Generalized Stokes Resolvent Equations in an Infinite Layer with Mixed Boundary Conditions*

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Abstract

In this paper we prove unique solvability of the generalized Stokes resolvent equations in an infinite layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$, $n \geq 2$, in L^q-Sobolev spaces, $1 < q < \infty$, with slip boundary condition on the "upper boundary" $\partial\Omega^+ = \mathbb{R}^{n-1} \times \{1\}$ and non-slip boundary condition on the "lower boundary" $\partial\Omega^- = \mathbb{R}^{n-1} \times \{-1\}$. The solution operator to the Stokes system will be expressed with the aid of the solution operators of the Laplace resolvent equation and a Mikhlin multiplier operator acting on the boundary. The present result is the first step to establish an L^q-theory for the free boundary value problem studied by Beale [8] and Sylvester [21] in L^2-spaces.

Key words: Stokes equations, free boundary value problems, boundary value problems for pseudodifferential operators

AMS-Classification: 35 Q 30, 76 D 07, 35 R 35, 35 S 15

1 Introduction

Let $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$, $n \geq 2$, be an *infinite layer* and let $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. We study the generalized Stokes resolvent equations with mixed boundary conditions

$$(\lambda - \Delta)u + \nabla p = f \quad \text{in } \Omega_0, \tag{1.1}$$

$$\operatorname{div} u = g \quad \text{in } \Omega_0, \tag{1.2}$$

$$T_1^+(u,p) = a^+ \quad \text{on } \partial\Omega_0^+, \tag{1.3}$$

$$\gamma_0^- u = 0 \quad \text{on } \partial\Omega_0^-, \tag{1.4}$$

where

$$T_1^+(u,p) = \nu \cdot S(u) - \nu p|_{\partial\Omega_0^+}, \quad S(u) = \nabla u + (\nabla u)^T, \quad \gamma_0^- u = u|_{\partial\Omega_0^-},$$

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$\partial\Omega_0^\pm = \mathbb{R}^{n-1} \times \{\pm 1\}$ and ν denotes the exterior normal vector on $\partial\Omega_0$. This system arises in the study of an infinite ocean of water under the force of gravity leading to a free boundary value problem for the instationary Navier-Stokes equations. Passing to Lagrangian coordinates, linearizing the transformed system, and applying Laplace transformation, one ends up with the *generalized* Stokes system (1.1)-(1.4) in a *layer-like domain* $\Omega_\gamma := \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x') < x_n < \gamma^+(x')\}$, where γ^+ and γ^- are suitable functions describing the upper and the lower boundary. Having proved unique solvability of (1.1)-(1.4) of an infinite layer, the results can be extended to *asymptotically flat layers*, which are layer-like domains that are close to an infinite layer "at infinity", by means of perturbation arguments and cut-off techniques. This extension is carried out in [5]. Using pseudodifferential operator techniques, we will show in [6] the existence of a bounded H_∞-calculus of the associated (reduced) Stokes operator, see Remark 4.2, which implies the maximal regularity of the corresponding instationary (reduced) Stokes equations. Using these results, one can solve the free boundary value problem – studied up to now in L^2-Sobolev spaces in [8, 21] – in the setting of L^q-Sobolev spaces by a method similar to [19]. The advantage of the L^q-theory is that the regularity assumptions can be reduced in comparison to [8, 21] since one can use the embedding $W^1_q(\Omega_0) \hookrightarrow L^\infty(\Omega_0)$ for $q > n$ instead of $W^m_2(\Omega_0) \hookrightarrow L^\infty(\Omega_0)$ for $m > \frac{n}{2}$, cf. [4] or [19] for bounded domains. Of course the L^q-theory is more demanding than the L^2-theory based on Hilbert space methods. Therefore it is divided into several parts.

Our main result is:

THEOREM 1.1 *Let* $1 < q < \infty$, $\delta \in (0, \pi)$, *and* $\lambda \in \Sigma_\delta \cup \{0\}$. *Then for every* $(f, g) \in L^q(\Omega_0)^n \times W^1_{q,\lambda}(\Omega_0)$ *and* $a^+ \in W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0^+)^n$ *there is a unique solution* $(u, p) \in W^2_{q,\lambda}(\Omega_0)^n \times W^1_q(\Omega_0)$ *of (1.1)-(1.4). Moreover,*

$$\begin{aligned} &(1 + |\lambda|)\|u\|_q + (1 + |\lambda|)^{\frac{1}{2}}\|\nabla u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q + \|\gamma_0^+ p\|_{1-\frac{1}{q},q,\lambda} \\ &\quad\leq\; C_\delta \Big(\|(f, \nabla g)\|_q + (1 + |\lambda|)^{\frac{1}{2}}\|g\|_q + (1 + |\lambda|)\|g\|_{{}_0W^{-1}_q} + \|a^+\|_{1-\frac{1}{q},q,\lambda}\Big) \qquad (1.5) \end{aligned}$$

uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$. *If additionally* $(f, g) \in L^r(\Omega_0)^n \times W^1_{r,\lambda}(\Omega_0)$ *and* $a^+ \in W^{1-\frac{1}{r}}_{r,\lambda}(\partial\Omega_0^+)^n$ *for an* $1 < r < \infty$, *then* $(u, p) \in W^2_{r,\lambda}(\Omega_0)^n \times W^1_r(\Omega_0)$.

Here $W^s_{q,\lambda}$ denotes a parameter-dependent variant of the usual Sobolev-Slobodeckij spaces. These variants and ${}_0W^{-1}_q$ are defined in Section 2.2 below. Moreover, $\Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \delta\}$.

The unique solvability of the system (1.1)-(1.2) with (pure) non-slip condition, $u|_{\partial\Omega_0} = 0$, has been studied by Wiegner [23] using explicit solution formulas obtained by partial Fourier transformation. Moreover, Abe and Shibata [1, 2] solved the Stokes resolvent equations, where $g = 0$, with non-slip boundary condition.

As in [23] we use partial Fourier transformation to calculate the solution operator, but we do not solve (1.1)-(1.4) directly. Using the approach of Grubb and Solonnikov

[15], we reduce the Stokes system to a pseudodifferential boundary value problem, which is called *reduced Stokes equations.* Using the idea of Grubb [13, Section 3], the solution operator can be expressed with aid of the solution operator of the Laplace resolvent equation with mixed Neumann-Dirichlet boundary condition and the inverse of a Mikhlin multiplier operator acting on the boundary. The latter inverse exists on $W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0)$, $1 < q < \infty$, iff the generalized or equivalently the reduced Stokes equations are uniquely solvable for $q = 2$. Therefore Theorem 1.1 can be reduced to the case $q = 2$, cf. Corollary 4.7 below.

The structure of the article is as follows:

In Section 2, we introduce basic notations, function spaces, and some fundamental results on scalar and operator-valued Mikhlin multiplier operators. In Section 3, we study the Laplace resolvent equation with mixed Neumann-Dirichlet boundary conditions, which is fundamental for the reduction of the generalized Stokes equations done in Section 4.1 and the reduction to the boundary in Section 4.2. As a byproduct of the results in Section 3, we obtain the Helmholtz decomposition of $L^q(\Omega_0)^n$ in a form with mixed boundary conditions, cf. Corollary 3.2 below. Finally in Section 4.3, we prove the unique solvability for $q = 2$, which implies Theorem 1.1 because of the results obtained by the reduction to the boundary, cf. Corollary 4.7 below.

Remark 1.2 The present approach can be adapted to the case of pure Dirichlet boundary conditions, which is done in [7, Section 5]. The same is true for all combinations of the boundary conditions studied in [15]. For all these boundary conditions there is an analogous reduction of the generalized Stokes equations, cf. [15, Section 4 and 5]. Since the corresponding reduced Stokes equations have the same structure, the reduction to the boundary done in Section 4.2 and all other arguments work by the same way, see also [13].

But there may arise some difficulties for the case $\lambda = 0$ which can be an exceptional case. In the case of pure Dirichlet boundary conditions, the equivalence of unique solvability of the reduced and the generalized Stokes equations does not hold for $\lambda = 0$ if the equations are considered in the L^q-Sobolev spaces used in [23] and in the present contribution. In these spaces the generalized Stokes equations are uniquely solvable although the reduced Stokes equations are not, cf. [7, Remark 5.4].

2 Preliminaries

2.1 Notation

In the following $\mathbb{N}$ denotes the set of natural numbers (without 0), $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ are the sets of integers, real numbers, and complex numbers, respectively. If $\alpha \in \mathbb{N}_0^n$ is a multi-index, $|\alpha| := \alpha_1 + \ldots + \alpha_n$, $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_n!$. Moreover, $x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$ and $D_x^\alpha := D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, where $D_{x_j} = \frac{1}{i}\partial_{x_j}$ and $\partial_{x_j} f$ is the partial derivative with respect to x_j. For $s \in \mathbb{R}$ let $[s]$ be the largest integer $\leq s$ and set $\{s\} := s - [s] \in [0, 1)$.

If $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^n$, then

$$\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}, \quad \langle \lambda; \xi \rangle := (1+|\lambda|+|\xi|^2)^{\frac{1}{2}}, \quad |\lambda; \xi| := (|\lambda|+|\xi|^2)^{\frac{1}{2}},$$

where $|\xi|$ is the Euclidean length of ξ. Moreover, $\Sigma_\delta := \{z \in \mathbb{C} : |\arg z| < \delta\}$, $\delta \in (0,\pi)$, where $\arg z \in (-\pi, \pi]$. We will use the simple inequalities

$$c_\delta(|\lambda|^{\frac{1}{2}}+s) \leq \left|(\lambda+s^2)^{\frac{1}{2}}\right| \leq C_\delta(|\lambda|^{\frac{1}{2}}+s), \tag{2.1}$$

$$\mathrm{Re}(\lambda+s^2)^{\frac{1}{2}} \geq c_\delta|(\lambda+s^2)^{\frac{1}{2}}|, \tag{2.2}$$

which hold uniformly in $\lambda \in \Sigma_\delta, s \geq 0$, where $(\lambda+s^2)^{\frac{1}{2}}$ is defined as the unique square root of $\lambda+s^2$ in $\Sigma_{\delta/2}$.

If $M \subseteq \mathbb{R}^n$ is measurable and X is a Banach space, then $L^q(M)$, $1 \leq q \leq \infty$ denotes the usual Lebesgue-space and $L^q(M;X)$ its vector-valued variant. Moreover, if $\omega\colon M \to \mathbb{R}$ is a measurable function and $\omega(x) > 0$ a.e., then $L^q(M;\omega)$ indicates the Lebesgue-space with respect to the measure $d\mu = \omega(x)dx$. For an open set $\Omega \subset \mathbb{R}^n$ let $L^q_{loc}(\overline{\Omega})$, $1 \leq q \leq \infty$, be the vector space of all measurable functions $f\colon \overline{\Omega} \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, such that $f \in L^q(B \cap \overline{\Omega})$ for all balls B with $B \cap \overline{\Omega} \neq \emptyset$. Moreover, $\mathcal{S}(\mathbb{R}^n)$ is the set of all smooth and rapidly decreasing function on $\mathbb{R}^n$ and $\mathcal{S}(\mathbb{R}^n;X)$ its vector-valued variant. If $\Omega \subseteq \mathbb{R}^n$ is a domain, $C_0^\infty(\Omega)$ is the set of all smooth functions such that $\operatorname{supp} f \subset \Omega$ is compact. Furthermore,

$$C^\infty_{(0)}(\overline{\Omega}) := \{f\colon \overline{\Omega} \to \mathbb{R} : f = u|_{\overline{\Omega}}, \text{where } u \in C_0^\infty(\mathbb{R}^n)\}.$$

The dual of a topological vector space V is denoted by V'. If $v \in V$ and $v' \in V'$, then $\langle v, v' \rangle := v'(v)$ is the duality product. If $A\colon V \to W$ is a continuous linear operator, $A'\colon W' \to V'$ denotes its adjoint. Moreover, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are the range and the kernel of A, resp.

