Locally convex root graded Lie algebras

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Abstract. In the present paper we start to build a bridge from the algebraic theory of root graded Lie algebras to the global Lie theory of infinite-dimensional Lie groups by showing how root graded Lie algebras can be defined and analyzed in the context of locally convex Lie algebras. Our main results concern the description of locally convex root graded Lie algebras in terms of a locally convex coordinate algebra and its universal covering algebra, which has to be defined appropriately in the topological context. Although the structure of the isogeny classes is much more complicated in the topological context, we give an explicit description of the universal covering Lie algebra which implies in particular that it depends only on the root system and the coordinate algebra. Not every root graded locally convex Lie algebra is integrable in the sense that it is the Lie algebra of a Lie group. In a forthcoming paper we will discuss criteria for the integrability of root graded Lie algebras.

Introduction

Let $\mathbb{K}$ be a field of characteristic zero and Δ a finite reduced irreducible root system. We write $\mathfrak{g}_\Delta$ for the corresponding finite-dimensional split simple $\mathbb{K}$-Lie algebra and fix a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\Delta$. In the algebraic context, a Lie algebra $\mathfrak{g}$ is said to be Δ-graded if it contains $\mathfrak{g}_\Delta$ and $\mathfrak{g}$ decomposes as follows as a direct sum of simultaneous $\operatorname{ad}\mathfrak{h}$-eigenspaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{g}_0 = \sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

It is easy to see that the latter requirement is equivalent to $\mathfrak{g}$ being generated by the *root spaces* $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, and that it implies in particular that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e., that $\mathfrak{g}$ is a perfect Lie algebra. Recall that two perfect Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are called *(centrally) isogenous* if $\mathfrak{g}_1/\mathfrak{z}(\mathfrak{g}_1) \cong \mathfrak{g}_2/\mathfrak{z}(\mathfrak{g}_2)$. A perfect Lie algebra $\mathfrak{g}$ has a unique universal central extension $\widetilde{\mathfrak{g}}$, called its universal covering algebra ([We95, Th. 7.9.2]). Two isogenous perfect Lie algebras have isomorphic universal central extensions, so that the *isogeny class of* $\mathfrak{g}$ consists of all quotients of $\widetilde{\mathfrak{g}}$ by central subspaces.

The systematic study of root graded Lie algebras was initiated by S. Berman and R. Moody in [BM92], where they studied Lie algebras graded by simply laced root systems, i.e., types A, D and E. The classification of Δ-graded Lie algebras proceeds in two steps. First one considers isogeny classes of Δ-graded Lie algebras and then describes the elements of a fixed isogeny class as quotients of the corresponding universal covering Lie algebra. Berman and Moody show that for a fixed simply laced root system of type Δ the isogeny classes are in one-to-one correspondence with certain classes of unital coordinate algebras which are

(1) commutative associate algebras for types D_r, $r \geq 4$, E_6, E_7 and E_8,

(2) associative algebras for type A_r, $r \geq 3$, and

(3) alternative algebras for type A_2.

The corresponding result for type A_1 is that the coordinate algebra is a Jordan algebra, which goes back to results of J. Tits ([Ti62]).

Corresponding results for non-simply laced root systems have been obtain by G. Benkart and E. Zelmanov in [BZ96], where they also deal with the A_1-case. In these cases the isogeny classes are determined by a class of coordinate algebras, which mostly is endowed with an involution, where the decomposition of the algebra into eigenspaces of the involution corresponds to the division of roots into short and long ones. Based on the observation that all root systems except E_8, F_4, and G_2 are 3-graded, E. Neher obtains in [Neh96] a uniform description of the coordinate algebras of 3-graded Lie algebras by Jordan theoretic methods. Neher's approach is based on the observation that if Δ is 3-graded, then each Δ-graded Lie algebra can also be considered as an A_1-graded Lie algebra, which leads to a unital Jordan algebra as coordinate algebra. Then one has to identify the types of Jordan algebras corresponding to the different root systems.

The classification of root graded Lie algebras was completed by B. Allison, G. Benkart and Y. Gao in [ABG00]. They give a uniform description of the isogeny classes as quotients of a unique Lie algebra $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$, depending only on the root system Δ and the coordinate algebra $\mathcal{A}$, by central subspaces. Their construction implies in particular the existence of a functor $\mathcal{A} \mapsto \widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ from the category of coordinate algebras associated to Δ to centrally closed Δ-graded Lie algebras.

Apart from split simple Lie algebras, there are two prominent classes of root graded Lie algebras, which have been studied in the literature from a different point of view. The first class are the affine Kac–Moody algebras which can be described as root graded Lie algebras ([Ka90, Ch. 6] and Example I.11 below). The other large class are the isotropic finite-dimensional simple Lie algebras $\mathfrak{g}$ over fields of characteristic zero. These Lie algebras have a restricted root decomposition with respect to a maximal toral subalgebra $\mathfrak{h}$ [1]. The corresponding root system Δ is irreducible, but it may also be non-reduced, i.e., of type BC_r ([Se76]). If it is reduced, then $\mathfrak{g}$ is Δ-graded in the sense defined above. In the general case, one needs the notion of BC_r-graded Lie algebras which has been developed by B. Allison, G. Benkart and Y. Gao in [ABG02]. Since three different root lengths occur in BC_r, we call the shortest ones the *short roots*, the longest ones the *extra-long roots*, and the other roots *long*. The main difference to the reduced case is that there cannot be any grading subalgebra of type BC_r, so that one has to distinguish between different types, where the grading subalgebra is either of type B_r (the short and the long roots), type C_r (the long and the extra-long roots), or of type D_r (the long roots).

The theory of root graded Lie algebras has a very geometric flavor because the coordinatization theorems for the different types of root systems are very similar to certain coordinatization results in synthetic geometry. That the Lie algebra $\mathfrak{g}$ under consideration is simple implies that the coordinate algebra is simple, too. In geometric contexts, in addition, the coordinate algebras are mostly division algebras or forms of division algebras. For a nice account on the geometry of groups corresponding to the root systems A_2, $B_2 \cong C_2$ and G_2 we refer to the memoir [Fa77] of J. R. Faulkner. Here type A_2 corresponds to generalized triangles, type B_2 to generalized quadrangles and G_2 to generalized hexagons.

An important motivation for the algebraic theory of root graded Lie algebras was to find a class of Lie algebras containing affine Kac–Moody algebras ([Ka90]), isotropic finite-dimensional simple Lie algebras ([Se76]), certain ones of Slodowy's intersection matrix algebras ([Sl86]), and extended affine Lie algebras (EALAs) ([AABGP97]), which can roughly be considered as those root graded Lie algebras with a root decomposition. Since a general structure theory of infinite-dimensional Lie algebras does not exist, it is important to single out large classes with a uniform structure theory. The class of root graded Lie algebras satisfies all these requirements in a very natural fashion. It is the main point of the present paper to show that root graded Lie algebras can also be dealt with in a natural fashion in a topological context, where it covers many important classes of Lie algebras, arising in such diverse contexts as mathematical physics, operator theory and geometry.

With the present paper we start a project which connects the rich theory of root graded Lie algebras, which has been developed so far on a purely algebraic level, to the theory of infinite-dimensional Lie groups. A *Lie group* G is a manifold modeled on a locally convex space $\mathfrak{g}$ which carries a group structure for which the multiplication and the inversion map are smooth ([Mi83], [Gl01a], [Ne02b]). Identifying elements of the tangent space $\mathfrak{g} := T_{\mathbf{1}}(G)$ of G in the identity $\mathbf{1}$

[1] We call a subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{g}$ *toral* if $\mathrm{ad}\, \mathfrak{t} \subseteq \mathrm{der}(\mathfrak{g})$ consists of diagonalizable endomorphisms.

with left invariant vector fields, we obtain on $\mathfrak{g}$ the structure of a *locally convex Lie algebra*, i.e., a Lie algebra which is a locally convex space and whose Lie bracket is continuous. Therefore the category of locally convex Lie algebras is the natural setup for the "infinitesimal part" of infinite-dimensional Lie theory. In addition, it is an important structural feature of locally convex spaces that they have natural tensor products.

In Section I we explain how the concept of a root graded Lie algebra can be adapted to the class of locally convex Lie algebras. The main difference to the algebraic concept is that one replaces the condition that $\sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ coincides with $\mathfrak{g}_0$ by the requirement that it is a dense subspace of $\mathfrak{g}_0$. This turns out to make the theory of locally convex root graded Lie algebras somewhat harder than the algebraic theory, but it is natural, as a closer inspection of the topological versions of the Lie algebras $\mathfrak{sl}_n(A)$ for locally convex associative algebras A shows. In Section I we also discuss some natural classes of "classical" locally convex root graded Lie algebras such as symplectic and orthogonal Lie algebras and the Tits–Kantor–Koecher–Lie algebras associated to Jordan algebras.

In Section II we undertake a detailed analysis of locally convex root graded Lie algebras. Here the main point is that the action of the grading subalgebra $\mathfrak{g}_\Delta$ on $\mathfrak{g}$ is semisimple with at most three isotypical components, into which $\mathfrak{g}$ decomposes topologically. The corresponding simple modules are the trivial module $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the adjoint module $\mathfrak{g}_\Delta$ and the simple module V_s whose highest weight is the maximal short root with respect to a positive system $\Delta^+ \subseteq \Delta$. In the algebraic context, the decomposition of $\mathfrak{g}$ is a direct consequence of Weyl's Theorem, but here we need that the isotypical projections are continuous operators, a fact which can be derived from the fact that they come from the center of the enveloping algebra $U(\mathfrak{g}_\Delta)$. The underlying algebraic arguments are provided in Appendix A. If A, B, resp., D, are the multiplicity spaces with respect to $\mathfrak{g}_\Delta$, V_s, resp., $\mathbb{K}$, then $\mathfrak{g}$ decomposes topologically as

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D.$$

A central point in our structural analysis is that the direct sum $\mathcal{A} := A \oplus B$ carries a natural (not necessarily associative) unital locally convex algebra structure, that D acts by derivations on $\mathcal{A}$, and that we have a continuous alternating map $\delta^D\colon \mathcal{A} \times \mathcal{A} \to D$ satisfying a certain cocycle condition. Here the type of the root system Δ dictates certain identities for the multiplication on $\mathcal{A}$, which leads to the coordinatization results mentioned above ([BM92], [BZ96] and [Neh96]). The main new point here is that $\mathcal{A}$ inherits a natural locally convex structure, that the multiplication is continuous and that all the related maps such as δ^D are continuous.

In the algebraic context, the coordinate algebra $\mathcal{A}$ and the root system Δ classify the isogeny classes. The isogeny class of $\mathfrak{g}$ contains a unique centrally closed Lie algebra $\widetilde{\mathfrak{g}}$ and a unique center-free Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. In the locally convex context, the situation is more subtle because we have to work with generalized central extensions instead of ordinary central extensions: a morphism $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of locally convex Lie algebras is called a *generalized central extension* if it has dense range and there exists a continuous bilinear map $b\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ for which $b \circ (q \times q)$ is the Lie bracket on $\widehat{\mathfrak{g}}$. The subtlety of this concept is that q need not be surjective and if it is surjective, it does not need to be a quotient map. Fortunately these difficulties are compensated by the nice fact that each *topologically perfect Lie algebra* $\mathfrak{g}$, meaning that the commutator algebra is dense, has a universal generalized central extension, called the *universal covering Lie algebra* $\widetilde{\mathfrak{g}}$. The basic results on generalized central extensions are developed in Section III.

In Section IV we apply this concept to locally convex root graded Lie algebras and show that the description of the universal covering Lie algebra can be translated from the algebraic context ([ABG00]) to the locally convex context without extra technical work. Here a central point is that for any generalized central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ the Lie algebra $\widehat{\mathfrak{g}}$ is Δ-graded if and only if $\mathfrak{g}$ is Δ-graded. This means that *generalized isogeny classes* contain a Δ-graded element if and only if they entirely consist of Δ-graded Lie algebras. Moreover, we show that the universal covering Lie algebra of a Δ-graded Lie algebra only depends on the root system Δ and the coordinate algebra $\mathcal{A}$. Therefore the universal covering Lie algebra deserves the name $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$, and it turns out that the assignment $\mathcal{A} \mapsto \widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ is functorial.

We thus obtain a locally convex version of isogeny classes. They still have the property that they contain a unique centrally closed member because all Lie algebras in the class have the same universal covering, but unfortunately there might be several center-free Lie algebras with the same universal covering. This is due to the fact that the Lie algebras of the class are obtained from the centrally closed Lie algebra $\widetilde{\mathfrak g}$ by generalized central extensions $q_{\mathfrak g}\colon \widetilde{\mathfrak g} \to \mathfrak g$. As $q_{\mathfrak g}$ is not necessarily a quotient map, the topology on $\mathfrak g$ is *not* determined by the topology on $\widetilde{\mathfrak g}$ (Example III.15, Example IV.16).

A Lie group G is said to be Δ*-graded* if its Lie algebra $\mathbf{L}(G)$ is Δ-graded. It is a natural question which root graded locally convex Lie algebras $\mathfrak g$ are *integrable* in the sense that they are the Lie algebra of a Lie group G. Although this question always has an affirmative answer if $\mathfrak g$ is finite-dimensional, it turns out to be a difficult problem to decide integrability for infinite-dimensional Lie algebras. These global questions will be pursued in another paper ([Ne03b], see also [Ne03a]). In Section V we give an outline of the global side of the theory and explain how it is related to K-theory and non-commutative geometry. One of the main points is that, in view of the results of Section V, it mainly boils down to showing that at least one member $\mathfrak g$ of an isogeny class is integrable and then analyze the situation for the universal covering Lie algebra $\widetilde{\mathfrak g}$.

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Preliminaries and notation

The theory of root graded Lie algebras is a subject with great aesthetic appeal and rich connections to many other fields of mathematics. We therefore tried to keep the exposition of the present paper as self-contained as possible to make it accessible to readers from different mathematical subcultures. In particular we include proofs for those results on the structure of the coordinate algebras which can be obtained by short elementary arguments; for the more refined structure theory related to the exceptional and the low rank algebras we refer to the literature. On the algebraic level we essentially build on the representation theory of finite-dimensional semisimple split Lie algebras (cf. [Dix74] or [Hum72]); the required Jordan theoretic results are elementary and provided in Appendices B and C. On the functional analytic level we do not need much more than some elementary facts on locally convex spaces such as the existence of the projective tensor product.

All locally convex spaces in this paper are vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If X and Y are locally convex spaces, then we write $\mathrm{Lin}(X,Y)$ for the space of continuous linear maps $X \to Y$.

A *locally convex algebra* $\mathcal{A}$ is a locally convex topological vector space together with a continuous bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. In particular a *locally convex Lie algebra* $\mathfrak g$ is a Lie algebra which is a locally convex space for which the Lie bracket is a continuous bilinear map $\mathfrak g \times \mathfrak g \to \mathfrak g$.

The assumption that the topological Lie algebras we consider are locally convex spaces is motivated by the fact that such Lie algebras arise naturally as Lie algebras of Lie groups and by the existence of tensor products, which will be used in Section III to construct the universal covering Lie algebra. Tensor products of locally convex spaces are defined as follows.

Let E and F be locally convex spaces. On the tensor product $E \otimes F$ there exists a natural locally convex topology, called the *projective topology*. It is defined by the seminorms

$$(p \otimes q)(x) = \inf \Big\{ \sum_{j=1}^n p(y_j) q(z_j) \colon x = \sum_j y_j \otimes z_j \Big\},$$

where p, resp., q are continuous seminorms on E, resp., F (cf. [Tr67, Prop. 43.4]). We write $E \otimes_\pi F$ for the locally convex space obtained by endowing $E \otimes F$ with the locally convex topology defined by this family of seminorms. It is called the *projective tensor product of E and F*. It has the universal property that for a locally convex space G the continuous bilinear maps $E \times F \to G$ are in one-to-one correspondence with the continuous linear maps $E \otimes_\pi F \to G$. We write $E \widehat{\otimes}_\pi F$

for the completion of the projective tensor product of E and F. If E and F are Fréchet spaces, their topology is defined by a countable family of seminorms, and this property is inherited by $E\widehat{\otimes}_\pi F$. Hence this space is also Fréchet.

If E and F are Fréchet spaces, then every element θ of the completion $E\widehat{\otimes}_\pi F$ can be written as $\theta = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$, where $\lambda \in \ell^1(\mathbb{N}, \mathbb{K})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ ([Tr67, Th. 45.1]). If, in addition, E and F are Banach spaces, then the tensor product of the two norms is a norm defining the topology on $E \otimes F$ and $E\widehat{\otimes}_\pi F$ also is a Banach space. For $\|\theta\| < 1$ we then obtain a representation with $\|\lambda\|_1 < 1$ and $\|x_n\|, \|y_n\| < 1$ for all $n \in \mathbb{N}$ ([Tr67, p.465]).

I. Root graded Lie algebras

In this section we introduce locally convex root graded Lie algebras. In the algebraic setting it is natural to require that root graded Lie algebras are generated by their root spaces, but in the topological context this condition would be unnaturally strong. Therefore it is weakened to the requirement that the root spaces generate the Lie algebra topologically. As we will see below, this weaker condition causes several difficulties which are not present in the algebraic setting, but this defect is compensated by the well behaved theory of generalized central extensions (see Section IV).

Definition I.1. Let Δ be a finite irreducible reduced root system and $\mathfrak{g}_\Delta$ the corresponding finite-dimensional complex simple Lie algebra.

A locally convex Lie algebra $\mathfrak{g}$ is said to be Δ*-graded* if the following conditions are satisfied:

(R1) $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$.

(R2) There exist elements $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \neq 0$, and a subspace $\mathfrak{h} \subseteq \mathfrak{g}_0$ with $\mathfrak{g}_\Delta \cong \mathfrak{h} + \sum_{\alpha\in\Delta} \mathbb{K} x_\alpha$.

(R3) For $\alpha \in \Delta \cup \{0\}$ we have $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \colon (\forall h \in \mathfrak{h})\, [h, x] = \alpha(x)h\}$, where we identify Δ with a subset of $\mathfrak{h}^*$.

(R4) $\sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is dense in $\mathfrak{g}_0$.

The subalgebra $\mathfrak{g}_\Delta$ of $\mathfrak{g}$ is called a *grading subalgebra.* We say that $\mathfrak{g}$ is *root graded* if $\mathfrak{g}$ is Δ-graded for some Δ.

A slight variation of the concept of a Δ-graded Lie algebra is obtained by replacing (R2) by

(R2') There exist a sub-root system $\Delta_0 \subseteq \Delta$ and elements $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_0$, and a subspace $\mathfrak{h} \subseteq \mathfrak{g}_0$ with $\mathfrak{g}_{\Delta_0} \cong \mathfrak{h} + \sum_{\alpha\in\Delta_0} \mathbb{K} x_\alpha$.

A Lie algebra satisfying (R1), (R2'), (R3) and (R4) is called (Δ, Δ_0)*-graded.* ■

Remark I.2. (a) Suppose that a locally convex Lie algebra $\mathfrak{g}$ satisfies (R1)-(R3). Then the subspace

$$\sum_{\alpha\in\Delta} \mathfrak{g}_\alpha + \sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

is invariant under each root space $\mathfrak{g}_\alpha$ and also under $\mathfrak{g}_0$, hence an ideal. Therefore its closure satisfies (R1)-(R4), hence is a Δ-graded Lie algebra.

(b) Sometimes one starts with the subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and the corresponding weight space decomposition, so that we have (R1) and (R3). Let Π be a basis of the root system $\Delta \subseteq \mathfrak{h}^*$ and $\check{\alpha}$, $\alpha \in \Delta$, the coroots. If there exist elements $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ for $\alpha \in \Pi$ such that $[x_\alpha, x_{-\alpha}] = \check{\alpha}$, then we consider the subalgebra $\mathfrak{g}_\Delta \subseteq \mathfrak{g}$ generated by $\{x_{\pm\alpha} \colon \alpha \in \Pi\}$. Then the weight decomposition of $\mathfrak{g}$ with weight set $\Delta \cup \{0\}$ easily implies that the generators $x_{\pm\alpha}$, $\alpha \in \Pi$, satisfy the Serre relations, and therefore that $\mathfrak{g}_\Delta$ is a split simple Lie algebra with root system Δ satisfying (R2).■

Remark I.3. (a) In the algebraic context one replaces (R4) by the requirement that $\mathfrak{g}_0 = \sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. This is equivalent to $\mathfrak{g}$ being generated by the spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta$.

(b) The concept of a Δ-graded Lie algebra can be defined over any field of characteristic 0. Here it already occurs in the classification theory of simple Lie algebras as follows. Let $\mathfrak{g}$ be

a simple Lie algebra which is *isotropic* in the sense that it contains non-zero elements x for which $\mathrm{ad}\, x$ is diagonalizable. The latter condition is equivalent to the existence of a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a maximal toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Then $\mathfrak{g}$ has an $\mathfrak{h}$-weight decomposition, and the corresponding set of weights $\Delta \subseteq \mathfrak{h}^*$ is a not necessarily reduced irreducible root system (cf. [Se76, pp.10/11]). If this root system is reduced, then one can use the method from Remark I.2(b) to show that $\mathfrak{g}$ is Δ-graded in the sense defined above. For restricted root systems of type BC_r this argument produces grading subalgebras of type C_r, hence (BC_r, C_r)-graded Lie algebras ([Se76]).
(c) (R4) implies in particular that $\mathfrak{g}$ is topologically perfect, i.e., that $\mathfrak{g}' := \overline{[\mathfrak{g}, \mathfrak{g}]} = \mathfrak{g}$.
(d) Suppose that $\mathfrak{g}$ is Δ-graded and

$$\mathfrak{d} \subseteq \mathrm{der}_\Delta(\mathfrak{g}) := \{D \in \mathrm{der}(\mathfrak{g}) \colon (\forall \alpha \in \Delta) D.\mathfrak{g}_\alpha \subseteq \mathfrak{g}_\alpha\}$$

is a Lie subalgebra with a locally convex structure for which the action $\mathfrak{d} \times \mathfrak{g} \to \mathfrak{g}$ is continuous. Then $\mathfrak{g} \rtimes \mathfrak{d}$ satisfies (R1)–(R3) with $(\mathfrak{g} \rtimes \mathfrak{d})_0 = \mathfrak{g}_0 \rtimes \mathfrak{d}$. ∎

Examples of root graded Lie algebras

Example I.4. Let Δ be a reduced finite root system and $\mathfrak{g}_\Delta$ be the corresponding simple split $\mathbb{K}$-Lie algebra. If A is a locally convex associative commutative algebra with unit $\mathbf{1}$, then $\mathfrak{g} := A \otimes \mathfrak{g}_\Delta$ is a locally convex Δ-graded Lie algebra with respect to the bracket

$$[a \otimes x, a' \otimes x'] := aa' \otimes [x, x'].$$

The embedding $\mathfrak{g}_\Delta \hookrightarrow \mathfrak{g}$ is given by $x \mapsto \mathbf{1} \otimes x$. ∎

Example I.5. Now let A be an associative unital locally convex algebra. Then the $(n \times n)$-matrix algebra $M_n(A) \cong A \otimes M_n(\mathbb{K})$ also is a locally convex associative algebra. We write $\mathfrak{gl}_n(A)$ for this algebra, endowed with the commutator bracket and

$$\mathfrak{g} := \overline{[\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]}$$

for the closure of the commutator algebra of $\mathfrak{gl}_n(A)$. We claim that this is an A_{n-1}-graded Lie algebra with grading subalgebra $\mathfrak{g}_\Delta = \mathbf{1} \otimes \mathfrak{sl}_n(\mathbb{K})$. It is clear that $\mathfrak{g}_\Delta$ is a subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{h} := \Big\{ \mathrm{diag}(x_1, \ldots, x_n) \colon x_1, \ldots, x_n \in \mathbb{K}, \sum_j x_j = 0 \Big\} \subseteq \mathfrak{g}_\Delta$$

denote the canonical Cartan subalgebra and define linear functionals ε_j on $\mathfrak{h}$ by

$$\varepsilon_j(\mathrm{diag}(x_1, \ldots, x_n)) = x_j.$$

Then the weight space decomposition of $\mathfrak{g}$ satisfies

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = A \otimes E_{ij}, \quad i \neq j,$$

where E_{ij} is the matrix with one non-zero entry 1 in position (i, j). From

$$[aE_{ij}, bE_{kl}] = ab\delta_{jk} E_{il} - ba\delta_{li} E_{kj}$$

we derive that

$$[aE_{ij}, bE_{ji}] = abE_{ii} - baE_{jj} \in [a, b] \otimes E_{ii} + A \otimes \mathfrak{sl}_n(\mathbb{K}) \in \frac{1}{n}[a, b] \otimes \mathbf{1} + A \otimes \mathfrak{sl}_n(\mathbb{K}).$$

In view of $A \otimes \mathfrak{sl}_n(\mathbb{K}) = [\mathfrak{g}_\Delta, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$, it is now easy to see that

$$\mathfrak{g}_0 = \Big\{ \operatorname{diag}(a_1, \ldots, a_n) : \sum_j a_j \in \overline{[A,A]} \Big\} = (A \otimes \mathfrak{h}) \oplus (\overline{[A,A]} \otimes \mathbf{1}).$$

From the formulas above, we also see that (R4) is satisfied, so that $\mathfrak{g}$ is an A_{n-1}-graded locally convex Lie algebra.

We have a natural non-commutative trace map

$$\operatorname{Tr}: \mathfrak{gl}_n(A) \to A/\overline{[A,A]}, \quad x \mapsto \Big[\sum_{j=1}^n x_{jj}\Big],$$

where $[a]$ denotes the class of $a \in A$ in $A/\overline{[A,A]}$. Then the discussion above implies that

$$\mathfrak{sl}_n(A) := \ker \operatorname{Tr} = \mathfrak{g} = (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus (\overline{[A,A]} \otimes \mathbf{1}).$$

To prepare the discussion in Section II below, we describe the Lie bracket in $\mathfrak{sl}_n(A)$ in terms of the above direct sum decomposition. First we note that in $\mathfrak{gl}_n(A)$ we have

$$[a \otimes x, a' \otimes x'] = aa' \otimes xx' - a'a \otimes x'x = \frac{aa' + a'a}{2} \otimes [x, x'] + \frac{1}{2}[a, a'] \otimes (xx' + x'x).$$

For $x, x' \in \mathfrak{sl}_n(\mathbb{K})$ we have

$$x * x' := xx' + x'x - 2\frac{\operatorname{tr}(xx')}{n}\mathbf{1} \in \mathfrak{sl}_n(\mathbb{K}),$$

so that for $a, a' \in A$ and $x, x' \in \mathfrak{sl}_n(\mathbb{K})$ we have

$$[a \otimes x, a' \otimes x'] = \Big(\frac{aa' + a'a}{2} \otimes [x, x'] + \frac{1}{2}[a, a'] \otimes x * x'\Big) + [a, a'] \otimes \frac{\operatorname{tr}(xx')}{n}\mathbf{1}, \tag{1.1}$$

according to the direct sum decomposition $\mathfrak{sl}_n(\mathbb{K}) = (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus ([A,A] \otimes \mathbf{1})$, and

$$[d \otimes \mathbf{1}, a \otimes x] = [d, a] \otimes x, \quad a, d \in A, x \in \mathfrak{sl}_n(\mathbb{K}).$$ ■

Remark I.6. A Lie algebra $\mathfrak{g}$ can be root graded in several different ways. Let $\mathfrak{s} \subseteq \mathfrak{g}$ be a subalgebra with $\mathfrak{s} = \operatorname{span}\{h, e, f\} \cong \mathfrak{sl}_2(\mathbb{K})$ and the relations

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

If $\operatorname{ad}_\mathfrak{g} h$ is diagonalizable with $\operatorname{Spec}(\operatorname{ad}_\mathfrak{g} h) = \{2, 0, -2\}$, then the eigenspaces of $\operatorname{ad}_\mathfrak{g} h$ yield on $\mathfrak{g}$ the structure of an A_1-grading with $\mathfrak{g}_\Delta := \mathfrak{s}$. This shows in particular that for any associative algebra A the Lie algebra $\mathfrak{sl}_n(A)$, $n \geq 3$, has many different A_1-gradings in addition to its natural A_{n-1}-grading. ■

Example I.7. Let $\mathcal{A}$ be a locally convex unital associative algebra with a continuous *involution* $\sigma: a \mapsto a^\sigma$, i.e., σ is a continuous involutive linear antiautomorphism:

$$(ab)^\sigma = b^\sigma a^\sigma \quad \text{and} \quad (a^\sigma)^\sigma = a, \quad a, b \in \mathcal{A}.$$

If $\sigma = \operatorname{id}_\mathcal{A}$, then $\mathcal{A}$ is commutative. We write

$$\mathcal{A}^{\pm\sigma} := \{a \in \mathcal{A} : a^\sigma = \pm a\}$$

and observe that $\mathcal{A} = \mathcal{A}^\sigma \oplus \mathcal{A}^{-\sigma}$, where $\mathcal{A}^\sigma$ is a subalgebra.

The involution σ extends in a natural way to an involution of the locally convex algebra $M_n(\mathcal{A})$ of $n\times n$-matrices with entries in $\mathcal{A}$ by $(x_{ij})^\sigma := (x_{ji}^\sigma)$. If $\sigma = \mathrm{id}_{\mathcal{A}}$, then $x^\sigma = x^\top$ is just the transposed matrix.
(a) Let $\mathbf{1}\in M_n(\mathcal{A})$ be the identity matrix and define

$$J := \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0}\end{pmatrix} \in M_{2n}(\mathcal{A}).$$

Then $J^2 = -\mathbf{1}$, and

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) := \{x \in \mathfrak{gl}_{2n}(\mathcal{A})\colon Jx^\sigma J^{-1} = -x\}$$

is a closed Lie subalgebra of $\mathfrak{gl}_{2n}(\mathcal{A})$. Writing x as a (2×2)-matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(M_n(\mathcal{A}))$, this means that

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = \Big\{ \begin{pmatrix} a & b \\ c & -a^\sigma\end{pmatrix} \in \mathfrak{gl}_{2n}(\mathcal{A})\colon b^\sigma = b, c^\sigma = c\Big\}.$$

For $\mathcal{A} = \mathbb{K}$ we have $\sigma = \mathrm{id}$, and we obtain $\mathfrak{sp}_{2n}(\mathbb{K},\mathrm{id}_{\mathbb{K}}) = \mathfrak{sp}_{2n}(\mathbb{K})$. With the identity element $\mathbf{1}\in\mathcal{A}$ we obtain an embedding $\mathbb{K}\cong\mathbb{K}\mathbf{1}\hookrightarrow\mathcal{A}$, and hence an embedding

$$\mathfrak{sp}_{2n}(\mathbb{K}) \hookrightarrow \mathfrak{sp}_{2n}(\mathcal{A},\sigma).$$

Let

$$\mathfrak{h} := \{\mathrm{diag}(x_1,\ldots,x_n,-x_1,\ldots,-x_n)\colon x_1,\ldots,x_n\in\mathbb{K}\}$$

denote the canonical Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbb{K})$. Then the $\mathfrak{h}$-weights with respect to the adjoint action of $\mathfrak{h}$ on $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)$ coincide with the set

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j\colon i,j = 1,\ldots,n\}$$

of roots of $\mathfrak{sp}_{2n}(\mathbb{K})$, where $\varepsilon_j(\mathrm{diag}(x_1,\ldots,x_n,-x_1,\ldots,-x_n)) = x_j$ for $j = 1,\ldots,n$. Typical root spaces are

$$\mathfrak{g}_{\varepsilon_i-\varepsilon_j} = \mathcal{A}\otimes(E_{ij} - E_{j+n,i+n}),\quad \mathfrak{g}_{\varepsilon_i+\varepsilon_j} = \{aE_{i,j+n} + a^\sigma E_{j,i+n}\colon a\in\mathcal{A}\},\ i\neq j,$$

$$\mathfrak{g}_{2\varepsilon_j} = \mathcal{A}^\sigma E_{j,j+n},\quad \text{and}\quad \mathfrak{g}_0 = \{\mathrm{diag}(a_1,\ldots,a_n,-a_1^\sigma,\ldots,-a_n^\sigma)\colon a_1,\ldots,a_n\in\mathcal{A}\}.$$

The centralizer of the subalgebra $\mathfrak{sp}_{2n}(\mathbb{K})$ is

$$\mathfrak{z}_{\mathfrak{sp}_{2n}(\mathcal{A},\sigma)}(\mathfrak{sp}_{2n}(\mathbb{K})) = \mathcal{A}^{-\sigma}\mathbf{1},$$

and therefore

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = [\mathfrak{sp}_{2n}(\mathbb{K}),\mathfrak{sp}_{2n}(\mathcal{A},\sigma)]\oplus\mathcal{A}^{-\sigma}\mathbf{1}.$$

From Example I.5 we know that a necessary condition for an element $a\mathbf{1}$ to be contained in the commutator algebra of $\mathfrak{gl}_{2n}(A)$ is $a\in\overline{[\mathcal{A},\mathcal{A}]}$. On the other hand, the embedding

$$\mathfrak{sl}_n(\mathcal{A})\hookrightarrow\mathfrak{sp}_{2n}(\mathcal{A},\sigma),\quad a\mapsto\begin{pmatrix} a & 0 \\ 0 & -a^\sigma\end{pmatrix}$$

implies that the elements

$$\begin{pmatrix} a & 0 \\ 0 & -a^\sigma\end{pmatrix},\quad a\in\overline{[\mathcal{A},\mathcal{A}]}$$

are contained in the closure $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ of the commutator algebra of $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)$. This proves that

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma)' = [\mathfrak{sp}_{2n}(\mathbb{K}),\mathfrak{sp}_{2n}(\mathcal{A},\sigma)]\oplus\overline{[\mathcal{A},\mathcal{A}]}^{-\sigma}\otimes\mathbf{1}.$$

Using Example I.5 again, we now obtain (R4), and therefore that $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ is a C_n-graded Lie algebra with grading subalgebra $\mathfrak{sp}_{2n}(\mathbb{K})$.

