Congruence Relations on Double Boolean Algebras

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Abstract. Double Boolean algebras form the variety generated by protoconcept algebras which are one of the fundamental structures of Contextual Logic. Every double Boolean algebra contains two Boolean algebras. In this paper it is shown that congruence relations on pure double Boolean algebras may be characterized by pairs consisting of an ideal in one Boolean algebra and a filter in the other. We explain how this characterization can be generalized for double Boolean algebras. Moreover, these results are applied to protoconcept algebras in order to obtain a direct decomposition in simple protoconcept algebras. Finally, it is shown that every finite subdirectly irreducible double Boolean algebra is simple.

The definition and investigation of double Boolean algebras arose from the development of Contextual Logic at TU Darmstadt during the last years. Contextual Logic is intended to be a mathematization of the traditional philosophical logic with its doctrines of concepts, judgments and conclusions. A survey of the basic ideas and results of this approach can be found in [Wi00b], for more detailed information see [Pr98], [GW99a], [Wi00a], [Kl01] and [Da02]. This paper focuses on congruence relations on double Boolean algebras. In the first section basic definitions and properties of double Boolean algebras, semiconcept algebras and protoconcept algebras are given. The second section deals with the congruence relations. It is divided into five parts. First, operational equivalence as the most basic non trivial congruence is introduced. The second part focuses on pure double Boolean algebras. It is shown how they can be characterized by pairs consisting of an ideal and a filter in the two Boolean algebras contained in the double Boolean algebra. Moreover, we explain how the necessary conditions can be reformulated for semiconcept algebras and for their contexts. As every double Boolean algebra contains a pure subalgebra, in the third part this characterization can be applied to double Boolean algebras in general. It is shown that a congruence relation on a double Boolean algebra is determined to a large extent by its restriction to the pure subalgebra. This yields a decomposition of finite protoconcept algebras in simple protoconcept algebras which is given in part four. In the fifth part it is shown that every finite subdirectly irreducible double Boolean algebra is simple. Finally, section three describes some perspectives for further research.

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1. Double Boolean Algebras and Protoconcept Algebras

Double Boolean algebras were introduced in [Wi00a].

Definition 1.1. *A* double Boolean algebra *is an algebra* $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ *of type (2,2,1,1,0,0), satisfying the equations*

1a) $(x \sqcap x) \sqcap y = x \sqcap y$	*1b)* $(x \sqcup x) \sqcup y = x \sqcup y$
2a) $x \sqcap y = y \sqcap x$	*2b)* $x \sqcup y = y \sqcup x$
3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	*3b)* $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
4a) $x \sqcap (x \sqcup y) = x \sqcap x$	*4b)* $x \sqcup (x \sqcap y) = x \sqcup x$
5a) $x \sqcap (x \dot\sqcup y) = x \sqcap x$	*5b)* $x \sqcup (x \dot\sqcap y) = x \sqcup x$
6a) $x \sqcap (y \dot\sqcup z) = (x \sqcap y) \dot\sqcup (x \sqcap z)$	*6b)* $x \sqcup (y \dot\sqcap z) = (x \sqcup y) \dot\sqcap (x \sqcup z)$
7a) $\neg\neg(x \sqcap y) = x \sqcap y$	*7b)* $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
8a) $\neg(x \sqcap x) = \neg x$	*8b)* $\lrcorner(x \sqcup x) = \lrcorner x$
9a) $x \sqcap \neg x = \bot$	*9b)* $x \sqcup \lrcorner x = \top$
10a) $\neg\bot = \top \sqcap \top$	*10b)* $\lrcorner\top = \bot \sqcup \bot$
11a) $\neg\top = \bot$	*11b)* $\lrcorner\bot = \top$

12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$

with the operations $\dot\sqcup, \dot\sqcap, \overline{\top}, \underline{\bot}$ *defined by*

$$\begin{aligned} x \dot\sqcup y &:= \neg(\neg x \sqcap \neg y) \\ x \dot\sqcap y &:= \lrcorner(\lrcorner x \sqcup \lrcorner y) \\ \overline{\top} &:= \neg\bot \\ \underline{\bot} &:= \lrcorner\top \end{aligned}$$

A pure double Boolean algebra *is a double Boolean algebra that satisfies the additional condition*

$$\textit{13)}\ x = x \sqcap x \text{ or } x = x \sqcup x.$$

To shorten notation we write $x_\sqcap$ for $x \sqcap x$ and $x_\sqcup$ for $x \sqcup x$, and define $D_\sqcap := \{x_\sqcap \mid x \in D\}$, $D_\sqcup := \{x_\sqcup \mid x \in D\}$ and $D_p := D_\sqcap \cup D_\sqcup$. The restriction of $\underline{D}$ to D_p is a pure subalgebra of $\underline{D}$.
We define a binary relation $\sqsubseteq$ on double Boolean algebras by:

$$x \sqsubseteq y :\Leftrightarrow x \sqcap y = x_\sqcap \text{ and } x \sqcup y = y_\sqcup$$

Some properties of double Boolean algebras were discussed in [HLSW00]:

Theorem 1.2. *Let* $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ *be a double Boolean algebra. Then the following conditions are satisfied:*

(1) $(D, \sqsubseteq)$ *is a quasi-ordered set.*
(2) $\underline{D_\sqcap} := (D_\sqcap, \sqcap, \dot\sqcup, \neg, \bot, \overline{\top})$ *is a Boolean algebra whose order relation is the restriction of* $\sqsubseteq$ *to* $D_\sqcap$.
(3) $\underline{D_\sqcup} := (D_\sqcup, \dot\sqcap, \sqcup, \lrcorner, \underline{\bot}, \top)$ *is a Boolean algebra whose order relation is the restriction of* $\sqsubseteq$ *to* $D_\sqcup$.
(4) $y \sqsubseteq x_\sqcap \Leftrightarrow y \sqsubseteq x$ *for* $x \in D$ *and* $y \in D_\sqcap$.
(5) $x_\sqcup \sqsubseteq y \Leftrightarrow x \sqsubseteq y$ *for* $x \in D$ *and* $y \in D_\sqcup$.
(6) $x \sqsubseteq y \Leftrightarrow x_\sqcap \sqsubseteq y_\sqcap$ *and* $x_\sqcup \sqsubseteq y_\sqcup$ *for* $x, y \in D$.

A class of examples for double Boolean algebras are so-called protoconcept algebras. Introduced in [Wi00a] as an extension of the theory of Formal Concept Analysis, they serve as the starting point for the inclusion of negation to contextual logic. For a detailed introduction to Formal Concept Analysis see [GW99b].

Definition 1.3. *A* formal context $\mathbb{K} := (G, M, I)$ *consists of two sets* G *and* M *and a relation* I *between* G *and* M*. The elements of* G *are called the* objects *and the elements of* M *are called the* attributes *of the context. In order to express that an object* g *is in relation* I *with an attribute* m*, we write* gIm *or* $(g, m) \in I$ *and read it as "the object* g has *the attribute* m*".*

Definition 1.4. *For a set* $A \subseteq G$ *of objects we define*

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\}$$

(the set of attributes common to the objects in A*). Correspondingly, for a set* B *of attributes we define*

$$B' := \{g \in G \mid gIm \text{ for all } m \in B\}$$

(the set of objects which have all attributes in B*).*

Definition 1.5. *A* protoconcept *of a formal context* $\mathbb{K} := (G, M, I)$ *is a pair* (A, B) *with* $A \subseteq G$ *and* $B \subseteq M$ *such that* $A' = B''$ *or, equivalently,* $A'' = B'$*. We denote the set of all protoconcepts of a context* $\mathbb{K}$ *by* $\mathfrak{P}(\mathbb{K})$ *and define on* $\mathfrak{P}(\mathbb{K})$ *operations* $\sqcap$, $\sqcup$, $\lrcorner$, $\neg$, $\top$ *and* $\bot$ *by:*

$$\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)') \\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)', B_1 \cap B_2) \\
\neg(A, B) &:= (G \setminus A, (G \setminus A)') \\
\lrcorner(A, B) &:= ((M \setminus B)', M \setminus B) \\
\top &:= (G, \emptyset) \\
\bot &:= (\emptyset, M)
\end{aligned}$$

The set of all protoconcepts of a context $\mathbb{K}$ *together with these operations is called the* protoconcept algebra *of* $\mathbb{K}$ *and denoted by* $\underline{\mathfrak{P}}(\mathbb{K})$*.*

For protoconcepts the quasi-order $\sqsubseteq$ is an order and

$$\begin{aligned}
(A_1, B_1) \sqsubseteq (A_2, B_2) \quad &:\Leftrightarrow \quad A_1 \subseteq A_2 \text{ and } B_1 \supseteq B_2 \\
&\Leftrightarrow \quad (A_1, B_1) \sqcap (A_2, B_2) = (A_1, B_1) \sqcap (A_1, B_1) \\
&\qquad \text{and } (A_1, B_1) \sqcup (A_2, B_2) = (A_2, B_2) \sqcup (A_2, B_2).
\end{aligned}$$