The Fourier transformation $\mathcal{F} = \mathcal{F}_{x \mapsto \xi}$ is defined as

$$\mathcal{F}_{x\mapsto\xi}[f](\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

for a suitable function $f\colon \mathbb{R}^n \to \mathbb{C}$ and $\mathcal{F}^{-1}$ denotes its inverse. If $x \in \mathbb{R}^n$, $n \geq 2$, then we use the decomposition $x = (x', x_n)$, where x' denotes the first $n-1$ components. Moreover, we will use the partial Fourier transformation $\tilde{f}(\xi', x_n) := \mathcal{F}_{x' \mapsto \xi'}[f](\xi', x_n)$.

Finally, $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$ and $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0,\infty)$. If f and g are defined on $\mathbb{R}^n$ and Ω_0, $r_{\Omega_0} f := f|_{\Omega_0}$ and $e_{\Omega_0} g$ denote the restriction to Ω_0 and extension by 0 to $\mathbb{R}^n$ of f, g, resp.

2.2 Parameter-Dependent Function Spaces

As in Grubb and Kokholm [14, Section 1], we introduce function spaces with parameter-dependent norm. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\lambda \in \mathbb{C}$. Then $W^m_{q,\lambda}(\Omega) = \{f \in L^q(\Omega) :$

$D^\alpha f \in L^q(\Omega), |\alpha| \le m\}$ is the usual Sobolev space normed by

$$\|f\|_{m,q,\lambda}^q := \sum_{|\alpha|\le m} (1+|\lambda|)^{q\frac{m-|\alpha|}{2}} \|D_x^\alpha f\|_q^q$$

for $m \in \mathbb{N}_0$, $1 < q < \infty$. Moreover, $W^m_{q,0,\lambda}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^m_{q,\lambda}(\Omega)$, $W^{-m}_{q,\lambda} := W^m_{q',0,\lambda}(\Omega)'$, where $\frac{1}{q}+\frac{1}{q'}=1$, its dual, and $W^m_q(\Omega) := W^m_{q,\lambda}|_{\lambda=0}$, $W^m_{q,0}(\Omega) := W^m_{q,0,\lambda}|_{\lambda=0}$ denote the Sobolev spaces equipped with the usual parameter-independent norm.

If $u \in W^m_q(\Omega)$, $m \ge 1+j$, $j \in \mathbb{N}_0$, then $\gamma_j u := \partial_\nu^j u|_{\partial\Omega}$, where ν denotes the exterior normal vector. Because of [14, Theorem 1.1], $\gamma_j \colon W^m_{q,\lambda}(\mathbb{R}^n_+) \to W^{m-j-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1})$ with $\|\gamma_j u\|_{m-j-\frac{1}{q},q,\lambda} \le C\|u\|_{m,q,\lambda}$ uniformly in $\lambda \in \mathbb{C}$, where

$$\begin{aligned} W^s_{q,\lambda}(\mathbb{R}^{n-1}) &:= B^s_{qq,\lambda}(\mathbb{R}^{n-1}) = \{a \in W^{[s]}_q(\mathbb{R}^{n-1}) \colon \|a\|_{s,q,\lambda} < \infty\}, \\ \|a\|^q_{s,q,\lambda} &:= (1+|\lambda|)^{\frac{\{s\}}{2}} \|a\|^q_{[s],q,\lambda} + \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|a(x)-a(y)|^q}{|x-y|^{n-1+\{s\}q}} dy dx \end{aligned}$$

for $s \in \mathbb{R}_+ \setminus \mathbb{N}_0$. Since $\partial\Omega_0$ is the disjoint union of $\partial\Omega_0^\pm$, we can identify a function $a \colon \partial\Omega_0 \to \mathbb{C}$ with its values on the upper boundary a^+ and the lower boundary a^-. Hence we identify the corresponding trace space $W^{m-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0)$ with $W^{m-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1}) \times W^{m-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1})$.

Lemma 2.1 *Let $1 < q < \infty$, $m \in \mathbb{N}$. Then*

$$(\gamma_0, \dots, \gamma_{m-1}) \colon W^m_{q,\lambda}(\Omega_0) \to \prod_{j=0}^{m-1} W^{m-j-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0)$$

is a surjective and continuous linear mapping with operator norm independent of λ. Moreover, $W^m_{q,0,\lambda}(\Omega_0) = \{f \in W^m_{q,\lambda}(\Omega_0) : \gamma_j f = 0$ for $j = 0, \dots, m-1\}$.

Proof: Using a partition of unity on $\overline{\Omega_0}$, the statements are easily reduced to the corresponding statements for the half-spaces $\mathbb{R}^n_+$, cf. [14, Theorem 1.1]. ■

Moreover, we define homogeneous and parameter-dependent variants of the Bessel potential and Besov spaces defined in [14]. Let

$$\dot{H}^s_{q,\lambda}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) \colon \mathcal{F}^{-1}\left[|\lambda;\xi|^s \hat{u}\right] \in L^q(\mathbb{R}^n)\right\}, \quad \|u\|_{\dot{H}^s_{q,\lambda}} := \left\||\lambda; D_x|^s u\right\|_q,$$

for $s \in \mathbb{R}$, $1 < q < \infty$, $\lambda \neq 0$, $\dot{W}^m_{q,\lambda}(\mathbb{R}^n) := \dot{H}^m_{q,\lambda}(\mathbb{R}^n)$ for $m \in \mathbb{Z}$,

$$\begin{aligned} \dot{B}^s_{q,\lambda}(\mathbb{R}^n) &:= \dot{B}^s_{qq,\lambda}(\mathbb{R}^n) := \{a \in \dot{W}^{\{s\}}_q(\mathbb{R}^n) \colon \|a\|_{\cdot,s,q,\lambda} < \infty\} \\ \|a\|^q_{\cdot,s,q,\lambda} &:= |\lambda|^{\frac{q[s]}{2}} \|a\|^q_{\dot{W}^{\{s\}}_{q,\lambda}} + \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|a(x)-a(y)|^q}{|x-y|^{n+q[s]}} dy dx, \end{aligned}$$

and $B^{-s}_{q,\lambda}(\mathbb{R}^n) := (B^s_{q,\lambda}(\mathbb{R}^n))'$, where $0 < s \notin \mathbb{N}$ and $1 < q < \infty$.

If $\lambda = 0$, then $\dot{H}^s_{q,\lambda}(\mathbb{R}^n)$, $\dot{W}^m_q(\mathbb{R}^n)$, and $dotB^s_{q,\lambda}$ are defined as the usual homogeneous Bessel potential $\dot{H}^s_q(\mathbb{R}^n)$, Sobolev $\dot{W}^s_q(\mathbb{R}^n)$, and Besov space $\dot{B}^s_q(\mathbb{R}^n)$, resp.; cf. Bergh and Löfström [9, Section 6.3] or Triebel [22] for the definition of the latter spaces. If $\lambda \neq 0$, the spaces coincide with the usual (non-homogeneous) spaces $H^s_q(\mathbb{R}^n)$, $B^s_q(\mathbb{R}^n)$, and $W^m_q(\mathbb{R}^n)$, respectively, as sets, but with different norms. These parameter-dependent spaces are well adapted to the Laplace resolvent equation in $\mathbb{R}^n_+$ and similar problems.

Using the scaling operator $(M_\lambda f)(x) = f(|\lambda|^{-\frac{1}{2}}x), \lambda \neq 0$, we can conclude similarly to [14, Section 1] that

$$\|u\|_{\dot{H}^s_{q,\lambda}} = |\lambda|^{-\frac{1}{2}(\frac{n}{q}+s)}\|M_\lambda f\|_{H^s_q}, \qquad \|u\|_{\dot{B}^s_{q,\lambda}} = |\lambda|^{-\frac{1}{2}(\frac{n}{q}+s)}\|M_\lambda f\|_{B^s_q}.$$

Hence we obtain as in [14, Section 1]:

Lemma 2.2 *Let $1 < q < \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, and $\theta \in (0,1)$. Then*

$$(\dot{B}^{s_0}_{q,\lambda}(\mathbb{R}^n), \dot{B}^{s_1}_{q,\lambda}(\mathbb{R}^n))_{\theta,q} = \dot{B}^s_{q,\lambda}(\mathbb{R}^n), \qquad (\dot{H}^{s_0}_{q,\lambda}(\mathbb{R}^n), \dot{H}^{s_1}_{q,\lambda}(\mathbb{R}^n))_{\theta,q} = \dot{B}^s_{q,\lambda}(\mathbb{R}^n),$$
$$(\dot{B}^{s_0}_{q,\lambda}(\mathbb{R}^n), \dot{B}^{s_1}_{q,\lambda}(\mathbb{R}^n))_{[\theta]} = \dot{B}^s_{q,\lambda}(\mathbb{R}^n), \qquad (\dot{H}^{s_0}_{q,\lambda}(\mathbb{R}^n), \dot{H}^{s_1}_{q,\lambda}(\mathbb{R}^n))_{[\theta]} = \dot{H}^s_{q,\lambda}(\mathbb{R}^n),$$

where $s = (1-\theta)s_0 + \theta s_1$ and $(.,.)_{\theta,q}$ denotes the real and $(.,.)_{[\theta]}$ the complex interpolation space, cf. [9]. Moreover, the norm of the interpolation spaces and the norm of $\dot{B}^s_{q,\lambda}(\mathbb{R}^n)$, $\dot{H}^s_{q,\lambda}(\mathbb{R}^n)$, resp., are equivalent with constants independent of $\lambda \neq 0$.

Because of the analogous interpolation properties of the usual homogeneous Besov and Bessel potential spaces, cf. [9, Chapter 6.3], the statements of the latter lemma are also true for $\lambda = 0$.

In order to consider mixed boundary conditions, we define

$$^0W^m_q(\Omega_0) := \{u \in W^m_q(\Omega_0) : \gamma^+_j u = 0, j = 0, \dots, m-1\},$$

where $\gamma^+_j u = \partial^j_\nu u|_{\partial\Omega^+_0}$, $1 < q < \infty$, and $m \in \mathbb{N}$. Analogously, $_0W^m_q(\Omega_0)$ is defined. Moreover,

$$^0W^{-m}_q(\Omega_0) := (_0W^m_{q'}(\Omega_0))', \quad \text{and} \quad {}_0W^{-m}_q(\Omega_0) := (^0W^m_{q'}(\Omega_0))'.$$

If $f \in L^q_{loc}(\overline{\Omega_0})$ and

$$\sup\{|(f,v)_{\Omega_0}| : v \in C^\infty_{(0)}(\overline{\Omega_0}) \cap {}_0W^1_{q'}(\Omega_0), \|\nabla v\|_{q'} = 1\} < \infty,$$

then f extends to a unique functional on $_0W^1_{q'}(\Omega_0)$. In this case we write $f \in L^q_{loc}(\overline{\Omega_0}) \cap {}^0W^{-1}_q(\Omega_0)$ for short.

