# On the hyperbolic symplectic geometry

Ralf Gramlich

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**Abstract**

The present article provides a new characterization of the geometry on the points and hyperbolic lines of a nondegenerate symplectic polar space. This characterization is accomplished by studying the familiy of subspaces obtained when considering the polars of all hyperbolic lines.

## 1 Introduction

The geometry on the points and hyperbolic lines of a nondegenerate symplectic polar space (or, short, **hyperbolic symplectic geometry**) is interesting for a number of reasons. One observation is the fact that every pair of intersecting hyperbolic lines spans a dual affine plane (sometimes also called a symplectic plane). With the exception of some small cases, this observation can actually be used to characterize the hyperbolic symplectic geometries, cf. Cuypers [2] and Hall [6]. In Section 2 we will give a short review of the achievements of Cuypers and Hall. Another observation is the 1-1 correspondence between the set of long root subgroups, resp. fundamental $SL_2$'s of a symplectic group and the points, resp. hyperbolic lines of the corresponding symplectic geometry. This correspondence is well-known, see e.g. [8], Chapter 2, Example 1.4. So, from a group-theoretic point of view, there also exists a natural interest in the hyperbolic symplectic geometry. This second observation motivated the author's research on this topic.

The first result focuses exclusively on the hyperbolic lines and their relative positions. More precisely, let $\mathbb{W}_{2n}(\mathbb{F})$ denote the polar space with respect to a nondegenerate symplectic polarity of $\mathbb{P}_{2n-1}(\mathbb{F})$, for $n \geq 1$ and $\mathbb{F}$ a field. The **hyperbolic line graph** $\mathbf{S}(\mathbb{W}_{2n}(\mathbb{F})) = \mathbf{S}_{2n}(\mathbb{F})$ then is the graph on the hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ where hyperbolic line $l$ and $m$ are adjacent (in symbols $l \perp m$) if and only if all points of $l$ are collinear (in $\mathbb{W}_{2n}(\mathbb{F})$) to all points of $m$, cf. Definition 3.1. Equivalently, $l \perp m$ if and only if $m$ is contained in the polar of $l$, or, equivalently, if and only if the corresponding fundamental $SL_2$'s commute.

A graph $\Gamma$ is **locally homogeneous** if and only if for any pair $\mathbf{x}$, $\mathbf{y}$ of vertices of $\Gamma$, the induced subgraphs $\Gamma(\mathbf{x})$ and $\Gamma(\mathbf{y})$ on the set of neighbors of $\mathbf{x}$, resp. $\mathbf{y}$ are isomorphic.

Such a graph $\Gamma$ is called **locally** $\Delta$, for some graph $\Delta$, if $\Gamma(\mathbf{x}) \cong \Delta$ for some, whence all, vertices $\mathbf{x}$ of $\Gamma$. It is easily seen (cf. Proposition 3.3) that the graph $\mathbf{S}_{2n}(\mathbb{F})$ is locally $\mathbf{S}_{2n-2}(\mathbb{F})$. Conversely, this property is characteristic for this graph for sufficiently large $n$:

**Theorem 1**
*Let $n \geq 4$, let $\mathbb{F}$ be a field, and let $\Gamma$ be a connected graph that is locally $\mathbf{S}_{2n}(\mathbb{F})$. Then $\Gamma$ is isomorphic to $\mathbf{S}_{2n+2}(\mathbb{F})$.*

The requirement in the preceding theorem that $\Gamma$ be connected comes from the fact that a graph is locally $\Delta$ if and only if each of its connected components is locally $\Delta$. So in fact, its primary role is to provide irreducibility.

In order to state the next theorem, we have to recall the notion of a **perp space**. This is a partial linear space $(\mathcal{P}, \mathcal{L})$ endowed with a symmetric relation $\perp \subseteq \mathcal{P} \times \mathcal{P}$ such that for every point $x$, whenever $p \neq q$ are points on a line $l$, the fact $x \perp p$ and $x \perp q$ implies $x \perp y$ for all $y \in l$.

We can think of the hyperbolic symplectic geometry as a perp space, by choosing $\perp$ to be the polar relation; two objects are related by $\perp$ if and only if one is in the polar of the other. Notice that this definition of $\perp$ is in accordance with our definition of $\perp$ above as the adjacency relation of the graph $\mathbf{S}_{2n}(\mathbb{F})$.

**Theorem 2**
*Let $n \geq 4$, let $\mathbb{F}$ be a field, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space in which for any line $k \in \mathcal{L}$ the space $k^\perp$ is isomorphic to the hyperbolic symplectic geometry of $\mathbb{W}_{2n}(\mathbb{F})$ with $l \perp m$ if and only if $m$ is in the polar of $l$ for (hyperbolic) lines $l$, $m$ inside $k^\perp$. If the graph $(\mathcal{L}, \perp)$ is connected, then $(\mathcal{P}, \mathcal{L}, \perp)$ is isomorphic to the hyperbolic symplectic geometry of $\mathbb{W}_{2n+2}(\mathbb{F})$.*

Rephrasing Theorem 2 yields the following theorem:

**Theorem 3**
*Let $n \geq 4$, let $(\mathcal{P}, \mathcal{L})$ be a partial linear space, and let $\perp$ denote non-collinearity in that space. Assume there exists a nondegenerate symplectic space of rank $n$ such that for all $k \in \mathcal{L}$ the set $k^\perp$ of all points and lines of $(\mathcal{P}, \mathcal{L})$ not collinear to $k$ is a subspace of $(\mathcal{P}, \mathcal{L})$ that is isomorphic to the hyperbolic geometry of that symplectic space. If the graph $(\mathcal{L}, \perp)$ is connected, then the space $(\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry on the points and hyperbolic lines of a nondegenerate symplectic polar space of rank $n+1$.*

We would like to point out that the lower bound on $n$ in our theorems is sharp. Indeed, the centralizer of a fundamental $SL_2$ in the exceptional group $F_4(\mathbb{F})$ is isomorphic to $Sp_6(\mathbb{F})$, cf. e.g. 7.18 of the third chapter of [8], so the corresponding constructions for the group $F_4(\mathbb{F})$ yield counterexamples to our theorems with $n = 3$.