The preceding description of the commutator algebra shows that each element $x = \begin{pmatrix} a & b \\ c & -a^\sigma \end{pmatrix} \in \mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ satisfies

$$\mathrm{tr}(x) = \mathrm{tr}(a - a^\sigma) = \mathrm{tr}(a) - \mathrm{tr}(a)^\sigma \in \overline{[\mathcal{A},\mathcal{A}]}.$$

That the latter condition is sufficient for x being contained in $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ follows from

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = [\mathfrak{sp}_{2n}(\mathbb{K}), \mathfrak{sp}_{2n}(\mathcal{A})] \oplus \mathcal{A}^{-\sigma} \otimes \mathbf{1}.$$

The Lie algebra $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)$ also has a natural 3-grading

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_+ \oplus \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_0 \oplus \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_-$$

with

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma)_\pm \cong \mathrm{Herm}_n(\mathcal{A},\sigma) := \{x \in M_n(\mathcal{A}) : x^\sigma = x\} \quad \text{and} \quad \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_0 \cong \mathfrak{gl}_n(\mathcal{A}),$$

obtained from the (2×2)-matrix structure.
(b) Now we consider the symmetric matrix

$$I := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \in M_{2n}(\mathcal{A}),$$

which satisfies $I^2 = \mathbf{1}$. We define the associate closed Lie subalgebra of $\mathfrak{gl}_{2n}(\mathcal{A})$ by

$$\mathfrak{o}_{n,n}(\mathcal{A},\sigma) := \{x \in \mathfrak{gl}_{2n}(\mathcal{A}) : I x^\sigma I^{-1} = -x\} = \Big\{ \begin{pmatrix} a & b \\ c & -a^\sigma \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathcal{A}) : b^\sigma = -b, c^\sigma = -c \Big\}.$$

For $\mathcal{A} = \mathbb{K}$ we have $\sigma = \mathrm{id}$, and we obtain $\mathfrak{o}_{n,n}(\mathbb{K}, \mathrm{id}_{\mathbb{K}}) = \mathfrak{o}_{n,n}(\mathbb{K})$. With the identity element $\mathbf{1} \in \mathcal{A}$ we obtain an embedding $\mathbb{K} \cong \mathbb{K}\mathbf{1} \hookrightarrow \mathcal{A}$, and hence an embedding

$$\mathfrak{o}_{n,n}(\mathbb{K}) \hookrightarrow \mathfrak{o}_{n,n}(\mathcal{A},\sigma).$$

Again,

$$\mathfrak{h} := \{\mathrm{diag}(x_1,\ldots,x_n,-x_1,\ldots,-x_n) : x_1,\ldots,x_n \in \mathbb{K}\}$$

is the canonical Cartan subalgebra of $\mathfrak{o}_{n,n}(\mathbb{K})$. The $\mathfrak{h}$-weights with respect to the adjoint action of $\mathfrak{h}$ on $\mathfrak{o}_{n,n}(\mathcal{A},\sigma)$ coincide with the set

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j : i,j = 1,\ldots,n\}.$$

Typical root spaces are

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathcal{A} \otimes (E_{ij} - E_{j+n,i+n}), \quad \mathfrak{g}_{\varepsilon_i+\varepsilon_j} = \{a E_{i,j+n} - a^\sigma E_{j,i+n} : a \in \mathcal{A}\},\ i \neq j,$$

$$\mathfrak{g}_{2\varepsilon_j} = \mathcal{A}^{-\sigma} E_{j,j+n}, \quad \text{and} \quad \mathfrak{g}_0 = \{\mathrm{diag}(a_1,\ldots,a_n,-a_1^\sigma,\ldots,-a_n^\sigma) : a_1,\ldots,a_n \in \mathcal{A}\}.$$

The root spaces $\mathfrak{g}_{2\varepsilon_j}$ are non-zero if and only if $\mathcal{A}^{-\sigma} \neq \{0\}$, which is equivalent to $\sigma \neq \mathrm{id}_{\mathcal{A}}$.

As in (a), we obtain

$$\mathfrak{z}_{\mathfrak{o}_{n,n}(\mathcal{A})}(\mathfrak{o}_{n,n}(\mathbb{K})) = \mathcal{A}^{-\sigma} \otimes \mathbf{1}, \quad \mathfrak{o}_{n,n}(\mathcal{A}) = [\mathfrak{o}_{n,n}(\mathbb{K}), \mathfrak{o}_{n,n}(\mathcal{A})] \oplus \mathcal{A}^{-\sigma} \otimes \mathbf{1},$$

and

$$\mathfrak{o}_{n,n}(\mathcal{A})' = [\mathfrak{o}_{n,n}(\mathbb{K}), \mathfrak{o}_{n,n}(\mathcal{A})] \oplus \overline{[\mathcal{A},\mathcal{A}]}^{-\sigma} \otimes \mathbf{1}.$$

If $\sigma_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$, then Δ is of type D_n, the root system of $\mathfrak{o}_{n,n}(\mathbb{K})$, and $\mathfrak{o}_{n,n}(\mathcal{A}) := \mathfrak{o}_{n,n}(\mathcal{A}, \mathrm{id}_{\mathcal{A}})$ is a D_n-graded Lie algebra. In this case $\mathcal{A} = \mathcal{A}^\sigma$, and

$$\mathfrak{o}_{n,n}(\mathcal{A}) \cong \mathcal{A} \otimes \mathfrak{o}_{n,n}(\mathbb{K}),$$

so that this case is also covered by Example I.4.

If $\sigma_{\mathcal{A}} \neq \mathrm{id}_{\mathcal{A}}$, then we obtain a (C_n, D_n)-graded Lie algebra with grading subalgebra $\mathfrak{o}_{n,n}(\mathbb{K})$ of type D_n. ■

Lemma I.8. *Let $\mathbb{K}$ be a field with $2 \in \mathbb{K}^\times$. For $x, y, z \in \mathfrak{sl}_2(\mathbb{K})$ we have the relations*

$$xy + yx = \operatorname{tr}(xy)\mathbf{1}, \tag{1.2}$$

and

$$[x, [y, z]] = 2\operatorname{tr}(xy)z - 2\operatorname{tr}(xz)y. \tag{1.3}$$

Proof. For $x \in \mathfrak{sl}_2(\mathbb{K})$ let

$$p(t) = \det(t\mathbf{1} - x) = t^2 - \operatorname{tr} x \cdot t + \det x = t^2 + \det x$$

denote the characteristic polynomial of x. Then the Cayley–Hamilton Theorem implies

$$0 = p(x) = x^2 + (\det x)\mathbf{1}.$$

On the other hand $-2\det x = \operatorname{tr} x^2$ follows by consideration of eigenvalues $\pm\lambda$ of x in a quadratic extension of $\mathbb{K}$. We therefore obtain $2x^2 - \operatorname{tr}(x^2)\mathbf{1} = 2x^2 + 2(\det x)\mathbf{1} = 0$. By polarization (taking derivatives in direction y), we obtain from $2x^2 = \operatorname{tr}(x^2)\mathbf{1}$ the relation $2xy + 2yx = \operatorname{tr}(xy + yx)\mathbf{1} = 2\operatorname{tr}(xy)\mathbf{1}$, which leads to

$$xy + yx = \operatorname{tr}(xy)\mathbf{1}.$$

We further get

$$\begin{aligned}\operatorname{tr}(xy)z - \operatorname{tr}(xz)y &= (xy + yx)z - y(xz + zx) = xyz - yzx = [x, yz] = \frac{1}{2}[x, [y, z] + (yz + zy)]\\ &= \frac{1}{2}[x, [y, z] + \operatorname{tr}(yz)\mathbf{1}] = \frac{1}{2}[x, [y, z]].\end{aligned}$$

■

Example I.9. (a) Let J be a locally convex Jordan algebra with identity $\mathbf{1}$ (cf. Appendix B). We endow the space $J \otimes J$ with the projective tensor product topology and define

$$\langle J, J\rangle := (J \otimes J)/I,$$

where $I \subseteq J \otimes J$ is the closed subspace generated by the elements of the form $a \otimes a$ and

$$ab \otimes c + bc \otimes a + ca \otimes b, \quad a, b, c \in J.$$

We write $\langle a, b\rangle$ for the image of $a \otimes b$ in $\langle J, J\rangle$. Then

$$\langle a, b\rangle = -\langle b, a\rangle \quad \text{and} \quad \langle ab, c\rangle + \langle bc, a\rangle + \langle ca, b\rangle = 0, \quad a, b, c \in J.$$

It follows in particular that $\langle \mathbf{1}, c\rangle + 2\langle c, \mathbf{1}\rangle = 0$, which implies $\langle \mathbf{1}, c\rangle = 0$ for each $c \in J$.

Let $L(a)b := ab$ denote the left multiplication in J. From the identity

$$[L(a), L(bc)] + [L(b), L(ca)] + [L(c), L(ab)] = 0$$

(Proposition B.2(1)) and the continuity of the maps $(a, b, x) \mapsto [L(a), L(b)].x$ we derive that the map

$$\delta_J : J \otimes J \to \operatorname{der}(J), \quad (a, b) \mapsto [L(a), L(b)]$$

(cf. Corollary B.3 for the fact that it maps into $\operatorname{der}(J)$) factors through a map

$$\delta_J : \langle J, J\rangle \to \operatorname{der}(J).$$

It therefore makes sense to define

$$\langle a,b\rangle.x := 2[L(a),L(b)].x, \quad a,b,x \in J. \tag{1.4}$$

We now define a bilinear continuous bracket on

$$\widetilde{\mathrm{TKK}}(J) := (J \otimes \mathfrak{sl}_2(\mathbb{K})) \oplus \langle J,J\rangle$$

by

$$[a\otimes x, a'\otimes x'] := aa \otimes [x,x'] + \langle a,a'\rangle \operatorname{tr}(xx'), \quad [\langle a,b\rangle, c\otimes x] := \langle a,b\rangle.c\otimes x$$
$$[\langle a,b\rangle, \langle c,d\rangle] := \langle \langle a,b\rangle.c, d\rangle + \langle c, \langle a,b\rangle.d\rangle.$$

The label TKK refers to Tits, Kantor and Koecher who studied the relation between Jordan algebras and Lie algebras from various viewpoints (see Appendices B and C). It is clear from the definitions that if we endow $\widetilde{\mathrm{TKK}}(J)$ with the natural locally convex topology turning it into a topological direct sum of $J\otimes \mathfrak{sl}_2(\mathbb{K})$ and $\langle J,J\rangle$, then $\widetilde{\mathrm{TKK}}(J)$ is a locally convex space with a continuous bracket. That the bracket is alternating follows for the $\langle J,J\rangle$-term from the calculation in Example III.10(3) below. To see that $\widetilde{\mathrm{TKK}}(J)$ is a Lie algebra, it remains to verify the Jacobi identity. The trilinear map

$$J(\alpha,\beta,\gamma) := [[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] =: \sum_{\text{cycl.}} [[\alpha,\beta],\gamma]$$

is alternating. Therefore we only have to show that it vanishes for entries in $J\otimes \mathfrak{sl}_2(\mathbb{K})$ and $\langle J,J\rangle$. The essential case is where all elements are in $J\otimes\mathfrak{sl}_2(\mathbb{K})$. In the last step of the following calculation we use Lemma I.8:

$$\begin{aligned}[][[a\otimes x, b\otimes y], c\otimes z] &= [ab\otimes [x,y] + \operatorname{tr}(xy)\langle a,b\rangle, c\otimes z]\\ &= (ab)c\otimes [[x,y],z] + \operatorname{tr}([x,y]z)\langle ab,c\rangle + \langle a,b\rangle.c\otimes \operatorname{tr}(xy)z\\ &= 2(ab)c\otimes(\operatorname{tr}(zy)x - \operatorname{tr}(zx)y) + \langle a,b\rangle.c\otimes\operatorname{tr}(xy)z + \operatorname{tr}([x,y]z)\langle ab,c\rangle.\end{aligned}$$

Now the vanishing of $J(a\otimes x, b\otimes y, c\otimes z)$ follows from

$$\sum_{\text{cycl.}} \operatorname{tr}([x,y]z)\langle ab,c\rangle = \operatorname{tr}([x,y]z)\sum_{\text{cycl.}}\langle ab,c\rangle = 0$$

and

$$(\langle a,b\rangle.c - 2(bc)a + 2(ca)b)\otimes \operatorname{tr}(xy)z = 0.$$

Note that this also explains the factor 2 in (1.4).

That the expression $J(\alpha,\beta,\gamma)$ vanishes if one entry is in $\langle J,J\rangle$ follows easily from the fact that $\delta(a,b) := 2[L(a),L(b)] \in \operatorname{der}(J)$. The case where two entries are in $\langle J,J\rangle$ corresponds to the relation

$$[\delta(a,b),\delta(c,d)] = \delta(\langle a,b\rangle.c,d) + \delta(c,\langle a,b\rangle.d)$$

in $\operatorname{der}(J)$, which in turn follows from the fact that for any $D\in\operatorname{der}(J)$ we have

$$\begin{aligned}[][D,\delta(c,d) &= 2[D,[L(c),L(d)]] = 2[[D,L(c)],L(d)] + 2[L(c),[D,L(d)]]\\ &= 2[L(D.c),L(d)] + 2[L(c),L(D.d)] = \delta(D.c,d) + \delta(c,D.d).\end{aligned}$$

The case where all entries of $J(\alpha,\beta,\gamma)$ are in $\langle J,J\rangle$ follows easily from the fact that the representation of $\operatorname{der}(J)$ on $J\otimes J$ factors through a Lie algebra representation on $\langle J,J\rangle$ given by $D.\langle a,b\rangle = \langle D.a,b\rangle + \langle a,D.b\rangle$. In this sense the latter three cases are direct consequences of the derivation property of the $\delta(a,b)$'s.

This proves that the bracket defined above is a Lie bracket on $\widetilde{\mathrm{TKK}}(J)$. The assignment $J\mapsto \widetilde{\mathrm{TKK}}(J)$ is functorial. It is clear that each derivation of J induces a natural derivation on

$\widetilde{\mathrm{TKK}}(J)$ and that each morphism of unital locally convex Jordan algebras $\varphi: J_1 \to J_2$ defines a morphism $\widetilde{\mathrm{TKK}}(J_1) \to \widetilde{\mathrm{TKK}}(J_2)$ of locally convex Lie algebras.

It is interesting to observe that in general tensor products $A \otimes \mathfrak{k}$ of an algebra A and a Lie algebra $\mathfrak{k}$ carry only a natural Lie algebra structure if A is commutative and associative (Example I.4). For more general algebras one has to add an extra space such as $\langle J, J\rangle$ for a Jordan algebra J and $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{K})$. The Jacobi identity for $\widetilde{\mathrm{TKK}}(J)$ very much relies on the identity for triple brackets in $\mathfrak{sl}_2(\mathbb{K})$ from Lemma I.8 and the definition of the action of $\langle a, b\rangle$ as $2[L(a), L(b)]$.

We have a natural embedding of $\mathfrak{sl}_2(\mathbb{K})$ into $\mathfrak{g}$ as $\mathfrak{g}_\Delta := \mathbb{K}\mathbf{1} \otimes \mathfrak{sl}_2(\mathbb{K})$. Let $h, e, f \in \mathfrak{sl}_2(\mathbb{K})$ be a basis with

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

Then $\mathfrak{h} = \mathbb{K}h$ is a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{K})$, and the corresponding eigenspace decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g}_2 = J \otimes e, \quad \mathfrak{g}_{-2} = J \otimes f \quad \text{and} \quad \mathfrak{g}_0 = J \otimes h \oplus \langle J, J\rangle.$$

In view of $[\mathfrak{g}_\Delta, \mathfrak{g}] = J \otimes \mathfrak{sl}_2(\mathbb{K})$, the formula for the bracket implies that $\langle J, J\rangle \subseteq [\mathfrak{g}, \mathfrak{g}]$, and hence that $\mathfrak{g}$ is an A_1-graded locally convex Lie algebra.
(b) If A is a locally convex unital associative algebra, then A also carries the structure of a locally convex unital Jordan algebra A_J with respect to the product

$$a \circ b := \frac{1}{2}(ab + ba).$$

It is interesting to compare $\widetilde{\mathrm{TKK}}(A_J)$ with the locally convex Lie algebra $\mathfrak{sl}_2(A)$ discussed in Example I.5, where we have seen that with respect to the decomposition

$$\mathfrak{sl}_2(A) = \big(A \otimes \mathfrak{sl}_2(\mathbb{K})\big) \oplus \big(\overline{[A, A]} \otimes \mathbf{1}\big),$$

the Lie bracket is given by

$$[a \otimes x, b \otimes y] = \frac{ab + ba}{2} \otimes [x, y] + \frac{1}{2}[a, b] \otimes x * y + [a, b] \otimes \frac{\mathrm{tr}(xy)}{n}\mathbf{1}.$$

In view of (1.2), we have $x * y = 0$, so that we obtain the simpler formula

$$[a \otimes x, b \otimes y] = (a \circ b) \otimes [x, y] + \frac{1}{2}[a, b] \otimes \mathrm{tr}(xy)\mathbf{1}.$$

Let $L_a(b) := ab$ and $R_a(b) := ba$. Then the left multiplication in the Jordan algebra is $L(a) = \frac{1}{2}(L_a + R_a)$, and therefore $\langle a, b\rangle$ acts on A_J as

$$2[L(a), L(b)] = \frac{1}{2}[L_a + R_a, L_b + R_b] = \frac{1}{2}([L_a, L_b] + [R_a, R_b]) = \frac{1}{2}(L_{[a,b]} - R_{[a,b]}) = \frac{1}{2}\,\mathrm{ad}([a, b]).$$

From this it easily follows that

$$\varphi: \widetilde{\mathrm{TKK}}(A) \to \mathfrak{sl}_2(A), \quad a \otimes x \mapsto a \otimes x, \quad \langle a, b\rangle \mapsto \frac{1}{2}[a, b] \otimes \mathbf{1}$$

defines a morphism of locally convex Lie algebras.

From the discussion of the examples in Section IV below, we will see that this homomorphism is in general neither injective nor surjective.
(c) From the continuity of the map

$$\langle J, J\rangle \times J \to J, \quad (\langle a, b\rangle, x) \mapsto \delta_J(a, b).x = \langle a, b\rangle.x$$

it follows that $\ker \delta_J$ is a closed subspace of $\langle J, J\rangle$. Hence the space $\mathrm{ider}(J) := \mathrm{im}(\delta_J) \cong \langle J, J\rangle / \ker(\delta_J)$ carries a natural locally convex topology as the quotient space $\langle J, J\rangle / \ker(\delta_J)$.

The closed subspace $\ker(\delta_J) \subseteq \langle J, J\rangle$ also is a closed ideal of $\widetilde{\mathrm{TKK}}(J)$. The quotient Lie algebra

$$\mathrm{TKK}(J) := \widetilde{\mathrm{TKK}}(J)/\ker(\delta_J) = (J \otimes \mathfrak{sl}_2(\mathbb{K})) \oplus \mathrm{ider}(J)$$

is called the *topological Tits–Kantor–Koecher–Lie algebra* associated to the locally convex unital Jordan algebra J. The bracket of this Lie algebra is given by

$$[a \otimes x, a' \otimes x'] := aa \otimes [x, x'] + 2\,\mathrm{tr}(xx')[L(a), L(a')], \quad [d, c \otimes x] := d.c \otimes x$$
$$[d, d'] := dd' - d'd.$$

Mostly $\mathrm{TKK}(J)$ is written in a different form, as $J \times \mathfrak{istr}(J) \times J$, where $\mathfrak{istr}(J) := L(J) + \mathrm{ider}(J)$ is the *inner structure Lie algebra of* J. The correspondence between the two pictures is given by the map

$$\Phi\colon \mathrm{TKK}(J) \to J \times \mathfrak{istr}(J) \times J, \quad a \otimes e + b \otimes h + c \otimes f + d \mapsto (a, 2L(b) + d, c).$$

To understand the bracket in the product picture, we observe that

$$(L(a) + [L(b), L(c)]).\mathbf{1} = a + b(c\mathbf{1}) - c(b\mathbf{1}) = a$$

implies

$$\mathfrak{istr}(J) = L(J) \oplus [L(J), L(J)] \cong J \oplus [L(J), L(J)].$$

For each derivation d of J we have $[d, L(a)] = L(d.a)$, which implies that

$$\sigma(L(x) + [L(y), L(z)]) = -L(x) + [L(y), L(z)]$$

defines an involutive Lie algebra automorphism on $\mathfrak{istr}(J)$. Now the bracket on $J \times \mathfrak{istr}(J) \times J$ can be described as

$$[(a, d, c), (a', d', c')]$$
$$= (d.a' - d'.a, 2L(ac') + 2[L(a), L(c')] - 2L(a'c) - 2[L(a'), L(c)], \sigma(d).c' - \sigma(d').c).$$

From this formula it is clear that the map $\tau(a, d, c) := (c, \sigma(d), a)$ defines an involutive automorphism of $\mathrm{TKK}(J)$. ■

Twisted loop algebras

There are also so-called twisted versions of the Lie algebras $A \otimes \mathfrak{g}_\Delta$ from Example I.4. The construction is based on the following observation.

Let $\mathfrak{k}$ be a split simple $\mathbb{K}$-Lie algebra, $\mathfrak{h}_\mathfrak{k} \subseteq \mathfrak{k}$ a splitting Cartan subalgebra, and Γ a group of automorphisms of $\mathfrak{k}$ fixing a regular element of $\mathfrak{k}$ in $\mathfrak{h}_\mathfrak{k}$. Typical groups of this type arise from the outer automorphisms of $\mathfrak{k}$, which can be realised by automorphisms of $\mathfrak{k}$ preserving the root decomposition and a positive system of roots (see Example I.10 below). Let $\mathfrak{k}^\Gamma$ denote the subalgebra of all elements of $\mathfrak{k}$ fixed by Γ. Then $\mathfrak{k}^\Gamma$ contains a regular element x_0 of $\mathfrak{h}_\mathfrak{k}$, and therefore Γ preserves $\mathfrak{z}_\mathfrak{k}(x_0) = \mathfrak{h}_\mathfrak{k}$. It follows in particular that Γ permutes the $\mathfrak{h}_\mathfrak{k}$-root spaces of $\mathfrak{k}$.

As $\mathfrak{h}^\Gamma := \mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}^\Gamma = \mathfrak{h}_\mathfrak{k}^\Gamma$ contains a regular element of $\mathfrak{k}$, it also is a splitting Cartan subalgebra of $\mathfrak{k}^\Gamma$. If $\Delta_\mathfrak{k}$ is the root system of $\mathfrak{k}$ and Δ_0 the root system of $\mathfrak{k}^\Gamma$, then clearly $\Delta_0 \subseteq \Delta_\mathfrak{k}|_{\mathfrak{h}^\Gamma}$, but it may happen that the latter set still is a root system.

Example I.10. Let Γ be a finite group of automorphisms of $\mathfrak{k}$ preserving the Cartan subalgebra $\mathfrak{h}_{\mathfrak{k}}$ and such that the action on the dual space preserves a positive system $\Delta_{\mathfrak{k}}^+$ of roots. By averaging over the orbit of an element $x \in \mathfrak{h}_{\mathfrak{k}}$ on which all positive roots are positive, we then obtain an element fixed by Γ on which all positive roots are positive, so that this element is regular in $\mathfrak{k}$.

Typical examples for this situation come from cyclic groups of diagram automorphisms:
(a) For type A_{2r-1} we have

$$\Delta_{\mathfrak{k}} = \{\pm(\varepsilon_i - \varepsilon_j) : i \neq j \in \{1, \ldots, 2r\}\}$$

on $\mathfrak{h}_{\mathfrak{k}} \cong \mathbb{K}^{2r}$. The non-trivial diagram automorphism σ is an involution satisfying

$$\sigma(x_1, \ldots, x_{2r}) = (-x_{2r}, \ldots, -x_1) \quad \text{and} \quad \sigma(\varepsilon_i) = -\varepsilon_{2r+1-i}.$$

We identify

$$\mathfrak{h}^\Gamma = \{(x_1, \ldots, x_r, -x_r, \ldots, -x_1) : x_i \in \mathbb{K}\}$$

with $\mathbb{K}^r$ by forgetting the last r entries. If $R: \mathfrak{h}_{\mathfrak{k}}^* \to (\mathfrak{h}^\Gamma)^*$ is the restriction map, then

$$\alpha_j := R(\varepsilon_j - \varepsilon_{j+1}), \quad j = 1, \ldots, r,$$

is a basis for the root system

$$R(\Delta_{\mathfrak{k}}) = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_j : 1 \leq j < i \leq r, 1 \leq j \leq r\}$$

of type C_r.
(b) For type D_{r+1}, $r \geq 4$, we have

$$\Delta_{\mathfrak{k}} = \{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j \in \{1, \ldots, r+1\}\}$$

on $\mathfrak{h}_{\mathfrak{k}} \cong \mathbb{K}^{r+1}$. A non-trivial diagram automorphism σ is the involution

$$\sigma(x_1, \ldots, x_{r+1}) = (x_1, \ldots, x_r, -x_{r+1}).$$

We identify $\mathfrak{h}^\Gamma = \{(x_1, \ldots, x_r, 0\}$ with $\mathbb{K}^r$ by forgetting the last entry. Then

$$R(\Delta_{\mathfrak{k}}) = \{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j \in \{1, \ldots, r\}\} \cup \{\varepsilon_j : j = 1, \ldots, r\}$$

is a root system of type B_r.
(c) For the triality automorphism of D_4 of order 3, we obtain a root system Δ_0 of type G_2.
(d) For the diagram involution of E_6 we obtain a root system Δ_0 of type F_4.

It is not hard to verify that in all cases $R(\Delta_{\mathfrak{k}})$ is the root system of $\mathfrak{k}_0$. ■

Now let $\mathfrak{k}$ and Γ be as above and assume, in addition, that $\mathfrak{k}^0$ is simple with root system Δ. We write $\mathfrak{g}_\Delta := \mathfrak{k}^\Gamma$, $\mathfrak{h} := \mathfrak{h}^\Gamma$ and assume that Δ coincides with $R(\Delta_{\mathfrak{k}})$, which is the case for all cyclic groups of diagram automorphisms.

Further let A be a locally convex commutative unital associative algebra on which Γ acts by continuous automorphisms. Then Γ also acts on the Lie algebra $A \otimes \mathfrak{k}$ via $\gamma.(a \otimes x) := \gamma.a \otimes \gamma.x$. We consider the Lie subalgebra

$$\mathfrak{g} := (A \otimes \mathfrak{k})^\Gamma$$

of Γ-fixed points in $A \otimes \mathfrak{k}$. We clearly have $\mathfrak{g} \supseteq A^\Gamma \otimes \mathfrak{g}_\Delta \supseteq \mathbf{1} \otimes \mathfrak{g}_\Delta$. Moreover, the action of $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}}^\Gamma$ on $A \otimes \mathfrak{k}$ commutes with the action of Γ, and our assumption implies that the $\mathfrak{h}$-weights of $\mathfrak{h}$ on $A \otimes \mathfrak{k}$ coincide with the root system Δ. This implies that $\mathfrak{g}$ satisfies (R1)–(R3) with respect to the subalgebra $\mathfrak{g}_\Delta$, and therefore that the closure of the subalgebra generated by the root spaces is Δ-graded.

Example I.11. This construction covers in particular all twisted loop algebras. In this case $A = C^\infty(\mathbb{T}, \mathbb{C})$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and if $\Gamma = \langle \sigma \rangle$ is generated by a diagram automorphism σ of order m, then we define the action of Γ on A by $\sigma(f)(z) = f(z\zeta)$, where ζ is a primitive m-th root of unity.

For Δ_k of type A_{2r-1}, D_{r+1}, E_6 and D_4, we thus obtain the twisted loop algebras of type $A_{2r-1}^{(2)}, D_{r+1}^{(2)}, E_6^{(2)}$ and $D_4^{(3)}$, and the corresponding root systems Δ are of type B_r, C_r, F_4 and G_2 ([Ka90]). ■

(Δ, Δ_0)-graded Lie algebras

Let Δ be a reduced irreducible root system and $\Delta_l \subseteq \Delta$ be the subset of long roots. Suppose that $\alpha, \beta \in \Delta_l$ with $\gamma := \alpha + \beta \in \Delta$. Then $\gamma \in \Delta_l$. Since α and β generate a subsystem of Δ whose rank is at most two, this can be verified by direct inspection of the cases A_2, $B_2 \cong C_2$ and G_2. Alternatively, we can observe that if $(\cdot,\cdot)$ denote the euclidean scalar product on $\mathrm{span}_{\mathbb{R}} \Delta \subseteq \mathfrak{h}^*$, then

$$\beta(\check{\alpha}) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 2\frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}}$$

equals $2 \cdot \cos\delta$, where δ is the angle between α and β. On the other hand $\beta(\check{\alpha}) \in \mathbb{Z}$, so that the only possible values are $\{0, \pm 1, \pm 2\}$, where ± 2 only arises for $\beta = \pm\alpha$ which is excluded if $\alpha + \beta \in \Delta$. Therefore

$$(\alpha,\alpha) \geq (\gamma,\gamma) = (\alpha,\alpha) + (\beta,\beta) + 2(\alpha,\beta) = 2(\alpha,\alpha) + 2(\alpha,\beta) = 2(\alpha,\alpha) \pm (\alpha,\alpha)$$

implies $(\alpha,\alpha) = (\gamma,\gamma)$, hence that γ is long.

We conclude that Δ_l satisfies

$$(\Delta_l + \Delta_l) \cap \Delta \subseteq \Delta_l,$$

and hence that we have an inclusion

$$\mathfrak{g}_{\Delta_l} \hookrightarrow \mathfrak{g}_\Delta.$$

It follows in particular that each Δ-graded Lie algebra $\mathfrak{g}$ can also be viewed as a (Δ, Δ_l)-graded Lie algebra and that each Δ-graded Lie algebra contains the Δ_l-graded Lie algebra

$$\mathfrak{g}_0 + \sum_{\alpha \in \Delta_l} \mathfrak{g}_\alpha.$$

The following table describes the systems Δ_l for the non-simply laced root systems.

Δ	B_r	C_r	F_4	G_2
Δ_l	D_r	$(A_1)^r$	D_4	A_2

In many cases the subalgebra $\mathfrak{g}_{\Delta_l}$ of $\mathfrak{g}_\Delta$ also has a description as the fixed point algebra of an automorphism γ fixing $\mathfrak{h}$ pointwise. Such an automorphism is given by a morphism

$$\chi \colon \mathbb{Z}[\Delta] \to \mathbb{K}^\times$$

of abelian groups via

$$\gamma.x_\alpha = \chi(\alpha) x_\alpha, \quad x_\alpha \in (\mathfrak{g}_\Delta)_\alpha.$$

For

$$\Delta = B_r = \{\pm(\varepsilon_i \pm \varepsilon_j) \colon i \neq j \in \{1, \ldots, r\}\} \cup \{\varepsilon_j \colon j = 1, \ldots, r\}$$

we define

$$\widetilde{\chi} \colon \mathbb{Z}[\Delta] \to \mathbb{Z}, \quad \sum_i n_i \varepsilon_i \mapsto \sum_i n_i.$$

Then

$$\widetilde{\chi}^{-1}(0) \cong A_{r-1}, \quad \Delta_s = \widetilde{\chi}^{-1}(2\mathbb{Z}+1) \quad \text{and} \quad \Delta_l = \widetilde{\chi}^{-1}(2\mathbb{Z}).$$

Therefore $\chi := (-1)^{\widetilde{\chi}}$ yields an involution γ_χ of $\mathfrak{g}_\Delta$ whose fixed point set is the subalgebra $\mathfrak{g}_{\Delta_l}$.

We likewise obtain for $\Delta = G_2$ a homomorphism $\widetilde{\chi} \colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ with

$$\Delta_l = \widetilde{\chi}^{-1}(3\mathbb{Z}).$$

If $1 \neq \zeta \in \mathbb{K}^\times$ satisfies $\zeta^3 = 1$, we then obtain via $\chi := \zeta^{\widetilde{\chi}}$ an automorphism γ_χ of order 3 whose fixed point set is $\mathfrak{g}_{\Delta_l} \cong \mathfrak{sl}_3(\mathbb{K})$.