If $u \in L^q(\Omega_0)^n$ with $\operatorname{div} u \in L^q_{loc}(\overline{\Omega_0}) \cap {}^0W^{-1}_q(\Omega_0)$, then we define the trace $\gamma^+_\nu u = \nu \cdot u|_{\partial\Omega^+_0} \in W^{-\frac{1}{q}}_q(\partial\Omega^+_0)$ as

$$\langle \gamma^+_\nu u, v \rangle = (u, \nabla V) + (\operatorname{div} u, V), \tag{2.3}$$

where $v \in W_{q'}^{1-\frac{1}{q'}}(\partial\Omega_0^+)$ and $V \in {}_0W^1_{q'}(\Omega_0)$ with $\gamma_0^+ V = v$. As in the case of the usual definition of the weak trace $\gamma_\nu u = \nu \cdot u|_{\partial\Omega}$, the definition does not depend on the choice of V. Moreover,

$$\|\gamma_\nu^+ u\|_{-\frac{1}{q},q} \leq C\left(\|u\|_q + \|\operatorname{div} u\|_{{}^0W_q^{-1}(\Omega_0)}\right). \tag{2.4}$$

In the same way we can define $\gamma_\nu^- u \in W_q^{-\frac{1}{q}}(\partial\Omega_0^-)$.

Lemma 2.3 *Let $1 < q < \infty$ and $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, be an infinite layer. Then*

$$\|u\|_{L^q(\Omega_0)} \leq C_q \|\nabla u\|_{L^q(\Omega_0)}$$

for all $u \in {}^0W^1_q(\Omega_0)$ and $u \in {}_0W^1_q(\Omega_0)$.

Proof: The lemma is an easy consequence of Poincaré's inequality on the interval $(-1, 1)$. ■

Moreover, we will need Korn's inequality since we also treat the boundary condition of second kind $T_1^+(u,p) = 0$, see Section 4 below.

Lemma 2.4 *Let $1 < q < \infty$ and $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, be an infinite layer. Then*

$$\|u\|_{1,q} \leq C_q \|S(u)\|_q$$

for all $u \in {}^0W^1_q(\Omega_0)^n$ and $u \in {}_0W^1_q(\Omega_0)^n$, where $S(u) = \nabla u + \nabla u^T$.

Proof: The proof is given in [8, Lemma 2.6]. ■

Lemma 2.5 *Let $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, be an infinite layer. Then*

$$\|\gamma_0 v\|_{\frac{1}{2},2,\lambda} \leq C\|\partial_n v\|_2^{\frac{1}{2}} \|((1+|\lambda|)v, \nabla' v)\|_2^{\frac{1}{2}} \qquad \textit{for all } v \in W^1_{2,\lambda}(\mathbb{R}^n_+),$$

$$\|\gamma_0^\pm v\|_{\frac{1}{2},2,\lambda} \leq C\|(v, \partial_n v)\|_2^{\frac{1}{2}} \|((1+|\lambda|)v, \nabla' v)\|_2^{\frac{1}{2}} \qquad \textit{for all } v \in W^1_{2,\lambda}(\Omega_0). \tag{2.5}$$

Proof: First let $v \in W^1_{2,\lambda}(\mathbb{R}^n_+)$. As in the parameter-independent case,

$$B^{\frac{1}{2}}_{2,\lambda}(\mathbb{R}^{n-1}) = H^{\frac{1}{2}}_{2,\lambda}(\mathbb{R}^{n-1}) := \{a \in \mathcal{S}'(\mathbb{R}^{n-1}) : \langle\lambda;\xi'\rangle^{\frac{1}{2}}\tilde{a}(\xi') \in L^2(\mathbb{R}^{n-1})\}$$

with equivalent norms, where the constants in the equivalence can be chosen independently of λ, cf. [14, (1.11)]. Then

$$\begin{aligned} \|\gamma_0 v\|^2_{\frac{1}{2},2,\lambda} &= \int_{\mathbb{R}^{n-1}} \langle\lambda;\xi'\rangle |\widetilde{v}(\xi',0)|^2 đ\xi' \\ &= -\int_{\mathbb{R}^{n-1}} 2\int_0^\infty (\partial_n \widetilde{v}(\xi',x_n))\langle\lambda;\xi'\rangle \widetilde{v}(\xi',x_n)\, dx_n\, đ\xi' \\ &\leq C\|\partial_n v\|_2 \|((1+|\lambda|)v, \nabla' v)\|_2 \end{aligned}$$

because of Plancherel's theorem. If $v \in W^1_{2,\lambda}(\Omega_0)$, the statement easily reduces to the statement for $\mathbb{R}^n_+$ by the use of suitable cut-off functions. ■

2.3 Mikhlin-Multiplier Operator

Definition 2.6 Let H_0 and H_1 be two Hilbert spaces, $d \in \mathbb{R}$, and k be the smallest integers $> \frac{n}{2}$. A function $m \in C^k(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(H_0, H_1))$ is called an $\mathcal{L}(H_0, H_1)$-valued *Mikhlin multiplier of order* d if it satisfies

$$\|D_\xi^\alpha m(\xi)\|_{\mathcal{L}(H_0,H_1)} \leq C|\xi|^{d-|\alpha|}, \qquad \xi \in \mathbb{R}^n \setminus \{0\},$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Moreover, we set

$$[m]_{\mathcal{M}}^{(d)} = \sup_{|\alpha|\leq k} \left\{ \|D_\xi^\alpha m(\xi)\|_{\mathcal{L}(H_0,H_1)} |\xi|^{-d+|\alpha|} : \xi \in \mathbb{R}^n \setminus \{0\} \right\}$$

and, if $d = 0$, $[m]_{\mathcal{M}} = [m]_{\mathcal{M}}^{(0)}$.

The Mikhlin multipliers of order 0 are the usual Mikhlin multipliers. If m is a Mikhlin multiplier, we denote by

$$m(D_x)u = \mathrm{OP}(m)u = \mathcal{F}^{-1}[m(\xi)\hat{u}(\xi)], \qquad u \in \mathcal{S}(\mathbb{R}^n; H_0),$$

the corresponding *multiplier operator*. If $m(\xi')$ is a Mikhlin multiplier in $n-1$ variables, then $\mathrm{OP}'(m) = m(D_{x'})$ denotes the associated operator.

THEOREM 2.7 (Vector-valued Mikhlin-multiplier theorem)
Let $1 < q < \infty$, H_0, H_1 be two Hilbert spaces, and let m be a Mikhlin multiplier (of order 0) with values in $\mathcal{L}(H_0, H_1)$. Then $m(D_x)$ extends to a bounded and linear operator $m(D_x)\colon L^q(\mathbb{R}^n; H_0) \to L^q(\mathbb{R}^n; H_1)$ with

$$\|m(D_x)u\|_{L^q(\mathbb{R}^n;H_1)} \leq C_q [m]_{\mathcal{M}} \|u\|_{L^q(\mathbb{R}^n;H_0)}$$

for all $u \in L^q(\mathbb{R}^n; H_0)$.

It is easy to observe that the product $m_1(\xi)m_2(\xi)$ of two Mikhlin multipliers $m_1(\xi)$ and $m_2(\xi)$ of order d_1 and d_2, resp., is a again a Mikhlin multiplier of order $d_1 + d_2$ if the product is defined. Then of course $OP(m_1)OP(m_2) = OP(m_1 m_2)$. Moreover, if m is a multiplier of order d, and if $m(\xi)^{-1} \in \mathcal{L}(H_1, H_0)$ exists and satisfies $\|m(\xi)^{-1}\|_{\mathcal{L}(H_1,H_0)} \leq C|\xi|^{-d}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then $m(\xi)^{-1}$ is an $\mathcal{L}(H_1, H_0)$-valued multiplier of order $-d$. This statement is a consequence of $\partial_j m^{-1}(\xi) = -m(\xi)^{-1}(\partial_j m(\xi))m(\xi)^{-1}$ and the chain rule. This yields a nice characterization:

Lemma 2.8 *Let m be an $\mathcal{L}(\mathbb{R}^k)$-valued Mikhlin multiplier of order 0 with $k \in \mathbb{N}$. Then $m(D_x) : L^q(\mathbb{R}^n; \mathbb{R}^k) \to L^q(\mathbb{R}^n; \mathbb{R}^k)$ is invertible for $1 < q < \infty$ iff it is invertible for $q = 2$. Moreover, if the operator is invertible, $m^{-1}(\xi)$ exists for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and is again a Mikhlin multiplier with $m(D_x)^{-1} = \mathrm{OP}(m^{-1}(\xi))$. If $\|m(.)^{-1}\|_\infty = \|m(D_x)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n;\mathbb{R}^k))} \leq R$ and $[m]_{\mathcal{M}} \leq R$ for $R > 0$, then $[m(.)^{-1}]_{\mathcal{M}} \leq C$, where C depends only on n, k, and R.*

Proof: Let $m(D_x)\colon L^2(\mathbb{R}^n;\mathbb{R}^k) \to L^2(\mathbb{R}^n;\mathbb{R}^k)$ be invertible. Then the inverse $m(D_x)^{-1}$ is again a translation invariant operator. Because of Stein [20, Chapter 2, 1.4, Proposition 2], $m(D_x)^{-1} = \mathrm{OP}(m'(\xi))$ for a bounded measurable function $m' : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^k)$. Since $m(D_x)\,\mathrm{OP}(m'(\xi)) = \mathrm{OP}(m(\xi)m'(\xi)) = I$, $m'(\xi) = m^{-1}(\xi)$ a.e. Hence $m^{-1}(\xi)$ is a bounded continuous function for $\xi \neq 0$, which implies, as seen above, that $m^{-1}(\xi)$ is a Mikhlin multiplier. Therefore $(m(D_x))^{-1}\colon L^q(\mathbb{R}^n;\mathbb{R}^k) \to L^q(\mathbb{R}^n;\mathbb{R}^k)$ exists. The converse is trivial. ■

In the following, we will deal with operators of the form

$$Ka := \mathrm{OP}'(\tilde{k}(\xi',x_n))a \quad \text{and} \quad Tf := \int_I \mathrm{OP}'(\tilde{t}(\xi',y_n))f(.,y_n)dy_n,$$

where a is a function defined on $\mathbb{R}^{n-1}$ and f is a function defined on $\mathbb{R}^{n-1}\times I$ with $I=\mathbb{R}_+$ or $I=(-1,1)$. Usually K and T will be called *Poisson operator* and *trace operator (of class* 0*)*, resp., and $\tilde{k}(\xi',x_n)$ and $\tilde{t}(\xi',y_n)$ are the so called *symbol-kernels* of K and T, resp. We also write $k(\xi',D_n)\colon \mathbb{C}\to L^2(\mathbb{R}_+)\colon a \mapsto \tilde{k}(\xi',x_n)a$ and $t(\xi',D_n)\colon L^2(\mathbb{R}_+) \to \mathbb{C}\colon f \mapsto \int_{-1}^1 \tilde{t}(\xi',y_n)f(y_n)dy_n$ for the corresponding operator-valued symbols.

A fundamental example is $K_\lambda := \mathrm{OP}'(e^{-(\lambda+|\xi'|^2)^{\frac12}x_n})$, which is the Poisson operator to the Laplace resolvent equation with Dirichlet boundary condition in $\mathbb{R}^n_+$. An important example of a trace operator is

$$T^{\pm}_{j,\lambda}f := \gamma^{\pm}_j(\lambda-\Delta)e_{\Omega_0}f = (-1)^j\int_{-1}^{1}\mathrm{OP}'\left(\frac{e^{-(\lambda+|\xi'|^2)^{\frac12}(1\mp y_n)}}{2(\lambda+|\xi'|^2)^{\frac12 1-j}}\right)f(.,y_n)dy_n, \qquad (2.6)$$

which is part of the resolvent of the Laplacian in Ω_0. Both examples can be considered as operator-valued Mikhlin multipliers because of the next lemma.

