Infinitesimal elastic-plastic Cosserat micropolar theory. Modelling and global existence in the rate-independent case.

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Abstract

In this contribution we investigate the regularizing properties of generalized continua of Cosserat micropolar type in the elasto-plastic case. We propose an extension of classical infinitesimal elasto-plasticity to include consistently non-dissipative micropolar effects.

It is shown that the new model is thermodynamically admissible and allows for unique, global in-time solution of the corresponding rate-independent initial boundary value problem. The method of choice are the Yosida-approximation and a passage to the limit.

Key words: plasticity, visco-plasticity, polar-materials, non-simple materials, solid mechanics, elliptic systems, variational methods.

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1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-algebra theory, i.e. $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \,|X^T = -X\}$ are skew symmetric second order tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \,|\operatorname{tr}[X] = 0\}$ are traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$.

For a second order tensor X we define the third order tensor $\mathfrak{h} = \mathrm{D}_{\mathrm{x}}X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in (\mathbb{M}^{3\times 3})^3$. For $\mathfrak{h}$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}[\mathfrak{h}^1], \operatorname{tr}[\mathfrak{h}^2], \operatorname{tr}[\mathfrak{h}^3]) \in \mathbb{R}^3$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$, respectively. Sometimes we use also $\partial_X W(X)$ to denote the first derivative of W with respect to X. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_{\circ}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions.

2 Introduction

This article addresses the modelling and mathematical analysis of **geometrically linear** generalized continua of **Cosserat micropolar** type in the elastic as well as elasto-plastic case. General continuum models involving **independent** rotations have been introduced by the Cosserat brothers [CC09]. In fact, their original motivation came from the theory of surfaces, where the moving three-frame (Gauss frame) had been used successfully.

Their development has been largely forgotten for decades only to be rediscovered in the beginning of the sixties [Osh55, Gün58, AK61, ES64, Eri68, Tou62, Tou64, GR64, MT62, Sch67, TN65]. At that time theoretical investigations of non-classical continuum theories were the main motivation [Krö68]. The Cosserat concept has been generalized in various directions, for an overview of these so called **microcontinuum** theories look at [EK76, Eri99, Cap89].

Among the first contributions extending the Cosserat framework to infinitesimal elasto-plasticity we have to mention [Saw67, Lip69, Bes74]. More recent infinitesimal elastic-plastic formulations have been investigated in [dB92, DSW93, IW98, RV96]. These models directly comprise joint elastic and plastic Cosserat effects. Lately, the models have been extended to a finite elastic-plastic setting as well, see e.g. [GT01, San99, Ste94, Gra03, FCS97] and references therein. Most of these extensions directly comprise joint elastic and plastic Cosserat effects as well but we pretend that their physical and mathematical significance is at present much more difficult to asses than models where Cosserat effects are restricted to the elastic response of the material [FCS97] and references therein. Our own contribution will be of the second type.

Apart from the theoretical development, the Cosserat type models are today increasingly advocated as a means to regularize the pathological mesh size dependence of localization computations where shear failure mechanisms [CH85, MV87, Müh89, BP91, Bar94] play a dominant role, for applications in plasticity see the non-exhaustive list [IW98, DSW93, RV96, dB91, dBS91, dB92]. The occurring mathematical difficulties reflect the physical fact that upon localization the validity limit of the classical models is reached. In models without any internal length the deformation should be homogeneous on the scale of a representative volume element of the material [MA91].

The incorporation of a length scale, which is natural in a Cosserat theory, in principal has the power to remove the mesh sensitivity. The presence of the internal length scale causes the localization zones to have finite width. However, the actual length scale of a material is difficult to establish experimentally and theoretically [Lak95] and remains basically an open question as is the determination of other additionally appearing material constants in the Cosserat framework. It is also not entirely clear, how the shear band width depends on the characteristic length.

The mathematical analysis of Cosserat micropolar models is at present restricted to the infinitesimal, linear elastic models, see e.g. [Duv70, HH69, Ghe74a, Ghe74b]. The elasto-plastic situation has not been dealt with mathematically to the best of our knowledge.

As far as classical rate-independent elasto-plasticity is concerned we remark that global existence for the displacement has been shown only in a very weak, measure-valued sense, while the stresses could be shown to remain in $L^2(\Omega)$. For this results we refer for example to [AL87, Che02, Tem86]. If hardening or viscosity is added, then global classical solution are found see e.g. [Alb98, Che01b, Che01a]. A complete theory for the classical rate-independent case remains elusive, see also the remarks in [Che02].

While the infinitesimal Cosserat micropolar elasto-plasticity model in its various versions is interesting mathematically in its own right we rather concentrate on its possible regularizing properties. We emphasize that our non-dissipative formulation seems to provide just the correct amount of regularization missing in the classical elasto-plastic problem. This being our main thrust, we do not investigate Cosserat models where additional Cosserat effects have been introduced for the plastic behaviour as well.

Our contribution is organized as follows: first, we review the basic concepts of the geometrically linear elastic Cosserat micropolar theories in a variational context and present various existence results.

The formulation is then consistently extended to infinitesimal elasto-plasticity with non-dissipative micropolar effects. The decisive stress tensor is nothing else than the linearized elastic Eshelby energy momentum tensor.

Subsequently, we mathematically study the obtained rate-independent case and show, by means of the Yosida approximation and a passage to the limit, that the rate-independent problem admits a unique, global in-time solution for displacements and microrotations in standard Sobolev spaces under fairly mild assumptions on the data.

3 The infinitesimal elastic Cosserat model

Let us start by recalling the infinitesimal Cosserat approach. First, in the purely elastic case, an infinitesimal Cosserat theory can be obtained by introducing the additive decomposition of the macroscopic displacement gradient ∇u into infinitesimal **microrotation** $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$ (infinitesimal Cosserat rotation tensor) and infinitesimal **micropolar stretch tensor** (or first Cosserat deformation tensor) $\overline{\varepsilon} \in \mathbb{M}^{3\times 3}$ with

$$\nabla u = \overline{\varepsilon} + \overline{A} \tag{3.1}$$

where $\overline{\varepsilon} \notin \mathrm{Sym}(3)$, such that (3.1) is not necessarily the decomposition of ∇u into infinitesimal continuum stretch $\mathrm{sym}(\nabla u)$ and infinitesimal continuum rotation $\mathrm{skew}(\nabla u)$.

In the quasistatic case, the Cosserat theory is then obtained from a variational principle [San99, p.51] or [Ste97] for the infinitesimal displacement $u : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the independent infinitesimal microrotation $\overline{A} : \overline{\Omega} \mapsto \mathfrak{so}(3, \mathbb{R})$:

$$\begin{aligned} \mathcal{E}(u, \overline{A}) = &\int_\Omega W(\nabla u, \overline{A}, \mathrm{D_x}\overline{A}) - \langle f, u\rangle - \langle M, \overline{A}\rangle \,\mathrm{dV} \\ &\qquad - \int_{\Gamma_S} \langle N, u\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}\rangle \,\mathrm{dS} \mapsto \min . \text{ w.r.t. } (u, \overline{A}), \\ \overline{A}_{|\Gamma} = \overline{A}_{\mathrm{d}}\,, &\quad u_{|\Gamma} = g(t,x) - x\,. \end{aligned} \tag{3.2}$$

Here W represents the elastic energy density and $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g, \overline{A}_{\mathrm{d}}$ for infinitesimal displacements and rotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. The classical volume force is denoted by f and the additional volume couple by M. Variation of the action $\mathcal{E}$ with respect to u yields the equation for linearized balance of linear momentum and variation of $\mathcal{E}$ with respect to $\overline{A}$ yields the linearized version of balance of angular momentum.

3.1 Infinitesimal elastic Cosserat theory

It remains to specify the analytic form of the energy density W. A linearized version of material frame-indifference implies the reduction

$$W(\nabla u, \overline{A}, \mathrm{D_x}\overline{A}) = W(\overline{\varepsilon}, \mathrm{D_x}\overline{A})\,, \tag{3.3}$$

and for infinitesimal displacements u and small curvature $\mathrm{D_x}\overline{A}$ a quadratic ansatz is appropriate:

$$W(\overline{\varepsilon}, \mathrm{D_x}\overline{A}) = W_{\mathrm{mp}}^{\mathrm{infin}}(\overline{\varepsilon}) + W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A})\,, \tag{3.4}$$

with an additive decomposition of the energy density into microstretch and curvature parts.

