95, Sim98, SH98, EGR90, CHM00, Mie00, OR99]. Different models have been compared numerically in [NW03]. Note that $F_p$ is not a plastic strain but rather a relaxed configuration: in a neighbourhood of a point, $F_p$ can be a rigid rotation, while the corresponding plastic strain $C_p - 1\!\!1 = F_p^T F_p - 1\!\!1$ vanishes.

To complete the phenomenological macroscopic elastic-viscoplastic Cosserat micropolar model we specify $f = \partial\mathcal{X}$ such that we retrieve the von Mises type 'incompressible' $J_2$ -visco-plasticity with **elastic domain** $\mathcal{E} := \{\Sigma_E | \, \|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| \leq \sigma_{\mathrm{y}}\}$ and yield stress $\sigma_{\mathrm{y}}$. To this end we take as viscoplastic potential $\mathcal{X} : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ of generalized Norton-Hoff overstress type[25] the following function:

$$\mathcal{X}(\Sigma_E) = \begin{cases} 0 & \Sigma_E \in \mathcal{E} \\ \frac{\bar{\sigma}_0}{(r+1)(k+1)\ \eta_p}\left(1 + \left[\frac{\|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| - \sigma_{\mathrm{y}}}{\bar{\sigma}_0}\right]^{r+1}\right)^{k+1} - \frac{\bar{\sigma}_0}{(r+1)(k+1)\ \eta_p} & \Sigma_E \notin \mathcal{E}\,, \end{cases} \tag{3.32}$$

where $\eta_p > 0$ is a relaxation time due to essentially plastic processes inside the grains, $r, k > 0$ and $\bar{\sigma}_0$ is a stress like material constant. This definition is consistent with the postulate of maximum plastic dissipation. The parameter $r$ allows to adjust the smoothness of the flow rule when passing the elastic boundary. A typical range for $k$ in engineering applications is $k \in \{0, \ldots, 80\}$. For $k \to \infty$ we recover formally ideal rate-independent plasticity. A simple calculation shows that the corresponding single valued subdifferential is given by

$$\partial_\Sigma \mathcal{X}(\Sigma_E) = \frac{1}{\eta_p}.\left(1 + \left[\frac{\|\operatorname{dev}\operatorname{sym}\Sigma_E\| - \sigma_{\mathrm{y}}}{\bar{\sigma_0}}\right]_+^{r+1}\right)^k \left[\frac{\|\operatorname{dev}\operatorname{sym}\Sigma_E\| - \sigma_{\mathrm{y}}}{\bar{\sigma_0}}\right]_+^r \frac{\operatorname{dev}\operatorname{sym}\Sigma_E}{\|\operatorname{dev}\operatorname{sym}\Sigma_E\|}\,. \tag{3.33}$$

The resulting model (3.31) is as close as possible to classical macroscopic elasto-plastic models, notably we did **not** introduce any plastic microrotation $\overline{R}_p$ together with a multiplicative decomposition of microrotations $\overline{R} = \overline{R_e} \cdot \overline{R}_p$, nor did we split artificially a total curvature $\mathfrak{K}$ into elastic and plastic parts as has been proposed in [DSW93, Ste94a, IW98, GT01, Gra03]. Such a decomposition represents an additional modelling assumption not necessarily related to elastic Cosserat effects. [26]

It is clear that $\Sigma_E$ will **not** be **symmetric** in general even under **isotropy** conditions. Thus the choice of $\operatorname{sym}(\Sigma_E)$ in the definition of the elastic domain $\mathcal{E}$ sets the **plastic spin** to **zero**, consistent with current classical isotropic macroscopic formulations for polycrystals. It is possible to incorporate hardening effects independent of the Cosserat framework in the standard, local phenomenological fashion.

We mention that for $\mu_c = 0$ we have

$$\overline{R}_e \operatorname{skew}(D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\, \overline{U}_e^T)\overline{R}_e^T = \operatorname{skew}(B_{\mathrm{approx}})\,, \tag{3.34}$$

i.e. the reaction stress in the Cosserat model (3.31) is the driving force of the **viscoelastic** evolution in [Nef03a, (P3)].

## 3.2 Elastic-viscoplastic Cosserat model for polycrystals with grain rotations

A formidable challenge for current research is to find tractable continuum models for crystalline materials at the same time capturing their physical essence and being geometrically exact. There are essentially two ways to proceed: either one starts from the better known single crystal case [CO92, OR99, ORS00] and computes a large array of single crystals in mutual contact or one enriches the classical description with new variables taking account of the microstructural evolution in an averaged sense. We follow the second line of thought.

In a polycrystal, single crystal grains are joined together along grain boundaries. The intergranular **grain boundary** is mainly **responsible** for the **elastic** and **viscous** response of the polycrystal while plastic effects are located predominantly inside each grain operating by slip and twinning along glide planes. The absence of any viscous grain boundary in single crystals explains why the plasticity of single crystals is traditionally modelled as rate-independent. Consistently, for a specimen made of a single crystal, relaxation effects are practically absent whereas the internal surface between grains, where frictional effects are dominant, increases for smaller grain size and leads to pronounced rate-dependent response already below a macroscopic yield limit [Nef03a]. Hence, a polycrystal is much more than a simple assembly of single crystals corroborating the fact that the small scale (single crystal) behaviour can be quite different from the bulk for non-linear heterogeneous material already for small loads.

In addition, depending on the size of the constituting grains, a polycrystal has different elasto-plastic properties. An account of the necessity for macroscopic problems to incorporate internal length scale effects into a model has been recently given in [WCZM02]. Polycrystalline copper has been made six times harder (apparent

[25] In finite plasticity, the question whether or not the plastic flow has a gradient structure seems to be of minor importance as far as mathematical existence results are concerned in sharp contrast to the infinitesimal case [Alb98, HR99]. However, the very feasibility of a time-incremental variational formulation [Mie00, OR99, CHM00] is contingent upon the potential structure.

[26] It is motivated, though, by the experimental observation [FMAH94] that geometrical size effects are becoming increasingly important in the plastic range. These size effects are explained on a microscale as being due to dislocation interactions. It seems therefore more natural to account for them directly by incorporating a dislocation density $\operatorname{Curl} F_p$ into the model and providing non-local flow rules of parabolic type for $F_p$. We will not pursue this issue here.

yield stress $\sigma_y{}^{\text{nano}} \geq 6\,\sigma_y{}^{\text{class}}$) by reducing the grain-diameter dramatically- a consequence of the Hall-Petch relation. This shows the general need for the incorporation of an internal length scale even when viewing the polycrystal macroscopically. [27]

Experimental evidence [DML91, LLT94] shows that the rotations of the individual grains e.g. in polycrystalline aluminium specimens may deviate considerably from the continuum rotation which must be viewed as orthogonal part of the average grain deformation gradient over some representative volume element. This picture lends itself most naturally to a treatment in a Cosserat context: we identify the averaged individual elastic rotations of grains with the elastic Cosserat microrotations and the orthogonal part of the averaged elastic deformation gradient with the elastic continuum rotation.[28]

Remaining in this context, it is clear, that a material, in which a substructure is allowed to rotate rather independently, should not become more rigid than a corresponding classical (equivalent, macroscopic) homogeneous material. Therefore, we conclude again, that $\mu_c = 0$ is a reasonable choice for a polycrystal treated on a macroscopic level.[29] If the related classical homogeneous description has the strain energy

$$W(U) = \mu\,\|U - 1\!\!1\|^2 + \frac{\lambda}{2}\text{tr}\,[U - 1\!\!1]^2 = \mu\,\|\,\text{sym}(U - 1\!\!1)\|^2 + \frac{\lambda}{2}\text{tr}\,[U - 1\!\!1]^2\,, \tag{3.35}$$

then after relaxing the constraint on the rotations to coincide with the continuum rotations we would rather expect the overall macroscopic strain energy due to macroscopic stretch to be smaller than the homogeneous one, i.e. $W_{\text{mp}}(\overline{U}) \leq W(U)$. Since in principle $\text{skew}(\overline{U})$ can be large, we conclude consistently $\mu_c = 0$.[30]

In the elasto-plastic theory the consequences on a macroscale induced by letting $\mu_c > 0$ are even more severe than in the elastic case: imagine a cyclic loading history which systematically leaves the elastic region. In general, we will obtain a time dependent inhomogeneous distribution of the plastic deformation $F_p(x)$. For $\mu_c > 0$, the so called **elastic trial step** [31] will be unconditionally stable irrespective of the accumulated spatial inhomogeneities of $F_p$ as long as $F_p \in L^\infty(\Omega, \text{GL}^+(3,\mathbb{R}))$, while $\mu_c = 0$ allows some sort of **elastic fatigue/softening/failure/fracture** since the positive definiteness of the elastic tangent stiffness matrix w.r.t. the reference configuration is affected by the spatial continuity properties of $F_p$. This apparent softening, namely the decrease of elastic moduli, is a well documented experimental fact. For a polycrystal, we then adopt the following picture: the plastic deformation $F_p$ represents on a macroscopic level the permanent material substructure, to be more precise, the permanent averaged cumulative plastic deformation of the individual grains due to slip and twinning. We may call therefore $F_p$ the average plastic grain transformation and $\overline{R}_e$ the average elastic grain rotation.

## 3.3 Infinitesimal elasto-plastic Cosserat model

If we assume that plastic deformations $F_p$ and elastic microrotations remain small, we can considerably simplify the problem (3.31). We expand $\overline{R}_e = 1\!\!1 + \overline{A}_e + \ldots\,,\ \overline{A}_e \in \mathfrak{so}(3)$, $\|\overline{A}_e\|^2 \ll 1$, $F_p = 1\!\!1 + p + \ldots\,,\ \|p\|^2 \ll 1$ and choose as elastic domain $\mathcal{E} := \{\Sigma_E |\ \|\text{dev}(\text{sym}\,\Sigma_E)\| \leq \sigma_y\}$ [32], then (3.31) reduces to the infinitesimal elasto-plastic system in variational form with **non-dissipative** Cosserat effects and reads

$$\begin{aligned}
&\int_\Omega \mu\,\|\varepsilon - \varepsilon_p\|^2 + \mu_c\,\|\,\text{skew}(\nabla u - \overline{A}_e)\|^2 + \frac{\lambda}{2}\,\text{tr}\,[\varepsilon]^2 + \mu\,\frac{L_c^2}{12}\,\|\nabla\,\text{axl}(\overline{A}_e)\|^2 - \langle f, u\rangle - \langle M, \overline{A}_e\rangle\,\text{dV} \\
&\qquad - \int_{\Gamma_S} \langle N, u\rangle\,\text{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}_e\rangle\,\text{dS} \mapsto \min.\quad \text{w.r.t. } (u, \overline{A}_e) \text{ at constant } \varepsilon_p \\
&\varepsilon_e = \varepsilon - \varepsilon_p, \quad \varepsilon(\nabla u(x)) = \tfrac{1}{2}(\nabla u^T + \nabla u), \quad \varepsilon_p = \tfrac{1}{2}(p^T + p), \\
&\dot{\varepsilon_p}(t) \in f\,(T_E), \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p) \\
&u_{|\partial\Omega}(t,x) = g_{\text{d}}(t,x) - x, \quad \overline{A}_{e|_\Gamma} = \text{skew}(\nabla g_{\text{d}}(t,x))_{|\Gamma}\,.
\end{aligned} \tag{3.36}$$

[27] Whether the incorporation of the necessary material length scale must be done in a Cosserat framework, cannot be decided. But the experimental evidence [FMAH94] suggests that size dependent hardening occurs predominantly under torsion (possible rotations), while in uniaxial tension (no rotations) strain gradients are negligible and length scale effects remain small.

[28] Observe that these two rotations do not coincide in general: the **averaged rotation** is understood to be the **best-approximating single rotation to a rotation field** defined over a representative volume element while the **continuum rotation** is the **orthogonal part** of the **averaged deformation gradient**. In the infinitesimal case, both so defined infinitesimal rotations coincide!

[29] The same argument put differently: Consider a wall made of concrete bricks and cast concrete, respectively. The cast wall will have more rigidity.

[30] "Polycrystalline ... type microstructures behave classically or nearly classically (in their elastic range)."[Lak95, p.22]

[31] or elastic predictor. In an operator-split method it amounts to 'freezing' the plastic evolution and to compute elastic equilibrium.

[32] no **infinitesimal plastic spin**: $p = \varepsilon_p$ and isochoric plasticity: $\text{tr}\,[\varepsilon_p] = 0$ and assume $\langle \varepsilon_p.\vec{n}, \vec{n}\rangle = 0$ on $\Gamma_S \cup \Gamma_C$.

The corresponding equilibrium system of equations for pure Dirichlet conditions and without external couples $M, M_c$ is given by (note that $\|\overline{A}_e\|^2 = 2\|\,\mathrm{axl}(\overline{A}_e)\|^2$ for $\overline{A}_e \in \mathfrak{so}(3,\mathbb{R})$)

$$
\begin{aligned}
0 &= \mathrm{Div}\,\sigma + f, \quad x \in \Omega \\
\sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - \overline{A}_e) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1 \\
0 &= \mu\frac{L_c^2}{12}\,\Delta\,\mathrm{axl}(\overline{A}_e) + \mu_c\ \mathrm{axl}((\mathrm{skew}(\nabla u) - \overline{A}_e) \\
\dot{\varepsilon_p}(t) &\in f\,(T_E), \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p) \\
u_{|_{\partial\Omega}}(t,x) &= g_{\mathrm{d}}(t,x) - x, \quad x \in \partial\Omega\,, \quad \overline{A}_{e|_{\partial\Omega}} = \mathrm{skew}(\nabla g_{\mathrm{d}}(t,x))_{|_{\partial\Omega}}\,. \\
\mathrm{tr}\,[\varepsilon_p(0)] &= 0\,, \quad \varepsilon_p(0) \in \mathrm{Sym}(3)\,.
\end{aligned}
\tag{3.37}
$$

It must be observed that this completely reduced set of equations is still intrinsically thermodynamically admissible. The model can also be obtained as limit case of models proposed in [IW98, Bes74, Lip69].

The infinitesimal model has already been completely justified as a non-local regularization ($\mu_c \to 0$) of classical ideal rate-independent plasticity in [NC03] using the methods exploited before in [Che98]. Precisely, it has been proved that (3.37) admits a unique global solution. The system (3.36) is therefore a reasonable regularization of classical plasticity in the sense that the system remains of second order and the plastic flow part is left unaltered compared to the traditional one.

In [DSW93, p.815] an elasto-plastic model based on the infinitesimal Cosserat theory with dissipative micropolar effects has been investigated. They show that $\mu_c > 0$ has a decisive influence [33] on localization effects essentially excluding mode II shear failure. Since our reduced model is non-dissipative it is difficult however, to transfer this insight directly. This remark finishes the modelling part of this contribution.

[33] Since the infinitesimal model is neither observer-invariant nor frame-indifferent it is not possible to conclude on physical grounds that $\mu_c > 0$ is necessary for the general theory. A model which is, strictly speaking, physically irrelevant cannot lead to stringent conclusions on the physical significance of some parameter. The infinitesimal model has only merits as a first approximation. We might wonder therefore about the status of mathematical investigations of the infinitesimal elasto-plastic models and the effort still put into the investigations of their intricacies (rate-independent limit), which are either irrelevant for the geometrically exact model or misleading. We conclude that their investigation is mainly of high inner - mathematical interest. This remark applies mutatis mutandis to the numerical treatment of classical infinitesimal elasto-plasticity. Here refined time-integration algorithms of higher order are investigated without leading to consequences for the numerical treatment of the exact theory, which is faced with completely different problems.

# 4 Mathematical analysis

## 4.1 Statement of the finite elastic Cosserat problem in variational form

Let us now return to the purely elastic Cosserat case. The problem has been posed in a variational form. The task is to find a pair $(\varphi, \overline{R}) \in \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ of deformation $\varphi$ and **independent microrotation** $\overline{R}$ satisfying

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi\rangle - \langle M, \overline{R}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}\rangle \,\mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}),\\
&\overline{U} = \overline{R}^T F, \quad \overline{R}_{|_\Gamma} = \overline{R}_{\mathrm{d}}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}} \qquad (4.38)\\
&W_{\mathrm{mp}}(\overline{U}) = \mu \,\|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\|\operatorname{skew}(\overline{U})\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 \, L_c^q \,\|\mathfrak{K}\|^q\right) \left(\alpha_5 \,\|\operatorname{sym}\mathfrak{K}\|^2 + \alpha_6 \,\|\operatorname{skew}\mathfrak{K}\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{K}]^2\right)^{\frac{1+p}{2}},\\
&\mathfrak{K} = \overline{R}^T \mathrm{D_x} \overline{R} = \left(\overline{R}^T \nabla(\overline{R}.e_1), \overline{R}^T \nabla(\overline{R}.e_2), \overline{R}^T \nabla(\overline{R}.e_3)\right), \quad \text{third order } \textbf{curvature tensor}.
\end{aligned}$$

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ depends on the deformation gradient $F = \nabla\varphi$ and microrotations $\overline{R}$ together with their space derivatives. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions $N$ are applied with $\Gamma \cap \Gamma_S = \emptyset$. The external volume force is $f$ and $M$ takes on the role of external volume couples. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples $M_c$ are applied with $\Gamma \cap \Gamma_C = \emptyset$. The parameters $\mu$, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the **Cosserat couple modulus** and $L_c > 0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$.

## 4.2 The different cases

We distinguish five completely different situations:

I: $\mu_{\mathbf{c}} > \mathbf{0}$, $\alpha_{\mathbf{4}} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{1}$, $\mathbf{q} \geq \mathbf{0}$, unconditional elastic macro-stability, local first order Cosserat micropolar, unqualified existence, microscopic specimens, non-zero Cosserat couple modulus. Fracture excluded.

II: $\mu_{\mathbf{c}} = \mathbf{0}$, $\alpha_{\mathbf{4}} > \mathbf{0}$, $\mathbf{p} \geq \mathbf{1}$, $\mathbf{q} > \mathbf{1}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded.

III: $\mu_{\mathbf{c}} = \infty$, $\alpha_{\mathbf{4}} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{1}$, $\mathbf{q} \geq \mathbf{0}$, unconditional elastic macro-stability, the constrained gradient Cosserat micropolar problem (indeterminate couple stress model (2.21)). Compatible Dirichlet boundary conditions: $\varphi_{|_\Gamma} = g_{\mathrm{d}}$, $\operatorname{polar}(\nabla\phi)_{|_\Gamma} = \operatorname{polar}(\nabla g_{\mathrm{d}})_{|_\Gamma}$.

IV: $\mu_{\mathbf{c}} = \mathbf{0}$, $\alpha_{\mathbf{4}} = \mathbf{0}$, $\mathbf{0} < \mathbf{p} \leq \mathbf{1}$, $\mathbf{q} = \mathbf{0}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $\varphi \notin W^{1,1}(\Omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including **fracture** in multiaxial situations.

V: $\mu_{\mathbf{c}} = \mathbf{0}, \mathbf{L_c} = \mathbf{0}$, elastic pre-stability, finite elasticity with free rotations and microstructure. Weak solutions of finite elasticity are stationary points of this minimization problem. Allowing for **sharp interfaces**.

We refer to $0 < p < 1$, $q \geq 0$ as the **sub-critical case**, $p = 1$, $q \geq 0$ as the **critical case** and $p \geq 1$, $q > 1$ as the **super-critical case**. We will mathematically treat the first three cases.

## 4.3 The coercive inequality

The decisive analytical tool for the treatment of case II (super-critical) is the following non-trivial coercive inequality:

**Theorem 4.1 (Extended 3D-Korn's first inequality)**
*Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then*

$$\exists\, c^+ > 0 \;\forall\, \phi \in H_\circ^{1,2}(\Omega, \Gamma): \quad \|\nabla\phi F_p^{-1}(x) + F_p^{-T}(x) \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+ \,\|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

**Proof.** The proof has been presented in [Nef02]. Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. However, in general, $F_p$ is **incompatible** giving rise to a **non-riemannian manifold** structure. Compare to [CG01] for an interpretation and the physical relevance of the quantity $\operatorname{Curl} F_p$. ■

Motivated by the investigations in [Nef02], it has been shown recently by my colleague W. Pompe [Pom03] that the extended Korn's inequality can be viewed as a special case of a general class of coercive inequalities for quadratic forms. He was able to show that indeed $F_p \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for (4.1) to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ and $\operatorname{Curl} F_p \in L^{3+\delta}(\Omega)$ should suffice, whereas $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [Pom03].

In view of the important role of the extended Korn's first inequality let us agree in saying that a material is **elastically pre-stable**, whenever

$$\begin{aligned}
&\exists\, H \in \mathbb{M}^{3\times3},\ H \neq 0: \quad D_F^2 W(x,F).(H,H) = 0 \qquad (4.39)\\
&\exists\, c^+ > 0\ \exists\, G \in \mathrm{GL}^+(3,\mathbb{R})\ \forall\, H \in \mathbb{M}^{3\times3}\ :\ D_F^2 W(x,F).(H,H) \geq c^+ \|G(x)^T H + H^T G(x)\|^2 .
\end{aligned}$$

In this terminology, infinitesimal classical elasticity is pre-stable with $G = 1\!\!1$ due to the classical Korn's first inequality and the extended Korn's first inequality links the smoothness of $G$ to the positive definiteness of the elastic tangent stiffness tensor.

## 4.4 The geometrically exact elastic Cosserat model

The following results are the first existence theorems for geometrically exact elastic Cosserat models known to the author:[34]

**Theorem 4.2 (Existence for 3D-finite elastic Cosserat model: case I.)**
*Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^1(\Omega,\mathbb{M}^{3\times3})$ and $M_c \in L^1(\Gamma_C,\mathbb{M}^{3\times3})$. Then (4.38) with material constants conforming to case I admits at least one minimizing solution pair $(\varphi,\overline{R}) \in H^1(\Omega,\mathbb{R}^3)\times W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$.*

**Proof.** We apply the direct methods of variations and consider for simplicity $N, M, M_c = 0$. With the prescription of $(g_{\mathrm{d}}, \overline{R}_{\mathrm{d}})$ it is clear that $I < \infty$ for some pair $(\varphi,\overline{R})$. Observe first that the micropolar curvature term $\mathfrak{K}$ controls $\overline{R} \in W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$, since $\|\mathfrak{K}\| = \|\overline{R}^T \mathrm{D_x}\overline{R}\| = \|\mathrm{D_x}\overline{R}\|$, pointwise and $\alpha_5,\alpha_6 > 0$.