Lemma 2.9 *Let $s>-\frac12$, $\delta\in(0,\pi)$, and $\zeta_\lambda = (\lambda+|\xi'|^2)^{\frac12}$. Then*

$$\left\| D^{\alpha'}_{\xi'} e^{-\zeta_\lambda x_n}\right\|_{L^2(\mathbb{R}_+;x_n^{2s})} \leq C_{\delta,\alpha',s}|\lambda;\xi'|^{-\frac12-s-|\alpha'|}$$

uniformly in $\lambda\in\Sigma_\delta\cup\{0\}$, $\xi'\in\mathbb{R}^{n-1}$ with $(\lambda,\xi)\neq 0$ for all $\alpha'\in\mathbb{N}_0^{n-1}$.

Proof: Obviously, $\tilde{k}_\lambda(\xi',x_n) = e^{-\zeta_\lambda x_n}$, $x_n\geq 0$, is quasi-homogeneous of degree -1 in the sense that $\tilde{k}_{r^2\lambda}(r\xi',\frac1r x_n) = \tilde{k}_\lambda(\xi',x_n)$ for all $r>0$. Hence $D^{\alpha'}_{\xi'}\tilde{k}_\lambda(\xi',x_n)$ is quasi-homogeneous of degree $-1-|\alpha'|$. More precisely, it is easy to verify that

$$D^{\alpha'}_{\xi'}\tilde{k}_\lambda(\xi',x_n) = \sum_{j=0}^{|\alpha'|} m_{\alpha',j}(\xi',\lambda)x_n^j e^{-\zeta_\lambda x_n},$$

where $m_{\alpha',j}(\xi',\lambda)$ are homogeneous functions of degree $-|\alpha'|+j$ in $(\lambda^{\frac12},\xi')$, which are smooth in $(\lambda,\xi')\in\overline{(\Sigma_\delta\times\mathbb{R}^{n-1})}\setminus\{0\}$. Therefore $|m_{\alpha',j}(\xi',\lambda)|\leq C_{\delta,\alpha'}|\lambda;\xi'|^{-|\alpha'|+j}$, which implies

$$\left\| D_{\xi'}^{\alpha'} e^{-\zeta_\lambda(1\pm x_n)} \right\|_{L^2(\mathbb{R}_+;x_n^{2s})}^2 \leq C_{\delta,\alpha'} \sum_{j=0}^{|\alpha'|} |\lambda;\xi'|^{-2|\alpha'|+2j} \int_0^\infty x_n^{2s+2j} e^{-c_\delta|\lambda;\xi'|x_n} dx_n \leq C_{\delta,\alpha'} |\lambda;\xi'|^{-1-2s-2|\alpha'|}$$

because of (2.1) and (2.2). ■

Corollary 2.10 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, $s > -\frac{1}{2}$, $\varepsilon \in (0, \frac{1}{q'})$, $b > 0$, and $\alpha \in \mathbb{N}_0$. Then $K_\lambda := \mathrm{OP}'(e^{-\zeta_\lambda x_n})$ and $D_x^\alpha K_\lambda$, $\alpha \in \mathbb{N}_0^n$, defined on $\mathcal{S}(\mathbb{R}^{n-1})$ extend to bounded operators*

$$K_\lambda \colon \dot{H}_{q,\lambda}^{-\frac{1}{2}-s}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^{n-1}; L^2(\mathbb{R}_+; x_n^{2s})), \quad K_\lambda \colon \dot{B}_{q,\lambda}^{-\frac{1}{q}}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^n_+),$$
$$D_x^\alpha K_\lambda \colon \dot{B}_{q,\lambda}^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^{n-1}; L^2(b,\infty)), \quad \varepsilon > 0.$$

Proof: The first statement is a direct consequence of Lemma 2.9 and the vector-valued Mikhlin multiplier theorem. The second part is obtained via interpolation similarly as in [14, Theorem 1.8]: Firstly, let $q \leq 2$. Then

$$(L^2(\mathbb{R}_+; x_n^{2s'}), L^2(\mathbb{R}_+; x_n^{2s}))_{\theta,q} \subseteq L^q(\mathbb{R}_+),$$

where $s' < \frac{1}{q} - \frac{1}{2} < s$ and $\theta = (\frac{1}{q} - \frac{1}{2} - s')/(s - s')$, cf. [14, Theorem 1.8]. Hence real interpolation yields the second continuity. Secondly, let $q > 2$. Since $\partial_n K = \mathrm{OP}'(e^{-\zeta_\lambda x_n})\,\mathrm{OP}'(-\zeta_\lambda)$ and $\mathrm{OP}'(-\zeta_\lambda) \colon \dot{H}_{q,\lambda}^{\frac{1}{2}}(\mathbb{R}^{n-1}) \to \dot{H}_{q,\lambda}^{-\frac{1}{2}}(\mathbb{R}^{n-1})$, we conclude

$$K \colon \dot{H}_{q,\lambda}^{\frac{1}{2}}(\mathbb{R}^{n-1}) \to L^q(\mathbb{R}^{n-1}; \dot{H}_2^1(\mathbb{R}_+)).$$

Because of

$$(\dot{H}_2^1(\mathbb{R}_+), L^2(\mathbb{R}_+))_{\frac{1}{q}-\frac{1}{2},q} = \dot{B}_{2,q}^{\frac{1}{2}-\frac{1}{q}}(\mathbb{R}_+) \subseteq \dot{B}_{q,2}^0(\mathbb{R}_+) \subseteq \dot{H}_q^0(\mathbb{R}_+) = L^q(\mathbb{R}_+),$$
$$(\dot{H}_{q,\lambda}^{\frac{1}{2}}(\mathbb{R}^{n-1}), \dot{H}_{q,\lambda}^{-\frac{1}{2}}(\mathbb{R}^{n-1}))_{\frac{1}{q}-\frac{1}{2},q} = \dot{B}_{q,\lambda}^{-\frac{1}{q}}(\mathbb{R}^{n-1}),$$

cf. Lemma 2.2 and the homogeneous counterpart of [9, Theorem 6.4.4], real interpolation yields the second part as before.

It is easy to see that $K_\lambda = \mathrm{OP}'(e^{-\zeta_\lambda(x_n-b/2)})\,\mathrm{OP}'(e^{-\zeta_\lambda b/2})$, where $\mathrm{OP}'(e^{-\zeta_\lambda b/2})$ is a smoothing operator, that $D_x^\alpha K_\lambda = \mathrm{OP}'(\xi^{\alpha'}(i\zeta_\lambda)^{\alpha_n})K_\lambda$, and that $L^2(\frac{b}{2},\infty;(x_n - \frac{b}{2})^{2\varepsilon}) \hookrightarrow L^2(b,\infty)$ if $\varepsilon > 0$. Hence we get from the previous statements by a simple translation in x_n

$$\|D_x^\alpha K_\lambda a\|_{L^q(\mathbb{R}^{n-1};L^2(b,\infty))} \leq C_\varepsilon \|D_x^\alpha K_\lambda a\|_{L^q(\mathbb{R}^{n-1};L^2(\frac{b}{2},\infty;(x_n-\frac{b}{2})^{2s}))} \leq C_{\varepsilon,\alpha,\delta} \|\mathrm{OP}'(e^{-\zeta_\lambda(x_n-b/2)})a\|_{L^q(\mathbb{R}^{n-1};L^2(\frac{b}{2},\infty;(x_n-\frac{b}{2})^{2s}))} \leq C_{\varepsilon,\alpha,\delta} \|a\|_{\dot{H}^{-\frac{1}{2}-s}(\mathbb{R}^{n-1})}.$$

Real interpolation with different values of ε finishes the proof. ■

3 Laplace Resolvent Equation in an Infinite Layer

We consider the Laplace resolvent equation with mixed Neumann-Dirichlet boundary conditions.

$$(\lambda - \Delta)u = f \quad \text{in } \Omega_0, \tag{3.1}$$
$$\gamma_1^+ u = a^+ \quad \text{on } \partial\Omega_0^+, \tag{3.2}$$
$$\gamma_0^- u = a^- \quad \text{on } \partial\Omega_0^-. \tag{3.3}$$

First let $f = 0$. Using partial Fourier transformation, a calculation yields $u(x', x_n) = \mathcal{F}^{-1}_{\xi' \mapsto x'}[k_{10,\lambda}(\xi', D_n)\tilde{a}(\xi')]$, where

$$k_{10,\lambda}(\xi', D_n)\tilde{a} := \frac{e^{-\zeta_\lambda(1-x_n)}}{1+e^{-4\zeta_\lambda}}\left(\frac{\tilde{a}^+}{\zeta_\lambda} + e^{-2\zeta_\lambda}\tilde{a}^-\right) + \frac{e^{-\zeta_\lambda(1+x_n)}}{1+e^{-4\zeta_\lambda}}\left(\tilde{a}^- - e^{-2\zeta_\lambda}\frac{\tilde{a}^+}{\zeta_\lambda}\right) \tag{3.4}$$

for given $a^\pm \in C_0^\infty(\mathbb{R}^{n-1})$ and with $\zeta_\lambda = (\lambda + |\xi'|^2)^{\frac{1}{2}}$. It is easy to see that $m_\lambda(\xi') := (1+e^{-4\zeta_\lambda})^{-1}$ is a Mikhlin multiplier with $[m_\lambda(\xi')]_{\mathcal{M}} \le C_\delta$ uniformly in $\lambda \in \Sigma_\delta \cup 0$, cf. [3, Proof of Lemma 4.1]. Therefore and because of Corollary 2.10, $K_{10,\lambda} = \mathrm{OP}'(k_{10,\lambda}(\xi', D_n))$ extends to a bounded operator

$$K_{10,\lambda}\colon \dot{B}^{m-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1}) \times \dot{B}^{m+1-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1}) \to r_{\Omega_0}\dot{W}^{m+1}_{q,\lambda}(\mathbb{R}^n) \tag{3.5}$$

for $m \in \mathbb{N}_0$. Note that $r_{\Omega_0}\dot{W}^{m+1}_q(\mathbb{R}^n) \neq \dot{W}^{m+1}_q(\Omega_0)$, cf. [7, Remarks 2.7] for details. If the boundary conditions in (3.2)-(3.3) are interchanged, we get the analogous result for the corresponding Poisson operator $K_{01,\lambda}$.

Now we consider a weak formulation, which will be fundamental for the reduction of the generalized Stokes equation in Section 4.1 below.

$$(\lambda - \Delta)u = -\operatorname{div} f \quad \text{in } \Omega_0, \tag{3.6}$$
$$\gamma_1^+ u = \gamma_\nu^+ f \quad \text{on } \partial\Omega_0^+, \tag{3.7}$$
$$\gamma_0^- u = 0 \quad \text{on } \partial\Omega_0^- \tag{3.8}$$

for $f \in L^q(\Omega_0)^n$, where (3.6) is understood in the sense of distributions, (3.8) shall hold in the sense of usual traces, and (3.7) is understood as $\gamma_\nu^+(\nabla u - f) = 0$, which is defined in (2.3). Because of the definition of $\gamma_\nu^+(\nabla u - f) = 0$, the system (3.6)-(3.8) is equivalent to the variational problem

$$\lambda(u, v) + (\nabla u, \nabla v) \;=\; \langle F, v\rangle \quad \text{for all } v \in {}_0W^1_{q'}(\Omega_0), \tag{3.9}$$

where $\langle F, v\rangle := (f, \nabla v), v \in {}_0W^1_{q'}(\Omega_0)$, is an element of ${}^0W^{-1}_q(\Omega_0)$.