In the isotropic case we assume for the stretch energy

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{infin}}(\overline{\varepsilon}) &= \mu\,\|\operatorname{sym}(\overline{\varepsilon})\|^2 + \mu_c\,\|\operatorname{skew}(\overline{\varepsilon})\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{\varepsilon})\right]^2 \\ &= \mu\,\|\operatorname{sym}\nabla u\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u) - \overline{A}\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\nabla u)\right]^2, \end{aligned} \tag{3.5}$$

where the **Cosserat couple modulus** $\mu_c \geq 0$ is an additional material parameter, complementing the two Lamè constants $\mu, \lambda > 0$.

For the curvature term we assume

$$W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A}) = \frac{2\mu+\lambda}{2}\frac{L_c^2}{12}\left(\alpha_5\,\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 + \alpha_6\,\|\operatorname{skew}\mathrm{D_x}\overline{A}\|^2 + \alpha_7\operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2\right). \tag{3.6}$$

Here, $L_c > 0$ with units of length introduces a specific internal characteristic length into the elastic formulation. In general we assume $\alpha_5 > 0$, $\alpha_6, \alpha_7 \geq 0$.

Two observations are essential. First, if $\mu_c = 0$, the infinitesimal problem completely decouples - the infinitesimal microrotations $\overline{A}$ have no influence at all on the macroscopic behaviour of the infinitesimal displacements and classical infinitesimal elasticity results.

Second, the choice $\alpha_6, \alpha_7 = 0$ is possible, since coercivity of the reduced curvature expression can still be concluded on account of the classical Korn's first inequality applied to $\operatorname{sym}\mathrm{D_x}\overline{A}$. [1]

In the limit of zero internal length scale $L_c = 0$, balance of angular momentum

$$D_{\overline{A}}W_{\mathrm{mp}}(\nabla u, \overline{A}) \in \mathrm{Sym} \Leftrightarrow D_{\overline{A}}W_{\mathrm{mp}}(\nabla u, \overline{A}) = 0\,, \tag{3.7}$$

implies already that infinitesimal continuum rotations and infinitesimal microrotations coincide: $\operatorname{skew}(\nabla u) = \overline{A}$, and this in turn is equivalent to the symmetry of the infinitesimal Cauchy stress σ or the so called **Boltzmann axiom**.

If we consider now $\mu_c > 0$, it is standard to prove that the corresponding minimization problem admits a unique minimizing pair $(u, \overline{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3))$. Existence results of this type have been obtained e.g. in [Duv70, HH69, Ghe74a, Ghe74b].

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[1] For $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ we have

$$\overline{A} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}, \operatorname{axl}(\overline{A}) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \nabla\operatorname{axl}(\overline{A}) = \begin{pmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{pmatrix}$$

$$\operatorname{sym}\nabla\operatorname{axl}(\overline{A}) = \begin{pmatrix} \alpha_x & \frac{\alpha_y+\beta_x}{2} & \frac{\alpha_z+\gamma_x}{2} \\ \frac{\alpha_y+\beta_x}{2} & \beta_y^2 & \frac{\beta_z+\gamma_y}{2} \\ \frac{\alpha_z+\gamma_x}{2} & \frac{\beta_z+\gamma_y}{2} & \gamma_z^2 \end{pmatrix}$$

$$\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2 = \alpha_x^2 + \beta_y^2 + \gamma_z^2 + \frac{(\alpha_y+\beta_x)^2}{2} + \frac{(\alpha_z+\gamma_x)^2}{2} + \frac{(\beta_z+\gamma_y)^2}{2}$$

$$\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 = \|\operatorname{sym}\nabla\overline{A}.e_1\|^2 + \|\operatorname{sym}\nabla\overline{A}.e_2\|^2 + \|\operatorname{sym}\nabla\overline{A}.e_3\|^2$$

$$= \frac{\alpha_x^2}{2} + \frac{\beta_x^2}{2} + \alpha_y^2 + \beta_z^2 + \frac{(\alpha_x+\beta_y)^2}{2} + \alpha_x^2 + \frac{\alpha_y^2}{2} + \gamma_y^2 + \gamma_z^2 + \frac{(\alpha_z+\gamma_x)^2}{2} + \beta_x^2 + \gamma_y^2 + \frac{\beta_z^2}{2} + \frac{\gamma_z^2}{2} + \frac{(\beta_y+\gamma_x)^2}{2}$$

Now it is easy to see that for some $c^+ > 0$ it holds $\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 \geq c^+\,\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2$ since $\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 = 0$ implies $\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2 = 0$. Hence, the standard Korn's inequality applied to $\|\operatorname{sym}\nabla\operatorname{axl}(\overline{A})\|^2$ yields unique existence.

Theorem 3.1 (Existence for infinitesimal elastic Cosserat model)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{A}_{\mathrm{d}} \in H^1(\Omega, \mathfrak{so}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M_c \in L^2(\Gamma_C, \mathfrak{so}(3,\mathbb{R}))$. Then models based on (3.5) and (3.6) admit a unique minimizing solution pair $(u, \overline{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3))$. The solution is smoother if the data are smoother.

Proof. We apply the direct methods of variations. First we observe that infimizing sequences $(u_k, \overline{A}_k)$ exist and

$$\begin{aligned}
\infty > \int_\Omega & W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u_k - \overline{A}_k) + W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A}_k) - \langle f, u_k\rangle \,\mathrm{dV} \\
&\geq \int_\Omega \mu_c \,\|\nabla u_k - \overline{A}_k\|^2 \,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \\
&= \int_\Omega \mu_c \,\|\,\mathrm{sym}(\nabla u_k - \overline{A}_k)\|^2 + \mu_c \,\|\,\mathrm{skew}(\nabla u_k - \overline{A}_k)\|^2 \,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \\
&\geq \int_\Omega \mu_c \,\|\,\mathrm{sym}\,\nabla u_k\|^2 \,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \\
&\geq \mu_c \, c_K \,\|u_k\|^2_{H^1(\Omega)} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)}\,,
\end{aligned} \tag{3.8}$$

showing that u_k is bounded in $H^1(\Omega)$. We have used that sym is orthogonal to skew and the classical Korn's first inequality together with the boundary conditions for u_k. Moreover, again by the classical Korn's first inequality (if $\alpha_6 = 0$) or directly pointwise, we obtain boundedness of $\overline{A}_k$ in $H^1(\Omega, \mathfrak{so}(3))$. We can choose a subsequence of $(u_k, \overline{A}_k)$ converging strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. By overall convexity of the energy density in $(\nabla u, \mathrm{D_x}\overline{A})$ the limit pair is a minimizer.

For the uniqueness we consider the second derivative of the strains

$$\begin{aligned}
& D^2_{(\nabla u, \overline{A})} W(\nabla u - \overline{A}).((\nabla\phi, \delta\overline{A}), (\nabla\phi, \delta\overline{A})) \geq \mu_c \|\nabla\phi - \delta\overline{A}\|^2 \\
&= \mu_c \|\,\mathrm{sym}\,\nabla\phi\|^2 + \mu_c \|\,\mathrm{skew}(\nabla\phi - \delta\overline{A})\|^2 \geq \mu_c \|\,\mathrm{sym}\,\nabla\phi\|^2\,.
\end{aligned} \tag{3.9}$$

By the classical Korn's first inequality we obtain uniform positivity of the second derivative upon integration. The functional is strictly convex, the solution is unique.