Note that the result of any operation in a protoconcept algebra is a protoconcept of the form (A, A') or (B', B). These protoconcepts are called $\sqcap$-*semiconcepts* or $\sqcup$-*semiconcepts*, respectively. The set of all $\sqcap$-semiconcepts of a protoconcept algebra $\underline{\mathfrak{P}}(\mathbb{K}) (= \underline{D})$ is denoted by $\mathfrak{P}(\mathbb{K})_\sqcap$ $(= D_\sqcap)$ and the set of all $\sqcup$-semiconcepts by $\mathfrak{P}(\mathbb{K})_\sqcup$ $(= D_\sqcup)$. As before, the set $\mathfrak{H}(\mathbb{K}) := \mathfrak{P}(\mathbb{K})_\sqcap \cup \mathfrak{P}(\mathbb{K})_\sqcup$ of all semiconcepts

	a	b	c
1	×		×
2		×	×
3	×	×	×

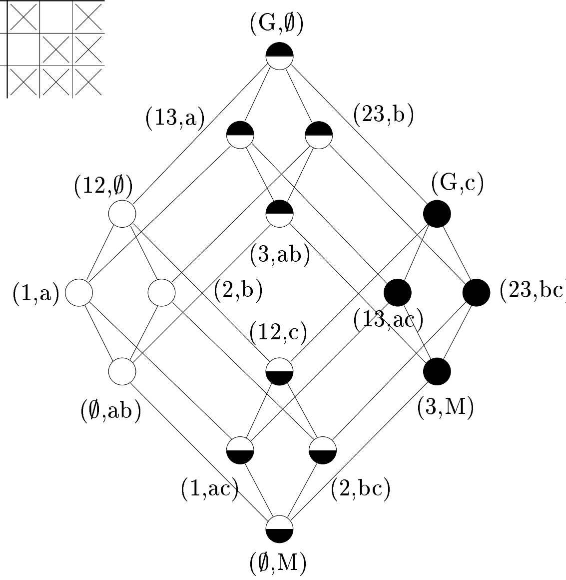


FIGURE 1. A context and its protoconcept algebra

of $\mathbb{K}$, together with the operations of $\underline{\mathfrak{P}}(\mathbb{K})$ is a subalgebra of $\underline{\mathfrak{P}}(\mathbb{K})$. We call this subalgebra the *semiconcept algebra* of the context $\mathbb{K}$.
An element of the intersection $\mathfrak{P}(\mathbb{K})_\sqcap \cap \mathfrak{P}(\mathbb{K})_\sqcup$ is called a *formal concept*. For a formal concept (A, B) holds $A' = B$ and $B' = A$.

Example 1.6. *Figure 1 depicts a context and its protoconcept algebra. The elements represented by filled circles are formal concepts. The circles with the upper half filled represent $\sqcup$-semiconcepts, those with the lower half filled represent $\sqcap$-semiconcepts.*

2. Congruence Relations on Double Boolean Algebras

2.1. Operational Equivalence. The most basic non-trivial congruence relation is operational equivalence:

Definition 2.1. *We say that two elements x, y of a double Boolean algebra $\underline{D}$ are* operationally equivalent *(or an* equivalent pair*) if and only if $x \sqsubseteq y$ and $y \sqsubseteq x$ and denote this by $x \mathrel{\underline{\sqcup}} y$.*

Lemma 2.2. *Let $\underline{D}$ be a double Boolean algebra. For elements $x, y \in \underline{D}$ holds:*

$$x \mathrel{\underline{\sqcup}} y \Leftrightarrow x_\sqcap = y_\sqcap \text{ and } x_\sqcup = y_\sqcup$$

Proof: This follows immediately from the definition of $\sqsubseteq$:

$$x \sqsubseteq y \text{ and } y \sqsubseteq x \Leftrightarrow x_\sqcap = x \sqcap y = y \sqcap x = y_\sqcap \text{ and } x_\sqcup = x \sqcup y = y \sqcup x = y_\sqcup$$

Definition 2.3. *A double Boolean algebra $\underline{D}$ satisfying for all $x, y \in \underline{D}$:*

$$x \sqsupseteq\!\!\!\sqsubseteq y (:\Leftrightarrow x \sqsubseteq y \text{ and } y \sqsubseteq x) \Rightarrow x = y$$

is called a contextual *double Boolean algebra.*

Theorem 2.4. *Let $\underline{D}$ be a double Boolean algebra, let $x, y \in \underline{D}$ be an equivalent pair and let $\Delta_{\underline{D}} := \{(x,x) \mid x \in D\}$. The relation $\Theta := \Delta_{\underline{D}} \cup \{(x,y),(y,x)\}$ is a congruence relation on $\underline{D}$.*

Proof: Obviously Θ is an equivalence relation. We have to show that $\neg x \,\Theta\, \neg y$, $\lrcorner x \,\Theta\, \lrcorner y$ and $a \sqcap x \,\Theta\, a \sqcap y$, $a \sqcup x \,\Theta\, a \sqcup y$ for arbitrary $a \in \underline{D}$. Lemma 2.2 yields $x_\sqcap = y_\sqcap$ and thus

$$\neg x \overset{8a}{=} \neg(x \sqcap x) = \neg(y \sqcap y) \overset{8a}{=} \neg y,$$

so $\neg x \,\Theta\, \neg y$, and dually $\lrcorner x \,\Theta\, \lrcorner y$. Likewise,

$$x \sqcap a \overset{1a}{=} (x \sqcap x) \sqcap a = (y \sqcap y) \sqcap a \overset{1a}{=} y \sqcap a$$

so $x \sqcap a \,\Theta\, y \sqcap a$ and dually $x \sqcup a \,\Theta\, y \sqcup a$. □

Corollary 2.5. *Operational equivalence $\sqsupseteq\!\!\!\sqsubseteq := \{(x,y) \in \underline{D} \times \underline{D} \mid x \sqsupseteq\!\!\!\sqsubseteq y\}$ is a congruence relation in every double Boolean algebra.*

Proof: It is easy to see that $\sqsupseteq\!\!\!\sqsubseteq$ is a reflexive and symmetric relation. In order to check the transitivity of $\sqsupseteq\!\!\!\sqsubseteq$ let $x \sqsupseteq\!\!\!\sqsubseteq y$ and $y \sqsupseteq\!\!\!\sqsubseteq z$. Lemma 2.2 yields $x_\sqcap = y_\sqcap = z_\sqcap$ and $x_\sqcup = y_\sqcup = z_\sqcup$, thus $x \sqsupseteq\!\!\!\sqsubseteq z$. The proof of Theorem 2.4 shows that from $x \sqsupseteq\!\!\!\sqsubseteq y$ follows $\neg x \,\Theta\, \neg y$, $\lrcorner x \,\Theta\, \lrcorner y$, $x \sqcap a \,\Theta\, y \sqcap a$ and $x \sqcup a \,\Theta\, y \sqcup a$. Therefore $\sqsupseteq\!\!\!\sqsubseteq$ is a congruence relation. □

2.2. Congruence Relations on pure Double Boolean Algebras. In this subsection we focus our investigation on pure double Boolean algebras because their factor algebras do not contain equivalent pairs. This restriction allows us to give a characterization of congruences on pure double Boolean algebras. This characterization will be extended to double Boolean algebras in general in the next subsection.

Lemma 2.6. *Let $\underline{D}$ be a pure double Boolean algebra and let Θ be a congruence relation on $\underline{D}$. Then the factor algebra $\underline{D}/\Theta$ is contextual.*

Proof: Let $x, y \in \underline{D}$, let $x_\sqcap \,\Theta\, y_\sqcap$ and let $x_\sqcup \,\Theta\, y_\sqcup$. In the cases $x, y \in \underline{D}_\sqcap$ and $x, y \in \underline{D}_\sqcup$ the result is immediate. Assume $x = x_\sqcap$ and $y = y_\sqcup$. We conclude from $x_\sqcup \,\Theta\, y_\sqcup = y$ that $x_\sqcup \sqcap x_\sqcup \,\Theta\, y_\sqcap$ and hence $x_\sqcup \,\Theta\, y_\sqcap$. This implies $x = x_\sqcap \,\Theta\, y_\sqcap \,\Theta\, x_\sqcup \,\Theta\, y_\sqcup = y$, hence the congruence classes of x and y in $\underline{D}/\Theta$ are equal. □

Note that if a double Boolean algebra $\underline{D}$ is not pure there always exists a proper congruence relation Θ such that $\underline{D}/\Theta$ consists only of equivalent pairs. To see this we set $\Theta := \Delta_{\underline{D}} \cup (D_p \times D_p)$. Then $x \,\Theta\, y$ holds for all $x, y \in D_p$. We conclude from $x_\sqcup$, $y_\sqcup$, $x_\sqcap$, $y_\sqcap \in D_p$ for arbitrary $x, y \in \underline{D}$:

$$x_\sqcup \,\Theta\, y_\sqcup \,\Theta\, x_\sqcap \,\Theta\, y_\sqcap.$$

So the congruence classes $[x]$ and $[y]$ of x and y form an equivalent pair in $\underline{D}/\Theta$. Moreover, $\underline{D}/\Theta$ contains at least two different congruence classes: For all elements $x, y \in D \setminus D_p$ and $z \in D_p$ holds $[x] \neq [y]$ and $[x] \neq [z]$. By assumption there exists at least one $x \in D \setminus D_p$. It follows that $\underline{D}/\Theta$ consists only of equivalent pairs.