Moreover, $\mathrm{SO}(3,\mathbb{R})$ is weakly closed in the topology of $W^{1,1+p}(\Omega)$. We omit to show that $I$ is bounded below: this will turn out not to be necessary. We may choose decreasing (infimizing) sequences of pairs $(\varphi^k, \overline{R}^k)$. The curvature contribution together with the appropriate boundary conditions and Poincarés inequality yields boundedness of $\overline{R}^k \subset W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$. We may extract a subsequence again denoted by $\overline{R}^k$ converging strongly in $L^{1+p}(\Omega)$ to an element $\tilde{\overline{R}} \in W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$ since $p > 0$ by assumption. Because $\mu_c > 0$, it is immediate that $\nabla\varphi^k = F^k$ is **bounded** in $L^2(\Omega,\mathbb{M}^{3\times3})$, **independent** of $\overline{R}^k$ on account of

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{R}^{k,T}F^k) &\geq \mu_c\, \|\overline{R}^{k,T}F^k - 1\!\!1\|^2 = \mu_c\, \left(\|F^k\|^2 - 2\langle \overline{R}^{k,T}F^k, 1\!\!1\rangle + 3\right)\\
&\geq \mu_c\left(\|F^k\|^2 - 2\sqrt{3}\|F^k\| + 3\right) , \qquad (4.40)
\end{aligned}$$

and

$$\begin{aligned}
\infty > \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) + W_{\mathrm{curv}}(\mathfrak{K}_k) - \langle f,\varphi_k\rangle\, \mathrm{dV} &\geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) - \langle f,\varphi_k\rangle\,\mathrm{dV}\\
&\geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k)\,\mathrm{dV} - \|f\|_{L^2(\Omega)}\,\|\varphi_k\|_{H^{1,2}(\Omega)} \qquad (4.41)\\
&\geq \mu_c\,\|\nabla\varphi_k\|^2 - 2\sqrt{3}\,\mu_c\,\|\varphi_k\|_{H^{1,2}(\Omega)} - \|f\|_{L^2(\Omega)}\,\|\varphi_k\|_{H^{1,2}(\Omega)} + 3\mu_c\\
&\geq \mu_c\,\|\nabla u_k\|^2 - C_1\,\|u_k\|_{H^{1,2}(\Omega)} + C_2 \geq \mu_c\, c^+\,\|u_k\|^2_{H^{1,2}(\Omega)} - C_1\,\|u_k\|_{H^{1,2}(\Omega)} + C_2 ,
\end{aligned}$$

[34]The proposed finite results determine the macroscopic deformation $\varphi \in H^1(\Omega,\mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure). If $\mu_c > 0$ fracture is effectively ruled out, which is unrealistic.

where we made use of the appropriate boundary conditions for $\varphi^k = x + u_k(x)$, and applied Poincarés inequality to $u_k$ since it has zero boundary values on $\Gamma$. This yields the boundedness of $\varphi^k$ in $H^1(\Omega, \mathbb{R}^3)$. Hence we may extract a subsequence, not relabelled, such that $\varphi^k \rightharpoonup \tilde{\varphi} \in H^1(\Omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of $(\varphi^k, \overline{R}^k)$ such that $\overline{U}_k = \overline{R}^{k,T} F^k$ converges weakly in $L^2(\Omega)$ to an element $\tilde{\overline{U}}$ on account of the boundedness of the stretch energy and $\mu_c > 0$.

For $p \geq 1$ we have as well that $\overline{R}^k$ converges indeed strongly in $L^2(\Omega)$ to an element $\tilde{\overline{R}} \in H^{1,2}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Thus $\overline{R}^{k,T} F^k$ converges weakly to $\tilde{\overline{R}}^T F$ in $L^1(\Omega)$. The weak limit in $L^1(\Omega)$ must coincide with the weak limit of $\overline{U}_k$ in $L^2(\Omega)$. Hence, $\tilde{\overline{U}} = \tilde{\overline{R}}^T \nabla\tilde{\varphi}$.

Since the total energy is convex in $(\overline{U}, \mathfrak{K})$ and $(\nabla\varphi, D\overline{R})$, we get

$$
\begin{aligned}
I(\tilde{\varphi}, \tilde{\overline{R}}) &= \int_\Omega W_{\mathrm{mp}}(\tilde{\overline{U}}) + W_{\mathrm{curv}}(\tilde{\mathfrak{K}}) - \langle f, \tilde{\varphi}\rangle \, \mathrm{dV} \\
&\leq \liminf_{k\to\infty} \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) + W_{\mathrm{curv}}(\mathfrak{K}_k) - \langle f, \varphi_k\rangle \, \mathrm{dV} = \lim_{k\to\infty} I(\varphi^k, \overline{R}^k) \,, \qquad (4.42)
\end{aligned}
$$

which implies that the limit pair is a minimizer. Note that the limit microrotations $\tilde{\overline{R}}$ may fail to be continuous if $p \leq 2$ (non-existence or limit case of Sobolev embedding). Moreover, uniqueness cannot be ascertained, since $\mathrm{SO}(3,\mathbb{R})$ is a nonlinear manifold (and the considered problem is indeed nonlinear), such that convex combinations of rotations are not rotations in general. Since the functional $I$ is differentiable the minimizing pair is a stationary point and therefore a solution of the field equations (2.12). Note again that the limit microrotations are trivial in $L^\infty(\Omega)$ but may fail to be continuously distributed in space. That under these unfavourable circumstances a minimizing solution may nevertheless be found is entirely due to $\mu_c > 0$ and $p \geq 1$. ■

We continue with the super-critical case appropriate for macroscopic situations and close to classical elasticity.

**Theorem 4.3 (Existence for 3D-finite elastic Cosserat model: case II.)**
*Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times3})$. Then (4.38) with material constants conforming to case II admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.*

**Proof.** We repeat the argument of case I. However, the boundedness of infimizing sequences is not immediately clear. Boundedness of the rotations $\overline{R}^k$ holds true in the space $W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ with $1+p+q > N = 3$, hence we may extract a subsequence, not relabelled, such that $\overline{R}^k$ converges strongly to $\hat{\overline{R}} \in C^0(\overline{\Omega}, \mathrm{SO}(3,\mathbb{R}))$ in the topology of $C^0(\overline{\Omega}, \mathrm{SO}(3,\mathbb{R}))$ on account of the Sobolev-embedding theorem. Along such strongly convergent sequence of rotations, the corresponding sequence of deformations $\varphi^k$ is also bounded in $H^1(\Omega, \mathbb{R}^3)$. However, this is not due to a basically simple pointwise estimate as in case I, but only true after integration over the domain: at face value we only control certain mixed symmetric expressions in the deformation gradient. More precisely, we have

$$
\begin{aligned}
\infty &> \int_\Omega W_{\mathrm{mp}}(\overline{U_k}) + W_{\mathrm{curv}}(\mathfrak{K}_k) - \langle f, \varphi_k\rangle \, \mathrm{dV} \geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) - \langle f, \varphi_k\rangle \, \mathrm{dV} \\
&\geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) \, \mathrm{dV} - \|f\|_{L^2(\Omega)} \, \|\varphi_k\|_{H^{1,2}(\Omega)} \\
&\geq \int_\Omega \frac{\mu}{4} \|\overline{R}_k^T \nabla\varphi_k + \nabla\varphi_k^T \overline{R}_k - 2 1\!\!1\|^2 \, \mathrm{dV} - \|f\|_{L^2(\Omega)} \, \|\varphi_k\|_{H^{1,2}(\Omega)} \qquad (4.43) \\
&\geq \int_\Omega \frac{\mu}{4} \|\overline{R}_k^T \nabla u_k + \nabla u_k^T \overline{R}_k\|^2 \, \mathrm{dV} - C_1 \, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \\
&= \int_\Omega \frac{\mu}{4} \|(\overline{R}_k - \hat{R} + \hat{R})^T \nabla u_k + \nabla u_k^T (\overline{R}_k - \hat{R} + \hat{R})\|^2 \, \mathrm{dV} - C_1 \, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \\
&\geq \int_\Omega \frac{\mu}{4} \underbrace{\|\hat{R}^T \nabla u_k + \nabla u_k^T \hat{R}\|^2}_{\text{combinations of derivatives}} \, \mathrm{dV} - C_3 \, \|\hat{R} - \overline{R}_k\|_\infty \, \|u_k\|^2_{H^{1,2}(\Omega)} \\
&\qquad - (C_1 + 2\, \|\hat{R} - \overline{R}_k\|_\infty) \, \|u_k\|_{H^{1,2}(\Omega)} + C_2
\end{aligned}
$$

$$\geq (\frac{\mu}{4} c_K - C_3 \|\hat{R} - \overline{R}_k\|_\infty) \|u_k\|^2_{H^{1,2}(\Omega)} - (C_1 + 2\|\hat{R} - \overline{R}_k\|_\infty) \|u_k\|_{H^{1,2}(\Omega)} + C_2 \,,$$

where we made use of the appropriate boundary conditions for $\varphi^k = x + u_k$ and applied the extended Korn's inequality (4.1) in the improved version of [Pom03] yielding the positive constant $c_K$ for the continuous microrotation $\hat{R}$. Since $\|\hat{R} - \overline{R}_k\|_\infty \to 0$ we conclude the boundedness of $u_k$ in $H^1(\Omega)$. Hence, $\varphi_k$ is bounded in $H^1(\Omega)$. Now we obtain that $\overline{U_k} \rightharpoonup \tilde{\overline{U}} = \hat{\overline{R}}^T \nabla\tilde{\varphi}$ by construction with the notations as in case I. The remainder proceeds as in case I. This finishes the argument. The limit microrotations $\hat{\overline{R}}$ are indeed found to be continuous. ■

**Theorem 4.4 (Existence for 3D-finite elastic Cosserat model with nonlinear volume part)** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\rm d} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times3})$. Assume furthermore that $W_{\rm mp}$ has the form*

$$W_{\rm mp}(\overline{U}) = \mu \,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\|\,\mathrm{skew}\,\overline{U}\|^2 + \beta_1 \,\|\,\mathrm{Cof}\,\overline{U}\|^2 - \beta_2 \,\ln\det[\overline{U}] + \beta_3\,(\det[\overline{U}] - 1)^2 \,, \qquad (4.44)$$

*with $\beta_1, \beta_2, \beta_3 > 0$. Then (4.38) with material constants conforming to case I (if $\mu_c > 0$) or conforming to case II (if $\mu_c = 0$) admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ and $\det[\nabla\varphi] > 0$ a.e.*

**Proof.** The argument of case I/II can be modified. We note first that the additional terms involving $\beta_i, i = 1,2,3$ are in fact independent of $\overline{R}$ and can be reduced to their dependence on $F$. The additional terms have no influence on the shear failure. Second, the additional terms are **polyconvex** [Bal77] w.r.t. $F$. Third, $W_{\rm mp} \to \infty$ for $\det[F] \to 0$. In addition to the properties of minimizing sequences in case II we obtain that $\mathrm{Cof}\,F_k \rightharpoonup \mathrm{Cof}\,F \in L^2(\Omega)$, $\det[F_k] \rightharpoonup \det[F] \in L^1(\Omega)$ due to separate control of the cofactor and determinant in the energy. The result follows by standard arguments. The minimizer may not be a solution of the corresponding Euler-Lagrange equations. Note that replacing $\mu \,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2$ with $\beta_0 \,\|\overline{U}\|^2$ would decouple the problem and remove the possibility of elastic shear failure. ■

**Corollary 4.5 (Existence for 3D-constrained Cosserat model: case III)** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\mathrm{polar}(\nabla g_{\rm d}) \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times3})$. Then problem (2.21) admits at least one minimizing solution $\varphi \in H^{1,2}(\Omega, \mathbb{R}^3)$.*

**Proof.** The proof mimics case I since the sequence of infimizing rotations $R_k$ is constrained to the orthogonal part $\mathrm{polar}(F_k)$ of the corresponding sequence of deformation gradients $F_k$. ■

**Remark 4.6**
*Complete higher regularity of $\varphi$ in the constrained Cosserat model, i.e. $\varphi \in H^{2,2}(\Omega, \mathbb{R}^3)$ cannot be ascertained in general since we only control certain second derivatives of $\varphi$ in the curvature term. One might wonder therefore, whether the additional $C^1$-continuity in treating the fourth order indeterminate couple stress problem numerically is worth the effort.*

**Theorem 4.7 (Existence for 3D-finite elastic Cosserat model with consistent boundary coupling)** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\mathrm{polar}(\nabla g_{\rm d}) \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times3})$. Then (4.38) with material constants conforming to case I/II and the* **consistent coupling condition**

$$\overline{R}_\Gamma = \mathrm{polar}(\nabla\varphi)_\Gamma \,, \qquad (4.45)$$

*admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.*

**Proof.** Basically, we repeat the argument of case I/II. For the prescription of $g_{\rm d}$ the energy is finite on the set of admissible functions verifying also the coupling condition. We choose minimizing sequences $(\varphi_k, \overline{R}_k)$. Since $\overline{R}_k \in \mathrm{SO}(3,\mathbb{R})$ it follows that $\|\overline{R}_k\| = \sqrt{3}$ independent of $k$, a distinguishing feature of true rotations. Hence $\overline{R}_k$ is bounded in the Sobolev space $W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ without independent prescription of boundary conditions. The remainder proceeds as in case I/II. ■

It is not immediately clear to us how to extend the method of proof to cover the elastic-plastic Cosserat case also. This topic will be left open for future research.

## 4.5 Partially linearized elastic Cosserat theory

If we assume small microrotations, i.e. $\overline{R} = 1\!\!1 + \overline{A} + \dots$, $\overline{A} \in \mathfrak{so}(3)$, $\|\overline{A}\| \ll 1$, then the micropolar stretch tensor may be expanded as follows

$$\overline{U} = \overline{R}^T F = (1\!\!1 + \overline{A} + \dots)^T (1\!\!1 + \nabla u) \approx 1\!\!1 + \nabla u - \overline{A} - \overline{A}\nabla u + \dots \,, \tag{4.46}$$

where $u$ is the (unrestricted) displacement but not the infinitesimal displacement from which we conclude the approximative expression of the stretch energy

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u, \overline{A}) &= \mu \,\|\, \mathrm{sym}\, \nabla u - \mathrm{sym}(\overline{A}\nabla u)\|^2 + \mu_c \,\|\, \mathrm{skew}(\nabla u) - \overline{A} - \mathrm{skew}(\overline{A}\nabla u)\|^2 \\ &\quad + \frac{\lambda}{2}\, \mathrm{tr}\left[\mathrm{sym}(\nabla u) - \mathrm{sym}(\overline{A}\nabla u)\right]^2 . \end{aligned} \tag{4.47}$$

The value $\mu_c = 0$ is still admissible, since the problem does not decouple, but if $\mu_c = 0$, the local coupling takes place only in the second order contribution $\overline{A}\nabla u$.

Since $\mathfrak{K} = \overline{R}^T \mathrm{D_x} \overline{R} = (1\!\!1 + \overline{A} + \dots)^T \mathrm{D_x}(1\!\!1 + \overline{A} + \dots) \approx \mathrm{D_x}\overline{A} + \overline{A}\,\mathrm{D_x}\overline{A} + \dots$ to first order we get as approximation for the curvature energy [35], based on (2.7) the expression

$$\begin{aligned} W_{\mathrm{curv}}^{\mathrm{small}}(\mathfrak{K}) &= \mu \frac{L_c^2}{12}\left(\alpha_5 \,\|\, \mathrm{sym}\, \mathrm{D_x}\overline{A}\|^2 + \alpha_6 \,\|\, \mathrm{skew}\, \mathrm{D_x}\overline{A}\|^2 + \alpha_7 \,\mathrm{tr}\left[\mathrm{D_x}\overline{A}\right]^2\right) \\ \overline{A}_{|\Gamma} &= \overline{A}_{\mathrm{d}} \,, \end{aligned} \tag{4.48}$$

together with the consistently reduced boundary condition for small rotations. For a mathematical treatment, the decisive simplification afforded by (4.48) is the treatment of $\overline{A}$ on the linear manifold $\mathfrak{so}(3)$ of skew-symmetric matrices instead of $\overline{R} \in \mathrm{SO}(3)$. The corresponding equation of balance of angular momentum

$$\begin{aligned} &\mathrm{skew}\left(-2\mu[\mathrm{sym}\, \nabla u - \mathrm{sym}(\overline{A}\nabla u)]\nabla u^T\right. \\ &\quad - 2\mu_c[\mathrm{skew}\, \nabla u - \overline{A} - \mathrm{skew}(\overline{A}\nabla u)](1\!\!1 + \nabla u)^T \\ &\quad \left.-\lambda\langle \mathrm{sym}\, \nabla u - \mathrm{sym}(\overline{A}\nabla u), 1\!\!1\rangle\, \nabla u^T\right) - \mathrm{skew}\left(\mathrm{Div}\, D_{\mathrm{D_x}\overline{A}} W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A})\right) - \mathrm{skew}(M) = 0_{\mathbb{M}^{3\times 3}} \,, \end{aligned} \tag{4.49}$$

can be written equivalently as

$$\frac{L_c^2}{12}\, \mathrm{Div}\, \mathfrak{M}.\mathrm{D_x}\overline{A} - \mathrm{skew}(M) = \widehat{M}(\overline{A}, \nabla u) := \mathrm{skew}\left(\dots\right), \tag{4.50}$$

with a (rearranging) linear operator $\mathfrak{M} : \mathfrak{T}(3) \mapsto \mathfrak{T}(3)$ and

$$\widehat{M}(\overline{A} + \hat{\overline{A}}, X) = \widehat{M}(\overline{A}, X) + \widehat{M}(\hat{\overline{A}}, X), \quad \|\widehat{M}(\overline{A}, X)\| \le (1 + C^+(\overline{A})) \cdot \|X\|^2 . \tag{4.51}$$

It is readily seen that (4.50) is a uniformly Legendre-Hadamard elliptic system with constant coefficients in $\overline{A}$ which is linear at given $\nabla u$.

For a model based on (4.47) and (4.48) the only mathematically interesting case left is the critical case $\mu_c = 0, q = 0, p = 1$ since otherwise the theorems treating the exact finite situations already apply. However, if $\mu_c = 0$ we have to make up for the loss of pointwise control in the stretch and the loss of regularity of $(1\!\!1 + \overline{A})$ if $\mathrm{D_x}\overline{A} \in L^2(\Omega)$ only. This suggests a slight modification of the problem on the level of the corresponding equilibrium system. We replace (regularize) $\widehat{M}$ in (4.50) with

$$\widehat{M}_\sharp^{ij}(\overline{A}, X) := \begin{cases} \widehat{M}^{ij}(\overline{A}, X) & |\widehat{M}^{ij}(\overline{A}, X)| \le K - 1 \\ \tilde{M}^{ij}(\overline{A}, X) & K - 1 \le |\widehat{M}^{ij}(\overline{A}, X)| \le K \\ K & |\widehat{M}^{ij}(\overline{A}, X)| > K \,, \end{cases} \tag{4.52}$$

such that $\widehat{M}_\sharp^{ij}(\overline{A}, X)$ is smooth to any order we need. It means physically that the components of the reaction stresses (couple stresses) are assumed to be essentially bounded by a constant $K > 0$. This is a consistent requirement with the other simplifying assumptions already made. The complete problem reads therefore:

$$\begin{aligned} &\int_\Omega W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u, \overline{A}) - \langle f, u\rangle - \langle M, \overline{A}\rangle \,\mathrm{dV} \mapsto \min .\ \text{w.r.t. } u \text{ at given } \overline{A} \\ &\qquad u_{|\partial\Omega}(x) = g_{\mathrm{d}}(x) - x \,, \quad \overline{A}_{|\partial\Omega} = \overline{A}_{\mathrm{d}} \\ &\frac{L_c^2}{12}\, \mathrm{Div}\, \mathfrak{M}.\mathrm{D_x}\overline{A} = \widehat{M}_\sharp(\overline{A}, \nabla u) - \mathrm{skew}(M) . \end{aligned} \tag{4.53}$$

We can prove the following result:

[35] The expansion of the curvature shows that we need not introduce a smallness assumption on the curvature itself, if we already assume that rotations are small.

**Theorem 4.8 (Existence for 3D-small rotation elastic Cosserat model)**
*Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary of class $C^3$ and assume for the boundary data $g_{\rm d} \in H^3(\Omega,\mathbb{R}^3)$ and $\overline{A}_{\rm d} \in H^{2,2}(\Omega,\mathfrak{so}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and $\operatorname{skew}(M) \in L^2(\Omega,\mathfrak{so}(3,\mathbb{R}))$. Then the system (4.53) with $\mu_c \geq 0, \alpha_6 \geq 0$ admits a, perhaps non-unique, solution pair $(u,\overline{A}) \in H^3(\Omega,\mathbb{R}^3)\times H^2(\Omega,\mathfrak{so}(3))$. The solution is smoother if the data are smoother.*

**Proof.** Since the exponent in the curvature energy is only two, we cannot mimic the variational argument of the last section since we have no easy way to obtain a strongly convergent subsequence $\overline{A}_k$ in a topology as strong as $C^0(\overline{\Omega})$. Instead, we define the following iteration scheme: given $(u_j,\overline{A}_j) \in H^{m+1}(\Omega,\mathbb{R}^3)\times H^m(\Omega,\mathfrak{so}(3))$ compute $(u_{j+1},\overline{A}_{j+1}) \in H^{m+1}(\Omega,\mathbb{R}^3)\times H^m(\Omega,\mathfrak{so}(3))$ such that

$$\begin{aligned}
&\int_\Omega W_{\rm mp}^{\rm small}(\nabla u_{j+1},\overline{A}_j) - \langle f,u_{j+1}\rangle - \langle M,\overline{A}_j\rangle \,{\rm dV} \mapsto \min .\ \text{w.r.t. } u_{j+1} \text{ at given } \overline{A}_j\\
&\qquad u_{j+1}{}_{|_{\partial\Omega}}(x) = g_{\rm d}(x) - x\,, \quad \overline{A}_{j+1}{}_{|_{\partial\Omega}} = \overline{A}_{\rm d} \qquad\qquad (4.54)\\
&\frac{L_c^2}{12}\,\operatorname{Div}\mathfrak{M}.{\rm D_x}\overline{A} = \widehat{M_\sharp}(\overline{A}_j,\nabla u_{j+1}) - \operatorname{skew}(M)\,.
\end{aligned}$$

We proceed to show that the sequence $(u_j,\overline{A}_j)_{j=1}^\infty$ is bounded in $H^{m+1}(\Omega,\mathbb{R}^3)\times H^m(\Omega,\mathfrak{so}(3))$ independent of $j$. To this end we note first that the sequence is uniquely determined if $\overline{A}_j \in C(\overline{\Omega},\mathfrak{so}(3))$ (extended Korn's first inequality). Based on sharp $L^2$-elliptic regularity [Ebe02] for systems with variable coefficients (Dirichlet conditions everywhere on $\partial\Omega$) for both equations separately yields for some yet unspecified pair $(m,r)\in\mathbb{N}\times\mathbb{N}$ the estimates [Nef03a]:

$$\begin{aligned}
\|u_{j+1}\|_{m+2,2} &\leq C(\Omega,\|\overline{A}_j\|_{m+1,2})\cdot\left(\|g\|_{m+2,2}+\|f\|_{m,2}+\|\overline{A}_j\|_{m+1,2}\right)\\
\|\overline{A}_{j+1}\|_{r+2,2} &\leq C(\Omega)\cdot\left(\|\overline{A}_{\rm d}\|_{r+2,2}+\|\widehat{M_\sharp}(\overline{A}_j,\nabla u_{j+1})\|_{r,2}+\|\operatorname{skew}(M)\|_{r,2}\right), \qquad (4.55)
\end{aligned}$$

if the solutions are unique, respectively. The constant in the first estimate is a polynomial in $\|\overline{A}_j\|_{m+1,2}$ and bounded above if the Legendre-Hadamard ellipticity constant of the related acoustic tensor is bounded away from zero. The algebraic estimate

$$D^2_{\nabla u}W_{\rm mp}^{\rm small}(\nabla u,\overline{A}).(\nabla\phi,\nabla\phi) \geq \mu\,\|(1\!\!1+\overline{A})^T\nabla\phi+\nabla\phi^T(1\!\!1+\overline{A})\|^2\,, \qquad (4.56)$$

implies (cf. (2.15)) that

$$\begin{aligned}
D^2_{\nabla u}W_{\rm mp}^{\rm small}(\nabla u,\overline{A}).(\xi\otimes\eta,\xi\otimes\eta) &\geq 2\,\mu\,\|(1\!\!1+\overline{A})^T\xi\otimes\eta\|^2 \qquad (4.57)\\
&\geq 2\mu\lambda_{\min}\left((1\!\!1+\overline{A})(1\!\!1+\overline{A})^T\right)\|\xi\|^2\,\|\eta\|^2\,.
\end{aligned}$$

Since $(1\!\!1+\overline{A})(1\!\!1+\overline{A})^T = 1\!\!1-\overline{A}^2$ and $\langle(1\!\!1-\overline{A}^2).v,v\rangle = \|v\|^2+\|\overline{A}.v\|^2 \geq \|v\|^2$, we conclude that $\lambda_{\min}\left((1\!\!1+\overline{A})(1\!\!1+\overline{A})^T\right)\geq 1$ and the ellipticity constant of the force balance equation is indeed uniform.

