Cutting Planes for the Optimisation of Gas Networks

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Abstract

This paper presents cutting planes which are useful or potentially useful for solving mixed integer programs that arise in the optimisation of gas networks. We consider polyhedra that are defining essential parts of the model and give an polynomial algorithm for the calculation of the set of vertices of such polyhedra implying a polynomial separation algorithm for the convex hull of the polyhedra can be developed.

Keywords: Mixed Integer Programming, Cutting Planes, Gas Optimisation, Piecewise Linear Functions

1 Introduction

The following studies of a special polyhedron have arisen from our researches on the problem of the Transient Technical Optimisation (TTO) in gas networks. A gas network basically consists of a set of compressors and valves that are connected by pipes. Since the gas pressure in the pipes decreases due to the friction in the pipes the compressors are used in order to increase the gas pressure again since the consumers want to get gas of a certain pressure value and quality. The task of the Transient Technical Optimisation is to optimise the drives of the gas and to set in the compressors cost-efficiently such that the required demands are satisfied. Modelling this problem will lead to a complex mixed integer nonlinear optimisation problem. We have approached it by approximating the nonlinearities (the most important nonlinear functions in this model describe the fuel gas consumption of the compressors and the pressure loss in the pipes) by piece-wise

linear functions leading to a huge mixed integer program. We want to solve the mixed integer program via a branch-and-cut algorithm. Therefore we have studied the polyhedral consequences of this model.
In this paper we present some new cutting planes for polyhedra that describe important substructures of the gas network model. We also point out how this knowledge can be generalised to more complex structures. Finally our preliminary computational results show the benefits when incorporating these cuts into a general mixed integer programming solver.

2 The Polyhedron

The above described problem of the Transient Technical Optimisation is evidently modelled in a graph $G = (V, E)$. The set E consists of the set of compressors, the set of valves and control valves and the set of pipes. The set V of nodes consists of the set of intersection points of the segments, the set of sources (the gas delivering points) and the set of sinks (which are the gas demanding points) of the gas network. We point out the most important kind of variables which are necessary to understand the succeeding ventilations. At first we introduce flow variables $q_e, e \in E$. These variables describe the mass flow of gas in each segment. Second we consider pressure variables $p_v, v \in V$. The pressure variables describe the pressure of the gas in each node. A very important principle is that the pressure at the end of all ingoing segments of a node must be equal the pressure at the beginning of all outgoing segments of the same node. This principle will be very important for our further discussions. There are a lot of other kinds of variables in the whole model, but for the here presented analysis of a polyhedron that describes only a special part of the whole model these two kinds of variables suffice.

Now let us shortly describe how the polyhedron under investigation comes upon in the global model. The physics of the gas is basically described by three partial differential equations. The momentum equation, the continuity equation and the energy equation. We focus on the momentum equation. Under some mild assumptions, which we do not want to discuss here, the momentum equation can be simplified to a nonlinear function of the following shape:

$$p_{out}^2 = p_{in}^2 - \mathrm{ff}\, q\, |q|, \qquad (*)$$

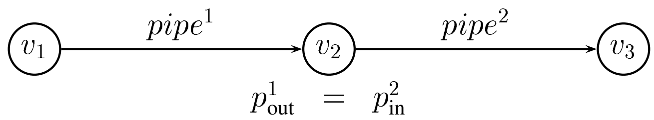

Figure 1: Sequence of pipes

where

$$\text{ff} = \text{ff}(p_{out},\, p_{in})$$

is the friction factor. After simplifying the friction factor to a constant we get $p_{out} = f(p_{in}, q)$, where p_{out} means the pressure at the end of the pipe, p_{in} means the pressure at the beginning of the pipe and q means the gas flow through the pipe. The other important nonlinearity is the fuel consumption of the compressors (which has to be minimised as mentioned in the introduction): The fuel consumption is described by a nonlinear function f of the form: $f = f(p_{in}, p_{out}, q)$. Here f describes the fuel consumption of the compressor, p_{in} the pressure of the gas at the beginning of the compressor, p_{out} the gas pressure which the compressor has to constitute at the endpoint of the compressor and q stands for the gas flow through the compressor. In order to come up with a mixed integer linear program these two nonlinear functions are approximated by suitable triangulations as pointed out in the following demonstrations.

The first substructure of the model we have studied are sequences of pipes. The situation is showed in Figure 1. We have already mentioned one important aspect of the model that the pressure p^1_{out} at the end of the ingoing pipe ($pipe^1$) must be equal the pressure p^2_{in} at the beginning of the outgoing pipe ($pipe^2$). We already know that p^1_{out} is a nonlinear function depending on the flow through the pipe and the pressure at the beginning of the pipe. We approximate the pressure loss in pipes by determing a **triangulation** of the 2-dimensional manifold describing the pressure loss in the pipes. We denote by Λ^{pipe} the set of grid points and by Y^{pipe} the set of triangles. We approximate the $2-$dimensional function $f(p_{in}, q)$ by linearising it within each triangle. Modelling this piecewise linear approximation results in the following non convex polyhedron:

$$
\begin{array}{rlcrcl}
P_\Delta = & \left\{ \binom{\lambda^1}{\lambda^2} \in \mathbb{R}^{|\Lambda^1|+|\Lambda^2|} \mid \right. & \sum_{j\in\Lambda^1} \lambda_j^1 & & = & 1 \\
& & & \sum_{j\in\Lambda^2} \lambda_j^2 & = & 1 \\
& & \sum_{j\in\Lambda^1} p_{\text{in},j}^1 \, \lambda_j^1 & - \sum_{j\in\Lambda^2} p_{\text{out},j}^2 \, \lambda_j^2 & = & 0 \\
& & \lambda_j^1, & \lambda_j^2 & \geq & 0 \\
& & \multicolumn{4}{l}{\lambda^1, \, \lambda^2 \text{ satisfy the triangle condition } \left.\right\},}
\end{array}
$$

where the triangle condition states that the set of $\lambda-$variables which are strictly positive must belong to grid points of a distinct triangle.
Figure 2 describes the situation of the polyhedron P_Δ: The numbers in the left triangulation (for the ingoing pipe) stand for the pressure values $p^1_{out,j}$ at the grid points $j \in \Lambda^{pipe}$ and the numbers in the right triangulation (for the outgoing pipe) stand for the pressure values $p^2_{in,i}$ at the grid points $i \in \Lambda^{pipe}$.

Let us consider a simple example (see Figure 3) for a little calculation. Here is $p^1_{out,1} = 10$, $p^1_{out,2} = 8$, $p^1_{out,3} = 4$ and so on, analogously we have for $p^2_{in,1} = p^2_{in,2} = p^2_{in,3} = 10, \ldots$, etc. Consider

$$\lambda_1^1 = \frac{1}{4}, \lambda_1^1 = 0, \lambda_1^1 = 0, \lambda_1^1 = \frac{1}{2}, \lambda_1^1 = \frac{1}{4}, \lambda_1^1 = 0$$

and

$$\lambda_1^2 = \frac{35}{100}, \lambda_1^2 = 0, \lambda_1^2 = 0, \lambda_1^2 = 0, \lambda_1^2 = \frac{65}{100}, \lambda_1^2 = 0.$$

This setting for the $\lambda-$variables fulfils all conditions, especially the triangle condition.
But if we take

$$\lambda_1^1 = \frac{1}{4}, \lambda_1^1 = 0, \lambda_1^1 = 0, \lambda_1^1 = \frac{1}{2}, \lambda_1^1 = \frac{1}{4}, \lambda_1^1 = 0$$

and

$$\lambda_1^2 = \frac{35}{100}, \lambda_1^2 = 0, \lambda_1^2 = 0, \lambda_1^2 = 0, \lambda_1^2 = 0, \lambda_1^2 = \frac{65}{100},$$

we see that the triangle condition is not satisfied since the nonzero variables λ_1^2 and λ_6^2 belong to two **different** triangles of the triangulation.

