Reduced and Generalized Stokes Resolvent Equations in Asymptotically Flat Layers, Part II: H_∞-Calculus

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Abstract

We study the generalized Stokes equations in asymptotically flat layers, which can be considered as compact perturbations of an infinite (flat) layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$. Besides standard non-slip boundary conditions, we consider a mixture of slip and non-slip boundary conditions on the upper and lower boundary, respectively. In this second part, we use pseudodifferential operator techniques to construct a parametrix to the reduced Stokes equations, which solves the system in L^q-Sobolev spaces, $1 < q < \infty$, modulo terms which get arbitrary small for large resolvent parameters λ. This parametrix can be analyzed to prove the existence of a bounded H_∞-calculus of the (reduced) Stokes operator.
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1 Introduction

We consider the generalized Stokes resolvent equations

$$(\lambda - \Delta)u + \nabla p = f \quad \text{in } \Omega_\gamma, \tag{1.1}$$

$$\operatorname{div} u = g \quad \text{in } \Omega_\gamma, \tag{1.2}$$

$$T_j^+(u,p) = a^+ \quad \text{on } \partial\Omega_\gamma^+, \tag{1.3}$$

$$u|_{\partial\Omega_\gamma^-} = 0 \quad \text{on } \partial\Omega_\gamma^- \tag{1.4}$$

with $j = 0$ or $j = 1$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, where

$$T_0^+(u,p) = u|_{\partial\Omega_\gamma^+}, \qquad T_1^+(u,p) = (\nu \cdot S(u) - \nu p)|_{\partial\Omega_\gamma^+}, \quad S(u) = \nabla u + (\nabla u)^T,$$

and $\lambda \in \Sigma_\delta \cup \{0\}$. Here $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, is an asymptotically flat layer with $C^{1,1}$-boundary, i.e.,

$$\Omega_\gamma = \{(x', x_n) \in \mathbb{R}^n : \gamma^+(x') < x_n < \gamma^-(x')\}$$

and $\gamma^\pm \to \pm 1$ and $\nabla\gamma^\pm, \nabla^2\gamma^\pm \to 0$ as $|x'| \to \infty$, cf. [3]. Moreover, $\partial\Omega_\gamma^\pm = \{(x', \gamma^\pm(x')) : x' \in \mathbb{R}^{n-1}\}$.

In [3, Section 3] it is proved that (1.1)-(1.4) are uniquely solvable (with the restriction $\lambda \neq 0$ if $j = 0$) if and only if the reduced Stokes equations

$$(\lambda - \Delta)u + G_{j0}u = f \qquad \text{in } \Omega_\gamma, \tag{1.5}$$

$$T_j'^+ u = a^+ \quad \text{on } \partial\Omega_\gamma^+, \tag{1.6}$$

$$u|_{\partial\Omega_\gamma^-} = 0 \qquad \text{on } \partial\Omega_\gamma^-, \tag{1.7}$$

where

$$G_{00}u = G_0 u = \nabla K_1 \nu \cdot (\Delta - \nabla\,\mathrm{div})u|_{\partial\Omega_\gamma}, \quad G_{10}u = \nabla K_{01} \begin{pmatrix} 2\partial_\nu u_\nu|_{\partial\Omega_\gamma^+} \\ \nu \cdot (\Delta - \nabla\,\mathrm{div})u|_{\partial\Omega_\gamma^-} \end{pmatrix},$$

$$T_0'^+ u = u|_{\partial\Omega_\gamma^+}, \quad (T_1'^+ u)_\tau = (\nu \cdot S(u))_\tau|_{\partial\Omega_\gamma^+}, \quad (T_1'^+ u)_\nu = \mathrm{div}\, u|_{\partial\Omega_\gamma^+},$$

are uniquely solvable (in suitable L^q-Sobolev spaces, see [3, Section 3] for details.) Here K_1 and K_{01} denote the Poisson operators for the Laplace equation ($\lambda = 0$).

The reduced system (1.5)-(1.7) fits well into the general calculus of parameter-dependent pseudodifferential boundary value problems developed by Grubb in [12]. In Grubb and Solonnikov [15], the authors used this approach and applied general results for parabolic boundary value problems to solve the instationary Navier-Stokes equations in anisotropic L^2-Sobolev in bounded smooth domains locally in time for various kinds of boundary conditions. Later this result was extended to L^q-Sobolev spaces, cf. [11], and smooth exterior domains, cf. [13].

In the following, we will use the calculus developed in [12] to construct a parametrix to the reduced Stokes system (1.5)-(1.7), which coincides with the exact solution operator modulo term which decay faster as $|\lambda| \to \infty$. Using this parametrix, we prove that the usual Stokes operator in the Dirichlet case and the reduced Stokes operator in the mixed case admit a bounded H_∞-calculus in the sense of McIntosh [18].

THEOREM 1.1 *Let $1 < q < \infty$, $\delta \in (0, \pi)$, and let $\Omega_\gamma \subseteq \mathbb{R}^n$ be an asymptotically flat layer with $C^{1,1}$-boundary. Moreover, let $A_q = -P_q\Delta$ be the Stokes operator and $A_{10} = -\Delta + G_{10}$ be the reduced Stokes operator with domains*

$$\mathcal{D}(A_q) = W_q^2(\Omega_\gamma)^n \cap W_{q,0}^1(\Omega_\gamma)^n \cap J_{q,0}(\Omega_\gamma),$$

$$\mathcal{D}(A_{10}) = \left\{u \in W_q^2(\Omega_\gamma)^n : T_1'^+ u = 0, u|_{\partial\Omega_\gamma^-} = 0\right\}, \textit{resp.},$$

cf. [3]. Then A_q and A_{10} admit a bounded H_∞-calculus with respect to δ on $X = J_{q,0}(\Omega_\gamma)$ and $X = L^q(\Omega_\gamma)^n$, resp. , i.e.,

$$h(A) = \frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + A)^{-1} d\lambda, \qquad A = A_q, A_{10}, \tag{1.8}$$

is a bounded operator on X and

$$\|h(A)\|_{\mathcal{L}(X)} \leq C_\delta \|h\|_\infty \tag{1.9}$$

for every $h \in H_\infty(\delta)$. Here $H_\infty(\delta)$ denotes the algebra of all bounded holomorphic functions $h\colon \Sigma_{\pi-\delta} \to \mathbb{C}$ and Γ is the negatively oriented boundary of $\Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} : |\arg| < \delta\}$.

The theorem implies the existence of *bounded imaginary powers* A^{iy}, $y \in \mathbb{R}$, of the Stokes and the reduced Stokes operator since $h_y(z) = z^{iy} \in H_\infty(\delta)$. Hence the result of Dore and Venni [7] implies the *maximal regularity* of the (reduced) Stokes operators.

The parametrix in Ω_γ is defined with the aid of the parametrix in *curved half-spaces* $\mathbb{R}^n_\gamma = \{(x', x_n) \in \mathbb{R}^n : x_n > \gamma(x')\}$ by means of a simple partition of unity. Here the parametrix in $\mathbb{R}^n_\gamma$ is constructed by transforming the equations to a system in $\mathbb{R}^n_+$ with variable coefficients and freezing coefficients. In order to construct the parametrix assuming only $C^{1,1}$-regularity of the boundary, it is necessary to modify the general theory in [12], which assumes smooth coefficients. This will be done by combining the techniques and results for pseudodifferential operators with non-smooth coefficients developed in [17, 23, 24] (in an operator-valued version) with known facts for the smooth coefficient case, cf. Sections 3 and 4 below. In order to prove the boundedness of (1.8), we have to analyze the symbols of the parametrix precisely. This is done by relating the symbol of the parametrix in $\mathbb{R}^n_\gamma$ to the symbols of the solution operators of the reduced Stokes equations in $\mathbb{R}^n_+$, cf. Section 5.3 below. The structure of the reduced Stokes equations enables us to consider the resolvent of the reduced Stokes operator as perturbation of the Laplace resolvent, cf. Section 5.1 below. Then we obtain the necessary estimates to prove boundedness of (1.8) in Section 5.4 below.

Remark 1.2 Note that the method presented here is not restricted to asymptotically flat domains. It has much in common with the first published proof that the Stokes operator in a bounded domain possesses bounded imaginary powers presented by Giga [9], which is also based on pseudodifferential operator techniques. Since the following proof uses the reduced Stokes equations, we can also deal with more general boundary conditions. In [9], the proof is presented in the case of a smooth bounded domain, but can be modified for the case of a $C^{2,\mu}$-boundary, $\mu > 0$.

An alternative method, using a perturbation theorem for the H_∞-calculus, can be found in Noll and Saal [19]. In the latter contribution the existence of a bounded H_∞-calculus for the Stokes operator in a bounded and exterior domain in $\mathbb{R}^n$, $n \geq 3$, with C^3-boundary is proved.

In the special case of an infinite layer, a more elementary proof that the Stokes operator possesses bounded imaginary powers based on Mikhlin multiplier techniques is presented in Abels [1].

2 Preliminaries

We will use the same notation and function spaces as in [3]. Additionally, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transformation,

$$\mathcal{F}[f](\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}[f](x) := \check{f}(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)đ\xi,$$

defined for a suitable function $f\colon \mathbb{R}^n \to \mathbb{C}$, where $đ\xi := (2\pi)^{-n}d\xi$. Note that in the following all integrals with respect to a phase ξ will be scaled by $(2\pi)^{-n}$ as above. Moreover, we will use partial Fourier transformation

$$\mathcal{F}_{x'\mapsto\xi'}[f](\xi', x_n) := \tilde{f}(\xi', x_n) := \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} f(x', x_n)dx'$$

and the conjugate Fourier transformation $\bar{\mathcal{F}}[f](\xi) = \mathcal{F}[f](-\xi)$.

Let $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$, and let $\langle D_x\rangle^s = \operatorname{OP}(\langle\xi\rangle^s) = \mathcal{F}^{-1}[\langle\xi\rangle^s\mathcal{F}[.]]$, $s \in \mathbb{R}$. Moreover, $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing smooth functions $f\colon \mathbb{R}^n \to \mathbb{C}$ and $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions. Recall that the *Bessel potential space* $H^s_q(\mathbb{R}^n)$, $1 < q < \infty$, $s \in \mathbb{R}$, is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\langle D_x\rangle^s f \in L^q(\mathbb{R}^n)$, with norm

$$\|f\|_{H^s_q} = \|\langle D_x\rangle^s f\|_{L^q}.$$

Moreover, $\mathcal{S}(\mathbb{R}^n; X)$ and $H^s_q(\mathbb{R}^n; X)$ denote the vector-valued variants, where X is a Banach space.

As in [14, 10], the space $H^s_q(\mathbb{R}^n_+) = r^+ H^s_q(\mathbb{R}^n)$ is defined as the space of all distributions of $H^s_q(\mathbb{R}^n)$ restricted to $\mathbb{R}^n_+$ equipped with the quotient norm and $H^s_{q;0}(\mathbb{R}^n_+)$ is defined as the space of all distributions of $H^s_q(\mathbb{R}^n)$ supported in $\overline{\mathbb{R}}^n_+$.

Let $\varphi \in C^\infty_0(\mathbb{R}^n)$ such that

$$\operatorname{supp}\varphi = \{\xi : 2^{-1} \le |\xi| \le 2\}, \quad \varphi(\xi) > 0 \text{ if } 2^{-1} < |\xi| < 2,$$
$$\textstyle\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1 \text{ when } \xi \neq 0,$$

and set $\varphi_0(\xi) = 1 - \sum_{k=1}^\infty \varphi(2^{-k}\xi)$. Then the *Besov space* $B^s_q(\mathbb{R}^n) = B^s_{q,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 \le q < \infty$, is defined as the spaces of all tempered distributions f with finite norm

$$\|f\|_{B^s_q} = \left(\|\varphi_0(D_x)f\|^q_{L^q} + \sum_{1\le k<\infty} 2^{sqk}\|\varphi(2^{-k}D_x)f\|^q_{L^q}\right)^{\frac{1}{q}}.$$

If $s > 0$ with $s \notin \mathbb{N}$, the Besov space $B^s_q(\mathbb{R}^n)$ coincides with the Sobolev-Slobodeckij space $W^s_q(\mathbb{R}^n)$.

We refer to [5, Chapter 6] for the interpolation properties of Besov and Bessel potential spaces. In particular, we use

$$(H^{s_0}_q(\mathbb{R}^n), H^{s_1}_q(\mathbb{R}^n))_{\theta,q} = B^s_q(\mathbb{R}^n) \tag{2.1}$$

for $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $1 < q < \infty$, and $s = (1-\theta)s_0 + \theta s_1$. Here $(.,.)_{\theta,q}$ denotes the real interpolation functor.

Finally, $C^s_*(\mathbb{R}^n) = B^s_\infty(\mathbb{R}^n)$, $s > 0$, denotes the Zygmund space, which consists of all functions f such that

$$\|f\|_{C^s_*} = \sup\{\|\varphi_0(D_x)f\|_{L^\infty}, 2^{ks}\|\varphi(2^{-k}D_x)f\|_{L^\infty} : k \in \mathbb{N}\} < \infty.$$

It coincides with the usual Hölder space $C^s(\mathbb{R}^n)$ for non-integer $s > 0$.

We also need the *weighted L^2-spaces*

$$L^2(\mathbb{R}_+, x_n^s) \;=\; \{u \in \mathcal{D}'(\mathbb{R}_+) : x_n^s u(x_n) \in L^2(\mathbb{R}_+)\}, \qquad s \in \mathbb{R},$$

and $L^2(\mathbb{R}, |x_n|^s)$, which is defined analogously. Note that, $(L^2(\mathbb{R}_+, x_n^s))' = L^2(\mathbb{R}_+, x_n^{-s})$ with respect to the $L^2(\mathbb{R}_+)$-scalar product and

$$(L^2(\mathbb{R}_+, x_n^{s_1}), L^2(\mathbb{R}_+, x_n^{s_2}))_{\theta,2} = L^2(\mathbb{R}_+, x_n^s) \tag{2.2}$$

for $\theta \in (0,1)$ and $s = (1-\theta)s_1 + \theta s_2$ because of [5, Theorem 5.4.1]. The analogous results hold for $L^2(\mathbb{R}, |x_n|^s)$.

Lemma 2.1 *1. Let $1 < q \leq 2$, $\delta' < \frac{1}{q} - \frac{1}{2} < \delta$, and $\theta = (\frac{1}{q} - \frac{1}{2} - \delta')/(\delta - \delta')$. Then*

$$(L^2(\mathbb{R}_+, x_n^{\delta'}), L^2(\mathbb{R}_+, x_n^{\delta}))_{\theta,q} \;\subseteq\; L^q(\mathbb{R}_+), \tag{2.3}$$

$$(H^{-\delta'}_{2;0}(\mathbb{R}_+), H^{-\delta}_{2;0}(\mathbb{R}_+))_{\theta,q} \;\supseteq\; L^q(\mathbb{R}_+). \tag{2.4}$$

2. Let $2 \leq q < \infty$, $\delta' < \frac{1}{2} - \frac{1}{q} < \delta$, and $\theta = (\frac{1}{2} - \frac{1}{q} - \delta')/(\delta - \delta')$. Then

$$(L^2(\mathbb{R}_+, x_n^{-\delta'}), L^2(\mathbb{R}_+, x_n^{-\delta}))_{\theta,q} \;\supseteq\; L^q(\mathbb{R}_+), \tag{2.5}$$

$$(H^{\delta'}_2(\mathbb{R}_+), H^{\delta}_2(\mathbb{R}_+))_{\theta,q} \;\subseteq\; L^q(\mathbb{R}_+). \tag{2.6}$$

Proof: The lemma was proved by Grubb and Kokholm [14, Theorem 1.8]. ■

3 Non-Smooth Pseudodifferential Operators

Let B be an arbitrary Banach space.

Definition 3.1 The symbol space $C^\tau_* S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; B)$, $\tau > 0$, $m \in \mathbb{R}$, is the set of all symbols $p\colon \mathbb{R}^n \times \mathbb{R}^n \to B$ that are in C^τ_* with respect to the first variable and smooth with respect to the second variable satisfying the estimates

$$\|D^\alpha_\xi p(.,\xi)\|_{C^\tau_*(\mathbb{R}^n;B)} \;\leq\; C_\alpha \langle\xi\rangle^{m-|\alpha|} \tag{3.1}$$

for all $\alpha \in \mathbb{N}_0^n$. Moreover, we define the semi-norms

$$|p|^{(m)}_k := \sup_{|\alpha|\leq k, \xi\in\mathbb{R}^n} \langle\xi\rangle^{|\alpha|-m}\|D^\alpha_\xi p(.,\xi)\|_{C^\tau_*(B)}, \qquad k \in \mathbb{N}.$$

The symbol space $C^{0,1}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; B)$ is defined in the same way with C^τ_* replaced by $C^{0,1}$. Note that $C^{0,1}(\mathbb{R}^n) \subset C^1_*(\mathbb{R}^n)$, hence $C^{0,1}S^m_{1,0} \subset C^1_* S^m_{1,0}$.

In the following we will only consider the case $B = \mathcal{L}(H_0, H_1)$ for some Hilbert spaces H_0 and H_1. Then given $p \in XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$, $X = C^\tau_*$ or $X = C^{0,1}$,

$$\begin{aligned} p(x, D_x)u &= \mathrm{OP}(p(x,\xi))u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi \quad \text{and} \\ p(D_x, x)u &= \mathrm{OP}(p(y,\xi))u = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(y,\xi)u(y)dy\, đ\xi \end{aligned} \tag{3.2}$$

for $u \in \mathcal{S}(\mathbb{R}^n; H_0)$ are the associated pseudodifferential operators in *L- and R-form*, resp; also called *x-form and y-from.* Here the second integral has to be understood as an oscillatory integral, cf. [17, Theorem 2.2].

We note that

$$(p(x, D_x)u, v)_{L^2(\mathbb{R}^n;H_1)} = (u, p'(D_x, x)v)_{L^2(\mathbb{R}^n;H_0)}$$

for all $u \in \mathcal{S}(\mathbb{R}^n; H_0)$, $v \in \mathcal{S}(\mathbb{R}^n; H_1)$, where $p'(x,\xi) = p(x,\xi)' \in \mathcal{L}(H_1, H_0)$ denotes the pointwise dual of the symbol p.

The following theorem is an operator-valued variant of [23, Proposition 2.1.D] and will be proved in the appendix.

THEOREM 3.2 *Let $\tau > 0$, $1 < q < \infty$, and $m \in \mathbb{R}$. If $p \in C^\tau_* S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$, then $p(x, D_x)$ and $p(D_x, x)$ extend to a bounded linear operators*

$$p(x, D_x)\colon H^{s+m}_q(\mathbb{R}^n; H_0) \to H^s_q(\mathbb{R}^n; H_1),\ p(D_x, x)\colon H^s_q(\mathbb{R}^n; H_0) \to H^{s-m}_q(\mathbb{R}^n; H_1)$$

for all $s \in \mathbb{R}$ with $-\tau < s < \tau$.

In the following we denote by $(p_1p_2)(x,\xi) = p_1(x,\xi) \circ p_2(x,\xi) \in \mathcal{L}(H_0, H_2)$ the pointwise composition of the symbols.