If $\alpha_6 > 0$, then the balance of angular momentum equation has a unique solution and $(4.55)_2$ is true as such. If $\alpha_6 = 0$, then we control only $\operatorname{sym}{\rm D_x}\overline{A}$. However, this is still enough to guarantee a unique solution. [36]

[36] For $\overline{A}\in\mathfrak{so}(3,\mathbb{R})$ we have

$$\begin{aligned}
&\overline{A} = \begin{pmatrix} 0 & \alpha & \beta\\ -\alpha & 0 & \gamma\\ -\beta & -\gamma & 0\end{pmatrix},\quad \operatorname{axl}(\overline{A}) = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix},\quad \nabla\operatorname{axl}(\overline{A}) = \begin{pmatrix}\alpha_x & \alpha_y & \alpha_z\\ \beta_x & \beta_y & \beta_z\\ \gamma_x & \gamma_y & \gamma_z\end{pmatrix},\quad \operatorname{sym}\nabla\operatorname{axl}(\overline{A}) = \begin{pmatrix}\alpha_x & \frac{\alpha_y+\beta_x}{2} & \frac{\alpha_z+\gamma_x}{2}\\ \frac{\alpha_y+\beta_x}{2} & \beta_y & \frac{\beta_z+\gamma_y}{2}\\ \frac{\alpha_z+\gamma_x}{2} & \frac{\beta_z+\gamma_y}{2} & \gamma_z\end{pmatrix},\\
&\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2 = \alpha_x^2+\beta_y^2+\gamma_z^2+\frac{(\alpha_y+\beta_x)^2}{2}+\frac{(\alpha_z+\gamma_x)^2}{2}+\frac{(\beta_z+\gamma_y)^2}{2} \qquad (4.58)\\
&\|\operatorname{sym}{\rm D_x}\overline{A}\|^2 = \|\operatorname{sym}\nabla\overline{A}.e_1\|^2+\|\operatorname{sym}\nabla\overline{A}.e_2\|^2+\|\operatorname{sym}\nabla\overline{A}.e_3\|^2\\
&\quad = \frac{\alpha_x^2}{2}+\frac{\beta_x^2}{2}+\alpha_y^2+\beta_z^2+\frac{(\alpha_x+\beta_y)^2}{2}+\alpha_x^2+\frac{\alpha_y^2}{2}+\gamma_y^2+\gamma_z^2+\frac{(\alpha_z+\gamma_x)^2}{2}+\beta_x^2+\gamma_y^2+\frac{\beta_z^2}{2}+\frac{\gamma_z^2}{2}+\frac{(\beta_y+\gamma_x)^2}{2}.
\end{aligned}$$

Now it is easy to see that for some $c^+ > 0$ it holds $\|\operatorname{sym}{\rm D_x}\overline{A}\|^2 \geq c^+\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2$ since $\|\operatorname{sym}{\rm D_x}\overline{A}\|^2 = 0$ implies $\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2 = 0$, algebraically. Hence, the standard Korn's inequality applied to $\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2$ yields unique existence.

Note that with the permutation $P = \begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix} \in {\rm O}(3)$ the **infinitesimal micropolar curvature tensor** is defined as $\kappa := \nabla P.\operatorname{axl}(\overline{A}) = P\cdot\nabla\operatorname{axl}(\overline{A})$ and it holds that $P.\operatorname{axl}(\operatorname{skew}(\nabla u)) = \frac{1}{2}\operatorname{curl} u$ and $\operatorname{tr}[\nabla P.\operatorname{axl}(\operatorname{skew}(\nabla u))] = \frac{1}{2}\operatorname{Div}\operatorname{curl} u = 0$. As a consequence, if we were to take $W_{\rm mp}^{\rm infin}(\kappa) = \frac{L_c^2}{12}\|\kappa\|^2$, then the corresponding infinitesimal couple-stress tensor $D_\kappa W_{\rm mp}^{\rm infin}(\kappa)$ is trace-free in the indeterminate couple stress theory. We make no explicit use of $\kappa$.

For $r = 0$ we obtain therefore that the sequence $\overline{A}_j$ is bounded independent of $j$ in the space $H^2(\Omega, \mathfrak{so}(3)) \subset C^{2-\frac{3}{2}}(\overline{\Omega}, \mathfrak{so}(3))$ which implies for $m = 1$ that the sequence $u_j$ is bounded independent of $j$ in the space $H^3(\Omega, \mathbb{R}^3)$. Extracting strongly convergent subsequences and letting $j \to \infty$ shows that the limit $(u_\infty, \overline{A}_\infty)$ is a solution of the modified field equation (4.53). This ends the argument. Note that due to the definition of $\widehat{M}_\sharp$, we are able to increase the regularity of the solution if the data are smoother. ■

Appreciating the method of proof, we see that the partially linearized model together with quadratic curvature energy is appropriate only for exceptionally 'smooth' situations. The additional provision of sufficiently smooth boundary and data and restriction to the pure Dirichlet case allows us to conclude the continuity of the infinitesimal microrotations $\overline{A}$ circumventing the direct use of Sobolev embedding theorems. Considering the reductions made, it is nevertheless our belief that the model (4.53) captures the essential features of the geometrically exact Cosserat micropolar framework in contrast to the subsequent infinitesimal models. However, it remains open whether a consistent boundary coupling in the sense of requiring $\overline{A}_{|\partial\Omega} = \operatorname{skew}(\nabla u)_\Gamma$ is possible for this model.

Rather for historical reasons and completeness we reconsider the classical infinitesimal Cosserat micropolar problem. Existence results have been obtained e.g. in [Ies71, Duv70, HH69, Ghe74a, Ghe74b].

## 4.6 Infinitesimal linear elastic Cosserat theory

If we finally assume infinitesimal microrotations and infinitesimal displacements, the micropolar stretch tensor may again be expanded as follows

$$\overline{U} = \overline{R}^T F = (1\!\!1 + \overline{A} + \ldots)^T (1\!\!1 + \nabla u) \approx 1\!\!1 + \nabla u - \overline{A} - \overline{A}\nabla u + \ldots . \tag{4.59}$$

Neglecting consistently the quadratic term $\overline{A}\nabla u$ in (4.47) yields the approximate expression for the stretch energy [37] (2.6)

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u - \overline{A}) &= \mu \,\|\operatorname{sym}(\nabla u - \overline{A})\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\nabla u - \overline{A})\right]^2 \\ &= \mu \,\|\operatorname{sym} \nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u) - \overline{A}\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\nabla u)\right]^2 , \end{aligned} \tag{4.60}$$

and for the curvature term

$$W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A}) = \mu \frac{L_c^2}{12} \left( \alpha_5 \,\|\operatorname{sym} \mathrm{D_x}\overline{A}\|^2 + \alpha_6 \,\|\operatorname{skew} \mathrm{D_x}\overline{A}\|^2 + \alpha_7 \operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2 \right) . \tag{4.61}$$

Two observations are essential. First, if $\mu_c = 0$, the infinitesimal problem completely decouples - the infinitesimal microrotations $\overline{A}$ have no influence whatsoever on the macroscopic behaviour of the infinitesimal displacements. We believe that this is a deficiency of the infinitesimal problem without much physical significance for the geometrically exact model. It has led perhaps to the erroneous common belief that $\mu_c > 0$ is regarded to be essential [38] for any true Cosserat micropolar theory as well.

And second, the choice $\alpha_6 = 0$ is possible, contrary to the finite case, since coercivity of the reduced curvature expression can still be concluded on account of the classical Korn's first inequality applied to $\operatorname{sym} \mathrm{D_x}\overline{A}$ as already shown. We mention also that contrary to the finite case there is no gap: balance of angular momentum without internal length scale and $\mu_c > 0$ yields

$$D_{\overline{A}} W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u, \overline{A}) \in \mathrm{Sym} \Leftrightarrow D_{\overline{A}} W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u, \overline{A}) = 0 \Leftrightarrow \operatorname{skew}(\nabla u) = \overline{A} . \tag{4.62}$$

This implies already that **infinitesimal continuum- and microrotations coincide**, and this in turn is equivalent to the symmetry of the infinitesimal Cauchy stress $\sigma$ or the **Boltzmann axiom**.

Hence the infinitesimal case rather inhibits our understanding of the general finite Cosserat micropolar problem.

[37] Traditionally, the infinitesimal model is obtained not as linearization but by directly assuming a split of the displacement gradient into infinitesimal micropolar stretch and infinitesimal microrotations: $\nabla u = \overline{\varepsilon} + \overline{A}$, where $\overline{\varepsilon}$ is not necessarily symmetric.

[38] In many treatments of the infinitesimal theory, e.g. [Kup79, p.34] the assumption $D^2 W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u - \overline{A}).(H,H) \geq c^+ \|H\|^2$ is explicitly introduced as being motivated on physical grounds rather then being necessary for a meaningful treatment of the infinitesimal Cosserat micropolar theory. It would exclude classical infinitesimal elasticity as a special case. Sometimes, a so called **coupling number** $N^2 = \frac{\mu_c}{\mu + \mu_c} \in [0,1]$ is introduced, which allows to compare different material moduli. $N = 0$ corresponds to classical infinitesimal elasticity, $N = 1$ corresponds to infinitesimal indeterminate couple stress theory.

The infinitesimal micropolar model in variational form is given by

$$\int_\Omega \mu \,\|\operatorname{sym}\nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\nabla u\right]^2 + \mu\,\frac{L_c^2}{12}\,\|\nabla \operatorname{axl}(\overline{A})\|^2 - \langle f, u\rangle - \langle M, \overline{A}\rangle \,\mathrm{dV}$$

$$- \int_{\Gamma_S} \langle N, u\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}\rangle \,\mathrm{dS} \mapsto \min. \quad \text{w.r.t. } (u, \overline{A}) \tag{4.63}$$

$$u_{|\partial\Omega}(t,x) = g_{\mathrm{d}}(t,x) - x, \quad x \in \partial\Omega\,, \quad \overline{A}_{|\partial\Omega} = \operatorname{skew}(\nabla g_{\mathrm{d}}(t,x))_{|\partial\Omega}\,.$$

The corresponding equilibrium system of equations for pure Dirichlet conditions and without external couples $M, M_c$ is given by (note that $\|\overline{A}\|^2 = 2\|\operatorname{axl}(\overline{A})\|^2$ for $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$)

$$\begin{aligned} 0 &= \operatorname{Div}\sigma + f, \quad x \in \Omega \\ \sigma &= 2\mu \operatorname{sym}\nabla u + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - \overline{A}) + \lambda \operatorname{tr}\left[\operatorname{sym}\nabla u\right]\cdot 1\!\!1 \\ 0 &= \mu\,\frac{L_c^2}{12}\,\Delta \operatorname{axl}(\overline{A}) + \mu_c \operatorname{axl}((\operatorname{skew}(\nabla u) - \overline{A})\,. \end{aligned} \tag{4.64}$$

If we consider nonetheless the only nontrivial case left open, $\mu_c > 0$, it is standard to prove that the corresponding minimization problem admits a unique minimizing pair $(u, \overline{A}) \in H^1(\Omega,\mathbb{R}^3) \times H^1(\Omega,\mathfrak{so}(3))$.

**Theorem 4.9 (Existence for 3D-infinitesimal elastic Cosserat model)**
*Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{A}_{\mathrm{d}} \in H^1(\Omega,\mathfrak{so}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^2(\Omega,\mathfrak{so}(3,\mathbb{R}))$ and $M_c \in L^2(\Gamma_C,\mathfrak{so}(3,\mathbb{R}))$. Then the model (4.63) admits a unique minimizing solution pair $(u, \overline{A}) \in H^1(\Omega,\mathbb{R}^3) \times H^1(\Omega,\mathfrak{so}(3))$. The solution is smoother if the data are smoother.*

**Proof.** We are again in a position to apply the direct methods of variations. Without loss of generality consider $N, M, M_c = 0$ and $\mu \geq \mu_c > 0$. First we observe that infimizing sequences $(u_k, \overline{A}_k)$ exist and

$$\begin{aligned} \infty > \int_\Omega W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u_k - \overline{A}_k) + W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A}_k) - \langle f, u_k\rangle \,\mathrm{dV} &\geq \int_\Omega \mu_c\,\|\nabla u_k - \overline{A}_k\|^2 \,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \qquad (4.65) \\ = \int_\Omega \mu_c\,\|\operatorname{sym}(\nabla u_k - \overline{A}_k)\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u_k - \overline{A}_k)\|^2 \,\mathrm{dV} &- \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \\ \geq \int_\Omega \mu_c\,\|\operatorname{sym}\nabla u_k\|^2 \,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} &\geq \mu_c\, c_K\,\|u_k\|^2_{H^1(\Omega)} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)}\,, \end{aligned}$$

showing that $u_k$ is bounded in $H^1(\Omega)$. We have used that sym is orthogonal to skew and the classical Korn's first inequality together with the boundary conditions for $u_k$. Moreover, again by the classical Korn's first inequality (if $\alpha_6 = 0$) or directly pointwise, we obtain boundedness of $\overline{A}_k$ in $H^1(\Omega,\mathfrak{so}(3))$. We can choose a subsequence of $(u_k, \overline{A}_k)$ converging strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. By overall convexity of the energy density in $(\nabla u, \mathrm{D_x}\overline{A})$ the limit pair is a minimizer.

For the uniqueness we consider the second derivative of the total strain energy $W = W_{\mathrm{mp}}^{\mathrm{infin}} + W_{\mathrm{curv}}^{\mathrm{small}}$ with respect to the complete argument

$$\begin{aligned} D^2_{(\nabla u,\overline{A})} W(\nabla u - \overline{A}).((\nabla\phi, \delta\overline{A}), (\nabla\phi, \delta\overline{A})) &\geq \mu_c\|\nabla\phi - \delta\overline{A}\|^2 \\ = \mu_c\|\operatorname{sym}\nabla\phi\|^2 + \mu_c\|\operatorname{skew}(\nabla\phi - \delta\overline{A})\|^2 &\geq \mu_c\|\operatorname{sym}\nabla\phi\|^2\,. \end{aligned} \tag{4.66}$$

By the classical Korn's first inequality we obtain uniform positivity of the second derivative upon integration. [39] The functional is strictly convex, the solution is unique.

Since the resulting field equations of force balance and balance of angular momentum are linear, uniformly elliptic with constants coefficients the standard elliptic regularity theory applies such that for pure Dirichlet boundary conditions the solution is the smoother the smoother the data. ■

[39] Assume $(\phi, \delta\overline{A}) \neq 0$, then $D^2_{(\nabla u,\overline{A})} W(\nabla u - \overline{A}).((\nabla\phi, \delta\overline{A}), (\nabla\phi, \delta\overline{A})) > 0$.

**Theorem 4.10 (Existence for 3D-infinitesimal elastic Cosserat model with consistent coupling)** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega, \mathbb{R}^3)$ and $\operatorname{skew}(\nabla g_{\rm d}) \in H^1(\Omega, \mathbb{M}^{3\times 3})$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^2(\Omega, \mathfrak{so}(3,\mathbb{R}))$ and $M_c \in L^2(\Gamma_C, \mathfrak{so}(3,\mathbb{R}))$. Then the model* (4.63) *with* **consistent coupling**

$$\overline{A}_{|\Gamma} = \operatorname{skew}(\nabla u(t,x))_{|\Gamma}\,, \tag{4.67}$$

*admits a unique minimizing solution pair $(u, \overline{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3))$. The solution is smoother if the data are smoother.*

**Proof.** Observe that with $g_{\rm d}$ as specified the energy is finite for consistent coupling. Next, note that infimizing sequences $(u_k, \overline{A}_k)$ exist and are bounded since the displacement can be controlled independent of $\overline{A}_k$ and the additional $L^2$ control of $\|\operatorname{skew}(\nabla u) - \overline{A}_k\|$ shows that $\overline{A}_k$ is as well $L^2$-controlled. Hence $(u_k, \overline{A}_k) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3))$, independent of $k$. This finishes the argument. The role played by exact rotations in the finite case is replaced by the role of $\mu_c > 0$. ■

The corresponding **infinitesimal gradient constrained Cosserat micropolar** model (infinitesimal indeterminate couple stress model) has the form (simplified curvature term: $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$)

$$\begin{aligned}
&\int_\Omega \mu\|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2 + \mu\,\frac{L_c^2}{12}\,\|\mathrm{D_x}\operatorname{skew}(\nabla u)\|^2 - \langle f, u\rangle\,\mathrm{dV}\\
&\qquad - \int_{\Gamma_S}\langle N, u\rangle\,\mathrm{dS} - \int_{\Gamma_C}\langle M_c, \operatorname{skew}(\nabla u)\rangle\,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } u \qquad (4.68)\\
&\sigma = \sigma^{\rm loc} + \sigma^{\rm hyper}\\
&\sigma^{\rm loc} = 2\,\mu\operatorname{sym}(\nabla u) + \lambda\operatorname{tr}[\operatorname{sym}(\nabla u)]\cdot 1\!\!1 \in \mathrm{Sym},\ \text{constitutive stress}\\
&\sigma^{\rm hyper} = -2\mu\,\frac{L_c^2}{12}\,\operatorname{axl}^{-1}\big(\operatorname{Div}\nabla\operatorname{axl}(\operatorname{skew}(\nabla u))\big) \in \mathfrak{so}(3,\mathbb{R})\\
&u_{|\partial\Omega}(x) = g_{\rm d}(x) - x\,,\quad \operatorname{skew}(\nabla u)_{|\partial\Omega} = \operatorname{skew}(\nabla g_{\rm d})_{|\partial\Omega}\,.
\end{aligned}$$

Using the same methods as before we obtain

**Theorem 4.11 (Existence for 3D-infinitesimal gradient case)** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary of class $C^1$ and assume for the boundary data $g_{\rm d} \in H^2(\Omega, \mathbb{R}^3)$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M_c \in L^2(\Gamma_C, \mathfrak{so}(3))$. Then the model* (4.68) *admits a unique minimizing solution $u \in H^1(\Omega) \cap \{\nabla \operatorname{curl} u \in L^2(\Omega)\}$.*

**Proof.** As before. See also [Duv70]. ■

# 5 On the choice of the curvature energy contribution

In the finite regime we have various choices for the assumed dependence of the elastic energy density on gradients of the microrotations. It seems that there is no universally correct term available, rather the applications aimed at dictate in some respect this form. We list several of these possibilities which can be useful:

1. The case exhibiting similarity to **plates** and **shells**:

$$W_{\rm curv}(\mathfrak{K}) = \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\left(\alpha_5\,\|\operatorname{sym}\mathfrak{K}\|^2 + \alpha_6\,\|\operatorname{skew}\mathfrak{K}\|^2 + \alpha_7\operatorname{tr}[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}, \tag{5.69}$$

for $\mu_c = \alpha_4 = \alpha_6 = 0$, $p = 1$, one component of $\mathfrak{K}$, namely $\overline{R}^T\nabla\overline{R}.e_3$ appears in the resulting curvature contribution obtained after expanding the underlying stretch energy with respect to deformations of the midsurface $m(x,y)$ of a plate assuming that $\varphi(x,y,z) = m(x,y) + z\,\overline{R}.e_3$ and integrating through the thickness. For infinitesimal rotations this may be reduced to

$$W_{\rm curv}(\mathrm{D_x}\overline{A}) = \mu\frac{L_c^2}{12}\left(\alpha_5\,\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 + \alpha_6\,\|\operatorname{skew}\mathrm{D_x}\overline{A}\|^2 + \alpha_7\operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2\right), \tag{5.70}$$

which yields pointwise control of $\mathrm{D_x}\overline{A}$ for $\alpha_5, \alpha_6 > 0$. If $\alpha_6 = 0$, non-local control of $\|\mathrm{D_x}\overline{A}\|_{L^2(\Omega)}$ is possible on account of Korn's classical first inequality.

2. The case for **almost rigid material** behaviour: i.e. $\nabla\varphi \approx \overline{R} \in \mathrm{SO}(3,\mathbb{R})$, then $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} \approx \mathrm{Sym}$, such that the former may be reduced to

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12}\,(1+\alpha_4\, L_c^q\, \|\mathfrak{K}\|^q)\,\left(\alpha_5\,\|\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,, \tag{5.71}$$

where pointwise control of the torsion-curvature tensor $\mathfrak{K}$ is possible.

3. The case accounting for **dislocations**: a measure accounting for incompatibility is the torsion tensor $\overline{R}^T \mathrm{Curl}\,\overline{R} \in \mathbb{M}^{3\times 3}$. Orthogonal to this expression is $\overline{R}^T.\mathrm{Div}\,\overline{R} \in \mathbb{R}^3$ (Helmholtz decomposition), a measure for the curvature. Thus $W_{\mathrm{curv}}$ may as well be assumed to consist of two independent parts

$$W_{\mathrm{tors,div}} = \mu \frac{L_c^{1+p}}{12}\left(\alpha_8 \|\overline{R}^T \mathrm{Curl}\,\overline{R}\|^2 + \alpha_9 \|\overline{R}^T.\mathrm{Div}\,\overline{R}\|^2\right)^{\frac{1+p}{2}}\,. \tag{5.72}$$

Non-local control of all first order derivatives of $\overline{R}$ is possible, if $\alpha_8, \alpha_9 > 0$. For infinitesimal incompatibility, $\overline{R} = 1\!\!1 + \overline{A} + \ldots$ and $\overline{R}^T \mathrm{Curl}\,\overline{R} \approx \mathrm{Curl}\,\overline{A}$ such that $\alpha_8 > 0, \alpha_9 \geq 0$ is sufficient for pointwise(!) control of $\mathrm{D_x}\overline{A}$ on account of the fact that $\mathrm{Curl}\,\overline{A}$ for $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ controls all first derivatives of $\overline{A}$. [40] While it is clear that we cannot control completely $\|\mathrm{D_x}\overline{R}\|$ if $\alpha_9 = 0$ in the finite case we might still stand a chance to show that $\alpha_8 > 0, \alpha_9 = 0$ is already sufficient for the variational problem. Counting derivatives in the finite case, we see that 9 independent entries in $\mathrm{Curl}\,\overline{R}$ together with 3 independent entries in $\mathrm{Div}\,\overline{R}$ control already the norm of a total of 27 derivatives of $R$. This is the object of ongoing research.