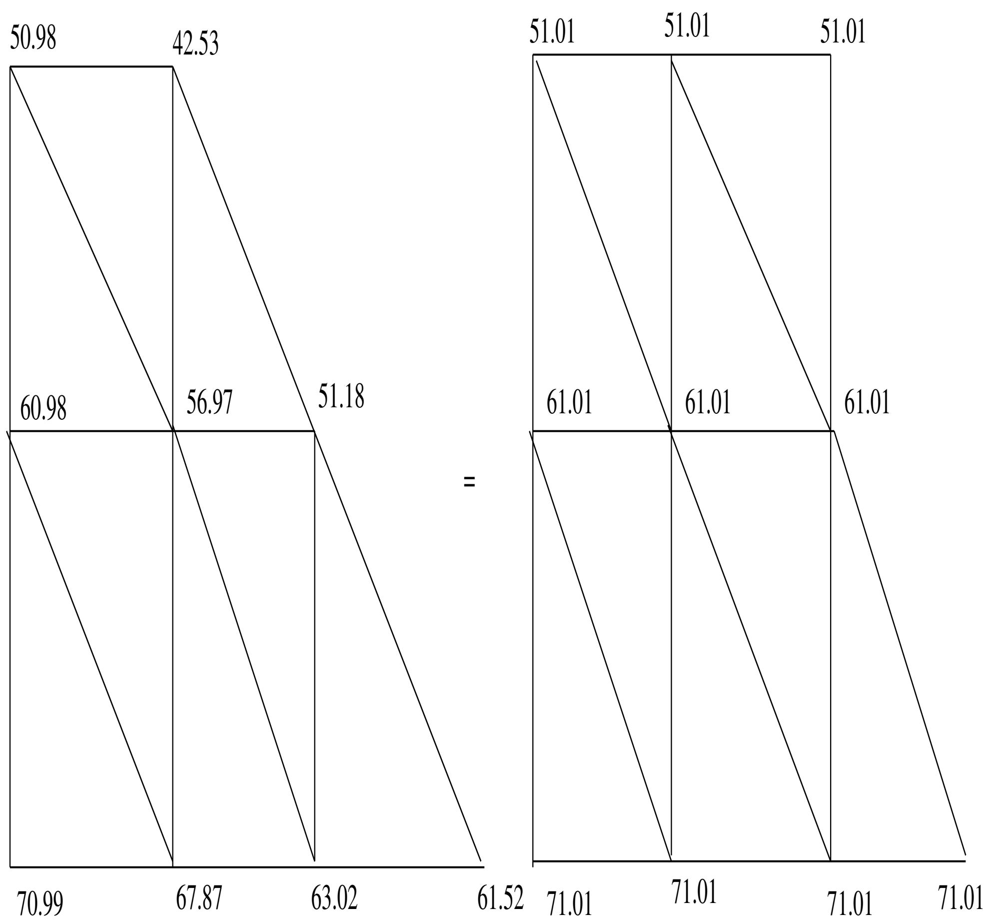

Figure 2: Typical triangulation of the pressure loss in a pipe

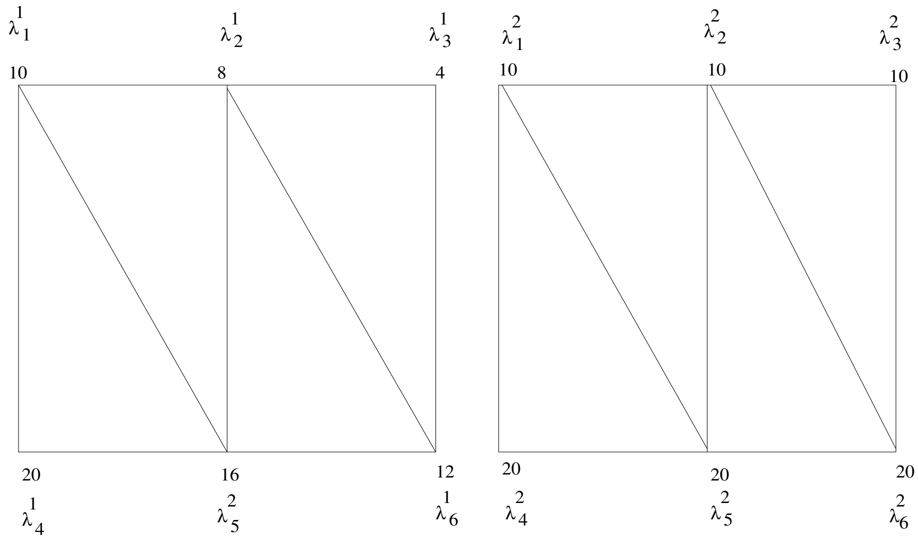

Figure 3: An easy example for a triangulation

In the following we want to generalise this approach to a more general setting. Clearly the sequence of two pipes is of course only the simplest case we are faced with. We want to examine the problem general, where we consider the case that we have an arbitrary number in of ingoing segments and an arbitrary number out of outgoing segments. A segment can now be either a pipe or a compressor. For every in- and outgoing segment we determine a certain triangulation. In the general case these triangulations do not need to consist only of such regular triangles as in Figure 2. The structure can be much more complicated. Perhaps we can not only consider triangles but also squares, pentagons, sexangles, heptagons and so on. Even arbitrary mixtures in the triangulations are possible although this is not interesting for a concrete gas network. And we do not only describe the pressure in the segments but also the gas flow in the segments. Very important for the general formulation is the first law of Kirchhoff which means that the sum of the ingoing gas flows must be equal to the sum of the outgoing gas flows. So in principle (see e.g. [1], [2]) we get the situation which is shown in Figure 4.

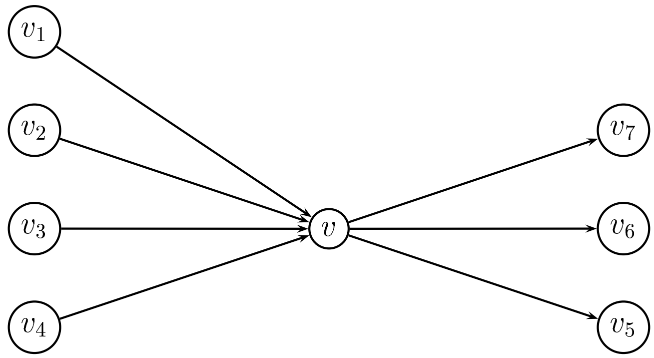

Figure 4: Ingoing and outgoing segments in a node

The requirements of the triangle conditions of $P_\triangle$ are now generalised in the following way:
The triangle conditions mean that for every segment only special combinations of $\lambda-$variables are allowed. For $P_\triangle$ this means that only $\lambda-$variables may be positive that belong to exactly one certain triangle. In the general case only the elements of special sets of $\lambda-$variables may not vanish (Indeed: the reader can recognise that our conditions are a generalised form of Special Ordered Sets (SOS) of type 2, see [3]). Before going into the details we need to fix some notation.

Notation

In this section we give some mathematical notation which is necessary in order to formalise and generalise the above approach.
Let $in \in \mathbb{N}$ be the number of ingoing segments and $out \in \mathbb{N}$ be the number of outgoing segments. A segment may be a pipe or a compressor but also the other types of segments, i.e., valves, control valves and connections can be included in this model. In the mathematical formulation of the model we are no longer bounded to the physical background of the model.
We define a set N^i of grid points for every segment $i \in \{1, 2, \dots, in + out\}$. W.l.o.g. we assume the ingoing segments to be $1, 2, \dots, in$ and the outgoing segments to be $in + 1, in + 2, \dots, in + out$. Furthermore we assume:

$$N^i \cap N^j = \emptyset \qquad \forall i \neq j.$$

We denote by

$$\mathcal{N} = \{N^i \mid i = 1, 2, \dots, in + out\}$$

the **list** of sets of grid points.
$\mathbb{R}^{N^i}$ denotes the $|N^i|-$dimensional vector space where the components are indexed by N^i and $\mathbb{R}^{\mathcal{N}}$ is defined as:

$$\mathbb{R}^{\mathcal{N}} = \bigotimes_{i=1}^{in+out} \mathbb{R}^{N^i}.$$

We remark that for $\lambda \in \mathbb{R}^{\mathcal{N}}$ we write

$$\lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^{in+out} \end{pmatrix}$$

with $\lambda^i \in \mathbb{R}^{N^i} \quad \forall i \in \{1, 2, \ldots, in + out\}$.
For a list $\mathcal{S}$ of sets of the form

$$\mathcal{S} = \{S^1, S^2, \ldots, S^{in+out}\}$$

we say for some index $j \in \bigcup_{i=1}^{in+out} N^i$:

$$j \in \mathcal{S} \quad \text{iff} \quad \exists\, i \in \{1, 2, \ldots, in + out\} \quad \text{with} \quad j \in S^i.$$

We define

$$\mathcal{S} \subseteq \mathcal{N} \quad \Leftrightarrow \quad \emptyset \neq S^i \subseteq N^i \quad \forall i \in \{1, 2, \ldots, in + out\}.$$

The cardinality of $\mathcal{S}$ is set to

$$|\mathcal{S}| = \sum_{i=1}^{in+out} |S^i|.$$

The characteristic vector of $\mathcal{S}$, which we denote by $\mathcal{X}^{\mathcal{S}} \in \mathbb{R}^{\mathcal{N}}$, is obtained by setting

$$\mathcal{X}^{\mathcal{S}}_j = \begin{cases} 1 & \text{, if } j \in \mathcal{S} \\ 0 & \text{else.} \end{cases}$$

For each $N^i, i \in \{1, 2, \ldots, in + out\}$ we define n_i subsets $N^i_k, k \in \{1, 2, \ldots, n_i\}$ with

$$N^i = \bigcup_{k=1}^{n_i} N^i_k \quad \text{and} \quad |N^i_k| \geq 2.$$

As an example: In Figure 2 holds $n_1 = 9$, $n_2 = 10$ and $|N^i_k| = 3$ for all i, k. We say that a vector $\lambda \in \mathbb{R}^{\mathcal{N}}, \lambda \geq 0$ satisfies the *set condition* if for all $i = 1, 2, \ldots, in + out$ there exists one $k_i \in \{1, 2, \ldots, n_i\}$ such that

$$\{j \in N^i | \lambda^i_j > 0\} \subseteq N^i_{k_i}.$$

In other words, the set condition holds if for all in- and outgoing segments the non vanishing $\lambda-$variables belong to exactly one of the subsets N^i_k. We say that $\mathcal{S}$ fullfills the set condition if $\mathcal{X}^{\mathcal{S}}$ fullfills the set condition.

