Fredholmness of convolution type operators

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Dedicated to the memory of Erhard Meister.

Abstract

We study the Fredholmness on $L^p(D)$ of operators of convolution type. Here D is an unbounded measurable domain in $\mathbb{R}^N$, and an operator A on $L^p(D)$ is of convolution type if it is constituted by operators of the form $aC(k)bI$ where $C(k)$ is the operator of convolution by the $L^1(\mathbb{R}^N)$-function k and where a and b are bounded and uniformly continuous functions. The domains under consideration include, for example, curved layers, curved cylinders, and cones with angular or cuspidal edges. The criterion for the Fredholmness of the operator A is formulated in terms of limit operators of A.

The topic of this paper is the Fredholmness of compressions of operators of convolution type. An operator of convolution type is an operator on $L^p(\mathbb{R}^N)$ which belongs to the smallest norm-closed subalgebra of $L(L^p(\mathbb{R}^N))$ which contains the identity operator, all compact operators, and all operators of the form $aC(k)bI$ where $C(k)$ is the operator of convolution by the $L^1(\mathbb{R}^N)$-function k and where a and b are bounded and uniformly continuous functions on $\mathbb{R}^N$. For an unbounded measurable subset D of $\mathbb{R}^N$, the compression of the operator A onto D is the operator

$$B := \chi_D A \chi_D I|_{L^p(D)} : L^p(D) \to L^p(D) \tag{1}$$

where χ_D is the characteristic function of the set D.

There are many papers which are devoted to the Fredholmness of compressions of operators of convolution type for concrete classes of coefficients a, b and concrete domains D; see, e.g., [2, 3, 5, 6, 7, 14]. For example, the multidimensional Wiener-Hopf operators

$$\chi_D(\gamma I + C(k))\chi_D I|_{L^p(D)} : L^p(D) \to L^p(D)$$

where $\gamma \in \mathbb{C}$ and $k \in L^1(\mathbb{R}^N)$ are considered in [3] for D being a half-space and in [14] in case D a cone in $\mathbb{R}^N$ with smooth cross section, whereas the quarter plane case is the topic of [2, 5, 7]. Operators on 3D wedge shaped domains are studied in [6].

We will consider the Fredholm problem for these operators, for example, in case when D is a curved layer, a curved cylinder, a cone with angular or cuspidal edges, or the epigraph of a certain function. In each of these cases, our solution to the Fredholm problem will be as follows. We associate with the operator B in (1) its so-called operator spectrum. This is a family of operators on $L^p(\mathbb{R}^N)$ which describes the behaviour of the operator B at infinity. Then the main result says that the operator B is Fredholm if and only if each operator in its operator spectrum is invertible, and if the norms of these inverses are uniformly bounded. Moreover, it turns out that in many cases (for example, if the coefficients a, b are slowly oscillating functions) the operators in the operator spectrum are much simpler objects than the operator B itself. This fact allows us to study their invertibility effectively.

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It will also turn out that the condition of *uniform* invertibility is redundant in many cases.

We will prove these Fredholm criteria by having recourse to the results of [8, 9]. In these papers we considered band-dominated operators on discrete l^p-spaces of sequences with values in a Banach space X. We showed that a band-dominated operator is invertible at infinity (in case X has finite dimension this simply means that the operator is Fredholm, i.e. that its kernel and cokernel have finite dimension) if the operators in its operator spectrum are uniformly invertible. Thus, the proof of the desired Fredholm criteria for compressions of operators of convolution type rests on two basic steps: we prove that a suitable discretization of that operator leads to a band-dominated operator on an discrete l^p-space, and we compute its operator spectrum.

The paper is organized as follows. In its first two sections we recall some basic definitions and facts on operators of convolution type as well as on band-dominated operators. Then we study discretizations of operators of convolution type. In particular, we show that the discretization of every convolution type operator B is band-dominated and that this discretization is invertible at infinity if and only if B is Fredholm. Finally, we specify the general Fredholm criterion to the concrete cases mentioned above where the operator spectrum can be explicitely computed.

Throughout this paper, we let $1 < p < \infty$, $q := p/(p-1)$, and N a positive integer.

1 Operators of convolution

In this section, we collect some basic facts on convolution operators on L^p-spaces. Theorem 1.2 goes back to [15], and the compactness of commutators of operators of multiplication by slowly oscillating function with convolution operators has been verified in [1]. Our presentation follows [11] where the results mentioned in this section are proved in the more general context of operators on locally compact groups.

Let $k \in L^1(\mathbb{R}^N)$ and $u \in L^p(\mathbb{R}^N)$. Then Young's inequality implies that the convolution

$$(k * u)(x) := \int_{\mathbb{R}^N} k(x-y)u(y)dy, \quad x \in \mathbb{R}^N, \tag{2}$$

belongs to $L^p(\mathbb{R}^N)$, and that $\|k * u\|_p \leq \|k\|_1 \, \|u\|_p$ ([10], IX.4). Hence, the operator $C(k)u := k * u$ of convolution by $k \in L^1(\mathbb{R}^N)$ acts boundedly on $L^p(\mathbb{R}^N)$, and

$$\|C(k)\|_{L^p(\mathbb{R}^N)} \leq \|k\|_1. \tag{3}$$

Let $\mathcal{C}_p$ denote the closure in $L(L^p(\mathbb{R}^N))$ of the set of all convolution operators $C(k)$ with kernels $k \in L^1(\mathbb{R}^N)$. Then $\mathcal{C}_p$ is a closed and commutative subalgebra of $L(L^p(\mathbb{R}^N))$ without identity. Its maximal ideal space can be identified with $\mathbb{R}^N$ (with its usual topology) in such a way that the Gelfand transform $\hat{C}$ of $C \in \mathcal{C}_p$ coincides with the Fourier transform of k if $C = C(k)$ (see [14]). Consequently, an operator $\gamma I + C$ in the unitization $\mathbb{C}I + \mathcal{C}_p$ of $\mathcal{C}_p$ is invertible if and only if

$$\inf_{\xi \in \mathbb{R}^N} |\gamma + \hat{C}(\xi)| > 0. \tag{4}$$

Note that $\gamma + \hat{C}$ is just the Gelfand transform of $\gamma I + C$ in $\mathbb{C}I + \mathcal{C}_p$.

A *semi-commutator* is an operator of the form $aC(k)$ or $C(k)aI$ where k is in $L^1(\mathbb{R}^N)$ and $a \in L^\infty(\mathbb{R}^N)$. The functions a for which the semi-commutators $aC(k)$ and $C(k)aI$ are compact for every function $k \in L^1(\mathbb{R}^N)$ can be characterized as follows.

Definition 1.1 *Let $Q_{SC}(\mathbb{R}^N)$ refer to the set of all functions $a \in L^\infty(\mathbb{R}^N)$ such that*

$$\limsup_{t\to\infty} \int_M |a(t+s)|\, ds = 0$$

for every compact subset M of $\mathbb{R}^N$.

For example, the class $Q_{SC}(\mathbb{R}^N)$ contains all functions $a \in L^\infty(\mathbb{R}^N)$ with

$$\lim_{R\to\infty} \text{ess sup}_{|x|\geq R} |a(x)| = 0$$

and, in particular, all compactly supported functions. The characteristic function of the set $\cup_{n\geq 2}[n - \frac{1}{n}, n + \frac{1}{n}] \subset \mathbb{R}$ is an example of a function in $Q_{SC}(\mathbb{R}^N)$ which does not vanish at infinity.

Theorem 1.2 *The following conditions are equivalent for a bounded measurable function a:*

(a) the operators BaI and aB are compact on $L^p(\mathbb{R}^N)$ for every $B \in \mathcal{C}_p$ and every $1 < p < \infty$,

(b) $a \in Q_{SC}(\mathbb{R}^N)$,

(c) There is a bounded open set $D \subset \mathbb{R}^N$ such that $\lim_{t\to\infty} \int_D |a(t+s)|\, ds = 0$.

Consequently, $Q_{SC}(\mathbb{R}^N)$ is a closed ideal in $L^\infty(\mathbb{R}^N)$.

The next goal is to characterize those functions $a \in L^\infty(\mathbb{R}^N)$ for which the *commutators* $aC(k) - C(k)aI$ are compact for every function $k \in L^1(\mathbb{R}^N)$. We start with defining two related subclasses of functions in $L^\infty(\mathbb{R}^N)$.

Definition 1.3 *Let $SO(\mathbb{R}^N)$ denote the set of all bounded continuous functions a on $\mathbb{R}^N$ such that, for every compact subset M of $\mathbb{R}^N$,*

$$\lim_{t\to\infty} \sup_{h\in M} |a(t) - a(t+h)| = 0.$$

The class $SO(\mathbb{R}^N)$ is a unital commutative C^*-subalgebra of $BUC(\mathbb{R}^N)$, the algebra of the bounded and uniformly continuous functions on $\mathbb{R}^N$. Functions in $SO(\mathbb{R}^N)$ are called *slowly oscillating* on $\mathbb{R}^N$. Examples of slowly oscillating functions are provided by the continuous functions which possess a finite limit at infinity and by the differentiable functions the derivative of which tends to zero at infinity.

Definition 1.4 *A function $a \in L^\infty(\mathbb{R}^N)$ belongs to the class $Q_C(\mathbb{R}^N)$ if, for every open and bounded subset M of $\mathbb{R}^N$, the function*

$$t \mapsto \int_M (a(t) - a(t+s))\, ds$$

lies in $Q_{SC}(\mathbb{R}^N)$.

The following result does not only solve the commutator problem; it moreover verifies the relation between the classes $Q_{SC}(\mathbb{R}^N)$, $SO(\mathbb{R}^N)$ and $Q_C(\mathbb{R}^N)$.

Theorem 1.5 *The following assertions are equivalent for $a \in L^\infty(\mathbb{R}^N)$:*

(a) the operators $BaI - aB$ are compact on $L^p(\mathbb{R}^N)$ for every $B \in \mathcal{C}_p$,

(b) the function a belongs to $Q_C(\mathbb{R}^N)$,

(c) the function a belongs to $Q_{SC}(\mathbb{R}^N) + SO(\mathbb{R}^N)$.

As a consequence one gets that $Q_C(\mathbb{R}^N) = Q_{SC}(\mathbb{R}^N) + SO(\mathbb{R}^N)$ is a unital commutative C^*-subalgebra of $L^\infty(\mathbb{R}^N)$ and that $Q_{SC}(\mathbb{R}^N)$ is a closed ideal of that algebra. Moreover, one can show that the intersection $Q_{SC}(\mathbb{R}^N) \cap SO(\mathbb{R}^N)$ consists of all continuous functions which tend to zero at infinity.

2 Band-dominated operators on l^p-spaces

Given a complex Banach space X, consider the Banach spaces $l^p(\mathbb{Z}^N, X)$ and $l^\infty(\mathbb{Z}^N, X)$ of all functions $f : \mathbb{Z}^N \to X$ such that

$$\|f\|_p^p := \sum_{x \in \mathbb{Z}^N} \|f(x)\|_X^p < \infty \quad \text{and} \quad \|f\|_\infty := \sup_{x \in \mathbb{Z}^N} \|f(x)\|_X < \infty,$$

respectively. Let E stand for one of the spaces $l^p(\mathbb{Z}^N, X)$ with $p \in (1, \infty)$. Every function $a \in l^\infty(\mathbb{Z}^N, L(X))$ gives rise to a multiplication operator on E on defining

$$(af)(x) := a(x)f(x), \quad x \in \mathbb{Z}^N.$$

We denote this operator by aI. Evidently, $aI \in L(E)$ and $\|aI\|_{L(E)} = \|a\|_\infty$. Finally, for $\alpha \in \mathbb{Z}^N$, let V_α refer to the shift operator

$$(V_\alpha f)(x) := f(x - \alpha), \quad x \in \mathbb{Z}^N,$$

which also belongs to $L(E)$ and has norm 1.

Definition 2.1 *A* band operator *on E is a finite sum of the form $\sum_\alpha a_\alpha V_\alpha$ where $\alpha \in \mathbb{Z}^N$ and $a_\alpha \in l^\infty(\mathbb{Z}^N, L(X))$. A* band-dominated operator *on E is the norm limit of a sequence of band operators.*

The band-dominated operators on E form a closed subalgebra of $L(E)$ which we denote by $\mathcal{A}_E$. One can show that an operator $A \in L(E)$ is band-dominated if and only if, for every function $\varphi \in BUC(\mathbb{R}^N)$,

$$\lim_{t \to 0} \|A\hat{\varphi}_{t,r}I - \hat{\varphi}_{t,r}A\|_{L(E))} = 0 \quad \text{uniformly with respect to } r \in \mathbb{R}^N$$

where, for r, t, $x \in \mathbb{R}^N$,

$$\varphi_{t,r}(x) := \varphi_t(x - r) \quad \text{and} \quad \varphi_t(x) := \varphi(tx) := \varphi(t_1x_1, \ldots, t_Nx_N)$$

and where $\hat{a}$ refers to the restriction of the function $a : \mathbb{R}^N \to \mathbb{C}$ onto $\mathbb{Z}^N$. For this and the following facts we refer to [8, 9].

For $n \geq 0$, define $\hat{P}_n : E \to E$ by

$$(\hat{P}_n f)(x) = \begin{cases} f(x) & \text{if} \quad |x|_\infty \leq n \\ 0 & \text{if} \quad |x|_\infty > n, \end{cases}$$

set $\hat{Q}_n := I - \hat{P}_n$, and let $\hat{\mathcal{P}}$ refer to the family $(\hat{P}_n)$.

Definition 2.2 *An operator $K \in L(E)$ is $\hat{\mathcal{P}}$-*compact *if*

$$\|K\hat{Q}_n\| \to 0 \quad \textit{and} \quad \|\hat{Q}_nK\| \to 0 \quad \textit{as}\; n \to \infty.$$

By $K(E, \hat{\mathcal{P}})$ we denote the set of all $\hat{\mathcal{P}}$-compact operators on E, and by $L(E, \hat{\mathcal{P}})$ the set of all operators $A \in L(E)$ for which both AK and KA are $\hat{\mathcal{P}}$-compact whenever K is $\hat{\mathcal{P}}$-compact.

It turns out that $L(E, \hat{\mathcal{P}})$ is a closed subalgebra of $L(E)$, $K(E, \hat{\mathcal{P}})$ a closed two-sided ideal of $L(E, \hat{\mathcal{P}})$, and $K(E, \hat{\mathcal{P}}) \subset \mathcal{A}_E \subset L(E, \hat{\mathcal{P}})$. Operators $A \in L(E, \hat{\mathcal{P}})$ for which the coset $A + K(E, \hat{\mathcal{P}})$ is invertible in the quotient algebra $L(E, \hat{\mathcal{P}})/K(E, \hat{\mathcal{P}})$ are called *$\hat{\mathcal{P}}$-Fredholm.* If X is a finite-dimensional space, then $L(E, \hat{\mathcal{P}}) = L(E)$, $K(E, \hat{\mathcal{P}})$ is the ideal of the compact operators on E, and the $\hat{\mathcal{P}}$-Fredholm operators are just the Fredholm operators in the common sense.