THEOREM 3.3 *Let $1 < q < \infty$, $m_1, m_2 \in \mathbb{R}$, $0 < \tau_1 \leq \tau_2$, and $s \in \mathbb{R}$ with $|s| < \tau_1$ and $|s + m_1| < \tau_2$. If $p_1 \in C^{\tau_1}_* S^{m_1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_1, H_2))$ and $p_2 \in C^{\tau_2}_* S^{m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$, then for every $0 < \theta \leq \frac{\tau_1}{1+\tau_1}$ with $s - \theta > -\tau_1$ and $s + m_1 - \theta > -\tau_2$*

$$(p_1p_2)(x, D_x) - p_1(x, D_x)p_2(x, D_x)\colon H^{s+m_1+m_2-\theta}_q(\mathbb{R}^n; H_0) \to H^s_q(\mathbb{R}^n; H_2)$$

is a bounded linear operator. Moreover, there is a $k \in \mathbb{N}$ such that

$$\|(p_1p_2)(x, D_x) - p_1(x, D_x)p_2(x, D_x)\| \leq C|p_1|^{(m_1)}_k |p_2|^{(m_2)}_k,$$

where $\|\cdot\|$ denotes the corresponding operator norm.

This theorem will also be proved in the appendix.

The latter theorem shows that the composition of $p_1(x, D_x)$ and $p_2(x, D_x)$ coincides with $(p_1 p_2)(x, D_x)$ modulo an operator of lower order in the sense of mapping properties in Bessel potential spaces. In the following parametrix construction the precise size of $\theta > 0$ does not matter and for given $s \in \mathbb{R}$ with $|s| < \tau_1$ and $|s + m_1| < \tau_2$, there are always some $\theta > 0$ which satisfy the assumption of the theorem.

Corollary 3.4 *Let $m \in \mathbb{R}$, $1 < q < \infty$, and $s \in \mathbb{R}$ such that $|s| < 1$ and $|s - m| < 1$. If $p \in C^{0,1}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$ and $a \in C^{0,1}(\mathbb{R}^n)$, then for every $0 < \theta \leq \frac{1}{2}$ with $s + \theta < 1$ and $s - m + \theta < 1$*

$$a(x)p(D_x, x) - (ap)(D_x, x) \colon H^s_q(\mathbb{R}^n; H_0) \to H^{s-m+\theta}_q(\mathbb{R}^n; H_2)$$

is a bounded operator.

Proof: The corollary is easily obtained from Theorem 3.3 by duality. ■

In the following we will restrict ourselves to the case of operators with Lipschitz-continuous coefficients; this is the case we need for the construction of the parametrix.

As in [12], the spectral parameter will be represented as $\lambda = \mu^2 e^{i\theta}$, $\theta \in (-\pi, \pi)$. Moreover, let $\rho = \rho(\xi', \mu) = \frac{\langle \xi' \rangle}{\langle \xi', \mu \rangle}$, where $\langle \xi', \mu \rangle = (1 + |\xi'|^2 + \mu^2)^{\frac{1}{2}}$ and $\mu \geq 0$, $\xi' \in \mathbb{R}^{n-1}$.

It is straightforward to define a non-smooth variant of these classes of parameter-dependent pseudodifferential operators studied in [12].

Definition 3.5 Let $d, \nu \in \mathbb{R}$. Then $C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^n \times \overline{\mathbb{R}}^{n+1}_+)$ is the space of all functions $p(x, \xi, \mu)$ smooth w.r.t. (ξ, μ) and Lipschitz-continuous in x such that

$$\|D^\alpha_\xi D^j_\mu p(., \xi, \mu)\|_{C^{0,1}} \quad \leq \quad C_{\alpha,j}(\rho(\xi, \mu)^{\nu - |\alpha|} + 1)\langle \xi, \mu \rangle^{d - |\alpha| - j}$$

uniformly in $(\xi, \mu) \in \overline{\mathbb{R}}^{n+1}_+$ and for all $\alpha \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$. Moreover, let

$$|p|^{(d,\nu)}_k = \sup_{|\alpha|, j \leq k, (\xi,\mu) \in \mathbb{R}^{n+1}_+} \|D^\alpha_\xi D^j_\mu p(., \xi, \mu)\|_{C^{0,1}}(\rho(\xi, \mu)^{\nu - |\alpha|} + 1)^{-1}\langle \xi, \mu \rangle^{-d + |\alpha| + j}$$

be the corresponding increasing sequence of semi-norms.

Recall that

$$(\rho(\xi, \mu)^\nu + 1)\langle \xi, \mu \rangle^d \simeq \begin{cases} \langle \xi, \mu \rangle^d & \text{if } \nu \geq 0, \\ \langle \xi \rangle^\nu \langle \xi, \mu \rangle^{d-\nu} & \text{if } \nu < 0. \end{cases}$$

Remark 3.6 Note that, if $p \in C^{0,1}S^{d,\nu}_{1,0}$ and $d' > d$, then $p \in C^{0,1}S^{d',\nu}_{1,0}$ with $|p|^{(d',\nu)}_k \leq \langle \mu \rangle^{d-d'}|p|^{(d,\nu)}_k$ for all $k \in \mathbb{N}_0$. Moreover, if $d \leq 0$, $\nu \geq 0$ and if we look at p as a parameter-independent symbol, then $|p(., \mu)|^{(d)}_k \leq C|p|^{(d,\nu)}_k$ uniformly in $\mu \in \overline{\mathbb{R}}_+$.

4 Pseudodifferential Boundary Value Problems with Non-Smooth Coefficients

We will now define a non-smooth version of parameter-dependent Green operators developed in [12].

We use the notation of [12] except that $\gamma_j u = \partial_n^j u|_{\partial\mathbb{R}^n_+}$. Recall that $\mathcal{H}_d$, $d \in \mathbb{Z}$, denotes the space of all smooth $f\colon \mathbb{R} \to \mathbb{C}$ which admit an asymptotic development $f(t) \sim s_d t^d + s_{d-1}t^{d-1} + \ldots$ in the sense that for all k, l, and $N \in \mathbb{N}_0$

$$\left|\partial_t^l \left[t^k f(t) - \sum_{j=d-N}^{d} s_j t^{j+k}\right]\right| \leq C_{k,l,N}(1+|t|)^{d-N-1+k-l} \quad \text{as } |t| \to \infty.$$

It is important that $\mathcal{H}_{-1} = \mathcal{H}^+ \oplus \mathcal{H}^-_{-1}$, where $\mathcal{H}^+$ and $\mathcal{H}^-_{-1}$ are the subspaces of all $f \in \mathcal{H}_{-1}$ which can be extended holomorphically to the lower resp. upper complex plane, and

$$\mathcal{H}^+ = \mathcal{F}[e^+\mathcal{S}(\overline{\mathbb{R}}_+)], \qquad \mathcal{H}^-_{-1} = \mathcal{F}[e^-\mathcal{S}(\overline{\mathbb{R}}_-)],$$

see [12, Chapter II, Section 2.2] for details. – Note that $f \in \mathcal{H}^+ \Leftrightarrow \overline{f} \in \mathcal{H}^-_{-1}$. – Moreover, $h^+ = \mathcal{F}e^+r^+\mathcal{F}^{-1}$ and $h^-_{-1} = \mathcal{F}e^-r^-\mathcal{F}^{-1}$ are continuous projections on $\mathcal{H}^+$ and $\mathcal{H}^-_{-1}$, resp. We use the convention $\mathcal{H}^-_r = \mathcal{H}^-_{-1} \oplus \mathbb{C}_r[t]$, $r \in \mathbb{N}_0$, where $\mathbb{C}_r[t]$ denotes the set of all complex polynomials of degree r. Moreover, $h_{-1}\colon \mathcal{H}_d \to \mathcal{H}_{-1}$ is the projection with range $\mathcal{H}_{-1}$ and kernel $\mathbb{C}_d[t]$. Finally, we note that $\mathcal{H}_d$, $d \in \mathbb{Z}$, $\mathcal{H}^+$, and $\mathcal{H}^-_r$, $r \in \mathbb{N}_0$, are nuclear spaces. Hence there exist unique complete tensor products $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_r$ and $\mathcal{H}_{-1}\hat{\otimes}\mathcal{H}_{-1}$.

We start with the definition of Poisson operators.

Definition 4.1 The space $C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^{n-1} \times \overline{\mathbb{R}}^n_+, \mathcal{H}^+)$, $d, \nu \in \mathbb{R}$, of *Poisson symbols* of degree d and regularity ν consists of functions $k(x', \xi', \xi_n, \mu) \in \mathcal{H}^+$ with respect to ξ_n which satisfy

$$\|D^\alpha_{\xi'} D^j_\mu h_{-1}(D^l_{\xi_n}\xi_n^{l'} k(.,\xi',.,\mu))\|_{C^{0,1}(\mathbb{R}^{n-1};L^2_{\xi_n})} \leq C(\rho^{\nu-[l-l']_+-|\alpha|}+1)\langle\xi',\mu\rangle^{d+\frac{1}{2}-l+l'-|\alpha|-j} \tag{4.1}$$

for all $\alpha' \in \mathbb{N}_0^{n-1}, j, l, l' \in \mathbb{N}_0$. If $k \in C^{0,1}S^{d-1,\nu}_{1,0}(\mathbb{R}^{n-1} \times \overline{\mathbb{R}}^n_+, \mathcal{H}^+)$, then

$$k(x', \mu, D_x)a = r^+\mathcal{F}^{-1}_{\xi\mapsto x}\left[k(x', \xi, \mu)\tilde{a}(\xi')\right], \qquad a \in \mathcal{S}(\mathbb{R}^{n-1}),$$

is the associated *Poisson operator* of order d and regularity ν in L-form.

Note that the *degree* of a Poisson symbol reflects the order of growth as $|(\xi,\mu)| \to \infty$ in contrast to the *order*, which reflects the mapping properties of the associated operator.

Alternatively, a Poisson operator can be described by its *symbol-kernel*:

$$k(x', \mu, D_x) \;=\; \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\tilde{k}(x', \xi, \mu, x_n)\tilde{a}(\xi')\right],$$

where $\tilde{k}(x',\xi',\mu,x_n)=\mathcal{F}^{-1}_{\xi_n\mapsto x_n}[k(x',\xi,\mu)]\in\mathcal{S}(\overline{\mathbb{R}}_+)$ w.r.t. x_n. Moreover, the *boundary symbol operators* $k(x',\xi',\mu,D_n)$ are defined as one-dimensional operators with symbols $k(x',\xi,\mu)$ for fixed (x',ξ').

Remark 4.2 Let $k\in C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+,\mathcal{H}^+)$ be (for simplicity) independent of μ. Then (4.1) and $\|\mathcal{F}[f]\|_{L^2(\mathbb{R}_+)}=\|f\|_{L^2(\mathbb{R})}$ for $f\in\mathcal{H}^+$ imply

$$\|D^\alpha_{\xi'}x_n^lD^{l'}_{x_n}k(x',\xi',D_n)\|_{\mathcal{L}(\mathbb{C},L^2(\mathbb{R}_+))}\leq C_{\alpha,l,l'}\langle\xi'\rangle^{d+\frac{1}{2}-l+l'-|\alpha|}.$$

In particular $k(x',\xi',D_n)$ is a $\mathcal{L}(\mathbb{C},L^2(\mathbb{R}_+))$-valued pseudodifferential operator of order $d+\frac{1}{2}$. Moreover, interpolation of the latter estimate for different values of $l,l'\in\mathbb{N}_0$ yields

$$\|D^\alpha_{\xi'}k(.,\xi',D_n)\|_{C^{0,1}(\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},L^2(\mathbb{R}_+,x_n^{2\delta})))}\leq C_{\alpha,\delta}\langle\xi'\rangle^{d+\frac{1}{2}-\delta-|\alpha|},\tag{4.2}$$

$$\|D^\alpha_{\xi'}k(.,\xi',D_n)\|_{C^{0,1}(\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},H^\delta_2(\mathbb{R}_+)))}\leq C_{\alpha,\delta}\langle\xi'\rangle^{d+\frac{1}{2}+\delta-|\alpha|}\tag{4.3}$$

for all $\delta\geq 0$ and $\alpha\in\mathbb{N}_0^{n-1}$, cf. (2.2) and [14, Section 3.3].

Considering a boundary symbol operator as operator-valued pseudodifferential operator, the corresponding operator in R-form $k(D_x,x',\mu)$ is defined as in (3.2).

Definition 4.3 Let $d,\nu\in\mathbb{R}$ and let $r\in\mathbb{N}_0$.

1. The space of trace symbols $C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+,\mathcal{H}^-_{r-1})$ of degree d and class r is the set of all

$$t(x',\xi',\xi_n,\mu)=\sum_{0\leq j\leq r-1}s_j(x',\xi',\mu)(i\xi_n)^j+t'(x',\xi',\xi_n,\mu),$$

with $\overline{t'(x',\xi,\mu)}\in C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+,\mathcal{H}^+)$ and $s_j(x',\xi',\mu)\in C^{0,1}S^{d-j,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+)$. The associated trace operator of order d in L-form is defined as

$$t(x',\mu,D_x)f=\sum_{j=0}^{r-1}s_j(x',\mu,D_{x'})\gamma_jf+\mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\int t'(x',\xi',\xi_n,\mu)\hat{f}(\xi)d\xi_n\right].$$

2. The space of singular Green symbols $C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ of degree d and class r consists of all functions

$$g(x',\xi',\xi_n,\eta_n,\mu)=\sum_{j=0}^{r-1}k_j(x',\xi',\xi_n,\mu)(i\eta_n)^j+g'(x',\xi',\xi_n,\eta_n,\mu)$$

such that $k_j(x',\xi',\xi_n,\mu)\in C^{0,1}S^{d-j,\nu}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n_+,\mathcal{H}^+)$ and $g'(x',\xi,\eta_n,\mu)\in\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$ with respect to (ξ_n,η_n) satisfying

$$\begin{aligned}&\|D^\alpha_{\xi'}D^j_\mu h_{-1,\xi_n}h_{-1,\eta_n}(D^k_{\xi_n}\xi_n^{k'}D^l_{\eta_n}\eta_n^{l'}g(x',\xi,\eta_n,\mu)\|_{C^{0,1}(\mathbb{R}^{n-1};L^2(\mathbb{R}^2))}\\&\leq C(\rho^{\nu-[k-k']_+-[l-l']_+-|\alpha|}+1)\langle\xi',\mu\rangle^{d+1-k+k'-l+l'-|\alpha|-j}\end{aligned}\tag{4.4}$$

for all $\alpha' \in \mathbb{N}_0^{n-1}, k, k', l, l', j \in \mathbb{N}_0$. If $g \in C^{0,1}S_{1,0}^{d-1,\nu}(\mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+^n, \mathcal{H}^+ \hat{\otimes} \mathcal{H}_r^-)$, then

$$g(x', \mu, D_x)f = \sum_{j=0}^{r-1} k_j(x', \mu, D_x)\gamma_j f + \mathcal{F}_{\xi \mapsto x}^{-1}\left[\int g'(x', \xi, \eta_n, \mu)\hat{f}(\xi', \eta_n)đ\eta_n\right],$$

$f \in \mathcal{S}(\overline{\mathbb{R}}_+^n)$, is the associated singular Green operator of order d, regularity ν, and class r in L-form.

Note that, if $t'(x', D_x)$ is a trace operator of class 0, then

$$(t(x', D_x)\varphi, \psi)_{\mathbb{R}^{n-1}} = (\varphi, k(D_x, x')\psi)_{\mathbb{R}_+^n}, \tag{4.5}$$

where $k(x', \xi) = \overline{t(x', \xi)}$ and $\varphi \in \mathcal{S}(\overline{\mathbb{R}}_+^n)$, $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$. Hence trace operators can be considered as duals of Poisson operators plus a sum of usual trace operators $s_j(x', \mu, D_{x'})\gamma_j$. Throughout the present contribution, the singular Green symbols will be products of Poisson and trace symbols.

We can also describe trace and singular Green operators with the aid of their *symbol-kernels*:

$$\begin{aligned} t(x', \mu, D_x) &= \sum_{j=0}^{r-1} s_j(x', \mu, D_{x'})\gamma_j f + \mathcal{F}_{\xi' \mapsto x'}^{-1}\left[\int_0^\infty \tilde{t}'(x', \xi', \mu, x_n)\tilde{f}(\xi', x_n)dx_n\right], \\ g(x', \mu, D_x) &= \sum_{j=0}^{r-1} k_j(x', \mu, D_x)\gamma_j f + \mathcal{F}_{\xi' \mapsto x'}^{-1}\left[\int_0^\infty \tilde{g}'(x', \xi', \mu, x_n, y_n)\hat{f}(\xi', y_n)dy_n\right], \end{aligned}$$

where $\tilde{t}'(x', \xi, \mu, x_n) = \bar{\mathcal{F}}_{\xi_n \mapsto x_n}^{-1}[t(x', \xi, \mu)] \in \mathcal{S}(\overline{\mathbb{R}}_+)$ w.r.t. x_n and $\tilde{g}'(x', \xi', \mu, x_n, y_n) = \mathcal{F}_{\xi_n \mapsto x_n}^{-1}\bar{\mathcal{F}}_{\eta_n \mapsto y_n}^{-1}[g(x', \xi, \eta_n, \mu)] \in \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ w.r.t. (x_n, y_n). Finally, the boundary symbol operators $t(x', \xi', \mu, D_n)$ and $g(x', \xi', \mu, D_n)$ and operators in R-form are defined in the same way as for the Poisson operator.

Definition 4.4 Let $p \in C^{0,1}S_{1,0}^{d,\nu}(\mathbb{R}^n \times \overline{\mathbb{R}}_+^{n+1})$, $d \in \mathbb{Z}$, $\nu \in \mathbb{R}$, that is independent of x_n. Then p satisfies the *transmission condition* if there are functions $s_{k,\alpha,j}$ smooth in (ξ', μ) and Lipschitz continuous in x' such that for any $\alpha \in \mathbb{N}_0^n$ and $l, j \in \mathbb{N}_0$

$$\left\| \xi_n^l D_\xi^\alpha D_\mu^j p(., \xi, \mu) - \sum_{k=-l}^{d-|\alpha|-j} s_{k,\alpha,j}(., \xi', \mu)\xi_n^{k+l} \right\|_X \leq C_{k,\alpha,j}\langle \xi', \mu\rangle^{m+1+l-|\alpha|-j}|\xi_n|^{-1}$$

when $|\xi_n| \geq \langle \xi', \mu\rangle$.

Definition 4.5 A Green operator (in L-form) of order $d \in \mathbb{Z}$, class $r \in \mathbb{N}_0$, and regularity $\nu \in \mathbb{R}$ with Lipschitz-continuous coefficients is defined as

$$a(x', \mu, D_x) = \begin{pmatrix} p(x', \mu, D_x)_+ + g(x', \mu, D_x) & k(x', \mu, D_x) \\ t(x', \mu, D_x) & s(x', \mu, D_{x'}) \end{pmatrix},$$

where $k(x',\mu,D_x)$, $t(x',\mu,D_x)$, and $g(x',\mu,D_x)$ are Poisson, trace, and singular Green operators of order d, regularity ν, and class r, $p(x',\mu,D_x)_+ = r^+p(x',\mu,D_x)e^+$, $p \in C^{0,1}S^{d,\nu}_{1,0}(\mathbb{R}^n \times \overline{\mathbb{R}}^n_+)$, is a truncated pseudodifferential operator satisfying the transmission condition and $s \in C^{0,1}S^{d-1,\nu}_{1,0}(\mathbb{R}^{n-1} \times \overline{\mathbb{R}}^n_+)$.

In the following we will often restrict ourselves to the case of parameter-independent symbols and operators. The corresponding symbol classes $C^{0,1}S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{K})$, $\mathcal{K} = \mathcal{H}^+, \mathcal{H}^-_{r-1}$, or $\mathcal{K} = \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$ are defined as above with the restriction that the symbols are independent of μ and the symbols estimates hold for $\mu = 0$.

Moreover, if f is a Poisson, trace, or singular Green symbol, then $|f|^{(d,\nu)}_k$, $k \in \mathbb{N}$, are the semi-norms (monotonically increasing in k) associated to (4.1), (4.4), resp., in the usual way, cf. Definitions 3.5 and 3.5. The semi-norms of parameter-independent symbols will be denoted by $|f|^{(d)}_k$.