Definition 2.7. *For double Boolean algebras $\underline{D}$ we define two functions* **t**: $\underline{D} \times \underline{D} \to \underline{D}$ *and* **b**: $\underline{D} \times \underline{D} \to \underline{D}$ *by:*

$$\mathrm{t}(x,y) := (x \sqcup \neg\!\!\lrcorner\, y) \bar{\sqcap} (\neg\!\!\lrcorner\, x \sqcup y),$$
$$\mathrm{b}(x,y) := (x \sqcap \neg y) \bar{\sqcup} (\neg x \sqcap y).$$

Lemma 2.8. *For a congruence relation Θ on a double Boolean algebra $\underline{D}$ and elements $x, y \in \underline{D}$ holds:*

$$x \,\Theta\, y \Rightarrow \mathrm{t}(x,y) \in [\top] \text{ and } \mathrm{b}(x,y) \in [\bot].$$

Proof: Let $x, y \in \underline{D}$ and $x \,\Theta\, y$. Then $\top \overset{9b}{=} x \sqcup \neg\!\!\lrcorner\, x \,\Theta\, x \sqcup \neg\!\!\lrcorner\, y$ and $\top \,\Theta\, y \sqcup \neg\!\!\lrcorner\, x$. Thus $\top = \top \bar{\sqcap} \top \,\Theta\, (x \sqcup \neg\!\!\lrcorner\, y) \bar{\sqcap} (y \sqcup \neg\!\!\lrcorner\, x) = \mathrm{t}(x,y)$. Duality gives $\mathrm{b}(x,y) \in [\bot]$. □

Theorem 2.9. *In a pure double Boolean algebra $\underline{D}$ holds:*

$$\mathrm{t}(x,y) \in [\top] \text{ and } \mathrm{b}(x,y) \in [\bot] \Leftrightarrow x \,\Theta\, y.$$

Proof: We conclude from $\mathrm{t}(x,y) = (x \sqcup \neg\!\!\lrcorner\, y) \bar{\sqcap} (y \sqcup \neg\!\!\lrcorner\, x) \,\Theta\, \top$ that

$$(x \sqcup \neg\!\!\lrcorner\, y) = (x \sqcup \neg\!\!\lrcorner\, y)_\sqcup \overset{5b}{=} (x \sqcup \neg\!\!\lrcorner\, y) \sqcup ((x \sqcup \neg\!\!\lrcorner\, y) \bar{\sqcap} (y \sqcup \neg\!\!\lrcorner\, x)) \,\Theta\, (x \sqcup \neg\!\!\lrcorner\, y) \sqcup \top = \top$$

and likewise $(y \sqcup \neg\!\!\lrcorner\, x) \,\Theta\, \top$. From $x_\sqcup, y_\sqcup$ and $\top \in \underline{D}_\sqcup$ follows:

$$\begin{aligned} x_\sqcup &= x_\sqcup \bar{\sqcap} \top \,\Theta\, x_\sqcup \bar{\sqcap} (\neg\!\!\lrcorner\, x \sqcup y) \overset{8b}{=} x_\sqcup \bar{\sqcap} (\neg\!\!\lrcorner\, x_\sqcup \sqcup y_\sqcup) = (x_\sqcup \bar{\sqcap} \neg\!\!\lrcorner\, x_\sqcup) \sqcup (x_\sqcup \bar{\sqcap} y_\sqcup) \\ &= x_\sqcup \bar{\sqcap} y_\sqcup = x \bar{\sqcap} y. \end{aligned}$$

Analogously we obtain $y_\sqcup \,\Theta\, x \bar{\sqcap} y$ and thus $y_\sqcup \,\Theta\, x_\sqcup$ and $x_\sqcap \,\Theta\, y_\sqcap$, i.e. $[x] \sqsubseteq\!\!\sqsupseteq [y]$. By Lemma 2.6 $\underline{D}/\Theta$ is contextual and we conclude $x \,\Theta\, y$. The converse implication was shown in Lemma 2.8. □

This theorem shows that in pure double Boolean algebras all congruence classes are determined by the classes of $\top$ and $\bot$. In addition, the functions $\mathrm{t}(x,y)$ and $\mathrm{b}(x,y)$ enable us to compute these congruence classes.

For an ideal I in $\underline{D}_\sqcap$ and a filter F in $\underline{D}_\sqcup$ we define

$$I^* := I \cup \{x \in \underline{D}_\sqcup \mid x_\sqcap \in I \text{ and } \neg\!\!\lrcorner\, x \in F\}$$

and

$$F^* := F \cup \{x \in \underline{D}_\sqcap \mid x_\sqcup \in F \text{ and } \neg x \in I\}.$$

Lemma 2.10. *Let $\underline{D}$ be a double Boolean algebra and Θ a congruence relation on $\underline{D}$. The set $F := [\top] \cap D_\sqcup$ is a filter in $\underline{D}_\sqcup$ and the set $I := [\bot] \cap D_\sqcap$ is an ideal in $\underline{D}_\sqcap$. If $\underline{D}$ is pure then $F^* = [\top]$ and $I^* = [\bot]$.*

Proof: It is easy to check that F is a filter. By definition holds $F \subseteq [\top]$. Now let $\underline{D}$ be a pure double Boolean algebra and let $x \in [\top] \setminus F$. It follows that $\top = \top \sqcup \top \,\Theta\, x_{\sqcup} \in [\top] \cap D_{\sqcup} = F$ and $\neg x \,\Theta\, \neg\top = \bot$, so $x \in F^*$. Conversely, let $x \in F^* \setminus F$. By definition holds $\neg x \in I \subseteq [\bot]$ and we obtain

$$x \sqcap x \overset{7a}{=} \neg(\neg(x \sqcap x)) \overset{8a}{=} \neg(\neg x) \,\Theta\, \neg\bot \overset{10a}{=} \top \sqcap \top.$$

We conclude from $x \sqcup x \in [\top] \cap D_{\sqcup}$ that $x \sqcup x \,\Theta\, \top = \top \sqcup \top$ so $x \sqsupseteq \top$ and Lemma 2.6 yields $x \,\Theta\, \top$. The rest follows dually. □

For a congruence relation Θ and sets F and I as in Lemma 2.10, obviously holds $\neg F^* \subseteq I$ and dually $\lrcorner I^* \subseteq F$. If we search for pairs (I, F) consisting of an ideal in $\underline{D}_{\sqcap}$ and a filter in $\underline{D}_{\sqcup}$ that can define a congruence relation we have to demand this condition. It follows from the definition of F^* and I^* that it is sufficient to demand $\neg F \subseteq I$ and $\lrcorner I \subseteq F$.

We need four more lemmas to prove the main theorem of this paper.

Lemma 2.11. *In a double Boolean algebra holds:*
(1) $x \sqcap y \sqsubseteq x \sqsubseteq x \sqcup y$
(2) the map $x \mapsto x \sqcap y$ *preserves* $\sqsubseteq$ *and* $\sqcap$,
(3) the map $x \mapsto x \sqcup y$ *preserves* $\sqsubseteq$ *and* $\sqcup$.

See [Wi00a] for a proof.

Lemma 2.12. *In double Boolean algebras holds:*
(1) For $x, y \in D_{\sqcup}$ *is* $x \sqcap y \sqsubseteq x \bar{\sqcap} y$
(2) For $x, y \in D_{\sqcap}$ *is* $x \bar{\sqcup} y \sqsubseteq x \sqcup y$

Proof: By 1.2.(5) holds $x \sqcap y \sqsubseteq x \bar{\sqcap} y \Leftrightarrow (x \sqcap y) \sqcup (x \sqcap y) \sqsubseteq x \bar{\sqcap} y$. Lemma 2.11.(1) and(3) yield $x \sqcap y \sqsubseteq x \Rightarrow (x \sqcap y)_{\sqcup} \sqsubseteq x \sqcup (x \sqcap y) \overset{4b}{=} x \sqcup x = x$. Likewise we obtain $(x \sqcap y)_{\sqcup} \sqsubseteq y$ and conclude from $(x \sqcap y)_{\sqcup} \in \underline{D}_{\sqcup}$ that $(x \sqcap y)_{\sqcup} \sqsubseteq x \bar{\sqcap} y$, hence (1). Dually we obtain (2). □

Lemma 2.13. *In a double Boolean algebra* $\underline{D}$ *holds for elements* $x, y \in D_{\sqcap}$: $(x \bar{\sqcup} y)_{\sqcup} = x \sqcup y$. *Dually, for* $x, y \in D_{\sqcup}$ *holds* $(x \bar{\sqcap} y)_{\sqcap} = x \sqcap y$.