Let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}^N$ which tend to infinity.

Definition 2.3 *Let $A \in L(E, \hat{\mathcal{P}})$ and $h \in \mathcal{H}$. The operator $A_h \in L(E)$ is called the* limit operator of A with respect to h *if*

$$\lim_{n\to\infty} \|(V_{-h(n)}AV_{h(n)} - A_h)\hat{P}_m\| = \lim_{n\to\infty} \|\hat{P}_m(V_{-h(n)}AV_{h(n)} - A_h)\| = 0 \qquad (5)$$

for every $\hat{P}_m \in \hat{\mathcal{P}}$. The set $\sigma_{op}(A)$ of all limit operators of A is called the operator spectrum *of A.*

Let finally refer $\mathcal{L}_E^{rich}$ to the set of all operators $A \in L(E, \mathcal{P})$ enjoying the following property: every sequence $h \in \mathcal{H}$ possesses a subsequence g for which the limit operator A_g exists. Set $\mathcal{A}_E^{rich} := \mathcal{A}_E \cap \mathcal{L}_E^{rich}$. Then the main result of [9] can be stated as follows.

Theorem 2.4 *An operator $A \in \mathcal{A}_E^{rich}$ is $\hat{\mathcal{P}}$-Fredholm if and only if all of its limit operators are invertible and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_{op}(A)\} < \infty. \qquad (6)$$

The condition of *uniform* invertibility can be weakened by employing local techniques. To describe some typical ideas and results we have to introduce some more notations. Let S^{N-1} denote the unit sphere $\{\eta \in \mathbb{R}^N : |\eta| = 1\}$ where $|\eta|$ stands for the Euklidean norm of η. Given a 'radius' $R > 0$, a 'direction' $\eta \in S^{N-1}$, and a neighborhood $U \subseteq S^{N-1}$ of η, we set

$$W_{R,U} := \{z \in \mathbb{Z}^N : |z| > R \text{ and } z/|z| \in U, \} \qquad (7)$$

and we call $W_{R,U}$ a *neighborhood at infinity of* η. A sequence $h \in \mathcal{H}$ is said to *tend into the direction of* $\eta \in S^{N-1}$ if, for every neighborhood at infinity W of η, there is an m_0 such that $h(m) \in W$ for all $m \geq m_0$.

Definition 2.5 *Let $\eta \in S^{N-1}$ and $A \in L(E, \hat{\mathcal{P}})$.*

(a) The local operator spectrum $\sigma_\eta(A)$ of A at η *is the set of all limit operators A_h of A with respect to sequences h tending into the direction of η.*

(b) The operator A is locally invertible at η *if there are operators $B, C \in L(E, \hat{\mathcal{P}})$ and a neighborhood at infinity W of η such that*

$$BA\hat{\chi}_W I = \hat{\chi}_W AC = \hat{\chi}_W I$$

where $\hat{\chi}_W$ refers to the characteristic function of W.

Theorem 2.6 *Let $A \in \mathcal{A}_E^{rich}$ and $\eta \in S^{N-1}$. Then the operator A is locally invertible at η if and only if all limit operators in $\sigma_\eta(A)$ are invertible and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty.$$

Corollary 2.7 *An operator $A \in \mathcal{A}_E^{rich}$ is $\hat{\mathcal{P}}$-Fredholm if and only if all of its limit operators are invertible, and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty \quad \textit{for all } \eta \in S^{N-1}.$$

This result is indeed a generalization of Theorem 2.4: It does not require that the suprema are uniformly bounded with respect to η.

3 Band-dominated operators on $L^p(\mathbb{R}^N)$ and their discretizations

$\mathcal{P}$-Fredholmness. We start with adapting the notion of $\hat{\mathcal{P}}$-Fredholmness introduced in the previous section to the context of L^p-spaces.

Let P_n stand for the operator of multiplication by the characteristic function of the cube $[-n, n]^N$ acting on $L^p(\mathbb{R}^N)$, and set $\mathcal{P} := (P_n)_{n=1}^\infty$ and $Q_n := I - P_n$. Further we introduce the set $K(L^p(\mathbb{R}^N), \mathcal{P})$ of the $\mathcal{P}$-compact operators, i.e., of the operators $K \in L(L^p(\mathbb{R}^N))$ such that

$$\lim_{n\to\infty} \|KQ_n\| = \lim_{n\to\infty} \|Q_n K\| = 0,$$

and the set $L(L^p(\mathbb{R}^N), \mathcal{P})$ of all operators $A \in L(L^p(\mathbb{R}^N))$ such that AK and KA are $\mathcal{P}$-compact whenever K is $\mathcal{P}$-compact. Then $L(L^p(\mathbb{R}^N), \mathcal{P})$ is a closed unital subalgebra of $L(L^p(\mathbb{R}^N))$ which contains $K(L^p(\mathbb{R}^N), \mathcal{P})$ as its closed ideal. Further, since both the operators P_n and their adjoints converge strongly to the identity operators on $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, respectively, one gets that $K(L^p(\mathbb{R}^N), \mathcal{P})$ contains the ideal $K(L^p(\mathbb{R}^N))$ of the compact operators on $L^p(\mathbb{R}^N)$ (but $K(L^p(\mathbb{R}^N), \mathcal{P})$ is strictly larger than $K(L^p(\mathbb{R}^N))$ since the operators P_n are not compact).

Our earlier definitions of generalized Fredholmness, invertibility at infinity and local invertibility at infinity specify as follows to the present context.

Definition 3.1 *The operator $A \in L(L^p(\mathbb{R}^N), \mathcal{P})$ is* $\mathcal{P}$-Fredholm *if the coset $A + K(L^p(\mathbb{R}^N), \mathcal{P})$ is invertible in the quotient algebra $L(L^p(\mathbb{R}^N), \mathcal{P})/K(L^p(\mathbb{R}^N), \mathcal{P})$, that is if there exist operators $B, C \in L(L^p(\mathbb{R}^N), \mathcal{P})$ such that*

$$BA - I \in K(L^p(\mathbb{R}^N), \mathcal{P}) \quad \textit{and} \quad AC - I \in K(L^p(\mathbb{R}^N), \mathcal{P}). \tag{8}$$

Equivalently, an operator $A \in L(L^p(\mathbb{R}^N), \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if it is *invertible at infinity* in the sense that there exist an $m \in \mathbb{N}$ and operators $B, C \in L(L^p(\mathbb{R}^N), \mathcal{P})$ such that

$$BAQ_m = Q_m \quad \text{and} \quad Q_m AC = Q_m.$$

Local invertibility. There is also an adequate notion of local invertibility at an infinitely distant point $\eta \in S^{N-1}$. Given $R > 0$ and a neighborhood $U \subseteq S^{N-1}$ of η, we set

$$V_{R,U} := \{x \in \mathbb{R}^N : |x| > R \text{ and } x/|x| \in U\}$$

and call $V_{R,U}$ again a *neighborhood at infinity of* η. Then an operator A is called *locally invertible at* η if there exist a neighborhood V at infinity of η and operators $B, C \in L(L^p(\mathbb{R}^N), \mathcal{P})$ such that

$$BA\chi_V I = \chi_V I \quad \text{and} \quad \chi_V AC = \chi_V I.$$

Shifts and limit operators. For $\alpha \in \mathbb{Z}^N$, we consider the operator

$$U_\alpha : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad (U_\alpha f)(t) := f(t - \alpha)$$

of shift by α. In accordance with the definitions from Section 2, we call the operator A_h a *limit operator of* $A \in L(L^p(\mathbb{R}^N), \mathcal{P})$ with respect to the sequence $h \in \mathcal{H}$ if

$$\lim_{m\to\infty} \|(U_{-h(m)} A U_{h(m)} - A_h)P_m\| = \lim_{m\to\infty} \|P_m(U_{-h(m)} A U_{h(m)} - A_h)\| = 0$$

for every $P_m \in \mathcal{P}$. The set $\sigma_{op}(A)$ of all limit operators of A is the *operator spectrum* of A. Further we denote by $\mathcal{L}_p^{rich}$ the subalgebra of $L(L^p(\mathbb{R}^N), \mathcal{P})$ which consists of all operators with rich operator spectrum. The latter means for an operator $A \in L(L^p(\mathbb{R}^N), \mathcal{P})$, that every sequence $h \in \mathcal{H}$ has a subsequence g such that the limit operator A_g with respect to g exists.

Discretization. Let χ_0 denote the characteristic function of the cube $I_0 := [0, 1)^N$, and set $X := L^p(I_0)$ and $E := l^p(\mathbb{Z}^N, X)$. Then the mapping G which maps the function $f \in L^p(\mathbb{R}^N)$ to the sequence

$$Gf = ((Gf)_\alpha)_{\alpha \in \mathbb{Z}^N} \quad \text{with} \quad (Gf)_\alpha := \chi_0 U_{-\alpha} f \tag{9}$$

is an isometry from $L^p(\mathbb{R}^N)$ onto $l^p(\mathbb{Z}^N, X)$, the inverse of which is given by

$$G^{-1} : u = (u_\alpha)_{\alpha \in \mathbb{Z}^N} \mapsto \sum_{\alpha \in \mathbb{Z}^N} U_\alpha u_\alpha \chi_0 \tag{10}$$

where the series converges in the norm in $L^p(\mathbb{R}^N)$. Thus, the mapping

$$\Gamma : L(L^p(\mathbb{R}^N)) \to L(l^p(\mathbb{Z}^N, X)), \quad A \mapsto GAG^{-1}$$

is an isometric algebra isomorphism. Obviously, $\Gamma(P_m)$ is the projection $\hat{P}_m$, and $\Gamma(U_\alpha)$ is the shift V_α, both introduced in Section 2.

Proposition 3.2 *The isometry Γ maps the ideal $K(L^p(\mathbb{R}^N), \mathcal{P})$ onto $K(E, \hat{\mathcal{P}})$ and the algebra $L(L^p(\mathbb{R}^N), \mathcal{P})$ onto $L(E, \hat{P})$.*

Proof. Since

$$\|K - KP_n\| = \|\Gamma(K - KP_n)\| = \|\Gamma(K) - \Gamma(K)\hat{P}_n\|$$

and $\|K - P_n K\| = \|\Gamma(K) - \hat{P}_n \Gamma(K)\|$, we get $\Gamma(K(L^p(\mathbb{R}^N), \mathcal{P})) = K(E, \hat{\mathcal{P}})$. Similarly, the second assertion follows if one takes into account that an operator $A \in L(L^p(\mathbb{R}^N))$ belongs to $L(L^p(\mathbb{R}^N), \mathcal{P})$ if and only if

$$\|P_k A Q_n\| \to 0 \quad \text{and} \quad \|Q_n A P_k\| \to 0 \quad \text{as } n \to \infty$$

and that an analogous result holds for operators on E. ∎

A consequence is that an operator $A \in L(L^p(\mathbb{R}^N), \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if $\Gamma(A)$ is $\hat{\mathcal{P}}$-Fredholm. A similar result holds for the local invertibility at $\eta \in S^{N-1}$. However, here the situation is a little bit more involved since, if $V \subseteq \mathbb{R}^N$ is a neighborhood at infinity of η, then $\Gamma(\chi_V I) \neq \hat{\chi}_{V \cap \mathbb{Z}^N} I$ in general. Nevertheless, the local invertibility of A at η is equivalent to that of $\Gamma(A)$, which can be seen as follows. Given a neighborhood $V \subseteq \mathbb{R}^N$ at infinity of η, there is evidently a neighborhood $W \subseteq \mathbb{Z}^N$ at infinity of η such that $\Gamma(\chi_V I)\hat{\chi}_W I = \hat{\chi}_W I$. Thus, if $BA\chi_V I = \chi_V I$, then

$$\Gamma(B)\Gamma(A)\Gamma(\chi_V I) = \Gamma(\chi_V I),$$

and after multiplication by $\hat{\chi}_W I$ from the right hand side we get

$$\Gamma(B)\Gamma(A)\hat{\chi}_W I = \hat{\chi}_W I,$$

whence the local invertibility at η of $\Gamma(A)$. The reverse implication follows similarly.

The next result shows that also the limit operators behave nicely under discretization.

Proposition 3.3 *Let $A \in L(L^p(\mathbb{R}^N), \mathcal{P})$ and $h \in \mathcal{H}$. Then the limit operator A_h of A exists (with respect to $\mathcal{P}$) if and only if the limit operator $(\Gamma(A))_h$ of $\Gamma(A)$ exists (with respect to $\Gamma(\mathcal{P}) = \hat{\mathcal{P}}$), and*

$$\Gamma(A_h) = (\Gamma(A))_h. \tag{11}$$

In particular, A belongs to $\mathcal{L}_p^{rich}$ if and only if $\Gamma(A)$ belongs to $\mathcal{L}_E^{rich}$.

Proof. Let the limit operator A_h of A exist, i.e. let

$$\lim_{n\to\infty} \|(U_{-h(n)}AU_{h(m)} - A_h)P_m\| = \lim_{n\to\infty} \|P_m(U_{-h(n)}AU_{h(m)} - A_h)\| = 0$$

for all m. Since $\Gamma(U_\alpha) = V_\alpha$ and $\Gamma(P_m) = \hat{P}_m$, and since Γ is an isometrical algebra isomorphism, we conclude that

$$\lim_{n\to\infty} \|\Gamma((U_{-h(n)}AU_{h(m)} - A_h)P_m)\| = \lim_{n\to\infty} \|(V_{-h(n)}\Gamma(A)V_{h(m)} - \Gamma(A_h))\hat{P}_m\| = 0$$

and, analogously, $\hat{P}_m(\|(V_{-h(n)}\Gamma(A)V_{h(m)} - \Gamma(A_h))\| \to 0$ for every m. Thus, the limit operator of $\Gamma(A)$ with respect to h exists and (11) holds. The reverse implication follows analogously. ■

In particular, an operator B belongs to the operator spectrum of A if and only if the operator $\Gamma(B)$ belongs to the operator spectrum of $\Gamma(A)$. An analogous relation holds for the local operator spectra.

Band-dominated operators on $L^p(\mathbb{R}^N)$. The following definition is motivated by the characterization of band-dominated operators on $l^p(\mathbb{Z}^N, X)$ mentioned in Section 2.

