Curtis-Phan-Tits Theory

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Abstract

In the present article we demonstrate that there is a relation between the Curtis-Tits theorem and Phan's theorems that goes well beyond the similarity in appearance. In particular, we present a geometric construction connecting those theorems and suggesting that Phan's theorems can be thought of as "twisted versions" of the Curtis-Tits theorem. The construction itself further suggests that Phan's theorems are only some of many possible such theorems. We make this explicit by presenting a new Phan-type theorem for the symplectic groups.

The work discussed in this article began as an attempt to provide a complete and clear proof of Phan's first theorem, together with a desire for a more geometric proof of the Curtis-Tits theorem. We were surprised, however, to find that our construction led to a unifying point of view on these two theorems, an unexpected bonus. Another remarkable observation is that the geometric constructions do not seem to depend on the finiteness of the field or the sphericality of the diagram. So the present article may be of interest not only to finite geometers and finite group-theorists but also to people interested in nonspherical twin buildings and Kac-Moody groups.

1 Introduction

Geometric methods in (finite) group theory have made tremendous advances since the classification of finite simple groups was first announced by Gorenstein. These methods have proven fruitful in simplifying some of the arguments needed in the classification. In particular, the connection between the universal cover of certain geometries and the universal completion of related amalgams can be used to simplify the identification of groups, considered as groups of automorphisms of these geometries.

An important step of the classification of finite simple groups as well as the revision of the classification, pursued by Gorenstein, Lyons, Solomon and others, is the identification of the "minimal counterexample" with one of the known simple groups. This follows the step of the local analysis. At this step, inside the minimal counterexample G one reconstructs one or more of the proper subgroups using the inductive assumption and available techniques. Thus the initial point of the identification is a set of subgroups of G that resemble the subgroups of a central extension $\hat{G}$ of some known simple group (referred to as the *target group*). The output of the identification step is the statement that G is isomorphic to a quotient of $\hat{G}$. Two of the most widely used identification tools are the Curtis-Tits theorem and Phan's first theorem.

The Curtis-Tits theorem allows the identification of G with a quotient of a universal Chevalley group $\hat{G}$ of twisted or untwisted type provided that G contains a system of subgroups identical to the system of appropiately chosen rank two Levi factors of $\hat{G}$. In particular in the case of the diagram A_n, the system in question consists of all the groups $SL(3,q)$ and $SL(2,q) \times SL(2,q)$ lying in $\hat{G}$ block-diagonally.

Phan's first theorem deals with the case $\hat{G} = SU(n+1,q^2)$ and the system of block-diagonal subgroups $SU(3,q^2)$ and $SU(2,q^2)\times SU(2,q^2)$. Thus Phan's theorem appears to be similar to the Curtis-Tits theorem in the case A_n. However, unlike the case of A_n, the block-diagonal $SU(3,q^2)$ and $SU(2,q^2) \times SU(2,q^2)$ are not Levi factors in $SU(n+1,q^2)$, so Phan's theorem is not a special case of the Curtis-Tits theorem.

It is one of the purposes of this paper to demonstrate that the relation between the Curtis-Tits theorem and Phan's theorem goes beyond a simple similarity in appearance. To this end we present a geometric construction revealing a deeper connection between these theorems. Our construction suggests that Phan's theorem is a "twisted" version of the Curtis-Tits theorem. Furthermore, from the point of view of this construction, Phan's theorems are just some of many possible Phan-type theorems. We stress this point by presenting a new such theorem dealing with the case $\hat{G} = Sp(2n,q)$ and a system of semisimple subgroups of rank two which again are not Levi factors (cf. [GHSh]). Moreover, the presented methods do not seem to depend on the finiteness of the field or the sphericality of the diagram. In fact, there already exists a Curtis-Tits-type theorem for certain Kac-Moody groups (cf. [M]), and we believe it to be an interesting problem to prove a Phan-type theorem for a suitable Kac-Moody group.

The structure of the paper is as follows. In Section 2 we introduce some notions from the areas of synthetic geometry, chamber systems and amalgams of groups. In Section 3 we discuss the proof of Phan's theorem from [BSh]. In Section 4 we introduce the language of buildings and twin buildings and present an overview of Mühlherr's geometric proof of the Curtis-Tits theorem from [M]. Finally in Section 5 we present our construction, discuss the new Phan-type theorem for $Sp(2n,q)$ from [GHSh], and propose more examples. Along the way we pose a number of open problems.

2 Geometries and amalgams

2.1 Geometries

A *pregeometry* over I is a set of elements Γ together with a type function t and a reflexive and symmetric incidence relation $\sim$. The type function maps Γ onto the type set I, and for any two elements $x, y \in \Gamma$ with $x \sim y$ and $t(x) = t(y)$ we have $x = y$. A *flag* in Γ is a set of pairwise-incident elements. Notice that the type function injects any flag into the type set. A *geometry* is a pregeometry such that t induces a bijection between any maximal flag of Γ and I.

The *residue* $\mathrm{res}_\Gamma(F)$ of a flag F in a geometry Γ is the set of elements from $\Gamma \setminus F$ that are incident to all elements of F. It follows that the residue $\mathrm{res}_\Gamma(F)$ is a geometry with the type set $I \setminus t(F)$. The *rank* of the geometry Γ is the cardinality of its type set I. We will only consider the case where I is finite. The rank of the residue of a flag F is called the *corank* of F. The geometry Γ is *connected* if the graph with vertex set Γ and edges given by $\sim$ is connected. In what follows all the geometries that we consider are connected. The geometry Γ is *residually connected* if the residue in Γ of every flag of corank at least 2 is connected. (Note that the property that the residue of every flag of corank 1 be non-empty—which usually is required as well for the definition of residual connectedness—follows from the fact that Γ is a geometry.)

An automorphism of a geometry Γ is a permutation of its elements that preserves type and incidence. The group of all automorphisms of Γ will be denoted by $\mathrm{Aut}\,\Gamma$. A subgroup $G \leq \mathrm{Aut}\,\Gamma$ acts *flag-transitively* on Γ if it is transitive on the set of maximal flags (strictly speaking *chamber-transitively*, but the notions of flag-transitivity and chamber-transitivity coincide for geometries). A geometry that possesses a flag-transitive automorphism group is also called flag-transitive.

A *parabolic subgroup* (or simply a *parabolic*) of G is the stabilizer in G of a non-empty flag F of Γ. The *rank* of the parabolic is the corank of F.

2.2 Simplicial complexes

A *simplicial complex* $\mathcal{S}$ is a pair (X, Δ) where X is a set and Δ is a collection of subsets of X such that if $A \in \Delta$ and $B \subset A$ then $B \in \Delta$. The subsets from Δ are called *simplices*.