This article is organized as follows. In Section 2 we will quickly review the characterizations of the hyperbolic symplectic geometries by Cuypers and by Hall. Section 3 is devoted to the study of graphs isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$. As a side result we determine the automorphism group of $\mathbf{S}_{2n}(\mathbb{F})$. In Section 4 we prove Theorem 1, whereas Theorems 2 and 3 are proved in Section 3.4. Actually, we also provide a proof for Statement 4 on perp spaces in the author's PhD thesis, cf. Theorem 5.1.

# 2 Existing Results

Several partial results have been known for quite some time, but the first complete result was obtained by Jonathan I. Hall in [6].

**Fact 2.1 (Hall [6], Main Theorem)**
*Let $(\mathcal{P}, \mathcal{L})$ be a finite, connected partial linear space in which each pair of intersecting lines lies in a subspace isomorphic to a dual affine plane. Assume that $(\mathcal{P}, \mathcal{L})$ contains at least two such planes. Then either*

(i) *for some prime power $q$ and some integer $n$ at least 3, the space $(\mathcal{P}, \mathcal{L})$ is isomorphic to the partial linear space of hyperbolic lines of a symplectic polar space embedded in the projective space $\mathbb{P}_n(q)$; or*

(ii) *all lines of $\mathcal{L}$ contain exactly three points.*

The partial linear spaces satisfying (ii) of Theorem 2.1 are called cotriangular spaces. Theorem 1 of [7] provides a complete classification of the spaces occuring in (ii), which reads as follows.

**Fact 2.2 (Hall [7], Theorem 1, Theorem 4)**
*Let $(\mathcal{P}, \mathcal{L})$ be a connected partial linear space all of whose lines contain exactly three points and in which every pair of intersecting lines lies in a subspace isomorphic to a dual affine plane (of order two). Then $(\mathcal{P}, \mathcal{L})$ is isomorphic to one of the following partial linear spaces:*

(i) *the geometry on the non-radical points and the hyperbolic lines of some symplectic space over $\mathbb{F}_2$;*

(ii) *the subgeometry of a space as in (i) on its non-singular points with respect to some quadratic form $q$ such that the symplectic form $f$ is obtained as $f(x, y) = q(x) + q(y) + q(x, y)$; or*

(iii) *the geometry defined as follows. Let $\Omega$ be a set of cardinality at least two and let $\Omega'$ be a set disjoint from $\Omega$. The points of the geometry are the finite subsets of $\Omega \cup \Omega'$ that intersect $\Omega$ in a set of cardinality two. The lines of the geometry are those triples $x_1$, $x_2$, $x_3$ of points with empty symmetric difference, i.e., $\left(\bigcup_{1\leq i\leq 3} x_i\right) \setminus \bigcup_{1\leq i,j\leq 3} (x_i \cap x_j) = \emptyset$.*

*If additionally* $\{y\} \cup \{p \in \mathcal{P} \mid p \in l, x \in l \in \mathcal{L}\} = \{x\} \cup \{p \in \mathcal{P} \mid p \in l, y \in l \in \mathcal{L}\}$ *implies* $x = y$ *for points* $x$, $y$, *then* $(\mathcal{P}, \mathcal{L})$ *is isomorphic to a geometry of Case* (i) *or* (ii) *with respect to a nondegenerate form or of Case* (iii) *with* $\Omega' = \emptyset$.

Hans Cuypers has proved a version of Hall's Main Theorem in [6] that includes infinite point orders and infinite dimensions but disregards point order 2.

**Fact 2.3 (Cuypers [2], Theorem 1.1)**
*Let* $(\mathcal{P}, \mathcal{L})$ *be a connected partial linear space in which any pair of intersecting lines is contained in a subspace isomorphic to a dual affine plane. Assume that* $(\mathcal{P}, \mathcal{L})$ *contains at least two such planes and a line with more than three points. Then* $(\mathcal{P}, \mathcal{L})$ *is isomorphic to the geometry on the non-radical points and the hyperbolic lines of a symplectic polar space embedded in some projective space of dimension at least* 3.

# 3 The hyperbolic line graph of symplectic spaces

In this section we study graphs isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$. Let us start with recalling the definition from the introduction. Throughout this section let $n \geq 1$ and let $\mathbb{F}$ be a field.

**Definition 3.1** Let $\mathbb{W}_{2n}(\mathbb{F})$ denote the polar space with respect to a nondegenerate symplectic polarity of $\mathbb{P}_{2n-1}(\mathbb{F})$. The **hyperbolic line graph** $\mathbf{S}(\mathbb{W}_{2n}(\mathbb{F})) = \mathbf{S}_{2n}(\mathbb{F})$ is the graph on the hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ where hyperbolic line $l$ and $m$ are adjacent (in symbols $l \perp m$) if and only if all points of $l$ are collinear (in $\mathbb{W}_{2n}(\mathbb{F})$) to all points of $m$. For sake of brevity, instead of $\mathbf{S}_{2n}(\mathbb{F})(\mathbf{x})$ for a vertex $\mathbf{x}$ we also write $\mathbf{x}^{\perp}$ to denote the induced subgraph of $\mathbf{S}_{2n}(\mathbb{F})$ on the set of neighbors of $\mathbf{x}$. Moreover, $X^{\perp} := \cap_{\mathbf{x} \in X} \mathbf{x}^{\perp}$ for a set of vertices $X$.

**Remark 3.2** In this section we always consider the symplectic space $\mathbb{W}_{2n}(\mathbb{F})$ embedded into its natural ambient space $\mathbb{P}_{2n-1}(\mathbb{F})$. Spans of geometrical objectcs of $\mathbb{W}_{2n}(\mathbb{F})$ are to be understood inside the ambient projective space and are denoted by $\langle \cdots \rangle_{\mathbb{P}}$.

**Proposition 3.3**
*Let* $n \geq 2$. *The graph* $\mathbf{S}_{2n}(\mathbb{F})$ *is connected; it has diameter two if* $n \geq 4$. *Moreover, it is locally* $\mathbf{S}_{2n-2}(\mathbb{F})$.