Problem I. Determine a systematic theory of (Δ, Δ_0)-graded Lie algebras for suitable classes of pairs (Δ, Δ_0). ■

II. The coordinate algebra of a root graded Lie algebra

After having seen various examples of root graded locally convex Lie algebras in Section I, we now take a more systematic look at the structure of root graded Lie algebras. The main point of the present section is to associate to a Δ-graded Lie algebra $\mathfrak{g}$ a locally convex algebra $\mathcal{A}$, its coordinate algebra, together with a continuous bilinear map $\delta_{\mathcal{A}}: \mathcal{A} \times \mathcal{A} \to \mathrm{der}(\mathcal{A})$. The type of this coordinate algebra (associative, alternative, Jordan etc.) and the map $\delta_{\mathcal{A}}$ is determined by the type of the root system Δ. We will see that, together with the centralizer D of $\mathfrak{g}_\Delta$ in $\mathfrak{g}$, which acts by derivations on $\mathcal{A}$, the algebra $\mathcal{A}$ and the map $\delta_{\mathcal{A}}$ completely encode the Lie bracket of $\mathfrak{g}$. These results will be refined in Section IV, where we discuss isogeny classes of locally convex root graded Lie algebras and show that the universal covering Lie algebra of $\mathfrak{g}$ is already determined by the pair $(\Delta, \mathcal{A})$, resp., $(\mathcal{A}, \delta_{\mathcal{A}})$.

The algebraic results of this section are known; new is only that they still remain true in the context of locally convex Lie algebras, which requires additions arguments in several places and, in addition, a more coordinate free approach, because in the topological context we can never argue with bases of vector spaces. We also tried to put an emphasis on those arguments which can be given for general root graded Lie algebras without any case by case analysis, as f.i. in Theorem II.13. We do not go into the details of the exceptional and the low-dimensional cases. For the arguments leading to the coordinate algebra, we essentially follow the expositions in [ABG00], [BZ96] (see also [Se76] which already contains many of the key ideas and arguments).

Let $\mathfrak{g}$ be a locally convex root graded Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathfrak{g}_\Delta$ a grading subalgebra. We consider the adjoint representation of $\mathfrak{g}_\Delta$ on $\mathfrak{g}$. From (R3) we immediately derive that $\mathfrak{g}$ is a $\mathfrak{g}_\Delta$-weight module in the sense that the action of $\mathfrak{h}$ is diagonalized by the Δ-grading. Moreover, the set of weights is $\Delta \cup \{0\}$ and therefore finite, so that Proposition A.2 leads to:

Theorem II.1. *The Lie algebra $\mathfrak{g}$ is a semisimple $\mathfrak{g}_\Delta$-weight module with respect to $\mathfrak{h}$. All simple submodules are finite-dimensional highest weight modules. There are only finitely many isotypic components $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$, and for each isotypic component the projection $p_i: \mathfrak{g} \to \mathfrak{g}_i$ can be realized by an element of the center of $U(\mathfrak{g}_\Delta)$. In particular, each p_i is continuous.* ■

Now we take a closer look at the isotypic components of the Lie algebra $\mathfrak{g}$. Let $\Delta_l \subseteq \Delta$ denote the subset of long roots and $\Delta_s \subseteq \Delta$ the subset of short roots, where we put $\Delta_l := \Delta$ if all roots have the same length. Then the Weyl group $\mathcal{W}$ of Δ acts transitively on the sets of short and long roots, so that it has at most three orbits in $\Delta \cup \{0\}$. Hence only three types of simple $\mathfrak{g}_\Delta$-modules may contribute to $\mathfrak{g}$. First we have the adjoint module $\mathfrak{g}_\Delta$, and each root vector in $\mathfrak{g}_\alpha$ for a long root α generates a highest weight module isomorphic to $\mathfrak{g}_\Delta$. Therefore the weight set of each other type of non-trivial simple $\mathfrak{g}_\Delta$-module occurring in $\mathfrak{g}$ must be smaller than $\Delta \cup \{0\}$, which already implies that it coincides with $\Delta_s \cup \{0\}$. The corresponding simple $\mathfrak{g}_\Delta$-module is the *small adjoint module* $V_s \cong L(\lambda_s, \mathfrak{g}_\Delta)$, i.e., the simple module whose highest weight is the highest short root λ_s with respect to a positive system Δ^+. In view of Theorem II.1, we therefore have a $\mathfrak{g}_\Delta$-module decomposition

$$\mathfrak{g} \cong (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D, \tag{2.1}$$

where

$$A := \mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g}), \quad B := \mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s, \mathfrak{g}), \quad \text{and} \quad D := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_\Delta) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathbb{K}, \mathfrak{g})$$

are multiplicity spaces. We have

$$\mathfrak{g}_\alpha \cong \begin{cases} A & \text{for } \alpha \in \Delta_l \\ A \oplus B & \text{for } \alpha \in \Delta_s. \end{cases}$$

Our next goal is to construct an algebra structure on the topological direct sum $\mathcal{A} := A \oplus B$. This *coordinate algebra* will turn turn out be an important structural feature of $\mathfrak{g}$.

For each finite-dimensional $\mathfrak{g}_\Delta$-module M the space $\mathrm{Hom}_{\mathfrak{g}_\Delta}(M, \mathfrak{g})$ is a closed subspace of $\mathrm{Hom}(M, \mathfrak{g}) \cong M^* \otimes \mathfrak{g} \cong \mathfrak{g}^{\dim M}$, hence inherits a natural locally convex topology from the one on $\mathfrak{g}$, and the evaluation map

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(M, \mathfrak{g}) \otimes M \to \mathfrak{g}, \quad \varphi \otimes m \mapsto \varphi(m)$$

is an embedding of locally convex spaces onto the M-isotypic component of $\mathfrak{g}$. In this sense we think of $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$ as topological subspaces of $\mathfrak{g}$. We conclude that the addition map

$$(A \otimes \mathfrak{g}_\Delta) \times (B \otimes V_s) \times D \to \mathfrak{g}, \quad (a \otimes x, b \otimes y, d) \mapsto a \otimes x + b \otimes y + d$$

is a continuous bijection of locally convex spaces. That its inverse is also continuous follows from Theorem II.1 which ensures that the isotypic projections of $\mathfrak{g}$ are continuous linear maps. Therefore the decomposition (2.1) is a direct sum decomposition of locally convex spaces. If $\mathfrak{g}$ is a Fréchet space, we do not have to use Theorem II.1 because we can argue with the Open Mapping Theorem.

It is clear that the subspace $D = \mathfrak{z}_\mathfrak{g}(\mathfrak{g}_\Delta)$ is a closed Lie subalgebra. To obtain an algebra structure on $A \oplus B$. The following lemma is crucial for our analysis.

Lemma II.2. *Let M_j, $j = 1, 2, 3$, be finite-dimensional simple $\mathfrak{g}_\Delta$-modules and V_j, $j = 1, 2, 3$, locally convex spaces considered as trivial $\mathfrak{g}_\Delta$-modules. We consider the locally convex spaces $V_j \otimes M_j$ as $\mathfrak{g}_\Delta$-modules. Let $\beta_1, \ldots, \beta_k$ be a basis of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(M_1 \otimes M_2, M_3)$ and*

$$\alpha: V_1 \otimes M_1 \times V_2 \otimes M_2 \to V_3 \otimes M_3$$

a continuous equivariant bilinear map. Then there exist continuous bilinear maps

$$\gamma_1, \ldots, \gamma_k: V_1 \times V_2 \to V_3$$

with

$$\alpha(v_1 \otimes m_1, v_2 \otimes m_2) = \sum_{i=1}^k \gamma_i(v_1, v_2) \otimes \beta_i(m_1, m_2).$$

Proof. Fix $v_1 \in V_1$ and $v_2 \in V_2$. Then the map

$$\alpha_{v_1,v_2}: (m_1, m_2) \mapsto \alpha(v_1 \otimes m_1, v_2 \otimes m_2)$$

is an equivariant bilinear map $M_1 \times M_2 \to V_3 \otimes M_3$. As the image of α_{v_1,v_2} is finite-dimensional, there exist $w_1, \ldots, w_m \in V_3$ such that

$$\alpha_{v_1,v_2} = \sum_{j=1}^m \sum_{i=1}^k w_j \otimes \beta_i = \sum_{i=1}^k \sum_{j=1}^m w_j \otimes \beta_i.$$

This show that there are bilinear maps $\gamma_1, \ldots, \gamma_k: V_1 \times V_2 \to V_3$ with $\alpha = \sum_{i=1}^k \gamma_i \otimes \beta_i$. For each i there exists an element $a_i := \sum_j m_1^j \otimes m_2^j \in M_1 \otimes M_2$ with $\beta_i(a_i) \neq 0$ and $\beta_j(a_i) = 0$ for $i \neq j$. Then

$$\sum_j \alpha(v_1 \otimes m_1^j, v_2 \otimes m_2^j) = \gamma_i(v_1, v_2) \otimes \beta_i(a_i)$$

shows that each map γ_i is continuous. ■

Remark II.3. If $M_1 := \mathfrak{g}_\Delta$, $M_2 := V_s$, $M_3 = \mathbb{K}$ and $V_i := \mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i, \mathfrak{g})$, then the Lie bracket on $\mathfrak{g}$ induces a family of $\mathfrak{g}_\Delta$-equivariant continuous bilinear maps

$$V_i \otimes M_i \times V_j \otimes M_j \to M_k \otimes V_k.$$

To apply Lemma II.2, we therefore have to analyze the spaces $\mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i \otimes M_j, M_k)$.

The case $3 \in \{i,j\}$ is trivial because $D = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_\Delta)$ commutes with the action of $\mathfrak{g}_\Delta$, so that the bracket map induces continuous bilinear maps

$$D \times A \to A, \quad (d,a) \mapsto d.a \quad \text{and} \quad D \times B \to B, \quad (d,b) \mapsto d.b$$

with

$$[d, a \otimes x] = d.a \otimes x \quad \text{and} \quad [d, b \otimes y] = d.b \otimes y.$$

Interpreting A as the space $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g})$, the action of D on this space corresponds to

$$d.\varphi := (\mathrm{ad}\, d) \circ \varphi,$$

and likewise for $B = \mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s, \mathfrak{g})$.

We may therefore assume that $i, j \in \{1,2\}$. For $k = 3$, i.e., $M_k = \mathbb{K}$, the space

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i \otimes M_j, \mathbb{K}) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i, M_j^*)$$

is trivial for $i \neq j$ because M_1 and M_2 have different dimensions. For $M_1 = \mathfrak{g}_\Delta$ we have

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, \mathbb{K}) = \mathbb{K}\kappa,$$

where κ is the Cartan-Killing form. As V_s and V_s^* have the same weight set $\Delta_s = -\Delta_s$, they are isomorphic, and [Bou90, Ch. VII, §7, no. 5, Prop. 12] implies that

$$\dim \mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, \mathbb{K}) = \mathbb{K}\kappa_{V_s}$$

for a non-zero invariant symmetric bilinear form κ_{V_s} on V_s. The symmetry of the form follows from the fact that the highest weight λ_s of V_s is an integral linear combination of the base roots of Δ. ■

The complete information on the relevant Hom-spaces is given in Theorem II.6 below. We have to prepare the statement of this theorem with the discussion of some special cases.

Definition II.4. (a) On the space $M_n(\mathbb{K})$ of $n \times n$-matrices the matrix product is equivariant with respect to the adjoint action of the Lie algebra $\mathfrak{gl}_n(\mathbb{K})$. Hence the product $(x,y) \mapsto xy + yx$ does also have this property, and therefore the map

$$\mathfrak{sl}_n(\mathbb{K}) \times \mathfrak{sl}_n(\mathbb{K}) \to \mathfrak{sl}_n(\mathbb{K}), \quad (x,y) \mapsto x * y := xy + yx - \frac{2\,\mathrm{tr}(xy)}{n}\mathbf{1}$$

is equivariant with respect to the adjoint action of $\mathfrak{sl}_n(\mathbb{K})$. In the following $x * y$ will always denote this product.

(b) Let Ω be the non-degenerate alternating form on $\mathbb{K}^{2r}$ given by $\Omega(x,y) = (x,y)J(x,y)^\top$, where $J = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ (cf. Example I.7). For $X^\sharp := JX^\top J^{-1}$ we then have

$$\mathfrak{sp}_{2r}(\mathbb{K}) \cong \{X \in \mathfrak{gl}_{2r}(\mathbb{K}) : X^\sharp = -X\} \quad \text{and} \quad V_s \cong \{X \in \mathfrak{gl}_{2r}(\mathbb{K}) : X^\sharp = X, \mathrm{tr}\, X = 0\}.$$

This follows easily by decomposing $\mathfrak{gl}_{2r}(\mathbb{K})$ into weight spaces with respect to a Cartan subalgebra of $\mathfrak{sp}_{2r}(\mathbb{K})$. Here we use $(XY)^\sharp = Y^\sharp X^\sharp$ to see that V_s is invariant under brackets with $\mathfrak{sp}_{2r}(\mathbb{K})$ and satisfies $[V_s, V_s] \subseteq \mathfrak{sp}_{2r}(\mathbb{K})$. Moreover, the $*$-product restricts to $\mathfrak{sp}_{2r}(\mathbb{K})$-equivariant symmetric bilinear maps

$$\beta_{\mathfrak{g}}^V : \mathfrak{sp}_{2r}(\mathbb{K}) \times \mathfrak{sp}_{2r}(\mathbb{K}) \to V_s \quad \text{and} \quad \beta_V^V : V_s \times V_s \to V_s.$$ ■

Remark II.5. For $\Delta = A_r$, $r \geq 2$, the product $*$ is an equivariant symmetric product on $\mathfrak{g}_\Delta = \mathfrak{sl}_{r+1}(\mathbb{K})$. Of course, the same formula also yields for $r = 1$ a symmetric product, but in this case we have $x * y = 0$ (Lemma I.8). ■

Theorem II.6. *For the* Hom *-spaces of the different kinds of Lie algebras we have:*

(1) *For* Δ *not of type* A_r, $r \geq 2$, *the space* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, \mathfrak{g}_\Delta)$ *is one-dimensional and generated by the Lie bracket. For* Δ *of type* A_r, $r \geq 2$, *this space is two-dimensional and a second generator is the symmetric product* $*$ *on* $\mathfrak{g}_\Delta \cong \mathfrak{sl}_{r+1}(\mathbb{K})$.

(2) *If* Δ *is not of type* C_r, $r \geq 2$, *then* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, V_s) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes V_s, \mathfrak{g}_\Delta) = \{0\}$. *For* Δ *of type* C_r, $r \geq 2$, *and* $\mathfrak{g}_\Delta \cong \mathfrak{sp}_{2r}(\mathbb{K})$ *the space* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, V_s)$ *is generated by the* $*$*-product.*

(3) $\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, \mathfrak{g}_\Delta) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes V_s, V_s)$ *is one-dimensional and generated by the module structure on* V_s. *For* Δ *of type* C_r, *a basis of the first space is given by the bracket map on* $\mathfrak{gl}_{2r}(\mathbb{K})$, *restricted to* V_s.

(4) $\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, V_s)$ *is one-dimensional for* C_n, $n \geq 3$, F_4 *and* G_2, *and vanishes for* B_n, $n \geq 2$. *For* Δ *of type* C_n, *a basis of this space is given by the* $*$*-product.*

Proof. All these statements follow from Definition II.4 and the explicit decomposition of the tensor products, which are worked out in detail in [Se76, §A.2] (see also the Appendix of [BZ96] for a list of the decompositions). ■

Before we turn to a more explicit description of the Lie bracket on $\mathfrak{g}$, we have to fix a notation for the basis elements of the Hom-spaces mentioned above.

Definition II.7. First we recall the symmetric invariant bilinear form κ_{V_s} on V_s from Remark II.3. Let $\beta_{\mathfrak{g}}^V$ be a basis element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, V_s)$ if this space is non-zero, and $\beta_{\mathfrak{g},V}^{\mathfrak{g}}$ the corresponding basis element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes V_s, \mathfrak{g}_\Delta)$ which is related to $\beta_{\mathfrak{g}}^V$ by the relation

$$\kappa_{V_s}(\beta_{\mathfrak{g}}^V(x,y), v) = \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v), y), \quad x, y \in \mathfrak{g}_\Delta, v \in V_s.$$

Let $\beta_V^{\mathfrak{g}}: V_s \otimes V_s \to \mathfrak{g}_\Delta$ be the equivariant map defined by

$$\kappa_{V_s}(x.v, v') = \kappa(\beta_V^{\mathfrak{g}}(v,v'), x), \quad v, v' \in V_s, x \in \mathfrak{g}_\Delta.$$

Then

$$\kappa_{V_s}(x.v, v') = -\kappa_{V_s}(v, x.v') = -\kappa_{V_s}(x.v', v)$$

(cf. Remark II.3 for the symmetry of κ_{V_s}) implies that $\beta_V^{\mathfrak{g}}$ is skew-symmetric . We further write β_V^V for a basis element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, V_s)$.

For Δ of type C_r, $r \geq 2$, we take

$$\kappa_{V_s}(v,w) = \theta \operatorname{tr}(vw),$$

where the factor $\theta = 2(r+1)$ is determined by $\kappa(x,y) = \theta \operatorname{tr}(xy)$ ([Bou90, Ch. VIII]). We further put

$$\beta_{\mathfrak{g}}^V(x,y) := x * y, \quad \beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v) = x * v, \quad \beta_V^{\mathfrak{g}}(v,w) = [v,w], \quad \beta_V^V(v,w) = v * w,$$

and observe that from the embedding $\mathfrak{sp}_{2r}(\mathbb{K}) \hookrightarrow \mathfrak{sl}_{2r}(\mathbb{K})$ we get for $v \in V_s$:

$$\begin{aligned}\kappa_{V_s}(\beta_{\mathfrak{g}}^V(x,y), v) &= \theta \operatorname{tr}(x * y, v) = \theta \operatorname{tr}(xy + yx, v) \\ &= \theta \operatorname{tr}(vx + xv, y) = \theta \operatorname{tr}(x * v, y) = \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v), y).\end{aligned}$$

This calculation implies that our special definitions for type C_r are compatible with the general requirements on the relation between $\beta_{\mathfrak{g}}^V$ and $\beta_{\mathfrak{g},V}^{\mathfrak{g}}$. ■

In view of Lemma II.2 and Theorem II.6, there exist continuous bilinear maps

$$\begin{aligned}&\gamma_\pm^A: A \times A \to A, \quad \gamma_A^B: A \times A \to B, \quad \gamma_{A,B}^A: A \times B \to A, \quad \gamma_{A,B}^B: A \times B \to B, \\ &\gamma_B^A: B \times B \to A, \quad \gamma_B^B: B \times B \to B, \quad \delta_A^D: A \times A \to D, \quad \delta_B^D: B \times B \to D,\end{aligned}$$

such that the Lie bracket on

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D$$

satisfies

(B1) $[a\otimes x, a'\otimes x'] = \gamma_+^A(a,a')\otimes[x,x'] + \gamma_-^A(a,a')\otimes x*x' + \gamma_A^B(a,a')\otimes\beta_{\mathfrak g}^V(x,x') + \kappa(x,x')\delta_A^D(a,a')$,

for $a, a' \in A, x, x' \in \mathfrak{g}_\Delta$,

(B2) $[a\otimes x, b\otimes v] = \gamma_{A,B}^A(a,b)\otimes\beta_{\mathfrak g,V}^{\mathfrak g}(x,v) + \gamma_{A,B}^B(a,b)\otimes x.v$, for $a\in A, b\in B, x\in\mathfrak{g}_\Delta, v\in V_s$,

and for $b, b' \in B$ and $v, v' \in V_s$:

(B3) $[b\otimes v, b'\otimes v'] = \gamma_B^A(b,b')\otimes\beta_V^{\mathfrak g}(v,v') + \gamma_B^B(b,b')\otimes\beta_V^V(v,v') + \kappa_{V_s}(v,v')\delta_B^D(b,b')$.

From the skew-symmetry of the Lie bracket and the symmetry of $*$, it follows that γ_+^A is symmetric and γ_-^A is alternating. Further the symmetry of κ and κ_{V_s} implies that δ_A^D and δ_B^D are alternating. The skew-symmetry of $\beta_V^{\mathfrak g}$ implies that γ_B^A is symmetric and likewise the symmetry of $\beta_{\mathfrak g}^V$ entails that γ_A^B is skew-symmetric.

If Δ is not of type $A_r, r \geq 2$, then we put $\gamma_-^A = 0$. In all cases where the β-map vanishes, we define the corresponding γ-map to be zero.

Definition II.8. (The coordinate algebra $\mathcal{A}$ of $\mathfrak{g}$) (a) On A we define an algebra structure by

$$ab := \gamma_+^A(a,b) + \gamma_-^A(a,b),$$

and observe that

$$\gamma_+^A(a,b) = \frac{ab+ba}{2} \quad \text{and} \quad \gamma_-^A(a,b) = \frac{ab-ba}{2}.$$

We define a (not necessarily associative) algebra structure on $\mathcal{A} := A \oplus B$ by defining the product on $A\times A$ by $\gamma_+^A + \gamma_-^A + \gamma_A^B$, on $A\times B$ by $\gamma_{A,B}^A + \gamma_{A,B}^B$, on $B\times B$ by $\gamma_B^A + \gamma_B^B$, and on $B\times A$ by

$$ba := \gamma_{A,B}^B(a,b) - \gamma_{A,B}^A(a,b) = ab - 2\gamma_{A,B}^A(a,b).$$

Then

$$\gamma_{A,B}^A(a,b) = \frac{1}{2}[a,b] = \frac{1}{2}(ab-ba) \quad \text{and} \quad \gamma_{A,B}^B(a,b) = \frac{1}{2}(ab+ba).$$

(b) The space $D = \mathfrak{z}_{\mathfrak g}(\mathfrak{g}_\Delta)$ is a Lie subalgebra of $\mathfrak{g}$ which acts by derivations on $\mathcal{A}$ preserving both subspaces A and B. This easily follows from the fact that the actions of D and $\mathfrak{g}_\Delta$ on $\mathfrak{g}$ commute.

We combine the two maps δ_A^D and δ_B^D to an alternating bilinear map

$$\delta^D\colon \mathcal{A}\times\mathcal{A} \to D, \quad (a+b, a'+b') \mapsto \delta_A^D(a,a') + \delta_B^D(b,b')$$

vanishing on $A\times B$. ■

Example II.9. Below we briefly explain how the relations (B1)–(B3) simplify for the two classes of Lie algebras that we obtain if we distinguish Lie algebras of type A_r or C_r and all others. In some sense the information is more explicit for A_r and C_r. We first discuss the other cases.

(a) For Δ not of type A_r, $r\geq 2$, we have $\gamma_-^A = 0$, and for Δ not of type C_r, $r \geq 2$, we have $\gamma_A^B = \gamma_{A,B}^A = 0$ (Theorem II.6.(2)). If these two conditions are satisfied, then the product on $\mathcal{A}$ is given by

$$\begin{aligned}(a,b)\cdot(a',b') &= (\gamma_+^A(a,a') + \gamma_B^A(b,b'), \gamma_{A,B}^B(a,b') + \gamma_{A,B}^B(a',b) + \gamma_B^B(b,b')) \\ &= (aa' + \gamma_B^A(b,b'), ab' + ba' + \gamma_B^B(b,b')).\end{aligned}$$

In this case the Lie bracket in $\mathfrak{g}$ can be written as

$$[a\otimes x, a'\otimes x'] = aa'\otimes[x,x'] + \kappa(x,x')\delta_A^D(a,a'), \quad a,a'\in A, x,x'\in\mathfrak{g}_\Delta,$$

$$[a \otimes x, b \otimes v] = ab \otimes x.v, \quad a \in A, b \in B, x \in \mathfrak{g}_\Delta, v \in V_s,$$

and

$$[b \otimes v, b' \otimes v'] = \gamma_B^A(b,b') \otimes \beta_V^{\mathfrak{g}}(v,v') + \gamma_B^B(b,b') \otimes \beta_V^V(v,v') + \kappa_{V_s}(v,v')\delta_B^D(b,b').$$

(b) If Δ is of type A_r, $r \geq 1$, then $B = \{0\}$ and $\mathcal{A} = A$.

For Δ of type C_r, $r \geq 2$, we have $\beta_V^V(v,v') = v * v'$, which is symmetric. Therefore γ_B^B is skew-symmetric. In view of

$$bb' = \gamma_B^A(b,b') + \gamma_B^B(b,b'),$$

this implies

$$\gamma_B^A(b,b') = \frac{bb' + b'b}{2} \quad \text{and} \quad \gamma_B^B(b,b') = \frac{1}{2}[b,b'] := \frac{bb' - b'b}{2}.$$

For $r = 2$ we have $\beta_V^V = 0$ and therefore $\gamma_B^B = 0$ (Theorem II.6(4)). In this case $C_2 \cong B_2$ implies that V_s can be viewed as the representation of $\mathfrak{so}_{3,2}(\mathbb{K})$ on $\mathbb{K}^5$.

In contrast to the formulas under (a), we have for Δ of type A_r and C_r the unifying formulas

$$\begin{aligned}[a \otimes x, a' \otimes x'] &= \frac{aa' + a'a}{2} \otimes [x,x'] + \underbrace{\gamma_-^A(a,a')}_{=0 \text{ for } C_r} \otimes x * x' + \underbrace{\gamma_A^B(a,a')}_{=0 \text{ for } A_r} \otimes x * x' + \kappa(x,x')\delta_A^D(a,a'), \\ &= \frac{aa' + a'a}{2} \otimes [x,x'] + \frac{1}{2}[a,a'] \otimes x * x' + \kappa(x,x')\delta_A^D(a,a')\end{aligned}$$

for $a, a' \in A, x, x' \in \mathfrak{g}_\Delta$, where we use that

$$[a,a'] = aa' - a'a = 2(\gamma_-^A + \gamma_A^B)(a,a'), \quad a, a' \in A.$$

We further have for C_r:

$$[a \otimes x, b \otimes v] = \frac{1}{2}[a,b] \otimes x * v + \frac{1}{2}(ab + ba) \otimes [x,v], \quad a \in A, b \in B, x \in \mathfrak{g}_\Delta, v \in V_s,$$

and

$$[b \otimes v, b' \otimes v'] = \frac{1}{2}(bb' + b'b) \otimes [v,v'] + \frac{1}{2}[b,b'] \otimes v * v' + \kappa_{V_s}(v,v')\delta_B^D(b,b'). \qquad \blacksquare$$

Remark II.10. (Involution on $\mathcal{A}$) On the space $\mathcal{A} = A \oplus B$ we have a natural continuous involution $\sigma(a,b) := (a,-b)$ with

$$A = \mathcal{A}^\sigma := \{a \in \mathcal{A} : a^\sigma = a\} \quad \text{and} \quad B = \mathcal{A}^{-\sigma} := \{a \in \mathcal{A} : a^\sigma = -a\}.$$

The map σ is an algebra involution, i.e., $\sigma(xx') = \sigma(x')\sigma(x)$ for $x, x' \in \mathcal{A}$, if and only if

(I1) $\sigma(aa') = a'a$ for $a, a' \in A$, i.e., $\gamma_-^A = 0$,

(I2) $\sigma(ab) = -ba$ for $a \in A$, $b \in B$, which is always the case because $[a,b] \in B$, and

(I3) $\sigma(bb') = b'b$ for $b, b' \in B$, which means that γ_B^A is symmetric and γ_B^B is skew-symmetric.

Condition (I1) is satisfied for any Δ not of type A_r, $r \geq 2$. For condition (I3), we recall that γ_B^A is symmetric because $\beta_V^{\mathfrak{g}}$ is skew-symmetric (Definition II.7). That γ_B^B is skew-symmetric means that β_V^V is symmetric, which is the case for Δ of type C_n, where $\beta_V^V(v,v') = v * v'$. It is also the case for Δ of type F_4, but not for type G_2, where it is the Malcev product on the pure octonions (cf. [ABG00, p.521]). $\blacksquare$

Remark II.11. (a) (The identity in $\mathcal{A}$) The inclusion $\mathfrak{g}_\Delta \hookrightarrow \mathfrak{g}$ is an element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g}) = A \subseteq \mathcal{A}$ which we call $\mathbf{1}$. It satisfies

$$[\mathbf{1} \otimes x, a \otimes y] = x.(a \otimes y) = a \otimes [x,y], \quad \text{and} \quad [\mathbf{1} \otimes x, b \otimes v] = b \otimes x.v.$$

This means that

$$\mathbf{1}a = a\mathbf{1} = a \quad \text{and} \quad \delta^D(\mathbf{1}, a) = 0 \quad \text{for all} \quad a \in \mathcal{A}.$$

In particular, $\mathbf{1}$ is an identity element in $\mathcal{A}$.

(b) The subspace A is a subalgebra of $\mathcal{A}$ if and only if $\gamma_A^B = 0$. If this map is non-zero, then $\beta_{\mathfrak{g}}^V \neq 0$ and Δ is of type C_r, $r \geq 2$ (Theorem II.6(2)). In all other cases A is a subalgebra of $\mathcal{A}$, and this subalgebra is commutative if and only if γ_-^A vanishes, which in turn is the case if Δ is not of type A_r or $C_r, r \geq 2$. $\blacksquare$

Remark II.12. (a) Axiom (R4) for a locally convex root graded Lie algebra is equivalent to the condition that the D-parts of the brackets $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ span a dense subspace of D. First we observe that only brackets of the type (B1) and (B3) have a non-zero D-part. Using the coordinate structure (B1)–(B3) of $\mathfrak{g}$, we can therefore translate (R4) into the fact that $\mathrm{im}(\delta^D_A) + \mathrm{im}(\delta^D_B) = \mathrm{im}(\delta^D)$ spans a dense subspace of D.
(b) Recall from Remark II.5 that for each root α we have $x_\alpha * x_{-\alpha} = 0$, and therefore, for all $a, a' \in A$, the simplification

$$[a \otimes x_\alpha, a' \otimes x_{-\alpha}] = \gamma^A_+(a,a') \otimes [x_\alpha, x_{-\alpha}] + \kappa(x_\alpha, x_{-\alpha})\delta^D_A(a,a').$$

Hence

$$[a \otimes x_\alpha, a' \otimes x_{-\alpha}] - [a' \otimes x_\alpha, a \otimes x_{-\alpha}] = 2\kappa(x_\alpha, x_{-\alpha})\delta^D_A(a,a'). \qquad \blacksquare$$

Theorem II.13. *The alternating map $\delta^D\colon \mathcal{A} \times \mathcal{A} \to D$ satisfies the cocycle condition*

$$\delta^D(aa', a'') + \delta^D(a'a'', a) + \delta^D(a''a, a') = 0, \quad a, a', a'' \in \mathcal{A}, \tag{2.2}$$

and

$$\delta^D(d.a, a') + \delta^D(a, d.a') = [d, \delta^D(a,a')] \quad d \in D, a, a' \in \mathcal{A}. \tag{2.3}$$

Proof. The plan of the proof is as follows. We will use the fact that (B1)–(B3) satisfy the Jacobi identity to obtain four relations for δ^D, which then will lead to the required cocycle condition for δ^D, where $0,1,2,3$ elements among a, a', a'' are contained in A, and the others in B.