A *morphism* from a complex $\mathcal{S} = (X, \Delta)$ to a complex $\mathcal{S}' = (X', \Delta')$ is a map between X and X' which takes simplices to simplices. The *star* of a simplex $A \in \Delta$ is the set of subsets $B \in \Delta$ such that $A \subseteq B$. A *covering* is a surjective morphism ϕ from $\mathcal{S}$ to $\mathcal{S}'$ such that for every $A \in \Delta$, the function ϕ maps the star of A bijectively onto the star of $\phi(A)$.

A *path* on a complex $\mathcal{S}$ is a sequence $x_0, x_1, \ldots, x_n$ of elements of X such that x_{i-1} and x_i are contained in a simplex for all $i = 1, \ldots, n$. We do not allow repetitions so $x_{i-1} \neq x_i$ for all i. The complex $\mathcal{S}$ is *connected* if every two elements of X can be connected by a path. The following two operations are called *elementary homotopies*: (a) substituting a subsequence x, y, x (a return) by just x, or (b) substituting a subsequence x, y, z, x (a triangle) by x, provided that x, y, z are all in the same simplex. Two paths are *homotopically equivalent* if they can be obtained from one another in a finite sequence of elementary homotopies. A *loop* is a closed path, that is, a path with $x_0 = x_n$. A loop is called *null-homotopic* if it is homotopically equivalent to the trivial path x_0. The *fundamental group* $\pi_1(\mathcal{S}, x)$ where $x \in X$ is the set of equivalence classes of loops based at x with respect to homotopy. The product is defined to be concatenation of loops. Notice that the fundamental group is independent of the choice of the base vertex x inside a fixed connected component. On the other hand, the fundamental group may vary for base vertices in distinct connected components. The coverings of $\mathcal{S}$, taken up to a certain natural equivalence, correspond bijectively to the subgroups of $\pi_1(\mathcal{S}, x)$. A connected complex $\mathcal{S}$ is called *simply connected* if it has no proper coverings, or, equivalently, if $\pi_1(\mathcal{S}, x) = 1$.

With every geometry Γ one can associate its *flag complex*. This is a simplicial complex defined on the set Γ whose simplices are the flags of Γ. We will say that Γ is *simply connected* if such is its flag complex.

2.3 Chamber systems

A *chamber system* over a type set I is a set $\mathcal{C}$ called the set of chambers, together with equivalence relations $\sim_i$, $i \in I$. For $i \in I$ and chambers $c, d \in \mathcal{C}$, we say that c and d are *i-adjacent* if $c \sim_i d$. The chambers c, d are *adjacent* if they are i-adjacent for some $i \in I$.

A chamber system is $\mathcal{C}$ is called *thick* if for every $i \in I$ and every chamber $c \in \mathcal{C}$, there are at least three chambers (c and two other chambers) i-adjacent to c. A chamber system is called *thin* if c is i-adjacent to exactly two chambers (itself and one other chamber) for all $i \in I$ and $c \in \mathcal{C}$.

If Γ is a geometry with type set I then one can construct a chamber system $\mathcal{C} = \mathcal{C}(\Gamma)$ over I as follows. The chambers are the maximal flags of Γ. Two maximal flags are i-adjacent if and only if they contain the same element of type j for all $j \in I \setminus \{i\}$. A chamber system is called *geometric* if it can be obtained in this way.

If Γ is residually connected, it can be recovered from the associated chamber system $\mathcal{C}(\Gamma)$ as follows: For $J \subseteq I$, a *J-cell* is an equivalence class of the minimal equivalence relation containing the relations $\sim_i$ for all $i \in J$. The poset of all cells ordered by reverse inclusion is naturally isomorphic to the poset of the flags of Γ ordered by inclusion. Under this isomorphism the cell corresponding to a flag F consists of all chambers (maximal flags) containing F. In particular the elements of type i of Γ will correspond to the $(I \setminus \{i\})$-cells.

2.4 Amalgams of groups

An *amalgam of groups* is a set $\mathcal{A} = \bigcup_{i \in I} G_i$ with a partial operation of multiplication such that

(A1) the restriction of the multiplication to every G_i makes G_i a group;

(A2) the product ab is defined if and only if $a, b \in G_i$ for some $i \in I$; and

(A3) $G_i \cap G_j$ is a subgroup of G_i and G_j for all $i, j \in I$.

A *completion* of an amalgam $\mathcal{A}$ is a group G together with a mapping ϕ from $\mathcal{A}$ to G such that the restriction of ϕ to every G_i is a homomorphism and $\phi(\mathcal{A})$

generates G. The universal completion of $\mathcal{A}$ is the group $U(\mathcal{A})$ with generators $\{t_s \mid s \in \mathcal{A}\}$ and relations $t_x t_y = t_{xy}$ whenever $x, y \in G_i$ for some i. The corresponding mapping is given by $x \mapsto t_x$. This is indeed the universal object in the category of completions of $\mathcal{A}$. We will normally identify the completion with just the group G and in this sense every completion is a quotient of the universal completion $U(\mathcal{A})$.

In terms of amalgams, the identification problem (see the introduction) amounts to finding the universal completions of certain amalgams arising in Chevalley groups. The result that connects amalgams and their completions with geometries is a lemma due to Jacques Tits. In the remainder of this section we will discuss this important result.

2.5 Tits' lemma

Given a geometry Γ and a flag-transitive group $G \leq \operatorname{Aut}\Gamma$, we can associate with them an amalgam $\mathcal{A}$ as follows. Let F be a maximal flag of Γ. Then $\mathcal{A} = \bigcup_{i \in I} G_i$ where G_i is the stabilizer in G of the element of type i from F. This $\mathcal{A}$ is called the *amalgam of maximal parabolics*. Notice that $\mathcal{A}$ is independent of the choice of F if we consider it up to isomorphism. If Γ is connected then $\mathcal{A}$ generates G so that G is a completion of $\mathcal{A}$.

Proposition 2.1 (Tits' lemma, Corollaire 1 of [T1]). *Let Γ be a connected geometry and let $G \leq \operatorname{Aut}\Gamma$ be a flag-transitive group of automorphisms. Moreover, let F be a maximal flag of Γ. Then G is the universal completion of the amalgam $\mathcal{A}$ of maximal parabolics with respect to F if and only if the geometry Γ is simply connected.*

This result reduces the problem of identifying the universal completion of certain amalgams to proving that the corresponding geometries are simply connected. As we have mentioned above, simple connectedness can be verified by proving that the fundamental group of the corresponding flag complex is trivial, that is, proving that every loop on that complex is null-homotopic.

3 Phan's theorem

3.1 History

The first of the identification theorems we shall discuss is Phan's first theorem. In 1975, Kok-Wee Phan gave a method for identifying an unknown group G as a quotient of the unitary group $SU(n+1, q^2)$ by finding a generating configuration of subgroups $SU(3, q^2)$ and $SU(2, q^2) \times SU(2, q^2)$ in G. We begin by looking at the configuration of subgroups in $SU(n+1, q^2)$ to motivate our later definition.