*Proof.* The first assertion is straightforward. The second is immediate from the fact that the set of points of $\mathbb{W}_{2n}(\mathbb{F})$ that are collinear to a given hyperbolic line $l$ spans a subspace isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$, whose hyperbolic lines are precisely those hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ that are in relation $\perp$ to the hyperbolic line $l$. □

**Lemma 3.4**
*Let* $n \geq 3$, *and let* $l$, $m$ *be distinct hyperbolic lines of* $\mathbb{W}_{2n}(\mathbb{F})$ *with* $\{l, m\}^{\perp} \neq \emptyset$. *Then any hyperbolic line contained in* $\{l, m\}^{\perp\perp}$ *is also contained in* $\langle l, m \rangle_{\mathbb{P}}$ *and vice versa.*

*Proof.* Let $p \in \langle l, m\rangle_{\mathbb{P}}$ be a point of $\mathbb{W}_{2n}(\mathbb{F})$. Then a vector that spans $p$ can be expressed as a linear combination of vectors spanning points on $l$ and $m$. But then points collinear to these are also collinear to $p$. Hence a hyperbolic line contained in $\langle l, m\rangle_{\mathbb{P}}$ is also contained in $\{l, m\}^{\perp\perp}$. Conversely, let $q$ be a point not contained in $\langle l, m\rangle_{\mathbb{P}}$. Note that in a symplectic space any hyperplane is singular, i.e., there exists a point having that hyperplane as its polar. The space $\langle l, m\rangle_{\mathbb{P}}$ has at most (projective) dimension three. Since $n \geq 3$, hyperplanes have at least dimension 4. Now consider the hyperplanes $\Pi_i$, $i \in I$ for some index set, of $\mathbb{P}_{2n-1}(\mathbb{F})$ containing $\langle l, m\rangle_{\mathbb{P}}$. Denote the corresponding points by $p_i$. If all of the $p_i$ were contained in $\langle l, m\rangle_{\mathbb{P}}$, then $\{l, m\}^{\perp} = \emptyset$ (for, a hyperbolic line of $\{l, m\}^{\perp} \cap \langle l, m\rangle_{\mathbb{P}}$ would have to be contained in the radical of $\langle l, m\rangle_{\mathbb{P}}$, which does not contain hyperbolic lines), whence there exists a $p_i$ outside $\langle l, m\rangle_{\mathbb{P}}$. Fix such a $p_i$ and choose a hyperline $\Lambda_i \subset \Pi_i$ with $\langle l, m\rangle_{\mathbb{P}} \subset \Lambda_i$ and $p_i, q \notin \Lambda_i$. Let $\Pi_j$ be any other hyperplane of $\mathbb{P}_{2n-1}(\mathbb{F})$ containing $\Lambda_i$. Since $p_i \notin \Pi_j$ we have $p_j \notin \Pi_i$ and $p_i$, $p_j$ are noncollinear. Moreover, at least one of $p_i$ and $p_j$ is not collinear with $q$ (because $q \notin \Pi_i \cap \Pi_j = \Lambda_i$) and we have found a hyperbolic line $p_i p_j$ contained in $\{l, m\}^{\perp}$ that ensures that no hyperbolic line containing $q$ is contained in $\{l, m\}^{\perp\perp}$. This finishes the proof, because $q$ has been chosen arbitrarily outside $\langle l, m\rangle_{\mathbb{P}}$. □

**Notation 3.5** Let $X$ be a subspace of $\mathbb{W}_{2n}(\mathbb{F})$. Denote the set of all hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ contained in $X$ by $\mathbf{S}(X)$.

**Lemma 3.6**
*Let $n \geq 3$. Let $k$, $l$, $m$ be three hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ with $\{k, l, m\}^{\perp} \neq \emptyset$ that intersect in a common point. Then $\mathbf{S}(\langle k, l, m\rangle) = \{k, l, m\}^{\perp\perp}$.*

*Proof.* There exist hyperbolic lines $a$ and $b$ with $\langle a, b\rangle_{\mathbb{P}} = \langle k, l, m\rangle_{\mathbb{P}}$. Then by the preceding lemma we have $\mathbf{S}(\langle a, b\rangle_{\mathbb{P}}) = \{a, b\}^{\perp\perp}$. Finally, $\{a, b\}^{\perp\perp} = \{k, l, m\}^{\perp\perp}$ by $\langle a, b\rangle_{\mathbb{P}} = \langle k, l, m\rangle_{\mathbb{P}}$ and linear algebra. □

**Lemma 3.7**
*Let $n \geq 3$. Distinct hyperbolic lines $l$ and $m$ of $\mathbb{W}_{2n}(\mathbb{F})$ intersect if and only if the perp $\{l, m\}^{\perp}$ in $\mathbf{S}_{2n}(\mathbb{F})$ is non-empty and the double perp $\{l, m\}^{\perp\perp}$ in $\mathbf{S}_{2n}(\mathbb{F})$ does not contain adjacent vertices (with respect to $\perp$).*

*Proof.* Let $l$ and $m$ be two intersecting hyperbolic lines. First we will show that $\{l, m\}^{\perp} \neq \emptyset$. The space $\langle l, m\rangle_{\mathbb{P}}$ has (projective) dimension two. Hence its polar $\langle l, m\rangle_{\mathbb{P}}^{\pi}$ has dimension two or bigger, since $n \geq 3$. (We denote the polarity by $\pi$.) If $n \geq 4$, then $\langle l, m\rangle_{\mathbb{P}}^{\pi}$ is not totally isotropic, so we find two noncollinear points in $\langle l, m\rangle_{\mathbb{P}}^{\pi}$, whence we also find a hyperbolic line adjacent to both $l$ and $m$. Now suppose $n = 3$. If $\langle l, m\rangle_{\mathbb{P}}^{\pi}$ does not contain a hyperbolic line, then it is totally singular and, because of the dimensions, equal to $\langle l, m\rangle_{\mathbb{P}}$. But $\langle l, m\rangle_{\mathbb{P}}$ is not totally singular, as it contains hyperbolic lines, a contradiction.