Since the resulting field equations of force balance and balance of angular momentum are linear, uniformly elliptic with constants coefficients the standard elliptic regularity theory applies such that for pure Dirichlet boundary conditions the solution is the smoother the smoother the data. ■

The corresponding **infinitesimal gradient constrained Cosserat micropolar** model (or **indeterminate couple stress** model) has the form (simplified curvature term:

$\alpha_5 = \alpha_6 = 1,\ \alpha_7 = 0$)

$$\int_\Omega \mu \|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2 + \frac{2\mu+\lambda}{2}\frac{L_c^2}{12}\|\mathrm{D_x}\operatorname{skew}(\nabla u)\|^2 - \langle f, u\rangle\,\mathrm{dV}$$

$$-\int_{\Gamma_S}\langle N, u\rangle\,\mathrm{dS} - \int_{\Gamma_C}\langle M_c, \operatorname{skew}(\nabla u)\rangle\,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } u \tag{3.10}$$

$$\sigma^{\mathrm{loc}} = 2\,\mu\operatorname{sym}(\nabla u) + \lambda\operatorname{tr}[\operatorname{sym}(\nabla u)]\cdot 1\!\!1 \in \operatorname{Sym},\ \text{constitutive stress}$$

$$u_{|\partial\Omega}(x) = g(x) - x\,, \quad \operatorname{skew}(\nabla u)_{|\partial\Omega} = \operatorname{skew}(\nabla g)_{|\partial\Omega}\,.$$

Using the same methods as before we obtain

Theorem 3.2 (Existence for infinitesimal gradient case)
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary of class C^1 and assume for the boundary data $g \in H^2(\Omega, \mathbb{R}^3)$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M_c \in L^2(\Gamma_C, \mathfrak{so}(3))$. Then a model based on (3.10) admits a unique minimizing solution $u \in H^1(\Omega) \cap \{\nabla \operatorname{curl} u \in L^2(\Omega)\}$, cf. [Duv70].

4 Infinitesimal Cosserat micropolar elasto-plasticity

4.1 Non-dissipative extension to micropolar elasto-plasticity

Now we extend the formulation of micropolar elasticity to cover infinitesimal elasto-plasticity as well. It should be clear that there exists various ways of obtaining such an extension, for an overview of the competing models we refer to the instructive survey article [FS03]. Incidentally, the Cosserats themselves [CC09, p.5] already envisaged the application of their general theory to plasticity and fracture. Without restricting generality we base the following considerations on a simplified curvature expression by setting $\alpha_5 = \alpha_6 = 1,\ \alpha_7 = 0$.

The basic idea of a **non-dissipative** extension is quite simple. Consider the addititive decomposition of the total micropolar stretch into elastic and plastic parts

$$\overline{\varepsilon} = \overline{\varepsilon}_e + \overline{\varepsilon}_p\,, \tag{4.11}$$

and assume that rotational effects remain elastic: $\overline{A}_e := \overline{A}$. Now we replace $\overline{\varepsilon}$ in (3.5) with $\overline{\varepsilon}_e$ which yields (note that $\|\mathrm{D_x}\overline{A}_e\|^2 = 2\|\nabla \operatorname{axl}(\overline{A}_e)\|^2$)

$$\int_\Omega \mu\,\|\operatorname{sym}\overline{\varepsilon}_e\|^2 + \mu_c\,\|\operatorname{skew}(\overline{\varepsilon}_e)\|^2 + \frac{\lambda}{2}\operatorname{tr}[\overline{\varepsilon}_e]^2 + 2\frac{2\mu+\lambda}{2}\frac{L_c^2}{12}\|\nabla\operatorname{axl}(\overline{A}_e)\|^2\,\mathrm{dV} \tag{4.12}$$

$$\int_\Omega \mu\,\|\varepsilon - \operatorname{sym}\overline{\varepsilon}_p\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u - \overline{A}_e - \overline{\varepsilon}_p)\|^2 + \frac{\lambda}{2}\operatorname{tr}[\varepsilon - \overline{\varepsilon}_p]^2 + 2\frac{2\mu+\lambda}{2}\frac{L_c^2}{12}\|\nabla\operatorname{axl}(\overline{A}_e)\|^2\,\mathrm{dV}$$

as thermodynamic potential. We need to supply a consistent flow rule for $\overline{\varepsilon}_p$ (note again that $\overline{A}_e$ acts solely elastically). By choosing

$$\dot{\varepsilon_p}(t) \in f\,(T_E), \quad T_E := -\partial_{\overline{\varepsilon}_p} W_{\mathrm{mp}}^{\mathrm{infin}}(\overline{\varepsilon}_e)\,, \tag{4.13}$$
$$W_{\mathrm{mp}}^{\mathrm{infin}}(\overline{\varepsilon}_e) = \mu\,\|\,\mathrm{sym}\,\overline{\varepsilon}_e\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{\varepsilon}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\overline{\varepsilon}_e]^2\,,$$

with a constitutive multifunction f such that $\langle f(\Sigma), \Sigma\rangle \geq 0, \forall\,\Sigma \neq 0$ the **reduced dissipation inequality**

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(\overline{\varepsilon}_e, \overline{A}_e, \overline{\varepsilon}_p) \leq 0 \tag{4.14}$$

at fixed in time $(\nabla u, \overline{A}_e)$ is satisfied, thus ensuring the second law of thermodynamics.

For simplicity we choose the multifunction f to take trace free symmetric values only, i.e. $f\,(T_E) \in \mathrm{Sym}(3) \cap \mathfrak{sl}(3,\mathbb{R})$. This sets the **infinitesimal plastic spin** $\mathrm{skew}(\overline{\varepsilon}_p)$ to **zero** and restricts attention to incompressible plasticity as in classical formulations. Since then $\overline{\varepsilon}_p \in \mathrm{Sym}(3)$ we may identify $\overline{\varepsilon}_p = \varepsilon_p$. We have thus obtained our infinitesimal model:

4.2 Infinitesimal elasto-plastic Cosserat model

The infinitesimal system in variational form with non-dissipative Cosserat effects reads

$$\int_\Omega \mu\,\|\varepsilon - \varepsilon_p\|^2 + \mu_c\,\|\,\mathrm{skew}(\nabla u - \overline{A}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\varepsilon]^2 + 2\frac{2\mu+\lambda}{2}\,\frac{L_c^2}{12}\,\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2 - \langle f, u\rangle$$
$$-\,\langle M, \overline{A}_e\rangle\,\mathrm{dV} - \int_{\Gamma_S} \langle N, u\rangle\,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}_e\rangle\,\mathrm{dS} \mapsto \min. \quad \text{w.r.t. } (u, \overline{A}_e) \text{ at constant } \varepsilon_p$$
$$\dot{\varepsilon_p}(t) \in f\,(T_E), \qquad T_E = 2\mu\,(\varepsilon - \varepsilon_p) \tag{4.15}$$
$$u_{|\Gamma} = g(t,x) - x, \quad \overline{A}_{e|_\Gamma} = \mathrm{skew}(\nabla g(t,x))_{|\Gamma}\,.$$

The corresponding weak system of equations for pure Dirichlet conditions is given by (note that $\|\overline{A}_e\|^2 = 2\|\,\mathrm{axl}(\overline{A}_e)\|^2$ for $\overline{A}_e \in \mathfrak{so}(3,\mathbb{R})$)

$$\mathrm{Div}\,\sigma = -f, \quad x \in \Omega$$
$$\sigma = 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - \overline{A}_e) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1 \tag{4.16}$$
$$-\frac{2\mu+\lambda}{2}\,\frac{L_c^2}{12}\,\Delta\,\mathrm{axl}(\overline{A}_e) = \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u) - \overline{A}_e) + 2\,\mathrm{axl}(\mathrm{skew}(M))$$
$$\dot{\varepsilon_p}(t) \in f\,(T_E), \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p)$$
$$u_{|\partial\Omega}(t,x) = g(t,x) - x, \quad x \in \partial\Omega\,, \overline{A}_{e|_{\partial\Omega}} = \mathrm{skew}(\nabla g(t,x))_{|\partial\Omega}\,.$$
$$\mathrm{tr}\,[\varepsilon_p(0)] = 0\,, \quad \varepsilon_p(0) \in \mathrm{Sym}(3)\,.$$

We remark that the derivation of this model is intrinsically correct but that it can also be obtained as the linearized version of a corresponding geometrically exact model [Nef03] based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts, which, as it were, was prior to this linearized model.

In [DSW93, p.815] an elasto-plastic model based on the infinitesimal theory with dissipative Cosserat effects has been investigated by means of localized considerations. They show that the Cosserat couple modulus $\mu_c > 0$ has a decisive influence on localization effects, essentially excluding mode II shear failure. In light of our development with non-dissipative Cosserat effects, however, it is difficult to transfer this insight directly.