Lemma 3.1 *Let $1 < q, r < \infty$, $\delta \in (0, \pi)$, and $\lambda \in \Sigma_\delta \cup \{0\}$. Then for every $F \in {}^0W^{-1}_q(\Omega_0)$ there is a unique solution $u \in {}_0W^1_{q,\lambda}(\Omega_0)$ of (3.9), which satisfies*

$$(1 + |\lambda|)\|u\|_{{}^0W^{-1}_q} + \|u\|_{1,q,\lambda} \le C_{\delta,q}\|F\|_{{}^0W^{-1}_q} \tag{3.10}$$

uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$. If additionally $F \in {}^0W^{-1}_r(\Omega_0)$, then $u \in {}_0W^1_r(\Omega_0)$.

Proof: Identifying ${}^0W_q^{-1}(\Omega_0)$ with a closed subspace of $L^q(\Omega_0)^n$ and using the Hahn-Banach theorem, for $F \in {}^0W_q^{-1}(\Omega_0)$ there is an $f \in L^q(\Omega_0)^n$ such that $\langle F, v\rangle = (f, \nabla v)$ for all $v \in {}_0W_{q'}^1(\Omega_0)$ and $\|f\|_q \leq C\|F\|_{{}^0W_q^{-1}}$. Hence it is sufficient to prove the lemma for functionals of the latter form.

Let $g \in C_{(0)}^\infty(\overline{\Omega})$. Then we set

$$R_{10,\lambda} g \;\;:=\;\; r_{\Omega_0}(\lambda-\Delta)^{-1} e_{\Omega_0} g - K_{10,\lambda}\begin{pmatrix} \gamma_1^+(\lambda-\Delta)^{-1}e_{\Omega_0} g \\ \gamma_0^-(\lambda-\Delta)^{-1}e_{\Omega_0} g\end{pmatrix}.$$

$R_{01,\lambda}$ is defined analogously. If we set $v = -\partial_n R_{01,\lambda} g$, then v solves

$$\begin{aligned} (\lambda-\Delta)v &= -\partial_n g && \text{in } \Omega_0,\\ \gamma_1^+ v &= \gamma_0^+(-(\lambda-\Delta')v + g) = \gamma_0^+ g && \text{on } \partial\Omega_0^+,\\ \gamma_0^- v &= 0 && \text{on } \partial\Omega_0^-. \end{aligned}$$

Thus we define $u := -\operatorname{div}' R_{10,\lambda} f' - \partial_n R_{01,\lambda} f_n$ for $f \in C_{(0)}^\infty(\overline{\Omega_0})^n$. Using integration by parts, u solves (3.9).

Since $r_{\Omega_0}(\lambda-\Delta)^{-1}e_{\Omega_0}\colon L^q(\Omega_0) \to r_{\Omega_0}\dot{B}^2_{q,\lambda}(\mathbb{R}^n)$, $\gamma_j^\pm(\lambda-\Delta)^{-1}e_{\Omega_0} g \in \dot{W}_{q,\lambda}^{2-j-\frac{1}{q}}(\partial\Omega_0^\pm)$ for $j = 0, 1$. Thus, because of (3.5),

$$\|(|\lambda|^{\frac{1}{2}}u, \nabla u)\|_q \leq C_{q,\delta}\|f\|_q \leq C_{q,\delta}\|F\|_{{}^0W_q^{-1}}$$

uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$. Because of $\gamma_0^- u = 0$ and Poincaré's inequality $|\lambda|^{\frac{1}{2}}$ can be replaced by $(1+|\lambda|)^{\frac{1}{2}}$. Hence we can extend the solution operator by continuity such that (3.9) holds. Therefore the mapping

$$A_{\lambda,q}\colon {}_0W_q^1(\Omega_0) \to {}^0W_q^{-1}(\Omega_0)\colon u \mapsto \lambda(u,.) + (\nabla u, \nabla .)$$

is surjective for $1 < q < \infty$. Since $A'_{\lambda,q} = A_{\overline{\lambda},q'}$, $A_{\lambda,q}$ is also injective.

Moreover, (3.9) and the estimate of $\|\nabla u\|_q$ imply the estimate of $|\lambda|\|u\|_{{}^0W_q^{-1}}$. Finally, the regularity assertion holds since the solution operators for q and r coincide in the dense subset $C_{(0)}^\infty(\overline{\Omega_0})$. ■

Corollary 3.2 (Helmholtz decomposition)
Let $1 < q < \infty$, $n \geq 2$, and $\Omega_0 \subset \mathbb{R}^n$ be an infinite layer. Then there is a continuous projection $P_q\colon L^q(\Omega_0)^n \to L^q(\Omega_0)^n$ such that

$$\begin{aligned} \mathcal{R}(P_q) &= {}^0J_q(\Omega_0) := \{u \in L^q(\Omega_0)^n : \operatorname{div} u = 0, \gamma_1^+ u = 0\},\\ \mathcal{N}(P_q) &= {}_0G_q(\Omega_0) := \{\nabla p \in L^q(\Omega_0)^n : p \in {}_0W_q^1(\Omega_0)\}. \end{aligned}$$

Proof: The proof is an easy modification of the standard proof, cf. [18]. The projection P_q is defined as $P_q f = f - \nabla p$, where p is the solution of (3.6)-(3.8) for $\lambda = 0$. ■

Lemma 3.3 *Let $1 < q, r < \infty$, $\delta \in (0, \pi)$, $m \in \mathbb{N}_0$, and $\lambda \in \Sigma_\delta \cup \{0\}$. Then for every $f \in W^m_{q,\lambda}(\Omega_0)$ there is a unique solution $u = R_{10,\lambda} f \in W^{m+2}_{q,\lambda}(\Omega_0)$ of (3.1)-(3.3) with $a^\pm = 0$. Moreover, $\|u\|_{m+2,q,\lambda} \leq C_{\delta,m,q} \|f\|_{m,q,\lambda}$ uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$. If additionally $f \in W^m_{r,\lambda}(\Omega_0)$, then $u \in W^{m+2}_{r,\lambda}(\Omega_0)$.*

Proof: W.l.o.g. let $m = 0$. Because of Lemma 3.1, there is a unique solution $u \in {}_0W^1_q(\Omega_0)$ of (3.9) with right-hand side $\langle F, v\rangle := (f, v)$. Differentiating in tangential direction, it is easy to observe that $\partial_j u \in {}_0W^1_q(\Omega_0)$ solves (3.9) with right-hand side $\langle F_j, v\rangle := (f, \partial_j v)$, $j = 1, \dots, n-1$. Using (3.9), we obtain $-\partial_n^2 u = f - (\lambda - \Delta')u$ in the sense of distributions, where $f - (\lambda - \Delta')u \in L^q(\Omega_0)$. Hence $u \in W^2_q(\Omega_0)$. Since (3.9) implies $(\lambda - \Delta)u = f$ a.e. in Ω_0,

$$(\gamma_1^+ u, \gamma_0^+ v) = (\nabla u, \nabla v) + (\Delta u, v) = (f, v) - \lambda(u, v) + (\Delta u, v) = 0$$

for all $v \in {}_0W^1_{q'}(\Omega_0)$, which implies $\gamma_1^+ u = 0$.

■

Finally, for $\lambda = 0$, we compare the Poisson operator $K_{10} := K_{10,\lambda}|_{\lambda=0}$ with the localized parametrix

$$\widetilde{K}_{10} a \;:=\; \psi^+ K_1^+ a^+ + \psi^- K_0^- a^-, \tag{3.11}$$

where $K_j^\pm a^\pm = \mathrm{OP}'\left(|\xi'|^{-j} e^{-|\xi'|(1\mp x_n)}\right) a$, $j = 0, 1$, denotes the Poisson operator of the Dirichlet ($j = 0$) or the Neumann problem ($j = 1$), resp., in $\mathbb{R}^n_{<1} = \{(x', x_n) : x_n < 1\}$ or in $\mathbb{R}^n_{>-1} = \{(x', x_n) : x_n > -1\}$, resp. Furthermore, $\psi^\pm \in C^\infty_{(0)}([-1, 1])$ with $\psi^\pm(x_n) = 1$ if $\mathrm{dist}(x_n, \pm 1) \leq \frac{1}{2}$, $\mathrm{supp}\,\psi^+ \subset (-1, 1]$, and $\mathrm{supp}\,\psi^- \subset [-1, 1)$.

The following result, which will be needed for the analysis of the reduced Stokes equations in Ω_0, shows that the error of this localization is of lower order.

Lemma 3.4 *Let $0 < \varepsilon < \frac{1}{2}$, and let $\widetilde{K}_{10}$ be defined as above. Then*

$$\|(\nabla K_{10} - \nabla \widetilde{K}_{10}) a\|_2 \;\leq\; C \left(\|a^+\|_{\dot{B}_2^{-\frac{1}{2}-\varepsilon}} + \|a^-\|_{W_2^{\frac{1}{2}-\varepsilon}} \right)$$

for all $a^+ \in \dot{B}_2^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1})$ and $a^- \in W_2^{\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1})$.

Proof: First of all, note that $W_2^{\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1}) \hookrightarrow \dot{W}_2^{\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1})$ and $\dot{B}_2^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1}) \hookrightarrow W_2^{-\frac{1}{2}}(\mathbb{R}^{n-1})$. Hence $\nabla K_{10} a$ and $\widetilde{K}_{10} a$ are well defined.

We use that

$$\begin{aligned} -\Delta(K_{10} - \widetilde{K}_{10})a &= S\widetilde{K}_{10} a, \\ \gamma_1^+(K_{10} - \widetilde{K}_{10})a &= 0, \qquad \gamma_0^-(K_{10} - \widetilde{K}_{10})a = 0, \end{aligned}$$

where S is a differential operator of order 1 with coefficients supported in $\mathrm{supp}\,\nabla\psi^+ \cup \mathrm{supp}\,\nabla\psi^-$. Hence $\nabla(K_{10} - \widetilde{K}_{10})a = \nabla R_{10} S \widetilde{K}_{10} a$, where R_{10} is the solution operator of (3.1)-(3.3) with $a^\pm = 0$ and $\lambda = 0$. Since the coefficients of S are supported in $\mathbb{R}^{n-1} \times (-a, a)$ for an $a < 1$, Corollary 2.10 and Lemma 3.3 imply

$$\|\nabla(K_{10} - \widetilde{K}_{10})a\|_2 \le C\|S\widetilde{K}_{10}a\|_2 \le C\left(\|a^+\|_{\dot{B}_2^{-\frac{1}{2}-\varepsilon}} + \|a^-\|_{W_2^{\frac{1}{2}-\varepsilon}}\right).$$

■

Remark 3.5 It is obvious that all statements remain true if we interchange the boundary conditions of the upper and the lower boundary, i.e. consider mixed Dirichlet-Neumann conditions. The corresponding solution operators will be denoted by $K_{01,\lambda}$, $R_{01,\lambda}$, K_{01}, R_{01}, and $\widetilde{K}_{01}$.

Moreover, we can add an $a^- \in W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_0^-)$ in (3.8) or consider a general $(a^+, a^-) \in W_{q,\lambda}^{m+1-\frac{1}{q}}(\partial\Omega_0^+) \times W_{q,\lambda}^{m+2-\frac{1}{q}}(\partial\Omega_0^-)$ in Lemma 3.3 and get analogous statements.