Step 1: For $a, a', a'' \in A$ and $x, x', x'' \in \mathfrak{g}_\Delta$, we use (B1) to see that the D-component of

$$[[a \otimes x, a' \otimes x'], a'' \otimes x'']$$

is

$$\kappa([x,x'],x'')\delta^D_A(\gamma^A_+(a,a'),a'') + \kappa(x * x', x'')\delta^D_A(\gamma^A_-(a,a'),a''). \tag{2.4}$$

From the invariance and the symmetry of κ, we derive

$$\kappa([x,x'],x'') = \kappa(x,[x',x'']) = \kappa([x',x''],x),$$

i.e., the cyclic invariance of $\kappa([x,x'],x'')$. If Δ is not of type A_r, $r \geq 2$, then $x * x' = 0$, and the second summand in (2.4) vanishes. But for Δ of type A_r we have $\kappa(x,x') = 2(r+1)\,\mathrm{tr}(xx')$ and therefore

$$\kappa(x * x', x'') = 2(r+1)\,\mathrm{tr}\Big(\big(xx' + x'x - \frac{2\,\mathrm{tr}(xx')}{r+1}\mathbf{1}\big) \cdot x''\Big) = 2(r+1)\big(\mathrm{tr}(xx'x'') + \mathrm{tr}(x'xx'')\big).$$

Hence we get in all cases the cyclic invariance of $\kappa(x*x', x'')$. Therefore the Jacobi identity in $\mathfrak{g}$, applied to the D-components of the form (2.4), leads to

$$\begin{aligned} 0 &= \sum_{\text{cycl.}} \kappa([x,x'],x'')\delta^D_A(\gamma^A_+(a,a'),a'') + \kappa(x * x', x'')\delta^D_A(\gamma^A_-(a,a'),a'') \\ &= \kappa([x,x'],x'') \sum_{\text{cycl.}} \delta^D_A(\gamma^A_+(a,a'),a'') + \kappa(x * x', x'') \sum_{\text{cycl.}} \delta^D_A(\gamma^A_-(a,a'),a''). \end{aligned}$$

For $x \in \mathfrak{g}_\alpha$ and $x' \in \mathfrak{g}_{-\alpha}$ with $[x,x'] = \check{\alpha}$ we have $x * x' = 0$ (Remark II.5), and we thus obtain

$$\sum_{\text{cycl.}} \delta^D_A(\gamma^A_+(a,a'),a'') = 0.$$

Choosing x, x', x'' such that $\kappa(x{*}x', x'') \neq 0$, we also obtain $\sum_{\rm cycl.} \delta_A^D(\gamma_-^A(a,a'), a'') = 0$. Adding these two identities leads to

$$\sum_{\rm cycl.} \delta_A^D(aa', a'') = 0.$$

Step 2: For $a, a' \in A$, $b \in B$, and $x, x' \in \mathfrak{g}_\Delta$, $v \in V_s$, we get for the D-component of

$$0 = [[a \otimes x, a' \otimes x'], b \otimes v] + [[a' \otimes x', b \otimes v], a \otimes x] + [[b \otimes v, a \otimes x], a' \otimes x']$$

the relation

$$\begin{aligned} 0 &= \kappa_{V_s}(\beta_{\mathfrak g}^V(x,x'), v)\delta_B^D(\gamma_A^B(a,a'), b) + \kappa(\beta_{\mathfrak{g},V}^{\mathfrak g}(x', v), x)\delta_A^D(\gamma_{A,B}^A(a',b), a) \\ &\quad - \kappa(\beta_{\mathfrak{g},V}^{\mathfrak g}(x, v), x')\delta_A^D(\gamma_{A,B}^A(a,b), a') \\ &= \kappa(\beta_{\mathfrak{g},V}^{\mathfrak g}(x,v), x')\big(\delta_B^D(\gamma_A^B(a,a'), b) + \delta_A^D(\gamma_{A,B}^A(a',b), a) - \delta_A^D(\gamma_{A,B}^A(a,b), a')\big) \\ &= \kappa(\beta_{\mathfrak{g},V}^{\mathfrak g}(x,v), x')\big(\delta(aa', b) + \delta(a'b, a) + \delta(ba, a')\big) \end{aligned}$$

because δ^D vanishes on $A \times B$, the A-component $\gamma_{A,B}^A(a,b)$ of ab is skew-symmetric in a and b, and

$$\kappa(\beta_{\mathfrak{g},V}^{\mathfrak g}(x,v), x') = \kappa_{V_s}(\beta_{\mathfrak g}^V(x,x'), v)$$

is symmetric in x and x' (Definition II.7). We conclude that

$$\delta^D(aa', b) + \delta^D(a'b, a) + \delta^D(ba, a') = 0.$$

Step 3: For $a \in A$, $b, b' \in B$, and $x \in \mathfrak{g}_\Delta$, $v, v' \in V_s$, we get from the D-components of

$$0 = [[b \otimes v, b' \otimes v'], a \otimes x] + [[b' \otimes v', a \otimes x], b \otimes v] + [[a \otimes x, b \otimes v], b' \otimes v']$$

the relation

$$\begin{aligned} 0 &= \kappa(\beta_V^{\mathfrak g}(v,v'), x)\delta_A^D(\gamma_B^A(b,b'), a) - \kappa_{V_s}(x.v', v)\delta_B^D(\gamma_{A,B}^B(a,b'), b) \\ &\quad + \kappa_{V_s}(x.v, v')\delta_B^D(\gamma_{A,B}^B(a,b), b') \\ &= \kappa_{V_s}(x.v, v')\big(\delta_A^D(\gamma_B^A(b,b'), a) + \delta_B^D(\gamma_{A,B}^B(a,b'), b) + \delta_B^D(\gamma_{A,B}^B(a,b), b')\big) \\ &= \kappa_{V_s}(x.v, v')\big(\delta^D(bb', a) + \delta^D(b'a, b) + \delta^D(ab, b')\big) \end{aligned}$$

because δ^D vanishes on $A \times B$ and the B-component $\gamma_{A,B}^B(a,b)$ of ab is symmetric in a and b. We conclude that

$$0 = \delta^D(bb', a) + \delta^D(b'a, b) + \delta^D(ab, b').$$

Step 4: For $b, b', b'' \in A$ and $v, v', v'' \in V_s$, the D-component of $[[b \otimes v, b' \otimes v'], b'' \otimes v'']$ is

$$\kappa_{V_s}(\beta_V^V(v,v'), v'')\delta_B^D(\gamma_B^B(b,b'), b'').$$

We claim that $F(v,v',v'') := \kappa_{V_s}(\beta_V^V(v,v'), v'')$ satisfies

$$F(v,v',v'') = F(v',v'',v) \quad \text{for} \quad v, v', v'' \in V_s.$$

Fix $v', v'' \in V_s$. Then the map

$$V_s \to \mathbb{K}, \quad v \mapsto \kappa_{V_s}(\beta_V^V(v,v'), v'') = F(v,v',v'')$$

can be written as

$$V_s \to \mathbb{K}, \quad v \mapsto \kappa_{V_s}(T(v',v''), v)$$

for a unique element $T(v',v'') \in V_s$. From the $\mathfrak{g}_\Delta$-equivariance properties and the uniqueness, we derive that $T\colon V_s \times V_s \to V_s$ is $\mathfrak{g}_\Delta$-equivariant, hence of the form $\lambda\beta_V^V$ for some $\lambda \in \mathbb{K}$

(Theorem II.6). As F is symmetric in the first two arguments, F is an eigenvector for the action of S_3 on $\mathrm{Lin}(V \times V \times V, \mathbb{K})$. Then F is fixed by the commutator subgroup of S_3, hence fixed under cyclic rotations, and this implies $\lambda = 1$.

Therefore the Jacobi identity in $\mathfrak{g}$, applied to the D-components above, leads to

$$0 = \sum_{\text{cycl.}} \delta_B^D(\gamma_B^B(b,b'),b'') = \sum_{\text{cycl.}} \delta^D(bb',b'').$$

Combining all four case, we see that δ^D satisfies the cocycle identity (2.2) because the function

$$G\colon \mathcal{A}^3 \to D, \quad (a,b,c) \mapsto \delta^D(ab,c) + \delta^D(bc,a) + \delta^D(ca,b)$$

is cyclically invariant and trilinear, so that it suffices to verify it in the four cases we dealt with above.

To verify the relation (2.3), we first use (B1) and (B3) to see that a comparison of the D-components of the brackets

$$[d,[a\otimes x, a'\otimes x']] = [d.a\otimes x, a'\otimes x'] + [a\otimes x, d.a'\otimes x'], \quad a,a' \in A, x,x' \in \mathfrak{g}_\Delta$$

and

$$[d,[b\otimes v, b'\otimes v']] = [d.b\otimes v, b'\otimes v'] + [b\otimes v, d.b'\otimes v'], \quad b,b' \in B, v,v' \in V_s$$

leads to (2.3). ■

Definition II.14. Let $\mathfrak{g}$ be a Δ-graded Lie algebra. From the isotypic decomposition of $\mathfrak{g}$ with respect to $\mathfrak{g}_\Delta$, we then obtain three items which, in view of (B1)–(B3), completely encode the structure of $\mathfrak{g}$:

(1) the coordinate algebra $\mathcal{A} = A \oplus B$,

(2) the Lie algebra D and its representation by derivations on $\mathcal{A}$ preserving the subspaces A and B, and

(3) the cocycle $\delta^D\colon \mathcal{A}\times\mathcal{A} \to D$ (Theorem II.13).

All other data that enter the description of the bracket in $\mathfrak{g}$ only depends on the Lie algebra $\mathfrak{g}_\Delta$ and the module V_s (Theorem II.6). We therefore call the triple $(\mathcal{A}, D, \delta^D)$ the *coordinate structure of the Δ-graded Lie algebra $\mathfrak{g}$*. ■

Theorem II.15. *Let $\mathfrak{g}$ be a root graded Lie algebra with coordinate structure $(\mathcal{A}, D, \delta^D)$. Further let $\widehat{D}$ be a locally convex Lie algebra acting by derivations preserving A and B on $\mathcal{A}$, and*

$$\delta^{\widehat{D}}\colon \mathcal{A}\times\mathcal{A} \to \widehat{D}$$

a continuous bilinear map such that

(1) $\delta^{\widehat{D}}(aa',a'') + \delta^{\widehat{D}}(a'a'',a) + \delta^{\widehat{D}}(a''a,a') = 0$ *for* $a,a',a'' \in \mathcal{A}$,

(2) *the map* $\widehat{D}\times\mathcal{A} \to \mathcal{A}, (d,a) \mapsto d.a$ *is continuous,*

(3) $d.\delta^{\widehat{D}}(a,a') = \delta^{\widehat{D}}(d.a,a') + \delta^{\widehat{D}}(a,d.a')$ *for* $a,a' \in \mathcal{A}$, $d \in \widehat{D}$, *and*

(4) $\delta^{\widehat{D}}(a,a').a'' = \delta^D(a,a').a''$ *for* $a,a',a'' \in \mathcal{A}$, *and*

(5) $\delta^{\widehat{D}}(A\times B) = \{0\}$.

Then we obtain on

$$\widehat{\mathfrak{g}} := (A\otimes\mathfrak{g}_\Delta) \oplus (B\otimes V_s) \oplus \widehat{D}$$

a Lie bracket by

$$[d, a\otimes x + b\otimes v + d'] = d.a\otimes x + d.b\otimes v + [d,d'],$$

and

$$[a\otimes x, a'\otimes x'] = \gamma_+^A(a,a')\otimes[x,x'] + \gamma_-^A(a,a')\otimes x * x' + \gamma_A^B(a,a')\otimes\beta_{\mathfrak{g}}^V(x,x') + \kappa(x,x')\delta^{\widehat{D}}(a,a'),$$

$$[a\otimes x, b\otimes v] = \frac{ab-ba}{2}\otimes \beta^{\mathfrak g}_{\mathfrak g,V}(x,v) + \frac{ab+ba}{2}\otimes x.v,$$
$$[b\otimes v, b'\otimes v'] = \gamma_B^A(b,b')\otimes \beta^{\mathfrak g}_V(v,v') + \gamma^B_B(b,b')\otimes \beta^V_V(v,v') + \kappa_{V_s}(v,v')\delta^{\widehat D}(b,b').$$

If $\mathrm{im}(\delta^{\widehat D})$ *is dense in* $\widehat D$, *then* $\widehat{\mathfrak g}$ *is a* Δ*-graded Lie algebra with coordinate structure* $(\mathcal A, \widehat D, \delta^{\widehat D})$.

Proof. From the definition and condition (3) it directly follows that the operators $\mathrm{ad}\, d$, $d\in \widehat D$, are derivations for the bracket. Therefore it remains to verify the Jacobi identity for triples of elements in $A\otimes \mathfrak g_\Delta$ or $B\otimes V_s$. In view of (4) and the fact that the Jacobi identity is satisfied in $\mathfrak g$, it suffices to consider the $\widehat D$-components of triple brackets. Reading the proof of Theorem II.13 backwards, it is easy to see that (1) and (4), applied to the four cases corresponding to how many among the a, a', a'' are contained in A, resp., B, leads to the Jacobi identity for triple brackets of elements in $A\otimes\mathfrak g_\Delta$, resp., $B\otimes V_s$.

For this argument one has to observe that in the case $a,a',a''\in A$ the relation (1) for all a,a',a'' also implies

$$\sum_{\mathrm{cycl.}} \delta^{\widehat D}(\gamma^A_+(a,a'),a'') + \delta^{\widehat D}(\gamma^A_+(a',a''),a) + \delta^{\widehat D}(\gamma^A_+(a'',a),a')$$
$$= \delta^{\widehat D}(aa',a'') + \delta^{\widehat D}(a'a'',a) + \delta^{\widehat D}(a''a,a') + \delta^{\widehat D}(a'a,a'') + \delta^{\widehat D}(aa'',a') + \delta^{\widehat D}(a''a',a) = 0$$

and

$$\delta^{\widehat D}(\gamma^A_-(a,a'),a'') + \delta^{\widehat D}(\gamma^A_-(a',a''),a) + \delta^{\widehat D}(\gamma^A_-(a'',a),a')$$
$$= \delta^{\widehat D}(aa',a'') + \delta^{\widehat D}(a'a'',a) + \delta^{\widehat D}(a''a,a') - \delta^{\widehat D}(a'a,a'') - \delta^{\widehat D}(aa'',a') - \delta^{\widehat D}(a''a',a) = 0.$$

■

Examples II.16. We now take a second look at the examples in Section I.

(a) For the algebras of the type $\mathfrak g = A\otimes\mathfrak g_\Delta$ (Example I.4), it is clear that $\mathcal A = A$ is the corresponding coordinate algebra, and $B = D = \{0\}$.

(b) For $\mathfrak g = \mathfrak{sl}_n(A)$ (Example I.5), formula (1.1) for the bracket shows that $\mathcal A = A$ is the coordinate algebra of $\mathfrak g$, $D = \overline{[A,A]}\otimes \mathbf 1 \cong \overline{[A,A]}$, and

$$\delta^D(a,b) = \frac{1}{2n^2}[a,b]$$

because $\kappa(x,y) = 2n\,\mathrm{tr}(xy)$ for $x,y\in\mathfrak{sl}_n(\mathbb K)$.

(c) For $\mathfrak g = \mathfrak{sp}_{2n}(\mathcal A,\sigma)$ (Example I.7), which is of type C_n, we see with the formula in Example II.9(b) that $A = \mathcal A^\sigma$, $B = \mathcal A^{-\sigma}$, $D = \overline{[A,A]}^{-\sigma}\otimes\mathbf 1 \cong \overline{[A,A]}^{-\sigma}$, and that $\mathcal A$ is the coordinate algebra. In this case we have $\gamma^B_A = 0$ because $A = \mathcal A^\sigma$ is a subalgebra of $\mathcal A$.

From $\kappa(x,y) = \theta\,\mathrm{tr}(xy)$, $\kappa_{V_s}(x,y) = \theta\,\mathrm{tr}(xy)$ $(\theta = 2(n+1))$, and

$$\kappa(x,x')\delta^D_A(a,a') = [a,a']\otimes\frac{\mathrm{tr}(xx')}{2n}\mathbf 1 \quad\text{and}\quad \kappa_{V_s}(v,v')\delta^D_B(b,b') = [b,b']\otimes\frac{\mathrm{tr}(vv')}{2n}\mathbf 1,$$

we get

$$\delta^D(\alpha,\beta) = \frac{1}{2\theta n}\frac{1}{2}([\alpha,\beta]-[\alpha,\beta]^\sigma)\otimes\mathbf 1 = \frac{1}{4\theta n}([\alpha,\beta]+[\alpha^\sigma,\beta^\sigma])\otimes\mathbf 1,$$

because

$$[a+b,a'+b'] = \underbrace{[a,a']+[b,b']}_{\in\mathcal A^{-\sigma}} + \underbrace{[a,b']+[b,a']}_{\in\mathcal A^\sigma}, \quad a\in\mathcal A^\sigma, b\in\mathcal A^{-\sigma}.$$

(d) For $\mathfrak g = \mathrm{TKK}(J)$ for a Jordan algebra J (Example I.9), we also see directly from the definition that J is the coordinate algebra of $\mathfrak g$ and $D = \langle J,J\rangle$. We have $\kappa(x,y) = 4\,\mathrm{tr}(xy)$ for $x,y\in\mathfrak{sl}_2(\mathbb K)$, and therefore

$$\delta^D(a,b) = \delta_J(a,b) = \frac14\langle a,b\rangle.$$ ■

The following proposition deals with the special case where B is trivial and the root system is not of type A_r. In this case it contains complete information on the possibilities of the coordinate algebra. For the root systems Δ of type D_r, $r\geq 4$, and E_r, it provides a full description of all Δ-graded Lie algebras (cf. [BM92] for the algebraic version of this result).

Proposition II.17. (a) *If $B = \{0\}$ and Δ is not of type A_r, $r \geq 1$, then the bracket of $\mathfrak{g}$ is of the form*

$$[a \otimes x, a' \otimes x'] = ab \otimes [x, x'] + \kappa(x, x')\delta^D(a, a'),$$

where A is a commutative associative unital algebra and D is central in $\mathfrak{g}$, i.e., D acts trivially on A.

(b) *If, conversely, $\widehat{D}$ is a locally convex space, A a locally convex unital commutative associative algebra and the continuous alternating bilinear map $\delta^{\widehat{D}}: A \times A \to \widehat{D}$ satisfies*

$$\delta^{\widehat{D}}(aa', a'') + \delta^{\widehat{D}}(a'a'', a) + \delta^{\widehat{D}}(a''a, a') = 0, \quad a, a', a'' \in A,$$

then

$$\widehat{\mathfrak{g}} := (A \otimes \mathfrak{g}_\Delta) \oplus \widehat{D}$$

is a Lie algebra with respect to the bracket

$$[a \otimes x + d, a' \otimes x' + d'] = aa' \otimes [x, x'] + \kappa(x, x')\delta^{\widehat{D}}(a, a').$$

Proof. (a) Our assumption that Δ is not of type A_1 means that $\dim \mathfrak{h} \geq 2$, so that there exist roots α and β with $\beta \neq \pm\alpha$. Moreover, the exclusion of A_r, $r \geq 2$, implies $\gamma_-^A = 0$, so that by consideration of the $A \otimes \mathfrak{g}_\Delta$-component of the cyclic sum $\sum_{\text{cycl.}}[[a \otimes x, a' \otimes x'], a'' \otimes x'']$, the Jacobi identity in $\mathfrak{g}$ implies

$$\sum_{\text{cycl.}} (aa')a'' \otimes [[x, x'], x''] + \delta^D(a, a').a'' \otimes \kappa(x, x')x'' = 0 \tag{2.6}$$

for $a, a', a'' \in A$ and $x, x', x'' \in \mathfrak{g}_\Delta$.

Let $x \in \mathfrak{g}_\alpha$, $x' \in \mathfrak{g}_\beta$, and $x'' \in \mathfrak{h}$. Then $\kappa(x, x') = \kappa(x', x'') = \kappa(x'', x) = 0$, and therefore

$$\begin{aligned}
&(aa')a'' \otimes [[x, x'], x''] + (a'a'')a \otimes [[x', x''], x] + (a''a)a' \otimes [[x'', x], x'] \\
&= -(\alpha + \beta)(x'')(aa')a'' \otimes [x, x'] - \beta(x'')(a'a'')a \otimes [x', x] + \alpha(x'')(a''a)a' \otimes [x, x'] \\
&= \big(-(\alpha + \beta)(x'')(aa')a'' + \beta(x'')(a'a'')a + \alpha(x'')(a''a)a' \big) \otimes [x, x'].
\end{aligned}$$

For $\beta(x'') = 0$ and $\alpha(x'') = 1$, we now get

$$(aa')a'' = (a'a'')a = a(a'a'').$$

Therefore the commutative algebra A is associative.

It remains to see that D is central. We consider the identity (2.6) with $x \in \mathfrak{g}_\alpha$, $x' \in \mathfrak{g}_{-\alpha}$ and $x'' = \check{\alpha}$. Then $\kappa(x, x') \neq 0 = \kappa(x, x'') = \kappa(x', x'')$. Further

$$\sum_{\text{cycl.}} (aa')a'' \otimes [[x, x'], x''] = (aa')a'' \otimes \sum_{\text{cycl.}} [[x, x'], x''] = 0$$

follows from the fact that A is commutative and associative, and the Jacobi identity in $\mathfrak{g}_\Delta$. Hence (2.6) leads to $\delta^D(a, a').a'' = 0$. This means that $\delta^D(A, A)$ is central in $\mathfrak{g}$, and since this set spans a dense subspace of D (Remark II.12(a)), the subalgebra D of $\mathfrak{g}$ is central.

(b) For the converse, we first observe that the map

$$\omega: (A \otimes \mathfrak{g}_\Delta) \times (A \otimes \mathfrak{g}_\Delta) \to \widehat{D}, \quad \omega(a \otimes x, a' \otimes x') \to \kappa(x, x')\delta^{\widehat{D}}(a, a')$$

is a Lie algebra cocycle because

$$\sum_{\text{cycl.}} \omega([a \otimes x, a' \otimes x'], a'' \otimes x'') = \sum_{\text{cycl.}} \kappa([x, x'], x'')\delta^{\widehat{D}}(aa', a'') = \kappa([x, x'], x'') \sum_{\text{cycl.}} \delta^{\widehat{D}}(aa', a'') = 0.$$

From this the Jacobi identity of $\widehat{\mathfrak{g}}$ follows easily, and the map $\widehat{\mathfrak{g}} \to A \otimes \mathfrak{g}_\Delta$ with kernel $\widehat{D}$ defines a central extension of the Lie algebra $A \otimes \mathfrak{g}_\Delta$ by $\widehat{D}$ (cf. Example I.4). ■

Definition II.18. (The Weyl group of $\mathfrak{g}$) Let $\alpha \in \Delta$ and $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ with $[x_\alpha, x_{-\alpha}] = \check{\alpha}$. We consider the automorphism

$$\sigma_\alpha := e^{\mathrm{ad} x_\alpha} e^{-\mathrm{ad}\, x_{-\alpha}} e^{\mathrm{ad}\, x_\alpha} \in \mathrm{Aut}(\mathfrak{g})$$

which is defined because the operators $\mathrm{ad}\, x_{\pm\alpha}$ are nilpotent. If $h \in \ker\alpha \subseteq \mathfrak{h}$, then h commutes with $x_{\pm\alpha}$, so that $\sigma_\alpha.h = h$. We claim that $\sigma_\alpha.\check{\alpha} = -\check{\alpha}$.

In $\mathrm{SL}_2(\mathbb{K})$ we have

$$S := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As $\sigma_\alpha|_{\mathfrak{g}_\Delta}$ corresponds to conjugation with S in $\mathfrak{sl}_2(\mathbb{K})$, we obtain

$$\sigma_\alpha.\check{\alpha} = -\check{\alpha}, \quad \sigma_\alpha.x_\alpha = -x_{-\alpha} \quad \text{and} \quad \sigma_\alpha.x_{-\alpha} = -x_\alpha.$$

We conclude that $\sigma_\alpha|_{\mathfrak{h}}$ coincides with the reflection in the hyperplane $\check{\alpha}^\perp$:

$$\sigma_\alpha(h) = h - \alpha(h)\check{\alpha} \quad \text{for } h \in \mathfrak{h}$$

(cf. [MP95, Props. 4.1.3, 6.1.8]). The corresponding reflection on $\mathfrak{h}^*$ is given by

$$r_\alpha: \mathfrak{h}^* \to \mathfrak{h}^*, \quad \beta \mapsto \beta - \beta(\check{\alpha})\alpha.$$

This leads to

$$\sigma_\alpha(\mathfrak{g}_\beta) = \mathfrak{g}_{r_\alpha.\beta}, \quad \beta \in \Delta \cup \{0\}.$$

We call

$$\mathcal{W} := \langle r_\alpha : \alpha \in \Delta\rangle \subseteq \mathrm{GL}(\mathfrak{h})$$

the *Weyl group of* $\mathfrak{g}$.

From the preceding calculation we obtain in particular that $\sigma_\alpha \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) := N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h}) := \{\varphi \in \mathrm{Aut}(\mathfrak{g}) : \varphi(\mathfrak{h}) = \mathfrak{h}\}$. This group contains the subgroup

$$Z_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h}) = \{\varphi \in \mathrm{Aut}(\mathfrak{g}) : \varphi|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\} \cong \mathrm{Hom}(\mathbb{Z}[\Delta], \mathbb{K}^\times) \cong (\mathbb{K}^\times)^r.$$

We therefore have a group extension

$$(\mathbb{K}^\times)^r \hookrightarrow \widehat{\mathcal{W}} \twoheadrightarrow \mathcal{W},$$

where $\widehat{\mathcal{W}} \subseteq \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ is the inverse image of $\mathcal{W}$ under the restriction homomorphism to $\mathfrak{h}$. This extension does not split for $\Delta(\check{\Delta}) \not\subseteq 2\mathbb{Z}$ because in this case there exists a root α with $1 \in \Delta(\check{\alpha})$, which implies that σ_α is of order 4. ■

Example II.19. (cf. [Ti62]) We take a closer look at the case $\Delta = A_1 = \{\pm\alpha\}$. We write

$$\mathfrak{g}_\Delta = \mathrm{span}\{\check{\alpha}, x_\alpha, x_{-\alpha}\}$$

with

$$x_\alpha \in \mathfrak{g}_\alpha, \quad x_{-\alpha} \in \mathfrak{g}_{-\alpha}, \quad \check{\alpha} = [x_\alpha, x_{-\alpha}].$$

Then formula (B1) for the product on A leads to

$$[a \otimes x_\alpha, [\mathbf{1} \otimes x_{-\alpha}, b \otimes x_\alpha]] = [a \otimes x_\alpha, -b \otimes h] = ab \otimes [h, x_\alpha] = 2ab \otimes x_\alpha,$$

and hence to

$$ab \otimes x_\alpha = \frac{1}{2}[a \otimes x_\alpha, [\mathbf{1} \otimes x_{-\alpha}, b \otimes x_\alpha]] = \frac{1}{2}[a \otimes x_\alpha, [x_{-\alpha}, b \otimes x_\alpha]].$$

Identifying A via the map $a \mapsto a \otimes x_\alpha$ with $\mathfrak{g}_\alpha$, the product on A is given by

$$ab := \frac{1}{2}[a, [x_{-\alpha}, b]].$$

We recall from Definition II.18 the automorphism σ_α of $\mathfrak{g}$. From the $\mathfrak{g}_\Delta$-module decomposition of $\mathfrak{g}$ it follows directly that $\sigma_\alpha^2 = \mathrm{id}_\mathfrak{g}$ because the restriction of σ_α to $\mathfrak{g}_\Delta$ is an involution. Moreover, $\sigma_\alpha(x_\alpha) = -x_{-\alpha}$. To see that the product on $\mathfrak{g}_\alpha$ defines a Jordan algebra structure on A, we first observe that Theorem C.3 implies that

$$\{x, y, z\} := \frac{1}{2}[[x, \sigma_\alpha.y], z]$$

defines a Jordan triple structure, and hence that $ab = \{a, -x_\alpha, b\}$ defines a Jordan algebra structure by Theorem C.4(b).

The quadratic operators of the Jordan triple structure are given by

$$P(x).y = \{x, y, x\} = -\frac{1}{2}(\mathrm{ad}\, x)^2 \circ \sigma_\alpha.y.$$

We claim that

$$P(-x_\alpha) = -\frac{1}{2}(\mathrm{ad}\, x_\alpha)^2 \circ \sigma_\alpha = -\mathrm{id}_{\mathfrak{g}_\alpha}.$$

Since the action of $\mathrm{ad}\, x_\alpha$ and σ_α is given by the $\mathfrak{g}_\Delta$-module structure of $\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus D$, the claim follows from

$$-\frac{1}{2}(\mathrm{ad}\, x_\alpha)^2 \circ \sigma_\alpha.x_\alpha = \frac{1}{2}(\mathrm{ad}\, x_\alpha)^2.x_{-\alpha} = \frac{1}{2}[x_\alpha, \check{\alpha}] = -x_\alpha.$$

We now conclude from Theorem C.4(b) that the Jordan triple structure associated to the Jordan algebra structure is given by $-\{\cdot, \cdot, \cdot\}$.

This permits us to determine δ_A. First we recall that

$$[a \otimes x_\alpha, a' \otimes x_{-\alpha}] = aa' \otimes \check{\alpha} + \delta^D(a, a')\kappa(x_\alpha, x_{-\alpha}) = aa' \otimes \check{\alpha} + 4\delta^D(a, a'),$$

which leads to

$$\begin{aligned} &2(aa')a'' \otimes x_\alpha + 4\delta_A(a, a').a'' \otimes x_\alpha \\ &= \big[[a \otimes x_\alpha, a' \otimes x_{-\alpha}], a'' \otimes x_\alpha\big] \\ &= -\big[[a \otimes x_\alpha, \sigma_\alpha(a' \otimes x_{-\alpha})], a'' \otimes x_\alpha\big] = -2\{a, a', a''\} \otimes x_\alpha \\ &= 2((aa')a'' + a(a'a'') - a'(aa'')) \otimes x_\alpha. \end{aligned}$$

From that we immediately get

$$\delta_A(a, a') = \frac{1}{2}[L_a, L_{a'}].$$ ■

The following theorem contains some refined information on the type of the coordinate algebras. We define

$$\delta_\mathcal{A}(\alpha, \beta).\gamma := \delta^D(\alpha, \beta).\gamma, \quad \alpha, \beta, \gamma \in \mathcal{A}.$$

Theorem II.20. (Coordinatization Theorem) *The coordinate algebra $\mathcal{A}$ of a Δ-graded Lie algebra $\mathfrak{g}$ is:*

(1) *a Jordan algebra for Δ of type A_1, and*

$$\delta_\mathcal{A}(\alpha, \beta) = \frac{1}{2}[L_\alpha, L_\beta].$$

(2) *an alternative algebra for Δ of type A_2, and*

$$\delta_\mathcal{A}(\alpha, \beta) = \frac{1}{3}(L_{[\alpha,\beta]} - R_{[\alpha,\beta]} - 3[L_\alpha, R_\beta]).$$

(3) *an associative algebra for* Δ *of type* A_r, $r \geq 3$, *and*

$$\delta_{\mathcal{A}}(\alpha, \beta) = \frac{1}{r+1}\,\mathrm{ad}[\alpha, \beta].$$

(4) *an associative commutative algebra for* Δ *of type* D_r, $r \geq 4$, *and* E_6, E_7 *and* E_8, *and* $\delta_{\mathcal{A}}(\alpha, \beta) = 0$.

(5) *an associative algebra* $(\mathcal{A}, \sigma)$ *with involution for* Δ *of type* C_r, $r \geq 4$, *and*

$$\delta_{\mathcal{A}}(\alpha, \beta) = \frac{1}{4r}(\mathrm{ad}[\alpha, \beta] + \mathrm{ad}[\alpha^\sigma, \beta^\sigma]).$$

(6) *a Jordan algebra associated to a symmetric bilinear form* $\beta: B \times B \to A$ *for* Δ *of type* B_r, $r \geq 3$, *and* $\delta_{\mathcal{A}}(\alpha, \beta) = -[L_\alpha, L_\beta]$.

Proof. (1) follows from the discussion in Example II.19 (see also [Ti62] and [BZ96]).
(2)–(4) [BM92]; see also Appendix B for some information on alternative algebras and Proposition II.17 for a proof of (4).
(5), (6) [BZ96] (cf. Lemma B.7 for Jordan algebras associated to symmetric bilinear forms). ∎

The scalar factors in the formulas for $\delta_{\mathcal{A}}$ are due to the normalization of the invariant bilinear forms κ and κ_{V_s}.

For the details on the coordinate algebras for Δ of type C_3 (an alternative algebra with involution containing A in the associative center (the nucleus), i.e., left, resp., right multiplications with elements of A commute with all other right, resp., left multiplications), C_2 (a Peirce half space of a unital Jordan algebra containing a triangle), F_4 (an alternative algebra over A with normalized trace mapping satisfying the Cayley–Hamilton identity ch_2) and G_2 (a Jordan algebra over A with a normalized trace mapping satisfying the Cayley-Hamilton identity ch_3), we refer to [ABG00], [BZ96] and [Neh96]. For all these types of coordinate algebras one has natural derivations $\delta_{\mathcal{A}}(\alpha, \beta)$ given by explicit formulas.

III. Universal covering Lie algebras and isogeny classes

In this section we discuss the concept of a generalized central extension of a locally convex Lie algebra. It generalizes central extensions $\widehat{\mathfrak{g}} \to \mathfrak{g}$, i.e., quotient maps with central kernel. Its main advantage is that it permits us to construct for a topologically perfect locally convex Lie algebra $\mathfrak{g}$ a universal generalized central extension $q_{\mathfrak{g}}: \widetilde{\mathfrak{g}} \to \mathfrak{g}$. This is remarkable because universal central extensions do not always exist, not even for topologically perfect Banach–Lie algebras.

Definition III.1. Let $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ be locally convex Lie algebras. A continuous Lie algebra homomorphism $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ with dense range is called a *generalized central extension* if there exists a continuous bilinear map $b: \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ with

$$b(q(x), q(y)) = [x, y] \quad \text{for} \quad x, y \in \mathfrak{g}. \tag{3.1}$$

We observe that, since q has dense range, the map b is uniquely determined by (3.1). ∎

Remark III.2. If $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a quotient homomorphism of locally convex Lie algebras with central kernel, i.e., a *central extension*, then $q \times q: \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{g}$ also is a quotient map. Therefore the Lie bracket of $\widehat{\mathfrak{g}}$ factors through a continuous bilinear map $b: \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $b(q(x), q(y)) = [x, y]$ for $x, y \in \widehat{\mathfrak{g}}$, showing that q is a generalized central extension of $\mathfrak{g}$. ∎

Definition III.3. (a) Let $\mathfrak{z}$ be a locally convex space and $\mathfrak{g}$ a locally convex Lie algebra. A *continuous $\mathfrak{z}$-valued Lie algebra 2-cocycle* is a continuous skew-symmetric bilinear function $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ satisfying

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0, \quad x,y,z \in \mathfrak{g}.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha \in \mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ with $\omega(x,y) = \alpha([x,y])$ for all $x,y \in \mathfrak{g}$. We write $Z^2(\mathfrak{g},\mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2(\mathfrak{g},\mathfrak{z})$ for the subspace of coboundaries. We define the *second continuous Lie algebra cohomology space* as

$$H^2(\mathfrak{g},\mathfrak{z}) := Z^2(\mathfrak{g},\mathfrak{z})/B^2(\mathfrak{g},\mathfrak{z}).$$

(b) If ω is a continuous $\mathfrak{z}$-valued 2-cocycle on $\mathfrak{g}$, then we write $\mathfrak{g} \oplus_\omega \mathfrak{z}$ for the locally convex Lie algebra whose underlying locally convex space is the topological product $\mathfrak{g} \times \mathfrak{z}$, and whose bracket is defined by

$$[(x,z),(x',z')] = \big([x,x'],\omega(x,x')\big).$$

Then $q\colon \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}, (x,z) \mapsto x$ is a central extension and $\sigma\colon \mathfrak{g} \to \mathfrak{g} \oplus_\omega \mathfrak{z}, x \mapsto (x,0)$ is a continuous linear section of q. ■

Lemma III.4. *For a generalized central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with the corresponding map b the following assertions hold:*

(1) $[x,y] = q(b(x,y))$ *for all* $x,y \in \mathfrak{g}$.