4.3 Mathematical analysis of the infinitesimal model

For brevity of notation we write in this part A instead of $\overline{A}_e$ and c instead of the positive constant $\frac{2\mu+\lambda}{2}\frac{L_c^2}{12}$. Moreover, we study general Dirichlet boundary conditions, this means that the boundary data for the displacement and for the microrotation may be prescribed independently. Without loss of generality we consider $M = 0$.

The goal of this subsection is to prove that the following initial boundary-value problem

$$\begin{aligned}
\operatorname{Div}\sigma &= -f\,, \\
\sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}[\varepsilon]\cdot 1\!\!1\,, \\
-c\,\Delta\operatorname{axl}(A) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u) - A)\,, \\
\dot\varepsilon_p &\in \mathcal{f}(T_E)\,, \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,, \\
u_{|\partial\Omega} &= u_D\,, \quad A_{|\partial\Omega} = A_D\,, \quad \varepsilon_p(0) = \varepsilon_p^0\,,
\end{aligned} \tag{4.17}$$

possesses global in time L^2-solutions, assuming that the given data $f, u_D, A_D, \varepsilon_p^0$ satisfy some natural restrictions and $\mathcal{f} : D(\mathcal{f}) \subset \operatorname{Sym}(3) \to \mathcal{P}(\operatorname{Sym}(3))$ is a maximal monotone mapping with $0 \in \mathcal{f}(0)$. This mapping defines the maximal monotone operator $\mathcal{f} : L^2(\Omega, \operatorname{Sym}(3)) \to \mathcal{P}(L^2(\Omega, \operatorname{Sym}(3)))$ with the domain $\mathcal{D}(\mathcal{f}) = \{T \in L^2(\Omega, \operatorname{Sym}(3)) : T(x) \in D(\mathcal{f})$ a.e. in Ω and there exists $S \in L^2(\Omega, \operatorname{Sym}(3))$ with $s(x) \in \mathcal{f}(T(x))$ a.e.in $\Omega\}$. System (4.17) contains only one physical nonlinearity, the constitutive multifunction $\mathcal{f}$, which is assumed **maximal monotone**. Such a nonlinear mapping can be approximated nicely by single-valued, global Lipschitz functions $\mathcal{f}_\eta$, in the literature called the **Yosida approximation** (see for example [AC84]). Hence, our idea is quite natural: we rewrite (4.17) with $\mathcal{f}_\eta$ instead of $\mathcal{f}$ and try to pass to the limit $\eta \to 0^+$.

Thus, for all $\eta > 0$ we study first the following **approximated** initial boundary-value problem

$$\begin{aligned}
\operatorname{Div}\sigma^\eta &= -f\,, \\
\sigma^\eta &= 2\mu\,(\varepsilon^\eta - \varepsilon_p^\eta) + 2\,\mu_c\,(\operatorname{skew}(\nabla u^\eta) - A) + \lambda\operatorname{tr}[\varepsilon^\eta]\cdot 1\!\!1\,, \\
-c\,\Delta\operatorname{axl}(A^\eta) &= -\mu_c\operatorname{axl}(A^\eta) + \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u^\eta))\,, \\
\dot\varepsilon_p^\eta &= \mathcal{f}_\eta(T_E^\eta)\,, \quad T_E^\eta = 2\mu\,(\varepsilon^\eta - \varepsilon_p^\eta)\,, \\
u^\eta_{|\partial\Omega} &= u_D\,, \quad A^\eta_{|\partial\Omega} = A_D\,, \quad \varepsilon_p^\eta(0) = \varepsilon_p^0\,.
\end{aligned} \tag{4.18}$$

Theorem 4.1 (Global existence and uniqueness for approximated problem) *Let us assume that the given data possess the following regularity: for all times $T > 0$*

$$f \in C([0,T], L^2(\Omega, \mathbb{R}^3))\,,\ u_D \in C([0,T], H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3))\,,\ A_D \in C([0,T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3,\mathbb{R})))$$

and the initial data ε_p^0 belongs to $L^2(\Omega, \mathrm{Sym}(3))$. Then the approximated problem has a global in time, unique solution $(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)$ with the regularity

$$\begin{aligned} &u^\eta \in C([0,T], H^1(\Omega, \mathbb{R}^3))\,, && \varepsilon^\eta \in C([0,T], L^2(\Omega, \mathrm{Sym}(3)))\,,\\ &\varepsilon_p^\eta \in C^1([0,T], L^2(\Omega, \mathrm{Sym}(3)))\,, && A^\eta \in C([0,T], H^2(\Omega, \mathfrak{so}(3,\mathbb{R})))\,. \end{aligned}$$

If the given data are more regular in time, more precisely if

$$\begin{aligned} &\dot f \in C([0,T], L^2(\Omega, \mathbb{R}^3))\,,\ \dot u_D \in C([0,T], H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3))\,,\\ &\dot A_D \in C([0,T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3,\mathbb{R})))\,, \end{aligned} \tag{4.19}$$

then the unique solution is also C^1 in time.

Proof. We give a sketch of the proof, which is otherwise standard. Note that the approximated system of equations contains only global Lipschitz nonlinearities. Hence, we use Banach´s Fix Point Theorem. For a fixed time $T > 0$ let us denote by X the Banach space $C([0,T], L^2(\Omega, \mathrm{Sym}(3)))$. We define an operator $P : X \to X$ as follows: for $\varepsilon \in X$ we solve the integral equation

$$\varepsilon_p(t) = \int_0^t f_\eta(2\mu\,(\varepsilon(\tau) - \varepsilon_p(\tau)))\,d\tau + \varepsilon_p^0\,. \tag{4.20}$$

By the regularity of f_η it follows that this equation is uniquely solvable in X. Then for the solution ε_p we study the following elliptic boundary-value problem

$$\begin{aligned} &\mathrm{Div}\left(2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1\right) = -f\,,\\ &-c\,\Delta\,\mathrm{axl}(A) = -\mu_c\ \mathrm{axl}(A) + \mu_c\ \mathrm{axl}(\mathrm{skew}(\nabla u))\,,\\ &u_{|\partial\Omega} = u_D\,, \quad A_{|\partial\Omega} = A_D \end{aligned}$$

for the pair (u, A) of unknown functions. This problem has a unique solution u with the regularity $C([0,T], H^1(\Omega, \mathbb{R}^3))$ and $A \in C([0,T], H^2(\Omega, \mathfrak{so}(3,\mathbb{R})))$. Finally, we set

$$P(\varepsilon) = \frac{1}{2}(\nabla u + \nabla^T u)\,.$$

It is not difficult to see that for short times T, the operator P is a contraction. Moreover, the contraction constant depends on the Lipschitz constant of the function f_η and on T only. Hence, for small T the mapping P possesses a unique fix point in X and this function defines a local in time solution of the approximated system. Next, using the fact that the length of the existence interval does not depend on the given data we may extend the solution with the same time step and obtain a global in time, unique solution. Finally, we see that the solution ε_p is even more regular in time, this means $\varepsilon_p \in C^1([0,T], L^2(\Omega, \mathrm{Sym}(3)))$. Then for given data satisfying (4.19) we conclude that the solution is C^1 in time. ■

The main idea of the last proof was based on the global Lipschitz property of the nonlinear function f_η. However, we did not yet use the physical structure of the problem. Next, we prove that the energy associated with the problem is bounded independently of the parameter η. The energy is defined by

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A)(t) = \int_\Omega \Big(\mu\|\varepsilon - \varepsilon_p\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[\varepsilon]^2 + \mu_c\|\,\mathrm{skew}(\nabla u) - A\|^2 + 2c\,\|\nabla\,\mathrm{axl}(A)\|^2\Big)\, dx\,.$$

This function is elastically **coercive** with respect to ∇u: for $\lambda > 0$ the part $\mathrm{tr}\,[\varepsilon]^2$ yields the boundedness of the divergence $\mathrm{Div}\, u$ of the displacement u and the term $\|\,\mathrm{skew}(\nabla u) - A\|^2$ together with the control of $\nabla\,\mathrm{axl}(A)$ implies the boundedness of the rotation $\mathrm{curl}\, u$ of u. This property of the energy is the crucial one in our existence theory. In classical rate-independent plasticity, $\mathrm{curl}\, u$ is not controlled.