The space $\langle l, m\rangle_{\mathbb{P}}$ is a projective plane, and the hyperbolic lines contained in which are precisely those of $\{l, m\}^{\perp\perp}$, by Lemma 3.4. If this plane contains two adjacent hyperbolic lines $a\mathbb{F}+b\mathbb{F}$ and $c\mathbb{F}+d\mathbb{F}$ then $(a, b) = (a, a\alpha_1+c\alpha_2+d\alpha_3) = (a, a)\alpha_1+(a, c)\alpha_2+(a, d)\alpha_3 = 0$ (where $(\cdot,\cdot)$ denotes the alternating bilinear form), a contradiction to the fact that $a\mathbb{F}+b\mathbb{F}$ is a hyperbolic line. Conversely, suppose $l$ and $m$ are non-intersecting hyperbolic lines. Then $\langle l, m\rangle_{\mathbb{P}}$ is a projective 3-space and $\langle l, m\rangle_{\mathbb{P}} \cap \mathbb{W}_{2n}(\mathbb{F})$ is a nondegenerate symplectic space (the direct sum of two disjoint hyperbolic lines) or has a projective line as its radical (and hence the space is the direct sum of a hyperbolic line and a non-intersecting singular line). In both cases $\langle l, m\rangle_{\mathbb{P}}$ contains adjacent hyperbolic lines. We may assume $\{l, m\}^{\perp} \neq \emptyset$, and the claim follows from Lemma 3.4. □

We now want to recover the points of the polar space as pencils of hyperbolic lines. Three mutually intersecting hyperbolic lines $k$, $l$, $m$ intersect in one point if there exists a fourth hyperbolic line $j$ that intersects with the first three and spans a projective 3-space with two of them. In terms of double perps this means that $k$, $l$ and $m$ are intersecting in one point if there exists a hyperbolic line $j$ with $\{k, l\}^{\perp\perp} = \mathbf{S}(\langle k, l\rangle_{\mathbb{P}}) \subsetneq \mathbf{S}(\langle j, k, l\rangle_{\mathbb{P}}) = \{j, k, l\}^{\perp\perp}$. The former equality is due to Lemma 3.4, the latter is due to Lemma 3.6. The only problem is to ensure that $\{k, l\}^{\perp} \neq \emptyset \neq \{j, k, l\}^{\perp}$. The first inequality has been shown in Lemma 3.7, the second will be handled by the following lemma. More precisely, we show that we can choose $j$ in such a way that $\{j, k, l\}^{\perp} \neq \emptyset$ holds.

**Lemma 3.8**
*Let $n \geq 3$. For distinct intersecting hyperbolic lines $l$ and $m$ of $\mathbb{W}_{2n}(\mathbb{F})$ there exists a hyperbolic line $j$ that intersects $l$ and $m$ such that $\langle j, l, m\rangle_{\mathbb{P}}$ has projective dimension 3 and $\{j, l, m\}^{\perp}$ in $\mathbf{S}_{2n}(\mathbb{F})$ is non-empty.*

*Proof.* Consider the plane $\langle l, m\rangle_{\mathbb{P}}$. It contains a point $x$ as radical, which lies on neither $l$ nor $m$. The space $l^{\pi}$ in $\mathbb{W}_{2n}(\mathbb{F})$ is isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$ and contains a point $y$ that is not collinear with $x$, because $\mathbb{W}_{2n}(\mathbb{F})$ is nondegenerate. Therefore $\langle l, xy\rangle_{\mathbb{P}}$ is a nondegenerate symplectic 3-space, a symplectic generalized quadrangle, and $\{l, xy\}^{\perp} \neq \emptyset$ as $n \geq 3$. Thus we are done, if we can find a point $p$ of $\langle l, xy\rangle_{\mathbb{P}}$ with $\langle l, m, p\rangle_{\mathbb{P}} = \langle l, xy\rangle_{\mathbb{P}}$ that is not collinear with $q := l \cap m$. But this point $p$ exists since $\langle l, m\rangle_{\mathbb{P}} \subset \langle l, xy\rangle_{\mathbb{P}}$ and $\langle l, xy\rangle_{\mathbb{P}}$ is nondegenerate, so we can choose $j$ to be the hyperbolic line $pq$. □

**Definition 3.9** Let $n \geq 3$. Following Lemma 3.7, distinct vertices $l$, $m$ of a graph $\Gamma$ isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$ are said to **intersect** if $\{l, m\}^{\perp} \neq \emptyset$ and the double perp $\{l, m\}^{\perp\perp}$ in $\Gamma$ does not contain adjacent vertices. In view of the paragraph before Lemma 3.8 three mutually intersecting vertices $k$, $l$, $m$ of $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$ are said to **intersect in one point** if there exists a vertex $j$ of $\Gamma$ that intersects $k$, $l$, and $m$ and that has the property that $\{j, k, l\}^{\perp} \neq \emptyset$ and $\{k, l\}^{\perp\perp} = \mathbf{S}(\langle k, l\rangle_{\mathbb{P}}) \subsetneq \mathbf{S}(\langle j, k, l\rangle_{\mathbb{P}}) = \{j, k, l\}^{\perp\perp}$.

An **interior point** of a graph $\Gamma$ isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$ is a maximal set of mutually intersecting vertices of $\Gamma$ any three elements of which intersect in one point. Denote the set of all interior points of $\Gamma$ by $\mathcal{P}$. Furthermore, an **interior hyperbolic line** of $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$ is a vertex of $\Gamma$. The set of interior hyperbolic lines of $\Gamma$ is denoted by $\mathcal{H}$.

By the preceding lemmas we have the following.

**Proposition 3.10**
*Let $n \geq 3$, and let $\Gamma$ be isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$. The geometry $(\mathcal{P}, \mathcal{H}, \supset)$ on the interior points and interior hyperbolic lines of $\Gamma$ is isomorphic to the geometry on points and hyperbolic lines of the symplectic space $\mathbb{W}_{2n}(\mathbb{F})$.* □

The space $(\mathcal{P}, \mathcal{H})$ is called the **interior hyperbolic space** on $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$.