Remarks 4.6 1. As in Remark 3.6, $|f|^{(d+\varepsilon,\nu)}_k \leq \langle\mu\rangle^{-\varepsilon}|f|^{(d,\nu)}_k$, $\varepsilon > 0$.

2. If f is a parameter-dependent Poisson or trace symbol of degree $d \leq -\frac{1}{2}$, regularity ν (and class r), then $f(.,\mu)$, $\mu \geq 0$ fixed, is a parameter-independent symbol of the same degree and class with $|f(.,\mu)|^{(d)}_k \leq |f|^{(d,\nu)}_k$ uniformly in $\mu > 0$. The same is true for parameter-dependent singular Green symbols of degree $d \leq -1$.

3. Conversely, if $k \in C^{0,1}S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{H}^+)$, then $k \in C^{0,1}S^{d-1,d-\frac{1}{2}}_{1,0}(\mathbb{R}^{n-1} \times \overline{\mathbb{R}}^n_+, \mathcal{H}^+)$ if considered parameter-dependent, cf. [12, Proposition 2.3.14]. Moreover, every parameter-independent differential trace symbol is a trace symbol of the same order and class and with regularity ∞.

Remark 4.7 Freezing x', the boundary symbol operator $a(x',\xi',\mu,D_n)$ belongs to the class studied in [12]. Thus, if we assume that the truncated pseudodifferential operator in $a(x',\xi',\mu,D_n)$ is actually a differential operator, the composition of $a(x',\xi',\mu,D_n)$ with a second boundary symbol operator $a'(x',\xi',\mu,D_n)$ is a boundary symbol operator of order $d'' = d + d'$, class $r'' = [r + d']_+$, and regularity $\nu'' = \min\{\nu,\nu',\nu+\nu'\}$, cf. [12, Theorem 2.6.1]. Here d,r,ν (resp. d',r',ν') denote the order, class, and regularity of a (resp. a').

Moreover, we note that a symbol f is in one of the pseudodifferential, Poisson, trace, or singular Green symbol classes $C^{0,1}S^{d,\nu}_{1,0}$ iff the symbol $f(x',.)$ with frozen $x' \in \mathbb{R}^{n-1}$ is in the corresponding smooth class $S^{d,\nu}_{1,0}$ and the semi-norms satisfy

$$|f(x',.)|^{(d,\nu)}_k \leq C_k \qquad |f(x',.) - f(y',.)|^{(d,\nu)}_k \leq C'_k|x'-y'|$$

uniformly in $x', y' \in \mathbb{R}^{n-1}$ and for all $k \in \mathbb{N}$.

Since composition of boundary symbol operators is continuous with respect to the semi-norms we have proved that

$$a(x',\xi',D_n) \circ a'(x',\xi',D_n) = a''(x',\xi',D_n),$$

where a'' is a non-smooth Green symbol of order d'', class r'', and regularity ν'', defined above, with coefficients in $C^{0,1}$.

Finally, we note that in our cases the regularities ν, ν' will be positive. Hence the composition will have regularity $\nu'' = \min\{\nu, \nu'\} > 0$, which is essential for the parametrix construction.

THEOREM 4.8 *Let $1 < q < \infty$.*

1. *If $k \in C^{0,1}S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{H}^+)$, $d \in \mathbb{R}$, is a Poisson symbol of order d, then*

$$k(x', D_x) \colon B_q^{d-\frac{1}{q}}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^n_+) \quad ,$$
$$k(D_x, x') \colon B_q^{d-\frac{1}{q}}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^n_+) \quad \text{if } \left| d - \frac{1}{q} \right| < 1, \text{ resp.}$$

   are continuous operators.

2. *Let $t \in C^{0,1}S^{d}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{H}^-_{-1})$, $d \in \mathbb{R}$, be a trace operator of order d and class 0. Then $t(x', D_x)$ and $t(D_x, x')$ extend to bounded operators*

$$t(x', D_x) \colon L^q(\mathbb{R}^n_+) \to B_q^{-d-\frac{1}{q}}(\mathbb{R}^{n-1}) \qquad \text{if } \left| d + \frac{1}{q} \right| < 1,$$
$$t(D_x, x') \colon L^q(\mathbb{R}^n_+) \to B_q^{-d-\frac{1}{q}}(\mathbb{R}^{n-1}), \text{ resp.}$$

3. *Let $g \in C^{0,1}S^{-m-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n}, \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$, $m \in \mathbb{N}_0$, be a singular Green operator of order $-m$ and class 0. Then $g(D_x, x')$ extends to a bounded operator*

$$g(D_x, x') \colon L^q(\mathbb{R}^n_+) \to W^m_q(\mathbb{R}^n_+).$$

All operators depend continuously on the symbols with respect to the operator norm and the symbol semi-norms.

Proof: The proof is carried out with the same method as in [14, Section 4.1] using the interpolation inclusions of Lemma 2.1

1. First let $1 < q \leq 2$. Then (4.2) and Theorem 3.2 imply

$$k(x', D_x), k(D_x, x') \colon H_q^{d-\frac{1}{2}-\delta}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^{n-1}; L^2(\mathbb{R}_+, x_n^{2\delta})) \qquad \delta \geq 0 \quad (4.6)$$

   under the restriction $|d - \frac{1}{2} - \delta| < 1$ for the operator in x-form. Hence interpolation with different values of δ and Lemma 2.1 yield the first statement in the case $1 < q \leq 2$. The case $2 \leq q < \infty$ is proved in the same way using (4.3) instead of (4.2).

2. The mapping properties of $t(x', D_x)$ and $t(D_x, x')$ can be obtained by duality using (4.5) and the analogous statement for $t(D_x, x')$ and $k(x', D_x)$.

3. We can assume w.l.o.g. $m = 0$. In the same way as in Remark 4.2, one can obtain the following estimates by interpolation of the singular Green symbol estimates in (4.3):

$$\begin{aligned} \|D_{\xi'}^{\alpha'} g(.,\xi',D_n)\|_{C_*^\tau(\mathbb{R}^{n-1};\mathcal{L}(L^2(\mathbb{R}_+,x_n^{-2\delta}),H_2^\delta(\mathbb{R}_+)))} &\leq C_{\alpha',\delta}\langle\xi'\rangle^{-|\alpha'|}, \\ \|D_{\xi'}^{\alpha'} g(.,\xi',D_n)\|_{C_*^\tau(\mathbb{R}^{n-1};\mathcal{L}(H_{2;0}^{-\delta}(\mathbb{R}_+),L^2(\mathbb{R}_+,x_n^{2\delta})))} &\leq C_{\alpha',\delta}\langle\xi'\rangle^{-|\alpha'|} \end{aligned}$$

for all $\delta \geq 0$. Hence application of Theorem 3.2 and Lemma 2.1 proves the last part of the lemma.

■

Remark 4.9 Since multiplication of a Poisson symbol-kernel $\tilde{k}(x',\xi',x_n)$ with x_n reduces the order by 1, cf. (4.1), it is a consequence of the latter theorem that

$$k(D_x,x')\colon B_q^{s-\frac{1}{q}}(\mathbb{R}^{n-1}) \to W_q^m(\mathbb{R}^n \times (\varepsilon,\infty))$$

for all $s \in \mathbb{R}$ with $s - \frac{1}{q} > -1$, $m \in \mathbb{N}_0$, and $\varepsilon > 0$.

Moreover, using $L^2(0,b,x_n^{2\delta}) \hookrightarrow L^1(0,b)$ for $\delta < \frac{1}{2}$ and (4.6),

$$k(D_x,x')\colon B_q^{d-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^{n-1},L^1(0,b)) \quad \text{if } \left|d - \frac{1}{q} - \varepsilon\right| < 1,$$

for all $0 < \varepsilon < \frac{1}{q'}$ and all $b \in \mathbb{R}_+$.

The following lemma summarizes the results concerning composition of non-smooth pseudodifferential operators which we need in Section 5.

Lemma 4.10 *Let $1 < q < \infty$, $d_1 \in \mathbb{N}_0$, and $r \in \mathbb{N}_0$. Moreover, let $p_1(x',D_x)$ be a differential operator and let $t(x',D_x)$ be a differential trace operator both of order d_1 with $C^{0,1}$-coefficients and of class r.*

1. *Let $k(x',\xi) \in C^{0,1}S_{1,0}^{d_2-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{H}^+)$, $d_2 \in \mathbb{R}$. If $|d_1+d_2-\frac{1}{q}| < 1$, then there are $\varepsilon > 0$ such that*

   $$p_1(x',D_x)k(D_x,x') - (p_1k)(D_x,x')\colon B_q^{d_1+d_2-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^n_+).$$

   Moreover, if $s \in (-1,1)$ such that $|s+d_1+d_2| < 1$, then

   $$t(x',D_x)k(D_x,x') - (tk)(D_x,x')\colon B_q^{s+d_1+d_2-\varepsilon}(\mathbb{R}^{n-1}) \to B_q^s(\mathbb{R}^{n-1})$$

   for an $\varepsilon > 0$.

2. *Let $g(x',\xi) \in C^{0,1}S_{1,0}^{-d_1-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$. Then*

   $$p_1(x',D_x)g(D_x,x') - (p_1g)(D_x,x')\colon L^q(\mathbb{R}^n_+) \to L^q(\mathbb{R}^n_+)$$

   with operator norm bounded by $C|p_1|_k^{(d_1)}|g|_k^{(-d_1-1+\varepsilon)}$ for some $\varepsilon, C > 0$, $k \in \mathbb{N}$.

3. *Let* $p_2(x,\xi)\in C^{0,1}S^{-d_1}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$. *Then*

$$p_1(x',D_x)p_2(D_x,x)_+-(p_1p_2)(D_x,x)_+\colon L^q(\mathbb{R}^n_+)\to L^q(\mathbb{R}^n_+)$$

with operator-norm bounded by $C|p_1|^{(d_1)}_k|p_2|^{(-d_1+\varepsilon)}_k$ *for some* $\varepsilon,C>0$, $k\in\mathbb{N}_0$.

4. *If* $k\in C^{0,1}S^{-d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{H}^+)$, $a\in C^{0,1}(\mathbb{R}^{n-1})$, *and* $t\in C^{0,1}S^d_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{H}^-_{-1})$, $d\in\mathbb{R}$, *with* $|d+\frac{1}{q}|<1$, *then*

$$k(D_x,x')t(D_x,x')-(kt)(D_x,x')\colon L^q(\mathbb{R}^n_+)\to L^q(\mathbb{R}^n_+),$$
$$a(x')t(D_x,x')-(at)(D_x,x')\colon L^q(\mathbb{R}^n_+)\to W_q^{-d-\frac{1}{q}}(\mathbb{R}^{n-1}),$$

where the operator norm is bounded by $C|k|^{(-d-1)}_l|t|^{(d+\varepsilon)}_l$ *and* $C\|a\|_{C^{0,1}}|t|^{(d+\varepsilon)}_l$, *resp., for some* $\varepsilon,C>0$, $l\in\mathbb{N}$.

Proof: We assume w.l.o.g. $d_1=0$. Then $p_1(x',D_x)=a(x')$ and $t(x',D_x)=a(x')\gamma_0$ for an $a\in C^{0,1}(\mathbb{R}^{n-1})$. Moreover, we only give the details for $1<q\le 2$ since the case $q\ge 2$ is treated in the same way.

1. Since $k(x',\xi',D_n)\in C^{0,1}S^{d_2-\frac{1}{2}-\delta}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(\mathbb{C},L^2(\mathbb{R}_+,x_n^\delta))$ for $0\le\delta<\frac{1}{2}$, cf. (4.2), we get by Corollary 3.4 that

   $$a(x')k(D_x,x')-(ak)(D_x,x')\colon B_q^{d_2-\frac{1}{2}-\delta-\varepsilon}(\mathbb{R}^{n-1})\to L^q(\mathbb{R}^{n-1},L^2(\mathbb{R}_+,x_n^\delta))$$

   for $\varepsilon>0$ (depending on δ). Hence

   $$a(x')k(D_x,x')-(ak)(D_x,x')\colon B_q^{d_2-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1})\to L^q(\mathbb{R}^n_+)$$

   by Lemma 2.1 for $\varepsilon>0$. Moreover, because of (4.3) and $\|f\|_\infty\le\|f\|_2^{\frac{1}{2}}\|f'\|_2^{\frac{1}{2}}$ for every $f\in\mathcal{S}(\overline{\mathbb{R}}_+)$, $\gamma_0k(x',\xi',D_n)\in S^{d_2}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Hence

   $$a(x')\gamma_0k(D_x,x')-(a\gamma_0k)(D_x,x')\colon B_q^{s+d_2-\varepsilon}(\mathbb{R}^{n-1})\to B^s_q(\mathbb{R}^{n-1})$$

   for an $\varepsilon>0$ because of Corollary 3.4 and (2.1).

2. Since $d_2=-d_1=0$, we look at $g(D_x,x')$ as a singular Green operator of order $0<\varepsilon\le\frac{1}{2}$. Hence $g(x',\xi',D_n)\in C^{0,1}S^\varepsilon_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{L}(H_2^{-\delta}(\mathbb{R}_+),L^2(\mathbb{R}_+,x_n^\delta)))$ for $0\le\delta<\frac{1}{2}$, cf. proof of Theorem 4.8.3. Therefore

   $$a(x')g(D_x,x')-(ag)(D_x,x')\colon L^q(\mathbb{R}^n_+)\to L^q(\mathbb{R}^n_+)$$

   with operator norm bounded by $C\|a\|_{0,1}|g|^{(-1+\varepsilon)}_k$ for $\varepsilon\in(0,\frac{1}{2}]$.

3./4. The last two parts of the lemma are proved analogously.

■

5 Parametrix Construction and H_∞-Calculus

5.1 The Model Operators of the Reduced Stokes Equations in $\mathbb{R}^n_+$

In this section we discuss the structure of the boundary symbol operators of the reduced Stokes equations in $\mathbb{R}^n_+$ – the "model operators".

For $\lambda = \mu^2 e^{i\theta}$, $\mu \geq 0$, $\theta \in (-\pi, \pi)$, let

$$a_j^r(\xi', \mu, D_n) = \begin{pmatrix} \mu^2 e^{i\theta} + |\xi'|^2 + D_n^2 + k_j^r(\xi', D_n) t_j^r(\xi', D_n) \\ t_j'(\xi', D_n) \end{pmatrix},$$

$j = 0, 1$, $\theta \in (-\pi, \pi)$, be the model operator of the reduced Stokes equations, where

$$k_0^r(\xi', D_n)u = -e^{-[\xi']x_n} \begin{pmatrix} \frac{i\xi'}{[\xi']} \\ -1 \end{pmatrix} i\xi'^T, \quad k_1^r(\xi', D_n)u = e^{-[\xi']x_n} \begin{pmatrix} i\xi' \\ -[\xi'] \end{pmatrix},$$

$$t_0^r(\xi', D_n)u = \partial_n u'(0), \qquad t_1^r(\xi', D_n)u = 2\partial_n u_n(0),$$

$$t_0'(\xi', D_n)u = \gamma_0 u, \qquad t_1'(\xi', D_n)u, = \begin{pmatrix} i\xi' u_n(0) + \partial_n u'(0) \\ i\xi' \cdot u'(0) + \partial_n u_n(0) \end{pmatrix}.$$

Here [.] denotes a smooth function with $[\xi'] = |\xi'|$ if $|\xi'| \geq 1$ and $[\xi'] \geq \frac{1}{2}$ if $|\xi'| < 1$.

In [15, Theorem 6.1] it was shown that the system of the reduced Stokes equations is parameter-elliptic for arbitrary $\theta \in (-\pi, \pi)$, see [12, Definition 3.1.2.] for the definition of parameter-ellipticity. This result implies:

Lemma 5.1 *Let $\theta \in (-\pi, \pi)$ and let $a_j^r(\xi', \mu, D_n)$, $j = 0, 1$, be defined as above. Then there is a $c_0 > 0$ such that*

$$a_j^r(\xi', \mu, D_n) : H_2^2(\mathbb{R}_+)^n \to L^2(\mathbb{R}_+)^n \times \mathbb{C}^n$$

is bijective for all $|(\xi', \mu)| \geq c_0$. Moreover, $a_j^r(\xi', \mu, D_n)^{-1}$ is a boundary symbol operator of order -2, class 0, and regularity $\frac{1}{2}$.

Proof: The first statement is a direct implication of [12, Proposition 3.1.3] and [12, Lemma 3.1.1]. The second statement is a consequence of [12, Theorem 3.2.2]. ■

Remark 5.2 Since $a_j^r(\xi', \mu, D_n)$ depends continuously on $\theta \in (-\pi, \pi)$, there is a constant c_0' such that $a_j(\xi', \mu, D_n)$ is invertible for all $|(\xi', \mu)| \geq c_0'$ and $\theta \in [-\delta, \delta]$ for every fixed $\delta \in (0, \pi)$.

Moreover, let

$$a_j(\xi', \mu, D_n) = \begin{pmatrix} \mu^2 e^{i\theta} + |\xi'|^2 + D_n^2 \\ \gamma_j \end{pmatrix} : H_2^2(\mathbb{R}_+) \to \begin{matrix} L^2(\mathbb{R}_+) \\ \times \\ \mathbb{C} \end{matrix}$$

$j=0,1,\ \theta\in(-\pi,\pi)$, be the model operator of the Laplace resolvent with Dirichlet or Neumann boundary condition. It is well-known that $a_j(\xi',\mu,D_n)$ is bijective for all $(\xi',\mu)\in\overline{\mathbb{R}}^n_+\setminus\{0\}$ and that

$$
\begin{aligned}
a_j^{-1}(\xi',\mu,D_n) &= \begin{pmatrix} r_j(\xi',\mu,D_n) & k_j(\xi',\mu,D_n) \end{pmatrix}, && (5.1)\\
r_j(\xi',\mu,D_n) &= p(\xi',\mu,D_n)_+ - k_j(\xi',\mu,D_n)t_j(\xi',\mu,D_n), && (5.2)\\
p(\xi,\mu) &= (e^{i\theta}\mu^2+|\xi|^2)^{-1}, && (5.3)\\
k_j(\xi',\mu,D_n)g &= \frac{e^{-\zeta_\mu x_n}}{\zeta_\mu{}^j}g,\quad t_j(\xi',\mu,D_n)f=\int_0^\infty \frac{(-1)^j e^{-\zeta_\mu y_n}}{2\zeta_\mu{}^{1-j}}f(y_n)dy_n, && (5.4)
\end{aligned}
$$

where $\zeta_\mu=(e^{i\theta}\mu^2+|\xi'|^2)^{\frac{1}{2}}$. As usual, we obtain a boundary symbol operator of order -2, class 0, and regularity ∞ if we smooth the symbols of $a_j^{-1}(\xi',\mu,D_n)$ for $|(\xi',\mu)|\le 1$. The smoothed operator will again be denoted by $a_j^{-1}(\xi',\mu,D_n)$. Moreover, we use the convention $a_{j,\lambda}(\xi',D_n)=a_j(\xi',\mu,D_n)$, $p_\lambda(\xi)=p(\xi,\mu)$ etc., where $\lambda=\mu^2e^{i\theta}$.