Now we define a polyhedron P by

$$P = \{\lambda \in \mathbb{R}^{\mathcal{N}} | A\lambda = b, \lambda \geq 0\},$$

where $A \in \mathbb{R}^{M \times \mathcal{N}}, b \in \mathbb{R}^M$ for some finite set M. We will say something about the cardinality of the set M in the next subsection when we discuss the special structure of the matrix A.
Let us continue with the following definition:
For $A \in \mathbb{R}^{M \times \mathcal{N}}$ with

$$A = (a_{ij})_{\substack{\text{i=1, \ldots ,m} \\ \text{j=1, \ldots ,n}}}$$

and a subset $J \subseteq \{1, 2, \ldots, n\}$ we define

$$A_J = (a_{ij})_{\substack{i \in M \\ j \in J}}$$

Here $m = |M|$ and $n = |\mathcal{N}|$. Analogously we define for $\lambda \in \mathbb{R}^{\mathcal{N}}$ and a subset $J \subseteq \{1, 2, \ldots, n\}$

$$\lambda_J = (\lambda_j)_{j \in J}.$$

At the end of this section let us define the so called zero-extension:

For $x \in \mathbb{R}^{\mathcal{S}}$ with $\mathcal{S} \subseteq \mathcal{N}$ we define the **zero-extension** $x^0(\mathcal{S}) \in \mathbb{R}^{\mathcal{N}}$ of x by

$$x^0(\mathcal{S}) \;=\; \begin{cases} x_i & \text{, if } i \in \mathcal{S}, \\ 0 & \text{, if } i \in \mathcal{N} \setminus \mathcal{S}. \end{cases}$$

At the end of this section we define a list $\mathcal{S}$ to be a subset of another list $\bar{\mathcal{S}}$:

Let

$$\mathcal{S} = \{S^1, S^2, \ldots, S^{in+out}\}$$

and

$$\bar{\mathcal{S}} = \{\bar{\mathcal{S}}^1, \bar{\mathcal{S}}^2, \ldots, \bar{\mathcal{S}}^{in+out}\}$$

two sets (in the sense of this section). We define:

$$\begin{aligned} \mathcal{S} \subseteq \bar{\mathcal{S}} \quad :\Leftrightarrow \quad & S^1 \subseteq \bar{S}^1 \\ & S^2 \subseteq \bar{S}^2 \\ & \vdots \\ & S^{in+out} \subseteq \bar{S}^{in+out} \end{aligned}$$

3 The Problem

Using the above notation we are now ready to introduce the polyhedron we are going to investigate in this paper. Remember that we want to model the situation that there are in ingoing and out outgoing segments at some node in the gas network So we consider a polyhedron P with the following structure:

$$P = \{\lambda \in \mathbb{R}^{\mathcal{N}} \,|\, A\lambda = b, \lambda \geq 0\,, \lambda \text{ satisfies the set conditions}\}.$$

We remark that from our introductory examples it is easy to see that this polyhedron in general is **not** convex.

The special form of the matrix A and the vector b for the general case becomes

$$A = \begin{pmatrix}
(\mathbb{1}^1)^T & & & & & & & \\
& (\mathbb{1}^2)^T & & & & & & \\
& & \ddots & & & & & \\
& & & (\mathbb{1}^{\mathsf{in}})^T & & & & \\
& & & & (\mathbb{1}^{\mathsf{in+1}})^T & & & \\
& & & & & (\mathbb{1}^{\mathsf{in+2}})^T & & \\
& & & & & & \ddots & \\
& & & & & & & (\mathbb{1}^{\mathsf{in+out}})^T \\
(p^1)^T & & & & -(p^{in+1})^T & & & \\
(p^1)^T & & & & & -(p^{in+2})^T & & \\
\vdots & & & & & & \ddots & \\
(p^1)^T & & & & & & & -(p^{in+out})^T \\
& (p^2)^T & & & -(p^{in+1})^T & & & \\
& (p^2)^T & & & & -(p^{in+2})^T & & \\
& \vdots & & & & & \ddots & \\
& (p^2)^T & & & & & & -(p^{in+out})^T \\
& & \ddots & & \vdots & \vdots & \vdots & \vdots \\
& & & (p^{in})^T & -(p^{in+1})^T & & & \\
& & & (p^{in})^T & & -(p^{in+2})^T & & \\
& & & \vdots & & & \ddots & \\
& & & (p^{in})^T & & & & -(p^{in+out})^T \\
(q^1)^T & (q^2)^T & \dots & (q^in)^T & -(q^{in+1})^T & -(q^{in+2})^T & \dots & -(q^{in+out})^T
\end{pmatrix}$$

The entries $p^i \in \mathbb{R}_+^{N^i}, i \in \{1, 2, \dots, in + out\}$ are vectors describing the pressure at the end of the ingoing and at the beginning of the outgoing segments. Analogously the vectors $q^i \in \mathbb{R}_+^{N^i}, i \in \{1, 2, \dots, in + out\}$ describe the gas flow in the in- and outgoing segments. These vectors are used to formulate the mentioned first law of Kirchhoff.

The form of the vector b is

$$b = \begin{pmatrix} \mathbb{1}_{\mathsf{in+out}} \\ 0_{in\ out+1} \end{pmatrix}$$

$\mathbb{1}^{\mathsf{i}} \in \mathbb{R}^{\mathsf{N^i}}$ denotes the vector of all ones. In addition $\mathbb{1}_{\mathsf{m}}$ denotes the vector of all ones in $\mathbb{R}^m$ and 0_m the zero vector in $\mathbb{R}^m$. Let us shortly describe the structure of

the matrix A more detailed:
The first in rows describe the sum of the $\lambda-$variables of each ingoing segment. Analogously the next out rows describe the sum of the $\lambda-$variables of each outgoing segment. All sums must be one. In each node there must be a certain pressure. So the rows $in + out + 1$ up to $in + 2out$ describe that the pressure at the end of the first segment must be equal the pressure at the beginning of the outgoing segments. The rows $in + 2out + 1$ up to $(in + 1)(out + 1) - 1$ describe the same situation for the other ingoing segments combined with the outgoing segments. The last row describes the gas flow in the distinct segments. The gas flow in the outgoing segments is multiplied by -1 because the sum of the gas flows of the ingoing segments must be equal the sum of the gas flows of the outgoing segments. It is easy to see that the matrix A and the vectors λ and b are generalisations of the first discussed situation of one ingoing and one outgoing segment.
As a side remark we want to mention that there are some additional types of segments in a gas network, for example valves, control valves and connections without pressure loss or fuel gas consumption (i.e., there no nonlinear function has to be linearised). In the situation that such an additional segment is an essential part of a subsystem of the gas network also this types of segments can be modelled. Here the vectors for the pressure p or the gas flow q reduce to vectors that are elements from $\mathbb{R}^1$ (i.e., in this case the set of grid points for such a segment consists only of one element. Here it is very important to know that it is our aim that we want to cut off LP-solutions, so we can set for this types of segments the pressure and flow values that are calculated in the last iteration. This solution then can be cut off.) because such a segment can in every iteration be interpreted with constant pressure and constant flow and so can be modelled via one single $\lambda-$variable which then has to be one. So the generality of the model is ensured.
When we do not want to include the first law of Kirchhoff, i.e., the gas flow preservation equation in this model, we forget about the last line in $A\lambda = b$. The rang of the Matrix A reduces by one in this case.