**Corollary 3.11**
*Let $n \geq 3$ and let $\mathbb{F}$ be a field. Then the automorphism group of $\mathbf{S}_{2n}(\mathbb{F})$ is isomorphic to the automorphism group of $\mathbb{W}_{2n}(\mathbb{F})$.* □

# 4 Locally hyperbolic line graphs

In this section let $n \geq 4$, let $\mathbb{F}$ be a field, and let $\Gamma$ be a connected graph that is locally $\mathbf{S}_{2n}(\mathbb{F})$. By the preceding section we can reconstruct the interior hyperbolic space from the graph $\mathbf{x}^{\perp}$ for any vertex $\mathbf{x} \in \Gamma$. Such a hyperbolic space on a perp is called **local** as is any object that belongs to such a space. To avoid confusion we will index any such local object by the vertex of $\Gamma$ whose perp it is defined on. We would like to point out that in the preceding section we only reconstruct the objects of the polar geometry, and not the objects of any ambient projective space. However, the singular lines are easily reconstructed from the set of points and the (non-)collinearity relation given by the hyperbolic lines. On the other hand, the set of points together with the set of singular and hyperbolic lines forms a projective space, in which we will embed the symplectic geometry. All spans of objects of the symplectic geometry are to be understood inside this projective space.

It will turn out that $\Gamma$ is isomorphic to $\mathbf{S}_{2n+2}(\mathbb{F})$. To obtain this result we will construct a global geometry on $\Gamma$ from the interior hyperbolic spaces on the perps, which will be shown to be isomorphic to the hyperbolic space of some symplectic polar space (using the characterizations by Cuypers and by Hall), whose hyperbolic line graph is isomorphic to $\Gamma$.

**Lemma 4.1**
*Consider $\Gamma$ as a two-dimensional simplicial complex whose two-simplices are its triangles. Let $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ be a chain of vertices in $\Gamma$. Then there exists a chain $\mathbf{w} \perp \mathbf{x}_1 \perp \mathbf{y}_1 \perp \mathbf{z}$ that is homotopically equivalent to the former chain of vertices with $\mathbf{w}^{\perp} \cap \mathbf{x}_1^{\perp} \cap \mathbf{y}_1^{\perp} \cap \mathbf{z}^{\perp} \neq \emptyset$. In particular, the diameter of $\Gamma$ (as a graph) is two, and $\Gamma$ is simply connected.*

*Proof.* Let us start with proving the first claim. We will distiguish the cases $n \geq 6$, $n = 5$, and $n = 4$. Suppose $n \geq 6$. Then the perp $\mathbf{y}^\perp$ is isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$, which can be endowed with the interior polar space isomorphic to $\mathbb{W}_{2n}(\mathbb{F})$ living in a projective space $\mathbb{P}_{2n-1}(\mathbb{F})$. By Lemma 3.3, the intersections $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ are isomorphic to $\mathbf{S}_{2n-2}(\mathbb{F})$ and can be endowed with interior polar spaces isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$ that are subspaces of the interior polar space on $\mathbf{y}^\perp$. These subspaces live in hyperlines of the projective space $\mathbb{P}_{2n-1}(\mathbb{F})$. Therefore the intersection $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ is a subspace of the interior polar space on $\mathbf{y}^\perp$ living in a subspace of $\mathbb{P}_{2n-1}(\mathbb{F})$ of projective codimension at most three. The polar space on $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ can also be considered as a subspace of the interior polar space on $\mathbf{x}^\perp$. The intersection $\mathbf{w}^\perp \cap \mathbf{x}^\perp$ admits an interior polar space isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$, as above. Now $\mathbf{w}^\perp \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ can be considered as the intersection of the interior polar space on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$ with the polar space on $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. The dimensions of the projective spaces are at least $2n-3$ and $2n-5$, whence the dimension of the intersection is at least $2n-7 \geq n-1$, since $n \geq 6$. But the largest totally isotropic subspace of the interior projective space on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$ has (projective) dimension $n-2$ and we can find a hyperbolic line in $\mathbf{w}^\perp \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$.

Now suppose $n = 4$. In $\mathbf{x}^\perp$ the vertices $\mathbf{w}$ and $\mathbf{y}$ correspond to hyperbolic lines, which we denote by $w_\mathbf{x}$, respectively $y_\mathbf{x}$, their respective polars are denoted by $w_\mathbf{x}^\pi$ and $y_\mathbf{x}^\pi$. In $\mathbf{y}^\perp$, the hyperbolic lines induced by $\mathbf{x}$ and $\mathbf{z}$ are denoted by $x_\mathbf{y}$ and $z_\mathbf{y}$. The intersection of their polars $x_\mathbf{y}^\pi \cap z_\mathbf{y}^\pi$ translates to a subspace of $\mathbf{x}^\perp$, which we denote by $U_\mathbf{x}$. Note that $U_\mathbf{x} \subset y_\mathbf{x}^\pi$. There are three different cases: rank $w_\mathbf{x}^\pi \cap y_\mathbf{x}^\pi = 4$ and rank $U_\mathbf{x} = 4$; rank $w_\mathbf{x}^\pi \cap y_\mathbf{x}^\pi = 2$ and rank $U_\mathbf{x} = 4$; rank $w_\mathbf{x}^\pi \cap y_\mathbf{x}^\pi = 2$ and rank $U_\mathbf{x} = 2$. (We can omit the natural fourth case by reversing the labeling of the chain $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$.)