4. The traditional representation based on the **reduced second order curvature tensor** $\hat{\mathfrak{K}}$:

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12}\left(\alpha_5\,\|\,\mathrm{sym}\,\hat{\mathfrak{K}}\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\hat{\mathfrak{K}}\|^2 + \alpha_7\,\mathrm{tr}\,\left[\hat{\mathfrak{K}}\right]^2\right)^{\frac{1+p}{2}}\,, \tag{5.73}$$

where $\hat{\mathfrak{K}} := \left(\mathrm{axl}(\overline{R}^T\partial_x\overline{R})|\,\mathrm{axl}(\overline{R}^T\partial_y\overline{R})|\,\mathrm{axl}(\overline{R}^T\partial_z\overline{R}\right) \in \mathbb{M}^{3\times 3}$. For small curvature this is further reduced to $(\mathrm{axl}(\partial_x\overline{A})|\,\mathrm{axl}(\partial_y\overline{A})|\,\mathrm{axl}(\partial_y\overline{A})) = \nabla\,\mathrm{axl}(\overline{A}) = P\cdot\kappa$ with $\kappa$ the **infinitesimal micropolar curvature tensor** and the permutation matrix $P$ as in (4.58).

# 6 The quasiconvex hull and relaxation

Since we have investigated the salient regularizing features of the elastic Cosserat approach which in principle should furnish a mesh-independent solution for related numerical implementations it seems worthwhile to contrast this with another well known successful computational method [Lam02], based on quasiconvexity, to obtain mesh-independent results for stress-strain relations. [41]

The elastic free energy density $W : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ is **quasiconvex**, if

$$\forall\,\hat{F} \in \mathbb{M}^{3\times 3}\,:\; |\Omega|\cdot W(\hat{F}) \leq \int_\Omega W(\hat{F} + \nabla\phi(x))\,\mathrm{dV} \quad \forall\,\phi \in C_0^\infty(\Omega,\mathbb{R}^3)\,. \tag{6.74}$$

This implies notably that the homogeneous deformation is a global minimizer to its own boundary conditions. For a given free energy density $W : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ the quasiconvex hull $QW$ is defined as the largest quasiconvex function below $W$, i.e.

$$QW := \sup\{\tilde{W} \leq W \;:\; \tilde{W} \text{ is quasiconvex}\}\,. \tag{6.75}$$

If $W$ is locally bounded and Borel measurable, another equivalent characterisation [Dac89] is

$$QW(F) := \inf\left\{\frac{1}{|\Omega|}\int_\Omega W(F + \nabla\phi(x))\,\mathrm{dV} \;:\; \phi \in C_0^\infty(\Omega,\mathbb{R}^3)\right\}, \tag{6.76}$$

and the infimum is independent of $\Omega$. The resulting **relaxed** function is always Legendre-Hadamard elliptic,

$$D_F^2[QW](F).(\xi\otimes\eta, \xi\otimes\eta) \geq 0\,, \tag{6.77}$$

[40] In fact, for $A \in \mathfrak{so}(3,\mathbb{R})$ it holds that $\mathrm{D_x}A = L.\mathrm{Curl}\,A$, with $L$ a linear mapping.

[41] In many practical cases, not a complete mesh-independent result for stresses and deformations is needed, but only a mesh-independent result for the stresses is looked for. The relaxed functional serves this restricted purpose. Average values of stresses are computed, the geometry of deformation is only considered in a probabilistic sense.

and the quasiconvex hull $QW$ is weakly lower semi continuous.

In most cases it is not possible to derive analytical formulae for $QW$, however, there is one notable exception. If $W(F) = \hat{W}(U)$ is a convex function in $U$ and bounded below, then a general result in [Pip94] implies[42] that

$$QW(F) = \inf_{P \in \mathrm{PSym}} \hat{W}(U + P) . \tag{6.78}$$

Let us apply this result to $W(F) = \|U - 1\!\!1\|^2$ which is easily seen to be not Legendre-Hadamard elliptic over the whole range of admissible deformations $F \in \mathrm{GL}^+(3, \mathbb{R})$ but to satisfy the Baker-Ericksen inequalities. The loss of ellipticity is due to the behaviour of the orthogonal part $R = \mathrm{polar}(F)$ of $F$. We obtain

$$QW(F) = \sum_{\lambda_i \geq 1} |\lambda_i - 1|^2 , \tag{6.79}$$

where $\lambda_i \geq 0$ are the eigenvalues of $U$. The resulting function is Legendre-Hadamard elliptic but not uniformly, since compression has zero energy. Moreover, the linearization of the relaxed density $QW(F)$ does not coincide with the linearization of $W(F)$ which is uniformly Legendre-Hadamard elliptic in a neighbourhood of $F = 1\!\!1$. It is well known that in a neighbourhood of the identity, the nonlinear unmodified problem is well-posed [Val88], also under compression. It is therefore apparent that replacing $W$ by its quasiconvex hull would modify the physical nature of the problem. Special care should therefore be exerted when taking $QW$ also as a means of regularization, especially when the analytical form of $W$ is not known, as is often the case in incremental elasto-plasticity.

# 7 Discussion and concluding remarks

A finite Cosserat model has been introduced and is consistently extended to elasto-viscoplasticity where Cosserat effects remain non-dissipative. Various reduced forms of the model are introduced. For elasticity, it is motivated that the Cosserat ansatz with **independent** rotations is especially suited in conjunction with energies quadratic in the micropolar stretch tensor $\overline{U} = \overline{R}^T F$, in which case the rotations are indeed the only essential nonlinearity left in the problem.

Our constitutive analysis suggests that the Cosserat ansatz with weak local coupling (Cosserat couple modulus $\mu_c = 0$) leading to a stretch energy density for small elastic strains of the form $W_{\mathrm{mp}}(\overline{U}) = \mu \, \| \mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\lambda}{2} \mathrm{tr} \left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2$ provides just the correct amount of regularization needed for classical macroscopic energies quadratic in the continuum stretch since loss of Legendre-Hadamard ellipticity in the equation of linear momentum can be traced back to the presence of the continuum rotations and is removed by taking independent variations with respect to these rotations.

A competing method of regularization, namely the computation of the quasiconvex hull $QW_{\mathrm{mp}}$ of the energy, which also restores the Legendre-Hadamard ellipticity leads to unphysical behaviour under compression in certain cases: the material shows no resistance under compression. Hence, the quasiconvexification is a useful computational tool to reduce the influence of localizations but can hardly be regarded as generally admissible without admitting a modification of the underlying physics which is however, sometimes intended, especially in elasto-plasticity, where complete local relaxation is arguably the optimal response. In the elastic case, we prefer to augment the underlying physics by introducing a length scale.

A delicate interplay between stretch and curvature terms allows under reasonable physical assumptions to establish the existence of minimizers or stationary points of the corresponding elastic action functional. The sub-critical case (including the true fracture case IV) $\mu_c = 0$, $\alpha_4 > 0$, $0 < p < 1$, $q \geq 0$ in (2.7) and the critical case $\mu_c = 0$, $\alpha_4 > 0$, $p = 1$, $q = 0$ in (2.7) and $\mu_c = 0$, $\alpha_8 > 0$, $\alpha_9 = 0$, $p \geq 1$ in (5.72) are not settled and leave a wide field of challenging purely mathematical problems.

Depending on the applications aimed at, the Cosserat couple modulus $\mu_c$ should either be very large (microscopic specimens) or zero (macroscopic specimens). The mathematical analysis reflects this dichotomy.

The different reduced elastostatic Cosserat micropolar models have thus been shown to be completely justified. The finite macroscopic elastic-plastic case where Cosserat effects are assumed to be non-dissipative, however, is completely left open.

As another result of our investigation we note that extreme care should be exerted when determining material constants already for isotropic Cosserat micropolar models: the data fit should preferably be based on the geometrically exact model with the same number and type of parameters and not on the infinitesimal model which degenerates for admitted values of material parameters which however, are to be fitted. If this is

[42] The proof in [Pip94] is based on $W(F) = \hat{W}(C)$, $\hat{W}$ convex in $C$. With appropriate changes it carries over to the representation in $U$. We remark that the loss of ellipticity of $\|C - 1\!\!1\|^2$ is already occurring in uniaxial compression. Clearly a deficiency of the Green strains not shared by the Biot strains which loose ellipticity in biaxial situations only.

done it might turn out that for specific macroscopic situations we are left with the problem of finding just the characteristic internal length $L_c$ while the Cosserat couple modulus $\mu_c$, already orders of magnitudes smaller than the classical shear modulus $\mu$ in many cases, drops completely out. In the macroscopic case, we favour therefore an essentially reduced **three-parameter isotropic Cosserat micropolar** theory with **independent** microrotations: the two classical Lamé constants $\mu, \lambda$, one additional internal characteristic length scale $L_c > 0$ and $(\mu_c, \alpha_4, \alpha_5, \alpha_6, \alpha_7, p, q) = (0, 1, 1, 1, 0, 1, 1)$, thus discarding all other unnecessary material constants. This far reaching reduction will also facilitate renewed experimental identification of the length scale $L_c$ and its precise relation to the shear band width, obscured in the infinitesimal theory.

To conclude, we believe that the choice $\mu_c = 0$ represents a refreshing departure from traditional linear approaches; it reconciles experimental evidence on a macroscale and the possibility of fracture with the Cosserat model and leads to interesting new mathematical problems. It shows to furnish a natural method to physically regularize certain shear failure problems and it introduces experimentally observed second order size effects which seem to be present in nearly all materials. It is therefore hoped that the presented development will stimulate further mathematical research in this important field for the applications.

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# 9 Appendix A

## 9.1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $\Gamma$ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times3}$ the set of real $3 \times 3$ second order tensors, written with capital letters and by $\mathfrak{T}(3)$ the set of all third order tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times3} \,|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \,|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \,|X^TX = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \,|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times3} \,|\operatorname{tr}[X] = 0\}$ of traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$. $(v|\xi|\eta) \in \mathbb{M}^{3\times3}$ is the matrix composed of the columns $v, \xi, \eta \in \mathbb{R}^3$. We write the polar decomposition in the form $F = R\,U = \operatorname{polar}(F)\,U$ with $R = \operatorname{polar}(F)$ the orthogonal part of $F$. For a second order tensor $X$ we define the third order tensor $\mathfrak{h} = \mathrm{D}_{\mathrm{x}}X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3}$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}\left[\mathfrak{h}^1\right], \operatorname{tr}\left[\mathfrak{h}^2\right], \operatorname{tr}\left[\mathfrak{h}^3\right]) \in \mathbb{R}^3$. Moreover, for any second order tensor $X$ we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$ correspondingly. Quantities with a bar, e.g. the micropolar rotation $\overline{R}$, represent the micropolar replacement of the corresponding classical continuum rotation $R$. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$ and we use $\nabla$ in general only for column-vectors in $\mathbb{R}^3$. Furthermore, $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_FW(F).H$ and $D_F^2W(F).(H, H)$, respectively. Sometimes we use also $\partial_X W(X)$ to denote the first derivative of $W$ with respect to $X$. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. For $X \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ we define $\operatorname{Curl} X(x)$ and $\operatorname{Div} X(x)$ as the operation curl and Div applied row wise, respectively. For $\mathfrak{h} \in \mathfrak{T}(3)$ we define $\operatorname{Div}\mathfrak{h} = \left(\operatorname{Div}\mathfrak{h}^1|\operatorname{Div}\mathfrak{h}^2|\operatorname{Div}\mathfrak{h}^3\right)^T \in \mathbb{M}^{3\times3}$. We define $H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in $\Omega$. We use capital letters to denote possibly large positive constants, e.g. $C^+, K$ and lower case letters to denote possibly small positive constants, e.g. $c^+, d^+$. The smallest eigenvalue of a positive definite symmetric tensor $P$ is abbreviated by $\lambda_{\min}(P)$. Finally, w.r.t. abbreviates with respect to.

## 9.2 The Boltzmann axiom without internal length

**Lemma 9.1 (Limit rotations with zero internal length scale)**
*Define* $W_{\mathrm{mp}}$ *as*

$$W_{\mathrm{mp}}(\overline{U}) = \alpha_1\,\|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U})\|^2 + \alpha_3\operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2 \tag{9.80}$$

*as in* (2.6). *If* $\alpha_1 = \mu_c$ *and* $\operatorname{tr}\left[\overline{U}\right] < 3 + \frac{2\mu_c}{\alpha_3}$ *then* $D_{\overline{U}}W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \Rightarrow \left[\overline{U} \in \mathrm{Sym} \Leftrightarrow \overline{R} = \operatorname{polar}(F)\right]$. *Otherwise,* $D_{\overline{U}}W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ *alone does not imply* $\overline{U} \in \mathrm{Sym}$.

**Proof.** An argument relating to the general case of $W_{\mathrm{mp}}$ taken as an isotropic scalar valued function of $\overline{U}$ has been given e.g. in [San99, p.29] and [SB95]. However, no conditions on the coefficients or the magnitude of $\operatorname{tr}\left[\overline{U}\right]$ are involved, which raises some questions. Note first that $D_{\overline{U}}W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ is equivalent to

$$\operatorname{skew}\left((\alpha_1 - \mu_c)\,\overline{U}^T\overline{U}^T - 2\,\alpha_1\,\overline{U}^T + 2\,\alpha_3\operatorname{tr}\left[\overline{U} - 1\!\!1\right]\overline{U}^T\right) = 0\,. \tag{9.81}$$

We write $\overline{U} = \operatorname{sym}\overline{U} + \operatorname{skew}\overline{U} = S + A$. This yields in three steps

$$\begin{aligned}
\operatorname{skew}\left((\alpha_1 - \mu_c)\,(S + A)^T(S + A)^T - 2\,\alpha_1\,(S + A)^T + 2\,\alpha_3\operatorname{tr}[S - 1\!\!1]\,(S + A)^T\right) &= 0\\
\operatorname{skew}\left((\alpha_1 - \mu_c)\,(S^TS^T + S^TA^T + A^TS^T + A^TA^T) - 2\,\alpha_1\,A^T + 2\,\alpha_3\operatorname{tr}[S - 1\!\!1]\,A^T\right) &= 0\\
\operatorname{skew}\left((\alpha_1 - \mu_c)\,(SA^T + A^TS) - 2\,\alpha_1\,A^T + 2\,\alpha_3\operatorname{tr}[S - 1\!\!1]\,A^T\right) &= 0\,,
\end{aligned} \tag{9.82}$$

and since $A^2 \in \mathrm{Sym}(3)$ for $A \in \mathfrak{so}(3,\mathbb{R})$, this leads to the system

$$-(\alpha_1 - \mu_c)\,(SA + AS) + 2\,(\alpha_1 + \alpha_3\,\mathrm{tr}\,[S - 1\!\!1])\,A = 0\,, \tag{9.83}$$

which represents 3 linear equations for three unknowns in $A \in \mathfrak{so}(3,\mathbb{R})$. If $\alpha_1 = \mu_c$ and $\mathrm{tr}\,[\overline{U}] < 3 + \frac{2\mu_c}{\alpha_3}$ then $A = 0$ necessarily. A small algebraic argument shows that $SA + AS = 0$, $A \in \mathfrak{so}(3,\mathbb{R}), S \in \mathrm{Sym}$ implies $A = 0$ if $(d_1 + d_2)\,(d_2 + d_3)\,(d_1 + d_3) \neq 0$ for $d_i$ the eigenvalues of $S$. For the second statement, set $\mu_c = \mu = \alpha_1$, $\alpha_3 = \frac{\lambda}{2} = \mu$, which is in the range of actual material behaviour of macroscopic crystalline solids, and consider accordingly

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\overline{U} - 1\!\!1\|^2 + \mu\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]^2, \quad D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T = 2\mu\,\overline{U}\,\overline{U}^T - 2\mu\,\overline{U}^T + 2\mu\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]\overline{U}^T\,. \tag{9.84}$$

Now take

$$\overline{U} = \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{GL}^+(3,\mathbb{R}) \text{ if } |\alpha| \ll 1 \quad \text{and} \quad \overline{U} \notin \mathrm{Sym}\,. \tag{9.85}$$

The corresponding microrotation $\overline{R}$ and deformation $\varphi$ such that $\overline{U} = \overline{R}^T \nabla\varphi$ can be easily deduced. This implies $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T \in \mathrm{Sym}$ and balance of linear momentum is satisfied for zero loads since the quantities $(\overline{R}, \nabla\varphi)$ are homogeneous if appropriate boundary conditions are specified but $\overline{U} \notin \mathrm{Sym}$. ■

## 9.3 Macroscopic elastic shear failure

It is appropriate to define what we mean by **shear failure** in classical isotropic elasticity. Let $W(F)$ be the free elastic energy density of the bulk material. If for some $F \in \mathrm{GL}^+(3,\mathbb{R})$

$$\exists\, \xi, \eta \in \mathbb{R}^3 : \quad D^2 W(F).(\xi \otimes \eta, \xi \otimes \eta) < 0\,, \tag{9.86}$$

we say that the material fails or looses **Legendre-Hadamard ellipticity (LH)**, also called a **material instability**. [43] This failure can give rise to highly localized deformation patterns, subsumed under the notion of **microstructure**. Related is the possible emergence of discontinuous deformations since **Hadamard´s jump relations** are violated. However, loss of ellipticity may already occur for deformations which are not related to shear, e.g. uniaxial situations and pure dilations. Thus we say that $W$ suffers from **genuine elastic shear failure** whenever

$$\begin{aligned} &\exists\, F \in \mathrm{GL}^+(3,\mathbb{R}) \quad \exists\, \xi, \eta \in \mathbb{R}^3 : \quad D^2 W(F).(\xi \otimes \eta, \xi \otimes \eta) < 0\,, \quad \text{but} \\ &\forall\, F \in \mathrm{diag}(\lambda_1^+, \lambda_2^+, \lambda_3^+) \quad \forall\, \xi, \eta \in \mathbb{R}^3 : \quad D^2 W(F).(\xi \otimes \eta, \xi \otimes \eta) \geq 0\,. \end{aligned} \tag{9.87}$$

It seems that failure of a material on a macroscale other than shear failure is unphysical and rather due to the idiosyncrasy of the constitutive equations, as long as the bulk is modelled as elastic. In fact, Legendre-Hadamard ellipticity for $F = \mathrm{diag}(\lambda_1^+, \lambda_2^+, \lambda_3^+)$ implies immediately the **Baker-Ericksen (BE)** inequalities [MH83, p.19] and genuine elastic shear failure happens, if BE is satisfied but LH is violated.[44]

In this sense the following non exhaustive list of free energy terms should be avoided since they are not only failing under shear (already BE is not satisfied):

$$\|C - 1\!\!1\|^2,\ \langle C - 1\!\!1, 1\!\!1\rangle^2,\ \langle \ln C, 1\!\!1\rangle^2,\ \langle \ln C, 1\!\!1\rangle^2 + \|\,\mathrm{dev}\ln C\|^2,\ \langle \ln U, 1\!\!1\rangle^2,\ -\ln\det[F] + (\ln\det[F])^2,\ \|\frac{C}{\det[C]^{1/3}} - 1\!\!1\|^2\,. \tag{9.88}$$

Of course, combination with other terms could remove the problem. Terms which genuinely fail only in shear are e.g. $\|U - 1\!\!1\|^2$, $\langle U - 1\!\!1, 1\!\!1\rangle^2$ and $\|\frac{U}{\det[U]^{1/3}} - 1\!\!1\|^2$, $\mathrm{tr}\left[\frac{U}{\det[U]^{1/3}} - 1\!\!1\right]^2$.

## 9.4 Analytical investigation of incompressible elastic simple shear

In order to elucidate the proposed theory and to be able to validate numerical solutions we consider first the deformation of an **incompressible** homogeneous unit cube in simple shear at the upper and lower faces. Let $\Omega = [0,1] \times [0,1] \times [0,1]$ be the unit cube and impose the boundary conditions $g(x_1,x_2,0) = (x_1,x_2,0)^T$, $g(x_1,x_2,1) = (x_1 + \gamma, x_2, x_3)^T$, $0 \leq x_1, x_2 \leq 1$. The parameter $\gamma \geq 0$ is the amount of maximal shear at the upper face per unit length. In order to arrive at an analytically tractable one-dimensional problem we wish to find **energy minimizing** deformations of the form

$$\varphi(x_1,x_2,x_3) = \begin{pmatrix} x_1 + u(x_1,x_3) \\ x_2 \\ x_3 \end{pmatrix}, \ \nabla\varphi(x_1,x_2,x_3) = F = \begin{pmatrix} 1 + u_{x_1}(x_1,x_3) & 0 & u_{x_3}(x_1,x_3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.89}$$

with $u(x_1,0) = 0$, $u(x_1,1) = \gamma$. Effectively, we require, that initially horizontal material planes remain horizontal.[45] Incompressibility requires that $\det[F] = 1$ and implies $u_{x_1}(x_1,x_3) = 0$. The boundary conditions show then that $u$ must be constant in $x_1$-direction. Hence the reduced kinematics

$$\varphi(x_1,x_2,x_3) = \begin{pmatrix} x_1 + u(x_3) \\ x_2 \\ x_3 \end{pmatrix}, \ \nabla\varphi(x_1,x_2,x_3) = F = \begin{pmatrix} 1 & 0 & u'(x_3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.90}$$

[43] Material instability should be carefully distinguished from **geometrical instabilities** occurring in buckling or necking and which are fully consistent with Legendre-Hadamard ellipticity. In this sense, **polyconvex** materials are **unconditionally** materially stable and certainly appropriate for rubber and soft-tissues [SN02, HN03].

[44] One version of **BE** can be stated as follows: for $\lambda_i > 0$ the principal stretches, the free energy $\Phi(\lambda_1,\lambda_2,\lambda_3) := W(U)$ is separately convex in $\lambda_i$. No mathematical existence results based only on BE are known. Note also that BE is enough to effectively exclude phase-transformations, modelled with multi-well potentials.