3.1 The vertices of the polyhedron

Let us introduce the idea of calculating the vertices of the polyhedron before we describe the general case formally in the case of the polyhedron P_Δ: If we want to find a vertex we take one triangle from the triangulation of the ingoing pipe ($pipe^1$) and one triangle from the triangulation of the outgoing pipe ($pipe^2$). Hereto we choose some $\lambda-$variables from the $\lambda-$variables belonging to the selected triangles. Due to the triangle condition the non vanishing $\lambda-$variables at a vertex of $P_\triangle$ must belong to exactly one triangle of $pipe^1$ and one triangle of $pipe^2$. Concentrating on two triangles we investigate the extreme points for the selected $\lambda-$variables that fulfil the remaining properties of $P_\triangle$, i.e., if the sum of the selected $\lambda-$variables of $pipe^1$ and the sum of the selected $\lambda-$variables of $pipe^2$ are equal 1, if the pressure equation is fulfilled and of course all $\lambda-$variables we have selected must be nonnegative. We will show that this results in a vertex. By repeating this procedure for all possible selections of $\lambda-$variables we will see that we obtain all vertices of $P_\triangle$.

Now we give the formal algorithm how the vertices of the polyhedron P can be calculated: Let us begin with the following definition ($rg(A)$ denotes the rang of matrix A):

Definition 1 *We say a subset $\mathcal{S} \subseteq \mathcal{N}$ is feasible if*

- $|\mathcal{S}| \leq rg(A)$.
- *$\mathcal{S}$ satisfies the set condition.*

Algorithm 2

1. *Let $L = \emptyset$ be the list of all vertices of P.*
2. *For all feasible subsets $\mathcal{S} \subseteq \mathcal{N}$ do*
   (a) *Solve $A_\mathcal{S}\, \lambda_\mathcal{S} = b$.*
   (b) *If the system has a unique solution $\bar{\lambda}_\mathcal{S}$ with $\bar{\lambda}_\mathcal{S} \geq 0$, add the zero-extension of $\bar{\lambda}_\mathcal{S}$ to L.*

In the following we want to prove that this algorithm runs in polynomial time and computes all vertices of P. As a consequence we obtain that P has only polynomially many vertices.
But at first let us make the following

Remark 3 *The matrix A on page 11 can be simplified to:*

$$A = \begin{pmatrix}
(\mathbb{1}^1)^T & & & & & & & & & \\
& (\mathbb{1}^2)^T & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & \ddots & & & & & & \\
& & & & (\mathbb{1}^{\mathsf{in}})^T & & & & & \\
& & & & & (\mathbb{1}^{\mathsf{in}+1})^T & & & & \\
& & & & & & (\mathbb{1}^{\mathsf{in}+2})^T & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & (\mathbb{1}^{\mathsf{in}+\mathsf{out}})^T \\
(p^1)^T & & & & & -(p^{in+1})^T & & & & \\
(p^1)^T & & & & & & -(p^{in+2})^T & & & \\
\vdots & & & & & & & \ddots & & \\
\vdots & & & & & & & & \ddots & \\
(p^1)^T & & & & & & & & & -(p^{in+out})^T \\
(p^1)^T & -(p^2)^T & & & & & & & & \\
& (p^2)^T & -(p^3)^T & & & & & & & \\
& \ddots & \ddots & & & & & & & \\
& & \ddots & \ddots & & & & & & \\
& & & (p^{in-1})^T & -(p^{in})^T & & & & & \\
(q^1)^T & (q^2)^T & \dots & \dots & (q^{in})^T & -(q^{in+1})^T & -(q^{in+2})^T & \dots & \dots & -(q^{in+out})^T
\end{pmatrix}$$

and the vector b to

$$b = \begin{pmatrix} \mathbb{1}_{\mathsf{in}+\mathsf{out}} \\ 0_{in+out} \end{pmatrix}$$

From this we conclude that

$$in + out \leq rg(A) \leq 2(in + out)$$

Before we prove that the algorithm is correct we discuss the following

Lemma 4 *The described algorithm reduced by the postulation of the set condition can principally also be used in order to calculate the vertices of the polyhedron P without the set condition.*

This lemma is a direct consequence of well known results of linear programming, namely that the support of a vertex of a polyhedron $P = \{x \mid Ax = b, x \geq 0\}$ is at

most the number of rows of A. When we consider the polyhedron P without the set conditions this polyhedron is completely described by (in-) equalities and thus the above argument applies. The problem in the case of P (with set conditions) is that we do not know the complete description of the polyhedron in form of (in-)equalities and thus this simple argument cannot be used.
In our case we formulate the following

Theorem 5 *The above algorithm is correct, i.e., it calculates all vertices of the polyhedron P.*

Before we give the proof of Theorem 5 we formulate

Lemma 6 *Let $\mathcal{S}$ be a feasible set in the sense of Definition 1. If $A_\mathcal{S}\lambda_\mathcal{S} = 0_{|\mathcal{N}|}$ has a non-trivial solution then the zero-extension of a positive solution of $A_\mathcal{S}\lambda_\mathcal{S} = b$ is not a vertex.*

Proof 7 *Let $\bar{\lambda}_\mathcal{S}$ be a positive solution of $A_\mathcal{S}\lambda_\mathcal{S} = b$, i.e., it holds $A_\mathcal{S}\bar{\lambda}_\mathcal{S} = b$ with $\bar{\lambda}_\mathcal{S} > 0_{|\mathcal{S}|}$.*
Let $\bar{\lambda}$ be the zero extension of $\bar{\lambda}_\mathcal{S}$. We will show in the following that there exists an $\epsilon-$environment ($\epsilon > 0$) of $\bar{\lambda}$ which is contained in P. This shows that $\bar{\lambda}$ cannot be a vertex. We define for $\mathcal{S}$ a vector $\epsilon \in \mathbb{R}^\mathcal{S}$ with $\mathcal{S} \subseteq \mathcal{N}$ as follows:

From $A_\mathcal{S}\bar{\lambda}_\mathcal{S} = b$ and the condition $A_\mathcal{S}(\bar{\lambda}_\mathcal{S} + \epsilon) = b$ we get the following condition:

$$A_\mathcal{S}\epsilon = 0_{|\mathcal{N}|}$$

Obviously $\bar{\epsilon} = 0_{|\mathcal{S}|}$ is a solution (i.e., the above system is homogeneous). We know from the assumptions of Lemma 6 that $A_\mathcal{S}\epsilon = 0_{|\mathcal{N}|}$ has a nontrivial solution. Because of this we also know that the set of solutions of $A_\mathcal{S}\epsilon = 0_{|\mathcal{N}|}$ is a vector space (with nontrivial solutions). Therefore, there exists $\bar{\epsilon} \neq 0_{|\mathcal{S}|}$ such that $A_\mathcal{S}(\bar{\lambda}_\mathcal{S} + \bar{\epsilon}) = b$, with $\bar{\lambda}_\mathcal{S} + \bar{\epsilon} > 0_{|\mathcal{S}|}$ and $\bar{\lambda}_\mathcal{S} - \bar{\epsilon} > 0_{|\mathcal{S}|}$.

Now we built the zero-extension $(\bar{\lambda}_\mathcal{S} + \bar{\epsilon})^0(\mathcal{S})$ of $\bar{\lambda}_\mathcal{S} + \bar{\epsilon}$ and of course it holds $(\bar{\lambda}_\mathcal{S} + \bar{\epsilon})^0(\mathcal{S}) \in P$. Observe that for $\mathcal{S}$ all $\lambda-$variables must fulfil the set condition by construction.
Similarly, $A_\mathcal{S}(\bar{\lambda}_\mathcal{S} - \bar{\epsilon}) = A_\mathcal{S}\bar{\lambda}_\mathcal{S} - A_\mathcal{S}\bar{\epsilon} = A_\mathcal{S}\bar{\lambda}_\mathcal{S} - 0_{|\mathcal{N}|} = b$ and hence also $A(\bar{\lambda}_\mathcal{S} - \bar{\epsilon})^0(\mathcal{S}) = b$. We conclude $(\bar{\lambda}_\mathcal{S} - \bar{\epsilon})^0(\mathcal{S}) \in P$.
Finally,

$$\frac{1}{2}(\bar{\lambda}_\mathcal{S} + \bar{\epsilon}) + \frac{1}{2}(\bar{\lambda}_\mathcal{S} - \bar{\epsilon}) = \bar{\lambda}_\mathcal{S}.$$

and

$$\frac{1}{2}(\bar{\lambda}_{\mathcal{S}} + \bar{\epsilon})^0(\mathcal{S}) + \frac{1}{2}(\bar{\lambda}_{\mathcal{S}} - \bar{\epsilon})^0(\mathcal{S}) = (\bar{\lambda}_{\mathcal{S}})^0(\mathcal{S}) = \bar{\lambda}.$$

Since $\bar{\lambda}$ can be written as a convex sum of two other points of P it cannot be a vertex.