We can consider the model operator of the reduced Stokes resolvent equation as perturbation of $a_j(\xi',\mu,D_n)$:

$$
a_j^r(\xi',\mu,D_n) = a_j(\xi',\mu,D_n)+b_j(\xi',\mu,D_n),
$$

where

$$
b_j(\xi',\mu,D_n)=\begin{pmatrix} k_j^r(\xi',D_n)t_j^r(\xi',D_n)\\ t_j''(\xi',D_n)\end{pmatrix},\quad t_1''(\xi',D_n)u,=\begin{pmatrix} i\xi' u_n(0)\\ i\xi'\cdot u'(0)\end{pmatrix},
$$

and $t_0''(\xi',D_n)u=0$. As in [13, Section 3] and [2, Section 4.2], we get by an elementary calculation

$$
a_j^{r,-1}(\xi',\mu,D_n) = (I+a_j^{-1}(\xi',\mu,D_n)b_j(\xi',\mu,D_n))^{-1}a_j^{-1}(\xi',\mu,D_n),
$$

where

$$
\begin{aligned}
\left(I+a_0^{-1}(\xi',\mu,D_n)b_0(\xi',\mu,D_n)\right)^{-1} &= I-r_0(\xi',\mu,D_n)k_0^r(\xi',D_n)s_0(\xi',\mu)t_0^r(\xi',D_n),\\
s_0(\xi',\mu) &= (I+t_0^r(\xi',D_n)r_0(\xi',\mu,D_n)k_0^r(\xi',D_n))^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&\left(I+a_1^{-1}(\xi',\mu,D_n)b_1(\xi',\mu,D_n)\right)^{-1}\\
&\quad= I-(r_1(\xi',\mu,D_n)k_1^r(\xi',D_n),k_1(\xi',\mu,D_n))\,s_1(\xi',\mu)\begin{pmatrix} t_1^r(\xi',D_n)\\ t_1''(\xi',\mu,D_n)\end{pmatrix},\\
s_1(\xi',\mu) &= \left[I+\begin{pmatrix} t_1^r(.,D_n)\\ t_1''(.,D_n)\end{pmatrix}(r_1(.,D_n)k_1^r(.,D_n),k_1(.,D_n))\right]^{-1}.
\end{aligned}
$$

In view of the composition rules, $s_j\in S^{0,\frac{1}{2}}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n_+)\otimes\mathcal{L}(\mathbb{C}^N)$ with $N=n-1$ if $j=0$ and $N=2$ if $j=1$. Hence we obtain:

Lemma 5.3 *Let $a_j^r(\xi',\mu,D_n), a_j(\xi',\mu,D_n)$, $j=0,1$, be defined as above and $|(\xi',\mu)| \geq \max\{c_0,1\} > 0$, where c_0 is the same constant as in Lemma 5.1. Then*

$$a_j^{r,-1}(\xi',\mu,D_n) \quad = \quad a_j^{-1}(\xi',\mu,D_n) - b_j'(\xi',\mu,D_n) \tag{5.5}$$

with

$$b_0'(.,D_n) = r_0(.,D_n)k_0^r(.,D_n)s_0(.)t_0^r(.,D_n)\left(r_0(.,D_n),k_0(.,D_n)\right), \tag{5.6}$$

$$b_1'(.,D_n) = \left(r_1(.,D_n)k_1^r(.,D_n),k_1(.,D_n)\right)s_1(.)\begin{pmatrix} t_1^r(.,D_n) \\ t_1''(.,D_n)\end{pmatrix}\left(r_1(.,D_n),k_1(.,D_n)\right). \tag{5.7}$$

5.2 Coordinate Transformation

In this section we calculate the principal symbols of the operators in the reduced Stokes equations for the curved half-space $\mathbb{R}^n_\gamma$ after coordinate transformation to $\mathbb{R}^n_+$. The principal rule is that if $a(\xi)$ is the symbol of the corresponding operator in $\mathbb{R}^n_+$, then

$$\underline{a}(x',\xi) := a(A(x')\xi), \qquad x' \in \mathbb{R}^{n-1}, \xi \in \mathbb{R}^n, \tag{5.8}$$

is the principal symbol for the curved half-space, where $A(\underline{x}')$ depends on $\nabla'\gamma \in C^{0,1}(\mathbb{R}^{n-1})$, cf. Section 5.3 below.

Lemma 5.4 *Let $p(\xi,\mu) \in S^{m,\nu}_{1,0}(\mathbb{R}^n \times \overline{\mathbb{R}}^{n+1}_+)$, $m,\nu \in \mathbb{R}$, and $A \in C^{0,1}(\mathbb{R}^n)^{n\times n}$ with $A^{-1} \in C^{0,1}(\mathbb{R}^n)^{n\times n}$. Then $q(x,\xi,\mu) := p(A(x)\xi,\mu) \in C^{0,1}S^{m,\nu}_{1,0}(\mathbb{R}^n \times \overline{\mathbb{R}}^{n+1}_+)$, and for every $k \in \mathbb{N}_0$ there is a $k' \in \mathbb{N}_0$ such that $|q|_k^{(m,\nu)} \leq C|p|_{k'}^{(m,\nu)}$, where C depends only on $\|A\|_{C^{0,1}}, \|A^{-1}\|_{C^{0,1}}, k, m, \nu$, and n.*

Proof: The proof is carried out in a straightforward manner using

$$\begin{aligned} & p(A(x)\xi,\mu) - p(A(y)\xi,\mu) \\ & = \int_0^1 \nabla_\xi p(tA(x)\xi + (1-t)A(y)\xi,\mu)dt \cdot (A(x)-A(y))\xi, \end{aligned} \tag{5.9}$$

where $tA(x)+(1-t)A(y)$ is invertible for all $t\in[0,1]$ if $|A(x)-A(y)| \leq (2\|A^{-1}\|_\infty)^{-1}$. ■

The analogous statement for Poisson, trace, and singular Green symbols is as follows:

Lemma 5.5 *Let $f(\xi,\mu) \in S^{m,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+,\mathcal{K})$, $m,\nu\in\mathbb{R}$, $r\in\mathbb{N}_0$, where $\mathcal{K}=\mathcal{H}^+$ or $\mathcal{K}=\mathcal{H}^-_{r-1}$. Moreover, let $A(x') \in C^{0,1}(\mathbb{R}^{n-1})^{n\times n}$ such that $A^{-1}(x') \in C^{0,1}(\mathbb{R}^{n-1})^{n\times n}$ and A possesses the block structure*

$$A(x') = \begin{pmatrix} A'(x') & 0 \\ b^T(x') & c(x') \end{pmatrix} \qquad \text{with } c(x') > 0. \tag{5.10}$$

Then $g(x',\xi,\mu) := f(A(x')\xi,\mu) \in C^{0,1}S^{m,\nu}_{1,0}(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+,\mathcal{K})$*, and for every* $k\in\mathbb{N}_0$ *there is some* $k'\in\mathbb{N}_0$ *such that* $|g|^{(m,\nu)}_k \leq C(\|A\|_{C^{0,1}},\|A^{-1}\|_{C^{0,1}})|f|^{(m,\nu)}_{k'}$*. The same statement is true if we set* $\mathcal{K}=\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$ *and set* $g(x',\xi,\eta_n,\mu) := f(A(x')\xi, c(x')\eta_n + b^T(x')\xi',\mu)$*.*

Proof: Note that $h_{-1}f(ct+d) = (h_{-1}f)(ct+d)$ if $f\in\mathcal{H}$ and $c>0$. Since $c(x'), c^{-1}(x')$, and $b(x')$ are uniformly bounded and $(c\xi_n)^k = \sum_{l=0}^k \binom{k}{l}(c\xi_n + b^T\xi')^l(-b^T\xi')^{k-l}$, we get

$$\begin{aligned} &\|h_{-1}\xi_n^k\partial_{\xi_n}^{k'}(f(c\xi_n+b^T\xi'))\|_{L^2(\mathbb{R})} \\ \leq\;& C\left(\sum_{l=0}^k\binom{k}{l}|b^T\xi'|^{k-l}\|h_{-1}(c\xi_n+b^T\xi')^l(\partial_{\xi_n}^{k'}f)(c\xi_n+b^T\xi')\|_{L^2(\mathbb{R})}\right) \\ \leq\;& C\left(\sum_{l=0}^k\langle\xi'\rangle^{k-l}\|h_{-1}\xi_n^l\partial_{\xi_n}^{k'}f(\xi_n)\|_{L^2(\mathbb{R})}\right) \end{aligned} \tag{5.11}$$

for every $f\in\mathcal{H}_{-1}$, where the constant C depends only on the bounds of $c(x')$, $c^{-1}(x')$, and $b(x')$.

Now let $\alpha'\in\mathbb{N}_0^{n-1}$, $k,k',j\in\mathbb{N}_0$. If we set $\alpha=(\alpha',k')$, we have

$$D^\alpha_\xi D^j_\mu g(x',\xi,\mu) = \left((A^T(x')D_\xi)^\alpha D^j_\mu f\right)(x',A(x')\xi,\mu).$$

Combining this identity with (5.11), we conclude

$$\|h_{-1,\xi_n}\xi_n^k D^{k'}_{\xi_n}D^{\alpha'}_{\xi}D^j_\mu g(x',\xi,\mu)\|_{L^2(\mathbb{R})} \leq C(\rho(\xi',\mu)^{\nu-[k'-k]_+}+1)\langle\xi',\mu\rangle^{m-|\alpha'|-k'+k-j}.$$

In order to estimate $g(x',\xi,\mu)-g(y',\xi,\mu)$, we use an analogous identity to (5.9). Furthermore, the case $\mathcal{K}=\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$ is proved in the same way. ■

We note that the block structure (5.10) in the previous lemma is essential in order to guarantee that $g(x',\xi)\in\mathcal{K}$ with respect to ξ_n.

Finally, we have to analyze how the transformed boundary symbol operators behave under composition. Let $\kappa_{c,d}(\xi_n) := c\xi_n + d$, $\xi_n\in\mathbb{R}$, with $c>0, d\in\mathbb{R}$, and $\kappa^*_{c,d}(f)(\xi_n) = f(\kappa_{c,d}(\xi_n))$ for $f\in\mathcal{H}$. Then $\kappa^*_{c,d}$ is an algebra homomorphism, which maps $\mathcal{H}^+$, $\mathcal{H}^-_r$, and $\mathcal{H}_r$ into itself and for which

$$h^+\kappa^*_{c,d} = \kappa^*_{c,d}h^+, \qquad c\int^+\kappa^*_{c,d}(f)(\xi_n)\,d\xi_n = \int^+ f(\xi_n)\,d\xi_n \tag{5.12}$$

hold. Moreover, we set $\kappa^*_{c,d}(g)(\xi_n,\eta_n) = g(\kappa_{c,d}(\xi_n),\kappa_{c,d}(\eta_n))$ for $g\in\mathcal{H}^+_{\xi_n}\hat{\otimes}\mathcal{H}^-_{-1,\eta_n}$ and $\kappa^*_{c,d}(f,b) = (\kappa^*_{c,d}(f),b)$ for $(f,b)\in(\mathcal{H}^+\otimes\mathbb{C}^M)\times\mathbb{C}^N$.

Now let $a(D_n)$ be a one-dimensional (x_n-independent) Green operator and $p(\xi_n)$, $k(\xi_n)$, $t(\xi_n)$, $g(\xi_n,\eta_n)$, and s be its symbols. Then we define

$$a(cD_n+d) := \mathrm{OP}_n\begin{pmatrix}\kappa^*_{c,d}(p)_+ + c\kappa^*_{c,d}(g) & \kappa^*_{c,d}(k)\\ c\kappa^*_{c,d}(t) & s\end{pmatrix}. \tag{5.13}$$

Because of the composition rules for one-dimensional Green operators, cf. [6, Theorem 1.12] and (5.12),

$$a_1(cD_n + d) \circ a_2(cD_n + d) = (a_1 \circ a_2)(cD_n + d), \tag{5.14}$$

where $a_1 \circ a_2$ denotes the symbol of $a_1(D_n) \circ a_2(D_n)$. – Note that the factors c in (5.13) are necessary to obtain (5.14).

5.3 Symbols of the Reduced Stokes Equations in $\mathbb{R}^n_\gamma$

In contrast to [8, 9], we use a very simple coordinate transformation, which allows us to construct the parametrix in a domain with $C^{1,1}$-boundary, but does not preserve the normal direction on the boundary. Therefore we have to analyse the relation between the model operators and the Green operator of the transformed equations carefully.

Given $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$ let $\mathbb{R}^n_\gamma = \{x : x_n > \gamma(x')\}$ be a curved half-space, and let $F : \mathbb{R}^n_+ \to \mathbb{R}^n_\gamma$ be the coordinate transformation

$$x = F(\underline{x}) = \begin{pmatrix} \underline{x}' \\ \underline{x}_n + \gamma(\underline{x}') \end{pmatrix}, \qquad \underline{x} \in \mathbb{R}^n_+.$$

In this section we will denote the variables and operators corresponding to the original problem in $\mathbb{R}^n_\gamma$ by $x, \xi, \nabla, \ldots$ and of the transformed problem in $\mathbb{R}^n_+$ by $\underline{x}, \underline{\xi}, \underline{\nabla}, \ldots$. Similarly, $\underline{a}(\underline{x}', \underline{\xi})$ will indicate the symbols of the transformed problem and $a(\xi)$ the symbols of the model operator.

If $v\colon \mathbb{R}^n_\gamma \to \mathbb{C}$, we set $F^*(v)(\underline{x}) = v(F(\underline{x}))$. Moreover, let $F_0\colon \mathbb{R}^{n-1} \to \partial\mathbb{R}^n_\gamma : \underline{x}' \mapsto (\underline{x}', \gamma(\underline{x}'))$ and $F_0^*(v)(\underline{x}') = v(F_0(\underline{x}'))$. Furthermore, let $U = U(\underline{x}')$ be an orthonormal matrix which maps the exterior normal vector

$$\nu(\underline{x}') = \frac{1}{\sqrt{1 + |\nabla'\gamma(\underline{x}')|^2}} \begin{pmatrix} \nabla'\gamma(\underline{x}') \\ -1 \end{pmatrix}$$

on $\partial\mathbb{R}^n_\gamma$ at the point $(\underline{x}', \gamma(\underline{x}'))$ to $-e_n$. We need this orthonormal matrix to correct vector fields in such a way that the normal direction ν on $\partial\mathbb{R}^n_\gamma$ is mapped to the normal direction $-e_n$ on $\partial\mathbb{R}^n_+$. This modification is essential for preserving the structure of the boundary conditions and model operators.

Using this notation,

$$\nabla F^{*,-1} v = F^{*,-1}\,\mathrm{OP}(U^T(\underline{x}')A(\underline{x}')i\underline{\xi})v = F^{*,-1}U^T(\underline{x}')A(\underline{x}')\nabla v,$$

where $A(\underline{x}')\underline{\xi} = U_{\underline{x}'}(D_{\underline{x}}F(\underline{x}))^{-T}\underline{\xi}$ and $v \in C^1(\overline{\mathbb{R}}^n_\gamma)$. Then A has the structure needed to apply Lemma 5.5

$$A(\underline{x}')\underline{\xi} = U_{\underline{x}'} \begin{pmatrix} I' & -\nabla'\gamma(\underline{x}') \\ 0 & 1 \end{pmatrix} \underline{\xi} = \begin{pmatrix} A'(\underline{x}') & 0 \\ b(\underline{x}')^T & c(\underline{x}') \end{pmatrix} \underline{\xi}, \tag{5.15}$$

where $A'(\underline{x}')$ and $b(\underline{x}')$ depend smoothly on $\nabla'\gamma(\underline{x}')$ and $c(\underline{x}') = \sqrt{1 + |\nabla'\gamma(\underline{x}')|^2}$.

Lemma 5.6 *Let $v \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_\gamma)$ and $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_\gamma)^n$. Then*

$$F^*\nabla v = \underline{\nabla} F^* v, \quad F^* \operatorname{div} u = \underline{\operatorname{div}} F^* u, \quad F^*\Delta u = \underline{\Delta} F^* u + R_1 F^* u,$$
$$F_0^*\gamma_\nu u = \underline{\gamma_\nu} F^* u, \quad F_0^*\gamma_1 v = \underline{\gamma_1} F^* v, \quad F_0^* T_1' u = \underline{t}_1'(\underline{x}', D_x) F^* u,$$

where

1. $\underline{\nabla} = \mathrm{OP}(U^T(\underline{x}')A(\underline{x}')i\underline{\xi})$, $\underline{\operatorname{div}}\, u = \mathrm{OP}((A(\underline{x}')i\underline{\xi})^T U(\underline{x}'))u$, $\underline{\Delta} = -\mathrm{OP}(|A(\underline{x}')\underline{\xi}|^2)$, $\underline{\gamma_\nu} = -e_n \cdot \gamma_0 U(x')$, *and* $\underline{\gamma_1} = \underline{\gamma_\nu}\underline{\nabla} = -\gamma_0 \,\mathrm{OP}((A(\underline{x}')i\underline{\xi})_n)$.
2. R_1 *is a differential operator of order 1 with L^∞-coefficients.*
3. $\underline{t}_1'(\underline{x}', \underline{\xi}', D_n)u = -\gamma_0 U^T(\underline{x}') \begin{pmatrix} (A(\underline{x}')i\underline{\xi})_n I' & A'(\underline{x}')i\underline{\xi}' \\ (A'(\underline{x}')i\underline{\xi}')^T & (A(\underline{x}')i\underline{\xi})_n \end{pmatrix} U(\underline{x}')u.$

If additionally $\gamma_0 u = 0$, then

$$F_0^*\gamma_\nu(\Delta - \nabla \operatorname{div})u = \underline{t}(\underline{x}', D_x)F^* u + R_2 F^* u,$$

where $\underline{t}(\underline{x}', \xi', D_n)u = -(A'(\underline{x}')i\xi')^T \underline{\gamma_1}(\underline{x}', \underline{\xi}', D_n)(U(\underline{x}')u)'$ and $R_2\colon W^{2-\varepsilon}_q(\mathbb{R}^n_+)^n \to W_q^{-\frac{1}{q}}(\mathbb{R}^{n-1})$ is a bounded linear operator for every $\varepsilon \in (0, \frac{1}{q'})$.

Proof: The identities can be checked by elementary calculations. We only give the details for the last statement. If $\gamma_0 u = 0$, then

$$\begin{aligned} F_0^*\gamma_\nu(\Delta - \nabla \operatorname{div})u &= \underline{\gamma_\nu}(\underline{\Delta} - \underline{\nabla}\,\underline{\operatorname{div}})F^*u + \gamma_0 R_1 F^* u \\ &= -e_n \cdot \gamma_0 \,\mathrm{OP}((-|A(\underline{x}')\underline{\xi}|^2 + A(\underline{x}')i\underline{\xi}(A(\underline{x}')i\underline{\xi})^T)U(\underline{x}'))F^*u + \gamma_0 R' F^* u \\ &= \gamma_0 \,\mathrm{OP}((A(\underline{x}')i\underline{\xi})_n((A'(\underline{x}')i\underline{\xi}')^T, 0)U(\underline{x}'))F^*u + \gamma_0 R' F^* u, \end{aligned}$$

where R' is a differential operator of order 1 with L^∞-coefficients depending on $\underline{x}'$. Hence, if $\varepsilon \in (0, \frac{1}{q'})$,

$$\|\gamma_0 R' F^* u\|_{W_q^{-\frac{1}{q}}(\mathbb{R}^{n-1})} \le \|\gamma_0 R' F^* u\|_{L^q(\mathbb{R}^{n-1})} \le C\|\gamma_0 \nabla F^* u\|_{L^q(\mathbb{R}^{n-1})} \le C\|F^* u\|_{W_q^{2-\varepsilon}(\mathbb{R}^n_+)}.$$

■

Thus the coordinate transformation acts on the principal symbol as

$$a(\xi) \rightsquigarrow \underline{a}(\underline{x}', \underline{\xi}) = a(A(\underline{x}')\underline{\xi})$$

with an additional factor $U^T(\underline{x}')$ on the left if the range of the operator consists of vector fields and additional factor $U(\underline{x}')$ on the right if the domain of the operator consists of vector fields. Therefore we can express principal boundary symbol operators of the equations after coordinate transformation with the aid of model operators; e.g.

$$\underline{a}_{j,\lambda}(\underline{x}', \underline{\xi}', D_n) \;:=\; \begin{pmatrix} \lambda - \underline{\Delta} \\ \underline{\gamma_j}(\xi', D_n) \end{pmatrix} = \operatorname{diag}(1, c^{-1}) a_{j,\lambda}(A'(\underline{x}')\underline{\xi}', cD_n + b^T\underline{\xi}'),$$

where we need the correction factor c^{-1} since

$$\gamma_j(.,cD_n+d) = \mathrm{OP}_n(c\gamma_j(.,c\xi_n+d)) \tag{5.16}$$

due to (5.13). Note that the symbol of the trace operator $\gamma_j u = \partial_n^j u|_{x_n=0}$ is $\gamma_j(\xi) = (i\xi_n)^j$.