[45] These are not the most general possible deformations. The most general deformations are of the form $\varphi(x_1,x_2,x_3) = (x_1 + u(x_1,x_3), x_2, x_3 + v(x_1,x_3))^T$.

with $u(0) = 0$, $u(1) = \gamma$ suffices.[46] Accordingly we postulate microrotations $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and infinitesimal microrotations $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ of the type:

$$\overline{R}(x_1,x_2,x_3) = \begin{pmatrix} \cos\alpha(x_3) & 0 & \sin\alpha(x_3) \\ 0 & 1 & 0 \\ -\sin\alpha(x_3) & 0 & \cos\alpha(x_3) \end{pmatrix}, \quad \overline{A}(x_1,x_2,x_3) = \begin{pmatrix} 0 & 0 & \alpha(x_3) \\ 0 & 1 & 0 \\ -\alpha(x_3) & 0 & 0 \end{pmatrix}. \tag{9.91}$$

This implies that $\|\mathrm{D_x}\overline{R}\|^2 = \|\mathrm{D_x}\overline{A}\|^2 = 2\,|\alpha'(x_3)|^2$. In view of symmetry considerations we try to find solutions for the shear profile angle $\alpha : [0,1] \mapsto \mathbb{R}$ of the form $\alpha(1/2 + x) = \alpha(1/2 - x)$, implying that $\alpha^{(n)}(0) = (-1)^n\,\alpha^{(n)}(0)$. Explicit calculation shows that

$$\overline{R}^T F = \begin{pmatrix} \cos\alpha(x) & 0 & \cos\alpha(x)\cdot u'(x) - \sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha(x) & 0 & \sin\alpha(x)\cdot u'(x) + \cos\alpha(x) \end{pmatrix}, \ \mathrm{sym}\,\overline{R}^T F = \begin{pmatrix} \cos\alpha(x) & 0 & \frac{\cos\alpha(x)\cdot u'(x)}{2} \\ 0 & 1 & 0 \\ \frac{\cos\alpha(x)\cdot u'(x)}{2} & 0 & \sin\alpha(x)\cdot u'(x) + \cos\alpha(x) \end{pmatrix},$$

$$\mathrm{skew}\,\overline{R}^T F = \begin{pmatrix} 0 & 0 & \frac{\cos\alpha(x)\cdot u'(x) - 2\sin\alpha(x)}{2} \\ 0 & 0 & 0 \\ -\frac{\cos\alpha(x)\cdot u'(x) - 2\sin\alpha(x)}{2} & 0 & 0 \end{pmatrix}. \tag{9.92}$$

The energy[47] is given by

$$\int_\Omega \mu\,\|\mathrm{sym}\,\overline{R}^T F - 1\!\!1\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{R}^T F\|^2 + 2\mu\,L_c^2|\alpha'(x)|^2\,\mathrm{dV} \mapsto \min. \quad \text{w.r.t. } (u,\alpha). \tag{9.93}$$

$u(0) = 0$, $u(1) = \gamma$, Dirichlet boundary conditions for displacement

$\alpha(0) = \alpha(1) = 0$, classical **(rigid)** boundary conditions for microrotations

$0 = \mathrm{skew}(\overline{R}^T F)_{\{0,1\}} \Leftrightarrow 2\tan\alpha(0) = u'(0),\ 2\tan\alpha(1) = u'(1)$, **consistent coupling** boundary conditions

in the finite case[48] and

$$\int_\Omega \mu\,\|\mathrm{sym}\,F - 1\!\!1\|^2 + \mu_c\,\|\mathrm{skew}\,F - \overline{A}\|^2 + 2\mu\,L_c^2|\alpha'(x)|^2\,\mathrm{dV} \mapsto \min. \quad \text{w.r.t. } (u,\alpha). \tag{9.94}$$

$u(0) = 0$, $u(1) = \gamma$

$\alpha(0) = \alpha(1) = 0$, classical rigid boundary conditions

$0 = \big(\mathrm{skew}(F) - \overline{A}\big)_{\{0,1\}} \Leftrightarrow 2\alpha(0) = u'(0),\ 2\alpha(1) = u'(1)$, reduced consistent coupling boundary conditions,

in the infinitesimal case. The classical elasticity formulation in the finite case[49] is recovered, if $\mu_c = 0$, $L_c = 0$, $\mathrm{skew}(\overline{R}^T F) = 0 \Leftrightarrow 2\tan\alpha(x) = u'(x)$ and in the infinitesimal case, if $\mu_c = 0$, $L_c = 0$, $\big(\mathrm{skew}(F) - \overline{A}\big) = 0 \Leftrightarrow 2\alpha(x) = u'(x)$ and independent variation is performed w.r.t. to the displacement $u$ only.

In terms of $(u',\alpha)$ the finite energy expression is

$$\begin{aligned} W_{\text{finite}}(u',\alpha) &= \mu\left(2(\cos\alpha - 1)^2 + \frac{u'^2}{2} + \frac{\sin^2\alpha u'^2}{2} + 2(\cos\alpha - 1)\sin\alpha u'\right) \\ &\quad + \mu_c\left(\frac{\cos^2\alpha u'^2}{2} - 2\cos\alpha\sin\alpha u' + 2\sin^2\alpha\right) + 2\mu\,L_c^2|\alpha'|^2, \end{aligned} \tag{9.95}$$

and the **first order reduction** $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha$ for small $\alpha$ yields the corresponding infinitesimal energy expression

$$W_{\text{small}}(u',\alpha) = \mu\,\frac{u'^2}{2} + \mu_c\left(\frac{u'^2}{2} - 2\alpha\cdot u' + 2\alpha^2\right) + 2\mu\,L_c^2|\alpha'|^2 = \mu\,\frac{u'^2}{2} + 2\mu_c\left(\frac{u'}{2} - \alpha\right)^2 + 2\mu\,L_c^2|\alpha'|^2. \tag{9.96}$$

The second derivative of the energy in the infinitesimal case w.r.t. $(u',\alpha)$ is given by

$$D^2_{(u',\alpha)}W_{\text{small}}(u',\alpha).((\delta u,\delta\alpha),(\delta u,\delta\alpha)) = \mu\,|(\delta u)'|^2 + 2\mu_c\,|\frac{(\delta u)'}{2} - \delta\alpha|^2 + 4\mu\,L_c^2|(\delta\alpha)'|^2, \tag{9.97}$$

which shows that for **classical rigid Dirichlet boundary conditions** and for **consistent coupling** conditions the **solution** $(u,\alpha)$ of the **infinitesimal** problem is **unique**. Since the homogeneous deformation $u(x) = \gamma\cdot x$ together with constant shear angle $\alpha(x) = \frac{\gamma}{2}$ is always a solution for consistent coupling, it is the unique solution coinciding with the unique solution of the classical infinitesimal elasticity problem with shear stress at the upper face $\tau_{\text{lin}} = \mu\,\gamma$.

It is useful to consider as well a **second order reduction** of the energy: $\cos\alpha \approx 1 - \frac{\alpha^2}{2}$, $\sin\alpha \approx \alpha - \frac{\alpha^3}{3!}$. Skipping terms higher then order four we get

$$W_{\text{red}}(u',\alpha) == \mu\,\left(\frac{1+\alpha^2}{2}u'^2 + \frac{\alpha^4}{2} - \alpha^3\,u'\right) + 2\mu_c\,\left(\left(\frac{u'}{2} - \alpha\right)^2 - \frac{\alpha^2\,u'^2}{4} + \frac{2}{3}\alpha^3 u' - \frac{\alpha^4}{3}\right) + 2\mu\,L_c^2|\alpha'|^2. \tag{9.98}$$

[46]The considered problem is therefore the exact formulation of the **simple glide** in $e_1$-direction with amount $\gamma$ at the upper face of an infinite layer of material with unit height fixed at the bottom.

[47]The energy corresponds to the class studied in Theorem 4.4. Note that by the analytical methods proposed in section (4) we already know that minimum energy configurations in the finite case exist for both types of boundary conditions. In the one-dimensional case the coercivity $|\alpha'(x)|^2$ is enough to guarantee strong convergence of minimizing sequences of microrotations $(\sin\alpha_k, \cos\alpha_k)$ in the space of continuous functions due to Sobolev embedding theorems.

[48]It is a delicate matter to specify independent boundary conditions for the microrotations $\alpha$. Somehow it requires to know the solution of the boundary value problem in advance. In this sense such a classical rigid boundary condition is nothing but a first guess, perhaps useful as start value in a numerical scheme.

[49]The consistent reduction of this requirement for small $\alpha$ is $\alpha = \frac{u'}{2} - \frac{u'^2}{24} + \ldots$.

The corresponding reduced classical elasticity problem is obtained by

$$W_{\text{red}}^{\text{class}}(u') := W_{\text{red}|_{\mu_c=0,L_c=0}}(u', \frac{u'}{2}) = \mu\left(\frac{{u'}^2}{2} + \frac{{u'}^4}{32}\right). \tag{9.99}$$

The equilibrium equations in the fully finite case are obtained by taking free variations w.r.t. $(u, \alpha)$. Force balance amounts to

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} W_{\text{finite}}(u + t\phi, \alpha) \\
0 &= \big[-\mu\, u''(x) - \mu\left(2\sin\alpha(x)\cos\alpha(x)\alpha'(x)u'(x) + \sin^2\alpha(x)\, u''(x)\right) - 2\mu\left((\cos\alpha(x) - 1)\cos\alpha(x)\alpha'(x) - \sin^2\alpha(x)\alpha'(x)\right) \\
&\quad -\mu_c\left(-2\cos\alpha(x)\sin\alpha\alpha'(x)u'(x) + \cos^2\alpha(x)u''(x)\right) - 2\mu_c\left(-\sin^2\alpha(x)\alpha'(x) + \cos^2\alpha(x)\alpha'(x)\right)\big]\cdot\phi,
\end{aligned} \tag{9.100}$$

and balance of angular momentum is obtained from variation w.r.t. $\alpha$:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} W_{\text{finite}}(u, \alpha + t\,\delta\alpha) \\
&= -4\mu(\cos\alpha(x) - 1)\,\sin\alpha\,\delta\alpha + \frac{\mu}{2}\,2\sin\alpha(x)\,\cos\alpha(x)u'(x)^2\,\delta\alpha + 2\mu(\cos\alpha(x) - 1)\,\cos\alpha(x)\,u'(x)\,\delta\alpha(x) \\
&\quad + 2\mu(-\sin\alpha(x)\,\sin\alpha(x)u'(x)\delta\alpha(x) + 4\mu\,L_c^2\alpha'(x)(\delta\alpha)'(x) \\
&\quad + \mu_c\left(\frac{2\cos\alpha(x)(-\sin\alpha(x))}{2}\delta\alpha\, u'(x)^2 - 2\cos\alpha(x)\cos\alpha(x)\delta\alpha(x) + 2\sin\alpha(x)\sin\alpha(x)\delta\alpha u'(x) + 4\sin\alpha(x)\cos\alpha(x)\delta\alpha\right).
\end{aligned} \tag{9.101}$$

In the finite case, we evaluate the generated tangential shear stresses $\tau$ at the upper face where maximal shear occurs. They are

$$\tau = \langle S_1(F(1), \overline{R}(1)).e_3, e_1\rangle = \langle \overline{R}(1)\left(2\mu\ \text{sym}(\overline{R}^T F - 1\!\!1) + 2\mu_c\ \text{skew}(\overline{R}^T F)\right).e_3, e_1\rangle. \tag{9.102}$$

For consistent coupling conditions, the homogeneous deformation $u(x) = \gamma\, x$ and $\tan\alpha(0) = \tan\alpha(x) = \frac{\gamma}{2}$ is always a solution and leads to a **nonlinear, strictly monotone** shear response at the upper face of

$$\begin{aligned}
\tau &= \mu u'(1) + \mu\sin^2\alpha(1)u'(1) + 2\mu\sin\alpha(1)(\cos\alpha(1) - 1) \\
&= \mu\gamma + \mu\sin^2(\arctan\frac{\gamma}{2})\,\gamma + 2\mu\sin(\arctan\frac{\gamma}{2})\,(\cos(\arctan\frac{\gamma}{2}) - 1).
\end{aligned} \tag{9.103}$$

For the finite problem, at this point, we do not know whether the homogeneous response is the only one possible or realizes the minimum energy.

In order to investigate this point further, we simplify the two equilibrium equations by considering the second order expansions $\cos\alpha(x) = 1 - \frac{\alpha^2}{2} + \ldots$, $\sin\alpha = \alpha - \frac{\alpha^3}{6} + \ldots$ and keep terms up to order three in the variables $(u', \alpha)$. After partial integration such a reduction coincides with the system of equilibrium equations based directly on the reduced energy $W_{\text{red}}$:

$$\begin{aligned}
0 &= u'' + \frac{2(\mu - \mu_c)\,\alpha\alpha' u'}{\mu(1 + \alpha^2) + \mu_c(1 - \alpha^2)} + \frac{(4\mu_c - 3\mu)\,\alpha^2\alpha'}{\mu(1 + \alpha^2) + \mu_c(1 - \alpha^2)} - \frac{2\mu_c\alpha'}{\mu(1 + \alpha^2) + \mu_c(1 - \alpha^2)} \\
0 &= \mu\left(\frac{1}{2}\alpha^3 + \frac{1}{4}\alpha{u'}^2 - \frac{3}{4}\alpha^2 u' - L_c^2\alpha''\right) - \mu_c\left(\frac{1}{4}\alpha{u'}^2 + \frac{2}{3}\alpha^3 + \mu_c\alpha^2 u' + \left(\frac{u'}{2} - \alpha\right)\right) \\
u(0) &= 0,\ u(1) = \gamma \\
\alpha(0) &= \alpha(1) = 0, \quad \text{classical rigid boundary conditions} \\
2\alpha(0) &= u'(0),\ 2\alpha(1) = u'(1), \quad \text{reduced consistent boundary conditions}.
\end{aligned} \tag{9.104}$$

Observe that the homogeneous deformation remains a solution of the reduced system for consistent coupling.

It is natural to **require** that the solution of the coupled boundary value problem, notably the **shear profile angle** $\alpha$, is in fact **independent of** the **shear modulus** $\mu$ and the **Cosserat couple modulus** $\mu_c$ as in classical elasticity. This condition can only be met with $\mu_c = 0$ or $\mu_c = \mu$.

If $\mu_c = 0$ the corresponding system of balance equations based consistently on $W_{\text{red}}$ reduces further to

$$\begin{aligned}
0 &= \frac{2\alpha\alpha'}{1 + \alpha^2}\,u' - \frac{3\alpha^2\alpha'}{1 + \alpha^2} + u'' \\
0 &= \frac{1}{2}\alpha^3 + \frac{1}{4}\alpha{u'}^2 - \frac{3}{4}\alpha^2 u' - L_c^2\alpha'' \\
u(0) &= 0,\ u(1) = \gamma \\
\alpha(0) &= \alpha(1) = 0, \quad \text{classical rigid boundary conditions} \\
2\alpha(0) &= u'(0),\ 2\alpha(1) = u'(1), \quad \text{consistent coupling boundary conditions}.
\end{aligned} \tag{9.105}$$

Let us integrate the first equation of (9.105) at given $\alpha$ by means of the variations of constants formula. This shows that

$$u'(x) = e^{-\int_0^x \frac{2\alpha(t)\alpha'(t)}{1+\alpha^2(t)}\,\mathrm{d}t}\cdot\left(u'(0) + \int_0^x \frac{3\alpha^2\alpha'(t)}{1 + \alpha^2(t)}\,e^{\int_0^t \frac{2\alpha(s)\alpha'(s)}{1+\alpha^2(s)}\,\mathrm{d}s}\right)\mathrm{d}t = \frac{1 + \alpha^2(0)}{1 + \alpha^2(x)}\left(u'(0) + \frac{[\alpha^3(x) - \alpha^3(0)]}{1 + \alpha^2(0)}\right). \tag{9.106}$$

The last equation is the **integrated form** of the **force balance** equation. The Dirichlet boundary conditions for $u$ imply the additional integral condition

$$\gamma = \gamma - 0 = u(1) - u(0) = \int_0^1 u'(x)\,\mathrm{d}x = \int_0^1 \frac{1 + \alpha^2(0)}{1 + \alpha^2(x)}\left(u'(0) + \frac{[\alpha^3(x) - \alpha^3(0)]}{1 + \alpha^2(0)}\right)\mathrm{d}x. \tag{9.107}$$

Let us consider the limit case $L_c = 0$, disregarding the possible boundary values for $\alpha$ in a first approach. This corresponds to the case (4.2) **finite elasticity with free rotations and microstructure** of the classification (2.20). The second (now algebraic) equation of (9.105) can then be solved for $\alpha$ with the result that $\alpha$ can take on only three distinct values

$$\alpha(x) \in \{0,\ \frac{u'(x)}{2},\ u'(x)\}. \tag{9.108}$$

Reinserting the result into (9.106) shows that we have altogether only two different values for $u'(x)$ at our disposition, determined by three equations (two coinciding)

$$\left[1+\alpha^2(0)\right]u'(0)-\alpha^3(0)\right] = \begin{cases} u'(x) & \text{insert } \alpha(x)=0 \\ u'(x)+\frac{1}{8}\,u'(x)^3 & \text{insert } \alpha(x)=\frac{u'(x)}{2} \\ u'(x) & \text{insert } \alpha(x)=u'(x) \end{cases} . \tag{9.109}$$

Let us choose $u'(0)$, $\alpha(0)$ such that

$$\left[1+\alpha^2(0)\right]u'(0)-\alpha^3(0)\right] = \gamma+\delta^+, \quad \delta^+>0\,. \tag{9.110}$$

The value $\delta^+>0$ to be determined subsequently. Taking $\alpha(0)=\frac{u'(0)}{2}$ in (9.110) implies that $u'(0)+\frac{1}{8}\,u'(0)^3=\gamma+\delta^+$ and (9.109) reduces to

$$\gamma+\delta^+ = \begin{cases} u'(x) \\ u'(x)+\frac{1}{8}\,u'(x)^3 \end{cases} . \tag{9.111}$$

We take $\delta^+>0$ as the unique solution of the equation

$$(\gamma-\delta^+)+\frac{1}{8}(\gamma-\delta^+)^3=\gamma+\delta, \quad \text{if } \delta^+=c^+\cdot\gamma \Leftrightarrow (1-c^+)^3=\left(\frac{4}{\gamma}\right)^2 c^+\,. \tag{9.112}$$

For $\gamma\to\infty$ we have $\delta^+\to\gamma$, $c^+\to 1$ and $\delta^+=\delta^+(\gamma)$ is a monotone increasing function of $\gamma$. This implies

$$u'(x)=\begin{cases} \gamma_1=\gamma+\delta^+ \\ \gamma_2=\gamma-\delta^+ \end{cases} , \tag{9.113}$$

and notably $u'(0)=\gamma-\delta^+$. Now consider the family of straight lines

$$v_1(x)=\gamma_1\,x, \quad v_2(x)=\gamma_1\,x+(\gamma-\gamma_1), \quad v_3(x)=\gamma_2\,x, \quad v_4(x)=\gamma_2\,x+(\gamma-\gamma_2)\,. \tag{9.114}$$

A family of weak solutions of (9.105) with $L_c=0$ and $\alpha(0)=\frac{u'(0)}{2}$, $u'(0)=\gamma-\delta^+$ is given as a continuous combination of piecewise affine functions with slopes parallel to $v_i, i=1,\dots 4$ satisfying $u(0)=0$, $u(1)=\gamma$. This is the expected **microstructure**. The constructed solutions $u$ are absolutely continuous, but do not belong to $H^2([0,1],\mathbb{R})$.

Any point symmetric solution w.r.t. (1/2) must have $\alpha(1)=\frac{u'(1)}{2}$, $u'(1)=\gamma-\delta^+$. Symmetry, however, is not enough to single out a unique response. We choose that solution, which has the least number of weak discontinuity points. It is given by

$$u(x)=\begin{cases} (\gamma-\delta^+)\,x & 0\le x\le\frac{1}{2}-\frac{1}{4} \\ (\gamma+\delta^+)\,x-\frac{\delta^+}{2} & \frac{1}{2}-\frac{1}{4}\le\frac{1}{2}+\frac{1}{4} \\ (\gamma-\delta^+)\,x+\delta^+ & \frac{1}{2}+\frac{1}{4}\le x\le 1 \end{cases}, \quad \alpha(x)=\frac{u'(x)}{2}=\begin{cases} \gamma-\delta^+ & 0\le x\le\frac{1}{2}-\frac{1}{4} \\ \gamma+\delta^+ & \frac{1}{2}-\frac{1}{4}\le\frac{1}{2}+\frac{1}{4} \\ \gamma-\delta^+ & \frac{1}{2}+\frac{1}{4}\le x\le 1 \end{cases}, \tag{9.115}$$

showing the expected (sharp) S-type symmetric shear profile. [50] For the reduced formulation we evaluate the generated shear stress response at the upper face. They are

$$\tau_{\text{red}}=D_{u'}W_{\text{red}}(u'(1))=\mu\,\left(u'(1)+\alpha^2(1)u'(1)-\alpha^3(1)\right)=\mu\left(u'(1)+\left(\frac{u'(1)}{2}\right)^2u'(1)-\left(\frac{u'(1)}{2}\right)^3\right)\Rightarrow$$

$$\tau^{\text{hom}}_{\text{red}}=\mu\,(1+\frac{1}{8}\gamma^2)\,\gamma, \qquad \tau^{\text{micro}}_{\text{red}}=\mu\,(1+c^+(\gamma))\,\gamma\,, \tag{9.116}$$

with $c^+(\gamma)\in(0,1)$ from (9.112). This shows that the response with **microstructure** due to free rotations is always **weaker** (energetically favourable) than the still possible homogeneous response. [51]

In order to determine $L_c$ for a given material we consider the same material in different sample sizes with edge length $L_i>1$ and perform corresponding shear experiments. Due to scaling relations the different sizes can be transformed to the unit cube leading to a modified internal length $\frac{L_c}{L_i}$. The solution on the unit cube depends only on $\gamma$ and $\frac{L_c}{L_i}$. Hence the sequence of experiments leads to best-fitting values $\frac{L_c}{L_i}=C_i$, which implies that $L_c=\frac{1}{n}\sum_{i=1}^n L_i\,C_i$ is a good candidate for the real characteristic length. Note that no knowledge of size-independent material parameters is necessary to obtain a value for $L_c$ if $\mu_c=0$.

Let us contrast the foregoing result with a similar analysis of the infinitesimal Cosserat model with necessarily $\mu_c>0$. The system of balance equations is now given by

$$\begin{aligned} u''(x)&=2\frac{\mu_c}{\mu+\mu_c}\,\alpha'(x)=2\,N^2\,\alpha'(x) \\ \mu\,L_c^2\alpha''(x)&=\mu_c\,\left(\alpha(x)-\frac{u'(x)}{2}\right) \\ & u(0)=0,\ u(1)=\gamma \\ & \alpha(0)=\alpha(1)=0, \quad \text{classical rigid boundary conditions} \\ & 2\alpha(0)=u'(0),\ 2\alpha(1)=u'(1), \quad \text{consistent coupling boundary conditions}\,, \end{aligned} \tag{9.117}$$

with the traditional **Cosserat coupling number** $N^2=\frac{\mu_c}{\mu_c+\mu}$. Observe again that the homogeneous solution is the unique solution for consistent coupling.

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[50]It is surprising that the constructed solution satisfies the Boltzmann axiom: $\alpha=\frac{u'}{2}$ everywhere yet it does not coincide with the classical solution.

[51]The weak discontinuities inherent in this microstructure can be seen as a precursor to **fracture**. The road to fracture starts with homogeneous solutions, which turn into smooth inhomogeneous solutions $u\in H^2$, which degenerate into solutions with weak discontinuities $u\in H^1\setminus H^2$, which finally fail along glide planes with $u\notin W^{1,1}$.