We use the lemma in the following

Proof 8 *We show Theorem 5 in two steps. At first we show that all calculated points are vertices of P and then we show that there cannot exist other vertices of P.*

1) The calculated points are vertices of P.

From the first $in + out$ rows of A it is clear that for every segment at least one variable must be greater than zero. We define for a feasible subset $\mathcal{S} \subseteq \mathcal{N}$ and its characteristic vector $\mathcal{X}^{\mathcal{S}}$ the following inequality:

$$(\mathcal{X}^{\mathcal{S}})^T \lambda \leq in + out.$$

From the definition of P we see that this inequality is valid for P since the sum of all $\lambda-$variables of a point in P is always equal to $in + out$.

Let $\bar{\lambda} = \lambda^0(\mathcal{S}) \in P$ be the zero extension of $\lambda_{\mathcal{S}}$ calculated according the algorithm corresponding to $\mathcal{S}$. We show the following:

$$\{\bar{\lambda}\} = P \cap \{(\mathcal{X}^{\mathcal{S}})^T \lambda = in + out\}.$$

Since $\bar{\lambda} \in P$ by construction and $(\mathcal{X}^{\mathcal{S}})^T \bar{\lambda} = in + out$ by definition of $\mathcal{X}^{\mathcal{S}}$, the first inclusion $\{\bar{\lambda}\} \subseteq P \cap \{\mathcal{X}^{\mathcal{S}T} \lambda = in + out\}$ is trivial.

Now we show $\{\bar{\lambda}\} \supseteq P \cap \{(\mathcal{X}^{\mathcal{S}})^T \lambda = in + out\}$.
Suppose there exists another point

$$\tilde{\lambda} \in P \cap \{(\mathcal{X}^{\mathcal{S}})^T \lambda = in + out\} \setminus \{\bar{\lambda}\}$$

Observe that $\tilde{\lambda}_i = 0$ for all $i \notin \mathcal{S}$. This implies that $\tilde{\lambda}$ is another solution to $A_{\mathcal{S}} \lambda_{\mathcal{S}} = b$, a contradiction to the construction of $\bar{\lambda}$.

2) There are no other vertices of P.

We have seen in the first part of this proof that the constructed points are indeed vertices of P. From Lemma 6 it is now easy to see that there are no other vertices of P. W.l.o.g. we can reduce us to feasible sets $\mathcal{S}$ that produce a positive solution $\bar{\lambda}_{\mathcal{S}}$ of $A_{\mathcal{S}}\lambda_{\mathcal{S}} = b$ which is not unique. Now all is clear: In this case we can apply Lemma 6, because we can conclude that $A_{\mathcal{S}}\lambda_{\mathcal{S}} = 0_{|\mathcal{N}|}$ in this case must have a non-trivial solution. Let $\bar{\lambda} \in P$ be the zero-extension of $\bar{\lambda}_{\mathcal{S}}$. Thus $\bar{\lambda} \in P$ cannot be a vertex.

From the theory above we conclude that the non-convex polyhedron P can be written as a union of convex polytopes.

Example 9 *We consider a simple example in order to demonstrate the essential parts of the notation (not all elements because the notation is much more complex than the idea behind it ...) and to show a first example for the case of part 2b) of the previous proof. Let us consider the following case of polyhedron $P_{\triangle}$ (a picture is shown in Figure 5. According to the picture holds: $n_1 = n_2 = 1$ with $|N_1^1| = |N_1^2| = 3$):*

Let the matrix A be:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 15 & 10 & 10 & -10 & -10 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$p^1 = \begin{pmatrix} 15 \\ 10 \\ 10 \end{pmatrix},$$

$$p^2 = \begin{pmatrix} 10 \\ 10 \\ 20 \end{pmatrix}.$$

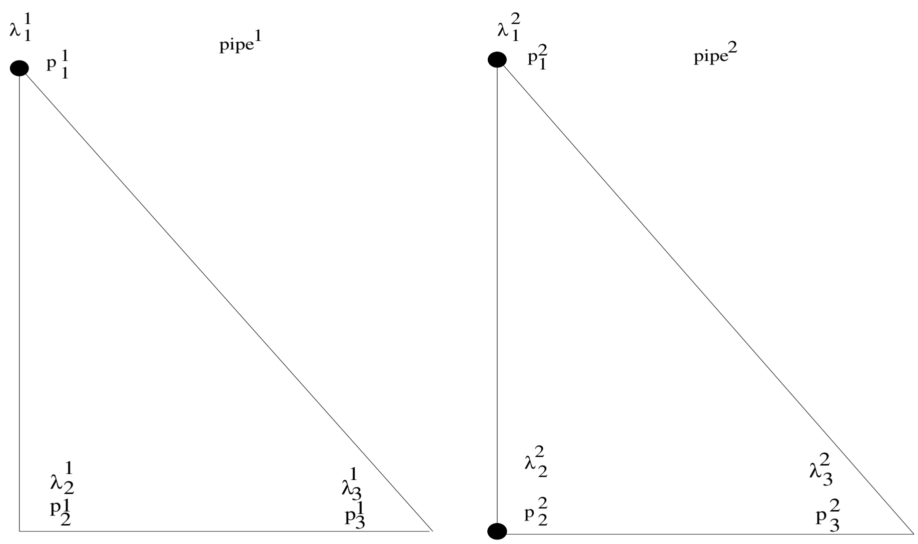

Figure 5: Building vertices of the polyhedron $P_\triangle$

Also

$$q^1 = q^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have already mentioned that if we do not want to model the gas flow preservation the last row of A can be omitted. We will do this from now on and in all upcoming examples. Because $rg(A) = 3$ we take as a first selection $\mathcal{S}_1 = \{S^1, S^2\}$ with $S^1 = \{1\}$ and $S^2 = \{4, 6\}$. Here $A_{\mathcal{S}}$ becomes

$$A_{\mathcal{S}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 15 & -10 & -20 \end{pmatrix}$$

and according to our algorithm we have to solve:

$$A_{\mathcal{S}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 15 & -10 & -20 \end{pmatrix} \begin{pmatrix} \lambda_1^1 \\ \lambda_1^2 \\ \lambda_3^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We get as a unique (and also nonnegative) solution:

$$\lambda_{\mathcal{S}} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

The zero-extension of $\lambda_{\mathcal{S}}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

is a vertex of P_Δ*. If we take as a second selection* $\mathcal{S}_2 = \{S^1, S^2\}$ *with* $S^1 = \{2\}$ *and* $S^2 = \{4, 5\}$ *we have to solve:*

$$A_{\mathcal{S}_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 10 & -10 & -10 \end{pmatrix} \begin{pmatrix} \lambda_2^1 \\ \lambda_1^2 \\ \lambda_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Here $rg(A_{\mathcal{S}_2}) = 2$ *and so we know from Theorem* 5 *that* $\mathcal{S}_2$ *does not lead to a vertex, since* $|\mathcal{S}_2| > 2$*. But if we reduce* $\mathcal{S}_2$ *to* $\mathcal{S}_3 = \{S^1, S^2\}$ *with* $S^1 = \{2\}$ *and* $S^2 = \{4\}$ *we have to solve*

$$A_{\mathcal{S}_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 10 & -10 \end{pmatrix} \begin{pmatrix} \lambda_2^1 \\ \lambda_1^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

and we get a unique (and nonnegative) solution

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which fullfills all demanded properties that we have pointed out in Algorithm 2*. Thus the zero-extension of this vector*

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

yields a vertex of P_Δ*. We will discuss the general case of* P_Δ *in the next example in a more detailed way.*

Let us come back to Algorithm 2 which we apply to polyhedron $P_\triangle$:

Example 10 *Let us now come back a second time to the case of one ingoing and one outgoing pipe described on page* 2 *and* 3*. The polyhedron* P *defined on page* 3 *reduces in this case to the polyhedron* $P_\triangle$*. We now want to describe a little bit formally the case that we have described numerically in the last example. The form of the matrix* A *generally reads:*

$$A = \begin{pmatrix} (\mathbb{1}^1)^T & \\ & (\mathbb{1}^2)^T \\ (p^1)^T & -(p^2)^T \end{pmatrix}$$

The form of the vectors λ *and* b *are*

$$\lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

It is easy to see that $2 \leq rg(A) \leq 3$ *and* $rg(A) = 2$ *iff there exist constants* c_1, c_2 *so that* $p^1 = c_1\mathbb{1}^1$ *and* $p^2 = c_2\mathbb{1}^2$*. If* $c_1 \neq c_2$ *the polyhedron is empty. In the case* $c_1 = c_2$ *we can easily describe the vertices of* $P_\triangle$*. From Algorithm* 2 *we know that we have to select feasible sets* $\mathcal{S}$ *with* $|S^1| = |S^2| = 1$*. All these possible feasible sets lead to a vertex of* $P_\triangle$ *in which the two selected* $\lambda-$*variables in* $\mathcal{S}$ *get the value* 1 *(and the not selected* $\lambda-$*variables in* $\mathcal{N} \setminus \mathcal{S}$ *are by construction* 0*).*
Now let $rg(A) = 3$*. So we can select a feasible set* $\mathcal{S}$ *with* $|S^1| + |S^2| = 3$ *and there are manifestly the following three possibilities:*