Proof: Let $x, y \in D_{\sqcap}$. Lemma 2.12 yields $x \sqsubseteq x \bar{\sqcup} y \sqsubseteq x \sqcup y$. From Lemma 2.11 it follows that

$$x \sqcup (x \bar{\sqcup} y) \sqsubseteq (x \bar{\sqcup} y) \sqcup (x \bar{\sqcup} y) \sqsubseteq (x \sqcup y) \sqcup (x \bar{\sqcup} y) = (x \sqcup y)_{\sqcup} = (x \sqcup y),$$

since $x \bar{\sqcup} y \sqsubseteq x \sqcup y$. We conclude from $y \sqsubseteq x \bar{\sqcup} y$ that $x \sqcup y \sqsubseteq x \sqcup (x \bar{\sqcup} y)$ and therefore $x \sqcup y \sqsubseteq (x \bar{\sqcup} y)_{\sqcup} \sqsubseteq x \sqcup y$. As $\sqsubseteq$ is an order in $\underline{D}_{\sqcup}$ we have equality. □

Lemma 2.14. *Let* $\underline{D}$ *be a double Boolean algebra, let* I *be an ideal in* $\underline{D}_{\sqcap}$ *and let* F *be a filter in* $\underline{D}_{\sqcup}$ *such that* $\neg F \subseteq I$ *and* $\lrcorner I \subseteq F$. *Then holds:*
(1) $x, y \in D_{\sqcap}$ *and* $b(x, y) \in I \Rightarrow t(x, y) \in F$,
(2) $x, y \in D_{\sqcup}$ *and* $t(x, y) \in F \Rightarrow b(x, y) \in I$.

<u>Proof:</u> We proof (1), then (2) follows dually. Let $x, y \in D_\sqcap$. Lemma 2.13 yields

$$x_\sqcup = ((x \sqcap y) \mathbin{\underline{\sqcup}} (x \sqcap \neg y))_\sqcup = (x \sqcap y) \sqcup (x \sqcap \neg y)$$

and $y_\sqcup = (x \sqcap y) \sqcup (\neg x \sqcap y)$. By assumption, both $(x \sqcap \neg y)$ and $(\neg x \sqcap y)$ are in I. From this we obtain

$$\begin{aligned}
\mathrm{t}(x,y) &= (x \sqcup \lrcorner y) \mathbin{\bar{\sqcap}} (y \sqcup \lrcorner x) \\
&= ((x \sqcap y) \sqcup (x \sqcap \neg y) \sqcup \lrcorner((x \sqcap y) \sqcup (\neg x \sqcap y))) \\
&\quad \mathbin{\bar{\sqcap}} ((x \sqcap y) \sqcup (\neg x \sqcap y) \sqcup \lrcorner((x \sqcap y) \sqcup (x \sqcap \neg y))) \\
&= ((x \sqcap \neg y) \sqcup (\lrcorner(x \sqcap y) \mathbin{\bar{\sqcap}} \lrcorner(\neg x \sqcap y)) \sqcup (x \sqcap y)) \\
&\quad \mathbin{\bar{\sqcap}} ((\neg x \sqcap y) \sqcup (\lrcorner(x \sqcap y) \mathbin{\bar{\sqcap}} \lrcorner(x \sqcap \neg y)) \sqcup (x \sqcap y)) \\
&= ((x \sqcap \neg y) \sqcup \lrcorner(\neg x \sqcap y) \sqcup (x \sqcap y)) \mathbin{\bar{\sqcap}} ((\neg x \sqcap y) \sqcup \lrcorner(x \sqcap \neg y) \sqcup (x \sqcap y)) \\
&= (x \sqcap y) \sqcup \mathrm{t}((x \sqcap \neg y), (\neg x \sqcap y)).
\end{aligned}$$

As for $u, v \in I$ holds $\lrcorner u, \lrcorner v \in F$, we have $\mathrm{t}(u,v) = (u \sqcup \lrcorner v) \mathbin{\bar{\sqcap}} (\lrcorner u \sqcup v) \in F$ and therefore

$$\mathrm{t}((x \sqcap \neg y), (\neg x \sqcap y)) \sqsubseteq (x \sqcap y) \sqcup \mathrm{t}((x \sqcap \neg y), (\neg x \sqcap y)) \in F.$$

Thus we have $\mathrm{t}(x,y) \in F$. □

Theorem 2.15. *Let $\underline{D}$ be a pure double Boolean algebra, let I be an ideal in $\underline{D}_\sqcap$ and let F be a filter in $\underline{D}_\sqcup$, such that $\neg F \subseteq I$ and $\lrcorner I \subseteq F$. Then*

$$x \,\Theta\, y :\Leftrightarrow \mathrm{t}(x,y) \in F \text{ and } \mathrm{b}(x,y) \in I$$

defines a congruence relation on $\underline{D}$. Moreover, $[\top] = F^*$ and $[\bot] = I^*$.

<u>Proof:</u> We first prove that Θ is an equivalence relation. Clearly, Θ is reflexive and symmetric. To check transitivity consider $x, y, z \in \underline{D}$ and $\mathrm{t}(x,y) \in F$, $\mathrm{t}(y,z) \in F$. Since F is a filter, $(x \sqcup \lrcorner y)$ and $(\lrcorner x \sqcup y)$ have to be in F. From $x_\sqcup = (x \sqcup y) \mathbin{\bar{\sqcap}} (x \sqcup \lrcorner y)$ we obtain

$$x \sqcup \lrcorner z = ((x \sqcup y) \mathbin{\bar{\sqcap}} (x \sqcup \lrcorner y)) \sqcup \lrcorner z = (x \sqcup (y \sqcup \lrcorner z)) \mathbin{\bar{\sqcap}} ((x \sqcup \lrcorner y) \sqcup \lrcorner z).$$

By assumption $y \sqcup \lrcorner z \in F$ and $x \sqcup \lrcorner y \in F$, hence $x \sqcup \lrcorner z \in F$ and $\lrcorner x \sqcup z \in F$ and we conclude $\mathrm{t}(x,z) \in F$. Dually it follows from $\mathrm{b}(x,y) \in I$ and $\mathrm{b}(y,z) \in I$ that $\mathrm{b}(x,z) \in I$. Thus Θ is an equivalence relation.
Lemma 2.14 yields that it is sufficient to check for $x, y, a \in \underline{D}$ and $x \,\Theta\, y$ that $\mathrm{b}(x \sqcap a, y \sqcap a) = \mathrm{b}(x,y) \sqcap a$ and $\mathrm{t}(x \sqcup a, y \sqcup a) = \mathrm{t}(x,y) \sqcup a$ in order to see that Θ respects the operations $\sqcap$ and $\sqcup$. This can be easily verified. Moreover, an easy computation shows that $\mathrm{b}(\neg x, \neg y) = \mathrm{b}(x,y)$ and dually $\mathrm{t}(\lrcorner x, \lrcorner y) = \mathrm{t}(x,y)$. Hence Θ is a congruence relation.
Finally, $F^* = [\top]$ and $I^* = [\bot]$ can be easily verified as well. □

Theorem 2.15 is the main result of this paper. It provides a characterization of all pairs consisting of an ideal in $\underline{D}_\sqcap$ and a filter in $\underline{D}_\sqcup$ that can generate a congruence relation on a pure double Boolean algebra $\underline{D}$.

Definition 2.16. *In a double Boolean algebra* $\underline{D}$ *we call a pair* (I, F) *where* I *is an ideal in* $\underline{D}_\sqcap$*,* F *is a filter in* $\underline{D}_\sqcup$ *and* $\neg F \subseteq I$*,* $\lrcorner I \subseteq F$ *a* congruence generating pair*.*

Lemma 2.17. *In a double Boolean algebra* $\underline{D}$*, it holds for all* $x, y \in \underline{D}$*:*

(1) $x \sqsubseteq y \Rightarrow \neg y \sqsubseteq \neg x$
(2) $x \sqsubseteq y \Rightarrow \lrcorner y \sqsubseteq \lrcorner x$

The proof of this lemma is straightforward.