(2) $[\mathfrak{g},\mathfrak{g}] \subseteq \mathrm{im}(q)$.

(3) $b \in Z^2(\mathfrak{g},\widehat{\mathfrak{g}})$, *i.e.,* $b([x,y],z) + b([y,z],x) + b([z,x],y) = 0$ *for* $x,y,z \in \mathfrak{g}$.

(4) *For* $x \in \mathfrak{g}$ *we define*

$$\widehat{\mathrm{ad}}(x)\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}, \quad y \mapsto b(x,q(y)).$$

Then $\widehat{\mathrm{ad}}$ defines a continuous representation of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$ by derivations for which q is equivariant with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$.

(5) *If $\widehat{\mathfrak{g}}$ is topologically perfect, then $q^{-1}(\mathfrak{z}(\mathfrak{g})) = \mathfrak{z}(\widehat{\mathfrak{g}})$.*

Proof. (1) If $x = q(a)$ and $y = q(b)$ holds for $a,b \in \widehat{\mathfrak{g}}$, then

$$[x,y] = [q(a),q(b)] = q([a,b]) = q(b(x,y)).$$

Therefore the Lie bracket on $\mathfrak{g}$ coincides on the dense subset $\mathrm{im}(q) \times \mathrm{im}(q)$ of $\mathfrak{g} \times \mathfrak{g}$ with the continuous map $q \circ b$, so that (1) follows from the continuity of both maps.

(2) follows from (1).

(3) In view of (3.1), the Jacobi identity in $\widehat{\mathfrak{g}}$ leads to

$$\begin{aligned} 0 &= [[x,y],z] + [[y,z],x] + [[z,x],y] \\ &= b(q([x,y]),q(z)) + b(q([y,z]),q(x)) + b(q([z,x]),q(y)) \\ &= b([q(x),q(y)],q(z)) + b([q(y),q(z)],q(x)) + b([q(z),q(x)],q(y)). \end{aligned}$$

Therefore the restriction of b to $\mathrm{im}(q)$ is a Lie algebra cocycle, and since $\mathrm{im}(q)$ is dense and b is continuous, it is a Lie algebra cocycle on $\mathfrak{g}$.

(4) First we observe that the bilinear map $\mathfrak{g} \times \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}, (x,y) \mapsto b(x,q(y))$ is continuous. Moreover, (1) implies

$$q(\widehat{\mathrm{ad}}(x).y) = q(b(x,q(y))) = [x,q(y)],$$

i.e., $q \circ \widehat{\mathrm{ad}}(x) = \mathrm{ad}\, x \circ q$.

From the cocycle identity

$$b([x,y],z) + b([y,z],x) + b([z,x],y) = 0, \quad x,y,z \in \mathfrak{g},$$

we derive in particular for $x \in \mathfrak{g}$ and $y, z \in \widehat{\mathfrak{g}}$:

$$\begin{aligned} 0 &= b([x, q(y)], q(z)) + b([q(y), q(z)], x) + b([q(z), x], q(y)) \\ &= b(q(\widehat{\mathrm{ad}}(x)y), q(z)) + b(q([y, z]), x) - b(q(\widehat{\mathrm{ad}}(x).z), q(y)) \\ &= [\widehat{\mathrm{ad}}(x)y, z] - \widehat{\mathrm{ad}}(x)[y, z] - [\widehat{\mathrm{ad}}(x)z, y]. \end{aligned}$$

Therefore each $\widehat{\mathrm{ad}}(x)$ is a derivation of $\widehat{\mathfrak{g}}$. On the other hand, the cocycle identity for b leads for $x, y \in \mathfrak{g}$ and $z \in \widehat{\mathfrak{g}}$ to

$$\begin{aligned} 0 &= b([x, y], q(z)) + b([y, q(z)], x) + b([q(z), x], y) \\ &= \widehat{\mathrm{ad}}([x, y])z + b(q(\widehat{\mathrm{ad}}(y)z), x) - b(q(\widehat{\mathrm{ad}}(x)z), y) = \widehat{\mathrm{ad}}([x, y])z - \widehat{\mathrm{ad}}(x)\widehat{\mathrm{ad}}(y)z + \widehat{\mathrm{ad}}(y)\widehat{\mathrm{ad}}(x)z, \end{aligned}$$

so that $\widehat{\mathrm{ad}}\colon \mathfrak{g} \to \mathrm{der}(\widehat{\mathfrak{g}})$ is a representation of $\mathfrak{g}$ by derivations of $\widehat{\mathfrak{g}}$, and the map q is equivariant with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$.

(5) Let $\widehat{\mathfrak{z}}(\mathfrak{g}) := q^{-1}(\mathfrak{z}(\mathfrak{g}))$. We first observe that $[\widehat{\mathfrak{z}}(\mathfrak{g}), \widehat{\mathfrak{g}}]$ is contained in $\ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$ because

$$q([\widehat{\mathfrak{z}}(\mathfrak{g}), \widehat{\mathfrak{g}}]) \subseteq [\mathfrak{z}(\mathfrak{g}), \mathfrak{g}] = \{0\}.$$

This leads to

$$[\widehat{\mathfrak{z}}(\mathfrak{g}), [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]] \subseteq [\widehat{\mathfrak{g}}, [\widehat{\mathfrak{z}}(\mathfrak{g}), \widehat{\mathfrak{g}}]] \subseteq [\widehat{\mathfrak{g}}, \ker q] = \{0\}.$$

If $\widehat{\mathfrak{g}}$ is topologically perfect, we obtain $\widehat{\mathfrak{z}}(\mathfrak{g}) \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$. The other inclusion follows from the density of the image of q. ■

The following proposition shows that generalized central extensions can be characterized as certain closed subalgebras of central extensions defined by cocycles.

Proposition III.5. (a) *If $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a generalized central extension and $b\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ the corresponding cocycle, then the map*

$$\psi\colon \widehat{\mathfrak{g}} \to \mathfrak{g} \oplus_b \widehat{\mathfrak{g}}, \quad x \mapsto (q(x), x)$$

is a is a topological embedding of $\widehat{\mathfrak{g}}$ onto a closed Lie subalgebra of $\mathfrak{g} \oplus_b \widehat{\mathfrak{g}}$.

(b) *If $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$ is a continuous 2-cocycle, $p\colon \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ the projection onto $\mathfrak{g}$ of the corresponding central extension, and $\widehat{\mathfrak{g}} \subseteq \mathfrak{g} \oplus_\omega \mathfrak{z}$ is a closed subalgebra for which $p(\widehat{\mathfrak{g}})$ is dense in $\mathfrak{g}$, then $q := p|_{\widehat{\mathfrak{g}}} : \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a generalized central extension with $b(x, y) = ([x, y], \omega(x, y))$ for $x, y \in \mathfrak{g}$.*

Proof. (a) We recall from Definition III.3 that the bracket in $\mathfrak{g} \oplus_b \widehat{\mathfrak{g}}$ is given by

$$[(x, y), (x', y')] = ([x, x'], b(x, x')).$$

Now

$$\begin{aligned} [\psi(x), \psi(x')] &= [(q(x), x), (q(x'), x')] = ([q(x), q(x')], b(q(x), q(x'))) \\ &= (q([x, x']), [x, x']) = \psi([x, x']) \end{aligned}$$

implies that the continuous linear map ψ is a morphism of Lie algebras. As the graph of continuous linear map q, the image of ψ is a closed subspace of $\mathfrak{g} \oplus_b \widehat{\mathfrak{g}}$, and the projection onto the second factor is a continuous linear map. Therefore ψ is a topological embedding onto a closed subalgebra.

(b) The range of q is dense by the assumption that $p(\widehat{\mathfrak{g}})$ is dense in $\mathfrak{g}$. It is also clear that $b \circ (p \times p)$ is the bracket on $\mathfrak{g} \oplus_\omega \mathfrak{z}$, but it remains to show that $\mathrm{im}(b) \subseteq \widehat{\mathfrak{g}}$.

For $x = q(x'), y = q(y')$ in $\mathrm{im}(q) = p(\widehat{\mathfrak{g}})$ we have

$$b(x, y) = b(q(x'), q(y')) = [x', y'] = ([x, y], \omega(x, y)) \in \widehat{\mathfrak{g}}.$$

Now the continuity of b, the density of $\mathrm{im}(q)$ in $\mathfrak{g}$, and the closedness of $\widehat{\mathfrak{g}}$ imply that $\mathrm{im}(b) \subseteq \widehat{\mathfrak{g}}$. ■

Full cyclic homology of locally convex algebras

In this subsection we define cyclic 1-cocycles for locally convex algebras $\mathcal{A}$ which are not necessarily associative. This includes in particular Lie algebras, where cyclic 1-cocycles are Lie algebra 2-cocycles. It also covers the more general coordinate algebras of root graded locally convex Lie algebras (see Section IV). In particular we associate to $\mathcal{A}$ a locally convex space $\langle \mathcal{A}, \mathcal{A}\rangle$ in such a way that continuous cyclic 1-cocycles are in one-to-one correspondence to linear maps on $\langle \mathcal{A}, \mathcal{A}\rangle$. Moreover, we will discuss a method to obtain Lie algebra structures on $\langle \mathcal{A}, \mathcal{A}\rangle$, which will be crucial in Section IV for the construction of the universal covering algebra of a root graded Lie algebra.

Definition III.6. (a) Let $\mathcal{A}$ be a locally convex algebra (not necessarily associative or with unit). We endow the tensor product $\mathcal{A} \otimes \mathcal{A}$ with the projective tensor product topology and denote this space by $\mathcal{A} \otimes_\pi \mathcal{A}$. Let

$$I := \overline{\operatorname{span}\{a \otimes a, ab \otimes c + bc \otimes a + ca \otimes b : a, b, c \in \mathcal{A}\}} \subseteq \mathcal{A} \otimes_\pi \mathcal{A}.$$

We define

$$\langle \mathcal{A}, \mathcal{A}\rangle := (\mathcal{A} \otimes_\pi \mathcal{A})/I,$$

endowed with the quotient topology, which turns it into a locally convex space. We write $\langle a, b\rangle$ for the image of $a \otimes b$ in the quotient space $\langle \mathcal{A}, \mathcal{A}\rangle$.

(b) Our definition of $\langle \mathcal{A}, \mathcal{A}\rangle$ in (a) is the one corresponding to the category of locally convex spaces, resp., algebras. In the category of complete locally convex spaces we write $\langle \mathcal{A}, \mathcal{A}\rangle$ for the completion of the quotient space $(\mathcal{A} \otimes_\pi \mathcal{A})/I$, and in the category of sequentially complete spaces for the smallest sequentially closed subspace of the completion, i.e., its sequential completion.

In the category of Fréchet spaces, the completed version of $\langle \mathcal{A}, \mathcal{A}\rangle$ can be obtained more directly by first replacing $\mathcal{A} \otimes_\pi \mathcal{A}$ by its completion $\mathcal{A}\widehat{\otimes}_\pi\mathcal{A}$. If $\overline{I}$ denotes the closure of I in the completion $\mathcal{A}\widehat{\otimes}_\pi\mathcal{A}$, then the quotient space $\mathcal{A}\widehat{\otimes}_\pi\mathcal{A}/\overline{I}$ is automatically complete, hence a Fréchet space.

(c) For a locally convex space $\mathfrak{z}$ the continuous linear maps $\langle \mathcal{A}, \mathcal{A}\rangle \to \mathfrak{z}$ correspond to those alternating continuous bilinear maps $\omega \colon \mathcal{A} \times \mathcal{A} \to \mathfrak{z}$ satisfying

$$\omega(ab, c) + \omega(bc, a) + \omega(bc, a) = 0, \quad a, b, c \in \mathcal{A}.$$

These maps are called *cyclic* 1*-cocycles.* We write $Z^1(\mathcal{A}, \mathfrak{z})$ for the space of continuous cyclic 1-cocycles $\mathcal{A} \times \mathcal{A} \to \mathfrak{z}$ and note that

$$Z^1(\mathcal{A}, \mathfrak{z}) \cong \operatorname{Lin}(\langle \mathcal{A}, \mathcal{A}\rangle, \mathfrak{z}).$$

The identity $\operatorname{id}_{\langle \mathcal{A}, \mathcal{A}\rangle}$ corresponds to the *universal cocycle*

$$\omega_u \colon \mathcal{A} \times \mathcal{A} \to \langle \mathcal{A}, \mathcal{A}\rangle, \quad (a, b) \mapsto \langle a, b\rangle. \qquad \blacksquare$$

Remark III.7. Lie algebra 2-cocycles $\omega \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ (Definition III.3) are the same as cyclic 1-cocycles of the algebra $\mathfrak{g}$.

In particular we have

$$Z^2(\mathfrak{g}, \mathfrak{z}) \cong \operatorname{Lin}(\langle \mathfrak{g}, \mathfrak{g}\rangle, \mathfrak{z})$$

for any locally convex space $\mathfrak{z}$. $\blacksquare$

Remark III.8. Let $\mathcal{A}$ be a locally convex associative algebra, $\mathcal{A}_L$ the corresponding Lie algebra with the commutator bracket $[a,b] = ab - ba$, and $\mathcal{A}_J$ the corresponding Jordan algebra with the product $a \circ b := \frac{1}{2}(ab + ba)$. In $\mathcal{A} \otimes \mathcal{A}$ we have the relations

$$[a,b] \otimes c + [b,c] \otimes a + [c,a] \otimes b = ab \otimes c + bc \otimes a + ca \otimes b - (ba \otimes c + cb \otimes a + ac \otimes b)$$

and

$$2(a \circ b \otimes c + b \circ c \otimes a + c \circ a \otimes b) = ab \otimes c + bc \otimes a + ca \otimes b + ba \otimes c + cb \otimes a + ac \otimes b.$$

Therefore we have natural continuous linear maps

$$\langle \mathcal{A}_L, \mathcal{A}_L \rangle \to \langle \mathcal{A}, \mathcal{A} \rangle, \quad \langle a, b \rangle \mapsto \langle a, b \rangle \quad \text{and} \quad \langle \mathcal{A}_J, \mathcal{A}_J \rangle \to \langle \mathcal{A}, \mathcal{A} \rangle, \quad \langle a, b \rangle \mapsto \langle a, b \rangle. \qquad \blacksquare$$

A remarkable point of the following proposition is that it applies without any assumption on the algebra $\mathcal{A}$, such as associativity etc.

Proposition III.9. *Let $\mathcal{A}$ be a locally convex algebra and*

$$\delta\colon \langle \mathcal{A}, \mathcal{A} \rangle \to \mathrm{der}(\mathcal{A}), \quad \langle a, b \rangle \mapsto \delta(a,b)$$

be a cyclic 1-cocycle for which the map $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (a,b,c) \mapsto \delta(a,b).c$ is continuous. As $\mathrm{der}(\mathcal{A})$ *acts naturally on $\langle \mathcal{A}, \mathcal{A} \rangle$ by*

$$d.\langle a, b \rangle = \langle d.a, b \rangle + \langle a, d.b \rangle, \quad d \in \mathrm{der}(\mathcal{A}), a, b \in \mathcal{A},$$

we obtain a well-defined continuous bilinear map

$$[\cdot,\cdot]\colon \langle \mathcal{A}, \mathcal{A} \rangle \times \langle \mathcal{A}, \mathcal{A} \rangle \to \langle \mathcal{A}, \mathcal{A} \rangle, \quad [\langle a, b \rangle, \langle c, d \rangle] \mapsto \delta(a,b).\langle c, d \rangle = \langle \delta(a,b).c, d \rangle + \langle c, \delta(a,b).d \rangle.$$

Suppose that
(1) $\delta(\delta(a,b).\langle c, d \rangle) = [\delta(a,b), \delta(c,d)]$, *and*
(2) $\delta(a,b).\langle c, d \rangle = -\delta(c,d).\langle a, b \rangle$ *for $a, b, c, d \in \mathcal{A}$.*
Then $[\cdot,\cdot]$ defines on $\langle \mathcal{A}, \mathcal{A} \rangle$ the structure of a locally convex Lie algebra and δ is a homomorphism of Lie algebras.

Proof. According to our continuity assumption on δ, the quadrilinear map

$$\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \langle \mathcal{A}, \mathcal{A} \rangle, \quad (a,b,c,d) \mapsto \delta(a,b).\langle c, d \rangle = \langle \delta(a,b).c, d \rangle + \langle c, \delta(a,b).d \rangle$$

is continuous. That δ is a cyclic cocycle implies that it factors through a continuous bilinear map

$$[\cdot,\cdot]\colon \langle \mathcal{A}, \mathcal{A} \rangle \times \langle \mathcal{A}, \mathcal{A} \rangle \to \langle \mathcal{A}, \mathcal{A} \rangle, \quad (\langle a, b \rangle, \langle c, d \rangle) \mapsto \delta(a,b).\langle c, d \rangle.$$

Condition (2) means that the bracket on $\langle \mathcal{A}, \mathcal{A} \rangle$ is alternating. In view of (1), the Jacobi identity follows from

$$\begin{aligned} [[\langle a, b \rangle, \langle c, d \rangle], \langle u, v \rangle] &= \delta(\delta(a,b).\langle c, d \rangle).\langle u, v \rangle = [\delta(a,b), \delta(c,d)].\langle u, v \rangle \\ &= [\langle a, b \rangle, [\langle c, d \rangle, \langle u, v \rangle]] - [\langle c, d \rangle, [\langle a, b \rangle, \langle u, v \rangle]]. \end{aligned}$$

Finally, we observe that (1) means that δ is a homomorphism of Lie algebras. $\blacksquare$

Example III.10. Typical examples where Proposition III.9 applies are
(1) Lie algebras: If $\mathfrak{g}$ is a locally convex Lie algebra and $\delta(x,y) = \mathrm{ad}[x,y]$, then the Jacobi identity implies that δ is a cocycle. That δ is equivariant with respect to the action of $\mathrm{der}(\mathfrak{g})$ follows for $d \in \mathrm{der}(\mathfrak{g})$ and $x,y \in \mathfrak{g}$ from

$$\delta(d.x,y) + \delta(x,d.y) = \mathrm{ad}([d.x,y] + [x,d.y]) = \mathrm{ad}(d.[x,y]) = [d,\mathrm{ad}[x,y]] = [d,\delta(x,y)].$$

We also have in $\langle \mathfrak{g}, \mathfrak{g}\rangle$:

$$\begin{aligned}\delta(x,y).\langle x',y'\rangle &= \langle [[x,y],x'],y'\rangle + \langle x',[[x,y],y']\rangle \\ &= -\langle [x',y'],[x,y]\rangle - \langle [y',[x,y]],x'\rangle + \langle x',[[x,y],y']\rangle = \langle [x,y],[x',y']\rangle,\end{aligned}$$

which implies $\delta(x,y).\langle x',y'\rangle = -\delta(x',y').\langle x,y\rangle$.
(2) Associative algebras: If $\mathcal{A}$ is an associative algebra, then the commutator bracket

$$\mathcal{A}\times\mathcal{A}\to\mathcal{A}, \quad (a,b)\mapsto [a,b] = ab - ba$$

is a cyclic cocycle because

$$[ab,c] + [bc,a] + [ca,b] = abc - cab + bca - abc + cab - bca = 0.$$

Therefore $\delta(x,y) = \mathrm{ad}[x,y]$ defines a cocycle $\mathcal{A}\times\mathcal{A}\to\mathrm{der}(\mathcal{A})$. That δ is equivariant with respect to the action of $\mathrm{der}(\mathcal{A})$ follows with the same calculations as in (1) above. Alternatively, we can observe that if $\mathcal{A}_L$ denotes the Lie algebra $\mathcal{A}$ with the commutator bracket, then $\langle\mathcal{A},\mathcal{A}\rangle$ is a quotient of $\langle\mathcal{A}_L,\mathcal{A}_L\rangle$ (Remark III.8).
(3) If $\mathcal{A}$ is a Jordan algebra and $\delta_{\mathcal{A}}(a,b) = [L(a),L(b)]$, then we have

$$\delta_{\mathcal{A}}(d.\langle a,b\rangle) = [d,\delta_{\mathcal{A}}(a,b)]$$

for all derivations $d\in\mathrm{der}(\mathcal{A})$, hence (1) in Proposition III.9. To verify (2), we calculate

$$\begin{aligned}\delta_{\mathcal{A}}(a,a').\langle b,b'\rangle &= \langle \delta_{\mathcal{A}}(a,a').b,b'\rangle + \langle b,\delta_{\mathcal{A}}(a,a').b'\rangle \\ &= \langle a(a'b) - a'(ab),b'\rangle + \langle b,a(a'b') - a'(ab')\rangle \\ &= \langle a(a'b),b'\rangle - \langle a'(ab),b'\rangle + \langle b,a(a'b')\rangle - \langle b,a'(ab')\rangle \\ &= -\langle (a'b)b',a\rangle - \langle b'a,a'b\rangle + \langle (ab)b',a'\rangle + \langle b'a',ab\rangle \\ &\quad - \langle a,(a'b')b\rangle - \langle a'b',ba\rangle + \langle a',(ab')b\rangle + \langle ab',ba'\rangle \\ &= -\langle b'(ba'),a\rangle - \langle b'a,a'b\rangle + \langle b'(ba),a'\rangle + \langle b'a',ab\rangle \\ &\quad - \langle a,b(b'a')\rangle - \langle b'a',ab\rangle + \langle a',b(b'a)\rangle + \langle b'a,a'b\rangle \\ &= -\langle b'(ba'),a\rangle + \langle b'(ba),a'\rangle - \langle a,b(b'a')\rangle + \langle a',b(b'a)\rangle \\ &= \langle \delta_{\mathcal{A}}(b',b).a,a'\rangle + \langle a,\delta_{\mathcal{A}}(b',b).a'\rangle \\ &= -\delta_{\mathcal{A}}(b,b').\langle a,a'\rangle.\end{aligned}$$

■

The universal covering of a locally convex Lie algebra

We call a generalized central extension $q_{\mathfrak{g}}\colon\widetilde{\mathfrak{g}}\to\mathfrak{g}$ of a locally convex Lie algebra $\mathfrak{g}$ *universal* if for any generalized central extension $q\colon\widehat{\mathfrak{g}}\to\mathfrak{g}$ there exists a unique morphism of locally convex Lie algebras $\alpha\colon\widetilde{\mathfrak{g}}\to\widehat{\mathfrak{g}}$ with $q\circ\alpha = q_{\mathfrak{g}}$.

Theorem III.11. *A locally convex Lie algebra $\mathfrak{g}$ has a universal generalized central extension if and only if it is topologically perfect. If this is the case, then the universal generalized central extension is given by the natural Lie algebra structure on $\widetilde{\mathfrak{g}} := \langle \mathfrak{g}, \mathfrak{g} \rangle$ satisfying*

$$[\langle x, x' \rangle, \langle y, y' \rangle] = \langle [x, x'], [y, y'] \rangle \qquad \text{for} \quad x, x', y, y' \in \mathfrak{g}, \tag{3.2}$$

and the natural homomorphism

$$q_{\mathfrak{g}}: \widetilde{\mathfrak{g}} \to \mathfrak{g}, \quad \langle x, y \rangle \mapsto [x, y]$$

is given by the Lie bracket on $\mathfrak{g}$.

Proof. Suppose first that $q_{\mathfrak{g}}: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a universal generalized central extension. We consider the trivial central extension $\widehat{\mathfrak{g}} := \mathfrak{g} \times \mathbb{K}$ with $q(x, t) = x$. According to the universal property, there exists a unique morphism of locally convex Lie algebras $\alpha: \widetilde{\mathfrak{g}} \to \mathfrak{g} \times \mathbb{K}$ with $q \circ \alpha = q_{\mathfrak{g}}$. For each Lie algebra homomorphism $\beta: \widetilde{\mathfrak{g}} \to \mathbb{K}$ the sum $\alpha + \beta: \widetilde{\mathfrak{g}} \to \mathfrak{g} \times \mathbb{K}$ also is a homomorphism of Lie algebras with $q \circ (\alpha + \beta) = q_{\mathfrak{g}}$. Hence the uniqueness implies that $\beta = 0$. That all morphisms $\widetilde{\mathfrak{g}} \to \mathbb{K}$ are trivial means that $\widetilde{\mathfrak{g}}$ is topologically perfect, and therefore $\mathfrak{g}$ is topologically perfect.

Conversely, we assume that $\mathfrak{g}$ is topologically perfect and construct a universal generalized central extension. Using Proposition III.9 and Example III.10(1), we see that $\langle \mathfrak{g}, \mathfrak{g} \rangle$ carries a locally convex Lie algebra structure with

$$[\langle x, y \rangle, \langle z, u \rangle] = \langle [x, y], [z, u] \rangle, \quad x, y, z, u \in \mathfrak{g}.$$

Next we observe that $\mathrm{im}(q_{\mathfrak{g}})$ is dense because $[\mathfrak{g}, \mathfrak{g}]$ is dense in $\mathfrak{g}$. The corresponding bracket map on $\widetilde{\mathfrak{g}}$ is given by the universal cocycle

$$\omega_u: \mathfrak{g} \times \mathfrak{g} \to \widetilde{\mathfrak{g}}, \quad (x, y) \mapsto \langle x, y \rangle.$$

In fact, for $x, x', y, y' \in \mathfrak{g}$ we have

$$\omega_u(q_{\mathfrak{g}}(\langle x, x' \rangle), q_{\mathfrak{g}}(\langle y, y' \rangle)) = \omega_u([x, x'], [y, y']) = \langle [x, x'], [y, y'] \rangle = [\langle x, x' \rangle, \langle y, y' \rangle].$$

Since the elements of the form $\langle x, x' \rangle$ span a dense subspace of $\widetilde{\mathfrak{g}}$, equation (3.1) holds for $q = q_{\mathfrak{g}}$.

Now let $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ be another generalized central extension with the corresponding map $b: \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$. Then Lemma III.4(3) implies the existence of a unique continuous linear map $\alpha: \widetilde{\mathfrak{g}} = \langle \mathfrak{g}, \mathfrak{g} \rangle \to \widehat{\mathfrak{g}}$ with

$$b(x, y) = \alpha(\langle x, y \rangle), \quad x, y \in \mathfrak{g}.$$

For $x = q(a)$, $x' = q(a')$, $y = q(b)$ and $y' = q(b')$ we then have

$$\begin{aligned} \alpha([\langle x, x' \rangle, \langle y, y' \rangle]) &= \alpha(\langle [x, x'], [y, y'] \rangle) = b([x, x'], [y, y']) = b(q([a, a']), q([b, b'])) \\ &= [[a, a'], [b, b']] = [b(x, x'), b(y, y')] = [\alpha(\langle x, x' \rangle), \alpha(\langle y, y' \rangle)]. \end{aligned}$$

Now the fact that $\mathrm{im}(q)$ is dense in $\mathfrak{g}$ implies that α is a homomorphism of Lie algebras. Further,

$$q(\alpha(\langle x, y \rangle)) = q(b(x, y)) = [x, y] = q_{\mathfrak{g}}(\langle x, y \rangle),$$

again with the density of $\mathrm{im}(q)$ in $\mathfrak{g}$, leads to $q \circ \alpha = q_{\mathfrak{g}}$.

To see that α is unique, we first observe that $\widetilde{\mathfrak{g}}$ is topologically perfect because $\mathfrak{g}$ is topologically perfect. If $\beta: \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ is another homomorphism with $q \circ \beta = q_{\mathfrak{g}}$, then $\gamma := \beta - \alpha$ is a continuous linear map $\widetilde{\mathfrak{g}} \to \ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$. Moreover,

$$\begin{aligned} \gamma([x, y]) &= \beta([x, y]) - \alpha([x, y]) = [\beta(x), \beta(y)] - [\alpha(x), \alpha(y)] \\ &= [\beta(x) - \alpha(x), \beta(y)] + [\alpha(x), \beta(y)] - [\alpha(x), \alpha(y)] \\ &= [\gamma(x), \beta(y)] + [\alpha(x), \gamma(y)] = 0 \end{aligned}$$

because the values of γ are central. Now $\gamma = 0$ follows from the topological perfectness of $\widetilde{\mathfrak{g}}$. ■

Definition III.12. For a topologically perfect locally convex Lie algebra $\mathfrak{g}$ the Lie algebra $\widetilde{\mathfrak{g}} = \langle \mathfrak{g}, \mathfrak{g} \rangle$ is called the *universal generalized central extension of* $\mathfrak{g}$ or the *(topological) universal covering Lie algebra of* $\mathfrak{g}$.

We call two topologically perfect Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ *centrally isogenous* if $\widetilde{\mathfrak{g}}_1 \cong \widetilde{\mathfrak{g}}_2$.

In the category of sequentially complete, resp., complete locally convex Lie algebras we define $\widetilde{\mathfrak{g}}$ as $\langle \mathfrak{g}, \mathfrak{g} \rangle$ in the sense of Definition III.6(b). Then the same arguments as in the proof of Theorem III.11 show that $\widetilde{\mathfrak{g}}$ is a universal generalized central extension in the corresponding category. ■

We call a central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of a locally convex Lie algebra $\mathfrak{g}$ *universal* if for any central extension $q'\colon \widehat{\mathfrak{g}}' \to \mathfrak{g}$ there exists a unique morphism of locally convex Lie algebras $\alpha\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}'$ with $q' \circ \alpha = q$. The following corollary clarifies the relation between universal central extensions and generalized universal central extensions. In particular it implies that the existence of a universal central extension is a quite rare phenomenon.

Corollary III.13. *A locally convex Lie algebra* $\mathfrak{g}$ *has a universal central extension if and only if it is topologically perfect and the universal covering map* $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ *is a quotient map. Then* $q_{\mathfrak{g}}$ *is a universal central extension.*

Proof. Suppose first that $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a universal central extension. Then the same argument as in the proof of Theorem III.11 implies that $\widehat{\mathfrak{g}}$ is topologically perfect, which implies that $\mathfrak{g}$ is topologicall perfect. Therefore the universal generalized central extension $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ exists by Theorem III.11. Its universal property implies the existence of a unique morphism $\widetilde{q}\colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ with $q \circ \widetilde{q} = q_{\mathfrak{g}}$. If $\widehat{b}\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ is the unique continuous bilinear map for which $\widehat{b} \circ (q \times q)$ is the bracket on $\widehat{\mathfrak{g}}$, the construction in the proof of Theorem III.11 implies that

$$\widetilde{q} \circ \omega_u = \widehat{b}$$

for the universal cocycle $\omega_u(x, y) = \langle x, y \rangle$.

Now let $q_u\colon \mathfrak{g} \oplus_{\omega_u} \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the central extension of $\mathfrak{g}$ by $\widetilde{\mathfrak{g}}$, considered as an abelian Lie algebra, defined by the universal cocycle. Then the universal property of $\widehat{\mathfrak{g}}$ implies the existence of a unique morphism

$$\psi\colon \widehat{\mathfrak{g}} \to \mathfrak{g} \oplus_{\omega_u} \widetilde{\mathfrak{g}}$$

with $q_u \circ \psi = q$. This means that $\psi(x) = (q(x), \alpha(x))$, where $\alpha\colon \widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ is a continuous linear map. That ψ is a Lie algebra homomorphism means that

$$(q([x, y]), \alpha([x, y])) = \psi([x, y]) = [\psi(x), \psi(y)] = ([q(x), q(y)], \langle q(x), q(y) \rangle),$$

which implies that

$$\alpha(\widehat{b}(q(x), q(y))) = \alpha([x, y]) = \langle q(x), q(y) \rangle, \quad x, y \in \widehat{\mathfrak{g}},$$

and hence

$$\alpha \circ \widehat{b} = \omega_u.$$

For the continuous linear maps $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ corresponding to these cocycles, we obtain

$$\alpha \circ \widetilde{q} = \mathrm{id}_{\widetilde{\mathfrak{g}}}.$$

We also have

$$\widetilde{q} \circ \alpha \circ \widehat{b} = \widetilde{q} \circ \omega_u = \widehat{b},$$

and since $\mathrm{im}(\widehat{b})$ spans a dense subspace of the topologically perfect Lie algebra $\widehat{\mathfrak{g}}$, it follows that

$$\widetilde{q} \circ \alpha = \mathrm{id}_{\widehat{\mathfrak{g}}}.$$

Therefore $\widetilde{q}$ is an isomorphism of locally convex spaces, hence an isomorphism of locally convex Lie algebras, and this implies that $q_{\mathfrak{g}}$ is a central extension.

If, conversely, $\mathfrak{g}$ is topologically perfect and $q_{\mathfrak{g}}$ is a central extension, its universal property as a generalized central extension implies that it is a universal central extension. ■

Comparing the construction above with the universal central extensions investigated in [Ne02c], it appears that generalized central extensions are more natural in the topological context because one does not have to struggle with the problem that closed subspaces of locally convex spaces do not always have closed complements, which causes many problems if one works only with central extensions defined by cocycles (cf. Definition III.3). Moreover, universal generalized central extensions do always exist for topologically perfect locally convex algebras, whereas there are Banach–Lie algebras which do not admit a universal central extension ([Ne01, Ex. II.18, III.9] and Proposition III.18 below, combined with Corollary III.13). The typical example is the Lie algebra of Hilbert–Schmidt operators on an infinite-dimensional Hilbert space discussed in some detail below.

We now address the question for which Lie algebra the universal covering morphism $q_{\mathfrak g}\colon \widetilde{\mathfrak g} \to \mathfrak g$ is an isomorphism. At the end of this section we will in particular describe examples, where $q_{\widetilde{\mathfrak g}}\colon \widetilde{\widetilde{\mathfrak g}} \to \widetilde{\mathfrak g}$ is not an isomorphism.