- *Select one* $\lambda-$*variable from* N^1 *and one* $\lambda-$*variable from* N^2 *and try to solve the resulting linear equality system (cf. the case when* $rg(A) = 2$ *above). Here is* $|S^1| = |S^2| = 1$*. The principle situation is as in Figure 6.*

  If $p_1^1 = p_1^2$ *we get a vertex for which* $\lambda_1^1 = \lambda_1^2 = 1$ *and the other remaining* $\lambda-$*variables are zero, otherwise we don't get a vertex.*

- *Select the feasible set* $\mathcal{S}$ *such that one* $\lambda-$*variable from the ingoing pipe and two* $\lambda-$*variables from the outgoing pipe are chosen, i.e., formally it holds* $|S^1| = 1, |S^2| = 2$*. The situation is pointed out in Figure 7.*

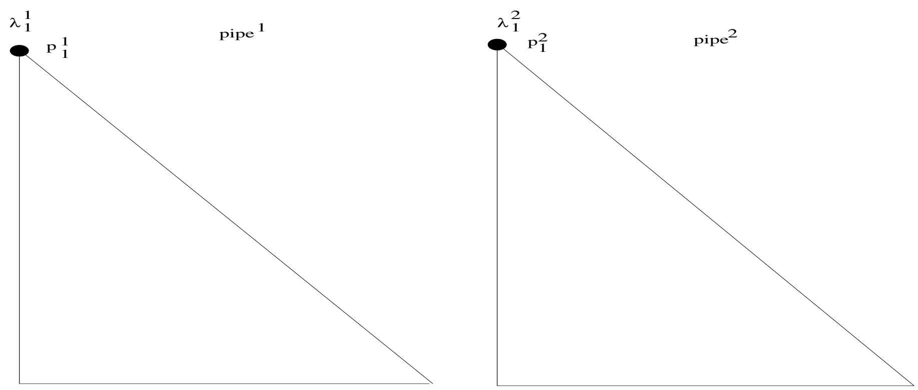

Figure 6: Vertices for the polyhedron P_Δ with a selection of two $\lambda-$variables

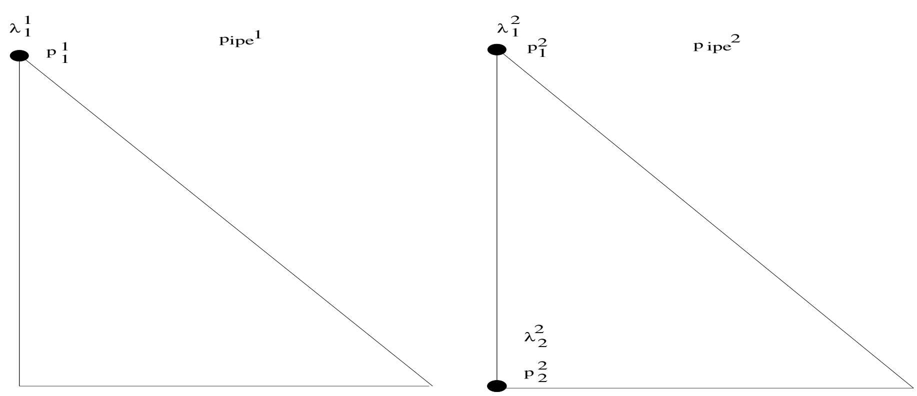

Figure 7: Vertices for the polyhedron P_Δ with a selection of three $\lambda-$variables

If $p^2_1 \leq p^1_1 \leq p^2_2$ or $p^2_2 \leq p^1_1 \leq p^2_1$ (see the figure) and $p^2_1 \neq p^2_2$ we can construct a vertex for which holds (cf. Algorithm 2):

$$\lambda^1_1 = 1 \tag{1}$$

$$\lambda^2_1 = \frac{p^1_1 - p^2_2}{p^2_1 - p^2_2} \tag{2}$$

$$\lambda^2_2 = \frac{p^2_1 - p^1_1}{p^2_1 - p^2_2} \tag{3}$$

The remaining $\lambda-$variables are again set to zero.

Proof 11 *This is an easy consequence from our algorithm. The reduced linear equation system (see $A_{\mathcal{S}}\lambda_{\mathcal{S}} = b$) reads*

$$\begin{array}{ccccccc} \lambda^1_1 & & & & & = & 1 \\ & & \lambda^2_1 & + & \lambda^2_2 & = & 1 \\ \lambda^1_1 p^1_1 & - & \lambda^2_1 p^2_1 & - & \lambda^2_2 p^2_2 & = & 0 \end{array}$$

Now we see that under our assumptions

$$det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ p^1_1 & -p^2_1 & -p^2_2 \end{pmatrix} = p^2_1 - p^2_2 \neq 0.$$

And an easy calculation shows that $\lambda^1_1, \lambda^2_1, \lambda^2_2$ are the unique non-negative solution of the above linear equation system $A_{\mathcal{S}}\lambda_{\mathcal{S}} = b$.
Therefore (built again the zero-extension) we have constructed a vertex of P_Δ.

- *Select a feasible set $\mathcal{S}$ with two selected $\lambda-$variables from N^1 and one $\lambda-$variable from N^2, in which case $|S^1| = 2, |S^2| = 1$.*
  The calculation of the non vanishing values of the vertex is analogous to the previous case.

Only these three types of feasible sets $\mathcal{S}$ possibly result in vertices of $P_\triangle$ because $rg(A) = 3$. We will see an example for a numerical calculation in the next subsection.

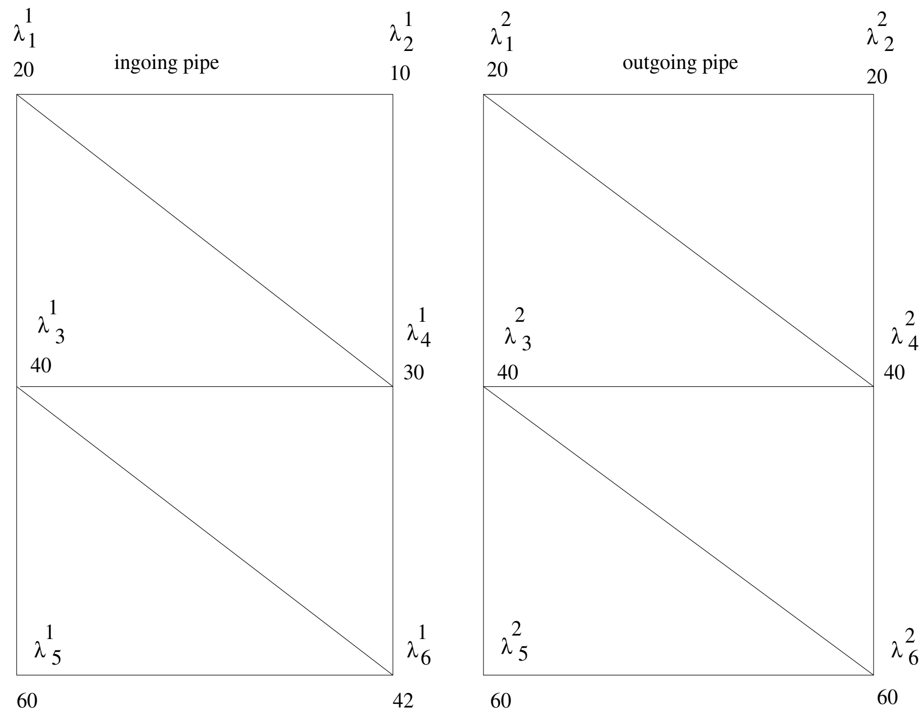

Figure 8: Example for comparing vertices and facets

3.2 The construction of cuts and the separation algorithm

The algorithm we have described above can now be used to construct cutting planes for our MIP model. Unfortunately, we do not know the facets that are describing P, since they are relatively difficult to describe even in quite easy situations like the sequence of two pipes described at the beginning of the paper.
Here we give a simple example for this situation. Clearly in more complicated or in realistic situations for the Transient Technical Optimisation the problem of describing the facets normally becomes bigger and bigger.