Suppose $n \geq 2$ and suppose q is a prime power. Consider $G = SU(n+1, q^2)$ and let $U_i \cong SU(2, q^2)$, $i = 1, 2, \ldots, n$, be the subgroups of G corresponding to the (2×2)-blocks along the main diagonal. Let D_i be the diagonal subgroup in U_i. Notice that D_i is a maximal torus of U_i of size $q+1$. When $q \neq 2$, the group G is generated by the subgroups U_i, and the following hold:

(P1) if $|i-j|>1$ then $[x,y]=1$ for all $x \in U_i$ and $y \in U_j$;

(P2) if $|i-j|=1$ then $\langle U_i, U_j \rangle$ is isomorphic to $SU(3,q^2)$; and

(P3) $[x,y]=1$ for all $x \in D_i$ and $y \in D_j$,

for $1 \leq i,j \leq n$. Suppose now G is an arbitrary group containing a system of subgroups $U_i \cong SU(2,q^2)$, and suppose a maximal torus D_i of size $q+1$ is chosen in each U_i. If the conditions (P1)–(P3) above hold true for G, we will say that G contains a *Phan system of rank n*. In [Ph1] Kok-Wee Phan proved the following result:

Theorem 3.1. *If G contains a Phan system of rank n at least two with $q>4$, then G is isomorphic to a factor group of $SU(n+1,q^2)$.*

Phan's proof of this result, however, is somewhat incomplete. Much of the proof is calculation-based, and many of these calculations are left to the reader. Moreover, while Phan apparently deals with the question of what the Phan system generates if the amalgam $\mathcal{A}$ formed by the subgroups $U_{ij} = \langle U_i, U_j \rangle$ is exactly as in $SU(n+1,q^2)$, he never addresses the question of the uniqueness of $\mathcal{A}$. Unfortunately, this is crucial. Indeed nothing in the conditions (P1)–(P3) tells us right away that $\mathcal{A}$ must be as in $SU(n+1,q^2)$. Potentially there may be many such amalgams and then G can be a quotient of the universal completion of any one of those amalgams. Thus the proof of the uniqueness of $\mathcal{A}$ must be an important part of the proof of Phan's theorem.

3.2 Strategy

Let us assume for now that the uniqueness of $\mathcal{A}$ is known so that $\mathcal{A}$ can be identified with the amalgam formed by block-diagonal subgroups $SU(3,q^2)$ and $SU(2,q^2) \times SU(2,q^2)$ of $\hat{G} = SU(n+1,q^2)$. Under this assumption, what remains to be shown is that the universal completion of $\mathcal{A}$ coincides with $\hat{G}$. A natural way to show this is via Tits' lemma.

In order to apply Tits' lemma we need a geometry on which G acts flag-transitively, so that $\mathcal{A}$ is (or at least, is related to) the corresponding amalgam of maximal parabolics. Such a geometry has, in fact, already appeared in the literature (e.g. see [A]). This geometry, $\mathcal{N} = \mathcal{N}(n+1,q^2)$, is defined as follows. Let V be the $(n+1)$-dimensional unitary space over $GF(q^2)$. The elements of $\mathcal{N}$ are the proper non-singular subspaces U of V. The type of U is its dimension; incidence is defined by containment. Fixing an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ in V, we make $\hat{G}$ act on $\mathcal{N}$, and it is easy to see that this action is flag-transitive. The next key fact is that $\mathcal{N}$ is almost always simply connected. We defer the exact statement and a discussion of the proof until the next subsection. For now let us just mention that the case where $q>3$ is odd was first proven in [D].

Once $\mathcal{N}$ is known to be simply connected, Tits' lemma implies that $\hat{G}$ is the universal completion of the amalgam $\hat{\mathcal{A}}$ of maximal parabolics associated with

$\mathcal{N}$. Choosing the maximal flag consisting of all the subspaces $U_i = \langle e_1, \ldots, e_i \rangle$, the amalgam $\hat{\mathcal{A}}$ is a union of block-diagonal subgroups

$$(GU(i, q^2) \times GU(n + 1 - i, q^2))^+.$$

(The plus indicates that within this direct product we only take matrices with determinant equal to one.) In particular, $\mathcal{A}$ is completely contained in $\hat{\mathcal{A}}$. Unfortunately, however, $\mathcal{A}$ is not equal to $\hat{\mathcal{A}}$. Consequently we have to do more work.

Let G be the universal completion of $\mathcal{A}$. Since $\mathcal{A}$ is contained in $\hat{\mathcal{A}}$ and generates $\hat{G}$ and, thus, the universal completion of $\hat{\mathcal{A}}$ is a completion of $\mathcal{A}$, the group $\hat{G} \cong SU(n + 1, q^2)$ is a quotient of G. Thus it suffices to show that G cannot be larger than $\hat{G}$. We accomplish this by extending $\mathcal{A}$ to a copy of $\hat{\mathcal{A}}$ inside G. (This implies that G is in turn a quotient of $\hat{G}$.)

Recall that in each U_i we have a torus D_i of order $q + 1$. Let $D = \prod D_i$ (we are working inside G). We show that D is in fact the direct product of the D_i's and that $U_{ij}D$ is isomorphic to the full rank 2 parabolic from $\hat{\mathcal{A}}$. Furthermore, the union of the subgroups $U_{ij}D$ in G produces an amalgam isomorphic to the full amalgam $\hat{\mathcal{A}}_2$ of rank 2 parabolics. The remaining part is easy, as we inductively extend every $\hat{\mathcal{A}}_s$ to $\hat{\mathcal{A}}_{s+1}$ using the case $s = 2$ as a base of induction. Notice that the simple connectedness of $\mathcal{N} = \mathcal{N}(s + 2, q)$ is used in extending $\hat{\mathcal{A}}_s$.

At this point we turn to the question of how the simple connectedness is proven.

3.3 Simple Connectedness

Recall that simple connectedness can be shown by proving that every loop of the flag complex of $\mathcal{N}$ is null-homotopic. Fixing a base point x to be a point (an element of type 1), a standard technique is to reduce every loop of $\mathcal{N}$ to a loop in the point-line incidence graph (lines are elements of type 2). This technique requires that the geometry in question contains sufficiently many connected residues, which is the case for the geometry $\mathcal{N}$.

Lemma 3.2. *Every loop starting at some point x is homotopic to a loop that is fully contained in the point-line incidence graph.*

Furthermore, every loop in the latter graph can be understood as a loop in the collinearity graph Σ of $\mathcal{N}$. The vertices of Σ are the points of $\mathcal{N}$ and two points are adjacent if and only if they are collinear (*i.e.*, incident to a common line).