Proposition III.14. *For a topologically perfect locally convex Lie algebra* $\mathfrak g$ *the following are equivalent:*

(1) $q_{\mathfrak g}\colon \widetilde{\mathfrak g} \to \mathfrak g$ *is an isomorphism of Lie algebras.*

(2) $H^2(\mathfrak g, \mathfrak z) = \{0\}$ *for each locally convex space* $\mathfrak z$*.*

Proof. (1) $\Rightarrow$ (2): Let $\omega \in Z^2(\mathfrak g, \mathfrak z)$ be a continuous Lie algebra cocycle $\mathfrak g \times \mathfrak g \to \mathfrak z$. According to Remark III.7, there exists a continuous linear map $\alpha\colon \widetilde{\mathfrak g} \to \mathfrak z$ with

$$\omega(x,y) = \alpha(\langle x, y\rangle) = \alpha \circ q_{\mathfrak g}^{-1}([x,y])$$

for $x, y \in \mathfrak g$, and this means that ω is a coboundary.

(2) $\Rightarrow$ (1): The triviality of $H^2(\mathfrak g, \widetilde{\mathfrak g})$ implies that there exists a continuous linear map $\alpha\colon \mathfrak g \to \widetilde{\mathfrak g}$ with

$$\langle x, y\rangle = \alpha([x,y]), \quad x, y \in \mathfrak g. \tag{3.3}$$

Then

$$(q_{\mathfrak g} \circ \alpha)([x,y]) = q_{\mathfrak g}(\langle x, y\rangle) = [x,y],$$

so that the density of $[\mathfrak g, \mathfrak g]$ in $\mathfrak g$ leads to $q_{\mathfrak g} \circ \alpha = \mathrm{id}_{\mathfrak g}$. On the other hand, (3.3) can also be read as $\alpha \circ q_{\mathfrak g} = \mathrm{id}_{\widetilde{\mathfrak g}}$. Therefore $q_{\mathfrak g}$ is an isomorphism of locally convex spaces, hence of locally convex Lie algebras. ■

A topologically perfect locally convex Lie algebra satisfying the two equivalent conditions of Proposition III.14 is called *centrally closed.* This means that $\mathfrak g$ is its own universal covering algebra, or, equivalently, that the Lie bracket $\mathfrak g \times \mathfrak g \to \mathfrak g$ is a universal Lie algebra cocycle.

Remark III.15. (a) Let $\mathfrak g_1, \mathfrak g_2$ and $\mathfrak g_3$ be topologically perfect locally convex Lie algebras and $q_1\colon \mathfrak g_1 \to \mathfrak g_2$, $q_2\colon \mathfrak g_2 \to \mathfrak g_3$ generalized central extensions. Then $q := q_2 \circ q_1\colon \mathfrak g_1 \to \mathfrak g_3$ is a morphism of locally convex Lie algebras with dense range. Moreover, Lemma III.4(5) implies that

$$\ker q = q_1^{-1}(\ker q_2) \subseteq q_1^{-1}(\mathfrak z(\mathfrak g_2)) = \mathfrak z(\mathfrak g_1).$$

Unfortunately, we cannot conclude in general that q is a generalized central extension. The bilinear map $b_1\colon \mathfrak g_2 \times \mathfrak g_2 \to \mathfrak g_1$ for which $b_1 \circ (q_1 \times q_1)$ is the Lie bracket of $\mathfrak g_1$ is a Lie algebra cocycle, which implies that

$$b_1(\ker q_2, \mathfrak g_2) \subseteq b_1(\mathfrak z(\mathfrak g_2), \overline{[\mathfrak g_2, \mathfrak g_2]}) = \{0\}.$$

Therefore b_1 factors through a bilinear map

$$b\colon \mathrm{im}(q_2) \times \mathrm{im}(q_2) \to \mathfrak g_1, \quad (q_2(x), q_2(y)) \mapsto b_1(x,y)$$

with

$$b(q(x), q(y)) = b_1(q_1(x), q_1(y)) = [x, y], \quad x, y \in \mathfrak{g}_1.$$

If b is continuous, it extends to a continuous bilinear map $\mathfrak{g}_3 \times \mathfrak{g}_3 \to \mathfrak{g}_1$ with the required properties, and q is a generalized central extension, but unfortunately, there is no reason for this to be the case.

(b) If q_2 is a quotient map, i.e., a central extension, then b is continuous. This shows that in the context of topologically perfect locally convex Lie algebras a generalized central extension of a central extension is a generalized central extension. This means in particular that if the universal covering map $q_{\mathfrak{g}}: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a quotient map, then $\widetilde{\mathfrak{g}}$ is centrally closed. ■

Lemma III.16. *Let H be a Hilbert space and $\mathfrak{sl}_0(H)$ the Lie algebra of all continuous finite rank operators of zero trace on H. For each derivation*

$$\Delta: \mathfrak{sl}_0(H) \to \mathfrak{sl}_0(H)$$

there exists a continous operator $D \in B(H)$ with $\Delta(x) = [D, x]$ for each $x \in \mathfrak{sl}_0(H)$. The operator D is unique up to an element in $\mathbb{K}\mathbf{1}$.

Proof. ([dlH72]) **Step 1:** For each finite subset F of $\mathfrak{sl}_0(H)$ there exists a finite-dimensional subspace $E \subseteq H$ such that

$$F \subseteq \mathfrak{sl}(E) := \{\varphi \in \mathfrak{sl}_0(H): \varphi(E) \subseteq E, \varphi(E^\perp) = \{0\}\}.$$

The Lie algebra $\mathfrak{sl}(E) \cong \mathfrak{sl}_{|E|}(\mathbb{K})$ is simple and the restriction Δ_E of Δ to $\mathfrak{sl}(E)$ is a linear map $\mathfrak{sl}(E) \to \mathfrak{sl}_0(H)$ satisfying

$$\Delta_E([x, y]) = [\Delta_E(x), y] + [x, \Delta_E(y)].$$

This means that $\Delta_E \in Z^1(\mathfrak{sl}(E), \mathfrak{sl}_0(H))$, where $\mathfrak{sl}(E)$ acts on $\mathfrak{sl}_0(H)$ by the adjoint action. Since this action turns $\mathfrak{sl}_0(H)$ into a locally finite module, Lemma A.3 implies that the cocycle Δ_E is trivial, i.e., there exists an element $D_E \in \mathfrak{sl}_0(H)$ with $\Delta_E(x) = [D_E, x]$ for all $x \in \mathfrak{sl}(E)$. Suppose that D_E' is another element in $\mathfrak{sl}_0(H)$ with this property. Then we write

$$D_E - D_E' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as a block matrix according to the decomposition $H = E \oplus E^\perp$. As $D_E - D_E'$ commutes with $\mathfrak{sl}(E)$, it preserves the subspaces $\mathfrak{sl}(E).H = E$ and $E^\perp = \{x \in H: \mathfrak{sl}(E).x = \{0\}\}$. Therefore $b = c = 0$, and $a \in \mathbb{K}\mathrm{id}_E$. This proves that $D_E|_E - D_E'|_E \in \mathbb{K}\mathrm{id}_E$. If we require, in addition, $D_E.v \perp v$ for some non-zero vector $v \in E$, then the restriction of D_E to E is uniquely determined.

Step 2: We may assume that $\dim H \geq 2$, otherwise the assertion is trivial. Fix $0 \neq v \in H$. As in Step 1, we find for each finite-dimensional subspace $E \subseteq H$ an operator D_E as above with $D_E.v \perp v$. For $E \subseteq E'$ the operator $D_{E'}$ also satisfies $D_{E'}.v \perp v$ and $\Delta_E(x) = [D_{E'}, x]$ for $x \in \mathfrak{sl}(E) \subseteq \mathfrak{sl}(E')$. Therefore $D_{E'}|_E = D_E$, so that we obtain a well-defined operator

$$D: H \to H, \quad D.w := D_E.w \quad \text{for} \quad w \in E.$$

This operator satisfies

$$\Delta(x) = [D, x] \quad \text{for all} \quad x \in \mathfrak{sl}_0(H).$$

Step 3: D is continuous: For $x, y \in H$ we consider the rank-one-operator $P_{x,y}.v = \langle v, y\rangle x$. Then $\mathrm{tr}\, P_{x,y} = \langle x, y\rangle$ vanishes if $x \perp y$. Then $P_{x,y} \in \mathfrak{sl}_0(H)$, and

$$[D, P_{x,y}](v) = P_{D.x,y}.v - \langle D.v, y\rangle x.$$

As for each $y \in H$ there exists an element x orthogonal to y, it follows that all functionals

$$v \mapsto \langle D.v, y\rangle$$

are continuous, i.e., that the adjoint operator D^* of the unbounded operator D is everywhere defined, and therefore that D has a closed graph ([Ne99, Th. A.II.8]). Now the Closed Graph Theorem implies that D is continuous.

Step 4: Uniqueness: We have to show that if an operator D on H commutes with $\mathfrak{sl}_0(H)$, then it is a multiple of the identity. The condition $[D, P_{x,y}] = 0$ for $x \perp y$ implies that

$$\langle v, y\rangle D.x = \langle D.v, y\rangle x, \quad v \in H.$$

It follows in particular that each $x \in H$ is an eigenvector, and hence that $D \in \mathbb{K}\mathbf{1}$. ■

Definition III.17. Let H be an infinite-dimensional Hilbert space. For each $p \in [1,\infty]$ we write $B_p(H)$ for the corresponding Schatten ideal in $B(H)$, where $B_\infty(H)$ denotes the space of compact operators (cf. [dlH72], [GGK00]). Each operator $A \in B_p(H)$ is compact, and if we write the non-zero eigenvalues of the positive operator $\sqrt{A^*A}$ (counted with multiplicity) in a sequence $(\lambda_n)_{n\in\mathbb{N}}$ (which might also contain zeros), the norm on $B_p(H)$ is given by

$$\|A\|_p = \Big(\sum_{n\in\mathbb{N}} \lambda_n^p\Big)^{\frac{1}{p}}.$$

According to [GGK00, Th. IV.11.2], we then have the estimate

$$\|AB\|_p \leq \|A\|_{p_1}\|B\|_{p_2} \quad \text{for} \quad \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}.$$

It follows in particular that each $B_p(H)$ is a Banach algebra. We also have

$$\|ABC\| \leq \|A\|\|B\|_p\|C\|, \quad B \in B_p(H), A, C \in B(H).$$

For $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$B_p(H)' \cong B_q(H),$$

where the pairing is induced by the trace $\langle x,y\rangle = \mathrm{tr}(xy)$. Here we use that $B_p(H)B_q(H) \subseteq B_1(H)$, and that the trace extends to a continuous linear functional $\mathrm{tr}\colon B_1(H) \to \mathbb{K}$ (cf. [dlH72, p.113]). We have

$$B_1(H) \subseteq B_p(H) \subseteq B_{p'}(H) \subseteq B_\infty(H)$$

for $p \leq p'$.

For $p = 1$ the elements of $B_1(H)$ are the *trace class operators* and for $p = 2$ the elements of $B_2(H)$ are the *Hilbert-Schmidt operators*. As the trace is a continuous linear functional on $B_1(H)$ vanishing on all commutators, the subspace

$$\mathfrak{sl}(H) := \{x \in B_1(H)\colon \mathrm{tr}\, x = 0\}$$

is a Lie algebra hyperplane ideal. ■

Proposition III.18. *Let $\mathfrak{gl}_p(H)$ be the Banach-Lie algebra obtained from $B_p(H)$ with the commutator bracket. Then $\mathfrak{gl}_p(H)$ is topologically perfect if and only if $p > 1$. The universal covering map is given by the inclusions maps*

$$\mathfrak{sl}(H) \hookrightarrow \mathfrak{gl}_p(H) \quad \text{for } 1 < p \leq 2, \qquad \text{and} \qquad \mathfrak{gl}_{\frac{p}{2}}(H) \hookrightarrow \mathfrak{gl}_p(H) \quad \text{for } p > 2.$$

Proof. That $\mathfrak{gl}_1(H)$ is not topologically perfect follows from the fact that the trace vanishes on all brackets. Assume that $p > 1$. Then an elementary argument with diagonal matrices implies that $\mathfrak{sl}_0(H)$ is dense in $B_p(H)$ with respect to $\|\cdot\|_p$. Since $\mathfrak{sl}_0(H)$ is a perfect Lie algebra, $\mathfrak{gl}_p(H)$ is topologically perfect.

Let $\omega\colon \mathfrak{gl}_p(H) \times \mathfrak{gl}_p(H) \to \mathbb{K}$ be a continuous Lie algebra cocycle. Then there exists a unique continuous linear map

$$\Delta\colon \mathfrak{gl}_p(H) \to \mathfrak{gl}_q(H) \cong \mathfrak{gl}_p(H)'$$

with $\mathrm{tr}(\Delta(x)y) = \omega(x,y)$ for all $x, y \in \mathfrak{gl}_p(H)$, and the cocycle identity for ω implies that Δ is a derivation, i.e.,

$$\Delta([x,y]) = [\Delta(x), y] + [x, \Delta(y)], \quad x, y \in \mathfrak{gl}_p(H).$$

The Lie algebra $\mathfrak{sl}_0(H)$ is a perfect ideal in $\mathfrak{gl}(H)$ and hence in each $\mathfrak{gl}_p(H)$. Therefore it is invariant under Δ, and Lemma III.16 implies the existence of a continuous operator

$D \in B(H)$ with $\Delta(x) = [D,x]$ for all $x \in \mathfrak{sl}_0(H)$. As both sides describe continuous linear maps $\mathfrak{gl}_p(H) \to \mathfrak{gl}(H)$ which coincide on the dense subspace $\mathfrak{sl}_0(H)$, we have $\Delta = \mathrm{ad}\, D$ on $\mathfrak{gl}_p(H)$.

For $1 \leq p \leq 2$ we have $q \geq 2 \geq p$, so that each bounded operator $D \in B(H)$ satisfies $\mathrm{ad}\, D(\mathfrak{gl}_p(H)) \subseteq \mathfrak{gl}_p(H) \subseteq \mathfrak{gl}_q(H)$. For $p > 2$ the dual space $\mathfrak{gl}_q(H)$ is a proper subspace of $\mathfrak{gl}_p(H)$, and it is shown in [dlH72, p.141] that

$$\{D \in \mathfrak{gl}(H) : [D, \mathfrak{gl}_p(H)] \subseteq \mathfrak{gl}_q(H)\} = \mathfrak{gl}_r(H) \quad \text{for} \quad \frac{1}{r} = \frac{1}{q} - \frac{1}{p} = 1 - \frac{2}{p} = \frac{p-2}{p}.$$

The cocycle associated to an operator D is given by

$$\omega(x,y) = \mathrm{tr}([D,x]y) = \mathrm{tr}(D[x,y]), \quad x, y \in \mathfrak{gl}_p(H).$$

That the trace on the right hand side makes sense follows from $B_p(H)B_p(H) \subseteq B_1(H)$ for $p \leq 2$ and $B_p(H)B_p(H) \subseteq B_{\frac{p}{2}}(H)$ and $D \in B_{\frac{p}{2}}(H)'$ for $p > 2$.

For $p \leq 2$ we have

$$[\mathfrak{gl}_p(H), \mathfrak{gl}_p(H)] \subseteq [\mathfrak{gl}_2(H), \mathfrak{gl}_2(H)] \subseteq \overline{[\mathfrak{sl}_0(H), \mathfrak{sl}_0(H)]} = \overline{\mathfrak{sl}_0(H)} = \mathfrak{sl}(H),$$

where the closure refers to the trace norm $\|\cdot\|_1$. An operator $D \in \mathfrak{gl}(H) \cong \mathfrak{gl}_1(H)'$ represents the cocycle 0 if and only if it is orthogonal to the hyperplane $\mathfrak{sl}(H)$, which means that $D \in \mathbb{K}\mathbf{1}$. For $p > 2$ an operator $D \in \mathfrak{gl}_r(H)$ is never a multiple of $\mathbf{1}$, so that we obtain

$$Z^2(\mathfrak{gl}_p(H), \mathbb{K}) \cong \begin{cases} \mathfrak{pgl}(H) := \mathfrak{gl}(H)/\mathbb{K}\mathbf{1} & \text{for } 1 \leq p \leq 2 \\ \mathfrak{gl}_{\frac{p}{2}}(H)' \cong \mathfrak{gl}_r(H) & \text{for } 2 < p. \end{cases} \tag{3.4}$$

Now let $q(\langle x,y \rangle) = [x,y]$ denote the bracket map

$$q \colon \widetilde{\mathfrak{gl}}_p(H) \cong \langle \mathfrak{gl}_p(H), \mathfrak{gl}_p(H) \rangle \to \begin{cases} \mathfrak{sl}(H) & \text{for } 1 \leq p \leq 2 \\ \mathfrak{gl}_{\frac{p}{2}}(H) & \text{for } 2 < p. \end{cases}$$

Then q is a continuous morphism of Banach–Lie algebras. Further

$$Z^2(\mathfrak{gl}_p(H), \mathbb{K}) \cong \mathrm{Lin}(\widetilde{\mathfrak{gl}}_p(H), \mathbb{K}),$$

and (3.4) imply that the adjoint map q^* is bijective. That q^* is injective implies that q has dense range and the surjectivity of q^* implies in particular that q is injective. Further the Closed Range Theorem ([Ru73, Th. 4.14]) implies that the image of q is closed, and hence that q is bijective. Finally the Open Mapping Theorem implies that q is an isomorphism. ∎

Remark III.19. From the preceding proposition, we obtain in particular examples of Lie algebras where the universal covering algebra is not centrally closed. For example each $\mathfrak{gl}_p(H)$ with $p > 2$ has this property. For $p < 2 \leq 4$ we have

$$\widetilde{\mathfrak{gl}}_p(H) \cong \mathfrak{gl}_{\frac{p}{2}}(H) \quad \text{and} \quad \widetilde{\widetilde{\mathfrak{gl}}_p(H)} \cong \mathfrak{sl}(H),$$

but for $2^k < p \leq 2^{k+1}$ we need to pass $k+1$-times to the universal covering Lie algebra until we reach $\mathfrak{sl}(H)$ which is centrally closed. ∎

In Section IV below we shall see many other concrete examples of universal central extensions, when we discuss root graded locally convex Lie algebras. In particular, we shall see that universal coverings of root graded Lie algebras are always centrally closed.

IV. Universal coverings of locally convex root graded Lie algebras

In this section we describe the universal covering Lie algebra of a locally convex root graded Lie algebra. In particular, we shall see that it only depends on the root system and the coordinate algebra. Several results in this section are topological versions of algebraic results in [ABG00]. A key point is that the concept of a generalized central extensions provides the natural framework to translate the algebraic structure of the universal covering algebra into the locally convex context.

Proposition IV.1. *Let $q: \widehat{\mathfrak g} \to \mathfrak g$ be a generalized central extension for which $\widehat{\mathfrak g}$ is topologically perfect. If $\mathfrak g$ is Δ-graded, then $\widehat{\mathfrak g}$ is Δ-graded and vice versa.*

Proof. (a) First we assume that $\mathfrak g$ is Δ-graded. On $\widehat{\mathfrak g}$ we consider the $\mathfrak g_\Delta$-module structure given by $\widehat{\mathrm{ad}}$ (Lemma III.4). Then the corestriction $\widehat{\mathfrak g} \to \mathrm{im}(q)$ is an extension of the locally finite $\mathfrak g_\Delta$-module $\mathrm{im}(q)$ by the trivial module $\ker q$, hence a trivial extension (Proposition A.4). It follows in particular that $\widehat{\mathfrak g}$ is an $\mathfrak h$-weight module. The weights occurring in this module are identical with those occurring in $\mathrm{im}(q) \supseteq [\mathfrak g, \mathfrak g]$ (Lemma III.4(1)). This implies that we have an $\mathfrak h$-weight decomposition

$$\widehat{\mathfrak g} = \widehat{\mathfrak g}_0 \oplus \bigoplus_{\alpha\in\Delta} \widehat{\mathfrak g}_\alpha$$

with $q(\widehat{\mathfrak g}_\alpha) = \mathfrak g_\alpha$ for $\alpha \neq 0$. As the central Lie algebra extension $q^{-1}(\mathfrak g_\Delta) \twoheadrightarrow \mathfrak g_\Delta$ is trivial, its commutator algebra $\widehat{\mathfrak g}_\Delta$ is a subalgebra which is mapped by q isomorphically onto $\mathfrak g_\Delta$. Therefore (R1)–(R3) are satisfied for $\widehat{\mathfrak g}_\Delta$ as a grading subalgebra in $\widehat{\mathfrak g}$.

As the bracket in $\widehat{\mathfrak g}$ is given by $[x,y] = b(q(x), q(y))$, the topological perfectness of $\widehat{\mathfrak g}$ implies that the image of b spans a dense subspace of $\widehat{\mathfrak g}$. Therefore

$$b(\mathfrak g_0, \mathfrak g_0) + \sum_{0\neq\alpha} b(\mathfrak g_\alpha, \mathfrak g_{-\alpha}) = b(\mathfrak g_0, \mathfrak g_0) + \sum_{0\neq\alpha} [\widehat{\mathfrak g}_\alpha, \widehat{\mathfrak g}_{-\alpha}]$$

is dense in $\widehat{\mathfrak g}_0$. For $x_{\pm\alpha} \in \widehat{\mathfrak g}_{\pm\alpha}$ and $x_{\pm\beta} \in \widehat{\mathfrak g}_{\pm\beta}$ we further have

$$b([q(x_\alpha), q(x_{-\alpha})], [q(x_\beta), q(x_{-\beta})]) = [[x_\alpha, x_{-\alpha}], [x_\beta, x_{-\beta}]] \subseteq [\widehat{\mathfrak g}_0, [\widehat{\mathfrak g}_\beta, \widehat{\mathfrak g}_{-\beta}]] \subseteq [\widehat{\mathfrak g}_\beta, \widehat{\mathfrak g}_{-\beta}].$$

Hence

$$b([\mathfrak g_\alpha, \mathfrak g_{-\alpha}], [\mathfrak g_\beta, \mathfrak g_{-\beta}]) \subseteq [\widehat{\mathfrak g}_\beta, \widehat{\mathfrak g}_{-\beta}],$$

so that (R4) holds for $\mathfrak g$, and the relation $q(\widehat{\mathfrak g}_\alpha) = \mathfrak g_\alpha$ for $\alpha \neq 0$ imply that $b(\mathfrak g_0, \mathfrak g_0)$ is contained in the closure of the sum of the spaces $[\widehat{\mathfrak g}_\alpha, \widehat{\mathfrak g}_{-\alpha}]$, $\alpha \neq 0$. This implies (R4) for $\widehat{\mathfrak g}$.
(b) Now we assume that $\widehat{\mathfrak g}$ is Δ-graded with grading subalgebra $\widehat{\mathfrak g}_\Delta$. Then $\ker q \subseteq \mathfrak z(\widehat{\mathfrak g})$, so that $\mathfrak g_\Delta := q(\widehat{\mathfrak g}_\Delta) \cong \widehat{\mathfrak g}_\Delta$. Clearly $\mathfrak g$ carries a natural $\mathfrak g_\Delta$-module structure.

From $[\mathfrak g, \mathfrak g] \subseteq \mathrm{im}(q)$ (Lemma III.4(2)) we derive that $\mathfrak g/\mathrm{im}(q)$ is a trivial $\mathfrak g_\Delta$-module. Moreover, $\mathrm{im}(q) \cong \widehat{\mathfrak g}/\ker(q)$ is a locally finite $\mathfrak g_\Delta$-module. Therefore Proposition A.4 implies that $\mathfrak g$ is a locally finite $\mathfrak g_\Delta$-module which is a direct sum of $q(\widehat{\mathfrak g})$ and a trivial module Z. This immediately leads to a weight decomposition of $\mathfrak g$ with weight system Δ, and it is obvious that (R1)–(R3) are satisfied.

As $\mathfrak h$ acts on $\mathfrak g$ by continuous operators, the projection $\mathfrak g \to \mathfrak g_0$ along the sum of the other root spaces is continuous, so that the density of the image of q in $\mathfrak g$ implies that $q(\widehat{\mathfrak g}_0)$ is dense in $\mathfrak g_0$. We further have

$$[\mathfrak g_\alpha, \mathfrak g_{-\alpha}] = q(b(\mathfrak g_\alpha, \mathfrak g_{-\alpha})) = q(b(q(\widehat{\mathfrak g}_\alpha), q(\widehat{\mathfrak g}_{-\alpha}))) = q([\widehat{\mathfrak g}_\alpha, \widehat{\mathfrak g}_{-\alpha}]),$$

so that (R4) for $\widehat{\mathfrak g}$ implies (R4) for $\mathfrak g$. ■

Corollary IV.2. *If $\mathfrak g$ is Δ-graded with grading subalgebra $\mathfrak g_\Delta$, then $\mathfrak z(\mathfrak g) \subseteq \mathfrak z_{\mathfrak g}(\mathfrak g_\Delta) \subseteq \mathfrak z_{\mathfrak g}(\mathfrak h) = \mathfrak g_0$, and $\mathfrak g/\mathfrak z(\mathfrak g) \cong \mathrm{ad}\,\mathfrak g$ is a Δ-graded Lie algebra. The quotient map $\mathrm{ad}: \mathfrak g \to \mathfrak g/\mathfrak z(\mathfrak g)$ is a morphism of Δ-graded Lie algebras.* ■

Lemma IV.3. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be locally convex Δ-graded Lie algebras with coordinate structures $(\mathcal{A}_i = A_i \oplus B_i, D_i, \delta^{D_i})$ and $\eta_i\colon \mathfrak{g}_\Delta \to \mathfrak{g}$ the corresponding embeddings that we use to identify $\mathfrak{g}_\Delta$ with a subalgebra of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. If $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of locally convex Lie algebra with $\varphi \circ \eta_1 = \eta_2$, then there exist continuous linear maps*

$$\varphi_A\colon A_1 \to A_2, \quad \varphi_B\colon B_1 \to B_2 \quad \text{and} \quad \varphi_D\colon D_1 \to D_2$$

such that

$$\varphi(a \otimes x + b \otimes v + d) = \varphi_A(a) \otimes x + \varphi_B(b) \otimes v + \varphi_D(d) \tag{4.1}$$

for $a \in A_1, b \in B_1, d \in D_1, x \in \mathfrak{g}_\Delta$ and $v \in V_s$, and

$$\varphi_{\mathcal{A}} := \varphi_A \oplus \varphi_B\colon \mathcal{A}_1 \to \mathcal{A}_2$$

is a continuous algebra homomorphism with

$$\delta^{D_2} \circ (\varphi_{\mathcal{A}} \times \varphi_{\mathcal{A}}) = \varphi_D \circ \delta^{D_1}. \tag{4.2}$$

Proof. The condition $\varphi \circ \eta_1 = \eta_2$ means that φ is equivariant with respect to the representations of $\mathfrak{g}_\Delta$ on $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Identifying A_1 with $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g}_1)$, the equivariance of φ with respect to the $\mathfrak{g}_\Delta$ permits us to define $\varphi_A(a) := \varphi \circ a$. We likewise define φ_B and φ_D. Then (4.1) is satisfied. Now (4.2) and that $\varphi_{\mathcal{A}}$ defines an algebra homomorphism follow directly from (B1)–(B3), because the algebra structure on $\mathcal{A}_1$, resp., $\mathcal{A}_2$ is completely determined by the Lie bracket. ∎

Remark IV.4. The preceding lemma applies in particular to generalized central extensions $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$. In this case the proof of Proposition IV.1 implies that $q_{\mathcal{A}}$ is a topological isomorphism, hence an isomorphism of locally convex algebras. We therefore may assume that $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ have the same coordinate algebra $\mathcal{A}$. In this sense we write

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D \quad \text{and} \quad \widehat{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widehat{D},$$

and $q_D\colon \widehat{D} \to D$ is a map with dense range and $q_D \circ \delta^{\widehat{D}} = \delta^D$.

This applies in particular to the universal covering algebra, which we write as

$$\widetilde{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widetilde{D}.$$

In the following subsection we will see how $\widetilde{D}$ can be described directly in terms of the coordinate algebra $\mathcal{A}$ and $\delta_{\mathcal{A}}$. ∎

The universal covering of a Δ-graded locally convex Lie algebra

To describe the universal covering Lie algebra $\widetilde{\mathfrak{g}}$ of a locally convex root graded Lie algebra $\mathfrak{g}$, we first consider its coordinate structure $(\mathcal{A} = A \oplus B, D, \delta^D)$ (Definition II.14). We consider the locally convex space

$$\langle \mathcal{A}, \mathcal{A} \rangle^\sigma := \langle \mathcal{A}, \mathcal{A} \rangle / \overline{\langle A, B \rangle}$$

and write the image of $\langle a, b \rangle \in \langle \mathcal{A}, \mathcal{A} \rangle$ in $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma$ also as $\langle a, b \rangle$.

Theorem IV.5. *For each root system Δ, the corresponding coordinate algebra $\mathcal{A}$, and the natural map $\delta_{\mathcal{A}}: \mathcal{A} \times \mathcal{A} \to \mathrm{der}(\mathcal{A})$, the derivations $\delta_{\mathcal{A}}(a,b)$ preserve the subspace $\overline{\langle A, B\rangle}$ of $\langle \mathcal{A}, \mathcal{A}\rangle$, and we obtain on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ the structure of a locally convex Lie algebra by*

$$[\langle a, a'\rangle, \langle b, b'\rangle] := \delta_{\mathcal{A}}(a,a').\langle b, b'\rangle.$$

Proof. Since the map $\mathcal{A}^3 \to \mathcal{A}, (a,b,c) \mapsto \delta^D(a,b).c$ is continuous, and δ^D is a cyclic 1-cocycle vanishing on $A \times B$ (Theorem II.13), it defines a continuous linear map

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to D, \quad \langle a, b\rangle \mapsto \delta^D(a,b).$$

Now define

$$\delta_{\mathcal{A}}: \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \mathrm{der}(\mathcal{A}), \quad \delta_{\mathcal{A}}(a,b).c := \delta^D(a,b).c,$$

and observe that the bilinear map

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \times \mathcal{A} \to \mathcal{A}, \quad (\langle a, b\rangle, c) \mapsto \delta_{\mathcal{A}}(a,b).c$$

is continuous.

From (2.3) in Theorem II.13 we further derive that

$$\delta_{\mathcal{A}}(\delta_{\mathcal{A}}(a,b).\langle c, d\rangle) = \delta_{\mathcal{A}}(\delta_{\mathcal{A}}(a,b).c, d) + \delta_{\mathcal{A}}(c, \delta_{\mathcal{A}}(a,b).d) = [\delta_{\mathcal{A}}(a,b), \delta_{\mathcal{A}}(c,d)] \tag{4.3}$$

for $a, b, c, d \in \mathcal{A}$.

As the operators $\delta(a,b) \in \mathrm{der}(\mathcal{A})$ all preserve the subspaces A and B of $\mathcal{A}$, the subspace $\overline{\langle A, B\rangle} \subseteq \langle \mathcal{A}, \mathcal{A}\rangle$ is invariant under all these operators with respect to the natural action of $\mathrm{der}(\mathcal{A})$ on $\langle \mathcal{A}, \mathcal{A}\rangle$, and we therefore obtain a well-defined bracket on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ with

$$[\langle a, a'\rangle, \langle b, b'\rangle] := \delta_{\mathcal{A}}(a,a').\langle b, b'\rangle.$$

As in Proposition III.9, the Jacobi identity for this bracket is a direct consequence of (4.3). That the bracket is alternating is equivalent to the relation

$$\delta_{\mathcal{A}}(a,a').\langle b, b'\rangle = -\delta_{\mathcal{A}}(b,b').\langle a, a'\rangle \tag{4.4}$$

for $a, a', b, b' \in \mathcal{A}$. This relation can be verified case by case for the coordinate algebras associated to the different types of root systems (see [ABG00, p.521]; cf. also Theorem II.20 and the subsequent comments).

For the case where $\mathcal{A}$ is an associative or a Jordan algebra, (4.4) can be obtained as in Example III.10(2), (3). In this case we already have on $\langle \mathcal{A}, \mathcal{A}\rangle$ a natural Lie algebra structure, and since $\overline{\langle A, B\rangle}$ is invariant under the operators $\delta_{\mathcal{A}}(a,b)$, it is a Lie algebra ideal, so that $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ simply is the quotient Lie algebra. ∎

The following theorem is the locally convex version of the description of the universal covering Lie algebra (cf. [ABG00] for the algebraic case).

Theorem IV.6. *The Lie algebra*

$$\widetilde{\mathfrak{g}} := (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$$

with the Lie bracket given by

$$[d, a \otimes x + b \otimes v + d'] = d.a \otimes x + d.b \otimes v + [d, d'],$$

and

$$[a \otimes x, a' \otimes x'] = \gamma_+^A(a,a') \otimes [x, x'] + \gamma_-^A(a,a') \otimes x * x' + \gamma_A^B(a,a') \otimes \beta_{\mathfrak{g}}^V(x,x') + \kappa(x,x')\delta_{\mathcal{A}}(a,a'),$$

$$[a\otimes x, b\otimes v] = \frac{ab+ba}{2}\otimes \beta^{\mathfrak g}_{\mathfrak g,V}(x,v) + \frac{ab-ba}{2}\otimes x.v,,$$

$$[b\otimes v, b'\otimes v'] = \gamma^A_B(b,b')\otimes \beta^{\mathfrak g}_V(v,v') + \gamma^B_B(b,b')\otimes \beta^V_V(v,v') + \kappa_{V_s}(v,v')\delta_{\mathcal A}(b,b')$$

is a universal covering Lie algebra of $\mathfrak g$ *with universal covering map*

$$q_{\mathfrak g}(a\otimes x + b\otimes v + d) = a\otimes x + b\otimes v + \delta^D_{\mathcal A}(d),$$

where $\delta^D_{\mathcal A}(\langle a,b\rangle) = \delta^D(a,b)$ *for* $a,b\in\mathcal A$.