Theorem 4.2 (Energy estimate for the approximate sequence)
Let us assume that the given data satisfy (4.19) and $\{(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)\}$ is the solution of the approximate problem. Then for all times $T > 0$ there exists a positive constant $C(T)$ independent of η such that

$$\mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t) \le C(T) \quad \text{for all} \;\; t \in [0,T)\,. \tag{4.21}$$

Proof. Calculating the time derivative of the energy we obtain

$$\begin{aligned}\dot{\mathcal{E}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t) &= \int_\Omega \Big(2\mu\langle\varepsilon^\eta - \varepsilon_p^\eta, \dot\varepsilon^\eta - \dot\varepsilon_p^\eta\rangle + \lambda \mathrm{tr}\,[\varepsilon^\eta]\mathrm{tr}\,[\dot\varepsilon^\eta]\\
&+2\mu_c\langle \mathrm{skew}(\nabla u^\eta) - A^\eta, \mathrm{skew}(\nabla \dot u^\eta) - \dot A^\eta\rangle + 4c\langle\nabla\,\mathrm{axl}(A^\eta), \nabla\,\mathrm{axl}(\dot A^\eta)\rangle\Big)\, dx =\\
&- \int_\Omega \langle T_E^\eta, \dot\varepsilon_p^\eta\rangle dx + \int_\Omega \langle\sigma^\eta, \nabla \dot u^\eta\rangle dx - 2\mu_c\int_\Omega \langle \mathrm{skew}(\nabla u^\eta) - A^\eta, \dot A^\eta\rangle dx\\
&+4c\int_\Omega \langle\nabla\,\mathrm{axl}(A^\eta), \nabla\,\mathrm{axl}(\dot A^\eta)\rangle dx\,.\end{aligned}$$

The first integral on the right hand side of the last equality is nonnegative. In the second and in the fourth integral we integrate partially to obtain

$$\begin{aligned}\dot{\mathcal{E}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t) &\le \int_\Omega \langle f, \dot u^\eta\rangle dx + \int_{\partial\Omega} \langle \sigma^\eta.n, \dot u^\eta\rangle ds\\
&-4\mu_c \int_\Omega \langle \mathrm{axl}\,\mathrm{skew}(\nabla u^\eta) - \mathrm{axl}(A^\eta), \mathrm{axl}(\dot A^\eta)\rangle dx\\
&-4c\int_\Omega \langle \Delta\,\mathrm{axl}(A^\eta), \mathrm{axl}(\dot A^\eta)\rangle dx + 4c\int_{\partial\Omega}\langle\nabla\,\mathrm{axl}(A^\eta).n, \mathrm{axl}(\dot A^\eta)\rangle ds\,.\end{aligned}$$

Using the equation for the microrotations and the boundary conditions we finally have

$$\dot{\mathcal{E}}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(t)\le\int\limits_\Omega\langle f,\dot u^\eta\rangle dx+\int\limits_{\partial\Omega}\langle\sigma^\eta.n,\dot u_D\rangle ds$$
$$+4c\int\limits_{\partial\Omega}\langle\nabla\,\mathrm{axl}(A^\eta).n,\mathrm{axl}(\dot A_D)\rangle ds\,. \tag{4.22}$$

Note that the boundary integrals are defined in the sense of the duality between the spaces $H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)$ and $H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)$. Integrating (4.22) in time we arrive at the inequality

$$\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(t)\le\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(0)+\int\limits_0^t\int\limits_\Omega\langle f,\dot u^\eta\rangle dx$$
$$+\int\limits_0^t\int\limits_{\partial\Omega}\langle\sigma^\eta.n,\dot u_D\rangle ds+4c\int\limits_0^t\int\limits_{\partial\Omega}\langle\nabla\,\mathrm{axl}(A^\eta).n,\mathrm{axl}(\dot A_D)\rangle ds\,. \tag{4.23}$$

By the continuity with respect to time we conclude that the initial values $u^\eta(0),\varepsilon^\eta(0),A^\eta(0)$ are solutions of the following linear elliptic boundary-value problem

$$\begin{aligned}
&\mathrm{Div}\,\sigma^\eta(0)=-f\,,\\
&\sigma^\eta(0)=2\mu\,(\varepsilon^\eta(0)-\varepsilon_p^\eta(0))+2\,\mu_c\,(\mathrm{skew}(\nabla u^\eta(0))-A^\eta(0))+\lambda\,\mathrm{tr}\,[\varepsilon^\eta(0)]\cdot 1\!\!1\,,\\
&-c\Delta\,\mathrm{axl}(A^\eta(0))=-\mu_c\;\mathrm{axl}(A^\eta(0))+\mu_c\;\mathrm{axl}(\mathrm{skew}(\nabla u^\eta(0)))\,,\\
&u^\eta(0)_{|\partial\Omega}=u_D\,,\quad A^\eta(0)_{|\partial\Omega}=A_D\,,
\end{aligned} \tag{4.24}$$

where $\varepsilon^\eta(0)=1/2(\nabla u^\eta(0)+\nabla^T u^\eta(0))$. The unique solution of (4.24) satisfies $u^\eta(0)\in H^1(\Omega,\mathbb{R}^3)$, $\varepsilon^\eta(0)\in L^2(\Omega,\mathrm{Sym}(3)))$, $A^\eta(0)\in H^2(\Omega,\mathfrak{so}(3,\mathbb{R})))$ and it independent of η and the initial energy value $\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(0)$ is a constant. Next, we analyse the first integral from the right hand side of (4.23). Integrating partially in time we obtain

$$\int\limits_0^t\int\limits_\Omega\langle f,\dot u^\eta\rangle dx\,d\tau=-\int\limits_0^t\int\limits_\Omega\langle\dot f,u^\eta\rangle dx\,d\tau+\int\limits_\Omega\langle f(t),u^\eta(t)\rangle dx-\int\limits_\Omega\langle f(0),u^\eta(0)\rangle dx$$
$$\le\frac{1}{2}\int\limits_0^t\|\dot f\|_{L^2}^2d\tau+\frac{1}{2}\int\limits_0^t\|u^\eta\|_{L^2}^2d\tau+\|f(0)\|_{L^2}\|u^\eta(0)\|_{L^2}+\|f(t)\|_{L^2}\|u^\eta(t)\|_{L^2}\,.$$

By Poincaré´s inequality we conclude that

$$\|u^\eta(t)\|_{L^2}\le\|u^\eta(t)-\tilde u_D(t)\|_{L^2}+\|\tilde u_D(t)\|_{L^2}\le\mathrm{diam}(\Omega)(\|\nabla u^\eta(t)\|_{L^2}+\|\nabla\tilde u_D(t)\|_{L^2})+\|\tilde u_D(t)\|_{L^2}$$

where $\tilde u_D$ is a function from $H^1(\Omega,\mathbb{R}^3)$ with $\tilde u_{D|\partial\Omega}=u_D$. By the coercivity of the energy with respect to the gradient of u^η there exists a positive constant C_E independent of η

such that $\|\nabla u^\eta(t)\|_{L^2} \leq C_E\, \mathcal{E}^{\frac{1}{2}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t)$. Using the last results we have

$$\Big|\int_0^t \int_\Omega \langle f, \dot{u}^\eta \rangle dx\, d\tau\Big| \leq C \int_0^t \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(\tau)\, d\tau$$

$$+C\|f(t)\|_{L^2}\, \mathcal{E}^{\frac{1}{2}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t) + C(t) \qquad (4.25)$$

where the constants $C, C(t)$ do not depend on η and $C(t)$ depends on the given data only. The second integral in (4.23) is estimated as follows

$$\Big|\int_0^t \int_{\partial\Omega} \langle \sigma^\eta.n, \dot{u}_D \rangle ds\Big| \leq \int_0^t \|\sigma^\eta.n\|_{H^{-\frac{1}{2}}} \|\dot{u}_D\|_{H^{\frac{1}{2}}} d\tau$$

$$\leq \text{(by the trace theorem in the space } L^2(\operatorname{Div}) \text{ [Tem83, Chapter1])} \leq$$

$$C \int_0^t (\|\sigma^\eta\|_{L^2} + \|\operatorname{Div}\sigma^\eta\|_{L^2}) \|\dot{u}_D\|_{H^{\frac{1}{2}}} d\tau \leq C \int_0^t \|f\|_{L^2} \|\dot{u}_D\|_{H^{\frac{1}{2}}} d\tau \qquad (4.26)$$