4 Stokes Equations in an Infinite Layer

4.1 The Reduced Stokes Equations

In the following reduction, which is an adaption of the reduction used in [15], the pressure p will be expressed in dependence on the data (f, g) and the (unknown) solution u. Therefore we will end up with a pseudodifferential equation, where p is replaced by a non-local operator applied to u, which is also called *singular Green operator* in the theory of pseudodifferential boundary value problems, cf. [12].

Let $(u, p) \in W_{q,\lambda}^2(\Omega_0)^n \times W_{q,\lambda}^1(\Omega_0)$ be a solution of (1.1)-(1.4) with

$$f \in L^q(\Omega_0)^n,\ a^+ \in W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_0^+)^n,\ g \in W_{q,\lambda}^1(\Omega_0). \tag{4.1}$$

Applying $-\operatorname{div}$ and γ_ν^- to the equation (1.1) and using (1.2), p solves

$$\begin{aligned} -\Delta p &= -\operatorname{div} f + (\lambda - \Delta)g && \text{in } \Omega_0, \\ \gamma_0^+ p &= 2\gamma_1^+ u_\nu - a_\nu^+ && \text{on } \partial\Omega_0^+, \\ \gamma_1^- p &= \gamma_\nu^+ f && \text{on } \partial\Omega_0^-. \end{aligned}$$

Now we split $p = p_1 + p_2$ such that p_1 depends only on u and p_2 depends only on (f, g, a^+). Then we end up with the *reduced Stokes equations*

$$(\lambda - \Delta)u + G_{10}u = f_r \quad \text{in } \Omega_0, \tag{4.2}$$

$$T_1'^+ u = a_r^+ \quad \text{on } \partial\Omega_0^+, \tag{4.3}$$

$$\gamma_0^- u = 0 \quad \text{on } \partial\Omega_0^- \tag{4.4}$$

with

$$G_{10}u = \nabla K_{01} \begin{pmatrix} 2\gamma_1^+ u_\nu \\ \gamma_\nu^- (\Delta - \nabla \operatorname{div}) u \end{pmatrix}, \quad T_1'^+ u = \begin{pmatrix} (\gamma_0^+ \nu \cdot S(u))_\tau \\ \gamma_0^+ \operatorname{div} u \end{pmatrix}, \quad a_r^+ = \begin{pmatrix} a_\tau^+ \\ \gamma_0^+ g \end{pmatrix},$$

$$f_r = f - \nabla p_2 = f - \nabla R_{01}(-\operatorname{div} f + (\lambda - \Delta) g) - \nabla K_{01} \begin{pmatrix} -a_\nu^+ \\ \gamma_\nu^- f + \gamma_1^- g \end{pmatrix},$$

where $v = (v_\tau, v_\nu)$ denotes the decomposition of a vector field v defined on $\partial\Omega_0$ into the tangential and normal components. Moreover, K_{01} and R_{01} are defined as in Remark 3.5.

Because of Lemma 3.1,

$$\|\nabla p_1\|_q \leq C_q \left(\|\gamma_\nu^- (\Delta - \nabla \operatorname{div}) u\|_{-\frac{1}{q},q} + \|\gamma_1^+ u_\nu\|_{1-\frac{1}{q},q} \right), \tag{4.5}$$

$$\|\nabla p_2\|_q \leq C_q \left(\|f\|_q + \|g\|_{1,q,\lambda} + |\lambda| \|g\|_{{}^0W_q^{-1}(\Omega_0)} + \|a^+\|_{1-\frac{1}{q},q,\lambda} \right), \tag{4.6}$$

where we have used that $\langle F, v \rangle := ((\Delta - \nabla \operatorname{div})u, \nabla v) = \langle \gamma_\nu^- (\Delta - \nabla \operatorname{div}) u, \gamma_0^- v \rangle$ for all $v \in {}^0W_{q'}^1(\Omega_0)$ because of (2.3) and therefore $\|F\|_{{}^0W_q^{-1}} \leq C \|\gamma_\nu^- (\Delta - \nabla \operatorname{div}) u\|_{-\frac{1}{q},q}$.

The most important fact about this reduction is that we may drop the equation $\operatorname{div} u = g$: If u solves the equations (4.2)-(4.4) with f_r defined as above, then

$$\begin{aligned} (\lambda - \Delta) \operatorname{div} u &= (\lambda - \Delta) g && \text{in } \Omega_0, \\ \gamma_0^+ \operatorname{div} u &= \gamma_0^+ g && \text{on } \partial\Omega_0^+, \\ \gamma_1^- \operatorname{div} u &= \gamma_1^- g && \text{on } \partial\Omega_0^- \end{aligned}$$

because of the construction and the definition of the operators in the reduced Stokes equations. Since these equations are uniquely solvable, $\operatorname{div} u = g$.

Hence the generalized Stokes equations (1.1)-(1.4) with right-hand side as in (4.1) are uniquely solvable if the reduced Stokes equations (4.2)-(4.4) are uniquely solvable for $f_r \in L^q(\Omega_0)^n$ and $a_r^+ \in W_q^{1-\frac{1}{q}}(\partial\Omega_0^+)^n$. Moreover, if the solution u of the reduced Stokes equations can be estimated by

$$\|u\|_{2,q,\lambda} \leq C_{q,\delta} \left(\|f_r\|_q + \|a_r^+\|_{1-\frac{1}{q},q,\lambda} \right) \tag{4.7}$$

uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$, $\delta \in (0, \pi)$, then the solution (u, p) of the generalized Stokes equations satisfies (1.5) because of (4.5)-(4.6).

The converse implication is also true: If $f_r \in L^q(\Omega_0)^n$ and $a_r^+ \in W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_0^+)^n$, then we get a solution u of the reduced Stokes equations (4.2)-(4.4) as follows: We solve the generalized Stokes equations (1.1)-(1.4) with right-hand side (f, g, a^+) with $a^+ = ((a_r^+)_\tau, 0)$, where g is determined as solution of

$$\begin{aligned} (\lambda - \Delta) g &= \operatorname{div} f_r && \text{in } \Omega_0, \\ \gamma_0^+ g &= (a_r^+)_\nu && \text{on } \partial\Omega_0^+, \\ \gamma_1^- g &= \gamma_\nu^- f_r && \text{on } \partial\Omega_0^-. \end{aligned}$$

Then, because of Lemma 3.1,

$$(1+|\lambda|)\|g\|_{{}_0W_q^{-1}} + \|g\|_{1,q,\lambda} \le C_{q,\delta}\left(\|f_r\|_q + \|a_r^+\|_{1-\frac{1}{q},q,\lambda}\right)$$

and the solution (u,p) of the generalized Stokes equations satisfies

$$\|u\|_{2,q,\lambda} + \|\nabla p\|_q \le C_{q,\delta}\left(\|f\|_q + \|a_\nu^+\|_{1-\frac{1}{q},q,\lambda}\right) \qquad \text{for all } \lambda \in \Sigma_\delta \cup \{0\}.$$

Moreover, p solves $-\Delta p = 0$ with $\gamma_1^- p = \gamma_\nu^-(\Delta - \nabla \operatorname{div})u$ and $\gamma_0^+ p = 2\gamma_1^+ u_\nu$. Thus $\nabla p = G_{10}u$ and u solve the reduced Stokes equations. Hence we have proved:

Lemma 4.1 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, and $\lambda \in \Sigma_\delta \cup \{0\}$. Then the generalized Stokes equations (1.1)-(1.4) are uniquely solvable for given data as in (4.1) iff the reduced Stokes equations (4.2)-(4.4) are uniquely solvable for every $f_r \in L^q(\Omega_0)^n$ and $a_r^+ \in W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_0^+)^n$. Moreover, the solutions of the generalized Stokes equations satisfy (1.5) iff the solutions of the reduced Stokes equations satisfy (4.7).*

Remark 4.2 Because of (4.2)-(4.4), it is natural to define the *reduced Stokes operator* $A_q := -\Delta + G_{10}$ on $L^q(\Omega_0)^n$ with domain

$$\mathcal{D}(A_q) = \{u \in W_q^2(\Omega_\gamma)^n : \gamma_0^- u = 0, T_1'^+ u = 0\}.$$

Then Theorem 1.1 and Lemma 4.1 imply that A_q is invertible and is the generator of a bounded analytic semi-group. Moreover, it is proved in [6] that A_q admits a *bounded H_∞-calculus*, cf. [16]. Hence it possesses *bounded imaginary powers* and therefore has maximal regularity in L^q-Sobolev spaces due to Dore and Venni [10].

The reduction described above can be done in many types of domains. In the following we will need the unique solvability of the reduced Stokes equation in $\mathbb{R}^n_+$:

$$(\lambda - \Delta)u + G_j u = f \quad \text{in } \mathbb{R}^n_+, \tag{4.8}$$

$$T_j' u = a \quad \text{on } \partial\mathbb{R}^n_+ \tag{4.9}$$

for $j = 0, 1$ with

$$G_1 = \nabla K_0 2\gamma_1 u_\nu, \qquad T_1' u = \begin{pmatrix} (\gamma_0 \nu \cdot S(u))_\tau \\ \gamma_0 \operatorname{div} u \end{pmatrix},$$

$$G_0 = \nabla K_1 \gamma_\nu (\Delta - \nabla \operatorname{div})u, \qquad T_0' u = \gamma_0 u.$$

Lemma 4.3 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, $\lambda \in \Sigma_\delta$, and $j = 0$ or $j = 1$. Then for every $(f,a) \in L^q(\mathbb{R}^n_+)^n \times \dot{B}_{q,\lambda}^{2-j-\frac{1}{q}}(\mathbb{R}^{n-1})^n$ there is a unique solution $u \in \dot{W}_{q,\lambda}^2(\mathbb{R}^n_+)^n$ of the reduced Stokes equations (4.8)-(4.9). Moreover,*

$$|\lambda|\|u\|_q + |\lambda|^{\frac{1}{2}}\|\nabla u\|_q + \|\nabla^2 u\|_q \le C_\delta(\|f\|_q + \|a\|_{\cdot,2-j-\frac{1}{q},q,\lambda})$$

uniformly in $\lambda \in \Sigma_\delta$.

Proof: The lemma is a consequence of [15, Theorem 6.1]. It can also be proved using the unique solvability of the generalized Stokes equations in $\mathbb{R}^n_+$, cf. Farwig and Sohr [11, Theorem 1.3] for the Dirichlet case and Shibata and Shimizu [17, Theorem 4.3] for the case $j = 1$, and an analogous equivalence statement of Lemma 4.1. ■

4.2 Reduction to the Boundary

Using the product structure of the term G_{10} in the reduced Stokes equations, we will obtain a representation of the solution operator to the Stokes equations in terms of the solution operator of the Laplace resolvent equation and an operator acting only on the boundary. The idea of this reduction goes back to [13, Section 3].