Because of (5.14), the inverse of $\underline{a}_{j,\lambda}(\underline{x}',\underline{\xi}',D_n)$ exists for $|\xi'|\geq 1$ and

$$\begin{aligned} \underline{a}_{j,\lambda}(\underline{x}',\underline{\xi}',D_n)^{-1} &= a_{j,\lambda}^{-1}(A'(\underline{x}')\underline{\xi}',cD_n+b^T\underline{\xi}')\,\mathrm{diag}(1,c) \\ &= \begin{pmatrix} r_{j,\lambda}(A'(\underline{x}')\underline{\xi}',cD_n+d) & ck_{j,\lambda}(A'(\underline{x}')\underline{\xi}',cD_n+d)\end{pmatrix}, \end{aligned}$$

cf. (5.1). In particular, we get

$$\underline{k}_j(\underline{x}',\underline{\xi}',D_n) := \underline{k}_{j,\lambda}(\underline{x}',\underline{\xi}',D_n)|_{\lambda=0} = ck_j(A'(\underline{x}')\underline{\xi}',cD_n+b^T\underline{\xi}'), \tag{5.17}$$

where $k_j(\xi',D_n) = k_{j,\lambda}(\xi',D_n)|_{\lambda=0}$. Now we set

$$\underline{a}^r_{j,\lambda}(\underline{x}',\underline{\xi}',D_n) := \underline{a}_{j,\lambda}(\underline{x}',\underline{\xi}',D_n) + \begin{pmatrix} \underline{k}^r_j(\underline{x}',\underline{\xi}',D_n)\underline{t}^r_j(\underline{x}',\underline{\xi}',D_n) \\ \underline{t}''_j(\underline{x}',\underline{\xi}',D_n)\end{pmatrix}, \tag{5.18}$$

where

$$\underline{k}^r_0(\underline{x}',\underline{\xi}',D_n) := -U^T(x')i\begin{pmatrix} A'(\underline{x}')\underline{\xi}' \\ cD_n+d\end{pmatrix}\underline{k}_1(\underline{x}',\underline{\xi}',D_n)(A'(\underline{x}')i\underline{\xi}')^T \tag{5.19}$$

$$\underline{t}^r_0(\underline{x}',\underline{\xi}',D_n)u := \underline{\gamma}_1(\underline{x}',\underline{\xi}',D_n)(U(x')u)', \tag{5.20}$$

$$\underline{k}^r_1(\underline{x}',\underline{\xi}',D_n) := U^T(x')i\begin{pmatrix} A'(\underline{x}')\underline{\xi}' \\ cD_n+d\end{pmatrix}\underline{k}_0(\underline{x}',\underline{\xi}',D_n), \tag{5.21}$$

$$\underline{t}^r_1(\underline{x}',\underline{\xi}',D_n)u := -2\underline{\gamma}_1(\underline{x}',\underline{\xi}',D_n)(U(x')u)_n, \tag{5.22}$$

$$\underline{t}''_1(\underline{x}',\underline{\xi}',D_n)u := -U^T(x')\begin{pmatrix} 0 & \underline{\gamma}_0 A'(\underline{x}')\underline{\xi}' \\ \underline{\gamma}_0(A'(\underline{x}')\underline{\xi}')^T & 0\end{pmatrix}U(x')u, \tag{5.23}$$

and $\underline{t}''_0(\underline{x}',\underline{\xi}',D_n)=0$ with $d=b^T(\underline{x}')\underline{\xi}'$. Because of (5.14), (5.16), and (5.17), it easy to check that

$$\underline{k}^r_j(\underline{x}',\underline{\xi}',D_n)\underline{t}^r_j(\underline{x}',\underline{\xi}',D_n) = U^T(\underline{x}')(k^r_j t^r_j)(A(\underline{x}')\underline{\xi}',cD_n+d)U(\underline{x}') \tag{5.24}$$

for $j=0,1$. Therefore

$$\underline{a}^r_{j,\lambda}(\underline{x}',\underline{\xi}',D_n) = U^T(\underline{x}')\,\mathrm{diag}(1,c^{-1})a^r_{j,\lambda}(A'(\underline{x}')\underline{\xi}',cD_n+d)U(\underline{x}')$$

and

$$\underline{a}^{r,-1}_{j,\lambda}(\underline{x}',\underline{\xi}',D_n) = U^T(\underline{x}')a^{r,-1}_{j,\lambda}(A'(\underline{x}')\underline{\xi}',cD_n+d)\,\mathrm{diag}(1,c)U(\underline{x}'). \tag{5.25}$$

This is the essential formula for the construction of the parametrix.

We have to estimate the semi-norms of the transformed symbols. Because of (5.15) and $\nabla'\gamma\in C^{0,1}(\mathbb{R}^{n-1})$, we have $A(x')$, $A^{-1}(x')\in C^{0,1}(\mathbb{R}^{n-1})$. Thus the same holds for $A'(\underline{x}'), A'(\underline{x}')^{-1}$, $b(\underline{x}')$, $c(\underline{x}')$, and $c^{-1}(\underline{x}')$. Hence we can apply Lemma 5.4 and Lemma 5.5.

Corollary 5.7 *Let* $\underline{a}^r_{j,\lambda}(\underline{x}', \underline{\xi})$, $j = 0, 1$, *be the symbol to the transformed boundary symbol operator of the reduced Stokes equations defined in (5.18). Then* $\underline{a}^r_j(\underline{x}', \underline{\xi}, \mu) := \underline{a}^r_{j,\lambda}(\underline{x}', \underline{\xi})$, $\lambda = e^{i\theta}\mu^2$, *and* $\underline{a}^{r,-1}_j(\underline{x}', \underline{\xi}, \mu)$ *are Green symbols of order* $2, -2$, *respectively, regularity* $\frac{1}{2}$, *and* $C^{0,1}$*-smoothness in* $\underline{x}'$. *Moreover, the semi-norms of the symbols are uniformly bounded in* $\theta \in [-\delta, \delta]$ *for any* $\delta \in (0, \pi)$.

5.4 H_∞-Calculus for the Model Operators

In this section we will prove the basic estimates for the singular Green operators of the parametrix for the reduced Stokes equations on the boundary symbol operator level.

First of all recall that it is sufficient to proof (1.9) for all $h \in H(\delta)$, where $H(\delta)$ consists of all $h \in H_\infty(\delta)$ such that

$$|h(z)| \leq C \frac{|z|^s}{1 + |z|^{2s}}$$

for all $z \in \Sigma_{\pi-\delta}$ and some constants $C, s > 0$, cf. Amann, Hieber, and Simonett [4]. Moreover, since A_q and A_{10} are invertible, it is sufficient to estimate the Cauchy integral (1.8) for $\Gamma_R := \Gamma \setminus B_R(0)$, $R > 0$, instead of Γ.

We first consider the boundary symbol operators of the Laplace resolvent

$$r_{j,\lambda}(\xi', D_n)f \;=\; p_\lambda(\xi', D_n)_+ f - k_{j,\lambda}(\xi', D_n) t_{j,\lambda}(\xi', D_n) f, \quad j = 0, 1,$$

cf. (5.1)-(5.4), and the corresponding transformed boundary symbol operator

$$\underline{r}_{j,\lambda}(x', \xi', D_n) = r_{j,\lambda}(A'(x')\xi', cD_n + b(x')^T\xi').$$

The analysis of the pseudodifferential operator parts $p_\lambda(\xi', D_n)_+$ and $\underline{p}_\lambda(x', \xi', D_n)_+$ is done at the end of this section. The singular Green operator falls under the scope of the following lemma, which is similar to [9, Lemma 3] and [20, Lemma 3].

Lemma 5.8 *Let* $g_\lambda(x', \xi, \eta_n)$ *be a symbol which is Lipschitz continuous in* $x' \in \mathbb{R}^{n-1}$, *continuous in* $\lambda \in \Sigma_\delta \setminus B_R(0)$, $0 < \delta < \pi$, $R > 0$, *smooth in* $\xi' \in \mathbb{R}^{n-1}$, *and in* $\mathcal{H}_{-1}\hat{\otimes}\mathcal{H}_{-1}$ *with respect to* (ξ_n, η_n). *Moreover, we assume that the symbol-kernel* $\tilde{g}_\lambda(x', \xi', x_n, y_n) := \mathcal{F}^{-1}_{\xi_n \mapsto x_n} \bar{\mathcal{F}}^{-1}_{\eta_n \mapsto y_n}[g(x', \xi, \eta_n)]$ *satisfies*

$$\|D^{\alpha'}_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde{g}_\lambda(., \xi', x_n, y_n)\|_{C^{0,1}} \leq C_{\delta,\alpha'} \langle\xi'\rangle^{-|\alpha'|-k+k'-l+l'} \frac{e^{-c_\delta|\lambda|^{\frac{1}{2}}(|x_n|+|y_n|)}}{|\lambda|^{\frac{1}{2}}} \quad (5.26)$$

uniformly in $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \Sigma_\delta \setminus B_R(0)$, *and* $x_n, y_n \neq 0$ *for all* $\alpha' \in \mathbb{N}_0^{n-1}, k, k', l, l' \in \{0, 1\}$ *and a* $c_\delta > 0$. *Then*

$$\left\| \int_{\Gamma_R} h(-\lambda) D^{\alpha'}_{\xi'} g_\lambda(., \xi', D_n) d\lambda \right\|_{C^{0,1}(\mathbb{R}^{n-1}; \mathcal{L}(L^2(|x_n|^{-\delta'}), H_2^{\delta'}))} \leq C_{\delta,\delta',\alpha'} \langle\xi'\rangle^{-|\alpha'|} \|h\|_\infty \quad (5.27)$$

$$\left\| \int_{\Gamma_R} h(-\lambda) D^{\alpha'}_{\xi'} g_\lambda(., \xi', D_n) d\lambda \right\|_{C^{0,1}(\mathbb{R}^{n-1}; \mathcal{L}(H_2^{-\delta'}, L^2(|x_n|^{\delta'})))} \leq C_{\delta,\delta',\alpha'} \langle\xi'\rangle^{-|\alpha'|} \|h\|_\infty \quad (5.28)$$

uniformly in $\xi' \in \mathbb{R}^{n-1}$ *for all* $h \in H(\delta)$, $0 \le \delta' < \frac{1}{2}$, $\alpha' \in \mathbb{N}_0^{n-1}$, *where*

$$g_\lambda(x',\xi',D_n)f := \int_{\mathbb{R}} \tilde{g}_\lambda(x',\xi',x_n,y_n)f(y_n)dy_n.$$

Remark 5.9 Note that the "singular Green operators" considered in this lemma are operators acting on functions defined on $\mathbb{R}$ instead of $\mathbb{R}_+$.

If $g(.,\xi_n,\eta_n) \in \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$, then $\tilde{g}(.,x_n,y_n) = 0$ if $x_n < 0$ or $y_n < 0$. Moreover, if $f \in \mathcal{S}(\overline{\mathbb{R}}_+)$, then $r^+g_\lambda(.,D_n)e^+f$, where $g_\lambda(.,D_n)$ is defined as above, coincides with the usual definition of $g_\lambda(.,D_n)f$ as a singular Green operator.

Proof of Lemma 5.8: Because of (2.2), we can replace $k,l = 0,1$ by arbitrary numbers $a,b \in [0,1]$. Applying the modified estimate, we get for $f \in L^2(\mathbb{R};|x_n|^{2\delta'})$:

$$\begin{aligned}
&\int_{\Gamma_R} \left| D_{\xi'}^{\alpha'} D_{x_n}^{k'} \left(|y_n|^{-\delta'} g_\lambda\right)(x',\xi',D_n)(|y_n|^{\delta'}f(y_n))\right| d|\lambda| \\
\le\ & C_{\alpha',\beta'}\langle\xi'\rangle^{k'+\delta'-|\alpha'|}\int_{\mathbb{R}}\int_0^\infty \frac{e^{-c_\delta s^{\frac{1}{2}}(|x_n|+|y_n|)}}{s^{\frac{1}{2}}}ds\,||y_n|^{\delta'}f(y_n)|dy_n \\
\le\ & C_{\alpha',\beta'}\langle\xi'\rangle^{k'+\delta'-|\alpha'|}\int_{\mathbb{R}} \frac{||y_n|^{\delta'}f(y_n)|}{|x_n|+|y_n|}dy_n.
\end{aligned}$$

Since the integral operator with kernel $k(x_n,y_n) = \frac{1}{|x_n|+|y_n|}$ is continuous on $L^2(\mathbb{R})$, we get for $k' = 0,1$

$$\begin{aligned}
&\left\|\int_{\Gamma_R} h(-\lambda)D_{\xi'}^{\alpha'}g_\lambda(x',\xi',D_n)f(y_n)d\lambda\right\|_{H_2^{k'}(\mathbb{R}_\pm)} \\
&\le \|h\|_\infty \left\|\int_{\Gamma_R}\left|D_{\xi'}^{\alpha'}D_{x_n}^{k'}\left(|y_n|^{-\delta'}g_\lambda\right)(x',\xi',D_n)||y_n|^{\delta'}f(y_n)|\right|d|\lambda|\right\|_{L^2(\mathbb{R}_\pm)} \\
&\quad + \|h\|_\infty \left\|\int_{\Gamma_R}\left|D_{\xi'}^{\alpha'}\left(|y_n|^{-\delta'}g_\lambda\right)(x',\xi',D_n)||y_n|^{\delta'}f(y_n)|\right|d|\lambda|\right\|_{L^2(\mathbb{R}_\pm)} \\
&\le C_{\delta,\delta',\alpha',k'}\langle\xi'\rangle^{k'+\delta'-|\alpha'|}\|h\|_\infty\|f\|_{L^2(\mathbb{R};|y_n|^{2\delta'})},
\end{aligned}$$

and therefore

$$\left\|\int_{\Gamma_R} h(-\lambda)D_{\xi'}^{\alpha'}g_\lambda(x',\xi',D_n)f(y_n)d\lambda\right\|_{H_2^{-\delta'}(\mathbb{R}_\pm)} \le C_{\delta,\delta',\alpha'}\langle\xi'\rangle^{-|\alpha'|}\|h\|_\infty\|f\|_{L^2(\mathbb{R};|y_n|^{2\delta'})} \tag{5.29}$$

by complex interpolation. Since $e^\pm r^\pm\colon H_2^s(\mathbb{R}) \to H_2^s(\mathbb{R})$ is a continuous mapping if $|s| < \frac{1}{2}$, cf. [25, Lemma 2.10.2], $f \in H_2^{-\delta'}(\mathbb{R})$ iff $r^+f \in H_2^{-\delta'}(\mathbb{R}_+)$ and $r^-f \in H_2^{-\delta'}(\mathbb{R}_-)$. Moreover,

$$\|f\|_{H_2^{-\delta'}(\mathbb{R})} \simeq \left(\|r^+f\|_{H_2^{-\delta'}(\mathbb{R}_+)} + \|r^-f\|_{H_2^{-\delta'}(\mathbb{R}_-)}\right).$$

Hence we can replace $H_2^{-\delta'}(\mathbb{R}_\pm)$ by $H_2^{-\delta'}(\mathbb{R})$ on the left-hand side of (5.29).

The same estimate holds with $g_\lambda(x',\xi',D_n)$ replaced by $g_\lambda(x',\xi',D_n)-g_\lambda(y',\xi',D_n)$ and an additional multiplicative term $|x'-y'|$ on the right-hand sides of the estimates. Hence the estimate (5.27) is proved.

Passing to the (pointwise) adjoint, (5.28) follows from (5.27) by duality. ■

Remark 5.10 In our case, the symbol-kernel of g_λ will be of the form $\tilde{g}_\lambda(x',\xi',x_n,y_n) = \tilde{k}_\lambda(x',\xi',x_n)\tilde{t}_\lambda(x',\xi',y_n)$, where

$$\|D_{\xi'}^{\alpha'} x_n^m D_{x_n}^{m'} \tilde{k}_\lambda(.,\xi',x_n)\|_{C^{0,1}} \leq C_{\delta,\alpha'} \frac{e^{-c_\delta|\lambda|^{\frac{1}{2}}|x_n|}}{|\lambda|^{\frac{1}{2}}} \langle\xi'\rangle^{-|\alpha'|-m+m'} \tag{5.30}$$

$$\|D_{\xi'}^{\alpha'} y_n^m D_{y_n}^{m'} \tilde{t}_\lambda(.,\xi',y_n)\|_{C^{0,1}} \leq C_{\delta,\alpha'} e^{-c_\delta|\lambda|^{\frac{1}{2}}|y_n|} \langle\xi'\rangle^{-|\alpha'|-m+m'} \tag{5.31}$$

uniformly in $\xi' \in \mathbb{R}^{n-1}$, $x_n, y_n \neq 0$, $\lambda \in \Sigma_\delta$, $|\lambda| \geq 1$, and for $\alpha' \in \mathbb{N}_0^{n-1}$, $m, m' = 0, 1$. It is a consequence of [1, Lemma 3.5] that $\tilde{k}_{j,\lambda}\langle\lambda;\xi'\rangle^{j-1}$ and $\tilde{t}_{j,\lambda}\langle\lambda;\xi'\rangle^{1-j}$ satisfy (5.30) and (5.31), respectively. Because of

$$\mathcal{F}_{\xi_n}^{-1}[f(c\xi_n+d)](x_n) = e^{-idx_n}\mathcal{F}_{\xi_n}^{-1}[f(c\xi_n)](x_n) = \frac{e^{-idx_n}}{c}\mathcal{F}_{\xi_n}^{-1}[f](x_n/c),$$

and $A', b, c, c^{-1} \in C^{0,1}(\mathbb{R}^{n-1})$, the symbol-kernels of the transformed Poisson and trace operators $\underline{\tilde{k}}_{j,\lambda}(x',\xi',x_n)\langle\lambda;\xi'\rangle^{j-1}$ and $\underline{\tilde{t}}_{j,\lambda}(x',\xi',x_n)\langle\lambda;\xi'\rangle^{1-j}$ satisfy the same estimates.

Finally, we note that multiplication of the symbol-kernels with a pseudodifferential symbol $s_\lambda(\xi')$ of order 0 and regularity $\nu \geq 0$ does not disturb (5.30) and (5.31).