Let us assume we are in the second case. Then we can choose a point $p_\mathbf{x}$ in $U_\mathbf{x} \cap w_\mathbf{x}^\pi$ and a point $q_\mathbf{x}$ in $U_\mathbf{x} \backslash w_\mathbf{x}^\pi$ such that $p_\mathbf{x} q_\mathbf{x}$ has rank two. Picking linearly independent vectors $v_1, v_2 \in w_\mathbf{x}$, $v_3 \in p_\mathbf{x}$, $v_4 \in q_\mathbf{x}$ we obtain the following Gram matrix with respect to $v_1$, $v_2$, $v_3$, $v_4$

$$\begin{pmatrix} 0 & \alpha & 0 & \gamma \\ -\alpha & 0 & 0 & \delta \\ 0 & 0 & 0 & \beta \\ -\gamma & -\delta & -\beta & 0 \end{pmatrix},$$

whose determinant is independent of $\gamma$ and $\delta$. Moreover, as $\alpha \neq 0 \neq \beta$, the matrix $A$ has full rank. Notice that the hyperbolic line $p_\mathbf{x} q_\mathbf{x}$ corresponds to a vertex $\mathbf{l}$ adjacent to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, so that the chain $\mathbf{w} \perp \mathbf{x} \perp \mathbf{l} \perp \mathbf{z}$ belongs to the first case and is homotopically equivalent to $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$.

Let us assume we are in the third case. If $U_\mathbf{x} \cap w_\mathbf{x}^\pi$ is not contained in the radical of $U_\mathbf{x}$, then an argument as in the above paragraph yields a vertex $\mathbf{l}$ adjacent to $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ with rank $\langle w_\mathbf{x}, l_\mathbf{x} \rangle = 4$, and after reversing the labelling of the path we are in the second case. Hence let us assume that $U_\mathbf{x} \cap w_\mathbf{x}^\pi$ is contained in the radical of $U_\mathbf{x}$ and in the radical of $w_\mathbf{x}^\pi \cap y_\mathbf{x}^\pi$. Choose $l_\mathbf{x}$ to be any hyperbolic line in $U_\mathbf{x} \backslash w_\mathbf{x}^\pi$. Then $l_\mathbf{x}^\pi$ contains the radical $R$

of $U_{\mathbf{x}}$. On the other hand, the span $\langle w_{\mathbf{x}}, l_{\mathbf{x}} \rangle$ does not intersect $R$. The hyperbolic line $l_{\mathbf{x}}$ corresponds to a vertex $\mathbf{l}$ adjacent to $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Replacing the chain $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ by the chain $\mathbf{w} \perp \mathbf{x} \perp \mathbf{l} \perp \mathbf{z}$, we are in the case of the first part of this paragraph.

Consider the case rank $w_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi} = 4$ and rank $U_{\mathbf{x}} = 4$. Choose a hyperbolic line $l_{\mathbf{x}}$ in $U_{\mathbf{x}}$ and let it vary through all hyperbolic lines in $U_{\mathbf{x}}$. Then $l_{\mathbf{x}}^{\pi}$ varies around the space $U_{\mathbf{x}}^{\pi}$. One can show that for some choice of $l$, this hyperbolic line corresponds to a vertex $\mathbf{l}$ adjacent to $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, such that $\mathbf{w}^{\perp} \cap \mathbf{x}^{\perp} \cap \mathbf{l}^{\perp} \cap \mathbf{z}^{\perp} \neq \emptyset$.

Finally suppose $n = 5$. Use the notation from case $n = 4$ and suppose that $w_{\mathbf{x}}$ does not intersect $y_{\mathbf{x}}$, so that $\langle w_{\mathbf{x}}, y_{\mathbf{x}} \rangle$ has (projective) dimension three whereas its polar $w_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$ has dimension five. By arguments as in case $n = 4$ we can assume that $w_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$ is nondegenerate. But then $U_{\mathbf{x}} \cap w_{\mathbf{x}}^{\pi}$ cannot be totally isotropic (it has too large dimension inside $w_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$) and we are done. On the other hand, if $w_{\mathbf{x}}$ does intersect $y_{\mathbf{x}}$, then $U_{\mathbf{x}} \cap w_{\mathbf{x}}^{\pi}$ cannot be totally isotropic (it has too large dimension inside $w_{\mathbf{x}}^{\pi}$) and we are done as well.

For the other claims, let $\mathbf{w}$ and $\mathbf{z}$ be arbitrary vertices of $\Gamma$. Since $\Gamma$ is connected, there exists a path from $\mathbf{w}$ to $\mathbf{z}$. Induction on the length of such a path yields diameter two. A similar induction with $\mathbf{w} = \mathbf{z}$ yields simple connectedness. □

**Lemma 4.2**
*Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be vertices of $\Gamma$ and let $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$, $\mathbf{l}_4$ be vertices of $\mathbf{y}^{\perp}$ with $\mathbf{l}_1 \perp \mathbf{x} \perp \mathbf{l}_2$ and $\mathbf{l}_3 \perp \mathbf{z} \perp \mathbf{l}_4$. Moreover, assume that the $\mathbf{l}_i$, $1 \leq i \leq 4$, are contained in a common interior point of $\mathbf{y}^{\perp}$. Then there exist vertices $\mathbf{l}_5, \mathbf{l}_6 \in \mathbf{x}^{\perp} \cap \mathbf{z}^{\perp}$ such that $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_5$, $\mathbf{l}_6$, respectively $\mathbf{l}_3$, $\mathbf{l}_4$, $\mathbf{l}_5$, $\mathbf{l}_6$ are contained in a common interior point of $\mathbf{x}^{\perp}$, respectively $\mathbf{z}^{\perp}$.*

*Proof.* Left to the reader as an exercise. (Observe first that $\mathbf{x}^{\perp} \cap \mathbf{l}_1^{\perp} \cap \mathbf{y}^{\perp} \cap \mathbf{l}_2^{\perp} \neq \emptyset$, and hence reduce the problem to the case $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$. Then conduct local analysis of $\mathbf{y}^{\perp}$.) □

**Definition 4.3** Let $\mathbf{x}$, $\mathbf{y}$ be vertices of $\Gamma$ and let $p_{\mathbf{x}}$, $q_{\mathbf{y}}$ be local points of $\mathbf{x}^{\perp}$, respectively $\mathbf{y}^{\perp}$. We define $p_{\mathbf{x}} \approx^p q_{\mathbf{y}}$ if and only if there exist distinct vertices $\mathbf{l}_1, \mathbf{l}_2 \in p_{\mathbf{x}} \cap q_{\mathbf{y}}$. By the preceding lemma, the relation $\approx^p$ is an equivalence relation on the set of all local points. An equivalence class of $\approx^p$ is called a **global point**. Note that the intersection of a global point $p$ with a perp $\mathbf{x}^{\perp}$ is either empty or a local point $p_{\mathbf{x}}$. Denote set of global points of $\Gamma$ by $\mathcal{P}_{\Gamma}$. Additionally, let $\mathcal{H}_{\Gamma}$ be a copy of the set of vertices of $\Gamma$.