Let

$$A_{10,\lambda} = \begin{pmatrix} (\lambda - \Delta)I \\ \gamma_1^+ \end{pmatrix} : W^2_{q,\lambda}(\Omega_0)^n \cap {}_0W^1_{q,\lambda}(\Omega_0)^n \to \begin{matrix} L^q(\Omega_0)^n \\ \times \\ W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0^+)^n \end{matrix}$$

be the operator associated to the Laplace resolvent equation (3.1)-(3.3) with $a^- = 0$ (in n components). Then we can express the reduced Stokes equations (4.2)-(4.4) as perturbation of the Laplace resolvent equation

$$A^r_{10,\lambda} = A_{10,\lambda} + B_{10}$$

with

$$B_{10} = \begin{pmatrix} K^r_{10}T^r_{10} \\ T_1'^+ - \gamma_1^+ \end{pmatrix}, \quad K^r_{10}a = \nabla K_{01}\begin{pmatrix} a^+ \\ -\operatorname{div}' a^- \end{pmatrix}, \quad T^r_{10}u = \begin{pmatrix} 2\gamma_1^+ u_\nu \\ \gamma_1^- u' \end{pmatrix},$$

where we have used that $\gamma_\nu^-(\Delta - \nabla \operatorname{div})u = -\gamma_1^- \operatorname{div}' u'$ if $\gamma_0^- u = 0$. Here $\operatorname{div}' a^- = \partial_1 a_1^- + \ldots + \partial_{n-1}a_{n-1}^-$.

Remark 4.4 Note that we have defined K^r_{10} and T^r_{10} such that all operators are of order 1 in the sense of mapping properties in Sobolev-Slobodeckij spaces. Moreover, we note that $W^{1-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1}) \hookrightarrow W^{1-\frac{1}{q}}_q(\mathbb{R}^{n-1})$ with norms uniformly bounded in λ. Hence

$$K^r_{10} \colon W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0^+) \times W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0^-)^{n-1} \cong W^{1-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1})^n \to L^q(\Omega_0)^n$$

with norms uniformly bounded in λ.

Since $A^{-1}_{10,\lambda} = (R_{10,\lambda}, K_{10,\lambda})$ exists for all $\lambda \in \Sigma_\delta \cup \{0\}$, $A^r_{10,\lambda}$ is invertible iff

$$I + A^{-1}_{10,\lambda}B_{10} \equiv I + \mathcal{K}_{10,\lambda}\mathcal{T}_{10}$$

is invertible, where

$$\mathcal{K}_{10,\lambda}\begin{pmatrix} a \\ b^+ \end{pmatrix} = R_{10,\lambda}K^r_{10}a + K_{10,\lambda}(b^+,0)^T, \qquad \mathcal{T}_{10} = \begin{pmatrix} T^r_{10} \\ T'^+_1 - \gamma^+_1 \end{pmatrix}.$$

Now we use the following simple lemma, cf. [13, Lemma 3.1]:

Lemma 4.5 *Let V, W be vector spaces and let $A\colon V \to W$, $B\colon W \to V$ be linear mappings. Then $I + AB\colon W \to W$ is bijective if and only if $I + BA\colon V \to V$ is bijective. Moreover,*

$$(I+BA)^{-1} = I - B(I+AB)^{-1}A \quad \textit{and} \quad (I+AB)^{-1} = I - A(I+BA)^{-1}B$$

if the inverses exist.

Hence $I + \mathcal{K}_{10,\lambda}\mathcal{T}_{10}$ (and therefore $A^r_{10,\lambda}$) is invertible iff

$$\mathcal{S}_{10,\lambda} = I + \mathcal{T}_{10}\mathcal{K}_{10,\lambda}\colon W^{1-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1})^n \to W^{1-\frac{1}{q}}_{q,\lambda}(\mathbb{R}^{n-1})^n$$

is invertible. The crucial observation is:

Lemma 4.6 $\mathcal{S}_{10,\lambda} = \mathrm{OP}'(s_{10,\lambda}(\xi'))$ *is a Mikhlin multiplier operator with $[s_{10,\lambda}]_{\mathcal{M}} \leq C_\delta$ for all $\lambda \in \Sigma_\delta \cup \{0\}$, $\delta \in (0,\pi)$.*

Proof: This lemma relies on the fact that all operators can be considered as $\mathcal{L}(H_0, H_1)$-valued Mikhlin multipliers of a certain order for suitable Hilbert spaces H_0, H_1. Moreover, all Mikhlin multiplier norms are uniformly bounded in $\lambda \in \Sigma_\delta \cup \{0\}$.

We have

$$\mathcal{S}_{10,\lambda}\begin{pmatrix} a \\ b^+ \end{pmatrix} = \begin{pmatrix} a + T^r_{10}(R_{10,\lambda}K^r_{10}a + K_{10,\lambda}(b^+,0)) \\ b^+ + (T'^+_1 - \gamma^+_1)(R_{10,\lambda}K^r_{10}a + K_{10,\lambda}(b^+,0)) \end{pmatrix}$$

Let $T = T^r_{10}$ or $T = T'^+_1 - \gamma^+_1$. All entries in T consist of differential trace operators of order 1, which are of the form $\gamma^\pm_1 + OP'(a\cdot\xi')\gamma^\pm_0$, $a \in \mathbb{C}^{n-1}$. Therefore, it is easy to observe from (3.4) that $TK_{10,\lambda}(b^+,0) = \mathrm{OP}'(m_\lambda(\xi'))b^+$ with $[m_\lambda]_{\mathcal{M}} \leq C_\delta$. Moreover,

$$R_{10,\lambda}f = r_{\Omega_0}(\lambda-\Delta)^{-1}e_{\Omega_0}f - K_{10,\lambda}\begin{pmatrix} \gamma^+_1(\lambda-\Delta)^{-1}e_{\Omega_0}f \\ \gamma^-_0(\lambda-\Delta)^{-1}e_{\Omega_0}f, \end{pmatrix}.$$

Because of (2.6), $T(\lambda-\Delta)^{-1}e_{\Omega_0}$ has a symbol-kernel of the form $m_\lambda(\xi')e^{-\zeta_\lambda(1\pm y_n)}$ with $[m_\lambda]_{\mathcal{M}} \leq C_\delta$. The same is true for $TK_{10,\lambda}(\gamma^+_1(\lambda-\Delta)^{-1}e_{\Omega_0}, \gamma^-_0(\lambda-\Delta)^{-1}e_{\Omega_0})$ because of (2.6) and (3.4) again. Hence $TR_{10,\lambda} = \mathrm{OP}'(t_\lambda(\xi', D_n))$ with $[t_\lambda(.,D_n)]^{(-\frac{1}{2})}_{\mathcal{M}} \leq C_\delta$ as $\mathcal{L}(\mathbb{C}, L^2(-1,1))$-valued Mikhlin multiplier because of Lemma 2.9. Similarly, it is easy to observe with the aid of (3.4) for $\lambda = 0$ and Lemma 2.9 that K^r_{10} can be considered as $\mathcal{L}(\mathbb{C}, L^2(-1,1))$-valued multiplier operator with symbol $k^r_{10}(\xi', D_n)$ of order $\frac{1}{2}$. Hence $\mathcal{T}_{10}\mathcal{K}_{10,\lambda}$ is an matrix-valued Mikhlin multiplier of order 0 with

$[\mathcal{T}_{10}\mathcal{K}_{10,\lambda}]_{\mathcal{M}} \le C_\delta.$ ■

Therefore $\mathcal{S}_{10,\lambda}$ is an invertible mapping on $W_q^{1-\frac{1}{q}}(\mathbb{R}^{n-1})^N$ iff it is invertible on $L^q(\mathbb{R}^{n-1})^N$. Because of Lemma 2.8, this is the case iff the operator is invertible on $L^2(\mathbb{R}^{n-1})^N$. Moreover,

$$\|\mathcal{S}_{10,\lambda}^{-1}\|_{\mathcal{L}(W_q^{1-\frac{1}{q}}(\mathbb{R}^{n-1})^N)} \le C_{\delta,q}, \qquad \text{uniformly in } \lambda \in \Sigma_\delta \cup \{0\},$$

iff the statement is true for $q = 2$.

Corollary 4.7 *Theorem 1.1 holds for $1 < q, r < \infty$ iff it holds for $q = r = 2$.*

4.3 Unique Solvability for $q = 2$

We start by proving an a priori estimate.

Lemma 4.8 *Let $\delta \in (0, \pi)$ and $\lambda \in \Sigma_\delta \cup \{0\}$. If $(u, p) \in W^2_{2,\lambda}(\Omega_0)^n \times W^1_2(\Omega_0)$ is a solution of the generalized Stokes equations (1.1)-(1.4) with right-hand side $(f, g, a^+) \in L^2(\Omega_0)^n \times W^1_{2,\lambda}(\Omega_0) \times W^{\frac{1}{2}}_{2,\lambda}(\partial\Omega_0^+)^n$, then*

$$\begin{aligned} &\|u\|_{2,2,\lambda} + \|\nabla p\|_2 + \|\gamma_0^+ p\|_{\frac{1}{2},2,\lambda} \\ \le \;& C_\delta \left(\|f\|_2 + \|g\|_{1,2,\lambda} + (1 + |\lambda|)\|g\|_{{}_0W_2^{-1}(\Omega_0)} + \|a^+\|_{\frac{1}{2},2,\lambda} \right). \end{aligned} \tag{4.10}$$

Proof: If $a^+ \neq 0$, we can choose a function $v \in {}_0W^2_{2,\lambda}(\Omega_0)^n$ with $\gamma_0^+ v = 0$, $\gamma_1^+ v_\tau = a^+_\tau$, $2\gamma_1^+ v_\nu = a^+_\nu$, and $\|v\|_{2,2,\lambda} \le C\|a^+\|_{\frac{1}{2},2,\lambda}$ because of Lemma 2.1. Then $u - v$ is a solution of the generalized Stokes equations (1.1)-(1.4) with $T_1^+(u, p) = 0$. Hence we can assume w.l.o.g $a^+ = 0$.

Let $E(u, v) = \frac{1}{2}(S(u), S(v))$, where $S(u) = \nabla u + \nabla u^T$. Using the inherent symmetry and Green's formula

$$E(u, v) \;=\; (S(u), \nabla v) = -(\Delta u, v) - (\nabla \operatorname{div} u, v) + (\gamma_\nu S(u), \gamma_0 v). \tag{4.11}$$

Since (u, p) is a solution of the Stokes resolvent equation,

$$(f - \nabla g, u) \;=\; \lambda\|u\|_2^2 + E(u, u) - (p, g) - (\gamma_0(S(u) \cdot \nu - p\nu), \gamma_0 u)_{\partial\Omega_0}.$$

The last term vanishes, since $T_1^+(u, p) = \gamma_0^+(S(u)\cdot\nu - p\nu) = 0$ and $\gamma_0^- u = 0$. Therefore we obtain by using Korn's inequality, see Lemma 2.4, and $\max\{\operatorname{Re}\lambda, |\operatorname{Im}\lambda|\} \ge c_\delta|\lambda|$ for $\lambda \in \Sigma_\delta \cup \{0\}$,

$$(1 + |\lambda|)\|u\|_2^2 + \|\nabla u\|_2^2 \le C_\delta \left(\|f - \nabla g\|_2\|u\|_2 + |(g, p)| \right). \tag{4.12}$$

Let $\tilde p \in {}_0W^1_{2,\lambda}(\Omega_0)$ be an extension of $\gamma_0^+ p$ with $\|p\|_{1,2,\lambda} \le C\|\gamma_0^+ p\|_{\frac12,2,\lambda}$. Then

$$\begin{aligned}(1+|\lambda|)|(g,p)| &\le (1+|\lambda|)\|g\|_{{}_0W_2^{-1}}\|\nabla(p-\tilde p)\|_2 + (1+|\lambda|)^{\frac12}\|g\|_2(1+|\lambda|)^{\frac12}\|\tilde p\|_2 \\ &\le C\left((1+|\lambda|)\|g\|_{{}_0W_2^{-1}} + \|g\|_{1,2,\lambda}\right)\left(\|\nabla p\| + \|\gamma_0^+ p\|_{\frac12,2,\lambda}\right)\end{aligned}$$