A loop in Σ that is contained entirely within the residue of an element of $\mathcal{N}$ (such a loop is called *geometric*) is null-homotopic. Thus, proving that $\mathcal{N}$ is simply connected requires showing that every loop in Σ can be decomposed into a product of geometric loops. In fact, we only use geometric triangles for this.

The key fact that allows us to proceed is that, with few exceptions, Σ has diameter two. This gives us by induction that every loop in Σ is a product of loops of length up to five. Thus it suffices to show that every loop γ of length 3, 4, and 5 is null-homotopic. When n is large, one can always find a point that is perpendicular to all the points on γ. This produces a decomposition of γ into geometric triangles. Hence the claim is essentially obvious for large n. All the difficulty of the proof lies in the case of small n, where we resort to a case-by-case analysis and the proof at times becomes rather intricate.

We end this section with the exact statement.

Proposition 3.3. *The geometry $\mathcal{N} = \mathcal{N}(n+1, q^2)$ is simply connected if (n, q) is not one of $(3, 2)$ and $(3, 3)$.*

Our proof of this proposition is computer-free with the exception of the case $n = 5$ and $q = 2$, which was handled by Jon Dunlap using a Todd-Coxeter coset enumeration in GAP ([GAP]). Notice that neither one of the exceptions above is simply connected, so that the result is in a sense best possible.

3.4 Uniqueness of $\mathcal{A}$

Notice that Phan does not address the cases $q \leq 4$ at all. Furthermore his definitions do not even make sense for $q = 2$. We would like to include all possible cases in our theorem so we need to modify Phan's setup.

We say that a group G possesses a *weak Phan system* if G contains subgroups $U_i \cong SU(2, q^2)$, $i = 1, 2, \ldots, n$, and $U_{i,j}$, $1 \leq i < j \leq n$, so that the following hold:

(wP1) If $|i - j| > 1$ then $U_{i,j}$ is a central product of U_i and U_j;

(wP2) for $i = 1, 2, \ldots, n-1$, the groups U_i and U_{i+1} are contained in $U_{i,i+1}$, which is isomorphic to $SU(3, q^2)$ or $PSU(3, q^2)$; moreover, U_i and U_{i+1} form a standard pair in $U_{i,i+1}$; and

(wP3) the subgroups $U_{i,j}$, $1 \leq i < j \leq n$, generate G.

Here a *standard pair* in $SU(3, q^2)$ is a pair of subgroups $SU(2, q^2)$ conjugate as a pair to the two block-diagonal $SU(2, q^2)$'s. Standard pairs in $PSU(3, q^2)$ are defined as the images under the natural homomorphism of the standard pairs from $SU(3, q^2)$.

This definition leaves a lot of possibilities for the members of the amalgam $\mathcal{A} = \bigcup U_{ij}$. This produces a variety of amalgams and we are unable to make any claims of uniqueness in the general case. We call an amalgam $\mathcal{A}$ *unambiguous* if every U_{ij} is isomorphic to just $SU(3, q^2)$ or $SU(2, q^2) \times SU(2, q^2)$ (rather than a quotient of these groups). Using some "scissors-and-glue" methods, one can associate to every amalgam $\mathcal{A}$ of weak Phan type an unambiguous amalgam whose universal completion has $U(\mathcal{A})$ as a quotient. This reduces the analysis of $\mathcal{A}$ to the case where $\mathcal{A}$ is unambiguous. However even in this case we cannot claim uniqueness, and we have to impose another restriction. A *non-collapsing* amalgam is an amalgam such that $U(\mathcal{A}) \neq 1$ (this simple definition works in all cases except for $q = 2$; the latter case requires the stronger condition that every U_i embeds into $U(\mathcal{A})$). Clearly, from the point of view of Phan's theorem, we are only interested in the non-collapsing amalgams. It is interesting that although many unambiguous amalgams exist, only one of them is non-collapsing.

Proposition 3.4. *If $\mathcal{A} = \bigcup U_{ij}$ is unambiguous and non-collapsing, then it is isomorphic to the canonical amalgam of block-diagonal subgroups in $SU(n+1, q^2)$.*

We use the non-collapsing condition as follows. For $\epsilon = \pm 1$, define $D_i^\epsilon = N_{U_i}(U_{i+\epsilon})$. Note that this normalizer makes sense in $U_{i,i+\epsilon}$. Assuming that $\mathcal{A}$ is non-collapsing, we have a completion H in which every member of $\mathcal{A}$ embeds. Working in H we show that $D_i^{+1} = D_i^{-1}$ for all $i = 2, \ldots, n-1$. This extra condition makes $\mathcal{A}$ unique. It also enables us to introduce the tori $D_i = D_i^{+1} = D_i^{-1}$ as in Phan's original setup.

The main part of the uniqueness proof splits into the cases $n = 3$ and $n > 3$. In the first case we use Goldschmidt's Lemma 2.7 of [G] to prove that the amalgam of U_{12} and U_{23} with joint subgroup U_2 is unique up to isomorphism. To identify $\mathcal{A}$ we need to decide which subgroups of U_{12} and U_{23} can serve as U_1 and U_3. Once these subgroups are found, the remaining member U_{13} is added to $U_{12} \cup U_{23}$ as $U_1 \times U_3$.

The condition on U_1 and U_3 is that each must form a standard pair with U_2. It can be seen that U_2 acts transitively by conjugation on the candidates for U_1 and on candidates for U_3. Since conjugation by an element of U_2 is an automorphism of the amalgam $U_{12} \cup U_{23}$. Thus we can assume that U_1 is a fixed subgroup. We have many possibilities for U_3 thus leading to many amalgams. Fortunately we have the extra condition coming from our assumption that $\mathcal{A}$ is non-collapsing. This condition leaves only two candidates for U_3 and we complete the proof by finding an automorphism of $U_{12} \cup U_{23}$ that stabilizes U_1 and permutes the two candidates for U_3.

For the $n > 3$ case, we now appeal to induction using the case $n = 3$ as the base. In the end, combining all the above we obtain the following two theorems.

Theorem 3.5. *If G contains a weak Phan system of rank n at least three with $q > 3$, then G is isomorphic to a factor group of $SU(n+1, q^2)$.*

Theorem 3.6. *Suppose G contains a weak Phan system of rank n specified below with $q = 2$ or 3.*

(1) *Suppose $q = 3$, $n \geq 4$, and additionally, for $i = 1, 2, \ldots, n-2$, the subgroup generated by $U_{i,i+1}$ and $U_{i+1,i+2}$ is isomorphic to a factor group of $SU(4, 9)$. Then G is isomorphic to a factor group of $SU(n+1, 9)$.*

(2) *Suppose $q = 2$, $n \geq 5$ and, for $i = 1, 2, \ldots, n-3$, the subgroup generated by $U_{i,i+1}$, $U_{i+1,i+2}$ and $U_{i+2,i+3}$ is isomorphic to a factor group of $SU(5, 4)$. Then G is isomorphic to a factor group of $SU(n+1, 4)$.*

Notice that the extra conditions are due to the fact that, for $q \leq 3$ and small n, the geometry $\mathcal{N}$ is not simply connected. (And in one case, $n = 2$ and $q = 2$, it is not even connected.)