Theorem 2.18. *Let* (I, F) *be a congruence generating pair in a double Boolean algebra* $\underline{D}$*. If* I *is a principal ideal* $(i]$ *with greatest element* i *in* $\underline{D}_\sqcap$ *and if* F *is a principal filter* $[f)$ *with smallest element* f *in* $\underline{D}_\sqcup$*, then* $\overline{I} := \{j \in D_\sqcap \mid j \sqsubseteq \neg i\}$ *and* $\overline{F} := \{g \in D_\sqcup \mid \lrcorner f \sqsubseteq g\}$ *form another congruence generating pair* $(\overline{I}, \overline{F})$ *which we call the* orthogonal congruence generating pair*.*

Proof: By the definition of congruence generating pairs we have $\neg f \sqsubseteq i$ and $f \sqsubseteq \lrcorner i$. Lemma 2.17 gives $i_\sqcup = \lrcorner\lrcorner i \sqsubseteq \lrcorner f$ and therefore $\neg(\lrcorner f) \sqsubseteq \neg i_\sqcup$. From $i \sqsubseteq i_\sqcup$ we conclude that $\neg(\lrcorner f) \sqsubseteq \neg i_\sqcup \sqsubseteq \neg i$. Hence $\neg\overline{F} \subseteq \overline{I}$ and dually $\lrcorner\overline{I} \subseteq \overline{F}$. □

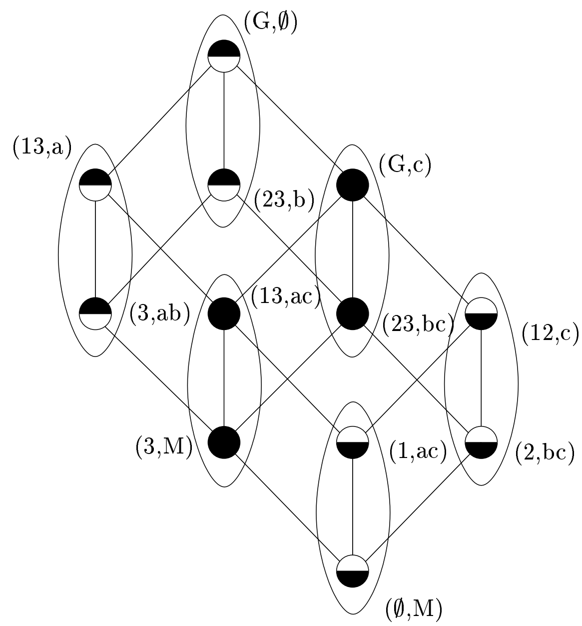


FIGURE 2. A congruence relation on a pure double Boolean algebra

In semiconcept algebras, $\underline{D}_\sqcap$ and $\underline{D}_\sqcup$ are isomorphic to the powerset lattices $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$ respectively, hence complete. Obviously, all ideals and filters are principal in finite semiconcept algebras. Thus, in this case there exists for every congruence generating pair (I, F) the orthogonal congruence generating pair $(\overline{I}, \overline{F})$.

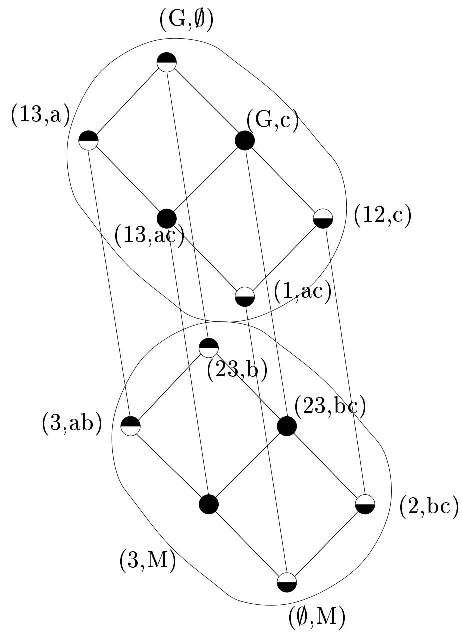


FIGURE 3. The congruence relation that is orthogonal to the one in Fig. 2

Moreover, in finite semiconcept algebras the congruence generating pairs can be found in the context $\mathbb{K}$:

Theorem 2.19. *In a finite semiconcept algebra* $\underline{\mathfrak{H}}(\mathbb{K})$ *with context* $\mathbb{K} := (G, M, I)$ *there is a one-to-one correspondence between the congruence generating pairs* (J, F) *in* $\underline{\mathfrak{H}}(\mathbb{K})$ *and pairs* $(\tilde{A}, \tilde{B}) \in \mathfrak{P}(G) \times \mathfrak{P}(M)$ *satisfying*

$$G \setminus \tilde{B}' \subseteq \tilde{A} \text{ and } M \setminus \tilde{A}' \subseteq \tilde{B}. \tag{1}$$

Proof: Let (J, F) be a congruence generating pair in $\underline{\mathfrak{H}}(\mathbb{K})$. Let a denote the greatest element of J in $\underline{\mathfrak{H}}(\mathbb{K})_\sqcap$ and let b denote the smallest element of F in $\underline{\mathfrak{H}}(\mathbb{K})_\sqcup$. As a and b are semiconcepts there exist sets $\tilde{A} \subseteq G$ and $\tilde{B} \subseteq M$ with $a = (\tilde{A}, \tilde{A}')$ and $b = (\tilde{B}', \tilde{B})$. We conclude from $\neg F \subseteq J$ that

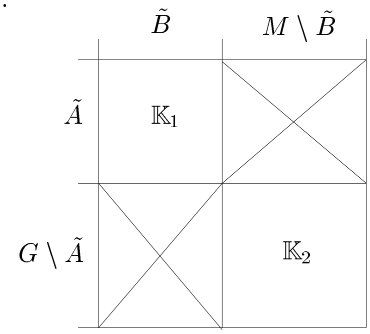
$$\begin{aligned} \neg(\tilde{B}', \tilde{B}) \sqsubseteq (\tilde{A}, \tilde{A}') \quad &\Leftrightarrow \quad (G \setminus \tilde{B}', (G \setminus \tilde{B}')') \sqsubseteq (\tilde{A}, \tilde{A}') \\ &\Leftrightarrow \quad G \setminus \tilde{B}' \subseteq \tilde{A} \end{aligned}$$

and dually $M \setminus \tilde{A}' \subseteq \tilde{B}$. The map $\phi : (I, F) \mapsto (\tilde{A}, \tilde{B})$ from the set of all congruence generating pairs in $\underline{\mathfrak{H}}(\mathbb{K})$ to $Q := \{(A, B) \in G \times M \mid G \setminus \tilde{B}' \subseteq \tilde{A} \text{ und } M \setminus \tilde{A} \subseteq \tilde{B}\}$ is injective. Conversely, we can define for each pair $(\tilde{A}, \tilde{B})$ satisfying (1) an ideal by $J := \{(A, A') \in \underline{\mathfrak{H}}(\mathbb{K})_\sqcap \mid A \subseteq \tilde{A}\}$ and a filter by $F := \{(B', B) \in \underline{\mathfrak{H}}(\mathbb{K})_\sqcup \mid B \subseteq \tilde{B}\}$. For an element $(A, A') \in J$ we obtain $\lrcorner(\tilde{A}, \tilde{A}') \sqsubseteq \lrcorner(A, A')$ and

$$(\tilde{B}', \tilde{B}) \sqsubseteq ((M \setminus \tilde{A}')', M \setminus \tilde{A}') = \lrcorner(\tilde{A}, \tilde{A}') \sqsubseteq \lrcorner(A, A').$$

Hence $\neg\neg J \subseteq F$ and dually $\neg F \subseteq J$, i.e. (J, F) is a congruence generating pair and the map ϕ is bijective. □

Note that in finite semiconcept algebras holds $\phi(\overline{J}, \overline{F}) = (G \setminus \tilde{A}, M \setminus \tilde{B})$. Hence, if for a finite semiconcept algebra the sets $\tilde{A}$, $\tilde{B}$ and their complements are nonempty then the context $\mathbb{K}$ is the direct sum of two subcontexts $\mathbb{K}_1$ and $\mathbb{K}_2$, i.e. for $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ we have $\mathbb{K} = (G_1 \dot\cup G_2, M_1 \dot\cup M_2, I_1 \dot\cup I_2 \cup (G_1 \times M_2) \cup (G_2 \times M_1))$.



Since $G \setminus \tilde{B} \subseteq \tilde{A}$, every object g that has not all attributes of $\tilde{B}$ is in $\tilde{A}$, and we have that $\tilde{B} \subseteq (M \setminus \tilde{A}')$. Dually, $\tilde{A} \subseteq (M \setminus \tilde{B}')$.
As congruence generating pairs are defined as pairs consisting of an ideal J in $\underline{D}_\sqcap$ and a filter F in $\underline{D}_\sqcup$, the congruence generating pairs of a double Boolean algebra $\underline{D}$ are determined by its pure subalgebra $\underline{D}_p$. Thus, if there exists a context $\mathbb{K} := (G, M, I)$ such that $\underline{D}_p \cong \underline{\mathfrak{H}}(\mathbb{K})$ and if $\phi : \underline{D}_p \to \underline{\mathfrak{H}}(\mathbb{K})$ is an isomorphism then ϕ maps congruence generating pairs in $\underline{D}_p$ to congruence generating pairs in $\underline{\mathfrak{H}}(\mathbb{K})$. Conversely, for every congruence generating pair (J, P) in $\underline{\mathfrak{H}}(\mathbb{K})$ the pair $(\phi^{-1}(J), \phi^{-1}(F))$ is a congruence generating pair in $\underline{D}_p$. Therefore, whenever $\underline{D}_p \cong \underline{\mathfrak{H}}(\mathbb{K})$ holds, the context $\mathbb{K}$ can be used to find the congruence generating pairs in $\underline{D}$. Since for protoconcept algebras holds $\underline{\mathfrak{P}}(\mathbb{K})_p = \underline{\mathfrak{H}}(\mathbb{K})$ this result will be useful in Section 2.4 where congruences on protoconcept algebras are investigated.