Because of Lemma 5.3 and (5.25),

$$\underline{r}^r_{j,\lambda}(x',\xi',D_n)f := \underline{a}^{r,-1}_{j,\lambda}(x',\xi',D_n)\begin{pmatrix} f \\ 0 \end{pmatrix} = \underline{r}_{j,\lambda}(x',\xi',D_n)f - \underline{g}'_{j,\lambda}(x',\xi',D_n)f. \tag{5.32}$$

Unfortunately, the symbol-kernel of $\underline{g}'_{j,\lambda}$ does not satisfy the (x_n, y_n-pointwise) estimate (5.26). The critical term in the additional singular Green operator is of the form $g_\lambda(x',\xi',D_n) = k_\lambda(x',\xi',D_n)t_\lambda(x',\xi',D_n)$ with

$$k_\lambda(x',\xi',D_n) = \underline{p}_\lambda(x',\xi',D_n)_+ k^r(x',\xi',D_n)$$

where k^r is a Poisson operators of order 1. The crucial observation is that $\underline{p}_\lambda(x',\xi',D_n)_+$ commutes with $k^r(x',\xi',D_n)$ in the following sense:

$$\begin{aligned} k_\lambda(x',\xi',D_n)a &= r^+\mathcal{F}^{-1}_{\xi_n\mapsto x_n}[k^r(x',\xi)\mathcal{F}_{x_n\mapsto\xi_n}[.]]\mathcal{F}^{-1}_{\xi_n\mapsto x_n}[\underline{p}_\lambda(x',\xi)a] \\ &= r^+ m_{k^r}(x',\xi',D_n)k_{\underline{p}_\lambda}(x',\xi',D_n)a \end{aligned}$$

where $m_{k^r}(x',\xi',D_n)$ is a one-dimensional multiplier operator depending on (x',ξ') with symbol $k^r(x',\xi)$ and $k_{\underline{p}_\lambda}(x',\xi',D_n)$ is a (generalized) Poisson operator with symbol $\underline{p}_\lambda(x',\xi)$, cf. Remark 5.9.

Lemma 5.11 *Let $m(x',\xi)$, be smooth in ξ and in $C^{0,1}(\mathbb{R}^{n-1})$ with respect to x' such that*

$$\sup_{\xi_n\in\mathbb{R}} \|D^{\alpha'}_{\xi'}\xi_n^k D^k_{\xi_n} m(.,\xi)\|_{C^{0,1}} \leq C_{\alpha',k}\langle\xi'\rangle^{-|\alpha'|} \tag{5.33}$$

for $\alpha'\in\mathbb{N}_0^{n-1}$, $k=0,1$. Let $m(x',\xi',D_n)\varphi = \mathcal{F}^{-1}_{\xi_n\mapsto x_n}[m(x',\xi)\hat{\varphi}(\xi_n)]$ be the corresponding multiplier operator. Moreover, let $g_\lambda(x',\xi)$ satisfy the assumptions of Lemma 5.8 and let $g'_\lambda(x',\xi',D_n) := m(x',\xi',D_n)g_\lambda(x',\xi',D_n)$. Then $g'_\lambda(x',\xi',D_n)$ satisfies the estimate (5.27).

Proof: Since m is λ-independent, we only have to show that $D^{\alpha'}_{\xi'}m(x',\xi',D_n)$ is continuous on $H=L^2(\mathbb{R};|x_n|^{2\delta'})$ and $H=H^{\delta'}_2(\mathbb{R})$, $|\delta'|<\frac{1}{2}$, and satisfies the estimate

$$\|D^{\alpha'}_{\xi'}m(x',\xi',D_n)\|_{C^{0,1}(\mathbb{R}^{n-1};\mathcal{L}(H))} \leq C_{\delta',\alpha'}\langle\xi'\rangle^{-|\alpha'|} \tag{5.34}$$

for every $\alpha'\in\mathbb{N}_0^{n-1}$, $j=0,1$.

The estimate (5.33) implies that $D^{\alpha'}_{\xi'}m(x',\xi',D_n)$ is a one-dimensional Mikhlin multiplier with respect to ξ_n satisfying

$$\begin{aligned}[D^{\alpha'}_{\xi'}m(x',\xi',.)]_{\mathcal{M}} &\leq C_{\alpha'}\langle\xi'\rangle^{-|\alpha'|} \quad \text{and}\\ [D^{\alpha'}_{\xi'}(m(x',\xi',.)-m(y',\xi',.))]_{\mathcal{M}} &\leq C_{\alpha'}\langle\xi'\rangle^{-|\alpha'|}|x'-y'|,\end{aligned}$$

where $[m]_{\mathcal{M}} = \sup_{\xi_n\in\mathbb{R},k=0,1}|\xi_n^k\partial^k_{\xi_n}m(\xi)|$. Since $|x_n|^{2\delta'}$ is a Muckenhoupt weight of class $\mathcal{A}_2$, cf. [22, Chapter V], iff $|\delta'|<\frac{1}{2}$, $m(x',\xi',D_n)$ is continuous on $L^2(\mathbb{R},|x_n|^{2\delta'})$ for $|\delta'|<\frac{1}{2}$; cf. [21] for an elementary proof. Moreover, the operator norm is bounded by $C[m]_{\mathcal{M}}$, where C depends on the weight $|x_n|^{2\delta'}$. Hence (5.34) holds in the case $H=L^2(\mathbb{R},|x_n|^{\delta'})$. The case $H=H^{\delta'}_2(\mathbb{R})$ is obvious since $m(x',\xi',D_n)$ commutes with $\langle D_n\rangle^s$, $s\in\mathbb{R}$. ■

Lemma 5.12 *Let $k^r\in C^{0,1}S^0_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathcal{H}^+)\otimes\mathcal{L}(\mathbb{C}^N,\mathbb{C}^n)$ and let $\underline{p}_\lambda(x',\xi',D_n)$ be as in Remark 5.10. Then*

$$\underline{p}_\lambda(x',\xi',D_n)_+k^r(x',\xi',D_n) = r^+m_{k^r}(x',\xi',D_n)k_{\underline{p}_\lambda}(x',\xi',D_n),$$

where $\tilde{k}_{\underline{p}_\lambda}(x',\xi',x_n)$ satisfies (5.30) and $m_{k^r}(x',\xi)$ satisfies the condition (5.33) of Lemma 5.11.

Proof: Using the estimate $\|f\|_\infty \leq C\|f\|_2^{\frac{1}{2}}\|f'\|_2^{\frac{1}{2}}$ for $f\in H^1_2(\mathbb{R})$ and (4.1), we conclude

$$\sup_{\xi_n\in\mathbb{R}}\|D^{\alpha'}_{\xi'}\xi_n^m D^m_{\xi_n}k^r(.,\xi)\|_{C^{0,1}} \leq C_{\alpha',m}\langle\xi'\rangle^{-|\alpha'|},$$

for all $m\in\mathbb{N}_0$, $\alpha'\in\mathbb{N}_0^{n-1}$. Hence (5.33) holds for $m_{k^r}(\xi)=k^r(\xi)$. ■

Using the previous results we get the following theorem:

THEOREM 5.13 *Let $\delta \in (0,\pi)$, $c_0 > 0$ be the constant in Lemma 5.1, and $\underline{r}^r_{j,\lambda}(x',\xi',D_n)$ be the transformed boundary symbol operator for the resolvent of the reduced Stokes operator defined in (5.32) with $j=0,1$. Then*

$$\underline{r}^r_{j,\lambda}(x',\xi',D_n) \;=\; \underline{p}_\lambda(x',\xi',D_n)_+ + \underline{g}^r_{j,\lambda}(x',\xi',D_n),$$

where $\underline{g}^r_{j,\lambda}$ satisfies (5.27). Hence

$$\left\| \int_{\Gamma_R} h(-\lambda)\underline{g}^r_{j,\lambda}(D_x,x')d\lambda \right\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} \le C_\delta \|h\|_\infty$$

for every $h \in H(\delta)$ and $R \ge \max\{c_0,1\}$.

Proof: We only consider the Dirichlet case $j=0$. The analysis of the case $j=1$ is done in the same way.

The additional singular Green operator in (5.32) is

$$\underline{g}'_{0,\lambda}(.,D_n) \;=\; \underline{r}_{0,\lambda}(.,D_n)\underline{k}^r_0(.,D_n)\underline{s}_{0,\lambda}\underline{t}^r_0(.,D_n)\underline{r}_{0,\lambda}(.,D_n)$$

because of Lemma 5.3 and (5.25). Then

$$\underline{r}_{0,\lambda}(.,D_n)\underline{k}^r_0(.,D_n) = \underline{p}_\lambda(.,D_n)_+\underline{k}^r_0(.,D_n) + \underline{k}_{0,\lambda}(.,D_n)\underline{t}_{0,\lambda}(.,D_n)\underline{k}^r_0(.,D_n),$$

where Lemma 5.12 can be applied to $\underline{p}_\lambda(.,D_n)_+\underline{k}^r_0(.,D_n)$. Moreover, $\underline{t}_{0,\lambda}(.,D_n)\underline{k}^r_0(.,D_n)$ is a pseudodifferential symbol of order 0 and regularity $\nu \ge 0$. Hence

$$\langle\lambda,\xi'\rangle^{-1}\tilde{k}_{0,\lambda}(x',\xi',x_n)\underline{t}_{0,\lambda}(.,D_n)\underline{k}^r_0(.,D_n)$$

satisfies (5.30) because of Remark 5.10. Similarly,

$$\underline{t}^r_0(.,D_n)\underline{r}_{0,\lambda}(.,D_n) = \underline{t}^r_0(.,D_n)\underline{p}_\lambda(.,D_n)_+ + \underline{t}^r_0(.,D_n)\underline{k}_{0,\lambda}(.,D_n)\underline{t}_{0,\lambda}(.,D_n),$$

where $\underline{t}^r_0(.,D_n)\underline{p}_\lambda(.,D_n)_+$ satisfies (5.31) since $\underline{t}^r_0(.,D_n)$ is a differential trace operator. Moreover, $\underline{t}^r_0(.,D_n)\underline{k}_{0,\lambda}(.,D_n)$ is a pseudodifferential symbol of order 0 and regularity $\frac{1}{2} \ge 0$. Altogether $\underline{g}'_{0,\lambda}(.,D_n)$ is the sum of operators satisfying the assumptions of Lemma 5.8 and Lemma 5.11.

The last statement is a consequence of Theorem 3.2 and Lemma 2.1. ■

It remains to estimate the pseudodifferential operator part of the parametrix.

Lemma 5.14 *Let $1 < q < \infty$, $R > 0$, and $\delta \in (0,\pi)$. Then $\underline{p}_\lambda(x,\xi) = (\lambda + |A(x)\xi|^2)^{-1}$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, with $A, A^{-1} \in C^{0,1}(\mathbb{R}^n)^{n\times n}$ satisfies*

$$\left\| \int_{\Gamma_R} h(-\lambda)D^\alpha_\xi \underline{p}_\lambda(.,\xi)d\lambda \right\|_{C^{0,1}} \le C_{\delta,R,\alpha}\|h\|_\infty \langle\xi\rangle^{-|\alpha|}$$

uniformly in $\xi \in \mathbb{R}^n$, for all $\alpha \in \mathbb{N}^n_0$ and $h \in H(\delta)$.

Proof: The function $\lambda \mapsto \underline{p}_\lambda(x,\xi)$ is meromorphic in $\mathbb{C}$ with a pole for $\lambda = -|A(x)\xi|^2$. Therefore the pole is contained in a compact interval $I \subset (-\infty, 0]$ if $|\xi| \leq 1$ and $x \in \mathbb{R}^n$. Using the homogeneity of $D_\xi^\alpha \underline{p}_\lambda(x,\xi)$ of order $-2-|\alpha|$, we get with $\xi = |\xi|\eta$ and $\lambda = |\xi|^2 z$

$$\int_{\Gamma_R} h(-\lambda) D_\xi^\alpha \underline{p}_\lambda(x,\xi) d\lambda = \int_{\Gamma_I \setminus B_R(0)} h(-|\xi|^2 z) D_\xi^\alpha \underline{p}_z(x,\eta) dz |\xi|^{-|\alpha|},$$

where Γ_I is a curve around the compact interval I with winding number 1 with respect to the each point in I. Using $\|D_\xi^\alpha \underline{p}_z(.,\eta)\|_{C^{0,1}} \leq C_\alpha$ for $z \in \Gamma_I$ and $|\eta| \leq 1$, we conclude that

$$\left\| \int_{\Gamma_R} h(-\lambda) D_\xi^\alpha \underline{p}_\lambda(.,\xi) d\lambda \right\|_{C^{0,1}} \leq C_\alpha |\Gamma_I| \|h\|_\infty \langle \xi \rangle^{-|\alpha|}$$

for $|\xi| \geq 1$. If $|\xi| < 1$, we estimate the integral in the same way as before but without using the substitution $\lambda = |\xi|^2 z$. ∎

5.5 Parametrix for the Poisson Operators

We have to estimate the difference of the Poisson operators K_j and their parametrices. In the first step we consider the case $\mathbb{R}^n_\gamma$, where $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$. Let

$$\widetilde{K}_j = F^{*,-1} \underline{k}_j(D_x, x') F_0^*, \qquad j = 0, 1,$$

where F^* and F_0^* are the same operators as in Section 5.2 and $\underline{k}_j(x', \xi', D_x)$ is defined as in (5.17).

Lemma 5.15 *Let $\mathbb{R}^n_\gamma$ be a curved half-space, $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$, and $\widetilde{K}_j$, $j = 0, 1$, be defined as above. Then $\widetilde{K}_j \colon W_q^{1-j-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma) \to W^1_q(\mathbb{R}^n_\gamma)$ and*

$$\begin{aligned} \Delta \widetilde{K}_j &= F^{*,-1}[R'_j + R''_j] F_0^* \\ \gamma_j \widetilde{K}_j &= I + S_j, \end{aligned}$$

where

1. $R'_j \colon W_q^{1-j-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^{n-1}; L^1(0,b))$ *with an arbitrary* $b \in \mathbb{R}_+$,
2. $R''_j \colon W_q^{1-j-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to H_q^{-1}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))$, *and*
3. $S_j \colon W_q^{1-j-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to W_q^{1-j-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma)$

are bounded linear operators for some $\varepsilon > 0$.

Proof: Let $|\delta| < \frac{1}{2}$ and $H_\delta = L^2(\mathbb{R}_+, x_n^\delta)$ if $\delta \geq 0$ and $H_\delta = H_2^{-\delta}(\mathbb{R}_+)$ if $\delta < 0$. Because of Lemma 5.5, $\underline{k}_j \in C^{0,1}S^{-j-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{H}^+)$. Since $|1-j-\frac{1}{q}| < 1$, we can apply Theorem 4.8 to conclude

$$\widetilde{K}_j \colon W_q^{1-j-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma) \to W^1_q(\mathbb{R}^n_\gamma).$$

Due to Lemma 5.6,

$$\Delta \widetilde{K}_j \quad = \quad F^{*,-1}\underline{\Delta}\, \underline{k}_j(D_x, x')F_0^* + F^{*,-1}R'\underline{k}_j(D_x,x')F_0^*,$$

where $R' = a(x')\cdot\nabla$, $a \in L^\infty(\mathbb{R}^{n-1})^n$. Hence Theorem 4.8 implies

$$R'\underline{k}_j(D_x, x') \colon B_q^{1-j-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^n; L^1(0,b))$$

for every $\varepsilon \in (0, \frac{1}{q'})$ and $b \in \mathbb{R}_+$ since $\nabla \underline{k}_j(D_x, x')$ is a Poisson operator of order $1-j$.

Since $\underline{\Delta} = \mathrm{OP}(-|A(x')\xi|^2)$ is a differential operator in x-form and

$$\underline{k}_j(\underline{x}', \xi', D_n) \in C^{0,1}S^{-j-\frac{1}{2}-\delta}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, H_\delta)),$$

cf. Remark 4.2, we can apply Corollary 3.4 and get for arbitrary $|\delta| < \frac{1}{2}$ that

$$R''_j := \underline{\Delta}\, \underline{k}_j(D_x, x') - \mathrm{OP}'(\mathrm{OP}_n(-|A(y')\xi|^2)\underline{k}_j(y', \xi', D_n))$$

satisfies

$$R''_j \colon H_q^{1-j-\frac{1}{2}-\delta-\varepsilon}(\mathbb{R}^{n-1}) \to H_q^{-1+\varepsilon'}(\mathbb{R}^{n-1}; H_\delta)$$

for some $\varepsilon, \varepsilon' > 0$. Hence $R''_j \colon H_q^{1-j-\frac{1}{2}-\delta-\varepsilon}(\mathbb{R}^{n-1}) \to H_q^{-1}(\mathbb{R}^{n-1}; H_\delta)$ and, because of Lemma 2.1,

$$R''_j \colon B_q^{1-j-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to H_q^{-1}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))$$

for some $\varepsilon > 0$. Since

$$\mathrm{OP}_n(h^+_{\xi_n}[|A(y')\xi|^2\underline{k}_j(y',\xi)]) = \mathrm{OP}_n(|A(y')\xi|^2)_+\underline{k}_j(y',\xi',D_n) = 0$$

for $|\xi'| \geq 1$, the Poisson symbol $h^+_{\xi_n}[|A(y')\xi|^2\underline{k}_j(y',\xi)]$ is of order $-\infty$. Hence

$$R'''_j := -\mathrm{OP}(h^+_{\xi_n}|A(y')\xi|^2\underline{k}_j(y',\xi)) \colon B_q^{1-j-\frac{1}{q}-\varepsilon}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^n_+)$$

for all $0 < \varepsilon < \frac{1}{q'}$. Thus we have proved 1. and 2.

Finally, if $j = 0$, then $\gamma_0\widetilde{K}_0 = I$ and $S_0 = 0$. If $j = 1$, then Lemma 4.10 yields

$$\begin{aligned}\gamma_1\widetilde{K}_1 &= F_0^{*,-1}\underline{\gamma_1}(x', D_x))\underline{k}_1(D_x,x')F_0^* \\ &= F_0^{*,-1}\,\mathrm{OP}'(\underline{\gamma_1}(y',\xi',D_n)\underline{k}_1(y',\xi',D_n))F_0^* + S_1'\end{aligned}$$

with $S_1' \colon B_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to B_q^{-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma)$. Since $\underline{\gamma_1}(y',\xi',D_n)\underline{k}_1(y',\xi',D_n) = I$ for $|\xi'| \geq 1$, $\gamma_1\widetilde{K}_1 = I + S_1$, where $S_1 \colon B_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to B_q^{-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma)$ for some $\varepsilon > 0$. ■

Now let $\Omega_\gamma \subset \mathbb{R}^n$ be an asymptotically flat $C^{1,1}$-domain and $\varepsilon > 0$ be a number such that $\varepsilon < \frac{1}{2}\operatorname{dist}(\partial\Omega^+_\gamma, \partial\Omega^-_\gamma)$. We choose cut-off functions $\varphi_\pm \in C^\infty(\overline{\Omega_\gamma})$ with

1. $\varphi_\pm(x) = 1$ for $\mathrm{dist}(x, \partial\Omega_\gamma^\pm) \le \varepsilon/3$, $0 \le \varphi_\pm \le 1$, and

2. $\varphi_\pm(x) = 0$ for $\mathrm{dist}(x, \partial\Omega_\gamma^\pm) \ge 2\varepsilon/3$.

Moreover, let $\psi_\pm \in C^\infty(\overline{\Omega_\gamma})$ such that

1. $\psi_\pm(x) = 1$ for $\mathrm{dist}(x, \partial\Omega_\gamma^\pm) \le 2\varepsilon/3$, $0 \le \psi_\pm \le 1$, and

2. $\psi_\pm(x) = 0$ for $\mathrm{dist}(x, \partial\Omega_\gamma^\pm) \ge \varepsilon$,

and let $\widetilde{K}_j^\pm$ denote the parametrices of the Poisson operators in $\mathbb{R}^n_{\gamma\pm}$ defined above. We will use

$$\widetilde{K}_1 a = \psi_+ \widetilde{K}_1^+ a^+ + \psi_- \widetilde{K}_1^- a^- \qquad \text{for } a \in W_q^{-\frac{1}{q}-s}(\partial\Omega_\gamma),$$
$$\widetilde{K}_{01} a = \psi_+ \widetilde{K}_0^+ a^+ + \psi_- \widetilde{K}_1^- a^- \qquad \text{for } a^+ \in W_q^{1-\frac{1}{q}-s}(\partial\Omega_\gamma), a^- \in W_q^{-\frac{1}{q}-s}(\partial\Omega_\gamma),$$

$s \in [0, \frac{1}{q'})$, as parametrices for K_0, K_{01}, respectively, in Ω_γ.