In order to find the unique nontrivial solution for rigid conditions, we integrate the first equation of (9.117) and get

$$2\,N^2[\alpha(x)-\alpha(0)] = u'(x)-u'(0), \quad \text{and} \quad 2\,N^2\int_0^1 \alpha(x)\,\mathrm{dx} - 2\,N^2\alpha(0) = u(1)-u(0)-u'(0) = \gamma - u'(0)\,, \tag{9.118}$$

where we have used the Dirichlet boundary conditions for the displacement $u$. This shows

$$u'(0) = \gamma - 2\,N^2\int_0^1 \alpha(x)\,\mathrm{dx}, \quad (\text{use } (9.118)_2 \text{ and } \alpha(0)=0) \Rightarrow \tag{9.119}$$

$$u'(x) = \gamma + 2\,N^2\left(\alpha(x) - \int_0^1 \alpha(x)\,\mathrm{dx}\right), \quad \text{classical rigid boundary conditions,} \Rightarrow u'(0)=u'(1)<\gamma\,.$$

Inserting the result for $u'(x)$ into balance of angular momentum and rearranging yields the linear second order differential equation

$$\alpha''(x) - \frac{N^2}{L_c^2}\alpha(x) = -\frac{\mu_c}{\mu\,L_c^2}\frac{\gamma}{2} + \frac{N^2\,\mu_c}{L_c^2\mu}\int_0^1 \alpha(x)\,\mathrm{dx}\,. \tag{9.120}$$

Differentiating w.r.t. $x$ once more yields the linear third order equation

$$\alpha'''(x) - \frac{N^2}{L_c^2}\alpha'(x) = 0 \Leftrightarrow -N^2\,\alpha'(x) + L_c^2\,\alpha'''(x) = 0\,. \tag{9.121}$$

The general solution of this differential equation in view of the exerted point symmetry of $\alpha$ w.r.t. $\frac{1}{2}$ is given by

$$\alpha(x) = \beta_1\cdot\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) + \beta_2\,. \tag{9.122}$$

For the rigid Dirichlet case we use now $\alpha(0)=\alpha(1)=0$. This implies that

$$0 = \alpha(0) = \beta_1\cdot\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + \beta_2 \Rightarrow \quad \alpha(x) = \beta_1\cdot\left[\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right]\,. \tag{9.123}$$

We calculate

$$\alpha''(x) = \beta_1\cdot\frac{N^2}{L_c^2}\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) \tag{9.124}$$

$$\int_0^1 \alpha(x)\,\mathrm{dx} = -\beta_1\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + \beta_1\frac{L_c}{N}\left[\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - \sinh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right] = 2\beta_1\frac{L_c}{N}\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - \beta_1\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)$$

Inserting the results into (9.120) we obtain

$$\beta_1 = \frac{-\gamma}{2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) - 4NL_c\,\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\,, \tag{9.125}$$

which yields the micropolar shear profile

$$\alpha(x) = -\frac{\left[\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right]\cdot\gamma}{2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) - 4NL_c\,\beta_1\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)} = \frac{\left[\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right]\cdot\gamma}{-2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + 4NL_c\,\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\,, \tag{9.126}$$

with the non-uniform behaviour $0\le\alpha(x)\to\frac{\gamma}{2}$ for $0<x<1$ and $N, L_c\to 0$. A physically acceptable (smooth) S-type symmetric shear profile is characterized by a steepest tangent of $u$ at $\frac{1}{2}$. This corresponds to a maximum of $\alpha$ at $\frac{1}{2}$. The sign of the denominator in the last formula is decisive: it should be negative. Therefore

$$0 > -2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + 4NL_c\,\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right) \Leftrightarrow 2N\,L_c < \coth\left(\frac{N}{L_c}[\frac{1}{2}]\right) \Leftrightarrow 1 > 2NL_c\tanh\left(\frac{N}{2L_c}\right) \tag{9.127}$$

which is always satisfied since $\tanh(x)<x, x>0$ and $0<N^2<1$. The tangential shear stresses are given by

$$\tau_{\text{small}} = \langle\sigma.e_3, e_1\rangle = \langle[2\mu\,\text{sym}(F-1\!\!1) + 2\mu_c(\text{skew}(F-1\!\!1) - \overline{A})].e_3, e_1\rangle = (\mu+\mu_c)\cdot u'(1) - 2\mu_c\,\alpha(1)$$

$$= (\mu+\mu_c)\cdot u'(0) = (\mu+\mu_c)\left(\gamma - 2N^2\int_0^1\alpha(x)\,\mathrm{dx}\right) = \mu\left(\frac{1}{1-2NL_c\tanh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\right)\cdot\gamma\,. \tag{9.128}$$

Expansion shows that for $N>0, L_c\to\infty$ (ever smaller samples)[52] it results in the limit $\tau_{\text{small}} = (\mu+\mu_c)\cdot\gamma$ and $u(x)=\gamma\,x$, the evaluated stresses $\tau_{\text{small}}$ are increased due to $\mu_c>0$ and the incompatibel rigid boundary prescription. For $0<N\ll 1$, $L_c>0$ we observe that $\tau_{\text{small}}\approx(\mu+\mu_c)\cdot\gamma$ and $u(x)\approx\gamma\,x$. In this case, it can be seen that the **Cosserat couple modulus** $\mu_c>0$ is in fact also **a measure of the influence of boundary conditions on the solution** and therefore **not a material parameter**. [53]

[52] Only a formal limit: the smallest sample size should be larger than the chosen $L_c>0$ of the unit cube, i.e. the smallest sample size must be larger than the smallest constituents of the material given as unit cube. Hence, if $L_c$ has any physical meaning, we should have $0\le L_c<1$.

[53] Consider any other independent (artificial) Dirichlet boundary condition for the shear angle $0\le\alpha(0)=\alpha(1)=a<\frac{\gamma}{2}$. The solution $u$ will produce a different shear stress response $\tau_{\text{small}} = \mu\,u'(1) + 2\mu_c\,(\frac{u'(1)}{2} - a)$ which, for different $L_c$, necessitates a modification of $\mu_c$ for the same material. In our example, this inconsistency can be avoided for consistent coupling but persists in the general case.

Similarly, it can be shown that $N \to 0$, $L_c > 0$ is possible and results in the classical response $\tau_{\rm small} = \mu \cdot \gamma$. Finally, $N > 0$, $L_c \to 0$ approaches the classical result as well. In all cases the micropolar response for rigid Dirichlet data is **stiffer** than the corresponding homogeneous classical response.

The computed micropolar displacement for rigid Dirichlet data is given by

$$u(x) = \int_0^x u'(s)\,{\rm ds} = \int_0^x \gamma + 2N^2\alpha(s) - 2N^2\int_0^1 \alpha(x)\,{\rm dx}\,{\rm ds} = \gamma\, x + 2N^2\int_0^x \alpha(s)\,{\rm ds} - 2N^2\, x\int_0^1 \alpha(x)\,{\rm dx}$$

$$= \gamma\left(\frac{\cosh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - 2NL_c\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\,x - NL_c\frac{\sinh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \sinh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)}{\cosh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - 2NL_c\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\right). \tag{9.129}$$

The complete solution of the problem in terms of the displacement $u$ is now a function of $\gamma$, $\frac{N}{L_c}$ and $N \cdot L_c$. Consider the same material given in different sample sizes of cubes with edge length $L_i > 1$. Due to scaling relations, we may transform the different sample sizes to the unit cube resulting in a modified internal length $\frac{L_c}{L_i}$ but identical values $\gamma$, $N$ by the assumption that $\mu_c$, hence $N$ is a **material parameter independent of size**. Performing a corresponding shear experiment on each sample size we obtain best-fitting values of $\frac{N}{(L_c/L_i)} = \hat{C}_i$ and $N\,\frac{L_c}{L_i} = \hat{D}_i$. If the infinitesimal micropolar model is correct this implies that $N^2 = \hat{C}_i \cdot \hat{D}_i$ independent[54] of $i$. A striking consequence of this development is that the assumed size-independent material parameter $N$ cannot be determined without prior knowledge of the characteristic length $L_c$ in contrast to the other elastic constants and vice-versa: the characteristic internal length $L_c$ can only be determined once $N$ is known. This is a problematic feature shared by all micropolar models with $\mu_c > 0$.

As for the homogeneous solution for consistent coupling: The tangential stresses are given by

$$\tau^{\rm hom}_{\rm small} == (\mu+\mu_c)\cdot u'(1) - 2\mu_c\,\alpha(1) = (\mu+\mu_c)\cdot u'(0) - 2\mu_c\alpha(0) = \mu\,\gamma\,. \tag{9.130}$$

Now we consider the **infinitesimal indeterminate couple stress** response in simple shear. The variational problem is easily obtained from (9.94) by identifying $\alpha(x) = \frac{u'(x)}{2}$ and taking free variations w.r.t. $u$ only. This results in the problem

$$\int_\Omega \mu\,\|\,{\rm sym}\,F - 1\!\!1\|^2 + \frac{\mu}{2}\,L_c^2|u''(x)|^2\,{\rm dV} \mapsto \min. \quad \text{w.r.t. } u, \quad u(0)=0,\ u(1)=\gamma\,, \tag{9.131}$$

$$\sigma = \sigma^{\rm loc} + \sigma^{\rm hyper}, \quad \tau^{\rm indet}_{\rm small} = \mu\,u'(x) - \mu\,L_c^2\,u'''(x)\,,$$

and the Euler-Lagrange equation of **fourth order** is given by

$$\begin{aligned} -u''(x) + L_c^2\,u^{(4)}(x) = 0, &\qquad u(0)=0,\ u(1)=\gamma \\ u'(0) = u'(1) = 0 &\quad \text{classical rigid condition} \\ u''(0) = u''(1) = 0 &\quad \text{natural boundary condition}\,. \end{aligned} \tag{9.132}$$

This equation coincides with (9.121) if we identify again $u' = \frac{\alpha}{2}$ and take $N \equiv 1$.

The general solution of (9.132) is given by

$$u(x) = b_1\,\sinh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right) + b_2\,\cosh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right) + b_3\,(x-\frac{1}{2}) + b_4\,. \tag{9.133}$$

Natural boundary conditions imply effectively $u(x) = \gamma\,x$ as unique homogeneous solution with shear stress response $\tau^{\rm indet,hom}_{\rm small} = \mu\,\gamma$. For rigid boundary conditions the constants are

$$b_1 = -\frac{\gamma\,L_c}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}, \quad b_2 = 0, \quad b_3 = -\frac{\gamma\,\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}, \quad b_4 = \frac{\gamma}{2} \tag{9.134}$$

and the unique solution of the rigid indeterminate couple stress problem is

$$\begin{aligned} u(x) &= \gamma\left(\frac{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\,x - L_c\frac{\sinh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right) - \sinh\left(\frac{1}{L_c}[-\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\right) \\ u'(x) &= \left(\frac{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - \cosh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\right)\cdot\gamma\,. \end{aligned} \tag{9.135}$$

The term $\frac{u'(x)}{2}$ does coincide with $\alpha(x)$ in (9.126) for $N \equiv 1$. The limit $L_c \to 0$ (ever larger samples) is possible, converging pointwise to the homogeneous solution, but the convergence is not uniform due to the appearance of a strong **boundary layer** caused by the incompatible rigid boundary prescription. For large $L_c$ the solution converges to a smooth S-type shear profile.

The shear stress response is given by

$$\tau^{\rm indet}_{\rm small} = \mu\,u'(1) - \mu\,L_c^2u'''(1) = -\mu L_c^2\left(\frac{b_1}{L_c^3}\cosh\left(\frac{1}{L_c}[1-\frac{1}{2}]\right) + \frac{b_2}{L_c^3}\cosh\left(\frac{1}{L_c}[1-\frac{1}{2}]\right)\right) \tag{9.136}$$

$$= -\mu\frac{b_1}{L_c}\cosh\left(\frac{1}{L_c}[1-\frac{1}{2}]\right) = \mu\,\frac{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\,\gamma = \mu\left(\frac{1}{1 - 2L_c\tanh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\right)\cdot\gamma\,,$$

coinciding with the **stiffer** shear stress response of the infinitesimal micropolar model for $N = 1$. Passage to the limit $L_c \to \infty$ (ever smaller samples) is not possible.[55] This implies as a rule for additional boundary conditions: boundary conditions should be such that in principal homogeneous solutions remain possible. The boundary conditions in a three-dimensional problem should not be the cause for nonhomogeneous response! This principle does not apply to plates and shells where boundary conditions appear naturally by a dimensional reduction process and carry physical information.

[54] It appears to us that $N^2 = \hat{C}_i \cdot \hat{D}_i$ independent of $i$ for different sizes is highly questionable.

[55] Since $\tanh x = x - \frac{x^3}{3} + \ldots$, for $L_c \to \infty$, then $\tau^{\rm indet}_{\rm small} \to \infty$, a severe shortcoming of the indeterminate couple stress model. This underlines the objections of Koiter against this model.

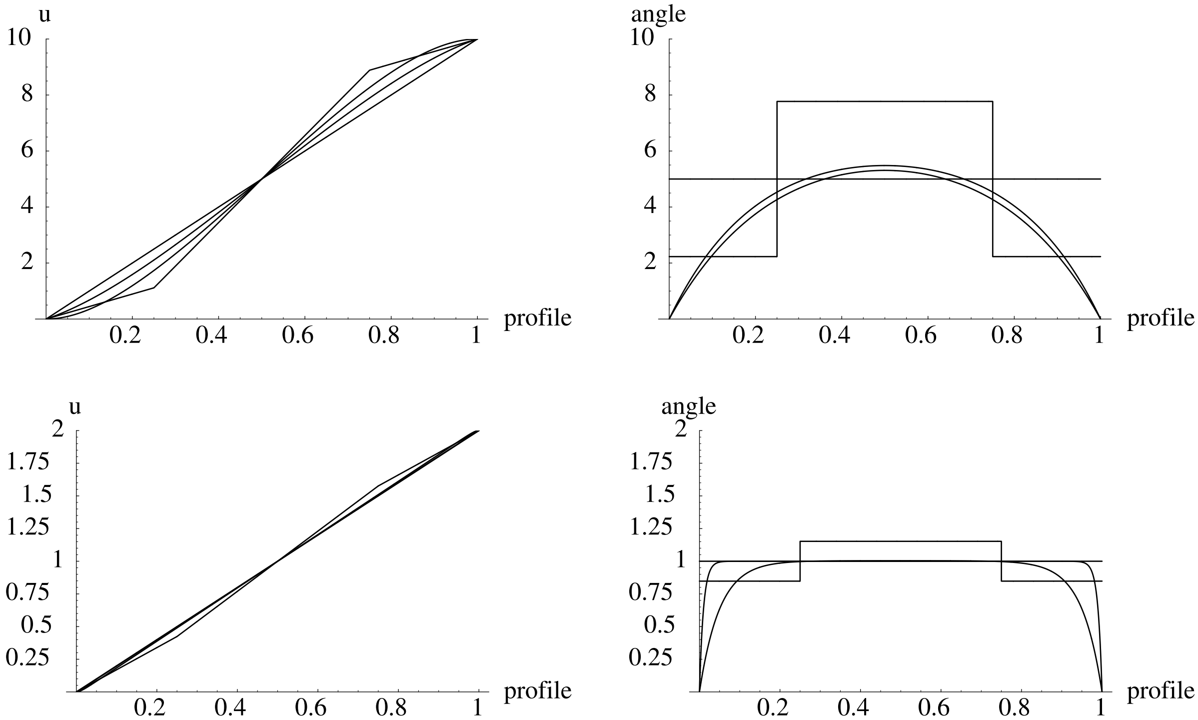


Figure 3: Computed shear profile $u$ and corresponding angle $\alpha$ for large (above) and small (below) values of maximal shear $\gamma$ in simple glide. Different curves: finite reduced problem with free rotations, infinitesimal micropolar and indeterminate couple stress model. $u(x) = \gamma\, x$, $\alpha(x) = \gamma/2$ the homogeneous classical response. The traditional infinitesimal Cosserat models with rigid boundary conditions and $\mu_c > 0$ show a strong **boundary layer**- not shared by the new relaxed model with $\mu_c = 0$ and consistent coupling. The new model allows for **sharp interfaces** for vanishing internal length $L_c$, indicating the onset of fracture.

## 9.5 Analytical investigations of classical incompressible elastic Biot-material in simple shear

Now we consider the same situation of simple glide in a classical elasticity context. The energy of the classical elastic Biot material is assumed to be of the form

$$\int_\Omega \mu \, \| \frac{U}{\det[U]^{(1/3)}} - 1\!\!1 \|^2 + \beta \left( \det[U] + \frac{1}{\det[U]} - 2 \right)^2 \mathrm{dx}\,, \tag{9.137}$$

which reduces to

$$\int_\Omega \mu \, \|U - 1\!\!1\|^2 \,\mathrm{dx} = \int_\Omega \mu \, \|R^T F - 1\!\!1\|^2 \,\mathrm{dx} = \int_\Omega \mu \, \|F - R\|^2 \,\mathrm{dx} = \int_\Omega \mu \left( \|F\|^2 - 2\langle F, R\rangle + 3 \right) \,\mathrm{dx} \tag{9.138}$$

for incompressible behaviour and $R \in \mathrm{SO}(3, \mathbb{R})$ represents the **continuum rotation**. For the assumed kinematics in simple glide, the continuum rotation has the form

$$R(x_1, x_2, x_3) = \begin{pmatrix} \cos\alpha(x_3) & 0 & \sin\alpha(x_3) \\ 0 & 1 & 0 \\ -\sin\alpha(x_3) & 0 & \cos\alpha(x_3) \end{pmatrix}, \tag{9.139}$$

with continuum rotation angle $\alpha \in [0, 2\pi)$. A simple calculation shows that

$$\sin\alpha(x) = \frac{u'(x)}{\sqrt{4 + (u'(x))^2}}, \qquad R(x) = \frac{1}{\sqrt{4 + (u'(x))^2}} \begin{pmatrix} 2 & 0 & u'(x) \\ 0 & \sqrt{4 + u'(x)^2} & 0 \\ -u'(x) & 0 & 2 \end{pmatrix}, \tag{9.140}$$

and the (surprisingly?) **convex** total energy is given by

$$\int_0^1 \mu \left( \|F\|^2 - 2\langle F, R(u')\rangle + 3 \right) \,\mathrm{dx} == \int_0^1 \mu \, \left( \sqrt{4 + u'(x)^2} - 1 \right)^2 - \mu \,\mathrm{dx}\,. \tag{9.141}$$

The Euler-Lagrange equation is given by

$$\forall \quad \phi \in C_0([0,1],\mathbb{R}): \qquad \int_0^1 \frac{\left(\sqrt{4+u'(x)^2}-1\right)}{\sqrt{4+u'(x)^2}}\, u'(x)\,\phi'(x)\,\mathrm{dx} = 0, \qquad \textbf{weak form} \tag{9.142}$$

$$u''(x)\cdot\left(1-\frac{1}{\sqrt{4+u'(x)^2}}+\frac{u'(x)^2}{\left(4+u'(x)^2\right)^{(3/2)}}\right) = 0, \qquad \textbf{differentiated form}$$

$$u(0) = 0,\; u(1) = \gamma,$$

showing that the homogeneous deformation is always a solution and hence, by strict convexity, the **unique solution** with **non-linear, strictly-monotone** shear stress response

$$\tau_{\text{finite}} = 2\,\mu\,\frac{\left(\sqrt{4+\gamma^2}-1\right)}{\sqrt{4+\gamma^2}}\cdot\gamma \quad (=\mu\,\gamma + o(\gamma))\;, \tag{9.143}$$

which coincides in fact with the shear response of the finite Cosserat model with consistent coupling evaluated for this homogeneous response. [56]

In contrast, from a three-dimensional viewpoint, the shear energy $\mu\,\|U-1\!\!1\|^2$ is **not quasiconvex** and **not Legendre-Hadamard elliptic** but satisfies the **Baker-Ericksen inequalities**. We consider therefore the behaviour of its **quasiconvexification** for the same assumed kinematics. It can be given explicitly. If $\lambda_i$ are the eigenvalues of $U$ we have

$$QW(F) = \mu \sum_{\lambda_i \geq 1} |\lambda_i - 1|^2\,. \tag{9.144}$$

In view of the underlying kinematics, the $\lambda_i$ can be calculated explicitly and we obtain

$$\lambda_1 = 1, \quad \lambda_2 = \frac{4+(u')^2-(u')\sqrt{4+(u')^2}}{2\sqrt{4+(u')^2}}, \quad \lambda_3 = \frac{4+(u')^2+(u')\sqrt{4+(u')^2}}{2\sqrt{4+(u')^2}}\,, \tag{9.145}$$

such that

$$QW(F) = \mu\sum_{\lambda_i\geq 1}|\lambda_i-1|^2 = \mu\,|\lambda_3-1|^2 = \mu\left(\frac{4+(u')^2+(u')\sqrt{4+(u')^2}}{2\sqrt{4+(u')^2}}-1\right)^2 = \mu\left(\frac{1}{2}\sqrt{4+(u')^2}+\frac{\gamma}{2}-1\right)^2. \tag{9.146}$$

The resulting formulation is again strictly convex and the homogeneous response is the unique minimizer of the corresponding minimization problem. However, the **shear response** is **considerably weaker** than the unmodified one for the same homogeneous solution, since

$$\tau^{\text{quasi}}_{\text{finite}} = D_{u'}\,[QW(F)]] = \mu\left(\frac{1}{2}\sqrt{4+\gamma^2}+\frac{\gamma}{2}-1\right)\cdot\left(\frac{\gamma}{\sqrt{4+\gamma^2}}+1\right)\,, \tag{9.147}$$

We note that in this specific example, quasiconvexification of the three-dimensional problem coincides with the rank-one convexifcation. However, quasiconvexification changes the stress/strain law already in situations, where convexity holds true for the assumed kinematics, i.e. where there is no imminent need for any change due to instabilities. This underlines the care, which has to be exerted when using the quasiconvex hull.

Let us summarize the obtained stress/strain behaviour in uniaxial shear for small amounts of shear. For consistent coupling we have:

$$\mu\,\gamma = \tau_{\text{lin}} = \tau_{\text{small}} = \tau^{\text{indet}}_{\text{small}} > \tau^{\text{quasi}}_{\text{class}}\,, \qquad \mu\,\gamma = \tau_{\text{lin}} < \tau^{\text{micro}}_{\text{red}} < \tau^{\text{hom}}_{\text{finite}} < \tau^{\text{hom}}_{\text{red}}\,, \tag{9.148}$$

where $\tau_{\text{lin}} < \tau^{\text{micro}}_{\text{red}} < \tau^{\text{hom}}_{\text{finite}} < \tau^{\text{hom}}_{\text{red}}$ have the same tangent in 0, but $\tau^{\text{quasi}}_{\text{class}}$ is weaker. For rigid Dirichlet data we obtain

$$\mu\,\gamma = \tau_{\text{lin}} < \tau_{\text{small}} < \tau^{\text{indet}}_{\text{small}}\,, \tag{9.149}$$

with artificially stiffer behaviour for arbitrary small shear.

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[56] To see only after some algebra.

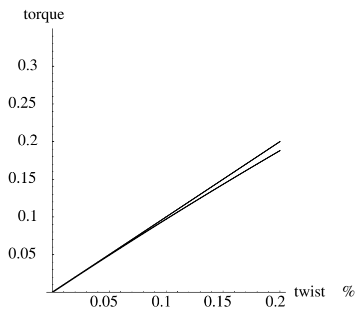


Figure 4: Parabolic behaviour for infinitesimal load.