The following table gives an impression how the complexity of the vertices and the facets of P_Δ increases with the increasing number of grid points.
The first row describes the number of triangles (the sum in the in- and outgoing pipe), the second row the number of $\lambda-$variables, the third row describes the number of vertices, the fourth the number facet-defining inequalities and the last row the maximal coefficient within the facet-defining inequalities.

Δ	λ	vertices	facets	max. coeff.
8	12	16	18	25
16	18	49	47	42
24	24	73	90	670
32	32	142	10492	50640

We have tested several other examples and usually we get the situation that if we add to P_Δ the first law of Kirchhoff the number of vertices and facets is lower than in the case above (this is clear because we have to fullfill more conditions) but the coefficients of the facets are getting worse (but this cannot be proved in general).

So we cannot yet calculate the facets until now but –blessing in disguise– we have seen that we can calculate the vertices of the polyhedron P. Now it is on the time to show what we can do with them.

In order to use the vertices it is first very important to see that in all interesting cases there are only polynomially many of vertices which we can calculate algorithmically in addition.

Lemma 12 *For the polyhedron P (with the usual definitions and notations as used before) exist numbers l, c such that the maximal number of vertices of P is less than or equal to cl^{in+out}.*

Proof 13 *Define a number l^* as*

$$l^* := \prod_{j=1}^{in+out} n_j \tag{4}$$

where the values $n_j, j = 1, 2, \ldots, in + out$ were defined as the number of subsets in which the set of $\lambda-$variables of the in- and outgoing segments are divided. It is clear that l^* is the number of possible combinations of subsets N_k^i from all in- and outgoing segments were from every segment exactly one subset according to Algorithm 2 is taken. We remark that in the special case P_Δ the values n_j are the number of triangles of the triangulation for the in- and outgoing pipe.
It is necessary for a vertex that the non vanishing $\lambda-$variables belong to exactly

one such subset for each segment. Let $m \leq rg(A)$ be the maximal number of non vanishing $\lambda-$variables as it was pointed out in Algorithm 2. Only in order to blow up the notation not too much we define for $j \in \{1, 2, \dots, in + out\}$ numbers N^j_{max}:

$$N^j_{max} := max\{|N^j_1|, |N^j_2|, \dots, |N^j_{n_j}|\}$$

Then take for $j \in \{1, 2, \dots, in + out\}$ positive (natural) numbers x_j and after that define a number c as:

$$c := \sum_{\sum_{j=1}^{in+out} x_j \leq m} \prod_{j=1}^{in+out} \binom{N^j_{max}}{x_j}$$

We remark that by construction $\sum_{j=1}^{in+out} x_j \geq in + out$.
The interpretation of c is as follows:
c is the maximal number of possible vertices for the selection of subsets in (4). This is clear because we sum over all selections of $\lambda-$variables (resp. the chosen subsets in $\mathcal{S}$) for which the number of selected $\lambda-$variables is not greater than m. Additionally the product of the binomial coefficients calculates the maximal number of possibilities how we can choose the $\sum_{j=1}^{in+out} x_j$ $\lambda-$variables out of the sets of $\lambda-$variables belonging to the selected feasible subsets. We conclude that the number of vertices cannot be greater than cl^*.
Define

$$l := max\{n_1, n_2, \dots, n_{in+out}\}$$

Summarising our argumentation we finally conclude that the number of vertices cannot be greater than cl^{in+out}.

Note that a trivial upper bound for c is

$$c = 2^{\sum_{j=1}^{in+out} N^j_{max}}.$$

But this value for c is a good deal worse than the (even not quite good) value we have given in Proof 13.

We see that in the case of the polyhedron $P_\triangle$ the polynomiality of Algorithm 2 follows since $m = 3$. Also the polynomiality of Algorithm 2 in the general case of polyhedron P follows since N^j_{max} and m are bounded from above. The estimation in the above lemma will be much bigger than the real number of vertices in the polyhedron. For the example in Figure 8 we obtain:

$$c = (\binom{3}{1}\binom{3}{1} + \binom{3}{1}\binom{3}{2} + \binom{3}{2}\binom{3}{1}) = 27$$

and $l = 4$, and thus the maximal number of vertices is $27 * 4^{1+1} = 432$. Indeed there are only 16 vertices. Although our estimation is very bad it suffices to show that the vertices can be calculated in polynomial time.
The number of vertices is usually noticeable lower than the maximal number of vertices. To give a reason for this consider the following

Lemma 14 *Let $\mathcal{S}, \bar{\mathcal{S}}$ be two feasible sets (of $\lambda-$variables) in Algorithm 2 with $\mathcal{S} \subseteq \bar{\mathcal{S}}$. If both sets lead to a vertex of P according to Algorithm 2 they are identical.*

Proof 15 *A vertex of P regarding to $\mathcal{S}$ is the zero-extension of a unique, nonnegative and not vanishing solution of $A_{\mathcal{S}}\lambda_{\mathcal{S}} = b$ (see the description of the algorithm). The same holds for $\bar{\mathcal{S}}$. Adding to the solution of $A_{\mathcal{S}}\lambda_{\mathcal{S}} = b$ the $\lambda-$variables of $\bar{S}^i \setminus S^i$ for all$i \in \{1, 2, \ldots, in + out\}$ which we set to zero. We get a solution of $A_{\bar{\mathcal{S}}}\lambda_{\bar{\mathcal{S}}} = b$. This solution must be the unique solution of $A_{\bar{\mathcal{S}}}\lambda_{\bar{\mathcal{S}}} = b$ by assumption. Analogously we argue when we start form a vertex calculated from $\bar{\mathcal{S}}$. If there would be a vertex belonging to the selection $\mathcal{S}$ we can conclude in the same way as above that the vertices must be equal.*

Lemma 14 has an interesting consequence: If we have found a vertex for a feasible set $\mathcal{S}$ (of a selection of $\lambda-$variables) it is not necessary to search for vertices in a superset of $\mathcal{S}$. Therefore we can start with the feasible sets in which we take exactly one $\lambda-$variable for each segment, i.e., $|S^i| = 1 \, \forall i \in \{1, 2, \ldots, in + out\}$, and then look for "bigger" (with respect to set inclusion) feasible sets of selected $\lambda-$variables. In this way we can find all needed vertices in a systematic way.
Another possibility is to start with feasible sets $\mathcal{S}$ for which $|\mathcal{S}| = rg(A)$ holds. If for such a set a vertex is found you do not need to search for a vertex in any subset of this set. This procedure starts from the "biggest" selections whereas the first one starts from the "smallest". In realistic cases (of course you can always construct some pathological cases) this strategy will find the vertices much faster as we have studied in the case of a sequence of two pipes where we modelled the gas flow equation. It turns out that in data sets from gas networks mostly $rg(A) = rg(A_{\mathcal{S}})$ holds for a feasible set $\mathcal{S}$.

For the polyhedron $P_\triangle$ Lemma 14 has a nice consequence for the maximal number of vertices:

Lemma 16 *An upper bound for the number of vertices of $P_\triangle$ is*

$$9\, n_1\, n_2$$

Proof 17 *We have described the possibilities for constructing vertices of the polyhedron $P_\triangle$ in Example 10. We have seen that for each choice of two triangles there are 9 possibilities for the selection of one $\lambda-$variable from the ingoing pipe and one $\lambda-$variable from the outgoing pipe, i.e., $|S^1| = |S^2| = 1$. Now we can easily see that from these feasible sets all other possible vertices that can result from all the other feasible sets of selections of $\lambda-$variables can be deduced via adding one $\lambda-$variable either in the chosen triangle of the ingoing pipe or the chosen triangle of the outgoing pipe. From this argument directly follows Lemma 16.*

Note from Lemma 12 we just obtain a bound of $27\, n_1\, n_2$, since

$$c = \binom{3}{1}\binom{3}{1} + \binom{3}{1}\binom{3}{2} + \binom{3}{2}\binom{3}{1} = 27$$

Let us come back now to our primal aim.
All the previous ventilations give us now the possibility to develop the following **separation algorithm** for P:

Let $v_1, \ldots, v_k$ be the constructed vertices for P (in realistic situations they can be calculated very fast as we have described above).
Let $\bar{\lambda}$ be an optimal LP-solution to be cut off. We look for a cut of the form $a^T x \leq \alpha$ by solving

$$\begin{aligned} z^* = max \quad & a^T \bar{\lambda} - \alpha \\ s.t. \quad & a^T v_i \leq \alpha \quad \forall i. \end{aligned}$$

Let $(\bar{a}, \alpha)$ be such that $\bar{a}^T \bar{\lambda} - \alpha = z^*$.

(a) $\bar{a}^T \lambda \leq \bar{\alpha}$ is valid for P.