Definition 3.4 *An operator $A \in L(L^p(\mathbb{R}^N))$ is* band-dominated *if, for every function $\varphi \in BUC(\mathbb{R}^N)$,*

$$\lim_{t\to 0} \|A\varphi_{t,r}I - \varphi_{t,r}A\|_{L(L^p(\mathbb{R}^N))} = 0 \quad \textit{uniformly with respect to } r \in \mathbb{R}^N. \tag{12}$$

The set of all band-dominated operators in $L(L^p(\mathbb{R}^N))$ will be denoted by $\mathcal{B}_p$, and we write $\mathcal{B}_p^{rich}$ instead of $\mathcal{B}_p \cap \mathcal{L}_p^{rich}$.

Clearly, $\mathcal{B}_p$ and $\mathcal{B}_p^{rich}$ are closed unital subalgebras of $L(L^p(\mathbb{R}^N))$, and the set $K(L^p(\mathbb{R}^N), \mathcal{P})$ is a closed two-sided ideal of both algebras. The latter can be checked, for example, by means of the following proposition.

Proposition 3.5 *$\Gamma(\mathcal{B}_p)$ coincides with the algebra $\mathcal{A}_E$ of the band-dominated operators on $E = l^p(\mathbb{Z}^N, L^p(I_0))$, and $\Gamma(\mathcal{B}_p^{rich}) = \mathcal{A}_E^{rich}$.*

Proof. If $A \in \mathcal{B}_p$ then, for every function $\varphi \in BUC(\mathbb{R}^N)$,

$$\lim_{t\to 0} \|[A, \varphi_{t,r}I]\|_{L(L^p(\mathbb{R}^N))} = 0$$

(with [., .] referring to the commutator) and, consequently,

$$\lim_{t\to 0} \|[\Gamma(A), \Gamma(\varphi_{t,r}I)]\|_{L(E)} = 0 \tag{13}$$

uniformly with respect to $r \in \mathbb{R}^N$. We claim that

$$\lim_{t\to 0} \|\hat{\varphi}_{t,r}I - \Gamma(\varphi_{t,r}I)\|_{L(E)} = 0 \tag{14}$$

uniformly with respect to $r \in \mathbb{R}^N$. Indeed,

$$\begin{aligned}
\sup_{r\in\mathbb{R}^N} \|(\hat{\varphi}_{t,r}I - \Gamma(\varphi_{t,r}I)\|_{L(E)} &= \sup_{r\in\mathbb{R}^N} \sup_{\alpha\in\mathbb{Z}^N} \sup_{x\in I_0} |\hat{\varphi}_{t,r}(\alpha) - (\Gamma(\varphi_{t,r}I)_\alpha)(x)| \\
&= \sup_{r\in\mathbb{R}^N} \sup_{\alpha\in\mathbb{Z}^N} \sup_{x\in I_0} |\varphi(t(\alpha - r)) - \varphi(t(x + \alpha - r))| \\
&\leq \sup_{\beta\in\mathbb{R}^N} \sup_{x\in I_0} |\varphi(t\beta) - \varphi(t(x + \beta))| \to 0
\end{aligned}$$

as $t \to 0$ due to the uniform continuity of φ. By (14) and (13),

$$\lim_{t\to 0} \|[\Gamma(A),\, \hat{\varphi}_{t,r} I]\|_{L(E)} = 0$$

uniformly with respect to $r \in \mathbb{R}^N$. Thus, $\Gamma(\mathcal{B}_p) \subseteq \mathcal{A}_E$. The reverse inclusion follows analogously. The second assertion is a consequence of the first one, together with Proposition 3.3. ∎

As an immediate consequence of Theorems 2.4 and 2.6 and of Propositions 3.3 and 3.5 we finally get the following result.

Theorem 3.6 *Let $A \in \mathcal{B}_p^{rich}$. Then the operator A is*
(a) locally invertible at point $\eta \in S^{N-1}$ if and only if all limit operators $A_h \in \sigma_\eta(A)$ are uniformly invertible.
(b) invertible at infinity if and only if, for every $\eta \in S^{N-1}$, all limit operators $A_h \in \sigma_\eta(A)$ are uniformly invertible.

4 Fredholmness of convolution type operators

Now we will apply the results of the preceding sections to examine the Fredholm properties of operators on $L^p(\mathbb{R}^N)$ which are constituted by convolution operators with kernels in $L^1(\mathbb{R}^N)$ and by operators of multiplication by functions in suitable subclasses of $L^\infty(\mathbb{R}^N)$.

4.1 Operators of convolution type

Given a subalgebra $\mathcal{E}$ of $L^\infty(\mathbb{R}^N)$, we let $\mathcal{A}(\mathcal{E},\, \mathcal{C}_p)$ denote the smallest closed subalgebra of $L(L^p(\mathbb{R}^N))$ which contains the identity operator, all compact operators, and all operators of the form

$$aKbI \quad \text{where} \quad a,\, b \in \mathcal{E} \text{ and } K \in \mathcal{C}_p, \tag{15}$$

and we call the elements of $\mathcal{A}(L^\infty(\mathbb{R}^N),\, \mathcal{C}_p)$ *convolution type operators*. Thus, every convolution type operator can be approximated as closely as desired by operators of the form

$$A := \gamma I + \sum \prod a_{ij} K_{ij} b_{ij} I + T \tag{16}$$

where $a_{ij},\, b_{ij} \in L^\infty(\mathbb{R}^N)$, $K_{ij} \in \mathcal{C}_p$, $\gamma \in \mathbb{C}$ and T is compact, and where the sum and all products are finite.

Proposition 4.1 $\mathcal{A}(L^\infty(\mathbb{R}^N),\, \mathcal{C}_p) \subseteq \mathcal{B}_p$.

The proof is based on the following norm estimate which is known as Schur's lemma ([16], Appendix A, Proposition 5.1).

Proposition 4.2 *Let l be a measurable function on $\mathbb{R}^N \times \mathbb{R}^N$ with*

$$M_1 := \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} |l(x,y)|dy < \infty \quad \text{and} \quad M_2 := \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} |l(x,y)|dx < \infty.$$

Then the operator

$$(Lu)(x) := \int_{\mathbb{R}^N} l(x,y)u(y)dy, \quad x \in \mathbb{R}^N$$

acts boundedly on $L^p(\mathbb{R}^N)$, and $\|L\|_{L(L^p(\mathbb{R}^N))} \le M_1^{1/q} M_2^{1/p}$.

Proof of Proposition 4.1. The algebra $\mathcal{B}_p$ contains the ideal $K(L^p(\mathbb{R}^N),\, \mathcal{P})$ and, hence, the ideal of the compact operators. Clearly, this algebra also contains all operators of multiplication by a bounded measurable function. Thus, and since $\mathcal{B}_p$ is a closed algebra, the result will follow once we have shown that $\mathcal{B}_p$ also contains a dense subset of $\mathcal{C}_p$. Actually, we will check that

$$\lim_{t\to 0} \sup_{h\in\mathbb{R}^N} \|[\varphi_{t,h} I,\, C(k)]\| = 0 \tag{17}$$

for every function $k \in L^1(\mathbb{R}^N)$ with compact support and every $\varphi \in BUC(\mathbb{R}^N)$. For definiteness, let the support of k be contained in a ball with center 0 and radius R. Since

$$([\varphi_{t,h} I,\, C(k)]u)(x) = \int_{\mathbb{R}^N} (\varphi_{t,h}(x) - \varphi_{t,h}(y))\, k(x-y) u(y)\, dy,$$

Proposition 4.2 implies

$$\begin{aligned} \|[\varphi_{t,h} I, C(k)]\|_{L(L^p)} &\le \|k\|_1 \sup_{x,\, y\in\mathbb{R}^N: |x-y|\le R} |\varphi_{t,h}(x) - \varphi_{t,h}(y)| \\ &= \|k\|_1 \sup_{x,\, y\in\mathbb{R}^N: |x-y|\le R} |\varphi(t(x-h)) - \varphi(t(y-h))|. \end{aligned}$$

For $|x-y| \le R$, we have

$$|t(x-h) - t(y-h)| \le |t|R \to 0 \quad \text{as} \quad t \to 0.$$

Since $\varphi \in BUC(\mathbb{R}^N)$, we obtain (17). ■

A striking property of operators of convolution type is that their $\mathcal{P}$-Fredholmness coincides with common Fredholmness.

Proposition 4.3 *An operator in $\mathcal{A}(L^\infty(\mathbb{R}^N),\, \mathcal{C}_p)$ is Fredholm if and only if it is $\mathcal{P}$-Fredholm.*

Proof. Let $\mathcal{J}$ refer to the closed ideal of $\mathcal{A}(L^\infty(\mathbb{R}^N),\, \mathcal{C}_p)$ which contains all operators in $\mathcal{C}_p$ and all compact operators. It is easy to check that, whenever $J \in \mathcal{J}$, the operator JP_k is compact for every k. Indeed, every operator $J \in \mathcal{J}$ can be approximated as closely as desired by a sum of a compact operator T and of products of operators of the form $aKbI$ where a and b are bounded measurable functions and $K \in \mathcal{C}_p$. Then TP_k is compact, and the compactness of $aKbP_k = aKP_kbI$ follows from Theorem 1.2.

Since P_k fails to be compact, we have $I \notin \mathcal{J}$, and the algebra $\mathcal{A}(L^\infty(\mathbb{R}^N),\, \mathcal{C}_p)$ decomposes into the direct sum $\mathbb{C}I + \mathcal{J}$. In particular, every operator A in this algebra can be uniquely written as $\gamma_A I + K_A$ where $\gamma_A \in \mathbb{C}$ and $K_A \in \mathcal{J}$, and it turns out that the mapping $A \mapsto \gamma_A$ is a continuous algebra homomorphism.

In the next step we will show that

$$\mathcal{J} \cap K(L^p(\mathbb{R}^N),\, \mathcal{P}) = K(L^p(\mathbb{R}^N)).$$

The inclusion $\supseteq$ follows from the definitions. If, conversely, $J \in \mathcal{J} \cap K(L^p(\mathbb{R}^N),\, \mathcal{P})$, then JP_k is compact for every k as we have just seen. On the other hand, since $J \in K(L^p(\mathbb{R}^N),\, \mathcal{P})$, one has $\|J - JP_k\| \to 0$ as $k \to \infty$. Thus, being the norm limit of compact operators, the operator J is compact.

Since $K(L^p(\mathbb{R}^N)) \subseteq K(L^p(\mathbb{R}^N),\, \mathcal{P})$, it is clear that every Fredholm operator is also $\mathcal{P}$-Fredholm. Let, conversely, $A \in \mathcal{A}(L^\infty(\mathbb{R}^N),\, \mathcal{C}_p)$ be a $\mathcal{P}$-Fredholm operator. Then there are an operator $L' \in \mathcal{B}_p$ and an operator $T \in K(L^p(\mathbb{R}^N),\, \mathcal{P})$ such that $L'A = I + T$. We claim that $\gamma_A \ne 0$. Contrary to what we want, assume that

$\gamma_A = 0$. Then $A \in \mathcal{J}$. Choose $m > 0$ and $n \in \mathbb{Z}^N$ such that $\|P_m U_{-n} T U_n P_m\| < 1/2$ (which can be done since T can be approximated by an operator of the form $P_k T$ as closely as desired). Then, by Neumann series, the right hand side of

$$P_m U_{-n} L' A U_n P_m = P_m + P_m U_{-n} T U_n P_m$$

is an invertible operator on the range of P_m, whence

$$P_m = (P_m + P_m U_{-n} T U_n P_m)^{-1} P_m U_{-n} L' A U_n P_m. \tag{18}$$

Since $U_n P_m U_{-n}$ is the operator of multiplication by a compactly supported function, the operator $A U_n P_m = A(U_n P_m U_{-n}) U_n$ and, hence, the operator on the right hand side of (18) are compact. But P_m is not compact, and this contradiction proves the claim.

Now write A as $\gamma_A I + K_A$ and set $L := K_A L' + I$. Then

$$LA - \gamma_A I = \gamma_A L' A - A L' A + A - \gamma_A I = (\gamma_A I - A)(L' A - I).$$

Since $L' A - I \in K(L^p(\mathbb{R}^N), \mathcal{P})$ and $\gamma_A I - A = K_A \in \mathcal{J}$, the operator $LA - \gamma_A I$ is compact. Similarly, one shows that $AR - \gamma_A I$ is compact for a certain operator $R \in \mathcal{B}_p$. Hence, and because of $\gamma_A \neq 0$, the operator A is Fredholm. ∎

Corollary 4.4 $\mathcal{A}(L^\infty(\mathbb{R}^N), \mathcal{C}_p) \cap K(L^p(\mathbb{R}^N), \mathcal{P}) = K(L^p(\mathbb{R}^N))$.

There are operators in $\mathcal{A}(L^\infty(\mathbb{R}^N), \mathcal{C}_p)$ which do not possess a rich operator spectrum. The next result identifies a subalgebra of $\mathcal{A}(L^\infty(\mathbb{R}^N), \mathcal{C}_p) \cap \mathcal{B}_p^{rich}$ which contains sufficiently many interesting operators.

Proposition 4.5 $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p) \subseteq \mathcal{B}_p^{rich}$.

Proof. It is easy to see that every compact operator T belongs to $\mathcal{B}_p^{rich}$ and that the limit operator T_h exists with respect to every sequence $h \in \mathcal{H}$ and is equal to zero.

Next, let $a \in BUC(\mathbb{R}^N)$, and let h be a sequence which tends to infinity. The family of all functions $x \mapsto a(x + h(m))$ is bounded and equicontinuous on every compact subset M of $\mathbb{R}^N$. Hence, by the Arzelà-Ascoli theorem, there are a subsequence g of h and a continuous bounded function a_h on $\mathbb{R}^N$ such that, for every compact $M \subset \mathbb{R}^N$,

$$\lim_{m\to\infty} \sup_{x\in M} |a(x + g(m)) - a_h(x)| = 0.$$

Thus, the operators $U_{-g(m)} a U_{g(m)}$ of multiplication by the function $x \mapsto a(x+g(m))$ converge $*$-strongly to the operator of multiplication by the function a_h.