4 The Curtis-Tits theorem

The following formulation of the Curtis-Tits theorem is taken from [GLS].

Theorem 4.1. *Let G be the universal version of a finite Chevalley group of (twisted) rank at least 3 with root system Σ, fundamental system Π, and root groups X_α, $\alpha \in \Sigma$. For each $J \subseteq \Pi$ let G_J be the subgroup of G generated by all root subgroups X_α, $\pm\alpha \in J$. Let D be the set of all subsets of Π with at most 2 elements. Then G is the universal completion of the amalgam $\bigcup_{J \in D} G_J$.*

We first discuss the similarities and differences between Phan's theorem and the Curtis-Tits theorem. Let us consider the case of the Chevalley group of type A_n, which is $G = SL(n+1, q)$. With the usual choice of the root subgroups in G, the subgroups $G_J = G_{ij}$ are the block-diagonal subgroups $SL(3, q)$ and $SL(2, q) \times SL(2, q)$, which we note is similar to the amalgam in Phan's theorem. The main difference between the two theorems is that the Curtis-Tits theorem merely claims that the universal completion of the known amalgam (the one found in $SL(n+1, q)$, *i.e.*, $\bigcup_{J \in D} G_J$) is $SL(n+1, q)$, while Phan's theorem makes a claim about the completion of an arbitrary Phan amalgam.

Clearly, as we are again trying to find the universal completion of an amalgam, Tits' lemma appears to be a natural tool for this task. To use it, one needs to find a suitable geometry on which G acts flag-transitively with the correct amalgam of maximal parabolics, and then prove that the geometry is simply connected. We begin by modifying the amalgam so as to replace the rank 2 subgroups, G_J, with the maximal ones. Consider the amalgam $\mathcal{A} = \bigcup_{\alpha \in \Pi} G_{\Pi \setminus \{\alpha\}}$. By induction on the rank, the Curtis-Tits theorem is equivalent to the following.

Theorem 4.2. *Under the assumptions of Theorem 4.1, the group G is the universal completion of the amalgam $\mathcal{A}$.*

In the rest of this section we will discuss a geometric proof of this theorem given by Mühlherr in [M].

Recall that a finite Chevalley group G acts on its natural finite geometry called a building. Let I be a set and M be a Coxeter matrix over I. Let (W, S) be the Coxeter system of type M, where $S = \{s_i \mid i \in I\}$. A *building of type M* is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a set and $\delta : \mathcal{C} \times \mathcal{C} \longrightarrow W$ is a distance function satisfying the following axioms. Let $x, y \in \mathcal{C}$ and $w = \delta(x, y)$. Then

(B1) $w = 1$ if and only if $x = y$;

(B2) if $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws; furthermore if $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$; and

(B3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

In this survey we will concentrate (unlike Mühlherr) on finite buildings, in which case the diagram is spherical, but a number of results that we state also apply to the non-finite case.

Given a building $\mathcal{B} = (\mathcal{C}, \delta)$ we can define a chamber system on the set of chambers $\mathcal{C}$ (we denote the chamber system by $\mathcal{C}$ as well) where two chambers c and d are i-adjacent if and only if $\delta(c,d) = s_i$. Conversely, the building $\mathcal{B}$ can be recovered from its chamber system $\mathcal{C}$. We will only consider those buildings $\mathcal{B}$ for which $\mathcal{C}$ is thick. If $\mathcal{B}$ is a building, its chamber system contains a class of thin subsystems called *apartments*. In an apartment Σ, for any $c \in \Sigma$ and $w \in W$, there is a unique chamber $d \in \Sigma$ such that $\delta(c,d) = w$. Every pair of chambers of $\mathcal{C}$ is contained in an apartment. Notice that the chamber system $\mathcal{C}$ defined by a building is always geometric. Let $\Gamma = \Gamma(\mathcal{B})$ be the corresponding geometry.

It is well known that Γ is simply connected. Unfortunately, we cannot use this to prove the Curtis-Tits theorem because it corresponds to the wrong amalgam. So we need to find a different geometry.

Given two buildings $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$, $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ of the same type M, a *codistance* (*twinning*) is a map $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \longrightarrow W$ such that the following axioms hold where $\epsilon = \pm$, $x \in \mathcal{C}_\epsilon, y \in \mathcal{C}_{-\epsilon}$ and $w = \delta_*(x,y)$:

(T1) $\delta_*(y,x) = w^{-1}$;

(T2) if $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y,z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x,z) = ws$; and

(T3) if $s \in S$, there exists $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y,z) = s \in S$ and $\delta_*(x,z) = ws$.

A *twin building* of type M is a triple $(\mathcal{B}_+, \mathcal{B}_-, \delta_*)$, where $\mathcal{B}_+$ and $\mathcal{B}_-$ are buildings of type M and δ_* is twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$.

Tits showed (cf. Proposition 1 of [T2]) that every spherical twin building can be obtained as follows from some building $\mathcal{B} = (\mathcal{C}, \delta)$ of the same type M. Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ be a copy of $\mathcal{B}$, define $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ as $(\mathcal{C}, w_0 \delta w_0)$, and let δ_* be defined as $w_0\delta$ and δw_0 on $\mathcal{C}_+ \times \mathcal{C}_-$ and $\mathcal{C}_- \times \mathcal{C}_+$ respectively. Here w_0 is the longest element of the Weyl group W.

Given a twin building $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$, one can define a chamber system $\mathrm{Opp}(\mathcal{T}) = \{(c_+, c_-) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(c_+, c_-) = 1_W\}$. Chambers $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$ with $\delta_*(x,y) = 1_W$ are called *opposite*, hence the notation. Note that $\mathrm{Opp}(\mathcal{T})$ is a geometric chamber system. Its corresponding geometry is denoted by Γ_{op} and is called the *opposites geometry*. It can be described as follows. Let Γ_+ and Γ_- be the building geometries that correspond to $\mathcal{B}_+$ and $\mathcal{B}_-$. Elements $x_+ \in \Gamma_+$ and $x_- \in \Gamma_-$ of the same type $i \in I$ are called *opposite* if they are contained in opposite maximal flags (*i.e.*, chambers). The elements of Γ_{op} of type i are pairs (x_+, x_-) of opposite elements of type i. Two pairs (x_+, x_-) and (x'_+, x'_-) are incident in Γ_{op} if both x_+ and x'_+ are incident in Γ_+ and x_- and x'_- are incident in Γ_-. Clearly, a pair $(c_+, c_-) \in \mathrm{Opp}(\mathcal{T})$ produces a maximal flag in Γ_{op}, and it can be shown that every maximal flag is obtained in this way.