2.3. Congruence Relations on Double Boolean Algebras. In this section we apply our results to double Boolean algebras in general. It is evident that for a double Boolean algebra $\underline{D}$ and a congruence relation Θ on $\underline{D}$ the restriction $\Theta|_{\underline{D}_p}$ of Θ to the pure subalgebra $\underline{D}_p$ is a congruence on $\underline{D}_p$. There are various ways to extend a congruence on $\underline{D}_p$ to $\underline{D}$, but Theorem 2.22 shows that they cannot differ very much.

Theorem 2.20. *If $\underline{D}$ is a double Boolean algebra and Θ a congruence relation on $\underline{D}_p$ then $\Theta' := \Theta \cup \Delta_{\underline{D}}$ is a congruence relation on $\underline{D}$.*

Proof: Clearly Θ' is an equivalence relation. For $(a, b) \in \Theta'$ holds $a = b$ or $a, b \in \underline{D}_p$. In any case we have that

$$a_\sqcap = a \sqcap a \,\Theta\, b \sqcap b = b_\sqcap$$

and therefore $(a_\sqcap, b_\sqcap) \in \Theta \subseteq \Theta'$. In the same manner we can see that $(a_\sqcup, b_\sqcup)$, $(\neg a, \neg b)$ and $(\lrcorner a, \lrcorner b)$ are elements of Θ'. $\square$

Obviously, Θ' is the smallest possible extension of Θ.

Theorem 2.21. *Let Θ be a congruence relation on $\underline{D}_p$. Then Θ^* defined by*

$$(x, y) \in \Theta^* :\Leftrightarrow \mathrm{t}(x, y) \in [\top]_\Theta \text{ and } \mathrm{b}(x, y) \in [\bot]_\Theta$$

is a congruence relation on $\underline{D}$ which extends Θ. The factor algebra $\underline{D}/\Theta^*$ is contextual.

Proof: Theorem 2.15 immediately yields that $\Theta^*|_{\underline{D}_p}$ is a congruence relation on $\underline{D}_p$. Moreover, $\Theta^*|_{\underline{D}_p} = \Theta$. Reflexivity and symmetry of Θ^* are easy to check. Now suppose $(x, y), (y, z) \in \Theta^*$. We have $\mathrm{t}(x, y) = \mathrm{t}(x_\sqcup, y_\sqcup) \in [\top]_\Theta$ and $\mathrm{t}(y, z) = \mathrm{t}(y_\sqcup, z_\sqcup) \in [\top]_\Theta$. Then Theorem 2.15 yields $x_\sqcup \,\Theta\, y_\sqcup \,\Theta\, z_\sqcup$. The transitivity of Θ implies $\mathrm{t}(x, z) = \mathrm{t}(x_\sqcup, z_\sqcup) \in [\top]_\Theta$. Dually we obtain $\mathrm{b}(y, z) \in [\bot]_\Theta$.
In order to see that Θ^* respects the operations we choose $(u, v), (x, y) \in \Theta^*$. As above $\mathrm{b}(u_\sqcup, v_\sqcup) \in [\bot]_\Theta$, and since $\mathrm{t}(u_\sqcup, v_\sqcup) \in [\top]_\Theta$ we have $u_\sqcup \,\Theta\, v_\sqcup$. Likewise, we have that $x_\sqcup \,\Theta\, y_\sqcup$ and therefore $u_\sqcup \sqcup x_\sqcup \,\Theta\, v_\sqcup \sqcup y_\sqcup$. This yields

$$u_\sqcup \sqcup x_\sqcup \,\Theta\, v_\sqcup \sqcup y_\sqcup \Rightarrow \mathrm{t}(u_\sqcup \sqcup x_\sqcup, v_\sqcup \sqcup y_\sqcup) \in [\top]_\Theta \Rightarrow \mathrm{t}(u \sqcup x, v \sqcup y) \in [\top]_\Theta$$

and

$$u_\sqcup \sqcup x_\sqcup \,\Theta\, v_\sqcup \sqcup y_\sqcup \Rightarrow \mathrm{b}(u_\sqcup \sqcup x_\sqcup \,\Theta\, v_\sqcup \sqcup y_\sqcup) \in [\bot]_\Theta \Rightarrow \mathrm{b}(u \sqcup x, v \sqcup y) \in [\bot]_\Theta,$$

hence $(u \sqcup x, v \sqcup y) \in \Theta^*$. In the same manner we find that the remaining operations are respected by Θ^*.
Finally, we show that $\underline{D}/\Theta^*$ contains no equivalent pairs. Let $x, y \in \underline{D}$ such that $[x]_{\Theta^*} \sqsupseteq\!\!\!\sqsubseteq [y]_{\Theta^*}$. We conclude from $[x_\sqcap]_{\Theta^*} = [y_\sqcap]_{\Theta^*}$ that $\mathrm{b}(x, y) \in [\bot]_{\Theta^*}$ and from $[x_\sqcup]_{\Theta^*} = [y_\sqcup]_{\Theta^*}$ that $\mathrm{t}(x, y) \in [\top]_{\Theta^*}$. This gives $[x]_{\Theta^*} = [y]_{\Theta^*}$ and $\underline{D}/\Theta^*$ contains no equivalent pairs. $\square$
The next theorem shows that the structure of a factor algebra depends to a large extent on the congruence relation on $\underline{D}_p$.

Theorem 2.22. *Let Θ, Ψ be two congruence relations on a double Boolean algebra $\underline{D}$ satisfying $\Theta|_{\underline{D}_p} = \Psi|_{\underline{D}_p}$. Then there exists a natural isomorphism*

$$\phi : (\underline{D}/\Theta)/\sqsupseteq\!\!\!\sqsubseteq \;\to (\underline{D}/\Psi)/\sqsupseteq\!\!\!\sqsubseteq, \quad \phi : [x]_{\Theta/\sqsupseteq\!\!\!\sqsubseteq} \mapsto [x]_{\Psi/\sqsupseteq\!\!\!\sqsubseteq}.$$

Proof: First we show that ϕ is indeed a map. Let $y \in [x]_{\Theta/\sqsupseteq\!\!\!\sqsubseteq}$. From $[y]_\Theta \sqsupseteq\!\!\!\sqsubseteq [x]_\Theta$ we conclude that

$$[x_\sqcap]_\Theta = ([x]_\Theta)_\sqcap = ([y]_\Theta)_\sqcap = [y_\sqcap]_\Theta$$
$$\text{and}$$
$$[x_\sqcup]_\Theta = ([x]_\Theta)_\sqcup = ([y]_\Theta)_\sqcup = [y_\sqcup]_\Theta.$$

Since $\Theta|_{\underline{D}_p} = \Psi|_{\underline{D}_p}$, the congruences $x_\sqcap \,\Theta\, y_\sqcap$ and $x_\sqcup \,\Theta\, y_\sqcup$ imply $x_\sqcap \,\Psi\, y_\sqcap$ and $x_\sqcup \,\Psi\, y_\sqcup$. This yields $[x_\sqcap]_\Psi = [y_\sqcap]_\Psi$ and $[x_\sqcup]_\Psi = [y_\sqcup]_\Psi$, i.e. $[x]_\Psi \sqsupseteq\!\!\!\sqsubseteq [y]_\Psi$ and $[x]_{\Psi/\sqsupseteq\!\!\!\sqsubseteq} = [y]_{\Psi/\sqsupseteq\!\!\!\sqsubseteq}$.

It is easy to verify that ϕ is a homomorphism. We only give the proof for $\sqcap$:

$$\begin{aligned}\phi([x]_{\Theta/\sqsubseteq} \sqcap [y]_{\Theta/\sqsubseteq}) &= \phi([x \sqcap y]_{\Theta/\sqsubseteq}) \\ &= [x \sqcap y]_{\Psi/\sqsubseteq} \\ &= [x]_{\Psi/\sqsubseteq} \sqcap [y]_{\Psi/\sqsubseteq} \\ &= \phi([x]_{\Theta/\sqsubseteq}) \sqcap \phi([y]_{\Theta/\sqsubseteq}).\end{aligned}$$

We obtain surjectivity from $\phi^{-1}([x]_{\Psi/\sqsubseteq}) = [x]_{\Theta/\sqsubseteq}$. Now let $[x]_{\Theta/\sqsubseteq} \neq [y]_{\Theta/\sqsubseteq}$. We have

$$\begin{aligned}[x_\sqcap]_\Theta \neq [y_\sqcap]_\Theta \text{ or } [x_\sqcup]_\Theta \neq [y_\sqcup]_\Theta &\Leftrightarrow \text{not } (x_\sqcap \,\Psi\, y_\sqcap) \text{ or not } (x_\sqcup \,\Psi\, y_\sqcup) \\ &\Leftrightarrow \text{not } ([x]_\Psi \sqsubseteq [y]_\Psi) \\ &\Leftrightarrow [x]_{\Psi/\sqsubseteq} \neq [y]_{\Psi/\sqsubseteq}.\end{aligned}$$

Hence ϕ is an isomorphism. □

Corollary 2.23. *Let Θ be a congruence relation on $\underline{D}$. Then*

$$(\underline{D}/\Theta)/\sqsubseteq \,\cong \underline{D}/(\Theta|_{\underline{D}_p})^*$$

2.4. Direct Decomposition of Protoconcept Algebras. In this section we apply our results to the special case of finite protoconcept algebras. This yields a direct decomposition to directly irreducible protoconcept algebras. Moreover, we show how the congruences on and the decompositions of these algebras correspond to structures and decompositions of their contexts.