$$+C \int_0^t \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(\tau)\, d\tau + C \int_0^t \|\dot{u}_D\|^2_{H^{\frac{1}{2}}} d\tau$$

where $C > 0$ does not depend on η. To estimate the last integral in (4.23) we use H^2-regularity of the microrotations

$$\Big|\int_0^t \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta).n, \operatorname{axl}(\dot{A}_D) \rangle ds\Big| \leq \int_0^t \|\nabla \operatorname{axl}(A^\eta).n\|_{H^{-\frac{1}{2}}} \|\operatorname{axl}(\dot{A}_D)\|_{H^{\frac{1}{2}}} d\tau$$

$$\leq C \int_0^t (\|\nabla \operatorname{axl}(A^\eta)\|_{L^2} + \|\Delta \operatorname{axl}(A^\eta)\|_{L^2}) \|\operatorname{axl}(\dot{A}_D)\|_{H^{\frac{1}{2}}} d\tau \qquad (4.27)$$

$$= C \int_0^t (\|\nabla \operatorname{axl}(A^\eta)\|_{L^2} + \frac{\mu_c}{c} \|\operatorname{skew}(\nabla u^\eta) - A^\eta\|_{L^2}) \|\operatorname{axl}(\dot{A}_D)\|_{H^{\frac{1}{2}}} d\tau$$

$$\leq \tilde{C} \int_0^t \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(\tau)\, d\tau + \tilde{C} \int_0^t \|\operatorname{axl}(\dot{A}_D)\|^2_{H^{\frac{1}{2}}} d\tau\,,$$

where again the constants $C, \tilde{C}$ do not depend on η. Inserting (4.25), (4.26) and (4.27) into (4.23) we obtain the following inequality

$$\mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t) \leq C_1 \|f(t)\|_{L^2}\, \mathcal{E}^{\frac{1}{2}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(t) + C_2 \int_0^t \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(\tau)\, d\tau + C_3(t),$$

where $C_1, C_2, C_3(t)$ do not depend on η and $C_3(t)$ depends on the given data only. Next, we separate in the first term on the right hand side the energy with a small factor and absorb this expression by the left hand side. Finally, the Lemma of Gronwall completes the proof. ■

The energy estimate proved in the last theorem yields boundedness of the stresses $\{\sigma^\eta\}$ in the space $L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))$ and of the microrotations $\{A^\eta\}$ in the space $L^\infty((0,T), H^1(\Omega, \mathfrak{so}(3,\mathbb{R})))$. Moreover, using that the energy controls the gradient of the displacement the sequence $\{u^\eta\}$ is bounded in the space $L^\infty((0,T), H^1(\Omega,\mathbb{R}^3))$ and consequently the sequence of strains $\{\varepsilon^\eta\}$ and the sequence of inelastic strains $\{\varepsilon_p^\eta\}$ are bounded in the space $L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))$. Hence, for a subsequence (again denoted using the superscript η) we have: for all $T > 0$

$$\begin{array}{lll}
\sigma^\eta \overset{*}{\rightharpoonup} \sigma & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
A^\eta \overset{*}{\rightharpoonup} A & \text{in} & L^\infty((0,T), H^1(\Omega, \mathfrak{so}(3,\mathbb{R})))\,, \\
u^\eta \overset{*}{\rightharpoonup} u & \text{in} & L^\infty((0,T), H^1(\Omega, \mathbb{R}^3))\,, \\
\varepsilon^\eta \overset{*}{\rightharpoonup} \varepsilon & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
\varepsilon_p^\eta \overset{*}{\rightharpoonup} \varepsilon_p & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))
\end{array}$$

and the limit functions satisfy

$$\varepsilon = \frac{1}{2}(\nabla u + \nabla^T u)\,, \quad \sigma = 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1\,.$$

Moreover, we see that the sequence $\{\mathrm{Div}\,\sigma^\eta\}$ is constant with respect to η and consequently bounded in the space $L^\infty((0,T), L^2(\Omega,\mathbb{R}^3))$ and the sequence $\{\Delta\,\mathrm{axl}(A^\eta)\}$ is bounded in the space $L^\infty((0,T), L^2(\Omega,\mathbb{R}^3))$. Using the closedness of the differential operators in Sobolev spaces the limit functions satisfy the system

$$\begin{aligned}
&\mathrm{Div}\,\sigma = -f\,, \\
&-c\Delta\,\mathrm{axl}(A) = -\mu_c\ \mathrm{axl}(A) + \mu_c\ \mathrm{axl}(\mathrm{skew}(\nabla u))\,, \\
&u_{|\partial\Omega} = u_D\,, \quad A_{|\partial\Omega} = A_D\,, \quad \varepsilon_p(0) = \varepsilon_p^0\,.
\end{aligned}$$

Thus, to end the existence theory for the infinitesimal elasto-plastic Cosserat model we should prove only that the limit functions satisfy the differential inclusion (4.17-4). The sequence $T_E^\eta = 2\mu(\varepsilon^\eta - \varepsilon_p^\eta)$ converges weakly-$*$ to $T_E = 2\mu\,(\varepsilon - \varepsilon_p)$ and the sequence $\int_0^t f_\eta(T_E^\eta)d\tau = \varepsilon_p^\eta - \varepsilon_p^0$ converges also weakly-$*$ to $\varepsilon_p - \varepsilon_p^0$. To conclude that the limit functions ε_p and T_E satisfy the differential inclusion we need estimates for the sequence $\{f_\eta(T_E^\eta)\}$. Hence, the next step in our existence theory is an estimate for time derivatives of the approximate sequence.

Theorem 4.3 (Energy estimate for time derivatives)
Suppose that the given data possess more time regularity as in the last theorem and satisfy additionally: for all times $T > 0$

$$\begin{gathered}
\ddot{f} \in L^2((0,T)\times\Omega,\mathbb{R}^3)\,,\ \ddot{u}_D \in L^2((0,T)\times\partial\Omega,\mathbb{R}^3)\,, \\
\ddot{A}_D \in L^2((0,T)\times\partial\Omega,\mathfrak{so}(3,\mathbb{R}))\,.
\end{gathered} \tag{4.28}$$

Moreover, assume that the initial data $\varepsilon_p^0 \in L^2(\Omega, \mathrm{Sym}(3))$ is chosen such that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu\,(\varepsilon(0) - \varepsilon_p^0)$ defined by system (4.24) *belongs to the domain of the maximal monotone operator f. Then there exists a positive constant $C(T)$ independent of the parameter η such that*

$$\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \le C(T) \quad \text{for all } \ t \in [0,T)\,.$$

Proof. For $h > 0$ let us denote by $(u_h^\eta(t), \varepsilon_h^\eta(t), \varepsilon_{p,h}^\eta(t), A_h^\eta(t))$ the shifted functions $(u^\eta(t+h), \varepsilon^\eta(t+h), \varepsilon_p^\eta(t+h), A^\eta(t+h))$ and calculate the energy evaluated on the differences $(u_h^\eta - u^\eta, \ldots)$. Then for the time derivative we have

$$\begin{aligned}
\dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) &= \int_\Omega 2\mu\, \langle \varepsilon_h^\eta - \varepsilon^\eta - \varepsilon_{p,h}^\eta + \varepsilon_p^\eta, \dot{\varepsilon}_h^\eta - \dot{\varepsilon}^\eta - \dot{\varepsilon}_{p,h}^\eta + \dot{\varepsilon}_p^\eta\rangle dx \\
&+2\mu_c \int_\Omega \langle \operatorname{skew}(\nabla u_h^\eta - \nabla u^\eta) - A_h^\eta + A^\eta, \operatorname{skew}(\nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta) - \dot{A}_h^\eta + \dot{A}^\eta\rangle dx \\
&+\lambda \int_\Omega \operatorname{tr}[\varepsilon_h^\eta - \varepsilon^\eta]\operatorname{tr}[\dot{\varepsilon}_h^\eta - \dot{\varepsilon}^\eta]\, dx + 4c\int_\Omega \langle \nabla \operatorname{axl}(A_h^\eta - A^\eta), \nabla \operatorname{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\rangle dx \qquad (4.29) \\
&= -\int_\Omega \langle T_{E,h}^\eta - T_E^\eta, \dot{\varepsilon}_{p,h}^\eta - \dot{\varepsilon}_p^\eta\rangle dx + \int_\Omega \langle \sigma_h^\eta - \sigma^\eta, \nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta\rangle dx \\
&+4\mu_c \int_\Omega \langle \operatorname{axl}\operatorname{skew}(\nabla u_h^\eta - \nabla u^\eta) - \operatorname{axl}(A_h^\eta - A^\eta), \operatorname{axl}\operatorname{skew}(\nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta) - \operatorname{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\rangle dx \\
&+4c\int_\Omega \langle \nabla \operatorname{axl}(A_h^\eta - A^\eta), \nabla \operatorname{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\rangle dx
\end{aligned}$$