We now give some examples.

Example 1a. Let $G \cong PSL(n+1,q)$, *i.e.*, M is of type A_n. Then the building geometry Γ is the projective space, whose elements of type i, $1 \le i \le n$, are all the i-dimensional subspaces in the corresponding $(n+1)$-dimesional vector space V. The geometries Γ_+ and Γ_- are isomorphic respectively to Γ and the dual geometry of Γ (same as Γ except that the type of the i-dimensional subspace is $n+1-i$). Elements (subspaces) $x_+ \in \Gamma_+$ and $x_- \in \Gamma_-$ of type i are opposite if they intersect trivially and thus form a direct sum decomposition $V = x_+ \oplus x_-$. It follows that these decompositions are the elements of Γ_{op}.

Example 2a. Let $G \cong PSp(2n,q)$, which corresponds to the diagram C_n. Then Γ is the geometry of all totally isotropic subspaces of a nondegenerate $2n$-dimensional symplectic space V. In this case, both Γ_+ and Γ_- are isomorphic to Γ. Two i-dimensional totally isotropic subspaces x_+ and x_- are opposite if x_- intersects trivially with the orthogonal complement of x_+. Such pairs (x_+, x_-) are the elements of Γ_{op}.

In general, if the twin building consists of two isomorphic parts $\mathcal{B}_+ \cong \mathcal{B} \cong \mathcal{B}_-$, which is the case for a spherical diagram, the automorphism group $\mathrm{Aut}\,(\mathcal{B})$ of the building acts on the twin building $\mathcal{T}$ by automorphisms, in particular, it preserves the opposition relation, and hence it also acts on Γ_{op}. It can be shown that the action of $\mathrm{Aut}\,(\mathcal{B})$ on the set of pairs of opposite chambers is transitive, thus it is flag-transitive on Γ_{op}. The stabilizers of the elements of a maximal flag of Γ_{op} are Levi factors in the maximal parabolic subgroups (in the sense of Chevalley groups) of G. The Levi factors differ from the members of the amalgam of Theorem 4.2 only by the Cartan subgroup. To be precise, the full Levi factors are the products of the subgroups $G_{\Pi\setminus\{\alpha\}}$ with the Cartan subgroup H. This is not a major impediment as the Cartan subgroup can be recovered piecewise from the initial amalgam $\mathcal{A}$. Therefore the Curtis-Tits theorem is equivalent to the following:

Theorem 4.3. *If $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ is a spherical twin building of rank at least three, then the geometry Γ_{op} is simply connected.*

This was proved by Mühlherr in [M] for twin buildings with arbitrary (that is, not only spherical) Coxeter matrix M. His proof is case-independent, short and elegant. The claim is derived directly from the axioms of twin buildings, properties of apartments in buildings, and certain connectivity properties of buildings. However his proof does not cover a number of exceptional (small field) cases where the connectivity fails. In particular, in the spherical case, the groups $G \cong Sp(2n,2)$ and $F_4(2)$ are not covered by his proof. In the nonspherical case Mühlherr has to exclude tree residues and rank 2 residues related to the buildings of type $B_2(2)$, ${}^2F_4(2)$, $G_2(2)$, and $G_2(3)$. Mühlherr remarks that in the nonspherical situation there appear to be counterexamples. Hence a general proof for all M may not be possible. In the spherical case we know by the original Curtis-Tits proof that there are no counterexamples. Thus the following seems to be an interesting problem.

Problem 1. *Generalize Mühlherr's proof to cover all spherical matrices M.*

As we have already noticed, the Curtis-Tits theorem is not concerned with the question of the uniqueness of the amalgam $\mathcal{A} = \bigcup_{\alpha\in\Pi} G_{\Pi\setminus\{\alpha\}}$. In our opinion this makes applying the Curtis-Tits theorem more complicated. Indeed, in order to apply it one has to show that inside the group G under consideration there is an exact copy of the amalgam $\mathcal{A}$. Thus it would be advantageous to strengthen the Curtis-Tits theorem by solving the following problem.

Problem 2. *Prove that any non-collapsing amalgam of groups isomorphic to $G_{\Pi\setminus\{\alpha\}}$ (with given isomorphism types of their intersections) is in fact isomorphic to $\mathcal{A}$.*

5 Flipflop geometries

We will start with an example.

Example 1b. Consider the situation of Example 1a, but change the field of definition to $GF(q^2)$, so that $G \cong PSL(n+1, q^2)$. Consider a unitary polarity σ, that is, an involutory isomorphism from Γ onto the dual of Γ (recall that these geometries have the same set of elements but different type functions) which is defined by a nondegenerate Hermitian form Φ on V. That is, σ sends every subspace of V to its orthogonal complement with respect to Φ. This σ produces an involutory automorphism of the twin building $\mathcal{T}$ that switches $\mathcal{C}_+$ and $\mathcal{C}_-$ (or else, Γ_+ and Γ_-). It is an automorphism in the sense that it transforms δ_+ into δ_- and *vice versa*, and preserves δ_*. Corresponding σ, there is an automorphism of G, which we will also denote σ. Consider $G_\sigma = C_G(\sigma)$ and $\Gamma_\sigma = \{(x_+, x_-) \in \Gamma_{op} \mid x_+^\sigma = x_-\}$. Then $G_\sigma \cong PSU(n+1, q^2)$ acts on Γ_σ. Notice that the elements of Γ_σ are of the form (x_+, x_-) where $x_- = x_+^\sigma = x_+^\perp$ and $V = x_+ \oplus x_- = x_+ \oplus x_+^\perp$. Thus, the mapping $(x_+, x_-) \mapsto x_+$ establishes an isomorphism between Γ_σ and the geometry of all proper nondegenerate subspaces of the unitary space V, as defined by Φ. This is exactly the geometry from Section 3 that was used for a new proof of Phan's first theorem.

This suggests the following general construction. Let $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin buiding. Consider an involutory automorphism σ of $\mathcal{T}$ with the following properties:

(F1) $\mathcal{C}_+^\sigma = \mathcal{C}_-$;

(F2) σ flips the distances, *i.e.*, $\delta_\epsilon(x, y) = \delta_{-\epsilon}(x^\sigma, y^\sigma)$ for $\epsilon = \pm$; and

(F3) σ preserves the codistance, *i.e.*, $\delta_*(x, y) = \delta_*(x^\sigma, y^\sigma)$.

We will additionally require that there be at least one chamber $c \in \mathcal{C}_\pm$ such that $\delta_*(c, c^\sigma) = 1_W$. Such σ's will be called *flips*.