Theorem 2.24. *Let $\underline{D} := \mathfrak{P}(\mathbb{K})$ be a finite protoconcept algebra with context $\mathbb{K} := (G, M, I)$ and let Θ be a congruence relation on $\underline{D}_p := \mathfrak{H}(\mathbb{K})$. Then the factor algebra $\underline{D}/\Theta^*$ is isomorphic to the protoconcept algebra of a subcontext.*

<u>Proof:</u> Let (I, F) be the congruence generating pair that corresponds to Θ and let $(\tilde{A}, \tilde{B})$ be the pair consisting of a set of objects and a set of attributes corresponding to (I, F) as in Theorem 2.19. Note that in protoconcept algebras the functions $\mathrm{t}(x, y)$ and $\mathrm{b}(x, y)$ calculate the symmetric difference of the sets of objects of x and y and the sets of attributes of x and y respectively. Therefore, two elements of $\underline{D}$ are in relation Θ if and only if their sets of objects differ only by elements of $\tilde{A}$ and their sets of attributes differ only by elements of $\tilde{B}$. We set $H := G \setminus \tilde{A}$, $N := M \setminus \tilde{B}$ and $\tilde{\mathbb{K}} = (H, N, I \cap (H \times N))$ and define

$$\begin{aligned}\phi : \quad & \underline{D}/\Theta^* \to \mathfrak{P}(\tilde{\mathbb{K}}), \\ \phi([(A, B)]_\Theta) \; & \mapsto (A \setminus \tilde{A}, B \setminus \tilde{B}).\end{aligned}$$

We prove that ϕ is an isomorphism. First we show that ϕ is well-defined, i.e. that $\phi([(A, B)]_\Theta)$ is in $\mathfrak{P}(\tilde{\mathbb{K}})$. Let $()^I$ denote the derivation in $\tilde{\mathbb{K}}$ and let $()'$ denote the derivation in $\mathbb{K}$. From $M \setminus \tilde{A}' \subseteq \tilde{B}$ (Theorem 2.19) we conclude that $N = \tilde{A}' \setminus \tilde{B}$

and dually $H = \tilde{B}' \setminus \tilde{A}$. If $(A, B) \in \mathfrak{P}(\mathbb{K})$ then $(A \setminus \tilde{A})^I = (A \setminus \tilde{A})' \setminus \tilde{B}$. This yields

$$\begin{aligned}(A \setminus \tilde{A})^I &= (A \setminus \tilde{A})^I \cap N \\ &= (A \setminus \tilde{A})^I \cap (\tilde{A}' \setminus \tilde{B}) \\ &= ((A \setminus \tilde{A})' \setminus \tilde{B}) \cap ((\tilde{A} \cap A)' \setminus \tilde{B}) \\ &= A' \setminus \tilde{B}\end{aligned}$$

and dually $(B \setminus \tilde{B})^I = B' \setminus \tilde{A}$. We obtain

$$\begin{aligned}(A \setminus \tilde{A})^{II} &= (A' \setminus \tilde{B})^I \\ &= (A' \setminus \tilde{B})' \setminus \tilde{A} \\ &= (A' \setminus \tilde{B})' \cap H \\ &= (A' \setminus \tilde{B})' \cap ((A' \cap \tilde{B})' \setminus \tilde{A}) \\ &= A'' \setminus \tilde{A} \\ &= (B \setminus \tilde{B})^I.\end{aligned}$$

hence $(A \setminus \tilde{A}, B \setminus \tilde{B})$ is a protoconcept in $\underline{\mathfrak{P}}(\tilde{\mathbb{K}})$. Now an easy computation yields that ϕ is a homomorphism. Finally, we show that ϕ is bijective. Suppose $[(A_1, B_1)]_\Theta \neq [(A_2, B_2)]_\Theta$. We conclude that $A_1 \setminus \tilde{A} \neq A_2 \setminus \tilde{A}$ or $B_1 \setminus \tilde{B} \neq B_2 \setminus \tilde{B}$, hence $\phi([(A_1, B_1)]_\Theta) \neq \phi([(A_2, B_2)]_\Theta)$. If (A, B) is a protoconcept in $\tilde{\mathbb{K}}$ consider

$$\begin{aligned}(A \cup \tilde{A})' &= A' \cap \tilde{A}' \\ &= (A^I \cup \tilde{B}) \cap (N \cup G') \\ &= A^I \cup G' \qquad \text{and} \\ (A \cup \tilde{A})'' &= (A^I \cup G')' \\ &= (A^I)' \cap G'' \\ &= (A^{II} \cup \tilde{A}) \cap G \\ &= A^{II} \cup \tilde{A}\end{aligned}$$

Since dually holds $B' = B^I \cup \tilde{A} = A^{II} \cup \tilde{A}$, the pair $(A \cup \tilde{A}, B)$ is a protoconcept in $\mathbb{K}$ satisfying $\phi(A \cup \tilde{A}, B) = (A, B)$. Therefore ϕ is bijective. □

Corollary 2.25. *If $\underline{D} := \underline{\mathfrak{P}}(\mathbb{K})$ is a finite protoconcept algebra with context $\mathbb{K} := (G, M, I)$ and if Θ is a congruence relation on $\underline{D}$ then $(\underline{D}/\Theta)/\sqcup$ is isomorphic to the protoconcept algebra of a subcontext.*

Theorem 2.26. *Let $\underline{D} := \underline{\mathfrak{P}}(\mathbb{K})$ be a finite protoconcept algebra with context $\mathbb{K} := (G, M, I)$ and let Θ^* be the congruence relation defined by a congruence generating pair such that $\underline{D}/\Theta^*$ is contextual. By Ψ^* we denote the congruence relation defined by $(\overline{I}, \overline{F})$ such that $\underline{D}/\Psi^*$ is contextual. Then*

$$\underline{D} \cong \underline{D}/\Theta^* \times \underline{D}/\Psi^*$$

<u>Proof:</u> As before, let $(\tilde{A}, \tilde{B}) \in \mathfrak{P}(G) \times \mathfrak{P}(M)$ correspond to (I, F). Then $(G \setminus \tilde{A}, M \setminus \tilde{B})$ corresponds to $(\overline{I}, \overline{F})$. Thus, $\mathbb{K}$ is the direct sum of $\mathbb{K}_1 := (G \setminus \tilde{A}, M \setminus \tilde{B}, I_1 := I \cap ((G \setminus \tilde{A}) \times (M \setminus \tilde{B})))$ and $\mathbb{K}_2 := (\tilde{A}, \tilde{B}, I_2 := I \cap (\tilde{A} \times \tilde{B})$. The preceding

theorem yields $\underline{D}/\Theta^* \cong \underline{\mathfrak{P}}(\mathbb{K}_1)$ and $\underline{D}/\Psi^* \cong \underline{\mathfrak{P}}(\mathbb{K}_2)$. Let π_Θ denote the projection $\underline{D} \to \underline{\mathfrak{P}}(\mathbb{K}_1)$ and let π_Ψ denote the projection $\underline{D} \to \underline{\mathfrak{P}}(\mathbb{K}_2)$. We claim that the following function is an isomorphism:

$$\begin{aligned} \phi : \underline{D} &\to \underline{\mathfrak{P}}(\mathbb{K}_1) \times \underline{\mathfrak{P}}(\mathbb{K}_2) \\ \phi(A,B) &= (\pi_\Theta(A,B), \pi_\Psi(A,B)) \\ &= ((A \setminus \tilde{A}, B \setminus \tilde{B}), (A \cap \tilde{A}, B \cap \tilde{B})) \end{aligned}$$

As π_Θ and π_Ψ are homomorphisms, so is ϕ. It is easy to check that ϕ is injective. Note that for a set of objects $C \subseteq G$ holds:

$$\begin{aligned} C' \cap \tilde{B} &= ((C \cap \tilde{A}) \cup (C \cap G \setminus \tilde{A}))' \cap \tilde{B} \\ &= (C \cap \tilde{A})' \cap (C \cap G \setminus \tilde{A})' \cap \tilde{B} \\ &= (C \cap \tilde{A})' \cap \tilde{B} \\ &= (C \cap \tilde{A})^{I_2} \end{aligned}$$

and likewise

$$C' \cap (M \setminus \tilde{B}) = (C \cap (G \setminus \tilde{A}))^{I_1}.$$

Dually, for $D \subseteq M$ we obtain

$$D' \cap \tilde{A} = (D \cap \tilde{B})^{I_2}$$

and

$$D' \cap (G \setminus \tilde{A}) = (D \cap (M \setminus \tilde{B}))^{I_1}.$$

Now consider $(A_1, B_1) \in \underline{\mathfrak{P}}(\mathbb{K}_1)$ and $(A_2, B_2) \in \underline{\mathfrak{P}}(\mathbb{K}_2)$. The inverse image of $((A_1, B_1), (A_2, B_2))$ is $\phi^{-1}((A_1, B_1), (A_2, B_2)) = (A_1 \cup A_2, B_1 \cup B_2)$. We set $C := A_1 \cup A_2$ and $D := B_1 \cup B_2)$ and obtain

$$\begin{aligned} C' &= (C' \cap \tilde{B}) \cup (C' \cap (M \setminus \tilde{B})) \\ &= (C \cap \tilde{A})^{I_2} \cup (C \cap (G \setminus \tilde{A}))^{I_1} \\ &= (A_2)^{I_2} \cup (A_1)^{I_1}. \end{aligned}$$

and, dually $D' = (B_2)^{I_2} \cup (B_1)^{I_1}$. This gives

$$\begin{aligned} C'' &= (C' \cap \tilde{B})^{I_2} \cup (C' \cap (M \setminus \tilde{B}))^{I_1} \\ &= A_2^{I_2 I_2} \cup A_1^{I_1 I_1} \\ &= B_2^{I_2} \cup B_1^{I_1} \\ &= D'. \end{aligned}$$

Hence (C, D) is a protoconcept in $\underline{D}$ and ϕ is surjective. □

Obviously, iteration of this decomposition yields a direct decomposition of finite protoconcept algebras in directly irreducible protoconcept algebras.