Proof. The Lie algebra $\langle\mathcal A,\mathcal A\rangle^\sigma$ together with the map $\delta_{\mathcal A}\colon\langle\mathcal A,\mathcal A\rangle^\sigma\to \mathrm{der}(\mathcal A)$ satisfy all assumptions of Theorem II.15, and we obtain on

$$\widetilde{\mathfrak g} := (A\otimes\mathfrak g_\Delta)\oplus(B\otimes V_s)\oplus\langle\mathcal A,\mathcal A\rangle^\sigma$$

a Lie bracket as described above. Now $\widetilde{\mathfrak g}$ is a Δ-graded Lie algebra with coordinate structure $(\mathcal A,\langle\mathcal A,\mathcal A\rangle^\sigma,\delta_{\mathcal A})$. Let $q\colon\widehat{\mathfrak g}\to\mathfrak g$ be a generalized central extension, where we write $\widehat{\mathfrak g}$ as

$$\widehat{\mathfrak g} = (A\otimes\mathfrak g_\Delta)\oplus(B\otimes V_s)\oplus\widehat D$$

(Remark IV.4). Then the corresponding map $\delta^{\widehat D}_{\mathcal A}\colon\langle\mathcal A,\mathcal A\rangle^\sigma\to\widehat D$ is a continuous homomorphism of Lie algebras because

$$\delta^{\widehat D}_{\mathcal A}([\langle a,b\rangle,\langle c,d\rangle]) = \delta^{\widehat D}_{\mathcal A}(\delta_{\mathcal A}(a,b).\langle c,d\rangle) = [\delta^{\widehat D}_{\mathcal A}(a,b),\delta^{\widehat D}_{\mathcal A}(c,d)]$$

(Theorem II.13). We now obtain a continuous linear map

$$\widetilde q\colon\widetilde{\mathfrak g}\to\widehat{\mathfrak g},\quad a\otimes x + b\otimes v + d\mapsto a\otimes x + b\otimes v + \delta^{\widehat D}_{\mathcal A}(d),$$

and (B1)–(B3) together with the relation $q_D\circ\delta^{\widehat D} = \delta^D$ (Lemma IV.3) imply that this map is a homomorphism of Lie algebras satisfying $q\circ\widetilde q = q_{\mathfrak g}$, where $q_{\mathfrak g}\colon\widetilde{\mathfrak g}\to\mathfrak g$ is the natural homomorphism induced by the Lie algebra homomorphism $\delta^D_{\mathcal A}\colon\langle\mathcal A,\mathcal A\rangle^\sigma\to D$. ■

Corollary IV.7. *If* $\mathfrak g$ *is a* Δ*-graded locally convex Lie algebra, then its universal covering Lie algebra* $\widetilde{\mathfrak g}$ *only depends on the pair* $(\mathcal A,\delta_{\mathcal A})$*, which in turn is completely determined by the coordinate algebra* $\mathcal A$ *and the type of* Δ*. If we write* $\widetilde{\mathfrak g}(\Delta,\mathcal A)$ *for* $\widetilde{\mathfrak g}$*, then the assignment*

$$\mathcal A\mapsto\widetilde{\mathfrak g}(\Delta,\mathcal A)$$

defines a functor from the category of locally convex algebras determined by the root system Δ *to the category of locally convex Lie algebras.* ■

Corollary IV.8. *Each Lie algebra* $\widetilde{\mathfrak g}(\Delta,\mathcal A)$*, i.e., the universal covering Lie algebra of a* Δ*-graded Lie algebra* $\mathfrak g$*, is centrally closed.*

Proof. From the explicit description of the universal covering Lie algebra $\widetilde{\mathfrak g}$ in Theorem IV.6 and the fact that it has the same coordinate algebra as $\mathfrak g$, it follows that the map $\widetilde{\widetilde{\mathfrak g}}\to\widetilde{\mathfrak g}$ is an isomorphism because for both algebras the D-part is $\langle\mathcal A,\mathcal A\rangle^\sigma$. ■

Lie algebra cocycles on root graded Lie algebras

Proposition IV.9. *Every continuous Lie algebra cocycle on a root graded Lie algebra $\mathfrak{g}$ is equivalent to a $\mathfrak{g}_\Delta$-invariant one.*

Proof. As a module of $\mathfrak{g}_\Delta$, the Lie algebra $\mathfrak{g}$ decomposes topologically as

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D,$$

and therefore

$$\mathfrak{g} \otimes \mathfrak{g} \cong (\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta) \otimes (A \otimes A) \oplus (\mathfrak{g}_\Delta \otimes M) \otimes (A \otimes B) + \cdots$$

is the decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ as a $\mathfrak{g}_\Delta$-module, where A, B and D are considered as trivial modules. We conclude that for each trivial locally convex $\mathfrak{g}_\Delta$-module $\mathfrak{z}$ we have

$$\mathrm{Lin}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{z}) \cong (\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta)^* \otimes \mathrm{Lin}(A \otimes A, \mathfrak{z}) \oplus (\mathfrak{g}_\Delta \otimes V_s)^* \otimes \mathrm{Lin}(A \otimes B, \mathfrak{z}) + \cdots$$

Since $\mathfrak{g}_\Delta$ and V_s are finite-dimensional, $\mathrm{Lin}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{z})$ is a locally finite $\mathfrak{g}_\Delta$-module, hence semisimple. This property is in particular inherited by the submodule $Z^2(\mathfrak{g}, \mathfrak{z}) \subseteq \mathrm{Lin}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{z})$ of continuous Lie algebra cocycles. Hence the decomposition into trivial and effective part yields

$$Z^2(\mathfrak{g}, \mathfrak{z}) = Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta} \oplus \mathfrak{g}_\Delta.Z^2(\mathfrak{g}, \mathfrak{z}).$$

For the representation ρ of $\mathfrak{g}$ on the space $C^2(\mathfrak{g}, \mathfrak{z})$ of continuous Lie algebra 2-cochains we have the Cartan formula

$$\rho(x) = i_x \circ d + d \circ i_x, \quad x \in \mathfrak{g},$$

which implies that on 2-cocycles we have $\rho(x).\omega = d(i_x.\omega)$ and hence $\mathfrak{g}.Z^2(\mathfrak{g}, \mathfrak{z}) \subseteq B^2(\mathfrak{g}, \mathfrak{z})$. We conclude that each element of $H^2(\mathfrak{g}, \mathfrak{z})$ has a $\mathfrak{g}_\Delta$-invariant representative. ∎

Proposition IV.10. *The invariant Lie algebra cocycles $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta}$ are in one-to-one correspondence with the elements of the space $\mathrm{Lin}(\langle \mathcal{A}, \mathcal{A}\rangle^\sigma, \mathfrak{z})$, where we obtain from $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta} \cong \mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})^{\mathfrak{g}_\Delta}$ a function $\omega_{\mathcal{A}}$ on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ by restricting to the subspace $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ of $\widetilde{\mathfrak{g}}$.*

The cocycle ω is a coboundary if and only if $\omega_{\mathcal{A}}$ can be written as $\alpha \circ \delta^D_{\mathcal{A}}$ for an $\alpha \in \mathrm{Lin}(D, \mathfrak{z})$, so that

$$H^2(\mathfrak{g}, \mathfrak{z}) \cong \mathrm{Lin}(\langle \mathcal{A}, \mathcal{A}\rangle^\sigma, \mathfrak{z}) / \mathrm{Lin}(D, \mathfrak{z}) \circ \delta^D_{\mathcal{A}}.$$

Proof. If $q_{\mathfrak{g}}: \widetilde{\mathfrak{g}} \cong \langle \mathfrak{g}, \mathfrak{g}\rangle \to \mathfrak{g}$ is the universal covering Lie algebra, then we have for each locally convex space $\mathfrak{z}$ a natural isomorphism $Z^2(\mathfrak{g}, \mathfrak{z}) \cong \mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})$ (Remark III.7). As $q_{\mathfrak{g}}$ is equivariant with respect to the action of $\mathfrak{g}_\Delta$, this leads to

$$Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta} \cong \mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})^{\mathfrak{g}_\Delta}$$

for the invariant Lie algebra cocycles. On the other hand

$$\widetilde{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$$

implies that $\mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})^{\mathfrak{g}_\Delta} \cong \mathrm{Lin}(\langle \mathcal{A}, \mathcal{A}\rangle^\sigma, \mathfrak{z})$.

If $\alpha \in \mathrm{Lin}(D, \mathfrak{z})$, then we extend α to a continuous linear map $\alpha_{\mathfrak{g}}: \mathfrak{g} \to \mathfrak{z}$ by zero on the subspaces $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$. Then $d\alpha(x, y) = \alpha([y, x])$ is a $\mathfrak{g}_\Delta$-invariant cocycle on $\mathfrak{g}$, and the corresponding function $(d\alpha)_{\widetilde{\mathfrak{g}}}$ on $\widetilde{\mathfrak{g}} \cong \langle \mathfrak{g}, \mathfrak{g}\rangle$ satisfies $(d\alpha)_{\widetilde{\mathfrak{g}}} = -\alpha \circ b_{\mathfrak{g}}$ which implies that

$$(d\alpha)_{\mathcal{A}} = -\alpha \circ b_{\mathfrak{g}}|_{\langle \mathcal{A}, \mathcal{A}\rangle^\sigma} = -\alpha \circ \delta^D_{\mathcal{A}}.$$

If, conversely, $\omega = d\alpha$ is a $\mathfrak{g}_\Delta$-invariant coboundary, then the same argument as in the proof of Proposition IV.7 implies that we may choose α as a $\mathfrak{g}_\Delta$-invariant function on $\mathfrak{g}$, which means that α vanishes on $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$, hence is of the form discussed above. We conclude that

$$\mathrm{Lin}(D, \mathfrak{z}) \circ \delta^D_{\mathcal{A}} \subseteq \mathrm{Lin}(\langle \mathcal{A}, \mathcal{A}\rangle^\sigma, \mathfrak{z})$$

corresponds to the $\mathfrak{g}_\Delta$-invariant coboundaries. This completes the proof. ∎

The preceding proposition describes the cohomology of $\mathfrak{g}$ with values in a trivial module $\mathfrak{z}$ in terms of the coordinate algebra. For the topological homology space we get

$$H_2(\mathfrak{g}) := \ker q_{\mathfrak{g}} \cong \ker \delta^D_{\mathcal{A}} \subseteq \langle \mathcal{A}, \mathcal{A}\rangle^\sigma,$$

which describes $H_2(\mathfrak{g})$ completely in terms of the coordinate algebra and D.

Definition IV.11. Motivated by the corresponding concept for associative algebras with involution (Appendix D), we define the *full skew dihedral homology of* $\mathcal{A}$, resp., the pair $(\mathcal{A}, \delta_{\mathcal{A}})$ as

$$HF(\mathcal{A}) := \ker \delta_{\mathcal{A}} \subseteq \langle \mathcal{A}, \mathcal{A}\rangle^\sigma. \qquad \blacksquare$$

Proposition IV.12. *If $\mathfrak{g}$ is a Δ-graded locally convex Lie algebra, then the centerfree Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is also Δ-graded with the same coordinate algebra and the same universal covering algebra, and*

$$H_2(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) \cong HF(\mathcal{A}).$$

Proof. The first two assertions follow from Corollary IV.2 and Remark III.15(b).

With respect to the $\mathfrak{g}_\Delta$-isotypical decomposition of $\mathfrak{g}$, we have

$$\mathfrak{z}(\mathfrak{g}) = \{d \in D \colon (\forall a \in \mathcal{A})\ d.a = 0\},$$

which implies that

$$H_2(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) = \ker q_{\mathfrak{g}/\mathfrak{z}(\mathfrak{g})} = q_{\mathfrak{g}}^{-1}(\mathfrak{z}(\mathfrak{g})) = \mathfrak{z}(\widetilde{\mathfrak{g}}) = \ker \delta_{\mathcal{A}} = HF(\mathcal{A}). \qquad \blacksquare$$

Example IV.13. (a) Let $n \geq 4$. If $\mathfrak{g} = \mathfrak{sl}_n(A)$ for a locally convex unital associative algebra, then the preceding considerations imply that

$$H_2(\mathfrak{sl}_n(A)) \cong HC_1(A) \quad \text{and} \quad H_2(\mathfrak{psl}_n(A)) \cong HF(A), \tag{4.5}$$

where

$$\mathfrak{psl}_n(A) := \mathfrak{sl}_n(A)/\mathfrak{z}(\mathfrak{sl}_n(A)) \cong \mathfrak{sl}_n(A)/(Z(A) \cap \overline{[A,A]}).$$

If $n = 3$, then $\mathfrak{g}$ is A_3-graded, and we have to consider A as an alternative algebra. Since A is associative, the left and right multiplications L_a and R_b on A commute, so that

$$L_{[a,b]} - R_{[a,b]} - 3[L_a, R_b] = \mathrm{ad}[a,b].$$

This implies that $\langle A, A\rangle$ carries the same Lie algebra structure, regardless of whether we consider it as an associative or an alternative algebra. We conclude that (4.5) remains true for $n = 3$.

For $n = 2$ the coordinate algebra of $\mathfrak{sl}_2(A)$ is the Jordan algebra $\mathcal{A} = A_J$ with the product $a \circ b = \frac{ab+ba}{2}$. Let $L_a(x) = ax$ and $R_a(x) = xa$ denote the left and right multiplications in the associative algebra A, and $L_a^J(x) = \frac{1}{2}(L_a + R_a)$ the left multiplication in the corresponding Jordan algebra. Then

$$8\delta_{A_J}(a,b) = 4[L_a^J, L_b^J] = [L_a + R_a, L_b + R_b] = [L_a, L_b] + [R_a, R_b] = L_{[a,b]} - R_{[a,b]} = \mathrm{ad}[a,b].$$

For $\mathfrak{g} = \mathfrak{sl}_2(A)$ we also have $D = \overline{[A,A]}$ and

$$\delta_{A_J}^D(a,b) = \frac{1}{2}[a,b]$$

(Example II.16(b)). We therefore obtain

$$H_2(\mathfrak{sl}_2(A)) \cong \ker \delta_{A_J}^D \quad \text{and} \quad H_2(\mathfrak{psl}_2(A)) \cong HF(A_J).$$

In the algebraic context, the preceding results have been obtained for $n = 2$ by Gao ([Gao93]), and for $n \geq 3$ by Kassel and Loday ([KL82]).

(b) For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathcal{A}, \sigma)$ (Example I.7, Example II.16(c)) the coordinate algebra is an associative algebra $\mathcal{A}$ with involution. For

$$\mathfrak{psp}_{2n}(\mathcal{A}, \sigma) := \mathfrak{sp}_{2n}(\mathcal{A}, \sigma)/\mathfrak{z}(\mathfrak{sp}_{2n}(\mathcal{A}, \sigma)),$$

we therefore obtain

$$H_2(\mathfrak{psp}_{2n}(\mathcal{A}, \sigma)) \cong HF(\mathcal{A})$$

and $H_2(\mathfrak{sp}_{2n}(\mathcal{A}, \sigma))$ is isomorphic to the kernel of the map

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \overline{[A,A]}^{-\sigma}, \quad \langle a, b\rangle \mapsto [a,b] + [a^\sigma, b^\sigma].$$

(c) If J is a Jordan algebra, then it follows from the construction in Example I.9 and our explicit description of the centrally closed Δ-graded Lie algebras in this section that $\widetilde{\mathrm{TKK}}(J)$ is centrally closed, hence the notation. In the sense of Corollary IV.7, we could also write $\widetilde{\mathrm{TKK}}(J) = \widetilde{\mathfrak{g}}(A_2, J)$. $\blacksquare$

Example IV.14. (a) Let $\mathcal{A}$ be an associative algebra with involution σ, $A := \mathcal{A}^\sigma$, $B := \mathcal{A}^{-\sigma}$, and consider the modified bracket map defined by

$$b_\sigma(x,y) := [x,y] - [x,y]^\sigma = [x,y] - [y^\sigma, x^\sigma] = [x,y] + [x^\sigma, y^\sigma].$$

Then b_σ defines a continuous linear map $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \mathcal{A}^{-\sigma}$, and

$$HD_1'(\mathcal{A},\sigma) := \ker b_\sigma \subseteq \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$$

is called the *first skew-dihedral homology space* of $(\mathcal{A},\sigma)$ (see Appendix D for more information on skew-dihedral homology). The corresponding full dihedral homology space is

$$HF(\mathcal{A}) = b_\sigma^{-1}(Z(\mathcal{A})) = \{x \in \langle \mathcal{A}, \mathcal{A}\rangle^\sigma : \mathrm{ad}(b_\sigma(x)) = 0\}.$$

(b) If $\mathcal{A} = A$ is an associative algebra, $B = \{0\}$, and $\delta_A(a,b) = \mathrm{ad}([a,b])$, then

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma = \langle A, A\rangle$$

with the Lie algebra structure

$$[\langle a,b\rangle, \langle c,d\rangle] = \langle [a,b], [c,d]\rangle$$

defined in Example III.10(2). If $b_A : \langle A, A\rangle \to A, \langle a,b\rangle \mapsto [a,b]$ is the commutator bracket, then

$$HC_1(A) := \ker b_A$$

is the *first cyclic homology of* A, and in this case the full skew-dihedral homology space is the *full cyclic homology space:*

$$HF(A) = b_A^{-1}(Z(A)) \supseteq HC_1(A),$$

where $Z(A)$ is the center of A.

By corestriction of the bracket map b_A, we obtain a generalized central extension of locally convex Lie algebras

$$HC_1(A) \hookrightarrow \langle A, A\rangle \to \overline{[A,A]}.$$

We also have a generalized central extension of locally convex Lie algebras

$$HF(A) \hookrightarrow \langle A, A\rangle \to \overline{[A,A]}/\big(Z(A) \cap \overline{[A,A]}\big).$$

(c) If A is commutative and associative, then $b_A = 0$, so that

$$HF(A) = HC_1(A) = \langle A, A\rangle.$$

A more direct description of this space can be given as follows. Let M be a locally convex A-module in the sense that the module structure $A \times M \to M$ is continuous. A *derivation* $D: A \to M$ is a continuous linear map with $D(ab) = a.D(b) + b.D(a)$ for $a, b \in A$. One can show that for each locally convex associative algebra there exists a *universal differential module* $\Omega^1(A)$, which is endowed with a derivation $d: A \to \Omega^1(A)$ which has the universal property that for each derivation $D: A \to M$ there exists a continuous linear module homomorphism $\varphi: \Omega^1(A) \to M$ with $\varphi \circ d = D$ (cf. [Ma02]). We consider the quotient space $\Omega^1(A)/\overline{dA}$ endowed with the locally convex quotient topology. Then we have a natural isomorphism

$$\langle A, A\rangle \to \Omega^1(A)/\overline{dA}, \quad \langle a,b\rangle \mapsto [a \cdot db]. \qquad \blacksquare$$

Example IV.15. In general it is not always easy to determine the space $HC_1(A)$ for a concrete commutative locally convex algebra. The following cases are of particular interest for applications: (1) $\Omega^1(A) = \{0\}$ for any commutative C^*-algebra A (Johnson, 1972; see [BD73, Prop. VI.14]).

(2) If M is a connected finite-dimensional smooth manifold and $A = C^\infty(M,\mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then A is a Fréchet algebra (a Fréchet space with continuous algebra multiplication). If $\Omega^1(M,\mathbb{K})$ is the space of smooth $\mathbb{K}$-valued 1-forms on M, then the differential

$$d: C^\infty(M,\mathbb{K}) \to \Omega^1(M,\mathbb{K}), \quad f \mapsto df$$

has the universal property, and therefore

$$\Omega^1(A) \cong \Omega^1(M,\mathbb{K}) \quad \text{and} \quad HC_1(A) \cong \Omega^1(M,\mathbb{K})/dC^\infty(M,\mathbb{K})$$

([Ma02]).

A similar result holds for the locally convex algebra $A = C^\infty_c(M,\mathbb{K})$ of smooth functions with compact support, endowed with the locally convex direct limit topology with respect to the Fréchet spaces $C^\infty_K(M,\mathbb{K})$ of all those functions whose support is contained in a fixed compact subset $K \subseteq M$. In this case we have

$$\Omega^1(A) \cong \Omega^1_c(M,\mathbb{K}) \quad \text{and} \quad HC_1(A) \cong \Omega^1_c(M,\mathbb{K})/dC^\infty_c(M,\mathbb{K})$$

([Ma02], [Ne02d]).

(3) If M is a complex manifold, then the algebra $A := \mathcal{O}(M)$ of $\mathbb{C}$-valued holomorphic functions is a Fréchet algebra with respect to the topology of uniform convergence on compact subsets of M. Assume that M can be realized as an open submanifold of a closed submanifold of some $\mathbb{C}^n$, i.e., as an open subset of a Stein manifold. Let $\Omega^1_{\mathcal{O}}(M)$ be the space of holomorphic 1-forms on M. Then it is shown in [NW03] that the differential

$$d: \mathcal{O}(M) \to \Omega^1_{\mathcal{O}}(M), \quad f \mapsto df$$

has the universal property, and therefore

$$\Omega^1(A) \cong \Omega^1_{\mathcal{O}}(M) \quad \text{and} \quad HC_1(A) \cong \Omega^1_{\mathcal{O}}(M)/d\mathcal{O}(M).$$ ■

Example IV.16. We construct two root graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which are isogenous, non-isomorphic, but have trivial center.

Let A be a locally convex associative unital algebra with $A = \overline{[A,A]} \oplus \mathbb{K}\mathbf{1}$. Then the center of

$$\mathfrak{sl}_n(A) \cong A \otimes \mathfrak{sl}_n(\mathbb{K}) \oplus \overline{[A,A]} \otimes \mathbf{1}$$

is trivial.

For the associative Banach algebra $B_2(H)$ of Hilbert-Schmidt operators on an infinite-dimensional Hilbert space H we consider the associated unital Banach algebra $A := B_2(H) + \mathbb{K}\mathbf{1}$. Then

$$\langle A, A\rangle = \langle B_2(H), B_2(H)\rangle$$

follows from $\langle A, \mathbf{1}\rangle = \{0\}$. If $\mathfrak{gl}_2(H) := B_2(H)_L$ is the Lie algebra obtained from $B_2(H)$ via the commutator bracket, then we have seen in Proposition III.18 that $\widetilde{\mathfrak{gl}}_2(H) = \langle \mathfrak{gl}_2(H), \mathfrak{gl}_2(H)\rangle \cong \mathfrak{sl}(H)$, and the universal Lie algebra cocycle is the commutator bracket

$$\omega_u: \mathfrak{gl}_2(H) \times \mathfrak{gl}_2(H) \to \mathfrak{sl}(H).$$

On the other hand the discussion in Example III.10(2) shows that the space $\langle B_2(H), B_2(H)\rangle$ obtained from the associative algebra structure is a quotient of $\langle \mathfrak{gl}_2(H), \mathfrak{gl}_2(H)\rangle$. As the bracket map $\mathfrak{q}_{\mathfrak{gl}_2(H)}: \langle \mathfrak{gl}_2(H), \mathfrak{gl}_2(H)\rangle \to \mathfrak{gl}_2(H)$ is injective, $\langle B_2(H), B_2(H)\rangle$ must be the quotient by the trivial subspace, and therefore the bracket map

$$\langle B_2(H), B_2(H)\rangle \to \mathfrak{sl}(H), \quad \langle a, b\rangle \mapsto [a,b]$$

is an isomorphism of Banach spaces.

Let $n \geq 3$. Then the natural morphism

$$\widetilde{\mathfrak{sl}}_n(A) \cong (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus \langle A, A\rangle \to \mathfrak{sl}_n(A)$$

is injective, and hence $\widetilde{\mathfrak{sl}}_n(A)$ has trivial center. As the map $\mathfrak{sl}(H) \to B_2(H)$ is not surjective, the two A_{n-1}-graded Lie algebras $\widetilde{\mathfrak{sl}}_n(A)$ and $\mathfrak{sl}_n(A)$ both have trivial center but are not isomorphic. ■

V. Perspectives: Root graded Lie groups

In this section we briefly discuss some aspects of the global Lie theory of root graded Lie algebras, namely root graded Lie groups.

An infinite-dimensional Lie group G is a manifold modeled on a locally convex space $\mathfrak{g}$ which carries a group structure for which the multiplication and the inversion map are smooth ([Mi83], [Gl01a], [Ne02b]). The space of left invariant vector fields on G is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. Identifying elements of the tangent space $\mathfrak{g} := T_{\mathbf{1}}(G)$ of G in the identity $\mathbf{1}$ with left invariant vector fields, we obtain on $\mathfrak{g}$ the structure of a *locally convex Lie algebra* $\mathbf{L}(G)$. That the so obtained Lie bracket on $\mathfrak{g}$ is continuous follows most easily from the observation that if we consider the group multiplication in local coordinates, where the identity element $\mathbf{1} \in G$ corresponds to $0 \in \mathfrak{g}$, then the first two terms of its Taylor expansion are given by

$$x * y = x + y + b(x,y) + \cdots,$$

where the quadratic term $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear with

$$[x,y] = b(x,y) - b(y,x).$$

We call a locally convex Lie algebra $\mathfrak{g}$ *integrable* if there exists a Lie group G with $\mathbf{L}(G) = \mathfrak{g}$. A Lie group G is said to be Δ*-graded* if its Lie algebra $\mathbf{L}(G)$ is Δ-graded. The question when a root graded Lie algebra $\mathfrak{g}$ is integrable can be quite difficult.

According to Lie's Third Theorem, every finite-dimensional Lie algebra is integrable, but this is no longer true for infinite-dimensional locally convex Lie algebras. If $\mathfrak{g}$ is a Banach–Lie algebra, then the Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ always is integrable. Let $\mathrm{PG}(\mathfrak{g})$ denote a corresponding connected Lie group. Then there is a natural homomorphism of abelian groups, called the *period homomorphism*

$$\mathrm{per}_{\mathfrak{g}}: \pi_2(\mathrm{PG}(\mathfrak{g})) \to \mathfrak{z}(\mathfrak{g}),$$

and $\mathfrak{g}$ is integrable if and only if the image of $\mathrm{per}_{\mathfrak{g}}$ is discrete. For general locally convex Lie algebras the situation is more complicated, but if $q: \widehat{\mathfrak{g}} \to \mathfrak{g} = \mathbf{L}(G)$ is a central extension with a sequentially complex locally convex space $\mathfrak{z}$ as kernel and a continuous linear section, then there is a period homomorphism

$$\mathrm{per}: \pi_2(G) \to \mathfrak{z},$$

and the existence of a Lie group $\widehat{G}$ with $\mathbf{L}(\widehat{G}) = \widehat{\mathfrak{g}}$ depends on the discreteness of the image of per ([Ne02a], [Ne03a]). For finite-dimensional groups these obstructions are vacuous because $\pi_2(G)$ always vanishes by a theorem of É. Cartan ([Mim95, Th. 3.7]).

For the class of root graded Banach–Lie algebras the situation can be described very well by period maps. In this case the Lie algebra $\mathfrak{g}$ is integrable if and only if the image of $\mathrm{per}_{\mathfrak{g}}$ is discrete. As the universal covering $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ also is a universal covering of $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \widetilde{\mathfrak{g}}/\mathfrak{z}(\widetilde{\mathfrak{g}})$ (Remark III.15), we obtain a similar criterion for the integrability of $\widetilde{\mathfrak{g}}$ via a period map

$$\mathrm{per}_{\widetilde{\mathfrak{g}}}: \pi_2(\mathrm{PG}(\mathfrak{g})) \to \mathfrak{z}(\widetilde{\mathfrak{g}}) = HF(\mathcal{A}),$$

where $\mathcal{A}$ is the coordinate algebra of $\mathfrak{g}$ and $HF(\mathcal{A})$ is its full skew-dihedral homology. If $\mathfrak{g}_1$ is a quotient of $\widetilde{\mathfrak{g}}$ by a central subspace and $\widetilde{\mathfrak{g}}$ is integrable, then $\mathfrak{g}_1$ is integrable if and only if the period map

$$\mathrm{per}_{\mathfrak{g}_1}: \pi_2(\mathrm{PG}(\mathfrak{g})) \to \mathfrak{z}(\mathfrak{g}_1)$$

obtained by composing $\mathrm{per}_{\widetilde{\mathfrak{g}}}$ with the natural map $\mathfrak{z}(\widetilde{\mathfrak{g}}) \to \mathfrak{z}(\mathfrak{g}_1)$ has discrete image.

For general locally convex root graded Lie algebras which are not Banach–Lie algebras the situation is less clear, but there are many important classes of locally convex root graded

Lie algebras, to which many results from the Banach context can be extended, namely the Lie algebras related to matrix algebras over continuous inverse algebras. A *unital continuous inverse algebra* (CIA) is a unital locally convex algebra A for which the unit group $A^\times$ is open and the inversion is a continuous map $A^\times \to A, a \mapsto a^{-1}$. Typical associated root graded Lie algebras are the A_{n-1}-graded Lie algebra $\mathfrak{sl}_n(A)$, and for a commutative CIA the Lie algebras of the type $\mathfrak{g} = A \otimes \mathfrak{g}_\Delta$ (cf. [Gl01b]). Further examples are the Lie algebras $\mathfrak{sp}_{2n}(A,\sigma)$ and $\mathfrak{o}_{n,n}(A,\sigma)$ discussed in Section I. For Jordan algebras the situation is more complicated, but in this context there also is a natural concept of a *continuous inverse Jordan algebra*, which is studied in [BN03], and can be applied to show that certain related Lie algebras are integrable.

Both classes lead to interesting questions in non-commutative geometry because for a sequentially complete CIA the discreteness of the image of the period map for $\widetilde{\mathfrak{sl}}_n(A)$ follows from the discreteness of the image of a natural homomorphism

$$P_A^3 : K_3(A) \to HC_1(A) \cong H_2(\mathfrak{sl}_n(A)),$$

where $K_3(A) := \lim\limits_{\longrightarrow} \pi_2(\mathrm{GL}_n(A))$ is the third topological K-group of the algebra A. If, in addition, A is complex, Bott periodicity implies that

$$K_3(A) \cong K_1(A) := \lim_{\longrightarrow} \pi_0(\mathrm{GL}_n(A)),$$

and the latter group is much better accessible. In particular, we get a period map

$$P_A^1 : K_1(A) \to HC_1(A).$$

One can show that this homomorphism is uniquely determined as a natural transformation between the functors K_1 and HC_1, which permits us to evaluate it for many concrete CIAs ([Ne03a]). If P_A has discrete image, then $\widetilde{\mathfrak{sl}}_n(A)$ is integrable, but the converse is not clear and might even be false. Nevertheless, one can construct certain Fréchet CIAs which are quantum tori of dimension three, for which the Lie algebra $\widetilde{\mathfrak{sl}}_n(A)$ is not integrable. For the details of these constructions we refer to [Ne03a].

There is also a purely algebraic approach to groups corresponding to root graded Lie algebras. Here we associate to a root graded Lie algebra $\mathfrak{g}$ the corresponding *projective group*

$$\mathrm{PG}^{\mathrm{alg}}(\mathfrak{g}) := \langle e^{\mathrm{ad}\, \mathfrak{g}_\alpha} : \alpha \in \Delta \rangle \subseteq \mathrm{Aut}(\mathfrak{g}).$$

As each derivation $\mathrm{ad}\, x$, $x \in \mathfrak{g}_\alpha$, of $\mathfrak{g}$ is nilpotent, the operator $e^{\mathrm{ad}\, x}$ is a well-defined automorphism of $\mathfrak{g}$ (cf. [Ti96], [Ze94]). The group $\mathrm{PG}^{\mathrm{alg}}(\mathfrak{g})$ can easily seen to be perfect, so that it has a universal covering group (a universal central extension) $\widetilde{\mathrm{G}}^{\mathrm{alg}}(\mathfrak{g})$. Let $\mathrm{PG}(\mathfrak{g})$ be a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. There are many interesting problems associated with these groups:

(1) Describe $\widetilde{\mathrm{G}}^{\mathrm{alg}}(\mathfrak{g})$ by generators and relations.
(2) Show that $\mathrm{PG}(\mathfrak{g})$ is a topologically perfect group. When is it perfect?
(3) Suppose that $\widetilde{\mathrm{G}}(\mathfrak{g})$ is a Lie group with Lie algebra $\widetilde{\mathfrak{g}}$. Describe the kernel of the universal covering $\widetilde{\mathrm{G}}(\mathfrak{g}) \to \mathrm{PG}(\mathfrak{g})$ in terms of the coordinate algebra.
(4) Is there a homomorphism $\mathrm{PG}^{\mathrm{alg}}(\mathfrak{g}) \to \mathrm{PG}(\mathfrak{g})$?
(5) Is there a homomorphism $\widetilde{\mathrm{G}}^{\mathrm{alg}}(\mathfrak{g}) \to \widetilde{\mathrm{G}}(\mathfrak{g})$?