Construct $\mathcal{C}_\sigma$ as the chamber system whose chambers are pairs (c, c^σ) that belong to $Opp(\mathcal{T})$. Note that by our assumption $\mathcal{C}_\sigma$ is non-empty. We do not know if $\mathcal{C}_\sigma$ is geometric in general, however this is the case in each of our examples except Example 5 (which we did not check but believe to be geometric). If $\mathcal{C}_\sigma$ is geometric, let Γ_σ denote the corresponding geometry. It will be referred to as the *flipflop geometry.*

In case of a spherical twin building, we can compute the action of σ on the Coxeter diagram of the building, as has been done in Section 3.3 of [Gr]. Indeed, using Tits' characterization of spherical twin buildings (Proposition 1 of [T2]), we have $\delta(c,d) = \delta_+(c,d) = \delta_-(c^\sigma, d^\sigma) = w_0\delta(c^\sigma, d^\sigma)w_0$. Therefore, the flip σ acts on the Coxeter diagram via conjugation with the longest word w_0 of the Weyl group. This gives the following characterization of a flip of a spherical twin building.

Proposition 5.1. *Let $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a spherical twin building. An adjacency-preserving involution σ that interchanges $\mathcal{B}_+$ and $\mathcal{B}_-$ and maps some chamber onto an opposite chamber is a flip if and only if the induced map $\hat{\sigma}$ on the building $\mathcal{B} = (\mathcal{C}, \delta)$ satisfies $\delta(c,d) = w_0\delta(c^{\hat{\sigma}}, d^{\hat{\sigma}})w_0$ for all chambers $c, d \in \mathcal{C}$ where w_0 is the longest word in the Weyl group W.*

Here are some additional examples.

Example 2b. Consider the situation of Example 2a, but with the field of definition of order q^2. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a hyperbolic basis of the symplectic space V. (So that $(e_i, f_j) = \delta_{ij}$.) Consider the semilinear transformation σ of V which is the composition of the linear transformation given by the Gram matrix of the form and the involutory field automorphism applied to the coordinates with respect to the above basis. It can be shown that σ produces a flip of $\mathcal{T}$. Furthermore, $\mathcal{C}_\sigma$ is geometric and $G_\sigma \cong PSp(2n,q)$ acts flag-transitively on the corresponding flipflop geometry Γ_σ. The geometry Γ_σ can be described as follows. For $u, v \in V$ let $((u,v)) = (u, v^\sigma)$, where $(\cdot,\cdot)$ is the symplectic form on V. Then $((\cdot,\cdot))$ is a nondegenerate Hermitian form. The flipflop geometry Γ_σ can be identified (via $(x_+, x_-) \mapsto x_+$) with the geometry of all subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ and, at the same time, nondegenerate with respect to $((\cdot,\cdot))$.

The configuration of Example 2b was looked at in [GHSh]. It is proved there that Γ_σ is almost always simply connected. Here is the main theorem from that paper.

Theorem 5.2. *The flipflop geometry Γ_σ described in Example 2b is simply connected if $n \geq 5$ or $n = 4$, $q \geq 3$ or $n = 3$, $q \geq 8$.*

We expect that some of the larger q's on this list of exceptions are there only because of the shortcomings of our particular proof, so that the final list of exceptions will be shorter.

The above theorem leads to a new "Phan-type" result on groups generated by subgroups $U_i \cong SU(2,q^2)$. Here we have that $\langle U_i, U_{i+1}\rangle \cong SU(3,q^2)$ for all $1 \leq i < n-1$, while $\langle U_{n-1}, U_n\rangle \cong Sp(4,q)$. As in Phan's original situation U_i and U_j with $|i-j| > 1$ commute elementwise. An amalgam of subgroups as indicated here is called a *Phan system of type C_n*. For the exact statements and other applications, see [GHSh]. We have to point out that the uniqueness of amalgams is not addressed in [GHSh] leaving the following an open problem.

Problem 3. *If q is sufficiently large prove that any non-collapsing Phan system of type C_n is in fact isomorphic to the canonical Phan system inside the group $Sp(2n, q)$.*

We expect that this problem can be solved by using the same methods as given in [BSh]. Consequently, for small q one first has to introduce the notion of a weak Phan system of type C_n as in Section 3 and then study unambiguous, non-collapsing weak Phan systems.

Example 3. For $G = PSO(2n, q^2, +)$ and $PSO(2n+1, q^2)$ (diagrams D_n and B_n, respectively) flips can be constructed by the same algorithm as in Example 2b, that is, σ can be defined as the composition of the linear transformation given by the Gram matrix, say, taken with respect to a hyperbolic basis (the actual requirement is that all entries of the Gram matrix must be in the subfield $GF(q)$) and the involutory field automorphism with respect to the same basis. In both cases we checked that this σ produces a flipflop geometry on which G_σ acts flag-transitively. While we have not obtained an exact result on the simple connectivity of Γ_σ, it is clear that Γ_σ is simply connected for all sufficiently large n and q, leading to new "Phan-type" theorems, cf. [BGHSh]. Notice that the D_n case here is likely to lead to Theorem 1.9 from Phan's second paper [Ph2]. This conjecture is underscored by our above observation (before Proposition 5.1) that a flip acts via conjugation with the longest word of the Weyl group on the diagram D_n. Indeed, for n even, Phan's target group is $Spin^+(q)$ (the universal Chevalley group of type $D_n(q)$) and conjugation with the longest word leaves the diagram invariant, while for n odd, Phan's target group is $Spin^-(q)$ (the universal Chevalley group of type ${}^2D_n(q^2)$) and conjugation with the longest word interchanges the two nodes representing the two classes of maximal totally singular subspaces. Another flip is induced by the linear transformation given by the Gram matrix with respect to a hyperbolic basis alone, without applying the involutory field automorphism.

Example 4. Now consider the group $G = PSO(2n, q, -)$ acting on the flag complex $\mathcal{C}$ of totally singular subspaces of a nondegenerate orthogonal form of $-$ type on the vector space V of dimension $2n$ over $GF(q)$. Choose two opposite chambers c and d of that flag complex and let $U := \langle c, d\rangle^\perp$ be the subspace of V that is perpendicular to both c and d. Fix a hyperbolic basis

$$\{e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}\}$$

of the vector space $c \oplus d$ such that $c = (\langle e_1\rangle, \ldots, \langle e_1, \ldots, e_{n-1}\rangle)$ and $d = (\langle f_1\rangle, \ldots, \langle f_1, \ldots, f_{n-1}\rangle)$ and, moreover, fix some orthogonal basis of U. Then there exists a linear map on V that preserves the form, maps e_i onto f_i and vice versa, and acts by scalar multiplication on each of the vectors of the orthogonal basis of U, e.g., the Gram matrix of the form with respect to the given basis. This linear map induces a flip σ of the twin building belonging to the flag complex $\mathcal{C}$. Notice, unlike Example 3, that we cannot compose this flip σ with an involutory field automorphism that acts entrywise on the vectors with respect to the given basis in order to obtain another flip, because this field automorphism would not commute with σ.