# 10 Appendix B

## 10.1 Koiter's remarks on couple stresses

It seems appropriate to cite also Koiter [Koi64]: "The predictions of the (classical elasticity) theory are usually in satisfactory agreement with careful experiments, if the stresses remain within the elastic limit of the material. Unfortunately the theory of elasticity apparently fails, however, to give an adequate description of the behaviour in fatigue of machine parts or other structural elements involving high stress concentrations. This failure can hardly be ascribed completely to inelastic behaviour of the material, because the endurance limit in fatigue is usually well below the macroscopic elastic limit of the material. A more likely explanation is that the classical (finite elasticity) theory is not adequate in the presence of large stress gradients. The latter explanation is entirely plausible in view of the discontinuous polycrystalline structure of actual engineering materials. It is also supported by evidence that the discrepancy between the theoretical predictions and fatigue test results is more marked for materials with a coarse grain structure. It would seem therefore that the idealized model of an elastic continuum is not quite appropriate for the analysis of stress and strain in an actual discontinuous polycrystalline material involving large stress gradients. It need hardly be argued, however, that a detailed analysis of the transmission of loads between the individual grains in a polycrystalline material would pose a formidable problem. Some idealisation, preferably in the form of a continuous model, is highly desirable in order to make the problem amenable to analysis. At first sight it might seem that this return to a continuous model would also imply a complete return to the classical theory of elasticity. It should be remembered, however, that we have already alluded to additional assumptions made in the classical theory, apart from the model of a continuum. The assumption in question is that the transmission of loads between the material on both sides of an infinitesimal surface element dS is described completely by a force vector $p\,\mathrm{dS}$ acting in the center of gravity of the surface element. We emphasize that this is an assumption which can neither be proved directly, nor disproved. It can only be tested by a confrontation of its predictions for measurable quantities with experiments. For most purposes it has indeed proved to be an appropriate assumption, resulting in satisfactory agreement between theoretical predictions and experimental evidence. The lack of agreement between theory and experiment on the effect of stress gradients, however, makes this assumption questionable at least in cases of large stress gradients. A quite natural generalisation of the classical theory of the elastic continuum is thus obtained, if we drop the additional assumption. (...) It turned out, however, that the magnitude of the effect of couple-stresses, required to explain quantitatively the effect of stress gradients in fatigue tests, was such that it could not easily have escaped attention in other careful experiments." and he continues [Koi64, p.41]: "We venture to conjecture that the stress gradient effect in fatigue cannot be described satisfactorily by allowing the presence of couple-stresses in an isotropic elastic medium." Brackets my addition.

It must be noted that Koiter came to reject the significant presence of couple stresses because he based his investigations on the so called indeterminate couple stress theory (2.21), which tends to maximize the influence of length scale effects. His arguments only show that this special constrained gradient theory cannot be based on experimental evidence. However, the main thrust of his comments remains valid. I have not been aware of Koiters contribution during the preparation of my main arguments, but it squares with my development

## 10.2 Some experimental results: Nonlinear relation near zero stress and size dependence

Usually, in engineering application, several simplifying assumptions are tacitly introduced. The first one is that the elastic behaviour of the structure can be neglected altogether leading to the rigid plasticity models and the second one, that the stress-strain relation for very small stress levels is indeed linear (generalized Hookes law). While the first simplification is evidently not appropriate the last simplification is generally assumed to be valid. However, it cannot be based on evidence. To the contrary, taking the vast amount of precise experimental data for metallic material compiled in [Bel73] seriously, we must conclude that there is no level of stress, such that the stress-strain relation is linear. Instead, practically all materials under infinitesimal loads (in torsion and tension) obey a parabolic relation of the form $\|\sigma\| = E\,\|\varepsilon\| - b\,\|\varepsilon\|^2$ with $E$ the initial zero stress modulus in tension and $b$ a positive constant [Bel73, p.127] and see the footnote in [Nad63, p.10]. There has never been made a serious attempt to understand this parabolic behaviour. Our Cosserat model offers an interpretation: since the initial tangent modulus is bounded, the case $\mu_c = \infty$ can be disposed of for all values of $L_c$. Because the initial tangent moduli determined from tension experiments are consistent with predictions based on the classical linear theory for torsion experiments, we must set $\mu_c = 0$. The point is that $\mu_c > 0$ would immediately change the elastic behaviour for very small stress levels in torsion. In this sense, $\mu_c$ is small strain determined. Whether the couple modulus is really zero, cannot be decided by experiment alone, given the unavoidable scatter in the experiment. In any case it would have to be very small.

The second order effect of rotations will intervene if we keep the geometrically exact structure of the model and lead to reduced tangent moduli in tension and torsion, with a more pronounced reduction in torsion, cf. (2.6). A difference of response in tension and torsion for small stress levels has been observed as early as 1857. Wertheim [Wer57] and Bauschinger [Bau81] observed (roughly) **linearity** in **tension** and **nonlinearity** in **torsion**. Moreover, to quote from [Bel73, p.89]: "For all the cylinders, Wertheim noticed that in addition to the nonlinearity of the relation between torque and angle, the amount of the departure from the linear approximation depended upon the length of the specimen. Only when very long specimens were compared could he achieve the independence of length assumed in the elementary theory."

In Wertheims own words [Wer57, p.281] he summarizes: "2. Les angles de torsion temporaires (elastic twist) ne sont pas rigoureusement proportionelles aux moments des couples; ils augment plus rapide que ceux-ci, et cet accroissement de l´angle moyen (empiric modulus in torsion) ... s´etend jusque´a la rupture ... 3. Ces angles temporaires ne sont pas rigoureusement proportionels aux longueurs; ramene a l´unite de longeur, suivant cette loi de proportionalite, on les trouve d´autant plus grand, tout egal du reste, que la piece soumise a l´experience a ete plus courte.

... cette proportionalite (classical infinitesimal linear elasticity) ne saurait etre considere que comme la limite vers laquelle tendent les angles a mesure que diminuent les intensites des couples et les dimension du corps qui sont perpendiculaire a l´axe de torsion."

## 10.3 Alternative representation of the micropolar stretch energy

For a **small elastic strain** theory, which should already cover many cases of physical interest we required that $W_{\mathrm{mp}}(\overline{U})$ is a non negative isotropic quadratic form with

$$W_{\mathrm{mp}}(\mathbb{1}) = 0, \quad D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})_{|_{\overline{U}=\mathbb{1}}} = 0\,. \tag{10.150}$$

The most general form of $W_{\mathrm{mp}}$ consistent with (2.5) is

$$\begin{aligned} W_{\mathrm{mp}}(\overline{U}) &= \alpha_1\, \|\mathrm{sym}(\overline{U}-\mathbb{1})\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U}-\mathbb{1})\|^2 + \alpha_3\, \mathrm{tr}\left[\mathrm{sym}(\overline{U}-\mathbb{1})\right]^2 \\ &= \frac{\alpha_1+\mu_c}{2}\, \|\overline{U}-\mathbb{1}\|^2 + \frac{\alpha_1-\mu_c}{2}\, \langle \overline{U}-\mathbb{1}, (\overline{U}-\mathbb{1})^T\rangle + \alpha_3\, \mathrm{tr}\left[\mathrm{sym}(\overline{U}-\mathbb{1})\right]^2, \\ &= -(2\alpha_1+6\alpha_3)\,\mathrm{tr}\left[\overline{U}\right] + \alpha_3\,\mathrm{tr}\left[\overline{U}\right]^2 + \frac{(\alpha_1-\mu_c)}{2}\,\mathrm{tr}\left[\overline{U}^2\right] + \frac{(\alpha_1+\mu_c)}{2}\,\mathrm{tr}\left[\overline{U}^T\overline{U}\right] + \mathrm{const.}, \end{aligned} \tag{10.151}$$

where $(2.6)_{2,3}$ is the form used in [Gra03, San99], respectively and we note the induced relations

$$\begin{aligned} D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) &= (\alpha_1+\mu_c)\,\overline{U} + (\alpha_1-\mu_c)\,\overline{U}^T - 2\,\alpha_1\,\mathbb{1} + 2\,\alpha_3\,\mathrm{tr}\left[\overline{U}-\mathbb{1}\right]\mathbb{1} \\ D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\cdot\overline{U}^T &= (\alpha_1+\mu_c)\,\overline{U}\,\overline{U}^T + (\alpha_1-\mu_c)\,\overline{U}^T\overline{U}^T - 2\,\alpha_1\,\overline{U}^T + 2\,\alpha_3\,\mathrm{tr}\left[\overline{U}-\mathbb{1}\right]\overline{U}^T \\ \mathrm{skew}\left(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\cdot\overline{U}^T\right) &= \mathrm{skew}\left((\alpha_1-\mu_c)\,\overline{U}^T\overline{U}^T - 2\,\alpha_1\,\overline{U}^T + 2\,\alpha_3\,\mathrm{tr}\left[\overline{U}-\mathbb{1}\right]\overline{U}^T\right)\,, \end{aligned} \tag{10.152}$$

with material constants $\alpha_i, i = 1,2,3$ such that $\alpha_1,\ 3\alpha_3-\alpha_1, \mu_c \geq 0$ from non negativity [Eri99].

## 10.4 Derivation of the geometrically exact micropolar balance equations

The derivation of $(2.12)_1$ is standard. For $(2.12)_2$ we consider simultaneously in each space point a one parameter group of rotations $\frac{\mathrm{d}}{\mathrm{dt}}\hat{R}(x,t) = A(x,t)\cdot\hat{R}(x,t),\ \hat{R}(x,0) = \overline{R}(x),\ \ A \in C_0^\infty(\Omega, \mathfrak{so}(3,\mathbb{R}))$. The stationarity condition is obtained from $\frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}} I(\varphi, \hat{R}(x,t)) = 0$. This yields three terms: the derivatives involving $W_{\mathrm{mp}}$ and $\langle M, \overline{R}\rangle$ are straightforward, using the definition of the one parameter group, and yield

$$\langle -\overline{R}\,\mathrm{skew}(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T)\,\overline{R}^T, A\rangle, \quad \langle -\overline{R}\,\mathrm{skew}(\overline{R}^T M)\,\overline{R}^T, A\rangle\,, \tag{10.153}$$

respectively, where $\langle\cdot,\cdot\rangle$ means additionally integration w.r.t. $x$. For the term containing the curvature part, we note

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}_{|_{t=0}} \int_\Omega W_{\mathrm{curv}}(\mathfrak{K}(x,t))\,\mathrm{dV} &= \sum_{i=1}^3 \langle \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3), \overline{R}^T\nabla(A\overline{R}.e_i) + (A\overline{R})^T\nabla(\overline{R}.e_i)\rangle_{\mathbb{M}^{3\times3}} \\ &= \sum_{i=1}^3 \langle \overline{R}\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3), \nabla(A\overline{R}.e_i)\rangle_{\mathbb{M}^{3\times3}} + \langle \overline{R}\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\overline{R}^T, A^T\rangle_{\mathbb{M}^{3\times3}} \\ &= \sum_{i=1}^3 -\langle \mathrm{Div}\left[\overline{R}\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\right], A\overline{R}.e_i\rangle_{\mathbb{R}^3} + \langle \overline{R}\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\overline{R}^T, A^T\rangle \\ &= -\langle \mathrm{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right], A\overline{R}\rangle_{\mathbb{M}^{3\times3}} + \langle \overline{R}\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\,\overline{R}^T\right), A^T\rangle \\ &= -\langle \mathrm{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{R}^T, A\rangle + \langle \overline{R}\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\overline{R}^T, A^T\rangle \\ &= -\langle \overline{R}\,\overline{R}^T\,\mathrm{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{R}^T, A\rangle - \langle \overline{R}\,\mathrm{skew}\left(\sum_{i=1}^3 \left(\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\right)\overline{R}^T, A\rangle \\ &= -\langle \overline{R}\,\mathrm{skew}\left(\overline{R}^T\,\mathrm{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\right)\overline{R}^T, A\rangle - \langle \overline{R}\,\mathrm{skew}\left(D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\mathfrak{K}^T\right)\overline{R}^T, A\rangle\,. \end{aligned} \tag{10.154}$$

Since $A \in C_0^\infty(\Omega, \mathfrak{so}(3,\mathbb{R}))$ is arbitrary, equation $(2.12)_2$ follows.

## 10.5 Scaling relations for Cosserat models

For completeness let us summarize the scaling relations appearing in a finite elastic Cosserat theory. Our goal is to relate the response of large and small samples of the same material and to asses the influence of the characteristic length $L_c$.

First, in our definition, the **characteristic length** $L_c$ is a given material parameter, corresponding to the **smallest discernable distance** to be accounted for in the model. A simple consequence is that geometrical dimensions $L$ of the bulk material must be larger than $L_c$, indeed for a continuum theory to apply $L$ should be significantly larger than $L_c$.

Now let $\Omega_L = [0, L] \times [0, L] \times [0, L]$ be the cube with edge length $L$, representing the bulk material. Consider a deformation $\varphi_L : \xi \in \Omega_L \mapsto \mathbb{R}^3$ and microrotation $\overline{R}_L(\xi) : \Omega_L \mapsto \mathrm{SO}(3, \mathbb{R})$ as solution of the simplified minimization problem

$$\int_{\xi\in\Omega_L} \mu \,\|\overline{R}_L^T(\xi)F(\xi) - 1\!\!1\|^2 + \mu\, L_c^q \,\|\mathrm{D}_\xi \overline{R}_L(\xi)\|^q \,\mathrm{d}\xi \mapsto \ \min . \text{ w.r.t. } (\varphi_L, \overline{R}_L) \tag{10.155}$$

The simple scaling transformation $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\zeta(x) = L \cdot x$ maps the unit cube $\Omega_1 = [0, 1] \times [0, 1] \times [0, 1]$ into $\Omega_L$. Defining the related deformation $\varphi : x \in \Omega_1 \mapsto \mathbb{R}^3$ and microrotation $\overline{R}(x) : \Omega_1 \mapsto \mathrm{SO}(3, \mathbb{R})$ as

$$\varphi(x) := \zeta^{-1}\left(\varphi_L(\zeta(x))\right), \quad \overline{R}(x) := \overline{R}_L(\zeta(x)), \tag{10.156}$$

shows

$$\begin{aligned} \mathrm{D_x} \overline{R}(x) &= \mathrm{D}_\xi \overline{R}_L(\zeta(x)) \cdot \nabla_x \zeta(x) = \mathrm{D}_\xi \overline{R}_L(\xi) \cdot L \\ \nabla_x \varphi(x) &= \frac{1}{L} \nabla_\xi \varphi_L(\zeta(x)) \nabla_x \zeta(x) = \nabla_\xi \varphi_L(\xi). \end{aligned} \tag{10.157}$$

Hence, the minimization problem can be transformed

$$\begin{aligned} &\int_{\xi\in\Omega_L} \mu \,\|\overline{R}_L^T(\xi)\nabla_\xi \varphi(\xi) - 1\!\!1\|^2 + \mu\, L_c^q \,\|\mathrm{D}_\xi \overline{R}_L(\xi)\|^q \,\mathrm{d}\xi = \int_{x\in\Omega_1} \mu \,\|\overline{R}^T(x)\nabla_x \varphi(x) - 1\!\!1\|^2 \det[\nabla_x \zeta(x)] + \mu\, L_c^q \,\|\frac{1}{L}\mathrm{D_x} \overline{R}(x)\|^q \det[\nabla_x \zeta(x)] \,\mathrm{dx} \\ &= \int_{x\in\Omega_1} \mu \,\|\overline{R}^T(x)\nabla_x \varphi(x) - 1\!\!1\|^2 \, L^3 + \mu\, L_c^q \, L^{3-q} \,\|\mathrm{D_x} \overline{R}(x)\|^q \,\mathrm{dx}, \end{aligned} \tag{10.158}$$

and we may consider the problem defined on $\Omega_1$:

$$\int_{x\in\Omega_1} \mu \,\|\overline{R}^T(x)\nabla_x \varphi(x) - 1\!\!1\|^2 + \mu\, L_c^q \, L^{3-q-3} \,\|\mathrm{D_x} \overline{R}(x)\|^q \,\mathrm{dx} \mapsto \ \min . \text{ w.r.t. } (\varphi, \overline{R}) \tag{10.159}$$

Comparison of different sample sizes is afforded by transformation to the unit cube repectively, e.g. we compare two samples of the same material with sizes $L_1 > L_2$. Transformation to the unit cube shows that the response of sample two is stiffer than the response of sample one.

It is plain to see that for $L$ large compared to $L_c$, the influence of the rotations will be small and in the limit $\frac{L_c}{L} \to 0$ classical behaviour results. Otherwise, the larger $\frac{L_c}{L} < 1$, the more pronounced the Cosserat effects become and a small sample is relatively stiffer than a large one.

## 10.6 Finite elasto-plastic Cosserat theory for small elastic rotations

Since we are at present not in a position to mathematically treat the geometrically exact elasto-plastic Cosserat model (3.31) it seems expedient to introduce a first partial reduction of the model which will allow an adequate analysis in the near future. We assume only that elastic rotations $\overline{R}_e$ remain small. Hence by expansion $\overline{R}_e = 1\!\!1 + \overline{A}_e + \ldots$, $\overline{A}_e \in \mathfrak{so}(3)$ and $\mathrm{D_x}\overline{R}_e \approx \mathrm{D_x}\overline{A}_e + \ldots$. Furthermore we take $\mu_c = 0$, $\alpha_4 = 0$, $p = 1$ and dispose of external volume and surface couples.

This simplification results in the following nonlinear system of coupled partial differential and evolution equations for the finite deformation $\varphi : [0, T] \times \overline{\Omega} \mapsto \mathbb{R}^3$, the plastic deformation $F_p : [0, T] \times \overline{\Omega} \mapsto \mathrm{GL}^+(3, \mathbb{R})$ and the independent Cosserat elastic "microrotation" $\overline{R}_e : [0, T] \times \overline{\Omega} \mapsto \mathrm{GL}^+(3)$

$$\begin{aligned} &\int_\Omega W_{\mathrm{mp}}(\overline{U}_e) \det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e) \det[F_p] - \langle f, \varphi\rangle \det[F_p] \,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle \,\|\operatorname{Cof} F_p.\vec{n}\| \,\mathrm{dS} \\ &\qquad \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}_e) \text{ at constant } F_p, \\ &\overline{R}_{e_{|\Gamma}} = 1\!\!1 + \begin{cases} \operatorname{skew}(\nabla g_{\mathrm{d}} \cdot F_p^{-1})_{|\Gamma}, \text{ rigid} \\ \operatorname{skew}(\nabla \varphi \cdot F_p^{-1})_{|\Gamma}, \text{ consistent} \end{cases} , \quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t) \\ &W_{\mathrm{mp}}(\overline{U}_e) = \mu \,\|\operatorname{sym}(\overline{U}_e - 1\!\!1)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\overline{U}_e - 1\!\!1)\right]^2, \quad W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\, \frac{L_c^2}{12}\, \|\mathfrak{K}_e\|^2 \\ &F_e = \nabla\varphi \cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \overline{R}_e = 1\!\!1 + \overline{A}_e \quad \mathfrak{K}_e = [\mathrm{D_x}\overline{A}_e(x) \cdot F_p^{-1}] \\ &\frac{\mathrm{d}}{\mathrm{dt}} \left[F_p^{-1}\right] \in -F_p^{-1} \cdot f(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}}. \end{aligned} \tag{10.160}$$

No assumptions on the magnitude of deformations or plastic deformation are introduced. We venture to say that local existence can be established along the lines of [Nef03b].

## 10.7 Partially linearized finite elasto-plastic Cosserat theory

A further reduction is achieved if we assume additionally that plastic deformations $F_p$ remain small.

To this end we proceed similar to (4.47) and write $\overline{R}_e = 1\!\!1 + \overline{A}_e + \dots$, $\overline{A}_e \in \mathfrak{so}(3)$, $\|\overline{A}_e\|^2 \ll 1$, $F_p = 1\!\!1 + p + \dots$, then the elastic micropolar stretch tensor $\overline{U}_e = \overline{R}_e^T F_e = \overline{R}_e^T F F_p^{-1}$ may be expanded as follows

$$\overline{U}_e = \overline{R}_e^T F F_p^{-1} = (1\!\!1 + \overline{A}_e + \dots)^T (1\!\!1 + \nabla u)(1\!\!1 - p + \dots) \approx 1\!\!1 + \nabla u - p - \overline{A}_e + \dots , \tag{10.161}$$

such that to leading order (in (4.47) by contrast we keep one order more)

$$W_{\rm mp}(\overline{U}_e) \approx \mu \,\| \operatorname{sym}(\nabla u - p)\|^2 + \mu_c \| \operatorname{skew}(\nabla u - p - \overline{A}_e)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\nabla u - p)\right]^2, \tag{10.162}$$

and the elastic micropolar curvature is expanded as

$$\mathfrak{K}_e = \overline{R}_e^T \left[ {\rm D_x} \overline{R}_e \cdot F_p^{-1} \right] = (1\!\!1 + \overline{A}_e + \dots)^T \left[ {\rm D_x}(1\!\!1 + \overline{A}_e + \dots)(1\!\!1 - p + \dots) \right] \approx {\rm D_x} \overline{A}_e (1\!\!1 - p) + \dots . \tag{10.163}$$

Introducing these reductions consistently into (3.31) we obtain the following (seemingly more complicated) system of coupled partial differential and evolution equations for the finite macroscopic displacement $u : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$, the infinitesimal plastic deformation $p : [0,T] \times \overline{\Omega} \mapsto \mathbb{M}^{3\times 3}$ and the independent infinitesimal Cosserat elastic microrotation $\overline{A}_e : [0,T] \times \overline{\Omega} \mapsto \mathfrak{so}(3)$

$$\begin{aligned}
&\int_\Omega W_{\rm mp}(\nabla u, p, \overline{A}_e) \left[1 + \operatorname{tr}[\varepsilon_p]\right] + W_{\rm curv}({\rm D_x}\overline{A}_e, p)\left[1 + \operatorname{tr}[\varepsilon_p]\right] - \langle f, u\rangle \left[1 + \operatorname{tr}[\varepsilon_p]\right] {\rm dV} \\
&\quad - \int_\Omega \langle M, \overline{A}_e\rangle \left[1 + \operatorname{tr}[\varepsilon_p]\right] {\rm dV} - \int_{\Gamma_S} \langle N, u\rangle \left[1 + \operatorname{tr}[\varepsilon_p]\right] \sqrt{1 - 2\langle \varepsilon_p.\vec{n}, \vec{n}\rangle}\, {\rm dS} \\
&\quad - \int_{\Gamma_C} \langle M_c, \overline{A}_e\rangle \left[1 + \operatorname{tr}[\varepsilon_p]\right] \sqrt{1 - 2\langle \varepsilon_p.\vec{n}, \vec{n}\rangle}\, {\rm dS} \mapsto \min . \quad \text{w.r.t. } (u, \overline{A}_e) \text{ at constant } p , \\
&W_{\rm mp}(\nabla u, p, \overline{A}_e) = \mu \| \operatorname{sym}(\nabla u - p)\|^2 + \mu_c \,\| \operatorname{skew}(\nabla u - p - \overline{A}_e)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\nabla u - p)\right]^2 \\
&W_{\rm curv}^{\rm small}({\rm D_x}\overline{A}_e, p) = \mu \frac{L_c^2}{12} \| {\rm D_x}\overline{A}_e (1\!\!1 - p)\|^2 \\
&\frac{\rm d}{\rm dt}\left[p(t)\right] \in \partial\chi(T_E), \quad T_E = -\partial_p \left[ \left[ W_{\rm mp}(\nabla u, p, \overline{A}_e) + W_{\rm curv}^{\rm small}({\rm D_x}\overline{A}_e, p) \right] \left[1 + \operatorname{tr}[\varepsilon_p]\right] \right] \\
&T_E = T_{E,\rm mp} + T_{E,\rm curv} , \quad \varepsilon_p = \tfrac{1}{2}(p^T + p) , \quad p(0) \in \mathbb{M}^{3\times 3} \\
&T_{E,\rm mp} = DW_{\rm mp}(\nabla u, p, \overline{A}_e) \left[1 + \operatorname{tr}[\varepsilon_p]\right] - W_{\rm mp}(\nabla u, p, \overline{A}_e)\, 1\!\!1 \\
&T_{E,\rm curv} = {\rm D_x}\overline{A}_e^T DW_{\rm curv}^{\rm small}({\rm D_x}\overline{A}_e, p) \left[1 + \operatorname{tr}[\varepsilon_p]\right] - W_{\rm curv}({\rm D_x}\overline{A}_e, p)\, 1\!\!1 \\
&u_{|_\Gamma}(t,x) = g_{\rm d}(t,x) - x, \quad x \in \Gamma , \quad \overline{A}_{e|_\Gamma}(t,x) = \operatorname{skew}(\nabla g_{\rm d}(t,x) - p(t,x))_{|_\Gamma} .
\end{aligned} \tag{10.164}$$