**Proposition 4.4**
*$(\mathcal{P}_{\Gamma}, \mathcal{H}_{\Gamma})$ is a connected partial linear space.*

*Proof.* Let $p$ and $q$ be two global points. Fix a vertex in each point, $\mathbf{p}$ and $\mathbf{q}$, say. By Lemma 4.1, there exists a vertex $\mathbf{x}$ adjacent to both $\mathbf{p}$ and $\mathbf{q}$. Hence there exist local counterparts $p_{\mathbf{x}}$ and $q_{\mathbf{x}}$. Connectedness of $(\mathcal{P}_{\Gamma}, \mathcal{H}_{\Gamma})$ now follows from connectedness of the interior hyperbolic space on $\mathbf{x}^{\perp}$. Moreover, two global points $p$, $q$ cannot intersect in more than one vertex, whence $(\mathcal{P}_{\Gamma}, \mathcal{H}_{\Gamma})$ is a partial linear space. For, if two global points would intersect in two vertices $\mathbf{x}$, $\mathbf{y}$, then there exists a vertex $\mathbf{z}$ adjacent to both $\mathbf{x}$, $\mathbf{y}$ by

Lemma 4.1. But then $p \cap \mathbf{z}^\perp$ and $q \cap \mathbf{z}^\perp$ are two local points that intersect in two vertices, a contradiction. □

**Proposition 4.5**
*The space* $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ *is isomorphic to the geometry of hyperbolic lines of a symplectic polar space* $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ *embedded in some projective space of dimension at least* 3.

*Proof.* Let $\mathbf{l}$ and $\mathbf{m}$ be two intersecting hyperbolic lines of $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$. By Lemma 4.1, there exists a vertex $\mathbf{k}$ of $\Gamma$ adjacent to both $\mathbf{l}$ and $\mathbf{m}$. Local analysis of $\mathbf{k}^\perp$ (or rather the interior hyperbolic space on it) shows that the intersecting lines $\mathbf{l}$ and $\mathbf{m}$ are contained in a dual affine plane. Certainly, $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ contains two such planes. If $\mathbb{F} \neq \mathbb{F}_2$, the space $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ contains a line with more than three points and the claim follows from Fact 2.3 with Proposition 4.4. If $\mathbb{F} = \mathbb{F}_2$, we can invoke Fact 2.1 and Fact 2.2. It remains to show that the geometry $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ does not belong to Cases (ii) or (iii) of Fact 2.2. Case (ii) is easily excluded, as locally all symplectic points occur, not only a subset of the symplectic points. Case (iii) is a bit more difficult. However, by the second statement of Fact 2.2, we obtain $\Omega' = \emptyset$. Indeed, for any pair $x$, $y$ of points of $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$, we find hyperbolic lines $\mathbf{l}$ incident with $x$ and $\mathbf{m}$ incident with $y$. By Lemma 4.1 there exists a hyperbolic line $\mathbf{k}$ that is adjacent to both $\mathbf{l}$ and $\mathbf{m}$ in $\Gamma$. Therefore, we can consider $x$, $y$ in some local space isomorphic to $\mathbb{W}_{2n}(\mathbb{F})$. But if $x \neq y$, then we find a point that lies on a common hyperbolic line with $x$, but not with $y$. Hence, in $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$, the equality $\{y\} \cup \{p \in \mathcal{P} \mid p \ni \mathbf{l}, x \ni \mathbf{l} \in \mathcal{L}\} = \{x\} \cup \{p \in \mathcal{P} \mid p \ni \mathbf{l}, y \ni \mathbf{l} \in \mathcal{L}\}$ implies $x = y$, and $\Omega' = \emptyset$. It follows from the size of $n$ that $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ cannot belong to Case (iii) either. The proposition is proved. □

**Proposition 4.6**
*The hyperbolic line graph of* $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ *is isomorphic to* $\Gamma$.

*Proof.* By definition the elements of $\mathcal{H}_\Gamma$ are precisely the vertices of $\Gamma$. The preceding proposition tells us that the elements of $\mathcal{H}_\Gamma$ are also precisely the hyperbolic lines of the symplectic space $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$, and we have a natural bijection between the hyperbolic lines of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ and the vertices of $\Gamma$, which preserves adjacency. □

**Proposition 4.7**
*The space* $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ *is isomorphic to the symplectic polar space* $\mathbb{W}_{2n+2}(\mathbb{F})$.

*Proof.* By Proposition 4.6, the hyperbolic line graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is isomorphic to $\Gamma$. Since $\Gamma$ is locally $\mathbf{S}_{2n}(\mathbb{F})$, this means for any hyperbolic line $\mathbf{l}$ that the subspace of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ consisting of all points collinear with all points of $l$ is isomorphic to $\mathbb{W}_{2n}(\mathbb{F})$. But the only symplectic polar space with that property is $\mathbb{W}_{2n+2}(\mathbb{F})$. The claim follows. □

Theorem 1 is now proved.

# 5 Perp spaces

In this section we will prove Theorems 2 and 3. Actually, we will give a proof of Statement 4 of the author's PhD thesis, which immediately implies Theorem 2 and Theorem 3 by our findings in earlier chapters.

Let us start with some definitions. From the introduction we recall that a **perp space** is a partial linear space $(\mathcal{P}, \mathcal{L})$ endowed with a symmetric relation $\perp \subseteq \mathcal{P} \times \mathcal{P}$ such that for every point $x$, whenever $p \neq q$ are points on a line $l$, the fact $x \perp p$ and $x \perp q$ implies $x \perp y$ for all $y \in l$. We set $l \perp m$, $l, m \in \mathcal{L}$ if $p \perp q$ for all $p \in l$, $q \in m$; similarly define $X \perp Y$ for other subsets $X$, $Y$ of $\mathcal{P}$. Moreover, in view of the following theorem, a graph $\Gamma$ is called **locally connected** if, for every any $\mathbf{x} \in \Gamma$, the induced subgraph $\Gamma(\mathbf{x})$ on all neighbors of $\mathbf{x}$ in $\Gamma$, is connected.