Therefore, multiplying (4.12) by $(1+|\lambda|)$ and using Young's inequality,

$$\begin{aligned}&(1+|\lambda|)^2\|u\|_2^2 + (1+|\lambda|)\|\nabla u\|_2^2 \\ &\le C_\delta\left(\|(f,\nabla g)\|_2^2 + \left((1+|\lambda|)\|g\|_{{}_0W_2^{-1}} + \|g\|_{1,2,\lambda}\right)\left(\|\nabla p\|_2 + \|\gamma_0^+ p\|_{\frac12,2,\lambda}\right)\right). \quad (4.13)\end{aligned}$$

If we set $v = \partial_i^2 u$, $i = 1,\dots,n-1$, in (4.11) and integrate by parts, we get

$$(1+|\lambda|)\|\partial_i u\|_2^2 + \|\nabla\partial_i u\|_2^2 \ \le\ C_\delta\left(\|(f,\nabla g)\|_2^2 + \|\partial_i g\|_2\|\partial_i p\|_2\right). \quad (4.14)$$

Expressing $p = p_1 + p_2$ by u and the data (f,g), cf. Section 4.1, using (4.5), (4.6), and the boundary conditions, we get

$$\begin{aligned}&\|\nabla p\|_2 + \|\gamma_0^+ p\|_{\frac12,2,\lambda} \\ &\le\ C_\delta\left(\|f\|_2 + \|g\|_{1,2,\lambda} + (1+|\lambda|)\|g\|_{{}_0W_2^{-1}} + \|\gamma_1^- \operatorname{div}' u'\|_{-\frac12,2} + \|\gamma_1 u_n\|_{\frac12,2,\lambda}\right) \\ &\le\ C_\delta\left(\|f\|_2 + \|g\|_{1,2,\lambda} + (1+|\lambda|)\|g\|_{{}_0W_2^{-1}} + \|\gamma_1 u\|_{\frac12,2,\lambda}\right). \quad (4.15)\end{aligned}$$

Moreover, because of Lemma 2.5,

$$\|\gamma_1 u\|_{\frac12,2,\lambda} \ \le\ C\|(\partial_n u, \partial_n^2 u)\|_2^{\frac12}\|((1+|\lambda|)^{\frac12}\partial_n u, \nabla'\partial_n u)\|_2^{\frac12}.$$

Combining this with $\partial_n^2 u = f - (\lambda - \Delta')u - \nabla p$ and (4.15), yields

$$\begin{aligned}&\|\nabla p\|_2 + \|\gamma_0^+ p\|_{\frac12,2,\lambda} \\ &\le\ C_\delta\left(\|f\|_2 + \|\nabla g\|_2 + (1+|\lambda|)\|g\|_{{}_0W_2^{-1}} + \|(|\lambda|u, (1+|\lambda|)^{\frac12}\partial_n u, \nabla\nabla' u)\|_2\right).\end{aligned}$$

Finally, if we combine the latter estimate with (4.13) and (4.14), we are led to

$$\begin{aligned}&\|u\|_{1,2,\lambda} + \|\nabla\nabla' u\|_2 + \|\nabla p\|_2 + \|\gamma_0^+ p\|_{\frac12,2,\lambda} \\ &\le\ C_\delta\left(\|f\|_2 + \|g\|_{1,2,\lambda} + (1+|\lambda|)\|g\|_{{}_0W_2^{-1}}\right).\end{aligned}$$

Using (1.1), we get the estimate for $\|\partial_n^2 u\|_2$. ■

In the next step we construct a solution operator for large λ.

Lemma 4.9 *Let $\delta \in (0,\pi)$. Then there is an $R > 0$ such that for every $\lambda \in \Sigma_\delta$ with $|\lambda| \ge R$ and for (f,g,a^+) as in (4.1) the generalized Stokes equations (1.1)-(1.4) have a unique solution $(u,p) \in W^2_{2,\lambda}(\Omega_0)^n \times W^1_{2,\lambda}(\Omega_0)$.*

Proof: As in the proof of Lemma 4.8 we can assume w.l.o.g. $a^+ = 0$. Let $R^-_{0,\lambda}$ be the solution operators of the reduced Stokes resolvent equations in $\mathbb{R}^{n-1} \times (-1, \infty)$ with Dirichlet boundary condition $\gamma_0^- u = 0$ and let $R^+_{1,\lambda}$ be the corresponding operators in $\mathbb{R}^{n-1} \times (-\infty, 1)$ for the boundary condition $T_1'^{\pm}(u) = 0$. Now we define

$$\widetilde{R}_{10,\lambda} f = \psi^+ R^+_{1,\lambda} \varphi^+ f + \psi^- R^-_{0,\lambda} \varphi^- f,$$

where φ^+, φ^- is a partition of unity for $[-1,1]$ with $\varphi^+ \equiv 1$ on $[\frac{1}{2}, 1]$ and $\varphi^- \equiv 1$ on $[-1, -\frac{1}{2}]$ and $\psi^\pm \in C^\infty_{(0)}([-1,1])$ with $\psi^\pm \equiv 1$ on $\operatorname{supp} \varphi^\pm$ and $0 \le \psi^\pm \le 1$. As usual we extend $\varphi^\pm f$ by zero where it is not defined.

Then $u = \widetilde{R}_{10,\lambda} f$ satisfies

$$(\lambda - \Delta) u + G_{10} u = f + S_{10} u + S'_{10} u$$

where $S_{10} u = -P^+ R^+_{1,\lambda} \varphi^+ f - P^- R^-_{0,\lambda} \varphi^- f$ with $P^\pm = 2\nabla \psi^\pm \cdot \nabla + \Delta \psi^\pm$ and

$$\begin{aligned} S'_{10} u &= \nabla K_{01}(2\gamma_1^+ u_\nu, -\gamma_1^- \operatorname{div}' u') - \psi^+ \nabla K_0^+ 2\gamma_1^+ u_\nu + \psi^- \nabla K_1^- \gamma_1^- \operatorname{div}' u' \\ &= \nabla(K_{01} - \widetilde{K}_{01})(2\gamma_1^+ u_\nu, -\gamma_1^- \operatorname{div}' u') + (\nabla \psi^+) K_0^+ 2\gamma_1^+ u_\nu - (\nabla \psi^-) K_1^- \gamma_1^- \operatorname{div}' u'. \end{aligned}$$

Here $\widetilde{K}_{01}$ is defined as in (3.11), see also Remark 3.5. Since $P^\pm$ is a differential operator of order 1 and because of Lemma 4.3, $\|S_{10} u\|_2 \le C_\delta |\lambda|^{-\frac{1}{2}} \|f\|_2$ for $\lambda \in \Sigma_\delta, |\lambda| \ge 1$. Due to Lemma 3.4,

$$\begin{aligned} \|\nabla(K_{01} - \widetilde{K}_{01})(2\gamma_1^+ u_\nu, -\gamma_1^- \operatorname{div}' u')\|_2 &\le C_\varepsilon \left(\|\gamma_1^+ u_\nu\|_{W_2^{\frac{1}{2}-2\varepsilon}} + \|\gamma_1^- \operatorname{div}' u'\|_{\dot{B}_2^{-\frac{1}{2}-2\varepsilon}} \right) \\ &\le C_\varepsilon \|\gamma_1 u\|_{W_2^{\frac{1}{2}-2\varepsilon}} \le C_\varepsilon \|u\|_{2-2\varepsilon,2} \end{aligned}$$

for $0 < \varepsilon < \frac{1}{4}$. Here $\|u\|_{2-2\varepsilon,2}$ denotes the $B_2^{2-2\varepsilon}(\Omega_0)$-norm of u. Since $|e^{-|\xi'|(1-x_n)}| \le 1$, we get $\sup_{|x_n| \le 1} \|(K_0^+ a)(., x_n)\|_2 \le \|a\|_2$. Thus

$$\|(\nabla \psi^+) \widetilde{K}_0^+ 2\gamma_1^+ u_\nu\|_2 \le C_\varepsilon \|\gamma_1^+ u_\nu\|_{\frac{1}{2}-2\varepsilon,2,\mathbb{R}^{n-1}} \le C_\varepsilon \|u\|_{2-2\varepsilon,2}.$$

Since $K_1^- \operatorname{div}' = K_0^- \operatorname{OP}'(i\xi'^T/|\xi'|)$, we get the same estimate with $(\nabla \psi^-) \tilde{K}_1^- \gamma_1^- \operatorname{div}' u'$ on the left-hand side. Because of $\|u\|_2 \le C_\delta |\lambda|^{-1} \|f\|_2$ and $\|u\|_{2,2} \le C_\delta \|f\|_2$, we get by real interpolation $\|S'_{01} u\|_2 \le C_\varepsilon \|u\|_{2-2\varepsilon,2} \le C_{\delta,\varepsilon} |\lambda|^{-\varepsilon} \|f\|_2$ uniformly in $\lambda \in \Sigma_\delta$, $|\lambda| \ge 1$, with $0 < \varepsilon < \frac{1}{4}$.

Hence there is an $R > 0$ such that $I + S_\lambda$ is invertible in $\mathcal{L}(L^q(\Omega_0))$ for $\lambda \in \Sigma_\delta$ with $|\lambda| \ge R$. Therefore $R_{10,\lambda} = \widetilde{R}_{10,\lambda}(I + S_\lambda)^{-1}$ is the solution operator of the reduced Stokes equation. Because of Lemma 4.1, the unique solvability of the reduced Stokes equations (for a fixed λ) is equivalent to unique solvability of the generalized Stokes equations. ■

Lemma 4.10 *Theorem 1.1 holds for $q = r = 2$.*

Proof: Lemma 4.8 implies that the linear operator $A_{10,\lambda}\colon X_{10,2} \to Y_{10,2}$,

$$\begin{aligned} A_{10,\lambda}(u,p) &:= ((\lambda-\Delta)u + \nabla p, \operatorname{div} u, T_1^+(u,p)),\\ X_{10,2} &:= (W^2_{2,\lambda}(\Omega_0)^n \cap {}_0W^1_{2,\lambda}(\Omega_0)^n) \times \{p \in \dot{W}^1_2(\Omega_0) : \gamma_0^+ p \in W^{\frac{1}{2}}_{2,\lambda}(\partial\Omega_0^+)\},\\ Y_{10,2} &:= L^2(\Omega_0)^n \times W^1_{2,\lambda}(\Omega_0) \times W^{\frac{1}{2}}_{2,\lambda}(\partial\Omega_0^+)^n, \end{aligned}$$

which corresponds to the generalized Stokes equations, is injective and has a closed range for every $\lambda \notin (-\infty,0)$. Hence it is a semi-Fredholm operator. Moreover, Lemma 4.9 implies that $A^{-1}_{10,\lambda}$ exists for all $\lambda \in \Sigma_\delta$, $|\lambda| \geq R$, for a suitable large $R > 0$. Because of the homotopy invariance of the Fredholm index, $A_{10,\lambda}$ has index 0 for every $\lambda \notin (-\infty,0)$. Since $A_{10,\lambda}$ is injective for every $\lambda \notin (-\infty,0)$ due to Lemma 4.8, $A_{10,\lambda}$ is invertible for these λ. Finally, (1.5) for $q=2$ is just the statement of Lemma 4.8. ■

Hence Theorem 1.1 is an immediate consequence of Corollary 4.7 and Lemma 4.10.

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