Let A be an operator of the form (16), but with $a_{ij}, b_{ij} \in BUC$. As we have just seen, given a sequence h tending to infinity, we can choose a subsequence g of h such that the operators $U_{-g(m)} a_{ij} U_{g(m)}$ and $U_{-g(m)} b_{ij} U_{g(m)}$ converge $*$-strongly to certain multiplication operators $(a_{ij})_h I$ and $(b_{ij})_h I$, respectively. Then

$$\begin{aligned} &U_{-g(m)} A U_{g(m)} P_k \\ &\quad = (\gamma I + \sum \prod (U_{-g(m)} a_{ij} U_{g(m)}) K_{ij} (U_{-g(m)} b_{ij} U_{g(m)}) + U_{-g(m)} T U_{g(m)}) P_k \\ &\quad = \gamma P_k + \sum \prod (U_{-g(m)} a_{ij} U_{g(m)}) K_{ij} P_k (U_{-g(m)} b_{ij} U_{g(m)}) + U_{-g(m)} T U_{g(m)} P_k \end{aligned}$$

converges in the norm to

$$\gamma P_k + \sum \prod (a_{ij})_h K_{ij} P_k (b_{ij})_h) I = (\gamma I + \sum \prod (a_{ij})_h K_{ij} (b_{ij})_h I) P_k$$

for every P_k and that the operators $K_{ij} P_k$ are compact due to Theorem 1.2).

Hence, all operators of the form (16) with $a_{ij}, b_{ij} \in BUC$ possess a rich operator spectrum. Since the operators of this form lie densely in $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$, and since $\mathcal{B}_p^{rich}$ is a closed algebra, this yields the assertion. ∎

4.2 Fredholmness

Due to Proposition 4.5, the operators in $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ are subject to Theorem 3.6. In combination with Proposition 4.3, we obtain the following result.

Theorem 4.6 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$. Then A is*
(a) Fredholm (i.e. locally invertible at infinity) if and only if all limit operators of A are uniformly invertible.
(b) locally invertible at the infinitely distant point $\eta \in S^{N-1}$ if and only if all operators in the local operator spectrum $\sigma_\eta(A)$ are uniformly invertible.

Corollary 4.7 *An operator $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ is a Fredholm operator if and only if, for each point $\eta \in S^{N-1}$, all operators in $\sigma_\eta(A)$ are uniformly invertible.*

We are going to specialize these results to operators with coefficients in certain subalgebras of $L^\infty(\mathbb{R}^N)$.

Slowly oscillating coefficients. Since slowly oscillating functions are uniformly continuous, one has $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p) \subseteq \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$, and Theorem 4.6 and its corollary apply to operators in the algebra $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$. Limit operators of operators in $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$ are of a particularly simple form which allows us to check their invertibility effectively via (4).

Proposition 4.8 *Every limit operator of an operator in $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$ lies in $\mathbb{C}I + \mathcal{C}_p$.*

Proof. Every operator in $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$ can be uniformly approximated by operators of the form (16) where $a_{ij}, b_{ij} \in SO(\mathbb{R}^N)$. If $K \in \mathcal{C}_p$ then, clearly, the limit operator K_h exists with respect to every sequence $h \in \mathcal{H}$, and $K_h = K$. Further, if T is compact, then the limit operator T_h also exists with respect to every sequence $h \in \mathcal{H}$, and $T_h = 0$. Thus, in view of the proof of Proposition 4.5, it remains to check the following: If $a \in SO(\mathbb{R}^N)$, and if $h \in \mathcal{H}$ is a sequence such that the operators of multiplication $U_{-h(n)} a U_{h(n)}$ converge *-strongly to $a_h I$ as $n \to \infty$, then a_h is a constant function. This can be done as follows. Let $a \in SO(\mathbb{R}^N)$. Then

$$\lim_{k\to\infty} (a(x' + h(k)) - a(x'' + h(k))) = 0$$

for all sequences h tending to infinity and for all $x', x'' \in \mathbb{R}^N$. Hence, if h is a sequence such that the limit operator $(aI)_h$ exists, then $\lim_{k\to\infty} a(x + h_k)$ is independent of $x \in \mathbb{R}^N$. ■

Corollary 4.9 *Let A be an operator of the form (16) with $a_{ij}, b_{ij} \in SO(\mathbb{R}^N)$. Then A is Fredholm if and only if all limit operators of A are invertible.*

Thus, the uniformity of the invertibility is not required.

Proof. We conclude from the previous proposition that every limit operator of A is a linear combination of the operators $\prod_{j=1}^{n_i} K_{ij}$ with $i = 1, \ldots, n$. Thus, $\sigma_{op}(A)$ lies in a finite dimensional subspace of $L(L^p(\mathbb{R}^N))$. Then a simple compactness argument yields the assertion. ■

Remark. The algebra $\mathcal{A}(Q_C(\mathbb{R}^N), \mathcal{C}_p)$ which is apparently larger than the algebra $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$ actually coincides with the latter algebra. Indeed, by Theorem 1.5, every operator aK with $a \in Q_C(\mathbb{R}^N)$ and $K \in \mathcal{C}_p$ is the sum of an operator $a_1 K$ with $a_1 \in SO(\mathbb{R}^N)$ and an operator $a_2 K$ with $a_2 \in Q_{SC}(\mathbb{R}^N)$. Since slowly oscillating functions are uniformly continuous and since $a_2 K$ is compact (Theorem 1.2), one has $aK \in \mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$.

Coefficients stabilizing at infinity. Theorem 4.6 and its corollary attain their most simple form for operators with coefficients which stabilize at infinity in the following sense. This class has been introduced in [4] in case $N = 1$.

Definition 4.10 *We say that the function* $a \in L^\infty(\mathbb{R}^N)$ stabilizes at infinity *if, for every infinitely distant point* $\eta \in S^{N-1}$*, there is a constant* $y \in \mathbb{C}$ *such that, for every* $\varepsilon > 0$*, there exists a neighborhood* $U = U_{\eta,\,\varepsilon}$ *at infinity of* η *such that*

$$\text{mes}\,\{x \in U_{\eta,\,\varepsilon} : |a(x) - y| > \varepsilon\} < \varepsilon. \tag{19}$$

The class of all functions which stabilize at infinity will be denoted by $L^\infty_{stab}(\mathbb{R}^N)$.

If a stabilizes at infinity and η is an infinitely distant point, then the constant y which satisfies (19) is uniquely determined. We denote it by $\hat{a}(\eta)$.

Lemma 4.11 *Let* $a \in L^\infty_{stab}(\mathbb{R}^N)$ *and* $\eta \in S^{N-1}$ *be an infinitely distant point. Then* $|\hat{a}(\eta)| \le \|a\|_\infty$.

Proof. Let $\varepsilon > 0$ and choose a neighborhood U of infinity such that

$$\text{mes}\,\{x \in U : |a(x) - \hat{a}(\eta)| > \varepsilon\} < \varepsilon.$$

Then

$$\text{mes}\,\{x \in U : |\,|a(x)| - |\hat{a}(\eta)|\,| > \varepsilon\} < \varepsilon.$$

Since the measure of U is infinite, there is a subset $M \subset U$ of measure 1 such that

$$|a(x)| - \varepsilon < |\hat{a}(\eta)| < |a(x)| + \varepsilon \quad \text{for all } x \in M.$$

This yields the assertion. ■

Theorem 4.12 $L^\infty_{stab}(\mathbb{R}^N)$ *is a* C^**-subalgebra of* $Q_C(\mathbb{R}^N)$.

Proof. First we will show that $L^\infty_{stab}(\mathbb{R}^N)$ is closed in $L^\infty(\mathbb{R}^N)$. Let $a_n \in L^\infty_{stab}(\mathbb{R}^N)$ and $a \in L^\infty(\mathbb{R}^N)$ such that $\lim \|a_n - a\|_\infty = 0$. Fix $\varepsilon > 0$, and choose $n_0 \in \mathbb{N}$ such that

$$\|a_n - a_m\|_\infty < \varepsilon \quad \text{for all } n,\, m \ge n_0.$$

Further, let $U_{\eta,\,\varepsilon,\,n}$ be a neighborhood at infinity of η such that

$$\text{mes}\,\{x \in U_{\eta,\,\varepsilon,\,n} : |a_n(x) - \widehat{a_n}(\eta)| > \varepsilon\} < \varepsilon,$$

and set

$$U'_{\eta,\,\varepsilon,\,n} := \{x \in U_{\eta,\,\varepsilon,\,n} : |a_n(x) - \widehat{a_n}(\eta)| \le \varepsilon\}.$$

Then, for $x \in U'_{\eta,\,\varepsilon,\,n} \cap U'_{\eta,\,\varepsilon,\,m}$ and $m,\, n > n_0$,

$$|\widehat{a_n}(\eta) - \widehat{a_m}(\eta)| \le |\widehat{a_n}(\eta) - a_n(x)| + |a_n(x) - a_m(x)| + |a_m(x) - \widehat{a_m}(\eta)| \le 3\,\varepsilon.$$

Thus, $(\widehat{a_n}(\eta))_{n\in\mathbb{N}}$ is a Cauchy sequence, and we let $\hat{a}(\eta)$ denote its limit.

Now we fix $n > n_0$ such that

$$\|a_n - a\|_\infty < \varepsilon/3 \quad \text{and} \quad |\widehat{a_n}(\eta) - \hat{a}(\eta)| < \varepsilon/3.$$

The estimate

$$|a_n(x) - \hat{a_n}(\eta)| \ge |a(x) - \hat{a}(\eta)| - |a(x) - a_n(x)| - |a_n(x) - \widehat{a_n}(\eta)|$$

implies that $|a_n(x) - \widehat{a_n}(\eta)| > \varepsilon/3$ whenever $|a(x) - \hat{a}(\eta)| > \varepsilon$. Since a_n stabilizes at infinity, there is a neighborhood $U_{\eta,\,\varepsilon/3,\,n}$ such that

$$\text{mes}\,\{x \in U_{\eta,\,\varepsilon/3,\,n} : |a_n(x) - \widehat{a_n}(\eta)| > \varepsilon/3\} < \varepsilon/3.$$

Thus,

$$\operatorname{mes}\{x \in U_{\eta,\,\varepsilon/3,\,n} : |a(x) - \hat{a}(\eta)| > \varepsilon\} < \varepsilon/3 < \varepsilon,$$

whence $a \in L^\infty_{stab}(\mathbb{R}^N)$.

In the next step we show that $L^\infty_{stab}(\mathbb{R}^N)$ is a *-algebra. The symmetry is obvious. Let $a, b \in L^\infty_{stab}(\mathbb{R}^N)$, and let η be an infinitely distant point. We choose neighborhoods at infinity $U_{\eta,\,\varepsilon/2,\,a}$ and $U_{\eta,\,\varepsilon/2,\,b}$ of η such that

$$\operatorname{mes}\{x \in U_{\eta,\,\varepsilon/2,\,a} : |a(x) - \hat{a}(\eta)| > \varepsilon/2\} < \varepsilon/2 \tag{20}$$

and

$$\operatorname{mes}\{x \in U_{\eta,\,\varepsilon/2,\,b} : |b(x) - \hat{b}(\eta)| > \varepsilon/2\} < \varepsilon/2. \tag{21}$$

Set $W_\eta := U_{\eta,\,\varepsilon/2,\,a} \cap U_{\eta,\,\varepsilon/2,\,b}$. Then W_η is a neighborhood at infinity of η, and it follows from

$$\begin{aligned}&\{x \in W_\eta : |a(x) + b(x) - \hat{a}(\eta) - \hat{b}(\eta)| > \varepsilon\}\\ &\quad \subseteq \{x \in W_\eta : |a(x) - \hat{a}(\eta)| > \varepsilon/2\} \cup \{x \in W_\eta : |b(x) - \hat{b}(\eta)| > \varepsilon/2\}\end{aligned}$$

and from (20), (21) that

$$\operatorname{mes}\{x \in W_\eta : |a(x) + b(x) - \hat{a}(\eta) - \hat{b}(\eta)| > \varepsilon\} < \varepsilon.$$

Thus, $a + b \in L^\infty_{stab}(\mathbb{R}^N)$ and

$$\widehat{(a+b)}(\eta) = \hat{a}(\eta) + \hat{b}(\eta) \quad \text{for all } \eta \in S^{N-1}.$$

In order to show that $ab \in L^\infty_{stab}(\mathbb{R}^N)$, too, we can assume that $a, b \neq 0$ (otherwise the assertion is obvious). Choose $m \in \mathbb{N}$ such that $m\|a\|_\infty > 1$ and $m\|b\|_\infty > 1$. Given an infinitely distant point η and $\varepsilon > 0$ a, choose neighborhoods at infinity of η such that

$$\operatorname{mes}\{x \in U_{\eta,\,a} : |a(x) - \hat{a}(\eta)| > \varepsilon/(2m\|b\|_\infty)\} < \varepsilon/(2m\|b\|_\infty)$$

and

$$\operatorname{mes}\{x \in U_{\eta,\,b} : |b(x) - \hat{b}(\eta)| > \varepsilon/(2m\|a\|_\infty)\} < \varepsilon/(2m\|a\|_\infty).$$

Set $W_\eta := U_{\eta,\,a} \cap U_{\eta,\,b}$. Then W_η is a neighborhood at infinity of η, and

$$\begin{aligned}&\operatorname{mes}\{x \in W_\eta : |(ab)(x) - \hat{a}(\eta)\hat{b}(\eta)| > \varepsilon\}\\
&\quad = \operatorname{mes}\{x \in W_\eta : |(a(x) - \hat{a}(\eta))b(x) + \hat{a}(\eta))(b(x) - \hat{b}(\eta))| > \varepsilon\}\\
&\quad \le \operatorname{mes}\{x \in W_\eta : |a(x) - \hat{a}(\eta)|\,\|b\|_\infty + \|a\|_\infty\,|b(x) - \hat{b}(\eta)| > \varepsilon\}\\
&\quad \le \operatorname{mes}\{x \in W_\eta : |a(x) - \hat{a}(\eta)|\,\|b\|_\infty > \varepsilon/2\}\\
&\qquad\quad + \operatorname{mes}\{x \in W_\eta : |b(x) - \hat{b}(\eta)|\,\|a\|_\infty > \varepsilon/2\}\\
&\quad \le \operatorname{mes}\{x \in U_{\eta,\,a} : |a(x) - \hat{a}(\eta)|\,\|b\|_\infty > \varepsilon/(2m)\}\\
&\qquad\quad + \operatorname{mes}\{x \in U_{\eta,\,b} : |b(x) - \hat{b}(\eta)|\,\|a\|_\infty > \varepsilon/(2m)\}\\
&\quad \le \operatorname{mes}\{x \in U_{\eta,\,a} : |a(x) - \hat{a}(\eta)| > \varepsilon/(2m\|b\|_\infty)\}\\
&\qquad\quad + \operatorname{mes}\{x \in U_{\eta,\,b} : |b(x) - \hat{b}(\eta)| > \varepsilon/(2m\|a\|_\infty)\}\\
&\quad < \varepsilon/(2m\|b\|_\infty) + \varepsilon/(2m\|a\|_\infty) < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Consequently, $ab \in L^\infty_{stab}(\mathbb{R}^N)$ and

$$\widehat{(ab)}(\eta) = \hat{a}(\eta)\,\hat{b}(\eta) \quad \text{for all } \eta \in S^{N-1}.$$