Example 5. Let G be the universal Chevalley group of type $E_6(q^2)$ and consider its 27-dimensional module V, a vector space over $GF(q^2)$. For sake of simplicity let us assume that q is not divisible by two. A vector $x \in V$ is represented by the triple $(x^{(1)}, x^{(2)}, x^{(3)})$ where $x^{(i)}$, $1 \leq i \leq 3$, is a (3×3)-matrix over $GF(q^2)$. The shadow space $E_{6,1}(q^2)$ can be described as the geometry on certain subspaces of V, cf. Section 5.2 of Cohen's Chapter 12 of [Bu]. There exists a nondegenerate bilinear form $(\cdot, \cdot)$ on V defined by

$$(x, y) = \operatorname{trace}(x^{(1)}y^{(1)} + x^{(2)}y^{(3)} + x^{(3)}y^{(2)}).$$

Define $g^\sharp \in \mathrm{GL}(V)$ by $(gx, g^\sharp y) = (x, y)$ for all $x, y \in V$. The map $\sharp : \mathrm{GL}(V) \to \mathrm{GL}(V) : g \mapsto g^\sharp$ induces an involutory automorphism α of the group G. This automorphism α in turn induces a correlation β of the geometry $E_{6,1}(q^2)$, *i.e.*, an incidence-preserving permutation of $E_{6,1}(q^2)$ that does not necessarily preserve types. In fact, β induces the involutory graph automorphism on the Coxeter diagram E_6. The composition of β and the involutory field automorphism acting entrywise on the representation $(x^{(1)}, x^{(2)}, x^{(3)})$ of any vector $x \in V$ induces a map σ on the corresponding twin building $\mathcal{T}$ that satisfies the axioms of a flip except that we did not check whether there exists a chamber that is mapped to an opposite chamber. We do, however, strongly believe that such a chamber exists. This observation is underscored by the fact that the centralizer in G of the composition of α and the involutory field automorphism equals ${}^2E_6(q^2)$ and, thus, the present setting is likely to lead to an alternative proof of Phan's Theorem 2.6 of [Ph2]. The correlation β can be expected to induce a flip as well.

We do not have a concrete example of a flip for an F_4 twin building, but we will discuss a general method for finding flips in the case where conjugation with the longest word of the Weyl group acts trivially on the diagram, which, for example, applies in the F_4 case. As a concrete example, one would hope to find a flip that centralizes the group $F_4(q)$ inside the group $F_4(q^2)$; the resulting flipflop geometry should admit the flipflop geometry of type B_3 from [BGHSh] and the flipflop geometry of type C_3 from [GHSh] as residues.

Let $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building. Define the *automorphism group* $\mathrm{Aut}(\mathcal{T})$ to be the set of all permutations α of $\mathcal{T}$ with

- $\delta_\epsilon(c, d) = \delta_\epsilon(c^\alpha, d^\alpha)$ for all $c, d \in \mathcal{C}_\epsilon$ if α preserves $\mathcal{C}_+$ and $\mathcal{C}_-$,
- $\delta_\epsilon(c, d) = \delta_{-\epsilon}(c^\alpha, d^\alpha)$ for all $c, d \in \mathcal{C}_\epsilon$ if α interchanges $\mathcal{C}_+$ and $\mathcal{C}_-$, and
- $\delta_*(c, d) = \delta_*(c^\alpha, d^\alpha)$ for all $c \in \mathcal{C}_\epsilon$, $d \in \mathcal{C}_{-\epsilon}$,

where $\epsilon = \pm$. Clearly, if $\alpha, \beta \in \mathrm{Aut}(\mathcal{T})$ both interchange $\mathcal{C}_+$ and $\mathcal{C}_-$ then their product $\alpha\beta$ preserves $\mathcal{C}_+$ and $\mathcal{C}_-$. So, $\mathrm{Aut}(\mathcal{T})$ is of the form $\mathrm{Aut}(\mathcal{B}).2$. If there exists a flip or any other distance-switching and codistance-preserving involution of $\mathcal{T}$, then $\mathrm{Aut}(\mathcal{T})$ even is a semidirect product.

Now suppose we have a spherical twin building with a Coxeter diagram such that conjugation with the longest word w_0 acts as the trivial automorphism on the diagram. Then the map τ assigning to each chamber c of $\mathcal{C}_\pm$ the unique chamber d of $\mathcal{C}_\mp$ with $\delta_*(c,d) = w_0$ (called the *closest* chamber to c) is contained in $\mathrm{Aut}\,(\mathcal{T})$. Moreover, τ commutes with any automorphism of $\mathcal{T}$ that preserves $\mathcal{C}_+$ and $\mathcal{C}_-$, so $\mathrm{Aut}\,(\mathcal{T})$ is even a direct product. This implies the following.

Proposition 5.3. *Let $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a spherical twin building such that conjugation with the longest word w_0 of the Weyl group acts trivially on its Coxeter diagram. Then $\mathrm{Aut}\,(\mathcal{T}) = \mathrm{Aut}\,(\mathcal{B}) \times \langle \tau \rangle$, where τ is the automorphism assigning to each chamber $c \in \mathcal{C}_\pm$ the unique closest chamber $d \in \mathcal{C}_\mp$. Moreover, any flip of $\mathcal{T}$ is the product $\alpha\tau$ for an involutory $\alpha \in \mathrm{Aut}\,(\mathcal{B})$ such that there exists a chamber $c \in \mathcal{C}$ with $\delta(c, c^\alpha) = w_0$. Conversely, every such $\alpha\tau$ is a flip.*

This partial result motivates the following problem.

Problem 4. *Classify all flips for all spherical twin buildings. For each flip investigate Γ_σ and its simple connectivity.*

Of course, it would be much nicer to have general building-theoretic arguments (Mühlherr's type) in place of a case-by-case analysis. In particular, this concerns showing that $\mathcal{C}_\sigma$ is always geometric.

Besides the spherical case the investigation of flips might be interesting for the nonspherical case as well.

Problem 5. *Find an interesting flip for a nonspherical twin building.*

A flip might be considered interesting if it either centralizes or flips an interesting geometry or if it has an interesting centralizer. Also, Mühlherr's proof of the Curtis-Tits theorem has established a Curtis-Tits-type theorem for certain Kac-Moody groups. It might be worth the effort to investigate whether interesting Phan-type theorems can be proved for Kac-Moody groups as well. A starting point for the search of flips of nonspherial twin buildings might be [B] on diagram automorphisms induced by certain root reflections.

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