Example 2.27. *We can write the context* $\mathbb{K}$ *from Example 1.6 as the direct sum of the contexts* $\mathbb{K}_0 := (\{1\}, \{b\}, \emptyset)$, $\mathbb{K}_1 := (\{2\}, \{a\}, \emptyset)$ *and* $\mathbb{K}_2 := (\{3\}, \{c\}, \{(3, c)\})$, *where* $\mathbb{K}_2$ *itself is the direct sum of* $\mathbb{K}_{2a} := (\{3\}, \emptyset, \emptyset)$ *and* $\mathbb{K}_{2b} := (\emptyset, \{c\}, \emptyset)$.

	b	a	c
1	$\mathbb{K}_0$	×	×
2	×	$\mathbb{K}_1$	×
3	×	×	$\mathbb{K}_2$

Thus the protoconcept algebra from Example 1.6 is the direct product of the protoconcept algebras in Figure 4.

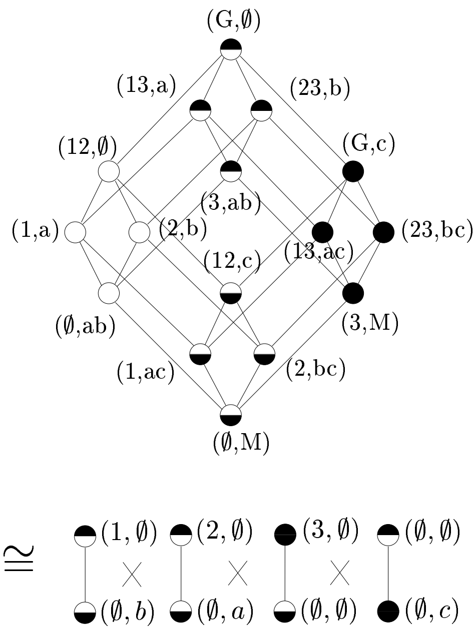


FIGURE 4. A decomposition of a protoconcept algebra in directly irreducible protoconcept algebras

2.5. Subdirectly Irreducible Double Boolean Algebras. For the more general case of finite double Boolean algebras we obtain a result similar to that for finite protoconcept algebras. While we have a direct decomposition of finite protoconcept algebras in simple protoconcept algebras, finite double Boolean algebras are subdirect products of simple double Boolean algebras.

Theorem 2.28. *A finite contextual double Boolean algebra $\underline{D}$ is subdirectly irreducible if and only if $\underline{D}$ is simple.*

Proof: Obviously, every simple double Boolean algebra is subdirectly irreducible. Let $\underline{D}$ be a finite contextual double Boolean algebra. Theorem 2.22 yields that if $\underline{D}$ is contextual and not simple then there exists a non-trivial congruence on $\underline{D}_p$.

Let (I,F) be a non-trivial congruence generating pair in $\underline{D}_p$, and let Θ be the congruence relation defined by

$$(x,y)\in\Theta \Leftrightarrow \mathrm{b}(x,y)\in I \text{ and } \mathrm{t}(x,y)\in F.$$

Since $\underline{D}$ is finite, (I,F) has an orthogonal congruence generating pair $(\overline{I},\overline{F})$. Let Ψ be the congruence relation defined by

$$(x,y)\in\Psi \Leftrightarrow \mathrm{b}(x,y)\in \overline{I} \text{ and } \mathrm{t}(x,y)\in \overline{F}.$$

We show that in the congruence lattice of $\underline{D}$ holds $\Theta\wedge\Psi=\Delta_{\underline{D}}$: Let $i:=\bigsqcup I$ denote the greatest element of I and let $f:=\bigsqcap F$ denote the smallest element of F. For $x,y\in\underline{D}$ with $x\,\Theta\,y$ and $x\,\Psi\,y$ we obtain

$$\begin{aligned} x\,\Theta\,y \quad &\Leftrightarrow \quad \mathrm{b}(x,y)\in I \text{ and } \mathrm{t}(x,y)\in F\\ &\Leftrightarrow \quad \mathrm{b}(x,y)\sqsubseteq i \text{ and } f\sqsubseteq \mathrm{t}(x,y)\\ &\text{and}\\ x\,\Psi\,y \quad &\Leftrightarrow \quad \mathrm{b}(x,y)\in \overline{I} \text{ and } \mathrm{t}(x,y)\in \overline{F}\\ &\Leftrightarrow \quad \mathrm{b}(x,y)\sqsubseteq \neg i \text{ and } \lrcorner f\sqsubseteq \mathrm{t}(x,y)\end{aligned}$$

Therefore we have that $\top=f\sqcup\lrcorner f\sqsubseteq \mathrm{t}(x,y)$ and $\mathrm{b}(x,y)\sqsubseteq i\sqcap\neg i=\bot$. This yields $\top=\mathrm{t}(x,y)=(x_\sqcup\sqcup\lrcorner y_\sqcup)\bar{\sqcap}(y_\sqcup\sqcup\lrcorner x_\sqcup)$, and since $\underline{D}_\sqcup$ is a Boolean algebra, we obtain $x_\sqcup=y_\sqcup$ and dually $x_\sqcap=y_\sqcap$. As $\underline{D}$ is contextual we conclude $x=y$ and thus $\Theta\wedge\Psi=\Delta_{\underline{D}}$. It follows that $\underline{D}$ is isomorphic to a subdirect product of $\underline{D}/\Theta$ and $\underline{D}/\Psi$ (cf. [G68] p.123). □

Theorem 2.29. *Let $\underline{D}$ be a double Boolean algebra. If $\underline{D}$ is not contextual, then $\underline{D}$ is not subdirectly irreducible.*

Proof: Let $x\sqsupseteq\!\!\!\sqsubseteq y$, $x\neq y$ be an equivalent pair in $\underline{D}$. We set $\Theta:=\Delta_{\underline{D}}\cup\{(x,y),(y,x)\}$ and $\Psi:=\{D\setminus\{y\}\times D\setminus\{y\}\}\cup\{(y,y)\}$ and show that Θ and Ψ are congruence relations satisfying $\Theta\wedge\Psi=\Delta_{\underline{D}}$. Obviously, Θ and Ψ are equivalence relations and Theorem 2.4 yields that Θ is a congruence relation. Note that from $x\sqsupseteq\!\!\!\sqsubseteq y$ and $x\neq y$ it follows that $y\neq y_\sqcap$ and $y\neq y_\sqcup$. This yields that for $a,b\in\underline{D}$ we have $a_\sqcap\,\Psi\,b_\sqcap\,\Psi\,b_\sqcup\,\Psi\,a_\sqcup$, hence Ψ is a congruence relation. Now assume that for $a,b\in\underline{D}$ holds $a\,\Theta\,b$ and $a\,\Psi\,b$. From $a\,\Psi\,b$ we conclude $a,b\in D\setminus\{y\}$ or $a=b=y$. Then $a\,\Theta\,b$ yields $a=b$, hence $\Theta\wedge\Psi=\Delta_{\underline{D}}$. □

This result immediately yields a generalization of Theorem 2.28:

Corollary 2.30. *A finite double Boolean algebra $\underline{D}$ is subdirectly irreducible if and only if $\underline{D}$ is simple.*

3. Further research

This paper is one of the first steps towards a theory of double Boolean algebras. Next steps in the field of construction and decomposition of such algebras should include the investigation of tolerance relations and tensor products.

As the number of elements of protoconcept algebras grows rapidly with increasing number of attributes and objects in the context, fast algorithms to compute the protoconcepts and good ways to obtain a diagrammatic representation are needed. The equational theory of double Boolean algebras and the investigation of free double Boolean algebras are of great importance for applications in Contextual Logic. This includes algorithmic solutions of word problems. Contextual Judgment Logic should be developed in parallel to Boolean Concept Logic, especially the theory of protoconcept graphs. This approach will benefit from insights into the theory of double Boolean algebras and inspire new investigation.

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