Here $T_E$ is the **reduced elastic Eshelby tensor**. Observe that the coupling in $T_{E,\rm curv}$ is of second order, otherwise the Cosserat contribution would not appear in the plastic flow part. This system is intrinsically thermodynamically admissible. Note that in the formal limit $\mu_c \to \infty$, the total infinitesimal continuum rotation splits additively into elastic infinitesimal microrotations $\overline{A}_e$ and **infinitesimal plastic spin** $\operatorname{skew}(p)$:

$$\operatorname{skew}(\nabla u) = \overline{A}_e + \operatorname{skew}(p) . \tag{10.165}$$

If we choose the elastic domain $\mathcal{E} := \{\Sigma_E \,|\, \|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| \le \sigma_{\rm y}\}$ [57], then the system further reduces to

$$\begin{aligned}
&\int_\Omega W_{\rm mp}(\nabla u, p, \overline{A}_e) + W_{\rm curv}({\rm D_x}\overline{A}_e, p) - \langle f, u\rangle - \langle M, \overline{A}_e\rangle \,{\rm dV} \\
&\quad - \int_{\Gamma_S} \langle N, u\rangle \,{\rm dS} - \int_{\Gamma_C} \langle M_c, \overline{A}_e\rangle \,{\rm dS} \mapsto \min . \quad \text{w.r.t. } (u, \overline{A}_e) \text{ at constant } \varepsilon_p , \\
&W_{\rm mp}(\nabla u, p, \overline{A}_e) = \mu \,\|\varepsilon - \varepsilon_p\|^2 + \mu_c \,\| \operatorname{skew}(\nabla u - \overline{A}_e)\|^2 + \frac{\lambda}{2} \operatorname{tr}[\varepsilon]^2 \\
&W_{\rm curv}^{\rm small}({\rm D_x}\overline{A}_e, \varepsilon_p) = \mu \frac{L_c^2}{12} \| {\rm D_x}\overline{A}_e (1\!\!1 - \varepsilon_p)\|^2 \\
&\varepsilon_e = \varepsilon - \varepsilon_p, \quad \varepsilon(\nabla u(x)) = \tfrac{1}{2}(\nabla u^T + \nabla u), \quad \varepsilon_p = \tfrac{1}{2}(p^T + p), \\
&\dot{\varepsilon_p}(t) \in \partial\chi(T_E), \quad T_E = -\partial_{\varepsilon_p} \left[ W_{\rm mp}(\nabla u, \varepsilon_p, \overline{A}_e) + W_{\rm curv}({\rm D_x}\overline{A}_e, \varepsilon_p) \right] \\
&T_E = T_{E,\rm mp} + T_{E,\rm curv}, \\
&T_{E,\rm mp} = -\partial_p W_{\rm mp}(\nabla u, p, \overline{A}_e) = 2\mu\,(\varepsilon - \varepsilon_p), \; T_{E,\rm curv} = {\rm D_x}\overline{A}_e^T DW_{\rm curv}^{\rm small}({\rm D_x}\overline{A}_e, p) \\
&u_{|_\Gamma}(t,x) = g_{\rm d}(t,x) - x, \quad x \in \Gamma , \quad \operatorname{tr}[\varepsilon_p(0)] = 0 , \quad \varepsilon_p(0) \in \operatorname{Sym}(3) \\
&\overline{A}_{e|_\Gamma}(t,x) = \operatorname{skew}(\nabla g_{\rm d}(t,x) - \varepsilon_p)_{|_\Gamma} = \operatorname{skew}(\nabla g_{\rm d}(t,x))_{|_\Gamma} .
\end{aligned} \tag{10.166}$$

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[57] no **infinitesimal plastic spin**: $p = \varepsilon_p$ and isochoric plasticity: $\operatorname{tr}[\varepsilon_p] = 0$ and assume $\langle \varepsilon_p.\vec{n}, \vec{n}\rangle = 0$ on $\Gamma_S \cup \Gamma_C$.

A simple calculation yields the relations $\|\mathrm{D_x}\overline{A}_e\|^2 = 2\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2$ and $\mathrm{D_x}\overline{A}_e^T\cdot\mathrm{D_x}\overline{A}_e(1\!\!1-\varepsilon_p)=\sum_{i=1}^3 \mathrm{D_x}\overline{A}_e^{i,T}\cdot\mathrm{D_x}\overline{A}_e^{i,T}(1\!\!1-\varepsilon_p) = 2\,[\nabla\,\mathrm{axl}(\overline{A}_e)]^T[\nabla\,\mathrm{axl}(\overline{A}_e)](1\!\!1-\varepsilon_p)$, where $\mathrm{axl}:\mathfrak{so}(3,\mathbb{R})\mapsto\mathbb{R}^3$ is the canonical identification of a skew-symmetric matrix with its axial vector. Therefore,

$$T_E = 2\mu\,(\varepsilon-\varepsilon_p) + (2\mu+\lambda)\,2\,\frac{L_c^2}{12}\,[\nabla\,\mathrm{axl}(\overline{A}_e)]^T[\nabla\,\mathrm{axl}(\overline{A}_e)](1\!\!1-\varepsilon_p)\,, \tag{10.167}$$

and we appreciate the role, the elastic Cosserat contribution takes in the elasto-plastic model: in first order it resembles a **softening** mechanism. A similar softening effect can be observed in the model of [Bes74, (11.5)], where, however, the kinematical description is different from ours. In spite of the softening effect, we expect that the system (10.166) admits a global in-time solution with slightly improved regularity in the rate-independent ideal plasticity case.

If we finally neglect all second order terms, we obtain the infinitesimal, geometrically linear model (3.36).

## 10.8 A remark on the elasto-plastic decomposition of the curvature tensor

There is some discussion in the literature as regards the suitable definition of an independent quantity of plastic curvature $\overline{\kappa}_p$. We did not advocate its use. Ehlers has already observed that $\overline{\kappa}_p$ cannot really be independent under a disguised consistency requirement. The argument runs as follows (only for the infinitesimal case for simplicity): the micropolar decomposition of the displacement gradient into microstrain $\overline{\varepsilon}$ and microrotation $\overline{A}$

$$\overline{\varepsilon} = \nabla u - \overline{A}\,, \tag{10.168}$$

shows that $\overline{\kappa} := \mathrm{Curl}\,\overline{A} = -\,\mathrm{Curl}\,\overline{\varepsilon}$. In infinitesimal micropolar elasto-plasticity the microstrain is additively decomposed into elastic and plastic parts: $\overline{\varepsilon}=\overline{\varepsilon}_e+\overline{\varepsilon}_p$. This yields

$$\overline{\varepsilon}_e+\overline{\varepsilon}_p=\overline{\varepsilon}=\nabla u-\overline{A},\Leftrightarrow\quad \mathrm{Curl}\,\overline{\varepsilon}_e+\mathrm{Curl}\,\overline{\varepsilon}_p = -\,\mathrm{Curl}\,\overline{A} = -\overline{\kappa}\,. \tag{10.169}$$

If we assume in addition that the total curvature splits as well additively into elastic and plastic parts, $\overline{\kappa}=\overline{\kappa}_e+\overline{\kappa}_p$, we are naturally led to assume by consistency with (10.169), that $\overline{\kappa}_e = -\,\mathrm{Curl}\,\overline{\varepsilon}_e$ and $\overline{\kappa}_p = -\,\mathrm{Curl}\,\overline{\varepsilon}_p$.

The corresponding thermodynamical dissipation potential

$$\int_\Omega \mu\,\|\mathrm{sym}\,\overline{\varepsilon}_e\|^2+\mu_c\,\|\mathrm{skew}(\overline{\varepsilon}_e)\|^2+\mu\,L_c^2\,\|\overline{\kappa}_e\|^2\,\mathrm{dV} = \int_\Omega \mu\,\|\mathrm{sym}\,\nabla u-\overline{\varepsilon}_p\|^2+\mu_c\,\|\mathrm{skew}(\nabla u-\overline{\varepsilon}_p-\overline{A})\|^2+\mu\,L_c^2\,\|\overline{\kappa}-\overline{\kappa}_p\|^2\,\mathrm{dV}$$

$$= \int_\Omega \mu\,\|\mathrm{sym}\,\nabla u-\overline{\varepsilon}_p\|^2+\mu_c\,\|\mathrm{skew}(\nabla u-\overline{\varepsilon}_p-\overline{A})\|^2+\mu\,L_c^2\,\|\,\mathrm{Curl}\,\overline{A}-\mathrm{Curl}\,\overline{\varepsilon}_p\|^2\,\mathrm{dV}\,, \tag{10.170}$$

with only remaining independent dissipative variable $\overline{\varepsilon}_p$ (instead of two independent variables $(\overline{\varepsilon}_p,\overline{\kappa}_p)$) should lead to a **parabolic** flow rule for $\overline{\varepsilon}_p$ due to a **nonlocal** evaluation (instead of a traditional flow rule of ordinary differential equations for $(\overline{\varepsilon}_p,\overline{\kappa}_p)$. While the consistency requirement (10.169) has already been postulated in [EDV98b], in their work they still use the flow rule coming from the ordinary differential approach.

## 10.9 Notes on parameter identification

We include this discussion because there seems to be some confusion on what significance certain material parameters appearing in a Cosserat context have. Here it suffices to consider only the infinitesimal case in equilibrium format

$$\begin{aligned} 0 &= \mathrm{Div}\,\sigma + f,\quad x\in\Omega \\ \sigma &= 2\mu\ \mathrm{sym}\,\nabla u + 2\,\mu_c\,(\mathrm{skew}(\nabla u)-\overline{A}) + \lambda\,\mathrm{tr}\,[\mathrm{sym}\,\nabla u]\cdot 1\!\!1 \\ 0 &= \mu\frac{L_c^2}{12}\,\Delta\,\mathrm{axl}(\overline{A}) + \mu_c\ \mathrm{axl}((\mathrm{skew}(\nabla u)-\overline{A}) \end{aligned} \tag{10.171}$$

$$u_{|\partial\Omega}(t,x) = g_\mathrm{d}(t,x)-x,\quad x\in\partial\Omega\,,\quad \overline{A}_{|\partial\Omega} = \mathrm{skew}(\nabla g_\mathrm{d}(t,x))_{|\partial\Omega}\,.$$

Suppose homogeneous boundary conditions are prescribed: for $B\in\mathrm{GL}(3,\mathbb{R})$ a constant matrix, we set $g_\mathrm{d}(x):=B.x$. Thus we are able e.g. to describe uniform traction, uniform compression or simple shear. It is clear that $u(x)=B.x-x$ and $\overline{A}(x)=\mathrm{skew}(B)$ satisfy the boundary conditions and equilibrium equations. They are also the unique solutions. However, for these unique solutions, the Cosserat mechanisms are not activiated and we may determine $\mu,\lambda$ as classical moduli independent of the couple modulus $\mu_c$. In any homogeneous situation, only classical mechanisms are involved. Turning this argument upside down, we conclude that classical infinitesimal elasticity is appropriate for homogeneous situations only.

In order to get some information on the value of $\mu_c$ and the length scale $L_c$ we need to perform experiments leading to **inhomogeneous** response. One of the simplest cases is torsion of a cylinder. A sequence of torsion experiments allows to determine $\mu_c$ and $L_c$ if analytical formula are available relating torque and twist and incorporating the appearing parameters. Such formulas exist, showing that torsion in a Cosserat material would be stiffer (and depending on the length of the specimen) than ought to be expected by calculations based on classical linear elasticity and the already determined classical coefficients.

Investigations to this end on many materials have been performed, with the deceiving result that $\mu_c$ should be set to zero, thus implying that the infinitesimal Cosserat model is not appropriate for a more realistic description than classical linear elasticity. However, this shows only that Cosserat effects, if they really exist, are second order effects, not discernable in a first linear approximation.

Since, however, the parameter $\mu_c$ appears as well in the geometrically exact description, the foregoing experiments have already shown conclusively that $\mu_c=0$ is the correct value for the finite theory. This possibility of $\mu_c=0$ together with a true finite Cosserat theory has been consistently overlooked by overemphasizing the linear model.

It remains to determine the length scale $L_c$. But now we cannot use the infinitesimal model and its solution formula in torsion since with $\mu_c=0$ no Cosserat effects remain. We are either in need of a solution formula for the torsion problem of the nonlinear model or we have to calculate the torsional response directly numerically. However, one simplification is possible: we do not really need the full nonlinear system, instead an intermediate model incorporating second order effects would suffice. Below, we will give such a model.

## 10.10 Stability of the homogeneous solution

Consider the simplified finite Cosserat problem with consistent coupling condition

$$\int_\Omega \mu\,\|\operatorname{sym}(\overline{R}^T F) - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{R}^T F)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T F - 1\!\!1)\right]^2 \mu\, L_c^2\,\|\mathrm{D_x}\overline{R}\|^2\,\mathrm{dV} \mapsto \min.\ \text{w.r.t.}\ (\varphi,\overline{R}),$$

$$\overline{R}_{|_\Gamma} = \operatorname{polar}(\nabla\varphi), \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}\,, \tag{10.172}$$

and impose homogeneous boundary conditions: $g_{\mathrm{d}}(x) = B.x$ with a constant matrix $B$. It is clear that the homogeneous solution $\varphi(x) = B.x$ and $\overline{R} = \operatorname{polar}(F) = \operatorname{polar}(B)$ solves the corresponding equilibrium equation and boundary conditions. Is it possible to conclude that this solution is also a (unique?) global minimizer of the energy? At least we surmise that the homogeneous solution is locally stable. No rigourous conclusion is possible at this stage of the investigation. Note that for the consistent coupling boundary condition we have the trivial estimate

$$\int_\Omega W(\nabla\varphi,\overline{R})\,\mathrm{dV} \le \int_\Omega W(B,\operatorname{polar}(B))\,\mathrm{dV} = W(B,\operatorname{polar}(B))\cdot|\Omega|\,. \tag{10.173}$$

Moreover, the homogeneous solution will be energetically more favourable, the higher the value of $L_c > 0$ (or the smaller the specimen is). The height of a potential well around the homogeneous solution should be strictly related to $L_c > 0$. If $L_c = \infty$, then the homogeneous solution is the only possible one.

The Cosserat model allows therefore in principal for inhomogeneous minimizers in situations where homogeneous stationary solutions are possible.

## 10.11 A simplified elasto-plastic model for easy numerical implementation

Here we propose a model, based on our development, which should cover the essential behaviour of the geometrically exact model while being slightly simplified in order to arrive at a reasonable numerical implementation. It should as well serve the purpose of finding the value of the characteristic length $L_c$ for zero Cosserat couple modulus $\mu_c = 0$.

Numerical implementations based on an infinitesimal system are already in use. Tentative calculations of geometrical exact equations have also been done. However, due to the nonlinear manifold structure of $\mathrm{SO}(3,\mathbb{R})$ the implementation is awkward and the performance of the finite codes is in general insufficient. In order to circumvent these problems right from the start we propose a **penalty** formulation for the treatment of the finite rotations. We augment the free energy with a penalty term $\frac{\aleph}{4}\,\|\overline{R}_e^T\overline{R}_e - 1\!\!1\|^2$ where $\aleph$ is not a material parameter but supposed to approach $\infty$ in order to adjust $\overline{R}$ to exact rotations.

The geometrical exact elasto-plastic Cosserat model was given by

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - \langle f,\varphi\rangle\det[F_p] - \langle M,\overline{R}_e\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N,\varphi\rangle\,\|\operatorname{Cof}F_p.\vec{n}\|\,\mathrm{dS}\\
&\qquad - \int_{\Gamma_C}\langle M_c,\overline{R}_e\rangle\,\|\operatorname{Cof}F_p.\vec{n}\|\,\mathrm{dS} \mapsto \min.\ \text{w.r.t.}\ (\varphi,\overline{R}_e)\ \text{at constant}\ F_p,\\
&\overline{R}_{e_{|_\Gamma}} = \operatorname{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|_\Gamma}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}(t)\\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\operatorname{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{U}_e - 1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}_e\|^q)\,\|\mathfrak{K}_e\|^{1+p}\\
&F_e = \nabla\varphi\cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = \overline{R}_e^T[\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]\\
&\frac{\mathrm{d}}{\mathrm{dt}}[F_p^{-1}] \in -F_p^{-1}\cdot f(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}}\\
&\Sigma_{E,\mathrm{mp}} = \overline{U}_e^T D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] - W_{\mathrm{mp}}(\overline{U}_e)\det[F_p]\,1\!\!1\\
&\Sigma_{E,\mathrm{curv}} = \mathfrak{K}_e^T D_{\mathfrak{K}_e}W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p]\,1\!\!1\\
&F_p^{-1}(0) = F_{p_0}^{-1}, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R})\,.
\end{aligned} \tag{10.174}$$

An immediate permitted simplification is obtained by setting $\alpha_4 = 0$, $p = 1$ and discarding external volume and surface couples. This implies already that we can reduce the consideration of $\mathfrak{K}_e$ to $\mathfrak{K}_e = [\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]$, the rotations do not appear explicitly in the curvature.

No we introduce the penalty term and relax the rotations in the sense that we only require $\overline{R}_e \in \mathbb{M}^{3\times 3}$:

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] + \frac{\aleph}{4}\,\|\overline{R}_e^T\overline{R}_e - 1\!\!1\|^2\det[F_p] - \langle f,\varphi\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N,\varphi\rangle\,\|\operatorname{Cof}F_p.\vec{n}\|\,\mathrm{dS}\\
&\qquad \mapsto \min.\ \text{w.r.t.}\ (\varphi,\overline{R}_e)\ \text{at constant}\ F_p,\\
&\overline{R}_{e_{|_\Gamma}} = \operatorname{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|_\Gamma}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}(t)\\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\operatorname{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{U}_e - 1\!\!1)\right]^2, \quad W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^2}{12}\,\|\mathfrak{K}_e\|^2\\
&F_e = \nabla\varphi\cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = [\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]\,.
\end{aligned} \tag{10.175}$$

While the formal structure is thus kept, the differences appear in the related Euler-Lagrange equations. The force balance equation (translational equilibrium) remains invariant. The balance of angular momentum (rotational equilibrium) is modified: taking free variations w.r.t. $\overline{R}_e \in \mathbb{M}^{3\times 3}$ yields the stationarity condition

$$0 = \langle D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\, F_e^T, \delta\overline{R}_e^T\rangle \det[F_p] + 2\mu \frac{L_c^2}{12} \langle \mathrm{D_x}\overline{R}_e \cdot F_p^{-1}, \mathrm{D_x}(\delta\overline{R}_e)\cdot F_p^{-1}\rangle \det[F_p] + \frac{\aleph}{2}\langle \overline{R}_e^T\overline{R}_e - 1\!\!1, (\delta\overline{R}_e)^T\overline{R}_e + \overline{R}_e^T(\delta\overline{R}_e)\rangle \det[F_p]$$

$$0 = \langle [D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\, F_e^T]^T, \delta\overline{R}_e\rangle - 2\mu \frac{L_c^2}{12}\langle \mathrm{Div}[\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}F_p^{-T}], \delta\overline{R}_e\rangle + \aleph\langle \overline{R}_e[\overline{R}_e^T\overline{R}_e - 1\!\!1], \delta\overline{R}_e\rangle\,. \tag{10.176}$$

Hence the strong form is given by

$$0 = 2\mu\,\frac{L_c^2}{12}\,\mathrm{Div}[\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}F_p^{-T}] - [D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\, F_e^T]^T - \aleph\overline{R}_e[\overline{R}_e^T\overline{R}_e - 1\!\!1]\,, \tag{10.177}$$

with $F_p^{-1}F_p^{-T}$ playing the role of a **plastic metric**. The complete penalized model reads

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] - \langle f,\varphi\rangle \det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N,\varphi\rangle\,\|\,\mathrm{Cof}\,F_p.\vec{n}\|\,\mathrm{dS} \mapsto \min.\ \text{w.r.t. } \varphi \text{ at constant } (\overline{R}_e, F_p),\\
&0 = 2\mu\,\frac{L_c^2}{12}\,\mathrm{Div}[\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}F_p^{-T}] - D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\, F_e^T - \aleph\overline{R}_e[\overline{R}_e^T\overline{R}_e - 1\!\!1]\\
&\overline{R}_{e_{|\Gamma}} = \mathrm{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|\Gamma}\,,\quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t)\\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\,\mathrm{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}_e - 1\!\!1)\right]^2,\quad W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^2}{12}\,\|\mathfrak{K}_e\|^2\\
&F_e = \nabla\varphi\cdot F_p^{-1},\quad \overline{U}_e = \overline{R}_e^T F_e,\quad \mathfrak{K}_e = [\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}]\\
&D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e) = 2\mu\ \mathrm{sym}(\overline{U}_e - 1\!\!1) + 2\mu_c\ \mathrm{skew}(\overline{U}_e) + \lambda\,\mathrm{tr}\left[\overline{U}_e - 1\!\!1\right]1\!\!1\\
&\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] \in -F_p^{-1}\cdot f(\Sigma_E),\quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}}\,.
\end{aligned} \tag{10.178}$$

The ensuing model is still **thermodynamical consistent** since the modifications affect only the elastic behaviour. A welcome feature of the penalized model is the fact that it remains **frame-indifferent**.

The significance of a computed solution can easily be checked by evaluating $\|\overline{R}_e^T\overline{R}_e - 1\!\!1\|$ and/or inserting the result into the exact Euler-Lagrange equations (2.12). It is also possible to make $\aleph$ a function of the residuum of the exact Euler-Lagrange equation.

If we augment the penalty term further with $\|\,\mathrm{skew}(F_e\overline{R}_e^T\|^2$, then in the limit $(\aleph\to\infty, L_c\to 0)$ we recover the classical result $\overline{R}_e = \mathrm{polar}(F_e)$ and we are close to the model investigated numerically in [NW03, M2].

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