It remains to show the inclusion $L^\infty_{stab}(\mathbb{R}^N) \subseteq Q_C(\mathbb{R}^N)$. Thus, if $a \in L^\infty_{stab}(\mathbb{R}^N)$, we have to show that, for every open bounded set $M \subset \mathbb{R}^N$, there is an open bounded set $D \subset \mathbb{R}^N$ such that

$$\lim_{t\to\infty} \int_D \left| \int_M (a(t+h) - a(t+h+s))\, ds \right| dh = 0 \tag{22}$$

(Definition 1.4). Let $M \subset \mathbb{R}^N$ be open and bounded, choose D as the open unit disk in $\mathbb{R}^N$, and let $d > 0$ be the radius of a disk with center 0 which contains $M + D$. Let further $\varepsilon > 0$. Then, for every infinitely distant point η, there is a neighborhood at infinity of η such that

$$\text{mes}\,\{x \in U_{\eta,\varepsilon} : |a(x) - \hat{a}(\eta)| > \varepsilon\} < \varepsilon.$$

Each neighborhood $U_{\eta,\varepsilon}$ is of the form

$$U_{\eta,\varepsilon} = \{y \in \mathbb{R}^N : |y| > R_{\eta,\varepsilon} \text{ and } y/|y| \in W_{\eta,\varepsilon}\}$$

where $R_{\eta,\varepsilon} \ge 0$ and $W_{\eta,\varepsilon} \subseteq S^{N-1}$ is an open neighborhood of η. In particular, $\{W_{\eta,\varepsilon}\}_{\eta\in S^{N-1}}$ is an open cover of the unit sphere, from which we can choose a finite subcover $\{W_{\eta_i,\varepsilon}\}_{i=1}^k$. Set

$$R_0 := \max\{R_{\eta_i,\varepsilon} : i = 1, \ldots, k\} + d.$$

Further, since the function $f : S^{N-1} \to \mathbb{R}^N$,

$$f(x) := \max\{\text{dist}\,(x, S^{N-1} \setminus W_{\eta_i,\varepsilon}) : i = 1, \ldots, k\},$$

is positive for every x (every x belongs to one of the sets $W_{\eta_i,\varepsilon}$) and continuous on the compact set S^{N-1}, there is a $\delta > 0$ such that $f(x) \ge \delta$ for all $x \in S^{N-1}$. Thus, for every $x \in S^{N-1}$, there is an $i \in \{1, \ldots, k\}$ such that

$$x \in W_{\eta_i,\varepsilon} \quad \text{and} \quad \text{dist}\,(x, \partial W_{\eta_i,\varepsilon}) \ge \delta.$$

Consequently, there is an $R_1 \ge R_0$ such that, for every $y \in \mathbb{R}^N$ with $|y| \ge R_1$, there is an $i \in \{1, \ldots, k\}$ such that

$$y \in U_{\eta_i,\varepsilon} \quad \text{and} \quad \text{dist}\,(y, \partial U_{\eta_i,\varepsilon}) \ge d.$$

Let now $t \in \mathbb{R}^N$ with $|t| \ge R_1$. By what we have just seen, there is an $i \in \{1, \ldots, k\}$ such that $t + D$ and $t + M + D$ are contained in $U_{\eta_i,\varepsilon}$. Thus,

$$\text{mes}\,\{x \in t + D : |a(x) - \hat{a}(\eta)| > \varepsilon\} < \varepsilon$$

and

$$\text{mes}\,\{x \in t + D + M : |a(x) - \hat{a}(\eta)| > \varepsilon\} < \varepsilon.$$

This implies

$$\begin{aligned}
&\int_D \left| \int_M (a(t+h) - a(t+h+s))\, ds \right| dh \\
&\quad \le \int_D \int_M |a(t+h) - \hat{a}(\eta)|\, ds\, dh + \int_D \int_M |a(t+h+s) - \hat{a}(\eta)|\, ds\, dh \\
&\quad \le \text{mes}\, D \int_{t+M} |a(h) - \hat{a}(\eta)|\, dh + \text{mes}\, D \int_{t+D+M} |a(h) - \hat{a}(\eta)|\, dh \\
&\quad \le \text{mes}\, D\, (\text{mes}\, M \cdot \varepsilon + 2\varepsilon\|a\|_\infty) + \text{mes}\, D\, (\text{mes}\,(D+M) \cdot \varepsilon + 2\varepsilon\|a\|_\infty) \\
&\quad \le \varepsilon\, \text{mes}\, D\, (\text{mes}\, M + \text{mes}\,(D+M) + 4\|a\|_\infty),
\end{aligned}$$

whence the assertion (22). ∎

Proposition 4.13 *Let $a \in L^\infty_{stab}(\mathbb{R}^N)$, and let h be a sequence which tends to infinity into the direction of $\eta \in S^{N-1}$. Then*

$$V_{-h(n)} a V_{h(n)} \to \hat{a}(\eta) I \quad \textit{strongly on } L^p(\mathbb{R}^N) \textit{ as } n \to \infty.$$

Proof. Given $\varepsilon > 0$, we find a neighborhood $U_{\eta,\varepsilon}$ at infinity of η such that

$$\text{mes}\,\{x \in U_{\eta,\varepsilon} : |a(x) - \hat{a}(\eta)| > \varepsilon\} < \varepsilon.$$

Let f be a continuous function with compact support. Then

$$\|(V_{-h(n)} a V_{h(n)} - \hat{a}(\eta)) f\|_p = \|(a - \hat{a}(\eta)) V_{h(n)} f\|_p.$$

Clearly, there exists an n_0 such that $\text{supp}\,(V_{h(n)} f) \subset U_{\eta,\varepsilon}$ for all $n \geq n_0$. Thus, if $n \geq n_0$, then

$$\|(V_{-h(n)} a V_{h(n)} - \hat{a}(\eta)) f\|_p \leq \varepsilon \|f\|_p + 2\|a\|_\infty \|u\|_\infty \varepsilon^{1/p}.$$

This proves the strong convergence on a dense subset of $L^p(\mathbb{R}^N)$. Since the operators $V_{-h(n)} a V_{h(n)}$ are uniformly bounded, we get the assertion. ∎

An obvious consequence of this proposition is that the local operator spectrum $\sigma_\eta(A)$ for operators $A \in \mathcal{A}(L^\infty_{stab}(\mathbb{R}^N), \mathcal{C}_p)$ is a singleton for every infinitely distant point $\eta \in S^{N-1}$, say $\sigma_\eta(A) = \{A_\eta\}$. Moreover, every limit operator A_η belongs to $\mathbb{C}I + \mathcal{C}_p$ since $\mathcal{A}(L^\infty_{stab}(\mathbb{R}^N), \mathcal{C}_p)$ is a subalgebra of $\mathcal{A}(SO(\mathbb{R}^N), \mathcal{C}_p)$, and by Proposition 4.8. Thus, the invertibility of A_η can be effectively checked via (4).

Corollary 4.14 *An operator $A \in \mathcal{A}(L^\infty_{stab}(\mathbb{R}^N), \mathcal{C}_p)$ is Fredholm if and only if every limit operator A_η (with $\eta \in S^{N-1}$) of A is invertible.*

5 Compressions of operators of convolution type

In this section we are going to study the Fredholm properties of compressions of operators of convolution type. If A is a linear bounded operator on $L^p(\mathbb{R}^N)$ and D is a measurable subset of $\mathbb{R}^N$, then the *compression of A onto D* is the operator

$$\chi_D A \chi_D I|_{L^p(\mathbb{R}^N)} : L^p(D) \to L^p(D).$$

The archetypical example is the *Wiener-Hopf operator* $W(k)$ on $L^p(\mathbb{R}^+)$ which is the compression of the convolution operator $\gamma I + C(k)$ with $k \in L^1(\mathbb{R})$ onto $\mathbb{R}^+$. Thus,

$$W(k) = \chi_+ (\gamma I + C(k)) \chi_+ I|_{L^p(\mathbb{R}^+)},$$

where χ_+ refers to the characteristic function of $\mathbb{R}^+$. Clearly, this operator is Fredholm on $L^p(\mathbb{R}^+)$ is and only if the operator $\gamma I + \chi_+ C(k) \chi_+ I$ is Fredholm on $L^p(\mathbb{R})$. Let f be the function with $f(x) = 0$ if $x < 0$, $f(x) = x$ on $[0, 1]$ and $f(x) = 1$ for $x > 1$. Then the function $\chi_+ - f$ has a compact support. Thus, the operator $\chi_+ C(k) \chi_+ I - f C(k) f I$ is compact on $L^p(\mathbb{R})$, and the operator $W(k)$ is Fredholm on $L^p(\mathbb{R}^+)$ if and only if the operator $\gamma I + f C(k) f I$ is Fredholm on $L^p(\mathbb{R})$. The latter operator is subject to Corollary 4.9 which says that this operator (hence, the Wiener-Hopf operator $W(k)$) is Fredholm if and only if the convolution operator $\gamma I + C(k)$ is invertible. This simple reduction is no longer possible for for compressions of operators onto more involved sets.

5.1 Compressions of operators in $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$

Let $1 < p < \infty$, and let D be a measurable subset of $\mathbb{R}^N$ whose associated multiplication operator belongs to $\mathcal{B}_p^{rich}$. Such subsets will be called *rich* .

We will consider compressions of operators in $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ onto the rich set D. Clearly, the compression $\chi_D A \chi_D I$ is invertible (Fredholm) on $L^p(D)$ if and only if its extension $(1 - \chi_D)I + \chi_D A \chi_D I$ is an invertible (Fredholm) operator on $L^p(\mathbb{R}^N)$. Each such extension can be considered as an element of the algebra $\mathcal{A}(BUC(\mathbb{R}^N), \chi_D, \mathcal{C}_p)$ which is the smallest closed subalgebra of the algebra $L(L^p(\mathbb{R}^N))$ which contains the algebra $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ as well as the multiplication operator $\chi_D I$. As a consequence of Proposition 4.5 we get

$$\mathcal{A}(BUC(\mathbb{R}^N), \chi_D, \mathcal{C}_p) \subseteq \mathcal{B}_p^{rich}, \tag{23}$$

and from Proposition 4.3 we conclude that an operator $A \in \mathcal{A}(BUC(\mathbb{R}^N), \chi_D, \mathcal{C}_p)$ is Fredholm if and only if it is $\mathcal{P}$-Fredholm. Thus, Theorem 3.6 (b) implies the following result.

Theorem 5.1 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$, and let D be a rich subset of $\mathbb{R}^N$. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if, for each point $\eta \in S^{N-1}$, all limit operators in $\sigma_\eta((1 - \chi_D)I + \chi_D A \chi_D I)$ are uniformly invertible on $L^p(\mathbb{R}^N)$.*

In the following subsections we will give some examples of unbounded rich domains D for which the limit operators of $\chi_D I$ can be explicitely calculated and for which, thus, explicit criteria for the Fredholmness of the compressions of operators from $\mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ onto D can be derived.

5.2 Compressions to a half space

Given a non-zero vector $a \in \mathbb{R}^N$, consider the half space

$$\mathbf{H}(a) := \{x \in \mathbb{R}^N : \langle x, a \rangle > 0\}. \tag{24}$$

Let further $h \in \mathcal{H}$ be a sequence which tends to infinity into the direction of $\eta \in S^{N-1}$. We distinguish several cases.

- If $\langle \eta, a \rangle > 0$, then $\langle h(n), a \rangle \to +\infty$, and the limit operator of $\chi_{\mathbf{H}(a)} I$ exists and is equal to the identity operator.
- If $\langle \eta, a \rangle < 0$, then $\langle h(n), a \rangle \to -\infty$, and the limit operator of $\chi_{\mathbf{H}(a)} I$ exists and is equal to the zero operator.
- If $\langle \eta, a \rangle = 0$, then h has a subsequence $g \in \mathcal{H}$ such that either the numbers $\langle g(n), a \rangle$ tend to $+\infty$, or to $-\infty$, or to a finite limit $b_g \in \mathbb{R}$. In each of these cases, the limit operator of $\chi_{\mathbf{H}(a)} I$ with respect to g exists, and it is equal to the identity operator in the first case, to the zero operator in the second case and to the operator of multiplication by the characteristic function of the shifted half space

  $$\mathbf{H}(a, b_g) := \{x \in \mathbb{R}^N : \langle x, a \rangle > -b_g\}$$

  in the third case.

Let $\mathcal{H}_\eta(A)$ stand for the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^{N-1}$ and for which the limit operator A_h exists. Further, we denote by $\mathcal{H}_{\eta,\infty}(A)$ and $\mathcal{H}_{\eta,b}(A)$ the set of all sequences $h \in \mathcal{H}_\eta(A)$ such that $\langle h(n), a \rangle \to \infty$ and $\langle h(n), a \rangle \to b \in \mathbb{R}^N$, respectively. Then Theorem 5.1 gives the following result.

Theorem 5.2 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ and $D = \mathbf{H}(a)$ with $a \in \mathbb{R}^N \setminus \{0\}$. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the following conditions are satisfied:*

(a) for each point $\eta \in S^{N-1}$ with $\langle \eta, a \rangle > 0$, the set $\{A_h : h \in \mathcal{H}_\eta(A)\}$ of limit operators of A is uniformly invertible.

(b) for each point $\eta \in S^{N-1}$ with $\langle \eta, a \rangle = 0$, the set $\{A_h : h \in \mathcal{H}_{\eta,\infty}(A)\}$ of limit operators of A is uniformly invertible.

(c) for each point $\eta \in S^{N-1}$ with $\langle \eta, a \rangle = 0$ and each $b \in \mathbb{R}^N$, the set

$$\{(1 - \chi_{\mathbf{H}(a,b)})I + \chi_{\mathbf{H}(a,b)} A_h \chi_{\mathbf{H}(a,b)} I : h \in \mathcal{H}_{\eta,b}(A)\}$$

of extended compressions of limit operators of A is uniformly invertible.

5.3 Compressions to curved half spaces

Let $N > 1$ and $f \in BUC(\mathbb{R}^{N-1})$. We consider the *curved half space*

$$\mathbf{P}(f) := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > f(x')\} \subseteq \mathbb{R}^N. \tag{25}$$

Let further $h \in \mathcal{H}$ be a sequence which tends to infinity into the direction of $\eta = (\eta', \eta_N) \in S^{N-1} \subset \mathbb{R}^{N-1} \times \mathbb{R}$. Again, we distinguish several cases.