Lemma 5.16 *Let $1 < q < \infty$, $\Omega_\gamma \subset \mathbb{R}^n$ be an asymptotically flat $C^{1,1}$-domain, K_1 be the Poisson operator of the Neumann problem in Ω_γ, and let K_{01} be the Poisson operator of the mixed Dirichlet-Neumann problem. Moreover, let $\widetilde{K}_1$ and $\widetilde{K}_{01}$ be defined as above. Then*

$$\|\nabla(K_1 - \widetilde{K}_1)\gamma_\nu(\Delta - \nabla\,\mathrm{div})u\|_q \le C\|u\|_{2-\varepsilon,q},$$
$$\|\nabla(K_{01} - \widetilde{K}_{01})a\|_q \le C(\|a^+\|_{1-\frac{1}{q}-\varepsilon,q} + \|a^-\|_{-\frac{1}{q}-\varepsilon,q})$$

for all $u \in W_q^{2-\varepsilon}(\Omega_\gamma)^n$, $(a^+, a^-) \in W_q^{1-\frac{1}{q}-\varepsilon}(\partial\Omega_\gamma^+) \times W_q^{-\frac{1}{q}-\varepsilon}(\partial\Omega_\gamma^-)$ and some $\varepsilon > 0$.

Proof: Let us first consider the mixed Dirichlet-Neumann case. Let $f \in L^{q'}(\Omega_\gamma)^n$ and let $f = f_0 + \nabla p$, $f_0 \in {}_0J_{q'}(\Omega_\gamma) = \{f \in L^{q'}(\Omega_\gamma)^n : \mathrm{div}\, f = 0, \gamma_\nu^- f = 0\}$, $p \in {}^0W^1_{q'}(\Omega_\gamma) = \{p \in W^1_{q'}(\Omega_\gamma) : \gamma_0^+ p = 0\}$, $\|(f_0, \nabla p)\|_{q'} \le C_q \|f\|_{q'}$, be its Helmholtz decomposition with mixed boundary data, cf. [3, Corollary A.3]. Then

$$\begin{aligned}(\nabla(K_{01} - \widetilde{K}_{01})a, f)_{\Omega_\gamma} &= (\gamma_0^+(K_{01} - \widetilde{K}_{01})a, \gamma_\nu^+ f_0)_{\partial\Omega_\gamma^+} + (\nabla(K_{01} - \widetilde{K}_{01})a, \nabla p)_{\Omega_\gamma} \\ &= (\Delta\widetilde{K}_{01}a, p)_{\Omega_\gamma} + (a^- - \gamma_1^- \widetilde{K}_1^- a^-, \gamma_0^+ p)_{\partial\Omega_\gamma^-}\end{aligned}$$

since $\mathrm{div}\, f_0 = 0$, $\gamma_\nu^- f_0 = 0$, $\gamma_0^+ K_{01} a = a^+ = \gamma_0^+ \widetilde{K}_{01} a$, $\Delta K_{01} a = 0, \gamma_1^- K_{01} = a^-$, $\gamma_1^- \widetilde{K}_{01} a = \gamma_1^- \widetilde{K}_1^- a^-$, and $\gamma_0^+ p = 0$. By construction,

$$\Delta\widetilde{K}_{01} a = \psi_+ \Delta\widetilde{K}_0^+ a^+ + \psi_- \Delta\widetilde{K}_1^- a^- + P_+ \widetilde{K}_0^+ a^+ + P_- \widetilde{K}_1^- a^-,$$

where $P_\pm$ are differential operators of order 1 with coefficients supported in $\mathrm{supp}\, \nabla\psi_\pm$ and $\mathrm{dist}(\mathrm{supp}\, \nabla\psi_\pm, \partial\Omega_\gamma) > 0$. Therefore $P_+ \widetilde{K}_0^+$ and $P_- \widetilde{K}_1^+$ are operators of order $-\infty$, cf. Remark 4.9, which implies that $P_+ \widetilde{K}_0 \colon W_q^{-s}(\partial\mathbb{R}^n_{\gamma+}) \to L^q(\mathbb{R}^n_{\gamma+})$ and $P_- \widetilde{K}_1 \colon W_q^{-s}(\partial\mathbb{R}^n_{\gamma-}) \to L^q(\mathbb{R}^n_{\gamma-})$ for all $|s| < 1$. Thus

$$(\Delta\widetilde{K}_{01} a, p)_{\Omega_\gamma} = (\Delta\widetilde{K}_0^+ a^+, \psi_+ p)_{\mathbb{R}^n_{\gamma+}} + (\Delta\widetilde{K}_1^- a^-, \psi_- p)_{\mathbb{R}^n_{\gamma-}} + (Ra, p)_{\Omega_\gamma},$$

where $|(Ra,p)| \leq C(\|a^+\|_{1-\frac{1}{q}-\varepsilon,q}+\|a^-\|_{-\frac{1}{q}-\varepsilon,q})\|f\|_{q'}$ for an $\varepsilon > 0$. Using Lemma 5.15,

$$(\Delta \widetilde{K}_0^+ a^+, \psi_+ p)_{\mathbb{R}^n_{\gamma+}} = (F^{*,-1} R_0' F_0^* a^+, \psi_+ p)_{\mathbb{R}^n_{\gamma+}} + (F^{*,-1} R_0'' F_0^* a^+, \psi_+ p)_{\mathbb{R}^n_{\gamma+}},$$

where

$$\begin{aligned}
\left|(F^{*,-1} R_0' F_0^* a^+, \psi_+ p)_{\mathbb{R}^n_{\gamma+}}\right| &\leq \left\|R_0' F_0^* a^+\right\|_{L^q(\mathbb{R}^{n-1};L^1(0,b))} \|F^*\psi_+ p\|_{L^{q'}(\mathbb{R}^{n-1};L^\infty(0,b))} \\
&\leq C\|a^+\|_{1-\frac{1}{q}-\varepsilon,q} \|F^*\psi_+ p\|_{W^1_{q'}(\mathbb{R}^n_+)} \leq C\|a^+\|_{1-\frac{1}{q}-\varepsilon,q}\|f\|_{q'}, \\
\left|(F^{*,-1} R_0'' F_0^* a^+, \psi_+ p)_{\mathbb{R}^n_{\gamma+}}\right| &\leq \|R_0'' F_0^* a^+\|_{H^{-1}_q(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \|F^*\psi_+ p\|_{H^1_{q'}(\mathbb{R}^{n-1};L^{q'}(\mathbb{R}_+))} \\
&\leq C\|a^+\|_{1-\frac{1}{q}-\varepsilon,q}\|f\|_{q'}.
\end{aligned}$$

Similarly, we conclude $\left|(\Delta \widetilde{K}_1^- a^-, \psi_- p)_{\mathbb{R}^n_{\gamma-}}\right| \leq C\|a^-\|_{-\frac{1}{q}-\varepsilon,q}\|f\|_{q'}$. Finally, because of Lemma 5.15, $a^- - \gamma_1^- \widetilde{K}_1^- a^- = -S_1 a^-$ with $S_1 \colon W_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_{\gamma-}) \to W_q^{-\frac{1}{q}}(\partial\mathbb{R}^n_{\gamma-})$, which proves the Lemma in the mixed Dirichlet-Neumann case.

In case of pure Neumann boundary conditions we use the usual Helmholtz decomposition $f = f_0 + \nabla p$ with $f_0 \in J_{q',0}(\Omega_\gamma) = \{f \in L^{q'}(\Omega_\gamma : \operatorname{div} f = 0, \gamma_\nu f = 0\}$ and $p \in \dot{W}^1_{q'}(\Omega_\gamma)$, cf. [3, Corollary A.3]. Then

$$(\nabla(K_1 - \widetilde{K}_1)a, f)_{\Omega_\gamma} = (\nabla(K_1 - \widetilde{K}_1)a, \nabla p)_{\Omega_\gamma}, \quad a = \gamma_\nu(\Delta - \nabla \operatorname{div})u,$$

since $(\nabla\tilde{p}, f_0) = 0$ for every $\tilde{p} \in \dot{W}^1_q(\Omega_\gamma)$. Since p is not in $L^{q'}(\Omega_\gamma)$ in general, we split $p = p_1 + p_2 \in \dot{W}^1_{q'}(\Omega_\gamma)$ with $p_1 \in W^1_{q'}(\Omega_\gamma)$, $p_2 = p_2(x') \in L^q_{loc}(\mathbb{R}^{n-1})$ with $\nabla' p_2 \in W^1_{q'}(\mathbb{R}^{n-1})$, satisfying $\|p_1\|_{1,q'}, \|\nabla' p_2\|_{1,q'} \leq C\|\nabla p\|_{q'}$, cf. [3, Remark 2.6.2]. Then we can prove as in the mixed case that

$$\left|(\nabla(K_1 - \widetilde{K}_1)a, \nabla p_1)_{\Omega_\gamma}\right| \leq C\|a\|_{W_q^{-\frac{1}{q}-\varepsilon}(\partial\Omega_\gamma)}\|f\|_{q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'}.$$

Therefore it remains to estimate $(\nabla(K_1 - \widetilde{K}_1)a, \nabla p_2)$, which can be done separately:

$$\begin{aligned}
|(\nabla K_1 a, \nabla p_2)| &= |(a, p_2)_{\partial\Omega_\gamma}| = |((\Delta - \nabla \operatorname{div})u, \nabla p_2)| \\
&\leq C\|(\Delta - \nabla \operatorname{div})u\|_{W_q^{-\varepsilon}(\Omega_\gamma)} \|\nabla' p_2\|_{W^\varepsilon_{q'}(\Omega_\gamma)} \leq C\|u\|_{2-\varepsilon,q}\|\nabla' p_2\|_{1,q'},
\end{aligned}$$

where $0 < \varepsilon < \frac{1}{q'}$, and

$$\begin{aligned}
&|(\nabla \widetilde{K}_1 a, \nabla p_2)| \\
&\quad \leq C\left(\|(\nabla \underline{k}_1(D_x, x')F^*_{0,+}a^+, \nabla \underline{k}_1(D_x, x')F^*_{0,-}a^-)\|_{L^q(\mathbb{R}^{n-1};L^1(0,b))}\right)\|\nabla' p_2\|_{q',\mathbb{R}^{n-1}} \\
&\quad \leq C\|a\|_{-\frac{1}{q}-\varepsilon,q}\|\nabla' p_2\|_{L^{q'}(\mathbb{R}^{n-1})}
\end{aligned}$$

for all $0 < \varepsilon < \frac{1}{q'}$ and a suitable large $b \in \mathbb{R}_+$ because of Remark 4.9. ■

5.6 Parametrix for the Reduced Stokes Equations

We first consider the case of a curved half-space $\mathbb{R}^n_\gamma$. In this case we define

$$R_{j,\lambda} \ := \ F^{*,-1}\,\mathrm{OP}'(\underline{r}^r_{j,\lambda}(\underline{y}',\underline{\xi}',D_n))F^*$$

as parametrix for the reduced Stokes equations, where $\underline{r}^r_{j,\lambda}$ is defined in (5.32). Because of Corollary 5.7, Theorem 3.2, and Theorem 4.8.3, $R_{j,\lambda}\colon L^q(\mathbb{R}^n_\gamma)^n \to W^2_q(\mathbb{R}^n_\gamma)^n$ with operator norm uniformly bounded in $\lambda \in \Sigma_\delta \cup \{0\}$, $\delta \in (0,\pi)$. Considering $\underline{r}^r_{j,\lambda}(\underline{y}',\underline{\xi})$ as Green symbol of order 0 with symbol semi-norms bounded by $C_\delta(1+|\lambda|)^{-1}$, cf. Remark 3.6 and Remark 4.6.1, we conclude $\|R_{j,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \le C_\delta(1+|\lambda|)^{-1}$. Hence

$$R_{j,\lambda}\colon L^q(\mathbb{R}^n_\gamma)^n \to W^2_{q,\lambda}(\mathbb{R}^n_\gamma)^n \tag{5.35}$$

with operator norm uniformly bounded in $\lambda \in \Sigma_\delta \cup \{0\}$, $\delta \in (0,\pi)$.

Lemma 5.17 *Let $\mathbb{R}^n_\gamma$ be a curved half-space with $C^{1,1}$-boundary, $\delta \in (0,\pi)$, and $R_{j,\lambda}$, $j = 0,1$, be defined as above. Then*

$$(\lambda - \Delta)R_{j,\lambda} = I - \widetilde{G}_{j,\lambda} + R'_{j,\lambda},$$

$\gamma_0 R_{0,\lambda} = 0$ if $j=0$ and $T'_1 R_{1,\lambda} = S_\lambda$ if $j = 1$, where

$$\widetilde{G}_{j,\lambda} \ = \ F^{*,-1}\,\mathrm{OP}'(\underline{k}^r_j(y',\xi',D_n)\underline{t}^r_j(y',\xi',D_n)\underline{r}^r_{j,\lambda}(y',\xi',D_n))F^*$$

and $\|R'_{j,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))}$, $\|S_\lambda\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma),W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_\gamma))} \le C_\delta(1+|\lambda|)^{-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$ for some $\varepsilon > 0$.

Proof: It is sufficient to prove the estimates of $R'_{j,\lambda}$ and S_λ for $|\lambda| \ge c_0$ where c_0 is the constant such that the model operator of the reduced Stokes equations is invertible, cf. Lemma 5.1.

Due to Lemma 5.6 and (5.35),

$$(\lambda - \Delta)R_{j,\lambda} \ = \ F^{*,-1}(\lambda - \underline{\Delta})\,\mathrm{OP}'(\underline{r}^r_{j,\lambda}(y',\xi',D_n))F^* + \widetilde{R}_\lambda$$

with $\widetilde{R}_\lambda = O((1+|\lambda|)^{-\frac{1}{2}})$. Because of Lemma 4.10 and the definition of $\underline{r}^r_{j,\lambda}$, we conclude

$$\begin{aligned}
&(\lambda - \underline{\Delta})\,\mathrm{OP}'(\underline{r}^r_{j,\lambda}(y',\xi',D_n)) \\
&\quad = \ \mathrm{OP}'(\mathrm{OP}_n(\lambda + |A(y')\xi|^2)_+ \underline{r}^r_{j,\lambda}(y',\xi',D_n)) + \widetilde{R}_\lambda \\
&\quad = \ I - \mathrm{OP}'(\underline{k}^r_j(y',\xi',D_n)\underline{t}^r_j(y',\xi',D_n)\underline{r}^r_{j,\lambda}(y',\xi',D_n)) + \widetilde{R}_\lambda,
\end{aligned}$$

where

$$\|\widetilde{R}_\lambda\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} \le C\left(\left|\underline{p}_\lambda\right|_k^{(-2+\varepsilon)} + \left|\underline{g}^r_{j,\lambda}\right|_k^{(-2+\varepsilon)}\right) \le C(1+|\lambda|)^{-\frac{\varepsilon}{2}}\left(\left|\underline{p}\right|_k^{(-2,0)} + \left|\underline{g}^r_j\right|_k^{(-2,0)}\right)$$

for an $\varepsilon > 0$ and $k \in \mathbb{N}$ due to Remark 3.6 and Remark 4.6.1.

Finally, $\gamma_0 R_{0,\lambda} f = F_0^{*,-1} \operatorname{OP}'(\gamma_0 \underline{r}^r_{0,\lambda}(y',\xi',D_n))F^* = 0$. Hence it remains to estimate $T'_1 R_{1,\lambda}$. Because of Lemma 5.6,

$$T'_1 R_{1,\lambda} = F_0^{*,-1} \underline{t}'_1(x',D_x)) \operatorname{OP}(\underline{r}^r_{1,\lambda}(y',\xi))F^*.$$

Since $\underline{t}'_1(x',D_x)$ consists of terms of the form $a(x')\gamma_0\partial_j$, $a \in C^{0,1}(\mathbb{R}^{n-1})$, and $\underline{r}^r_{1,\lambda}(y',\xi)$ can be considered as symbol of order -2 or $\frac{1}{q}$, Lemma 4.10.4 implies

$$\underline{t}'_1(x',D_x) \operatorname{OP}(\underline{r}^r_{1,\lambda}(y',\xi)) = (\underline{t}'_1 \underline{r}^r_{1,\lambda})(D_x,x') + \widetilde{R}_\lambda = \widetilde{R}_\lambda$$

with $\|\widetilde{R}_\lambda\|_{\mathcal{L}(L^q(\mathbb{R}^n_+),W^{1-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1}))} \leq C_{q,\delta}(1+|\lambda|)^{-\varepsilon}$ for an $\varepsilon > 0$. ■

Now let $\Omega_\gamma \subseteq \mathbb{R}^n$ be an asymptotically flat domain and let $\varphi_+,\varphi_-,\psi_+,\psi_-$ be defined as in the previous section. Moreover, let $\varphi_0 = 1-\varphi_+-\varphi_-$ and $\psi_0 \in C^\infty(\overline{\Omega_\gamma})$ with $\psi_0(x) \equiv 1$ on $\operatorname{supp}\varphi_0$ and $\operatorname{supp}\psi_0 \subset \Omega_\gamma$.

Then we set for $j = 0,1$

$$R_{j0,\lambda} = \psi_+ R^+_{j,\lambda}\varphi_+ + \psi_- R^-_{0,\lambda}\varphi_- + \psi_0 P_\lambda \varphi_0,$$

and $R_{0,\lambda} = R_{00,\lambda}$, where $R^\pm_{j,\lambda}$, $j = 0,1$, is the parametrix in the curved half-space $\mathbb{R}^n_{\gamma\pm}$ defined above and $P_\lambda = \operatorname{OP}((\lambda+|\xi|^2)^{-1})$.

Lemma 5.18 *Let $1 < q < \infty$ and $\delta \in (0,\pi)$. Then the operators defined above satisfy*

$$\begin{aligned}
(\lambda-\Delta+G_0)R_{0,\lambda}f &= (I+S_{1,\lambda})f && \text{in } \Omega_\gamma,\\
\gamma_0 R_{0,\lambda} f &= 0 && \text{on } \partial\Omega_\gamma,\\
(\lambda-\Delta+G_{10})R_{10,\lambda}f &= (I+S_{2,\lambda})f && \text{in } \Omega_\gamma,\\
T'^{+}_1 R_{10,\lambda} f &= S_{3,\lambda} f && \text{on } \partial\Omega^+_\gamma,\\
\gamma^-_0 R_{10,\lambda} f &= 0 && \text{on } \partial\Omega^-_\gamma
\end{aligned}$$

for $f \in L^q(\Omega_\gamma)^n$, where

$$\|S_{j,\lambda}\|_{\mathcal{L}(L^q(\Omega_\gamma))}, \|S_{3,\lambda}\|_{\mathcal{L}(L^q(\Omega_\gamma),W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_\gamma))} \leq C_{q,\delta}(1+|\lambda|)^{-\varepsilon}, \quad j = 1,2,$$

uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$ for an $\varepsilon > 0$.