It is an interesting project to clarify the precise relation between the Lie theoretic (analytic) approach to root graded groups and the algebraic one.

Appendix A. Some generalities on representations

In this section we collect some material on finite-dimensional representations of reductive Lie algebras, which is used in Sections II and III of this paper. All results in this appendix are valid over any field $\mathbb{K}$ of characteristic zero.

Let $\mathfrak{r}$ be a finite-dimensional split reductive Lie algebra over the field $\mathbb{K}$ of characteristic zero and $\mathfrak{h} \subseteq \mathfrak{r}$ a splitting Cartan subalgebra. We fix a positive system Δ^+ of roots of $\mathfrak{r}$ with respect to $\mathfrak{h}$ and write $L(\lambda)$ for the simple $\mathfrak{r}$-module of highest weight $\lambda \in \mathfrak{h}^*$ with respect to Δ^+. We write $Z := Z(U(\mathfrak{r}))$ for the center of the enveloping algebra $U(\mathfrak{r})$ of $\mathfrak{r}$. Recall that for each highest weight module V we have $\mathrm{End}_{\mathfrak{r}}(V) = \mathbb{K}\mathbf{1}$ because the highest weight space is one-dimensional and cyclic. Therefore Z acts by scalar multiples of the identity on $L(\lambda)$, and we obtain for each λ an algebra homomorphism $\chi_\lambda \colon Z \to \mathbb{K}$, the corresponding central character.

The following theorem permits us to see immediately that certain modules are locally finite. We call an $\mathfrak{r}$-module an $\mathfrak{h}$*-weight module* if it is the direct sum of the common $\mathfrak{h}$-eigenspaces. An $\mathfrak{h}$-weight module V of a split reductive Lie algebra $\mathfrak{r}$ is called *integrable* if for each $x_\alpha \in \mathfrak{r}_\alpha$ the operator $\mathrm{ad}\, x_\alpha$ is locally nilpotent.

Theorem A.1. *For an $\mathfrak{h}$-weight module V of the finite-dimensional split reductive Lie algebra $\mathfrak{r}$ with splitting Cartan subalgebra $\mathfrak{h}$ the following assertions hold:*

(1) *If V is integrable, then V is locally finite and semisimple.*

(2) *If $\mathrm{supp}(V) := \{\alpha \in \mathfrak{h}^* \colon V_\alpha \neq \{0\}\}$ is finite, then V is integrable.*

Proof. (1) Let V be an integrable $\mathfrak{r}$-module and $\Delta := \{\alpha_1, \ldots, \alpha_m\}$. Then

$$\mathfrak{r} = \mathfrak{h} \oplus \mathfrak{r}_{\alpha_1} \oplus \ldots \oplus \mathfrak{r}_{\alpha_m},$$

so that the Poincaré–Birkhoff–Witt Theorem implies

$$U(\mathfrak{r}) = U(\mathfrak{h}) U(\mathfrak{r}_{\alpha_1}) \cdots U(\mathfrak{r}_{\alpha_m}).$$

Since V is integrable, it is by definition a locally finite module for each of the one-dimensional Lie algebras $\mathfrak{r}_\alpha$, $\alpha \in \Delta$. Hence for each vector $v \in V$ we see inductively that the space

$$U(\mathfrak{r}_{\alpha_j}) \cdots U(\mathfrak{r}_{\alpha_m}).v$$

is finite-dimensional for $j = m, m-1, \ldots, 1$, and finally that $U(\mathfrak{r}).v$ is finite-dimensional. Therefore V is a locally finite $\mathfrak{r}$-module.

Let $F \subseteq V$ be a finite-dimensional submodule. Since F is a weight module, it is a direct sum of the common eigenspaces for $\mathfrak{z}(\mathfrak{r}) \subseteq \mathfrak{h}$, which are $\mathfrak{r}$-submodules. According to Weyl's Theorem, these common eigenspaces are semisimple modules of the semisimple Lie algebra $\mathfrak{r}' := [\mathfrak{r}, \mathfrak{r}]$, hence also of $\mathfrak{r} = \mathfrak{r}' + \mathfrak{z}(\mathfrak{r})$. Therefore F is a sum of simple submodules, and the same conclusion holds for the locally finite module V. As a sum of simple submodules, the module V is semisimple ([La93, XVII, §2]).

(2) If $\mathrm{supp}(V)$ is finite, then $x_\alpha.V_\beta \subseteq V_{\beta+\alpha}$ for $\beta \in \mathrm{supp}(V)$ and $\alpha \in \Delta$ imply that the root vectors x_α act as locally nilpotent operators on V. ■

The preceding theorem is a special case of a much deeper theorem on Kac–Moody algebras. According to the Kac–Peterson Theorem, each integrable module in category $\mathcal{O}$ is semisimple ([MP95, Th. 6.5.1]). This implies in particular that integrable modules of finite-dimensional split reductive Lie algebras are semisimple.

Proposition A.2. *Let V be an $\mathfrak{h}$-weight module of $\mathfrak{r}$ for which $\mathrm{supp}(V)$ is finite. Then the following assertions hold:*

(1) *V is a semisimple $\mathfrak{r}$-module with finitely many isotypic components $V_1, \ldots, V_n$.*

(2) *The simple submodules of V are finite-dimensional highest weight modules $L(\lambda_1), \ldots, L(\lambda_n)$.*

(3) *For each $j \in \{1, \ldots, n\}$ there exists a central element z_j in $U(\mathfrak{g}_\Delta)$ with $\chi_{\lambda_k}(z_j) = \delta_{jk}$. In particular, z_j acts on V as the projection onto the isotypic component V_j.*

Proof. (1), (2) First Theorem A.1 implies that V is semisimple. Moreover, each simple submodule is a finite-dimensional weight module, hence isomorphic to some $L(\lambda)$. As $\mathrm{supp}(V)$ is finite, there are only finitely many possibilities for the highest weights λ.
(3) According to Harish-Chandra's Theorem ([Dix74, Prop. 7.4.7]), for $\lambda, \mu \in \mathfrak{h}^*$ we have

$$\chi_\lambda = \chi_\mu \quad \Leftrightarrow \quad \mu + \rho \in \mathcal{W}.(\lambda + \rho),$$

where $\mathcal{W}$ is the Weyl group of $(\mathfrak{r}, \mathfrak{h})$ and $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. If $L(\lambda)$ and $L(\mu)$ are finite-dimensional, then λ and μ are dominant integral. Therefore $\lambda + \rho$ and $\mu + \rho$ are dominant, so that $\mu + \rho \in \mathcal{W}.(\lambda + \rho)$ implies $\lambda = \mu$. Hence two non-isomorphic finite-dimensional highest weight modules $L(\lambda)$ and $L(\mu)$ have different central characters.

This proves that the central characters $\chi_{\lambda_1}, \ldots, \chi_{\lambda_n}$ corresponding to the isotypic components of V are pairwise different. As the kernel of a character is a hyperplane ideal, this means that for $i \neq j$ we have

$$\ker \chi_{\lambda_i} + \ker \chi_{\lambda_j} = Z.$$

Now the Chinese Remainder Theorem ([La93, Th. II.2.1]) implies that the map

$$\chi: Z \to \mathbb{K}^n, \quad z \mapsto (\chi_{\lambda_1}(z), \ldots, \chi_{\lambda_n}(z))$$

is surjective. Finally (3) follows with $z_i := \chi^{-1}(e_i)$, where $e_1, \ldots, e_n \in \mathbb{K}^n$ are the standard basis vectors. ∎

For the following lemma, we recall the definition of Lie algebra cohomology from [We95].

Lemma A.3. *If $\mathfrak{s}$ is a finite-dimensional semisimple Lie algebra and V a locally finite $\mathfrak{s}$-module, then*

$$H^p(\mathfrak{s}, V) = \{0\} \quad \textit{for} \quad p = 1, 2.$$

Proof. As V is a direct sum of finite-dimensional modules V_j, $j \in J$, the relations

$$C^p(\mathfrak{s}, V) \cong \bigoplus_{j \in J} C^p(\mathfrak{s}, V_j) \quad \text{easily lead to} \quad H^p(\mathfrak{s}, V) \cong \bigoplus_{j \in J} H^p(\mathfrak{s}, V_j),$$

so that the assertion follows from the Whitehead Lemmas ([We95, Cor. 7.8.10/12]), saying that $H^p(\mathfrak{s}, V_j)$ vanishes for each j and $p = 1, 2$. ∎

Proposition A.4. *Let $\mathfrak{s}$ be a semisimple finite-dimensional Lie algebra $\mathfrak{s}$.*

(1) *Each extension $Z \hookrightarrow \widehat{M} \xrightarrow{q} M$ of a locally finite $\mathfrak{s}$-module M by a trivial module Z is trivial.*

(2) *Each extension $M \hookrightarrow \widehat{M} \xrightarrow{q} Z$ of a trivial $\mathfrak{s}$-module Z by a locally finite $\mathfrak{s}$-module M is trivial.*

Proof. (1) If $\widehat{M}$ is locally finite, then Weyl's Theorem implies that it is semisimple, and therefore that the extension of M by Z splits. Hence it suffices to show that $\widehat{M}$ is locally finite. Let $v \in \widehat{M}$. We have to show that v generates a finite-dimensional submodule. Since the $\mathfrak{s}$-submodule of M generated by $q(v)$ is finite-dimensional, we may replace M by this module and hence assume that M is finite-dimensional. Now

$$\mathrm{Ext}(M, Z) \cong H^1(\mathfrak{s}, \mathrm{Hom}(M, Z))$$

([We95, Ex. 7.4.5]), and $\mathrm{Hom}(M, Z) \cong M^* \otimes Z$ is a locally finite module, so that

$$H^1(\mathfrak{s}, \mathrm{Hom}(M, Z)) = \{0\}$$

(Lemma A.3). Therefore the module extension splits, and in particular $\widehat{M}$ is locally finite.

(2) First we show that $\widehat{M}$ is locally finite. Let $v \in \widehat{M}$. To see that v generates a finite-dimensional submodule, we may assume that Z is one-dimensional. Then $\mathrm{Hom}(Z, M) \cong M$ is a locally finite $\mathfrak{s}$-module, and the same argument as in (1) above implies that the extension $\widehat{M} \to Z$ is trivial. In particular, we conclude that $\widehat{M}$ is locally finite.

Returning to the general situation, we obtain from Weyl's Theorem that the locally finite module $\widehat{M}$ is semisimple, hence in particular that $\widehat{M} = \mathfrak{g}.\widehat{M} \oplus \widehat{M}^{\mathfrak{g}}$. As Z is trivial, we have $\mathfrak{g}.\widehat{M} \subseteq M$, so that each subspace of $\widehat{M}^{\mathfrak{g}}$ complementing $M \cap \widehat{M}^{\mathfrak{g}}$ yields a module complement to M. ∎

Appendix B. Jordan algebras and alternative algebras

In this appendix we collect some elementary results on Jordan algebras.

Jordan algebras

Definition B.1. A finite dimensional vector space J over a field $\mathbb{K}$ is said to be a *Jordan algebra* if it is endowed with a bilinear map $J \times J \to J$ satisfying:
(JA1) $xy = yx$.
(JA2) $x(x^2y) = x^2(xy)$. ∎

In this section J denotes a Jordan algebra and $(a,b) \mapsto L(a)b := ab = ba$ the multiplication of J. Then (JA2) means that

$$[L(a), L(a^2)] = 0 \quad \text{for all} \quad a \in J.$$

Proposition B.2. *For a Jordan algebra J over a field $\mathbb{K}$ with $\{2,3\} \subseteq \mathbb{K}^\times$ the following assertions hold for $x, y, z \in J$.*
(1) $[L(x), L(yz)] + [L(y), L(zx)] + [L(z), L(xy)] = 0$.
(2) $L(x(yz) - y(xz)) = [[L(x), L(y)], L(z)]$.

Proof. Passing to the first derivative of (JA2) with respect to x in the direction of z leads to

$$z(x^2y) + 2x((xz)y) = 2(xz)(xy) + x^2(zy)$$

for $x, y, z \in J$. Passing again to the derivative with respect to x in the direction of u leads to

$$z((xu)y) + u((xz)y) + x((uz)y) = (uz)(xy) + (xz)(uy) + (xu)(zy)$$

for $u, x, y, z \in J$. This means that

$$[L(z), L(xu)] + [L(u), L(xz)] + [L(x), L(uz)] = 0,$$

or

$$L(xy)L(z) + L(zx)L(y) + L(yz)L(x) = L(z)L(y)L(x) + L\big((zx)y\big) + L(x)L(y)L(z).$$

Note that the expression

$$L(xy)L(z) + L(zx)L(y) + L(yz)L(x)$$

is invariant under any permutation of x, y, z. By exchanging x and y and subtracting, we therefore obtain

$$[[L(x), L(y)], L(z)] = L\big((zy)x\big) - L\big((zx)y\big) = L\big(x(yz) - y(xz)\big).$$

∎

Corollary B.3. $[L(J), L(J)] \subseteq \mathrm{der}(J)$.

Proof. This means that for $x, y \in J$ the operator $D := [L(x), L(y)]$ is a derivation of J, which in turn means that

$$[D, L(z)] = L(D.z), \quad z \in J.$$

This is a reformulation of Proposition B.2(2). ∎

Jordan algebras associated to bilinear forms

Lemma B.4. *Let A be a commutative associative algebra, B an A-module and $\beta: B \times B \to A$ a symmetric bilinear form which is invariant in the sense that*

$$a\beta(b,b') = \beta(ab,b') = \beta(b,ab'), \quad a \in A, b, b' \in B.$$

Then $\mathcal{A} := A \oplus B$ is a Jordan algebra with respect to

$$(a,b)(a',b') := (aa' + \beta(b,b'), ab' + a'b).$$

Proof. First we note that

$$L(a,0)(a',b') = (aa',ab') \quad \text{and} \quad L(0,b)(a',b') = (\beta(b,b'),a'b).$$

The set $L(A,0) \subseteq \mathrm{End}(\mathcal{A})$ is commutative because A is a commutative algebra. Further

$$L(0,b)L(a,0)(a',b') = (\beta(b,ab'),aa'b) = L(a,0)L(0,b)(a',b')$$

implies that $L(A,0)$ commutes with $L(0,B)$, so that $L(A,0)$ is central in the subspace $L(\mathcal{A})$ of $\mathrm{End}(\mathcal{A})$.

It is clear that $\mathcal{A}$ is commutative. To see that it is a Jordan algebra, we have to verify that each $L(a,b)$ commutes with

$$L((a,b)^2) = L(a^2 + \beta(b,b), 2ab).$$

As $L(A,0)$ is central in $L(\mathcal{A})$, it suffices to show that $L(0,b)$ commutes with $L(0,ab)$, which follows from

$$\begin{aligned} L(0,b)L(0,ab)(x,y) &= L(0,b)(\beta(ab,y),xab) = (\beta(b,xab),\beta(ab,y)b) \\ &= (\beta(xb,ab),\beta(b,y)ab) = L(0,ab)(\beta(b,y),xb) = L(0,ab)L(0,b)(x,y). \end{aligned}$$

■

Alternative algebras

Lemma B.5. *Let A be a (non-associative) algebra. For $a,b,c \in A$ we define the associator*

$$(a,b,c) := (ab)c - a(bc).$$

Then the associator is an alternating function if and only if for $a,b \in A$ we have

$$a^2b = a(ab) \quad \text{and} \quad ab^2 = (ab)b. \tag{B.1}$$

Proof. First we assume that the associator is alternating. Then

$$a^2b - a(ab) = (a,a,b) = 0 \quad \text{and} \quad ab^2 - (ab)b = (a,b,b) = 0.$$

Suppose, conversely, that (B.1) is satisfied. The derivative of the function

$$f_c(a) := a^2c - a(ac)$$

in the direction of b is given by

$$df_c(a)(b) = (ab+ba)c - b(ac) - a(bc),$$

which leads to the identity

$$(a,b,c) = (ab)c - a(bc) = b(ac) - (ba)c = -(b,a,c).$$

We likewise obtain from $a(c^2) = (ac)c$ the identity

$$(a,b,c) = (ab)c - a(bc) = a(cb) - (ac)b = -(a,c,b).$$

As the group S_3 is generated by the transpositions (12) and (23), the associator is an alternating function. ■

We call an algebra A *alternative* if the conditions from Lemma B.5 are satisfied. For $L_a(b) := ab =: R_b(a)$ this means that

$$L_a^2 = L_{a^2} \quad \text{and} \quad R_{b^2} = R_b^2.$$

Theorem B.6. (Artin) *An algebra is alternative if every subalgebra generated by two elements is associative.*

Proof. In view of (B.1), the algebra A is alternative if any pair (a,b) of elements generates an associative subalgebra. For the converse we refer to [Sch66, Th. 3.1]. ■

Lemma B.7. *Each alternative algebra is a Jordan algebra with respect to* $a \circ b := \frac{1}{2}(ab + ba)$.

Proof. Let $L_a^J(b) := a \circ b$, $L_a(b) = ab$ and $R_a(b) := ba$. Since A is alternative, we have

$$0 = (a, b, a) = (ab)a - a(ba)$$

which means that $[L_a, R_a] = 0$. Therefore the associative subalgebra of $\mathrm{End}(A)$ generated by L_a and R_a is commutative. Since $L_a^J = \frac{1}{2}(L_a + R_a)$ commutes with

$$L_{a^2}^J = \tfrac{1}{2}\big(L_{a^2} + R_{a^2}\big) = \tfrac{1}{2}\big(L_a^2 + R_a^2\big),$$

$(A, \circ)$ is a Jordan algebra. ■

Appendix C. Jordan triple systems

The natural bridge between Lie algebras and Jordan algebras is formed by Jordan triple systems. In this appendix we briefly recall how this bridge works. We are using this correspondence in particular in Section III to see that for each A_1-graded Lie algebra the coordinate algebra is a Jordan algebra.

Definition C.1. (a) A finite dimensional vector space V over a field $\mathbb{K}$ is said to be *a Jordan triple system* (JTS) if it is endowed with a trilinear map $\{\cdot\}\colon V \times V \times V \to V$ satisfying:
(JT1) $\{x, y, z\} = \{z, y, x\}$.
(JT2) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$ for all $a, b, x, y, z \in V$.
For $x, y \in V$ we define the operator $x \square y$ by $(x \square y).z := \{x, y, z\}$ and put $P(x)(y) := \{x, y, x\}$. Then (JT2) is equivalent to

$$\text{(JT2}'\text{)} \qquad [a \square b, x \square y] = \big((a \square b).x\big) \square y - x \square \big((b \square a).y\big).$$

It follows in particular that the subspace $V \square V \subseteq \mathrm{End}_{\mathbb{K}}(V)$ spanned by the elements $x \square y$ is a Lie algebra. This Lie algebra is denoted $\mathfrak{istr}(V)$ and called the *inner structure algebra* of V.

If $2 \in \mathbb{K}^\times$, then (JT1) implies that the trilinear map $\{\cdot, \cdot, \cdot\}$ can be reconstructed from the quadratic maps $P(x)$ via polarization of $P(x).y = \{x, y, x\}$, i.e., by taking derivatives w.r.t. x in the direction of z. Therefore the Jordan triple structure is completely determined by the maps $P(x)$, $x \in V$. ■

Lemma C.2. *If* $3 \in \mathbb{K}^\times$ *and* $(V, \{\cdot, \cdot, \cdot\})$ *is a Jordan triple system, then the following formulas hold for* $x, y, z \in V$:
(1) $P(x).\{y, x, z\} = \{P(x).y, z, x\} = \{x, y, P(x).z\}$.
(2) $P(x)(y \square x) = (x \square y)P(x)$.
(3) $[P(x)P(y), x \square y] = 0$.

Proof. (1) From the Jordan triple identity

$$x \square y.\{a, b, c\} = \{x \square y.a, b, c\} - \{a, y \square x.b, c\} + \{a, b, x \square y.c\}$$

we derive

$$\begin{aligned}\{x,y,\{x,z,x\}\} &= \{\{x,y,x\},z,x\} - \{x,\{y,x,z\},x\} + \{x,z,\{x,y,x\}\}\\ &= 2\{\{x,y,x\},z,x\} - \{x,\{y,x,z\},x\}\\ &= 2\{x,y,\{x,z,x\}\} - 2\{x,\{y,x,z\},x\} + 2\{\{x,z,x\},y,x\} - \{x,\{y,x,z\},x\}\\ &= 4\{x,y,\{x,z,x\}\} - 3\{x,\{y,x,z\},x\}.\end{aligned}$$

This implies

$$3\{x,y,\{x,z,x\}\} = 3\{x,\{y,x,z\},x\},$$

so that $3 \in \mathbb{K}^\times$ leads to

$$\{x,y,\{x,z,x\}\} = \{x,\{y,x,z\},x\}.$$

This is the second equality we had to prove. The first one follows from the second one, which implies that $\{x,y,\{x,z,x\}\}$ is symmetric in y and z.
(2) follows directly from (1).
(3) is an immediate consequence of (2). ∎

Theorem C.3. (a) *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ is a 3-graded Lie algebra with an involutive automorphism τ satisfying $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j = 0, \pm 1$, then $V := \mathfrak{g}_1$ is a Jordan triple system with respect to $\{x,y,z\} := \big[[x,\tau.y],z\big]$.*
(b) *If, conversely, V is a Jordan triple system for which there exists an involution σ on $\mathfrak{istr}(V)$ with $\sigma(a\Box b) = -b\Box a$ for $a,b \in V$, then $\mathfrak{g} := V \times \mathfrak{istr}(V) \times V$ is a Lie algebra with respect to the bracket*

$$[(a,x,d),(a',x',d')] = (x.a' - x'.a, a\Box d' - a'\Box d + [x,x'], \sigma(x).d' - \sigma(x').d)$$

and $\tau(a,b,c) := (c,\sigma(b),a)$ is an involutive automorphism of $\mathfrak{g}$.

Proof. (a) Since $\mathfrak{g}$ is graded, we have $[\mathfrak{g}_1,\mathfrak{g}_1] = \{0\}$, and this implies that $[\mathrm{ad}\,x, \mathrm{ad}\,y] = 0$ for $x, y \in \mathfrak{g}_1$, hence (JT1). To verify (JT2), we first observe that $a\Box b = \mathrm{ad}[x,\tau.y]$. We have

$$\begin{aligned}\big[[a,\tau.b],[c,\tau.d]\big] &= \big[[[a,\tau.b],c],\tau.d\big] + \big[c,[[a,\tau.b],\tau.d]\big]\\ &= \big[[[a,\tau.b],c],\tau.d\big] + \big[c,\tau.[[\tau.a,b],d]\big]\\ &= \big[[[a,\tau.b],c],\tau.d\big] - \big[c,\tau.[[b,\tau.a],d]\big].\end{aligned}$$

Therefore (JT2) follows from

$$\begin{aligned}[a\Box b, c\Box d] &= \mathrm{ad}\,\big[[a,\tau.b],[c,\tau.d]\big] = \mathrm{ad}\,\big[[[a,\tau.b],c],\tau.d\big] - \mathrm{ad}\,\big[c,\tau.[[b,\tau.a],d]\big]\\ &= (a\Box b).c\Box d - c\Box(b\Box a).d.\end{aligned}$$

(b) One observes directly that τ is an involution preserving the bracket. It is clear that the bracket is skew symmetric, so that

$$J(x,y,z) := \big[[x,y],z\big] + \big[[y,z],x\big] + \big[[z,x],y\big]$$

is an alternating trilinear function on $\mathfrak{g}$. We have to show that J vanishes.

Let $\mathfrak{g}_1 := V \times \{(0,0)\}$, $\mathfrak{g}_0 = \{0\} \times \mathfrak{istr}(V) \times \{0\}$, and $\mathfrak{g}_{-1} := \{(0,0)\} \times V$. It is easy to check that $J(x,y,z) = 0$ if no entry is contained in $\mathfrak{g}_1$ or no entry is contained in $\mathfrak{g}_{-1}$. We identify $x \in V$ with $(x,0,0)$ and write $\widetilde{x} = (0,0,x)$ for the corresponding element of $\mathfrak{g}_{-1}$. Then we may assume that the first entry is $x \in \mathfrak{g}_1$ and the second one is $\widetilde{y} \in \mathfrak{g}_{-1}$. For $z \in V \cong \mathfrak{g}_1$ we then obtain

$$J(x,\widetilde{y},z) = \big[[\widetilde{y},z],x\big] + \big[[x,\widetilde{y}],z\big] = (x\Box y).z - (z\Box y).x = \{x,y,z\} - \{z,y,x\} = 0.$$

If $z \in \mathfrak{g}_{-1}$, the assertion follows from $\tau.J(x,\widetilde{y},z) = J(\tau.x,\tau.\widetilde{y},\tau.z) = 0$. Finally, let $z \in \mathfrak{g}_0$. We may assume that $z = a\Box b$. Then (JT2) implies that $[z, x\Box y] = [z,x]\Box y + x\Box\sigma(z).y$. This leads to

$$\begin{aligned}J(x,\widetilde{y},z) &= \big[[\widetilde{y},z],x\big] + \big[[z,x],\widetilde{y}\big] + \big[[x,\widetilde{y}],z\big] = -[(\sigma(z).y)\widetilde{\ },x] + [z.x,\widetilde{y}] + [x\Box y, z]\\ &= x\Box(\sigma(z).y) + (z.x)\Box y - [z, x\Box z] = 0.\end{aligned}$$

∎

We conclude this section with the connection between Jordan algebras and Jordan triple systems.

Theorem C.4. (a) *If J is a Jordan algebra, then J is a Jordan triple system with respect to*

$$(C.1) \qquad \{x,y,z\} = (xy)z + x(yz) - y(xz), \qquad i.e., \quad x\Box y = L(xy) + [L(x), L(y)],$$

where we write $L(x)y := xy$ for the left multiplications in J.
(b) *If V is a JTS and $a \in V$, then*

$$x \cdot_a y := \{x, a, y\}$$

defines on V the structure of a Jordan algebra. The Jordan triple structure determined by the Jordan product $\cdot_a$ is given by

$$\{x,y,z\}_a = \{x, \{a,y,a\}, z\} = \{x, P(a).y, z\}.$$

It coincides with the original one if $P(a) = \mathbf{1}$.
(c) *Let J be a Jordan algebra which we endow with the Jordan triple structure from* (a). *If $e \in J$ is an identity element, then $x \cdot_e y = xy$ reconstructs the Jordan algebra structure from the Jordan triple structure.*

Proof. (a) From (JA1) it immediately follows that (C.1) satisfies (JT1). The proof of (JT2) requires Lemma B.2.

In view of Corollary B.3, $D := [L(x), L(y)]$ is a derivation of J, so that

$$D.\{a,b,c\} = \{D.a, b, c\} + \{a, D.b, c\} + \{a, b, D.c\}.$$

Therefore (C.1) shows that to prove (JT2), it suffices to show that for each $x \in J$ we have

$$L(x).\{a,b,c\} = \{L(x).a, b, c\} - \{a, L(x).b, c\} + \{a, b, L(x).c\},$$

i.e.,

$$L(x).(a\Box b) = (xa)\Box b - a\Box(xb) + (a\Box b)L(x),$$

which in turn means that

$$\begin{aligned} &L(x)L(ab) + L(x)[L(a), L(b)] \\ &= L((xa)b) + [L(xa), L(b)] - L(a(bx)) - [L(a), L(xb)] + L(ab)L(x) + [L(a), L(b)]L(x), \end{aligned}$$

i.e.,

$$[L(x), L(ab)] + [L(a), L(xb)] + [L(b), L(ax)] = [[L(a), L(b)], L(x)] + L((xa)b) - L(a(bx)).$$

This identity follows from Lemma B.2, because both sides of this equation vanish separately.
(b) Put $xy := x \cdot_a y$, so that $L(x) = x\Box a$. The identity (JA1) follows directly from (JT1). To verify (JA2), we observe that

$$\begin{aligned} L(x^2).y &= \{\{x,a,x\}, a, y\} = \{y, a, \{x,a,x\}\} \\ &= \{\{y,a,x\}, a, x\} - \{x, \{a,y,a\}, x\} + \{x, a, \{y,a,x\}\} \\ &= 2(x\Box a)^2.y - P(x)P(a).y. \end{aligned}$$

Therefore Lemma C.2(3) implies

$$[L(x^2), L(x)] = [2(x\Box a)^2 - P(x)P(a), x\Box a] = [x\Box a, P(x)P(a)] = 0.$$

The quadratic operator $P^a(x)$ associated to the Jordan triple structure defined by $\cdot_a$ in the sense of (a) is given by

$$P^a(x) = 2L(x)^2 - L(x^2) = 2(x\Box a)^2 - \big(2(x\Box a)^2 - P(x)P(a)\big) = P(x)P(a).$$

Therefore the Jordan triple structure associated to $\cdot_a$ is given by $\{x,y,z\}_a = \{x, P(a).y, z\}$.
(c) is trivial. ∎

Appendix D. Skew dihedral cohomology

In this section we briefly recall the definition of skew dihedral cohomology of associative algebras, which is the background for the definition of the full skew-dihedral homology spaces defined in Section IV.

Definition D.1. Let $\mathcal{A}$ be a unital associative algebra and $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$ the $(n+1)$-fold tensor product of $\mathcal{A}$ with itself. We define a *boundary operator*

$$b_n: C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A}) \quad \text{for} \quad n \in \mathbb{N}$$

and $b_0: C_0(\mathcal{A}) \to \{0\}$ by

$$b_n(a_0 \otimes \ldots \otimes a_n) := \sum_{i=0}^{n-1}(-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Then $b_n b_{n+1} = 0$ for each $n \in \mathbb{N}_0$, and the corresponding homology spaces $HH_*(\mathcal{A})$ are called the *Hochschild homology of* $\mathcal{A}$. ■

Of particular interest for Lie algebras is the first Hochschild homology group $HH_1(\mathcal{A})$. The map $b_1: C_1(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A} \to C_0(\mathcal{A}) \cong \mathcal{A}$ is given by

$$b_1(x \otimes y) = xy - yx = [x, y],$$

so that $Z_1(\mathcal{A}) = \ker b \subseteq C_1(\mathcal{A})$ is the kernel of the bracket map. The space $B_1(\mathcal{A})$ of boundaries is spanned by elements of the type

$$b_2(x \otimes y \otimes z) = xy \otimes z - x \otimes yz + zx \otimes y.$$

Note in particular that $b_2(x \otimes \mathbf{1} \otimes \mathbf{1}) = x \otimes \mathbf{1}$, so that $\mathcal{A} \otimes \mathbf{1} \subseteq B_1(\mathcal{A})$.

Definition D.2. Let $(\mathcal{A}, \sigma)$ be an associative algebra with involution $\sigma: \mathcal{A} \to \mathcal{A}, a \mapsto a^\sigma$. Then we obtain a natural action of the dihedral group D_{n+1} on the space $C_n(\mathcal{A})$ as follows. We present D_{n+1} as the group generated by x_n and y_n subject to the relations

$$x_n^{n+1} = y_n^2 = \mathbf{1} \quad \text{and} \quad y_n x_n y_n^{-1} = x_n^{-1},$$

and define the action of x_n and y_n on $C_n(\mathcal{A})$ by

$$x_n(a_0 \otimes \ldots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$$

and

$$y_n(a_0 \otimes \ldots \otimes a_n) := -(-1)^{\frac{n(n+1)}{2}} a_0^\sigma \otimes a_n^\sigma \otimes a_{n-1}^\sigma \ldots \otimes a_2^\sigma \otimes a_1^\sigma.$$

These operators are compatible with the boundary operators in the sense that the operators b_n induce on the spaces $C_n'(\mathcal{A})$ of coinvariants for the D_{n+1}-action boundary operators

$$b_n': C_n'(\mathcal{A}) \to C_{n-1}'(\mathcal{A}).$$

The corresponding homology is called the *skew-dihedral homology* $HD_n'(\mathcal{A}, \sigma)$ *of the algebra with involution* $(\mathcal{A}, \sigma)$ (cf. [Lo98, 10.5.4; Th. 5.2.8]). ■

In the present paper we only need the space $HD_1'(\mathcal{A}, \sigma)$. We observe that

$$x_1.(a_0 \otimes a_1) = -a_1 \otimes a_0 \quad \text{and} \quad y_1.(a_0 \otimes a_1) = a_0^\sigma \otimes a_1^\sigma.$$

Writing the image of $a_0 \otimes a_1$ in $C_1'(\mathcal{A})$ as $\langle a, b\rangle$, this means that

$$\langle a_0, a_1\rangle = -\langle a_1, a_0\rangle = \langle a_0^\sigma, a_1^\sigma\rangle, \quad a_0, a_1 \in \mathcal{A}.$$

It follows in particular that $\langle \mathcal{A}^\sigma, \mathcal{A}^{-\sigma}\rangle = \{0\}$, and further that

$$C_1'(\mathcal{A}) \cong \Lambda^2(\mathcal{A}^\sigma) \oplus \Lambda^2(\mathcal{A}^{-\sigma}).$$

Moreover,

$$b_2'(\langle a_0, a_1, a_2\rangle) = \langle a_0 a_1, a_2\rangle - \langle a_0, a_1 a_2\rangle + \langle a_2 a_0, a_1\rangle = \langle a_0 a_1, a_2\rangle + \langle a_1 a_2, a_0\rangle + \langle a_2 a_0, a_1\rangle,$$

and these elements span the space $B_1'(\mathcal{A}) \subseteq C_1'(\mathcal{A})$ of skew-dihedral 1-boundaries.

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