- If $\eta_N > 0$, then the limit operator of $\chi_{\mathbf{P}(f)} I$ exists and is equal to the identity operator.
- If $\eta_N < 0$, then the limit operator of $\chi_{\mathbf{P}(f)} I$ exists and is equal to the zero operator.
- Now let $\eta_N = 0$. Then h has a subsequence $g \in \mathcal{H}$ such that either the numbers $g(n)_N$ tend to $+\infty$, or to $-\infty$, or that the sequence $(g(n)_N)_{n \geq 1}$ is bounded. In the first two cases, the limit operator of $\chi_{\mathbf{P}(f)} I$ with respect to g exists, and it is equal to the identity operator in the first case and to the zero operator in the second case. In the third case, there exists a subsequence k of g, a real number b_k and a function $f_k : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

  $$\lim_{n \to \infty} k(n)_N = b_k \quad \text{and} \quad \lim_{n \to \infty} f(x' + k(n)') = f_k(x')$$

  in the sense of the uniform convergence on compact subsets of $\mathbb{R}^{N-1}$. In this case, the limit operator of $\chi_{\mathbf{P}(f)} I$ exists, too, and it is equal to the operator of multiplication by the characteristic function of

  $$\mathbf{P}(f_k - b_k) = \{x \in \mathbb{R}^N : x_N > f_k(x') - b_k\}$$

  in the third case.

Let $\mathcal{H}_\eta(A)$ stand for the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^{N-1}$ and for which the limit operator A_h exists. Further, given a real number b and a function $g : \mathbb{R}^{N-1} \to \mathbb{R}$, we denote by $\mathcal{H}_{\eta,\infty}(A)$ and $\mathcal{H}_{\eta,g,b}(A)$ the set of all sequences $h \in \mathcal{H}_\eta(A)$ such that $h(n)_N \to \infty$ and

$$h(n)_N \to b \quad \text{and} \quad f(x' + h(n)') \to g(x')$$

uniformly on compact subsets of $\mathbb{R}^{N-1}$, respectively. If the set $\mathcal{H}_{\eta,g,b}(A)$ is not empty, then we call g a *limit function* with respect to η. Then Theorem 5.1 implies the following result.

Theorem 5.3 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ and $D = \mathbf{P}(f)$ with $f \in BUC(\mathbb{R}^{N-1})$. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the following conditions are satisfied:*

(a) for each point $\eta \in S^{N-1}$ with $\eta_N > 0$, the set $\{A_h : h \in \mathcal{H}_\eta(A)\}$ of limit operators of A is uniformly invertible.

(b) for each point $\eta \in S^{N-1}$ with $\eta_N = 0$, the set $\{A_h : h \in \mathcal{H}_{\eta, \infty}(A)\}$ of limit operators of A is uniformly invertible.

(c) for each point $\eta \in S^{N-1}$ with $\eta_N = 0$, each $b \in \mathbb{R}$, and each limit function $g : \mathbb{R}^{N-1} \to \mathbb{R}$, the set

$$\{(1 - \chi_{\mathbf{P}(g-b)})I + \chi_{\mathbf{P}(g-b)} A_h \chi_{\mathbf{P}(g-b)} I : h \in \mathcal{H}_{\eta, g, b}(A)\}$$

of extended compressions of limit operators of A is uniformly invertible.

This result gets a particular simple form if $f \in SO(\mathbb{R}^N)$. In the setting of assertion (c) of the theorem, this hypothesis implies that all functions g are constant (their possible values are just the partial limits of $f(x')$ as $x' \to \infty$). Thus, all possible limit domains $\mathcal{P}(g - b)$ are (uncurved) half spaces.

5.4 Compressions to curved layers

Let again $N > 1$, and let $f_1, f_2 \in BUC(\mathbb{R}^{N-1})$ be such that $f_1(x') < f_2(x')$ for all $x' \in \mathbb{R}^{N-1}$. Then we call the set

$$\mathbf{L}(f_1, f_2) := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : f_1(x') < x_N < f_2(x')\} \tag{26}$$

a *curved layer*. Let $h \in \mathcal{H}$ be a sequence which tends to infinity into the direction of $\eta \in S^{N-1}$. If $\eta_N \neq 0$, then the limit operator of $\chi_{\mathbf{L}(f_1, f_2)} I$ with respect to h exists and it is equal to 0. The same happens if $\eta_N = 0$ and the sequence $(h(n)_N)_{n \geq 1}$ tends to $\pm\infty$. Thus, the only non-trivial case is when $\eta_N = 0$ and the sequence $(h(n)_N)_{n \geq 1}$ is bounded. Then, as in the previous subsection, there is a subsequence k of h, a real number b_k as well as functions $f_{1k}, f_{2k} : \mathbb{R}^{N-1} \to \mathbb{R}$ such that the limit operator of $\chi_{\mathbf{L}(f_1, f_2)} I$ with respect to k exists and is equal to $\chi_{\mathbf{L}(f_{1k} - b_k, f_{2k} - b_k)} I$.

Let again $\mathcal{H}_\eta(A)$ stand for the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^{N-1}$ and for which the limit operator A_h exists, and denote by $\mathcal{H}_{\eta, g_1, g_2, b}(A)$ the set of all sequences $h \in \mathcal{H}_\eta(A)$ such that

$$h(n)_N \to b \quad \text{and} \quad f_i(x' + h(n)') \to g_i(x') \quad (i = 1, 2)$$

uniformly on compact subsets of $\mathbb{R}^{N-1}$.

Theorem 5.4 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ and $D = \mathbf{L}(f_1, f_2)$ with functions f_1, f_2 in $BUC(\mathbb{R}^{N-1})$ and $f_1 < f_2$. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if, for each point $\eta \in S^{N-1}$ with $\eta_N = 0$, each $b \in \mathbb{R}$, and all limit functions $g_1, g_2 : \mathbb{R}^{N-1} \to \mathbb{R}$, the set*

$$\{(1 - \chi_{\mathbf{L}(g_1 - b, g_2 - b)})I + \chi_{\mathbf{L}(g_1 - b, g_2 - b)} A_h \chi_{\mathbf{L}(g_1 - b, g_2 - b)} I : h \in \mathcal{H}_{\eta, g_1, g_2, b}(A)\}$$

of extended compressions of limit operators of A is uniformly invertible.

If $f_1, f_2 \in SO(\mathbb{R}^{N-1})$, then the functions f_{1k}, f_{2k} are constant, and $\mathbf{L}(f_{1k} - b_k, f_{2k} - b_k)$ is a usual layer bounded by two parallel planes.

Corollary 5.5 *In addition to the hypothesis from Theorem 5.4, let*

$$\lim_{x' \to \infty} (f_1(x') - f_2(x')) = 0.$$

Then all limit operators of $\chi_{\mathbf{L}(f_1, f_2)} I$ are zero, and the compression of A onto $\mathbf{L}(f_1, f_2)$ is Fredholm on $L^p(\mathbf{L}(f_1, f_2))$.

5.5 Compressions to curved cylinders

Let $N > 1$, $\Omega \subset \mathbb{R}^{N-1}$ be a bounded domain, and $f \in BUC(\mathbb{R})$ a positive function, and consider the *curved cylinder*

$$\mathbf{Z}_\Omega(f) := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x' \in f(x_N)\Omega\}. \tag{27}$$

Let $h \in \mathcal{H}$. If $h(n)' \to \infty$, then the limit operator of $\chi_{\mathbf{Z}_\Omega(f)} I$ with respect to h exists and is equal to the zero operator. Thus, nontrivial limit operators of $\chi_{\mathbf{Z}_\Omega(f)} I$ with respect to h exist only if the sequence $(h(n)')_{n\geq 1}$ is bounded and $h(n)_N \to \pm\infty$. In this case, there is a subsequence k of h, a point $b_k \in \mathbb{Z}^{N-1}$, and a function f_k on $\mathbb{R}$ such that

$$g(n)' \to b_k \quad \text{and} \quad f(x_N + k(n)_N) \to f_k(x_N) \quad \text{as } n \to \infty$$

uniformly on compact subsets of $\mathbb{R}$. Then the limit operator of $\chi_{\mathbf{Z}_\Omega(f)} I$ with respect to the sequence k exists, and it is equal to the operator of multiplication by the characteristic function of the shifted curved cylinder

$$\mathbf{Z}_\Omega(f_k, b_k) := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x' \in f_k(x_N)\Omega - b_k\}.$$

Let $\mathcal{H}_\eta(A)$ denote the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^{N-1}$ and for which the limit operator A_h exists, and write $\mathcal{H}_{\eta, g, b}(A)$ for the set of all sequences $h \in \mathcal{H}_\eta(A)$ such that

$$h(n)_N \to b \in \mathbb{Z}^{N-1} \quad \text{and} \quad f(x_N + h(n)_N) \to g(x_N)$$

uniformly on compact subsets of $\mathbb{R}$.

Theorem 5.6 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^N), \mathcal{C}_p)$ and $D = \mathbf{Z}_\Omega(f)$ with $f \in BUC(\mathbb{R})$. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if, for each point $\eta \in S^{N-1}$ with $\eta' = 0$, each $b \in \mathbb{Z}^{N-1}$, and all limit functions $g : \mathbb{R} \to \mathbb{R}$, the set*

$$\{(1 - \chi_{\mathbf{Z}_\Omega(g, b)})I + \chi_{\mathbf{Z}_\Omega(g, b)} A_h \chi_{\mathbf{Z}_\Omega(g, b)} I : h \in \mathcal{H}_{\eta, g, b}(A)\}$$

of extended compressions of limit operators of A is uniformly invertible.

If $f \in SO(\mathbb{R})$, then the function f_g is constant and, thus, $\mathbf{Z}_\Omega(f_g, b)$ is a usual straight cylinder.

Corollary 5.7 *In addition to the hypothesis from Theorem 5.6, let the ends of the cylinder be cuspidal, i.e. let*

$$\lim_{x_N \to \pm\infty} f(x_N) = 0.$$

Then all limit operators of $\chi_{\mathbf{Z}_\Omega(f)} I$ are zero, and the compression of A onto $\mathbf{Z}_\Omega(f)$ is Fredholm on $L^p(\mathbf{Z}_\Omega(f))$.

5.6 Compressions to cones with smooth cross section

Let $\Omega \subseteq \mathbb{R}^N$ be an open domain with C^1-boundary $\partial\Omega$ in case $N \geq 2$ or an open interval in $\mathbb{R}^1$. By $\mathbf{C}_\Omega$, we denote the cone in $\mathbb{R}^{N+1}$ generated by Ω,

$$\mathbf{C}_\Omega := \{(y, y_{N+1}) \in \mathbb{R}^N \times [0, \infty) : y \in y_{N+1}\Omega\}. \tag{28}$$

Given $x \in \mathbb{R}^N$, let $\eta_x \in S^N$ be the point which lies on the ray in $\mathbb{R}^{N+1}$ starting at the origin and passing through the point $(x, 1)$, i.e.

$$\eta_x = \frac{(x, 1)}{\sqrt{\|x\|^2 + 1}}.$$

Let $h \in \mathcal{H}$ be a sequence which tends to infinity into the direction of $\eta \in S^N$. Again there are two trivial cases: If η is not of the form η_x with some $x \in \overline{\Omega}$, then the limit operator of $\chi_{\mathbf{C}\Omega} I$ exists and is equal to the zero operator. If $\eta = \eta_x$ with $x \in \Omega$, then the limit operator of $\chi_{\mathbf{C}\Omega} I$ exists, too, and is equal to the identity operator.

Let now $x \in \partial\Omega$ and $\eta = \eta_x$. We denote by $T_x\Omega$ the tangential space and by ν_x the interior normal unit vector to $\partial\Omega$ at x. Further, we write $\mathbb{H}_x$ for the closed half space in $\mathbb{R}^N$ which is bounded by $T_x\Omega$ and for which ν_x is an interior normal unit vector to $\partial\mathbb{H}_x$ at x. Finally, we let $\mathbf{H}_x$ refer to the half space in $\mathbb{R}^{N+1}$ which is generated by $\mathbb{H}_x$,

$$\mathbf{H}_x := \{(y, y_{N+1}) \in \mathbb{R}^N \times \mathbb{R} : y \in \mathbb{H}_x + (y_{N+1} - 1)x\}.$$

Further, we write the sequence h as

$$h(n) := \alpha_n(\nu_x, 0) + (r_n, 0) + \beta_n(x, 1) \tag{29}$$

where $r_n \in T_x\Omega$ and $\alpha_n, \beta_n \in \mathbb{R}$. The following lemma claims the conditions under which the sequence (29) tends to infinity in the direction of η_x.