**Theorem 5.1**
*Let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space without isolated points and with $l^\perp \cong k^\perp$ (as perp spaces) for all $k, l \in \mathcal{L}$ such that*

(i) *the induced graph $(\mathcal{L}, \perp)$ is locally connected and locally recognizable;*

(ii) *the diameter of $(\mathcal{L}, \perp)$ is two; and*

(iii) *for all lines $k \perp l$, every point in the space $\{k, l\}^\perp$ is uniquely determined by its line pencil in $\{k, l\}^\perp$.*

*Then $(\mathcal{P}, \mathcal{L}, \perp)$ can be characterized, as a perp space, by the structure of the spaces $l^\perp$, $l \in \mathcal{L}$.*

*Proof.* Let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space satisfying the hypothesis of the theorem. The graph $(\mathcal{L}, \perp)$ is locally homogeneous, because $k^\perp \cong l^\perp$ for all $k, l \in \mathcal{L}$. Moreover, $(\mathcal{L}, \perp)$ is locally recognizable and connected (indeed, its diameter is two), so $(\mathcal{L}, \perp)$ is uniquely determined up to isomorphism. It suffices to recover the point set $\mathcal{P}$ and the incidence relation between points and lines. Let $p$ be any point of $(\mathcal{P}, \mathcal{L}, \perp)$. Then, as $p$ is not isolated, there exists a line $l$ containing $p$. Choose any line $k \perp l$, and we have $k \perp p$. Hence any point $p$ in $\mathcal{P}$ actually occurs in some local space $k^\perp$. Converely, let $p \in l^\perp$ and $q \in m^\perp$ be points in distinct local spaces. There is at most one possible isomorphism type for $(\mathcal{P}, \mathcal{L}, \perp)$, if it is determined, whether $p$ and $q$ actually describe the same point of $\mathcal{P}$ or not. Choose lines $a \ni p$ and $b \ni q$. Since the diameter of $(\mathcal{L}, \perp)$ is two, there exists a line $c$ adjacent to $a$ and $b$. Notice that $c \perp a$ implies $c \perp p$ and that $c \perp b$ implies $c \perp q$. But in $c^\perp$ the points $p$ and $q$ either coincide or they do not coincide. Either way it is determined whether $p$ and $q$ should be identical or distinct points of $\mathcal{P}$. Hence there exists at most one isomorphism type of perp spaces with a given local structure as in the hypothesis. We are done by the assumption of the existence of such a perp space. □

*Proof of Theorem 2.* Let $(\mathcal{P}, \mathcal{L}, \perp)$ be isomorphic to the hyperbolic symplectic geometry of $\mathbb{W}_{2n+2}(\mathbb{F})$. The graph $(\mathcal{L}, \perp)$ has diameter two and is locally connected by Proposition 3.3; it is locally recognizable by Theorem 1. The space $(\mathcal{P}, \mathcal{L}, \perp)$ also satisfies Hypothesis (iii) of Theorem 5.1 (by the reconstruction of the symplectic space from the hyperbolic line graph in Section 3). Application of the above theorem finishes the proof. □

Theorem 3 is just a rephrasing of Theorem 2.

# 6 Open problems

As indicated in the introduction, the graph on the commuting fundamental $SL_2$'s of a group of type $F_4(\mathbb{F})$ is also (connected and) locally $\mathbf{S}_6(\mathbb{F})$. It would be interested to extend the dimension of Theorem 1 to include a complete listing of all connected, locally $\mathbf{S}_6(\mathbb{F})$ graphs. It is my belief that the following is true:

**Conjecture**
*Let $\mathbb{F}$ be a field and let $\Gamma$ be a connected, locally $\mathbf{S}_6(\mathbb{F})$ graph. Then $\Gamma$ is isomorphic to $\mathbf{S}_8(\mathbb{F})$ or the graph on the commuting fundamental $SL_2$'s of $F_4(\mathbb{F})$.*

The reason for this conjecture is the following: first of all the diameter of $\Gamma$ should be quite small; it cannot have diameter two, however, as the graph coming from $F_4(\mathbb{F})$ is a counterexample. Then one should be able to conduct a case-by-case analysis depending on the diameter of $\Gamma$. For diameter two, $\Gamma$ should be isomorphic to $\mathbf{S}_8(\mathbb{F})$, as one should be able to control all planes.

# References

[1] A.M. Cohen, H. Cuypers, R. Gramlich, Local recognition of non-incident point-hyperplane graphs, submitted.

[2] H. Cuypers, Symplectic geometries, transvection groups, and modules, *J. Comb. Theory Ser. A* **65** (1994), pp. 39 – 59.

[3] H. Cuypers, Nonsingular points of polarities, internal report, Technische Universiteit Eindhoven 1999; in preparation as an article.

[4] R. Gramlich, *On Graphs, Geometries, and Groups of Lie Type*, PhD thesis, Technische Universiteit Eindhoven 2002.

[5] R. Gramlich, Line-hyperline pairs of projective spaces and fundamental subgroups of linear groups, submitted.

[6] J.I. Hall, The hyperbolic lines of finite symplectic spaces, *J. Combin. Theory Ser. A* **47** (1988), pp. 284 – 298.

[7] J.I. Hall, Graphs, geometry, 3-transpositions, and symplectic $\mathbb{F}_2$-transvection groups, *Proc. London Math. Soc.* **58** (1989), pp. 89 – 111.

[8] F.G. Timmesfeld, *Abstract root subgroups and simple groups of Lie type*, Birkhäuser, Basel 2001.

Address of the author:
TU Darmstadt
Fachbereich Mathematik / AG 5
Schloßgartenstraße 7
64289 Darmstadt

Germany

e-mail: `gramlich@mathematik.tu-darmstadt.de`