where $T_{E,h}^\eta(t) = T_E^\eta(t+h)$ and $\sigma_h^\eta(t) = \sigma^\eta(t+h)$. By the monotonicity of the Yosida approximation the first term on the right hand side of (4.29) is non positive. Similar to the energy estimate in Theorem 4.2 we integrate partially in the second and in the fourth integral and use the equation for microrotations. Hence, we arrive at the inequality

$$\begin{aligned}
\dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) &\le \int_\Omega \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta\rangle dx \\
+\int_{\partial\Omega} \langle (\sigma_h^\eta - \sigma^\eta).n, \dot{u}_{D,h} - \dot{u}_D\rangle ds &+ 4c\int_{\partial\Omega} \langle \nabla \operatorname{axl}(A_h^\eta - A^\eta).n, \operatorname{axl}(\dot{A}_{D,h} - \dot{A}_D\rangle ds \quad (4.30)
\end{aligned}$$

where $f_h(t) = f(t+h)$, $u_{D,h}(t) = u_D(t+h)$ and $A_{D,h}(t) = A_D(t+h)$. Next, we integrate (4.30) in time and obtain

$$\begin{aligned}
\mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) &\le \mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(0) \\
+\int_0^t \int_\Omega \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta\rangle dx\, d\tau &+ \int_0^t \int_{\partial\Omega} \langle (\sigma_h^\eta - \sigma^\eta).n, \dot{u}_{D,h} - \dot{u}_D\rangle ds\, d\tau \qquad (4.31) \\
&+4c\int_0^t \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A_h^\eta - A^\eta).n, \operatorname{axl}(\dot{A}_{D,h} - \dot{A}_D\rangle ds\, d\tau\,.
\end{aligned}$$

Before we divide (4.31) by h^2 we should shift in the integral terms the shift operator onto given data. We calculate this with details for the first integral only.

$$\int_0^t \int_\Omega \langle f_h - f, \dot u^\eta_h - \dot u^\eta \rangle dx\, d\tau = \int_\Omega \int_0^t \langle f(\tau+h) - f(\tau), \dot u^\eta(\tau+h)\rangle d\tau\, dx$$
$$- \int_\Omega \int_0^t \langle f(\tau+h) - f(\tau), \dot u^\eta(\tau)\rangle d\tau\, dx = (\tau + h = s \text{ in the first integral}) =$$
$$= \int_\Omega \int_h^{t+h} \langle f(s) - f(s-h), \dot u^\eta(s)\rangle ds\, dx - \int_\Omega \int_0^t \langle f(s+h) - f(s), \dot u^\eta(s)\rangle ds\, dx$$
$$= -\int_\Omega \int_h^{t+h} \langle f(s+h) - 2f(s) + f(s-h), \dot u^\eta(s)\rangle ds\, dx - \int_\Omega \int_0^h \langle f(s+h) - f(s), \dot u^\eta(s)\rangle dx\, ds$$
$$+ \int_\Omega \int_t^{t+h} \langle f(s+h) - f(s), \dot u^\eta(s)\rangle ds\, dx\,. \tag{4.32}$$

In the same manner we transform the second and the third integral term from (4.31). Next, we insert (4.32) and the results for other terms into (4.31), divide by h^2 and pass to the limit $h \to 0^+$. Hence, we conclude with the following inequality

$$\mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(t) \le \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(0) - \int_0^t \int_\Omega \langle \ddot f, \dot u^\eta\rangle dx\, d\tau$$
$$- \int_\Omega \langle \dot f(0), \dot u^\eta(0)\rangle dx + \int_\Omega \langle \dot f(t), \dot u^\eta(t)\rangle dx \tag{4.33}$$
$$- \int_0^t \int_{\partial\Omega} \langle \sigma^\eta.n, \ddot u_D\rangle ds\, d\tau - \int_{\partial\Omega} \langle \sigma^\eta(0).n, \dot u_D(0)\rangle ds + \int_{\partial\Omega} \langle \sigma^\eta(t).n, \dot u_D(t)\rangle ds$$
$$-4c \int_0^t \int_{\partial\Omega} \langle \nabla\, \mathrm{axl}(A^\eta).n, \mathrm{axl}(\ddot A_D)\rangle ds\, d\tau - 4c \int_{\partial\Omega} \langle \nabla\, \mathrm{axl}(A^\eta)(0).n, \mathrm{axl}(\dot A_D)(0)\rangle ds$$
$$+4c \int_{\partial\Omega} \langle \nabla\, \mathrm{axl}(A^\eta)(t).n, \mathrm{axl}(\dot A_D)(t)\rangle ds\,.$$

To obtain the initial energy for time derivatives we observe that $\dot\varepsilon^\eta_p(0) = f_\eta(T^\eta_E(0)) = f_\eta(T_E(0))$. By assumption $T_E(0) \in \mathcal{D}(f)$ we have that the sequence $\{f_\eta(T_E(0))\}$ is bounded in $L^2(\Omega, \mathrm{Sym}(3))$. The other initial values $\dot u^\eta(0), \dot\varepsilon^\eta(0)$ and $\dot A^\eta(0)$ are solutions of (4.24) with $\dot\varepsilon^\eta_p(0)$ instead of ε^0_p. Consequently, the initial energy for time derivatives is bounded. The integral term on the right hand side of (4.33) can be estimated in the same manner as in the proof of Theorem 4.2. Thus we arrive at the following inequality

$$\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \le C_1 \|\dot{f}(t)\|_{L^2}\, \mathcal{E}^{\frac{1}{2}}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) + C_2 \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau)\, d\tau + C_3(t)$$

where $C_1, C_2, C_3(t)$ do not depend on η and $C_3(t)$ depends on the given data only. Similar as in the proof of Theorem 4.2 this concludes the statement. ■

The energy estimate for time derivatives yields that the sequence $\{f_\eta(T_E^\eta)\}$ is bounded in $L^\infty(0,T), L^2(\Omega, \mathrm{Sym}(3)))$. Hence, we can select a subsequence (denoted again with the superscript η) with $f_\eta(T_E^\eta) \overset{*}{\rightharpoonup} f^*$ in $L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))$. This shows that the limit function $T_E = 2\mu\,(\varepsilon - \varepsilon_p)$ belongs to $\mathcal{D}(f)$. To end our existence theory we need only to prove that

$$f^*(t,x) \in f(T_E(t,x)) \quad \text{a.e. in } \ (0,T)\times\Omega\,. \tag{4.34}$$

From the definition of a maximal monotone operator it is easy to see that its graph is weakly-strongly closed. Thus we have to improve the weak convergence of the sequence $\{T_E^\eta\}$.