Proof: Since $\|R_{j0,\lambda}f\|_q \leq C_{q,\delta}(1+|\lambda|)^{-1}\|f\|_q$ and $\|R_{j0,\lambda}f\|_{2,q} \leq C_{q,\delta}\|f\|_q$, we have $\|R_{j0,\lambda}f\|_{2-\varepsilon,q} \leq C_{q,\delta}(1+|\lambda|)^{-\frac{\varepsilon}{2}}\|f\|_q$ for $\varepsilon \in (0,2]$, $j = 0,1$. Hence

$$\begin{aligned}
&(\lambda-\Delta)R_{j0,\lambda}\\
&= \psi_+(\lambda-\Delta)R^+_{j,\lambda}\varphi_+ + \psi_-(\lambda-\Delta)R^-_{0,\lambda}\varphi_- + \psi_0(\lambda-\Delta)P_\lambda\varphi_0 + O((1+|\lambda|)^{-\frac{1}{2}})\\
&= I - \psi_+\widetilde{G}^+_{j,\lambda}\varphi_+ - \psi_-\widetilde{G}^-_{0,\lambda}\varphi_- + O((1+|\lambda|)^{-\varepsilon}) \qquad (5.36)
\end{aligned}$$

for an $\varepsilon > 0$ because of Lemma 5.17. Hence it remains to estimate the differences

$$G_0 R_{0,\lambda} - \psi_+ \widetilde{G}^+_{0,\lambda}\varphi_+ - \psi_- \widetilde{G}^-_{0,\lambda}\varphi_- =: S_{1,\lambda},$$
$$G_{10} R_{10,\lambda} - \psi_+ \widetilde{G}^+_{1,\lambda}\varphi_+ - \psi_- \widetilde{G}^-_{0,\lambda}\varphi_- =: S_{2,\lambda},$$

and to estimate the boundary values.

Because of Lemma 5.16,

$$\|\nabla(K_1 - \widetilde{K}_1)\gamma_\nu(\Delta - \nabla\operatorname{div})R_{0,\lambda}f\|_q \le C_q\|R_{0,\lambda}f\|_{2-\varepsilon,q} \le C_{q,\delta}(1+|\lambda|)^{-\frac{\varepsilon}{2}}\|f\|_q,$$
$$\|\nabla(K_{01} - \widetilde{K}_{01})(2\gamma_1^+ u_\nu, \gamma_\nu^-(\Delta - \nabla\operatorname{div})u)\|_q \le C_q\|R_{10,\lambda}f\|_{2-\varepsilon,q} \le C_{q,\delta}(1+|\lambda|)^{-\frac{\varepsilon}{2}}\|f\|_q,$$

where $u = R_{10,\lambda}f$. Moreover,

$$\begin{aligned}
&\nabla\widetilde{K}_1\gamma_\nu(\Delta - \nabla\operatorname{div})R_{0,\lambda}f \\
&\quad= \nabla\widetilde{K}_1^+\gamma_\nu^+(\Delta - \nabla\operatorname{div})R^+_{0,\lambda}\varphi_+ f + \nabla\widetilde{K}_1^-\gamma_\nu^-(\Delta - \nabla\operatorname{div})R^-_{0,\lambda}\varphi_- f,\\
&\nabla\widetilde{K}_{01}(2\nu^+\cdot\gamma_1^+R_{10,\lambda}f, \gamma_\nu^-(\Delta - \nabla\operatorname{div})R_{10,\lambda}f)\\
&\quad= \nabla\widetilde{K}_0^+ 2\nu^+\cdot\gamma_1^+R^+_{1,\lambda}\varphi_+ f + \nabla\widetilde{K}_1^-\gamma_\nu^-(\Delta - \nabla\operatorname{div})R^-_{0,\lambda}\varphi_- f.
\end{aligned}$$

Then Lemma 4.10 yields

$$\begin{aligned}
\nabla\widetilde{K}_j^\pm &= F_\pm^{*,-1}\underline{\nabla}\,\underline{k}_j(D_x,x')F^*_{0,\pm}\\
&= F_\pm^{*,-1}\operatorname{OP}'\left(U^T(y')A(y')\begin{pmatrix} i\xi' \\ \partial_n\end{pmatrix}\underline{k}_j(y',\xi',D_n)\right)F^*_{0,\pm} + R',
\end{aligned}$$

where $R'\colon W_q^{1-j-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_{\gamma\pm}) \to L^q(\mathbb{R}^n_{\gamma\pm})^n$ for an $\varepsilon > 0$. Thus

$$\|R'\gamma_\nu^\pm(\Delta - \nabla\operatorname{div})R^\pm_{0,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_{\gamma\pm}))}, \|R'\nu^+\cdot\gamma_1^+R^+_{1,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_{\gamma+}))} \le C(1+|\lambda|)^{-\frac{\varepsilon}{2}}.$$

Because of Lemma 5.6,

$$\gamma_\nu^\pm(\Delta - \nabla\operatorname{div})u = F_{0,\pm}^{*,-1}\operatorname{OP}'((A'(x')i\xi')^T\underline{t}_0^r(x',\xi',D_n))F^*_\pm u + R''u$$

if $\gamma_0^\pm u = 0$ and $2\nu\cdot\gamma_1^\pm = F_{0,\pm}^{*,-1}\operatorname{OP}'(\underline{t}_1^r(x',\xi',D_n))F^*_\pm$. Here $(A'(x')i\xi')^T\underline{t}_0^r(x',\xi)$ and $\underline{t}_1^r(x',\xi)$ are differential trace symbols of order $d=2$, $d=1$, resp. and class 2 Therefore they are of the product form $a(x')\cdot t(D_x)$, where $a \in C^{0,1}(\mathbb{R}^{n-1})^N$ and $t(D_x)$ is a differential trace operator of order d and class 2 with constant coefficients. Since $\operatorname{OP}(k(y',\xi))a(x') = \operatorname{OP}(k(y',\xi)a(y'))$ and $t(D_x)\operatorname{OP}'(r(y',\xi',D_n)) = \operatorname{OP}'(t(\xi',D_n)r(y',\xi',D_n))$,

$$\begin{aligned}
&F^*_\pm\nabla\widetilde{K}_1^\pm\gamma_\nu^\pm(\Delta - \nabla\operatorname{div})R^\pm_{0,\lambda}F_\pm^{*,-1}\\
&\quad= \operatorname{OP}'\left(U^TA(y')\begin{pmatrix} i\xi' \\ \partial_n\end{pmatrix}\underline{k}_1(y',\xi',D_n)a(y')\right)\operatorname{OP}'(t(\xi',D_n)\underline{r}^r_{0,\lambda}(y',\xi',D_n))\\
&\qquad+ O((1+|\lambda|)^{-\varepsilon}),\\
&F^*_+\nabla\widetilde{K}_0^+2\nu^+\cdot\gamma_1^+R^+_{1,\lambda}F_+^{*,-1}\\
&\quad= \operatorname{OP}'\left(U^TA(y')\begin{pmatrix} i\xi' \\ \partial_n\end{pmatrix}\underline{k}_0(y',\xi',D_n)a(y')\right)\operatorname{OP}'(t(\xi',D_n)\underline{r}^r_{1,\lambda}(y',\xi',D_n))\\
&\qquad+ O((1+|\lambda|)^{-\varepsilon}).
\end{aligned}$$

Hence Lemma 4.10.4 yields

$$\begin{aligned}
&F_\pm^* \nabla \widetilde{K}_1^\pm \gamma_\nu^\pm (\Delta - \nabla \operatorname{div}) R_{0,\lambda}^\pm F_\pm^{*,-1} \\
&\quad = \operatorname{OP}'(\underline{k}_0^r(y',\xi',D_n)\underline{t}_0^r(y',\xi',D_n)\underline{r}_{0,\lambda}^r(y',\xi',D_n)) + O((1+|\lambda|)^{-\varepsilon}) \\
&F_+^* \nabla \widetilde{K}_0^+ 2\nu^+ \cdot \gamma_1^+ R_{1,\lambda}^+ F_+^{*,-1} \\
&\quad = \operatorname{OP}'(\underline{k}_1^r(y',\xi',D_n)\underline{t}_1^r(y',\xi',D_n)\underline{r}_{1,\lambda}^r(y',\xi',D_n)) + O((1+|\lambda|)^{-\varepsilon}),
\end{aligned}$$

where we have used that $t'_\lambda(x',\xi',D_n) := t(\xi',D_n)\underline{r}_{j,\lambda}^r(x',\xi',D_n)$ is a parameter-dependent trace operator of order $d-2$, class 0, and regularity $\frac{1}{2}$ because of the composition rules for boundary symbol operators, cf. Remarks 4.7. Therefore all error terms can be estimated by $C|t'_\lambda|_k^{(d-2+2\varepsilon)} \leq C(1+|\lambda|)^{-\varepsilon}|t'_\lambda|_k^{(d-2,\frac{1}{2})}$. Finally, the statements for the traces of $R_{0,\lambda}$ and $R_{10,\lambda}$ are direct consequences of Lemma 5.17. ■

Remark 5.19 Note that the latter lemma is used in the proof of [3, Lemma 4.3] and that we did not use the unique solvability of the reduced and generalized Stokes equations so far.

Proof of Theorem 1.1: Let

$$S'_{1,\lambda} f = (\lambda - \Delta + G_0)^{-1} S_{1,\lambda} f, \quad S'_{2,\lambda} f = (\lambda - \Delta + G_{10})^{-1} S_{2,\lambda} f + K_{10,\lambda}^r S_{3,\lambda} f,$$

where $K_{10,\lambda}^r$ denotes the solution operator of the reduced Stokes equation (1.5)-(1.7) with $j=1$ and right-hand side $f=0$ and $a^+ \in W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_\gamma^+)^n$. Then

$$(\lambda - \Delta + G_0)^{-1} = R_{0,\lambda} + S'_{1,\lambda}, \quad (\lambda - \Delta + G_{10})^{-1} = R_{10,\lambda} + S'_{2,\lambda}$$

with $\|S'_{j,\lambda}\|_{\mathcal{L}(L^q(\Omega_\gamma))} \leq C_{q,\delta}(1+|\lambda|)^{-1-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$, $|\lambda| \geq c > 0$ for every $c > 0$. Therefore $S'_{j,\lambda}$, $j=1,2$, corresponds to an absolutely integrable part in (1.8) and can be neglected, cf. [4, Lemma 2.1]. Since $-\Delta + G_{10}$ is invertible and because of Lemma 5.18, Theorem 5.13, and Lemma 5.14, we conclude that $-\Delta + G_{10}$ admits a bounded H_∞-calculus with respect to δ.

Finally, the Stokes operator admits a bounded H_∞-calculus with respect to δ since $A_q = -\Delta + G_0|_{J_{q,0}(\Omega_\gamma)}$, $(\lambda + A_q)^{-1} = (\lambda - \Delta + G_0)^{-1}|_{J_{q,0}(\Omega_\gamma)}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, cf. [3, Remark 3.3], and the Stokes operator is invertible. ■

A Non-Smooth Pseudodifferential Operators

For the proof of Theorem 3.2, we will proceed as in [23, §2.1] and have to verify that all statements remain true for operator-valued pseudodifferential operators.

For this purpose, we need the following more general class of non-smooth operator valued pseudodifferential operators, which generalizes the Hörmander classes $S_{1,\delta}^m$.

Definition A.1 The set $C^s_* S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0,H_1))$, $\delta \in [0,1]$, $s > 0$, $m \in \mathbb{R}$, is the set of all symbols $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(H_0,H_1)$ that are smooth in ξ and are in C^s_* with respect to x and satisfy the estimates

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \|D^\alpha_\xi p(x,\xi)\|_{\mathcal{L}(H_0,H_1)} &\leq C_\alpha \langle\xi\rangle^{m-|\alpha|} \\ \|D^\alpha_\xi p(.,\xi)\|_{C^s_*(\mathbb{R}^n;\mathcal{L}(H_0,H_1))} &\leq C_\alpha \langle\xi\rangle^{m-|\alpha|+s\delta} \end{aligned}$$

for all $\alpha \in \mathbb{N}^n_0$.

The corresponding operators are defined in the usual way.

The proof of Theorem 3.2 relies essentially on the following operator-valued variant of [23, Theorem 2.1.A.].

THEOREM A.2 *If* $r > 0$, $1 < q < \infty$, *and* $p \in C^r_* S^m_{1,1}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0,H_1))$, *then*

$$p(x,D_x)\colon H^{s+m}_q(\mathbb{R}^n; H_0) \to H^s_q(\mathbb{R}^n; H_1)$$

for all $0 < s < r$.

Proof: The proof is done in the same way as in [23] using elementary estimates and inequalities based on

$$C'_q \|u\|_{L^q(\mathbb{R}^n;H)} \leq \left\| \left(\sum_{j=0}^\infty \|\varphi_j(D_x)u\|^2_H \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^n;H)} \leq C_q \|u\|_{L^q(\mathbb{R}^n;H)}$$

for all $1 < q < \infty$, where $H = \mathbb{C}$ and $\varphi_j(\xi)$ are smooth functions such that $\operatorname{supp}\varphi_0 \subseteq B_1(0)$, $\operatorname{supp}\varphi_1 \subseteq \{\frac{1}{2} \leq |\xi| \leq 2\}$, $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for $j \geq 2$, and $\sum_{j=0}^\infty \varphi_j(\xi)^2 = 1$.

The latter estimate is also valid if H is a Hilbert space. In this case $\ell^2(\mathbb{N}_0; H)$ is again a Hilbert space and we can apply the vector-valued Mikhlin multiplier theorem as in the proof of the usual Littlewood-Paley estimate, cf. [23, §0.11.]. ■

First of all, the continuity of $p(D_x,x)$ in Theorem 3.2 can be reduced by duality to the statement for operators in L-form. Since $C^r_* S^m_{1,\delta} \subseteq C^r_* S^m_{1,1}$ for $\delta \in [0,1]$, the last theorem implies the statement of Theorem 3.2 for $s > 0$. For the proof in the case of $-r < s \leq 0$, we will use the technique of *symbol smoothing.*

Let $p \in C^r_* S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0,H_1))$, $r > 0$. If $r \notin \mathbb{N}$, then $C^r_* = C^r$ and there exists a decomposition

$$\begin{aligned} p(x,\xi) &= p^\#(x,\xi) + p^b(x,\xi) \qquad \text{with} \qquad \text{(A.1)} \\ p^\#(x,\xi) &= \sum_{k=0}^\infty \Psi_0(2^{-k(\gamma-\delta)}D_x)p(x,\xi)\psi_k(\xi), \end{aligned}$$

where $\gamma \in (\delta, 1]$ and $\Psi_0 \in C_0^\infty(\mathbb{R}^n)$ with $\Psi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\Psi_0(\xi) = 0$ for $|\xi| \geq 2$ and $\psi_k(\xi) := \Psi_0(2^{-k}\xi) - \Psi_0(2^{-k+1}\xi)$, $k \geq 1$, $\psi_0(\xi) := \Psi_0(\xi)$. For this decomposition we have

$$p^\#(x,\xi) \in S^m_{1,\gamma}, \qquad p^b(x,\xi) \in C^r_* S^{m-r(\gamma-\delta)}_{1,\gamma}, \tag{A.2}$$

cf. [24, Proposition 3.2.] or [23, §1.3.]. This decomposition easily carries over to the vector-valued case since it only uses the symbol estimates.

Using this decomposition we prove:

Proposition A.3 Let $1 < q < \infty$ and let $p \in C^r_* S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$, $m \in \mathbb{R}$, $\delta \in [0,1]$, $r > 0$. Then

$$p(x, D_x) \colon H^{s+m}_q(\mathbb{R}^n; H_0) \to H^s_q(\mathbb{R}^n; H_1)$$

for all $s \in \mathbb{R}$ with $-r(1-\delta) < s < r$.

Proof: The proof is just a modification of the proof of [23, Proposition 2.1.D]. ■

Now we are able to prove Theorem 3.3.

Proof of Theorem 3.3: Let $p_i \in C^{\tau_i}_* S^{m_i}_{1,0}$, $i = 1, 2$, as in the assumption of the theorem. We set $\delta_i := \frac{\theta}{\tau_i}$. Then

$$\delta_1 \geq \delta_2, \quad -\tau_1(1-\delta_1) = -\tau_1 + \theta < s, \quad -\tau_2(1-\delta_2) = -\tau_2 + \theta < s + m_1, \; 1 - \delta_i \geq \theta$$

since $\theta < \min\{\tau_1 + s, \tau_2 + s + m_1\}$ and $\theta \leq \frac{\tau_i}{1+\tau_i}$. Let $p_i(x,\xi) = p_i^\#(x,\xi) + p_i^b(x,\xi)$ such that $p_i^\# \in S^{m_i}_{1,\delta_i}$ and $p_i^b \in C^{\tau_i}_* S^{m_i-\theta}_{1,\delta_i}$. Then we get

$$\begin{aligned}
&(p_1p_2)(x,D_x) - p_1(x,D_x)p_2(x,D_x) \\
&\quad = (p_1^\# p_2^\#)(x,D_x) - p_1^\#(x,D_x)p_2^\#(x,D_x)(p_1^\# p_2^b)(x,D_x) - p_1^\#(x,D_x)p_2^b(x,D_x) + \\
&\qquad +(p_1^b p_2)(x,D_x) - p_1^b(x,D_x)p_2(x,D_x).
\end{aligned}$$

We will estimate each difference separately with the aid of Proposition A.3.

1. Due to the usual symbolic calculus $(p_1^\# p_2^\#)(x,D_x) - p_1^\#(x,D_x)p_2^\#(x,D_x)$ is a pseudodifferential operator with symbol in $S^{m_1+m_2-1+\delta_1}_{1,\delta_1}$ since $\delta_1 \geq \delta_2$ and $\delta_1 < 1$. (See e.g. [16, Chapter 2, Theorem 1.7].)

   Due to Proposition A.3 and $1 - \delta_1 \geq \theta$, we get the continuity

   $$(p_1^\# p_2^\#)(x,D_x) - p_1^\#(x,D_x)p_2^\#(x,D_x) \colon H^{s+m_1+m_2-\theta}_q(\mathbb{R}^n; H_0) \to H^s_q(\mathbb{R}^n; H_2).$$

2. Using Proposition A.3 again,

   $$p_2^b(x,D_x) \colon H^{s+m_1+m_2-\theta}_q(\mathbb{R}^n; H_0) \to H^{s+m_1}_q(\mathbb{R}^n; H_1)$$

since $-\tau_2(1-\delta_2) = -\tau_2 + \theta < s + m_1 < \tau_1$. Moreover, $p_1^\#(x,\xi)$ is a smooth symbol of order m_1. Therefore we conclude

$$p_1^\#(x,D_x)p_2^b(x,D_x)\colon H_q^{s+m_1+m_2-\theta}(\mathbb{R}^n;H_0) \to H_q^s(\mathbb{R}^n;H_2). \tag{A.3}$$

Considering $(p_1^\# p_2^b)(x,D_x)$, we observe that $p_1^\# p_2^b \in C_*^{\tau_1} S_{1,\delta_1}^{m_1+m_2-\theta}$. Hence we get the same mapping properties as in (A.3).

3. Since $p_1^b p_2 \in C_*^{\tau_1} S_{1,\delta_1}^{m_1+m_2-\theta}$, we get the same continuity as in (A.3). Finally,

$$\begin{aligned} p_1^b(x,D_x)&\colon H_q^{s+m_1-\theta}(\mathbb{R}^n;H_1) \to H_q^s(\mathbb{R}^n;H_2),\\ p_2(x,D_x)&\colon H_q^{s+m_1+m_2-\theta}(\mathbb{R}^n;H_0) \to H_q^{s+m_1-\theta}(\mathbb{R}^n;H_1) \end{aligned}$$

because of $-\tau_1(1-\delta_1) = -\tau_1+\theta < s < \tau_1$ and $-\tau_2 < s+m_1-\theta < \tau_2$.

■

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