Lemma 5.8 *The sequence h defined by* (29) *tends to infinity in in the direction of η_x if and only if $\beta_n \to +\infty$ and*

$$\alpha_n/\beta_n \to 0 \quad \textit{and} \quad r_n/\beta_n \to 0 \quad \textit{as} \quad n \to \infty. \tag{30}$$

Proof. The sequence h tends to infinity if and only if

$$|\alpha_n|^2 + \|r_n\|^2 + |\beta_n|^2 \to \infty, \tag{31}$$

and then it converges in the direction of η_x if and only if

$$\frac{(\alpha_n\nu_x + r_n + \beta_n x, \beta_n)}{\sqrt{\|\alpha_n\nu_x + r_n + \beta_n x\|^2 + |\beta_n|^2}} \to \frac{(x, 1)}{\sqrt{\|x\|^2 + 1}}. \tag{32}$$

The convergence of the last component of (32) tells us that $\beta_n > 0$ for all sufficiently large n. Thus, (31) implies

$$\frac{(\frac{\alpha_n}{\beta_n}\nu_x + \frac{r_n}{\beta_n} + x, 1)}{\sqrt{\|\frac{\alpha_n}{\beta_n}\nu_x + \frac{r_n}{\beta_n} + x\|^2 + 1}} \to \frac{(x, 1)}{\sqrt{\|x\|^2 + 1}}. \tag{33}$$

From the convergence of the last component of (33) we conclude that

$$\left\| \frac{\alpha_n}{\beta_n}\nu_x + \frac{r_n}{\beta_n} + x \right\| \to \|x\|.$$

This implies for the first component of (33) that

$$\frac{\alpha_n}{\beta_n}\nu_x + \frac{r_n}{\beta_n} + x \to x$$

whence (30) since $\nu_x \perp r_n$. Writing (31) as

$$\beta_n^2 \left(\left| \frac{\alpha_n}{\beta_n} \right|^2 + \left\| \frac{r_n}{\beta_n} \right\|^2 + 1 \right) \to \infty$$

and taking into account (30), we finally get $\beta_n \to +\infty$. The reverse implications can be checked similarly. ∎

In order to compute the limit operators into the direction of η_x for $x \in \partial\Omega$, we assume for simplicity that $x = 0$ (which can be reached by shifting Ω) and that $T_x\Omega = T_0\Omega = \mathbb{R}^{N-1} \times \{0\}$ (which can be reached by rotating the shifted Ω). Then, since Ω has a C^1-boundary, there is an open neighborhood $U \subseteq \mathbb{R}^{N-1}$ of 0, an open interval $I \subseteq \mathbb{R}$ which contains 0, and a continuously differentiable function $g : U \to I$ such that

$$\partial\Omega \cap (U \times I) = \{(x,\, g(x)) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in U\}$$

and

$$\Omega \cap (U \times I) = \{(x,\, x_N) \in U \times I : x_N > g(x)\}.$$

Thus, if $\beta > 0$, then the part of the boundary of $\beta\Omega$ which lies in $\beta U \times \beta I$ is just the graph of the function

$$\beta U \to \beta I, \quad x \mapsto \beta g(x/\beta).$$

Let h be as in (29), and assume that the limit

$$\delta^* := \lim(\beta_n g(r_n/\beta_n) - \alpha_n) \in \mathbb{R} \cup \{\pm\infty\}$$

exists (otherwise we pass to a suitable subsequence of h). Let further $d > 0$ and $K_d^N := [-d,\, d]^N$, and set $\mathbf{C}_{n,\Omega} := V_{-h(n)}\mathbf{C}_\Omega$. We consider the intersection of the shifted cone $\mathbf{C}_{n,\Omega}$ with $\mathbb{R}^N \times \{0\}$ and identify this intersection with a subset of $\mathbb{R}^N$. Since $(y+r_n)/\beta_n \in U$ for all $y \in K_d^{N-1}$ and for all sufficiently large n, the boundary of $\mathbf{C}_{n,\Omega} \cap (\mathbb{R}^N \times \{0\})$ can be locally described as the graph of the function

$$G_n : K_d^{N-1} \to \mathbb{R}, \quad y \mapsto \beta_n g((y + r_n)/\beta_n) - \alpha_n.$$

Then, for every $y \in K_d^{N-1}$, we have

$$\lim(G_n(y) - \delta^*) = \lim(G_n(y) - G_n(0))$$

with

$$\begin{aligned} |G_n(y) - G_n(0)| &\leq \max_{\xi\in[0,\, y]} \|G'(\xi)\|\, \|y - 0\| \\ &= \max_{\xi\in[0,\, y]} \|g'((\xi + r_n)/\beta_n)\|\, \|y\|. \end{aligned}$$

Since g is continuously differentiable with $g'(0) = 0$, and since

$$\|(\xi + r_n)/\beta_n\| \leq (d + \|r_n\|)/\beta_n \to 0$$

by Lemma 5.8, we conclude that $G_n(y) \to \delta^*$ for every $y \in K_d^{N-1}$. Thus, if $(y,\, y_N) \in K_d^N$, then

$$\chi_{\mathbf{C}_{n,\Omega}\cap(\mathbb{R}^N\times\{0\})}(y) \to \begin{cases} 1 & \text{if } \; y_N > \delta^* \\ 0 & \text{if } \; y_N < \delta^* \end{cases}$$

An analogous result holds of the sequence (β_n) is replaced by $(\beta_n + \beta')$ with $\beta' \in [-d,\, d]$. This shows that

$$\chi_{\mathbf{C}_{n,\Omega}}(y) \to \begin{cases} \chi_{\mathbb{R}^{N+1}}(y) = y & \text{if } \; \delta^* = -\infty \\ \chi_{\mathbf{H}_x + \delta^*(\nu_x,0)}(y) & \text{if } \; \delta^* \in \mathbb{R} \\ \chi_\emptyset(y) = 0 & \text{if } \; \delta^* = +\infty \end{cases}$$

almost everywhere on K_d^{N+1}. By the dominated convergence theorem, this implies that

$$\chi_{\mathbf{C}_{n,\Omega}}\, \chi_{K_d^{N+1}} \to \begin{cases} \chi_{K_d^{N+1}} & \text{if } \; \delta^* = -\infty \\ \chi_{\mathbf{H}_x + \delta^*(\nu_x,0)}\, \chi_{K_d^{N+1}} & \text{if } \; \delta^* \in \mathbb{R} \\ 0 & \text{if } \; \delta^* = +\infty \end{cases}$$

with respect to the L^1-norm and, hence, also with respect to every L^p-norm with $1 \le p < \infty$ (the occurring functions take values in $\{-1, 0, 1\}$ almost everywhere). Since d is arbitrary, this finally yields that

$$\chi_{\mathbf{C}_{n,\Omega}} I \to \begin{cases} I & \text{if} \quad \delta^* = -\infty \\ \chi_{\mathbf{H}_x + \delta^*(\nu_x, 0)} I & \text{if} \quad \delta^* \in \mathbb{R} \\ 0 & \text{if} \quad \delta^* = +\infty \end{cases}$$

strongly on $L^p(\mathbb{R}^{N+1})$.

Given $A \in \mathcal{A}(BUC(\mathbb{R}^{N+1}), \mathcal{C}_p(\mathbb{R}^{N+1}))$, let $\mathcal{H}_\eta(A)$ denote the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^N$ and for which the limit operator A_h exists. Further, write $\mathcal{H}_{\eta,-\infty}(A)$ and $\mathcal{H}_{\eta,\delta^*}(A)$ with $\delta^* \in \mathbb{R}$ for the set of all sequences $h \in \mathcal{H}_\eta(A)$ with

$$\lim(\beta_n g(r_n/\beta_n) - \alpha_n) = -\infty \quad \text{and} \quad \lim(\beta_n g(r_n/\beta_n) - \alpha_n) = \delta^*,$$

respectively. Finally, we abbreviate the shifted half space $\mathbf{H}_x + \delta^*(\nu_x, 0)$ to $\mathbf{H}_{x,\delta^*}$. Then Theorem 5.1 has the following consequence.

Theorem 5.9 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^{N+1}), \mathcal{C}_p(\mathbb{R}^{N+1}))$ and $D = \mathbf{C}_\Omega$ with $\Omega \in \mathbb{R}^N$ an open domain with C^1 boundary. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the following conditions are satisfied:*

(a) for each point $x \in \Omega$, the set $\{A_h : h \in \mathcal{H}_{\eta_x}(A)\}$ of limit operators of A is uniformly invertible.

(b) for each point $x \in \partial\Omega$, the set $\{A_h : h \in \mathcal{H}_{\eta_x,-\infty}(A)\}$ of limit operators of A is uniformly invertible.

(c) for each point $x \in \partial\Omega$, the set

$$\{(1 - \chi_{\mathbf{H}_{x,\delta^*}})I + \chi_{\mathbf{H}_{x,\delta^*}} A_h \chi_{\mathbf{H}_{x,\delta^*}} I : h \in \mathcal{H}_{\eta,\delta^*}(A), \ \delta^* \in \mathbb{R}\}$$

of extended compressions of limit operators of A is uniformly invertible.

We still mention some special situations in which the conditions of Theorem 5.9 take a very simple form.

Let $A \in \mathcal{A}(SO(\mathbb{R}^{N+1}), \mathcal{C}_p(\mathbb{R}^{N+1}))$. Then all limit operators of A belong to $\mathbb{C}I + \mathcal{C}_p(\mathbb{R}^{N+1})$. In this case, the invertibility of the compressions in condition (c) can be effectively checked. Let, for example, A_h be the operator $\gamma I + C(k)$ with $\gamma \in \mathbb{C}$ and $k \in L^1(\mathbb{R}^{N+1})$. Since $C(k)$ is shift invariant, the corresponding compression (c) is invertible if and only if the operator

$$(1 - \chi_{\mathbf{H}_x})I + \chi_{\mathbf{H}_x}(\gamma I + C(k))\chi_{\mathbf{H}_x} I \tag{34}$$

is invertible. Further, given an orthogonal mapping S on $\mathbb{R}^{N+1}$, we write R_S for the rotation operator $(R_S f)(x) = f(Sx)$, and for $k \in L^1(\mathbb{R}^{N+1})$, we let k_S be the function $k_S(x) = k(S^T x)$ with S^T referring to the transposed of S. Then convolution operators are rotation invariant in the sense that

$$C(k)R_S = R_S C(k_S).$$

Thus, if we choose S such that it rotates $\mathbf{H}_x$ to the half space $\mathbf{H} := \{(x_1, x) \in \mathbb{R} \times \mathbb{R}^N : x_1 \ge 0\}$, then the compression (34) is invertible if and only if the operator

$$(1 - \chi_{\mathbf{H}})I + \chi_{\mathbf{H}}(\gamma I + C(k_S))\chi_{\mathbf{H}} I \tag{35}$$

is invertible. Finally, the compression (35) is invertible if and only if the operator

$$(1 - \chi_{\mathbf{H}})I + (\gamma I + C(k_S))\chi_{\mathbf{H}} I$$

is invertible. This follows easily from the identities $AP+Q = (PAP+Q)(I+QAP)$ and $(I+QAP)^{-1} = I-QAP$ which hold for arbitrary operators A and idempotents P, Q with $P+Q = I$.

The same results hold if $C(k)$ is replaced by an arbitrary operator in $\mathcal{C}_p(\mathbb{R}^{N+1})$. For the invertibility of the resulting compressions, one has the following result from [14] (Theorem 1.4).

Theorem 5.10 *Let $B \in \mathbb{C}I + \mathcal{C}_p(\mathbb{R}^{N+1})$. Then the operator $(1-\chi_{\mathbf{H}})I + B\chi_{\mathbf{H}}I$ is invertible on $L^p(\mathbb{R}^{N+1})$ is and only if the operator B is invertible on $L^p(\mathbb{R}^{N+1})$.*

Note once more that the invertibility of $B \in \mathbb{C}I + \mathcal{C}_p(\mathbb{R}^{N+1})$ can be effectively checked via (4).

With these remarks, we get the following corollaries to Theorem 5.9.

Corollary 5.11 *Let $A \in \mathcal{A}(SO(\mathbb{R}^{N+1}), \mathcal{C}_p(\mathbb{R}^{N+1}))$ and $D = \mathbf{C}_\Omega$ with $\Omega \in \mathbb{R}^N$ an open domain with C^1 boundary. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if, for each point $x \in \overline{\Omega}$, the set $\{A_h : h \in \mathcal{H}_{\eta_x}(A)\}$ of limit operators of A (= the local operator spectrum at η_x) is uniformly invertible.*

Corollary 5.12 *Let $A \in \mathcal{A}(L^\infty_{stab}(\mathbb{R}^{N+1}), \mathcal{C}_p(\mathbb{R}^{N+1}))$ and $D = \mathbf{C}_\Omega$ with $\Omega \in \mathbb{R}^N$ an open domain with C^1 boundary. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if, for each point $x \in \overline{\Omega}$, the limit operator A_{η_x} of A is invertible.*

Remark. Let $f : [0, \infty) \to \mathbb{R}$ be a slowly oscillating function, and let $\Omega \in \mathbb{R}^N$ be an open domain with C^1 boundary. We consider the slowly oscillating cone,

$$\mathbf{C}_{\Omega,f} := \{(y, y_{N+1}) \in \mathbb{R}^N \times [0, \infty) : y \in (y_{N+1} + f(y_{N+1}))\Omega\}. \tag{36}$$

In a similar way as above, one can show that the limit operators of the multiplication operator $\chi_{\mathbf{C}_{\Omega,f}}I$ are the same as in case of the unperturbed cone $\mathbf{C}_\Omega$, and that the analogue of Theorem 5.9 holds.

5.7 Compressions to cones with edges

Here we are going to consider compressions of convolution operators to cones which are allowed to have a finite number of edges. For simplicity, we restrict ourselves to the case $N = 2$.

More precisely, we let Ω be an open domain in $\mathbb{R}^2$ the boundary $\partial\Omega$ of which is C^1 up to a finite set M of singular points (i.e. $\partial\Omega$ is not C^1 in any neighborhood of $x \in M$). For each point $x \in M$ we suppose that there are an open neighborhood $U_x \subseteq \mathbb{R}^2$ of x as well as two open domains $\Omega_{x,l}$ and $\Omega_{x,r}$ with C^1-boundary such that either

$$U_x \cap \Omega = U_x \cap (\Omega_{x,l} \cap \Omega_{x,r}) \tag{37}$$

or

$$U_x \cap \Omega = U_x \setminus (\Omega_{x,l} \cap \Omega_{x,r}). \tag{38}$$

If the tangent spaces $T_x\Omega_{x,l}$ and $T_x\Omega_{x,r}$ do not coincide, then we call x an outward angular point in case of (37) and an inward angular point in case of (38). If these tangent spaces coincide, then x is called an outward resp. inward cuspidal point.

As in the previous section, we consider the cone generated by Ω,

$$\mathbf{C}_\Omega := \{(y, y_3) \in \mathbb{R}^2 \times [0, \infty) : y \in y_3\Omega\} \tag{39}$$

and, for $x \in \mathbb{R}^2$ and $\delta \in \mathbb{R}$, the half spaces $\mathbf{H}_x$ and $\mathbf{H}_{x,\delta}$. Further we set $\mathbf{H}_{x,-\infty} := \mathbb{R}^3$ and $\mathbf{H}_{x,+\infty} := \emptyset$ and, for $\delta, \epsilon \in \mathbb{R} \cup \{\pm\infty\}$ and $x \in M$,

$$\mathbf{K}_{x,\delta,\epsilon} := \mathbf{H}_{x,\delta,l} \cap \mathbf{H}_{x,\epsilon,r}$$

where $\mathbf{H}_{x,\delta,l}$ and $\mathbf{H}_{x,\epsilon,r}$ are half spaces belonging to $\Omega_{x,l}$ and $\Omega_{x,r}$, respectively.

Proposition 5.13 *Let $x \in \mathbb{R}^2$. Then the local operator spectrum $\sigma_{\eta_x}(\chi_{\mathbf{C}_\Omega} I)$ is equal to*

(a) $\{0\}$ *if* $x \notin \overline{\Omega}$.

(b) $\{I\}$ *if* $x \in \Omega$.

(c) $\{\chi_{\mathbb{H}_{x,\,\delta}} I : \delta \in \mathbb{R} \cup \{\pm\infty\}\}$ *if* $x \in \partial\Omega \setminus M$.