Proof 18 *We know from the theory of linear optimisation that every feasible point of the polytope P can be combined as a convex combination of its*

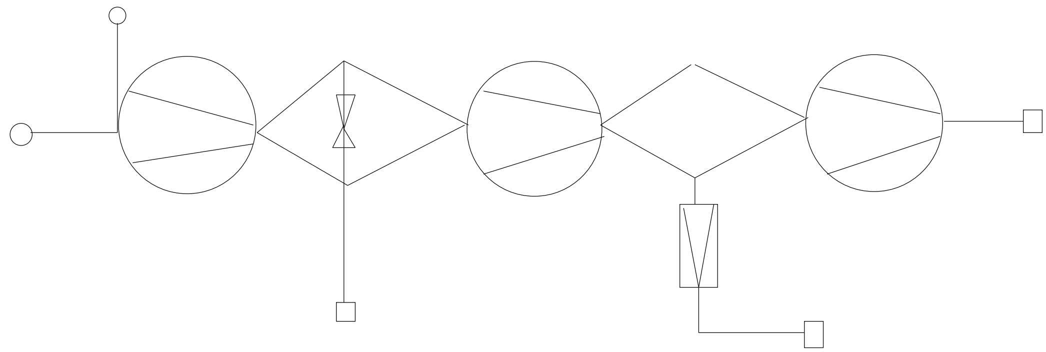

Figure 9: A simple gas network

vertices $v_1, v_2, \dots, v_k$. That is for $\lambda^* \in P$ there exist nonnegative real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ with $\sum_{i=1}^{k} \alpha_i = 1$ such that:

$$\lambda^* = \sum_{i=1}^{k} \alpha_i v_i$$

We then calculate:

$$\bar{a}^T \lambda^* = \bar{a}^T \sum_{i=1}^{k} \alpha_i v_i = \sum_{i=1}^{k} \alpha_i (\bar{a}^T v_i) \leq \sum_{i=1}^{k} \alpha_i \bar{\alpha} = \bar{\alpha} \sum_{i=1}^{k} \alpha_i = \bar{\alpha}.$$

Therefore $\bar{a}^T \lambda \leq \bar{\alpha}$ is valid for P.

(b) There exists a violated cut if and only if $z^* > 0$.

Proof 19 *If $z^* > 0$ then due to (a) $\bar{a}^T \lambda \leq \alpha$ is such a cut. On the other hand, suppose $\tilde{a}^T \lambda \leq \tilde{\alpha}$ is a valid inequality violated by $\bar{\lambda}$ then $z^* \geq \tilde{a}^T \bar{\lambda} - \bar{\alpha} > 0$.*

4 Computational Results

We have tested our implementation of the algorithm for the polyhedron $P_\triangle$ as it was described on page 2 for a gas network which consists of three compressors and ten pipes. This gas network is shown in Figure 9.

compressors ($\Box, y$)			pipes ($\triangle$)		Solution			
$p_{in,C}$	$p_{out,C}$	q_C	$p_{in,P}$	q_P	CPLEX cuts	User cuts	Opt val	sec
3	3	7	4	10	29	0	9.39	3.07
3	3	7	4*	10*	6	10	9.36	0.79
3	3	7	8	20	28	0	9.16	295.9
3	3	7	8*	20*	7	204	9.15	23.09

In our first formulation of the model we used the traditional way of the introduction of binary variables for modelling piecewise linear functions. That is we introduce for each triangle $i \in \Lambda$ a binary variable y_i and model the fact that all positive $\lambda-$variables must belong to the same triangle. The computational results for this model are indicated by y in the table above. The table shows our experiences of the computational progress when incorporating the polynomial separation algorithm instead of binary variables in a branch-and-cut algorithm (here the compressors are formulated with binary variables but the pipes are using already the cuts obtained from the separation algorithm).

$p_{in,C}$ is the number of grid points used for the pressure at the beginning of a compressor. $p_{in,C}$ analogously describes the number of grid points for the pressure at the end of a compressor. q_C is the number of grid points for the gas flow of the compressor. $p_{in,P}$ is the number of grid points used for pressure at the beginning of a pipe and q_P means the number of grid points for the gas flow in the pipe. In the rows in which the number of user cuts (constructed by the separation algorithm) is zero the problem was calculated by the formulation with binary variables. We see that the use of cuts constructed by the separation algorithm reduces the calculation time about factor 10. Let us give a short comment about our implementation: The LP-relaxations are calculated with CPLEX. We are working with the CPLEX cutcallback functions. Callbacks may be called repeatedly at various points during an optimization. Here we are looking in each LP-iteration for a cut we have calculated by the separation algorithm. This cut is added to the LP-relaxation of the problem.
Figure 10 shows the situation before using the constructed cuts. The solid lined pipes do not fulfill the triangle (set) conditions whereas the dotted pipes do. In Figure 11 we can see the situation after the use of the separation algorithm. We see that in Figure 11 still one pipe does not fulfill the triangle condition. The reason for this is that the polyhedron $P_\triangle$ (in general the polyhedron P) is not convex. So in some cases it can be possible that the solution to be cut off lies in the interior

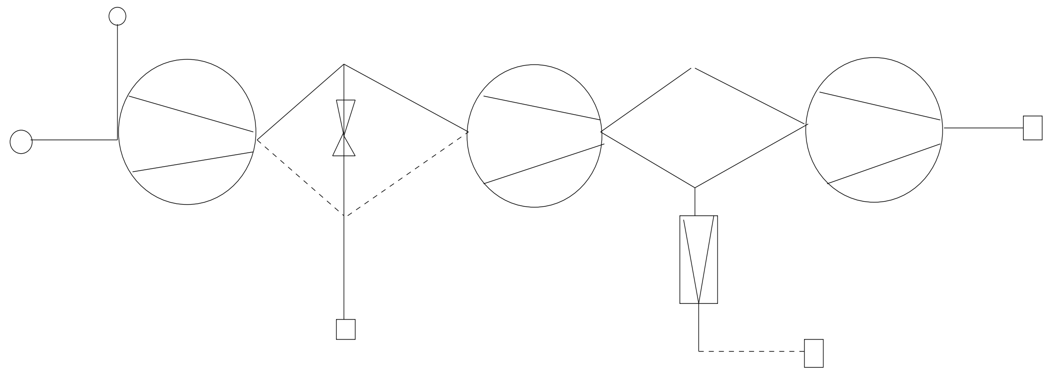

Figure 10: The testmodell before the separation algorithm

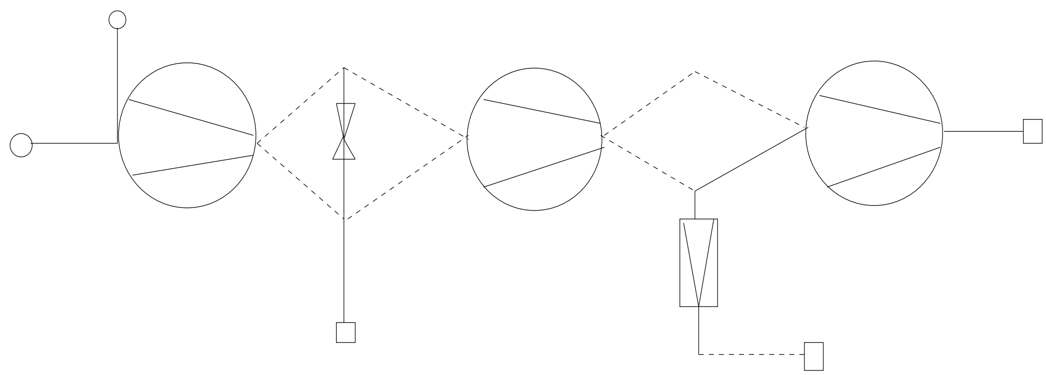

Figure 11: The testmodell after the separation algorithm

of the convex hull of the polyhedron P but not in P itself. Such points cannot be cut off by a valid inequality constructed by the separation algorithm.

5 Conclusions

Although we have not yet implemented the separation algorithm for compressors or for more complex subsystems than the sequence of two pipes the theoretical knowledge of the vertices and the separation algorithm gives us the possibility to extend our branch-and-cut algorithm to complexer gas networks and it is very supposable that in this way the solution time can be reduced significantly.

References

[1] SEKIRNJAK, E. (1998): *Mixed Integer Optimization for Gas Transmission and Distribution Systems.* INFORMS Meeting, Seattle, October 1998

[2] SEKIRNJAK, E. (2000): *Transiente Technische Optimierung (TTO) Prototyp* Technical Report for use in the PSI AG, Ruhrgas AG

[3] BEALE, E.L.M, TOMLIN, J.A. (1970): *Special Facilities in a General Mathematical Programming System for Nonconvex Problems Using Ordered Sets of Variables* Proceedings of the Fith International Conference on Operations Research, Tavistock Publications, pp. 447-454