Theorem 4.4 (Strong convergence of the stresses)
Let us assume that the given data satisfy all requirements of Theorem 4.3.3. Then $\mathcal{E}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) \to 0$ *for* $\eta, \nu \to 0^+$ *uniformly on bounded time intervals.*

Proof. Calculating the time derivative of the energy evaluated on the differences of two approximation steps we obtain

$$\begin{aligned}\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) &= 2\mu \int_\Omega \langle \varepsilon^\eta - \varepsilon^\nu - \varepsilon_p^\eta + \varepsilon_p^\nu, \dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu - \dot{\varepsilon}_p^\eta + \dot{\varepsilon}_p^\nu\rangle dx \\ &+\lambda \int_\Omega \mathrm{tr}\,[\varepsilon^\eta - \varepsilon^\nu]\mathrm{tr}\,[\dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu] dx + 4c \int_\Omega \langle \nabla\,\mathrm{axl}(A^\eta - A^\nu), \nabla\,\mathrm{axl}(\dot{A}^\eta - \dot{A}^\nu)\rangle dx \\ &+2\mu_c \int_\Omega \langle \mathrm{skew}(\nabla u^\eta - \nabla u^\nu) - A^\eta + A^\nu, \mathrm{skew}(\nabla \dot{u}^\eta - \nabla \dot{u}^\nu) - \dot{A}^\eta + \dot{A}^\nu\rangle dx\,.\end{aligned}$$

Using that the given data for both approximation steps are the same we conclude that

$$\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) = -\int_\Omega \langle T_E^\eta - T_E^\nu, f_\eta(T_E^\eta) - f_\nu(T_E^\nu)\rangle dx\,. \tag{4.35}$$

Next, we estimate the right hand side of (4.35). This estimation is a standard one in the theory of maximal monotone operators (compare with the proof of Theorem 1 p. 147 in [AC84]). Nevertheless, for completeness of the proof we insert it here. By definition of the Yosida approximation we have

$$f_i(T_E^i) \in f(J_i(T_E^i)) \quad \text{where } \ J_i(T_E^i) = T_E^i - i f_i(T_E^i) \ \text{ and } i = \eta, \nu \tag{4.36}$$

is the resolvent of the operator f. Hence, be (4.36) we have

$$-\int_\Omega \langle T_E^\eta - T_E^\nu, f_\eta(T_E^\eta) - f_\nu(T_E^\nu)\rangle dx = -\int_\Omega \langle J_\eta(T_E^\eta) - J_\nu(T_E^\nu), f_\eta(T_E^\eta) - f_\nu(T_E^\nu)\rangle dx$$
$$-\int_\Omega \langle \eta f_\eta(T_E^\eta) - \nu f_\nu(T_E^\nu), f_\eta(T_E^\eta) - f_\nu(T_E^\nu)\rangle dx$$
$$\leq \frac{\eta}{4}\|f_\nu(T_E^\nu)\|_{L^2}^2 + \frac{\nu}{4}\|f_\eta(T_E^\eta)\|_{L^2}^2 = \frac{\eta}{4}\|\dot\varepsilon_p^\nu\|_{L^2}^2 + \frac{\nu}{4}\|\dot\varepsilon_p^\eta\|_{L^2}^2 .$$

Inserting the last result into (4.35) and integrating in time we finally obtain

$$\mathcal{E}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) \leq \frac{t}{4}(\eta + \nu)C(T) \ \text{ for all } t \in [0, T)\,,$$

where the constant $C(T)$ is from the statement of Theorem 4.3. The last inequality immediately completes the proof. ■

Using (4.36) and the fact that the resolvent J_η is a global Lipschitz operator with the Lipschitz constant less or equal to 1, we see that the sequence $\{J_\eta(T_E^\eta)\}$ converges strongly to the function T_E (note that the sequence $\{f_\eta(T_E^\eta)\}$ is bounded). Thus, the weak limit $f_\eta(T_E^\eta) \overset{*}{\rightharpoonup} f^*$ belongs to the set $f(T_E)$ and the limit functions $(u, \varepsilon, \varepsilon_p, A)$ satisfy (4.17).

Theorem 4.5 (Uniqueness of solutions)
Let us assume that the given data $f, u_D, A_D, \varepsilon_p^0$ satisfy all requirements of Theorem 4.3 Then system (4.17) possesses a unique, global in time solution $(u, \varepsilon, \varepsilon_p, A)$.

Proof. Assume that $(u^1, \varepsilon^1, \varepsilon_p^1, A^1)$ and $(u^2, \varepsilon^2, \varepsilon_p^2, A^2)$ are two solutions of (4.17) for the same given data. Then for the energy function evaluated on differences of these solutions we have

$$\dot{\mathcal{E}}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_p^1 - \varepsilon_p^2, A^1 - A^2)(t) = 2\mu\int_\Omega \langle \varepsilon^1 - \varepsilon^2 - \varepsilon_p^1 + \varepsilon_p^2, \dot\varepsilon^1 - \dot\varepsilon^2 - \dot\varepsilon_p^1 + \dot\varepsilon_p^2\rangle dx$$
$$+\lambda\int_\Omega \mathrm{tr}\,[\varepsilon^\eta - \varepsilon^\nu]\mathrm{tr}\,[\dot\varepsilon^\eta - \dot\varepsilon^\nu]dx + 4c\int_\Omega \langle \nabla\,\mathrm{axl}(A^1 - A^2), \nabla\,\mathrm{axl}(\dot A^1 - \dot A^2)\rangle dx$$
$$+2\mu_c\int_\Omega \langle \mathrm{skew}(\nabla u^1 - \nabla u^2) - A^1 + A^2, \mathrm{skew}(\nabla \dot u^1 - \nabla \dot u^2) - \dot A^1 + \dot A^2\rangle dx$$
$$= -\int_\Omega \langle T_E^1 - T_E^2, \dot\varepsilon_p^1 - \dot\varepsilon_p^2\rangle dx \leq 0\,.$$

This yields immediately, that

$$\mathcal{E}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_p^1 - \varepsilon_p^2, A^1 - A^2)(t) \leq \mathcal{E}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_p^1 - \varepsilon_p^2, A^1 - A^2)(0) = 0$$

and the statement follows from coerciveness of the energy function. ■

Finally we formulate the existence theorem, which we have proved:

Theorem 4.6 (Existence for the infinitesimal elasto-plastic Cosserat model) *Suppose that the given data f, u_D, A_D satisfy: for all times $T > 0$*

$$
\begin{aligned}
& f \in C^1([0,T], L^2(\Omega, \mathbb{R}^3))\,, && \dddot{f} \in L^2((0,T) \times \Omega, \mathbb{R}^3)\,, \\
& u_D \in C^1([0,T], H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3))\,, && \ddot{u}_D \in L^2((0,T) \times \partial\Omega, \mathbb{R}^3)\,, \\
& A_D \in C^1([0,T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3,\mathbb{R})))\,, && \ddot{A}_D \in L^2((0,T) \times \partial\Omega, \mathfrak{so}(3,\mathbb{R}))\,.
\end{aligned}
$$

Moreover, assume that the initial data $\varepsilon_p^0 \in L^2(\Omega, \mathrm{Sym}(3))$ is chosen such that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu\,(\varepsilon(0) - \varepsilon_p^0)$ defined by system (4.24) belongs to the domain of the maximal monotone operator f. Then system (4.17) possesses a global in time, unique solution $(u, \varepsilon, \varepsilon_p, A)$ with the regularity: for all times $T > 0$

$$
\begin{aligned}
& u \in H^{1,\infty}((0,T), H^1(\Omega, \mathbb{R}^3))\,, && \varepsilon, \varepsilon_p \in H^{1,\infty}((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
& A \in H^{1,\infty}((0,T), H^2(\Omega, \mathfrak{so}(3,\mathbb{R})))\,.
\end{aligned}
$$

Remark In the analysis part we did not assume that the values of the constitutive multifunction f are trace free. This means that the existence theory developed so far works as well without this requirement. For constitutive multifunctions possessing trace free values, assuming additionally that the initial inelastic strain ε_p^0 is also trace free, we conclude that $\mathrm{tr}\,[\varepsilon_p](t) = 0$ during the whole deformation process.

Note that for the model to be well-posed in the rate-independent case, we did not need a so called **safe load** condition, otherwise unavoidable.

5 Discussion and concluding remarks

The infinitesimal Cosserat model has been extended to elasto-plasticity where Cosserat effects remain, in contrast to standard approaches, non-dissipative. As only difference to classical rate-independent infinitesimal plasticity we have introduced an additional infinitesimal microrotation $\overline{A}_e$, influencing only the elastic behaviour of the model. This minor change is shown to completely regularize the pathological behaviour of rate-independent classical plasticity theory. Decisive in our analysis is the observation that the microrotations provide an independent control of $\mathrm{curl}\, u$, otherwise not present in the theory. This extra resistance against elastic shear is also a welcome feature from a modelling and numerical point of view.

Since this modification of classical rate-independent plasticity is not operative in uniaxial tension/compression we may arguably say that the provided regularization is optimal. Numerical calculations based on this modification are "cheap", in the sense that the resulting system remains of second order.

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