(d) $\{\chi_{\mathbb{K}_{x,\,\delta,\,\epsilon}} I : \delta,\, \epsilon \in \mathbb{R} \cup \{\pm\infty\}\}$ *if* $x \in \partial\Omega \cap M$ *is an angular point.*

(e) $\{0\}$ *if* $x \in \partial\Omega \cap M$ *is an outward cuspidal point.*

(f) $\{I\}$ *if* $x \in \partial\Omega \cap M$ *is an inward cuspidal point.*

Proof. The proof for (a) – (c) is the same as in the previous section. The results of the previous section also show that the local operator spectrum $\sigma_{\eta_x}(\chi_{\mathbf{C}_\Omega})$ is contained in the set (d) if $x \in \partial\Omega \cap M$ is an angular point. That it is actually equal to this set can be seen as follows. Since x is an angular point, we can independently shift the half space $\mathbf{H}_{x,0,l}$ by a sequence which tends into the direction of η_x and comes closer and closer to $\partial\mathbf{H}_{x,0,r}$ and the half space $\mathbf{H}_{x,0,r}$ by a sequence which also tends into the direction of η_x and comes closer and closer to $\partial\mathbf{H}_{x,0,l}$. Since each of these sequences influences the associated limit operators of only one of the half spaces, we get any desired combination of shifts of the half spaces $\mathbf{H}_{x,0,l}$ and $\mathbf{H}_{x,0,r}$ in this way. This shows (d), and (e) and (f) can be proved as in the previous section. (See the discussion before Theorem 5.9. The obvious point is that, in case of a cuspidal point, the half spaces $\mathbf{H}_{x,0,l}$ and $\mathbf{H}_{x,0,r}$ cannot be shifted independently of each other.) ■

Given $A \in \mathcal{A}(BUC(\mathbb{R}^3),\, \mathcal{C}_p(\mathbb{R}^3))$, let $\mathcal{H}_\eta(A)$ denote the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^2$ and for which the limit operator A_h exists. Further, if $x \in \partial\Omega \setminus M$ and $\delta \in \mathbb{R} \cup \{\pm\infty\}$, write $\mathcal{H}_{x,\delta}(A)$ for the set of all sequences $h \in \mathcal{H}_{\eta_x}(A)$ such that the limit operator of $\chi_{\mathbf{C}_\Omega} I$ exists and is equal to $\chi_{\mathbb{H}_{x,\,\delta}} I$. Finally, if $x \in \partial\Omega \cap M$ is an angular point and $\delta,\, \epsilon \in \mathbb{R} \cup \{\pm\infty\}$, then let $\mathcal{H}_{x,\,\delta,\,\epsilon}(A)$ stand for the set of all sequences $h \in \mathcal{H}_{\eta_x}(A)$ such that the limit operator of $\chi_{\mathbf{C}_\Omega} I$ exists and is equal to $\chi_{\mathbb{K}_{x,\,\delta,\,\epsilon}} I$. With these notations, we have the following consequence of Theorem 5.1.

Theorem 5.14 *Let $A \in \mathcal{A}(BUC(\mathbb{R}^3),\, \mathcal{C}_p(\mathbb{R}^3))$ and $D = \mathbf{C}_\Omega$ with $\Omega \in \mathbb{R}^2$ an open domain with piecewise C^1 boundary as above. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the following conditions are satisfied:*

(a) *for each point $x \in \Omega$, the set $\{A_h : h \in \mathcal{H}_{\eta_x}(A)\}$ of limit operators of A is uniformly invertible.*

(b) *for each point $x \in \partial\Omega \setminus M$, the set*

$$\{(1 - \chi_{\mathbf{H}_{x,\delta}})I + \chi_{\mathbf{H}_{x,\,\delta}} A_h \chi_{\mathbf{H}_{x,\,\delta}} I : h \in \mathcal{H}_{x,\delta}(A),\ \delta \in \mathbb{R} \cup \{\pm\infty\}\}$$

of extended compressions of limit operators of A is uniformly invertible.

(c) *for each angular point $x \in \partial\Omega \cap M$, the set*

$$\{(1 - \chi_{\mathbf{K}_{x,\delta,\,\epsilon}})I + \chi_{\mathbf{K}_{x,\,\delta,\,\epsilon}} A_h \chi_{\mathbf{K}_{x,\,\delta,\,\epsilon}} I : h \in \mathcal{H}_{x,\,\delta,\,\epsilon}(A),\ \delta,\, \epsilon \in \mathbb{R} \cup \{\pm\infty\}\}$$

of extended compressions of limit operators of A is uniformly invertible.

(d) *for each inward cuspidal point $x \in \partial\Omega \cap M$, the set $\{A_h : h \in \mathcal{H}_{\eta_x}(A)\}$ of limit operators of A is uniformly invertible.*

Note that the conditions in (b) and (c) get a simpler form if one of the shift parameters δ and ϵ is $\pm\infty$. Let us also emphasize that, if x is an outward cuspidal point, then the local invertibility at η_x of the compression of A onto D is trivially satisfied since all limit operators of $(1-\chi_D)I + \chi_D A \chi_D I$ with respect to sequences which tend to infinity into the direction of η_x are equal to I.

Again we mention some special situations in which the conditions of Theorem 5.14 can be readily verified.

Corollary 5.15 *Let $A \in \mathcal{A}(SO(\mathbb{R}^3), \mathcal{C}_p(\mathbb{R}^3))$ and D be as in Theorem 5.14. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the following conditions are satisfied:*

(a) for each point $x \in \overline{\Omega}$ which is neither angular nor outward cuspidal, the set $\{A_h : h \in \mathcal{H}_{\eta_x}(A)\}$ of limit operators of A is uniformly invertible.

(b) for each angular point $x \in M$, the set

$$\{(1-\chi_{\mathbf{K}_{x,0,0}})I + \chi_{\mathbf{K}_{x,0,0}} A_h \chi_{\mathbf{K}_{x,0,0}} I : h \in \mathcal{H}_{x,\delta,\epsilon}(A),\ \delta, \epsilon \in \mathbb{R}\}$$

of extended compressions of limit operators of A is uniformly invertible.

(c) for each angular point $x \in M$, the set

$$\{A_h : h \in \mathcal{H}_{x,-\infty,\epsilon}(A) \cup \mathcal{H}_{x,\delta,-\infty}(A) \cup \mathcal{H}_{x,-\infty,-\infty}(A) : \delta, \epsilon \in \mathbb{R}\}$$

of limit operators of A is uniformly invertible.

Here we have used the shift invariance of the limit operators of A as well as Simonenko's Theorem 5.10 again.

Corollary 5.16 *Let $A \in \mathcal{A}(L^\infty_{stab}(\mathbb{R}^3), \mathcal{C}_p(\mathbb{R}^3))$ and D be as in Theorem 5.14. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the following conditions are satisfied:*

(a) for each point $x \in \overline{\Omega}$ which is neither angular nor outward cuspidal, the limit operator A_{η_x} of A is invertible.

(b) for each angular point $x \in M$, the extended compression

$$(1-\chi_{\mathbf{K}_{x,0,0}})I + \chi_{\mathbf{K}_{x,0,0}} A_{\eta_x} \chi_{\mathbf{K}_{x,0,0}}$$

of the limit operator A_h of A is invertible.

Indeed, this result follows from the fact that each local operator spectrum is a singleton under the hypothesis of the corollary. Furthermore, one shows by choosing suitable sequences tending to infinity that every operator in condition (c) of Corollary 5.15 is a limit operator of the operator in (b) (compare the proof of Theorem 2.33 in [9]). Thus, the invertibility of that operator already implies the invertibility of all operators in Corollary 5.15 (c).

5.8 Compressions to epigraphs of functions

We let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with

$$\lim_{t\to\pm\infty} f(t) = +\infty \quad \text{and} \quad \lim_{t\to\pm\infty} f'(t) = 0 \tag{40}$$

and consider its epigraph

$$\mathbf{E}_f := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > f(x_1)\}. \tag{41}$$

Let $h = (h_1, h_2) \in \mathcal{H}$ be a sequence which tends to infinity into the direction of $\eta = (\eta_1, \eta_2) \in S^1$. It is evident that the limit operator of $\chi_{\mathbf{E}_f} I$ exists and is equal to the identity operator if $\eta_2 > 0$, whereas the limit operator of $\chi_{\mathbf{E}_f} I$ exists and is equal to zero if $\eta_2 < 0$. Now let $\eta = (1, 0)$, and let h be a sequence for which the limit operator of $\chi_{\mathbf{E}_f}$ exists. We write

$$h_2(n) =: f(h_1(n)) + d_n$$

and choose a subsequence of h (which we denote by h again) such that the sequence (d_n) becomes convergent with limit $\delta \in \mathbb{R} \cup \{\pm\infty\}$. Further, for $\delta \in \mathbb{R} \cup \{\pm\infty\}$, we let

$$\mathbf{H}_\delta := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > \delta\}.$$

Then it is easy to check that the limit operator of $\chi_{\mathbf{E}_f} I$ coincides with $\chi_{\mathbf{H}_\delta} I$ and that, conversely, every operator of this form appears as a limit operator of $\chi_{\mathbf{E}_f} I$. The same holds for $\eta = (-1, 0)$.

Given $A \in \mathcal{A}(BUC(\mathbb{R}^2), \mathcal{C}_p(\mathbb{R}^2))$, let $\mathcal{H}_\eta(A)$ denote the set of all sequences $h \in \mathcal{H}$ which tend to infinity into the direction of $\eta \in S^1$ and for which the limit operator A_h exists. Further, for $\delta \in \mathbb{R} \cup \{\pm\infty\}$, write $\mathcal{H}_{\pm 1, \delta}(A)$ for the set of all sequences $h \in \mathcal{H}_{(\pm 1, 0)}(A)$ such that the limit operator of $\chi_{\mathbf{E}_f} I$ exists and is equal to $\chi_{\mathbb{H}_\delta} I$. Then Theorem 5.1 yields, for example, the following.

Theorem 5.17 *Let $A \in \mathcal{A}(L^\infty_{stab}(\mathbb{R}^2), \mathcal{C}_p(\mathbb{R}^2))$, and let $D = \mathbf{E}_f$ be the the epigraph of the function f satisfying* (40)*. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if, for each point $\eta = (\eta_1, \eta_2) \in S^1$ with $\eta_2 \geq 0$, the limit operator A_η of A is invertible.*

The proof is the same as for Corollaries 5.12 and 5.16. ■

Finally, let $f_\pm : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions with

$$\lim_{t \to +\infty} f_\pm(t) = \lim_{t \to -\infty} f_\pm(t) = \pm\infty \quad \text{and} \quad \lim_{t \to +\infty} f'_\pm(t) = \lim_{t \to -\infty} f'_\pm(t) = 0, \tag{42}$$

and let

$$\mathbf{E}_{f_+, f_-} := \{(x_1, x_2) \in \mathbb{R}^2 : f_-(x_1) < x_2 < f_+(x_1)\}. \tag{43}$$

As before one can check that every limit operator of $\chi_{\mathbf{E}_{f_+, f_-}} I$ is of the form $\chi_{\mathbf{H}_\delta} I$ with $\delta \in \mathbb{R} \cup \{\pm\infty\}$ and that, conversely, every operator of this form is a limit operator of $\chi_{\mathbf{H}_\delta} I$ if h tends into the direction of $(\pm 1, 0) \in S^1$. If h tends to infinity into the direction of $\eta \in S^1$ with $\eta_2 \neq 0$ then, necessarily, the limit operator of $\chi_{\mathbf{H}_\delta} I$ with respect to h exists and is equal to the zero operator.

Theorem 5.18 *Let $A \in \mathcal{A}(L^\infty_{stab}(\mathbb{R}^2), \mathcal{C}_p(\mathbb{R}^2))$, and let $D = \mathbf{E}_{f_+, f_-}$ with functions $f_\pm$ satisfying* (42)*. Then the compression of A onto D is Fredholm on $L^p(D)$ if and only if the limit operators A_η of A with $\eta = (\pm 1, 0) \in S^1$ are invertible.*

References

[1] H. O. Cordes, On compactness of commutators of multiplications and convolutions and boundedness of pseudodifferential operators. – J. Fctl. Anal. **18**(1975), 2, 115 – 131.

[2] R. G. Douglas, R. Howe, On the C^*-algebra of Toeplitz operators on the quarter plane. – Trans. Am. Math. Soc. **158**(1971), 203 – 217.

[3] L. S. Goldenstein, I. C. Gohberg, On a multidimensional integral equation on a half-space whose kernel is a function of the difference of the arguments, and on a discrete analogue of this equation. – Dokl. AN SSSR **131**(1960), 1, 4 – 12 (Russian, Engl. transl. Sov. Math. **1**(1960), 1, 173 – 176).

[4] N. Karapetiants, S. Samko, Equations with Involutive Operators. – Birkhäuser, Boston, Basel, Berlin, 2001.

[5] E. Meister, F.-O. Speck, Some multidimensional Wiener-Hopf equations with applications. – *In*: Trends in Applications of Pure Mathematics to Mechanics, Vol. II, Proc. Symp. Kozubnik, Poland, September 1977 (Ed. H. Zorski), Pitman, London et al. 1979, 217 – 262.

[6] E. Meister, F.-O. Speck, Wiener-Hopf operators on three-dimensional wedge-shaped regions. – Appl. Anal. **10**(1980), 31 – 45.

[7] E. Meister, F.-O. Speck, A contribution to the quarter-plane problem in diffraction theory. – J. Math. Anal. Appl. **130**(1988), 223 – 236.

[8] V. S. Rabinovich, S. Roch, B. Silbermann, Fredholm theory and finite section method for band-dominated operators. – Integral Equations Oper. Theory **30**(1998), 4, 452 – 495.

[9] V. S. Rabinovich, S. Roch, B. Silbermann, Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite sections. – Integral Equations Oper. Theory **40**(2001), 3, 342 – 381.

[10] M. Reed, B. Simon, Methods of Modern Mathematical Physics, II. – Academic Press, New York, San Francisco, London 1975.

[11] B. Ya. Shteinberg, Operators of convolution type on locally compact groups. – Rostov-na-Donu 1979, Dep. at VINITI 715-80, 65 p. (Russian).

[12] B. Ya. Shteinberg, On convolution operators on locally compact groups. – Funkts. Anal. prilozh. **15**(1981), 95 – 96 (Russian).

[13] B. Ya. Shteinberg, Boundedness and compactness of convolution operators on locally compact groups. – Mat. Zametki **38**(1985), 2, 278 – 292 (Russian).

[14] I. B. Simonenko, Operators of convolution type in cones. – Mat. Sbornik **74 (116)**(1967), 2, 567 – 586 (Russian, Engl. transl. Math. USSR-Sbornik **3**(1967), 2, 279 – 293).

[15] F.-O. Speck, Eine Erweiterung des Satzes von Rakovšik und ihre Anwendung in der Simonenko-Theorie. – Math. Ann. **228**(1977), 2, 93 – 100.

[16] M. E. Taylor, Partial Differential Equations I. – Springer-Verlag, New York 1996.

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